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Realizing Smiles: Options Pricing with Realized Volatility

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Abstract

We develop a discrete-time stochastic volatility option pricing model, which exploits the information contained in high-frequency data. The Realized Volatility (RV) is used as a proxy of the unobservable log-returns volatility. We model its dynamics by a simple but effective long-memory process: The Leverage Heterogeneous Auto-Regressive Gamma (HARGL) process. The discrete-time specification and the use of the RV allows us to easily estimate the model using observed historical data. Assuming a standard exponentially affine stochastic discount factor, we obtain a fully analytic change of measure. An extensive empirical analysis of S&P 500 index options illustrates that our approach significantly outperforms competing time-varying (i.e. GARCH-type) and stochastic volatility pricing models. The pricing improvement can be ascribed to: (i) The direct use of the RV, which provides a precise and fast-adapting measure of the unobserved underlying volatility; (ii) the specification of our model, which is able to accurately reproduce the implied volatility term structure.

Keywords: High-frequency, Realized Volatility, Option Pricing.

JEL Classification: C13, G12, G13

1 Introduction

Financial literature devoted large amount of theoretical effort to modeling the dynamics of financial assets returns and volatility under both the historical or statistical measure $\mathbb{P}$ (used for data description and forecasting) and the risk-neutral or martingale measure $\mathbb{Q}$ (applied for derivatives pricing). Two main classes of alternative parametric approaches have been developed to tackle the problem: Discrete-time models and continuous-time models. Each approach has its own advantages and disadvantages under the two measures. On the one hand, discrete-time models are usually more intuitive and easier to estimate, but they lead to challenging pricing formulas. On the other hand, continuous-time models entail more difficult estimation procedures, but they yield very convenient pricing formulas, especially in the affine case (see Duffie et al. (2000)).

Due primarily to mathematical tractability, the literature focusing on option pricing (modeling asset prices under the risk-neutral measure $\mathbb{Q}$) has been dominated traditionally by continuous-time processes. Stochastic and time-varying volatility models, such as those is Heston (1993), Duan (1995) and Heston and Nandi (2000), represent a first attempt to model the implied volatility (IV) curve. While those models are able to qualitatively reproduce the smile (i.e. excess kurtosis) and the smirk (i.e. negative skewness) observed in short term options, they fail to address these features quantitatively. As a result, they severely underprice out-of-the-money put options. To cope with this problem, a variety of models have been developed to include jumps in returns (see Merton (1976), Bates (1996), Bates (2000), Pan (2002), Huang (2004) and Bates (2006) in continuous-time and Maheu and McCurdy (2004), Duan et al. (2006) and Christoffersen et al. (2010b) in discrete-time) and jumps in volatility (see Eraker (2004), Eraker et al. (2003) and Broadie et al. (2007)). These models produce an IV surface which closely resembles the empirical one, especially for short maturities, but the surface tends to flatten out too rapidly for options having longer maturities. This is because the main ingredient of continuous-time stochastic volatility models is the Cox et al. (1985) (CIR) process, employed to describe the volatility dynamics. The drawback of the CIR is related to its exponential memory decay. As a result, the CIR is able neither to preserve the skewness nor to generate the kurtosis needed for pricing long maturity options. In spite of this disadvantage, the CIR model has the clear advantage of being an affine process, thus leading to semi-closed option pricing formulas (i.e. obtained through standard Fourier methods). So far, only marginal attention has been devoted to the long-term part of the IV surface, where the persistence of the volatility process plays a crucial role. Two exceptions are Comte et al. (2003) who employ a fractional stochastic volatility model and Carr and Wu (2004) who apply alpha-stable processes in order to slow-down the central limit theorem and to obtain negative skewness and excess kurtosis for long-maturity options.

On the other hand, models for the asset dynamics under the physical measure $\mathbb{P}$ have been primarily developed in discrete-time. The time-varying volatility models of the ARCH-GARCH families (Engle (1982), Bollerslev (1996), Glosten et al. (1993) and Nelson (1991)) have led the field and they partially explain the volatility dynamics. For GARCH-type models, negative skewness and excess of kurtosis can be induced by the well-known leverage effect, by using non-normal innovations (see, e.g. Christoffersen et al. (2006), Duan (1999) and Christoffersen et al. (2010b)), or by the inclusion of jumps (Christoffersen et al. (2008a)). Differently from the approaches under the martingale measure, a wide spectrum of models have been developed to reproduce the strong volatility persistence. Among those, the FIGARCH of Baillie et al. (1996) and the component GARCH of Engle and Lee (1999) provide good descriptions of the highly persistent volatility dynamics observed in the data. This feature has considerable implications for option pricing, as documented by Bollerslev and Mikkelsen (1996).

Moreover, a growing strand of literature advocates the presence of a multi-factors volatility structure. For instance, Li and Zhang (2010), using non-linear principal components analysis, find that two factors are needed to explain the variation in the IV surface, while Christoffersen et al. (2008b) employ a modified version of the two-factors component GARCH of Engle and Lee (1999) for options pricing. In addition, Adrian and Rosenberg (2007) show that a multi-component volatility model substantially improves the cross-sectional pricing of volatility risk.

All of these models treat the volatility as an hidden persistent component. Both important methodological simplifications and more flexible models can be obtained by using an observable proxy for the hidden volatility process. Nowadays, the availability of intra-day data has facilitated the use of the so called Realized Volatility (RV) for measuring (i.e., making observable) and forecasting volatility. The RV literature has grown and attracted significant attention in recent years. The key advantage of the RV is that it provides a precise nonparametric measure of daily volatility² which leads to simplicity in model estimation and superior forecasting performance. Andersen, Bollerslev, Diebold and Labys (2001), (2003) and Andersen, Bollerslev, Diebold and Ebens (2001) highlight strong persistence also in the RV dynamics.

Surprisingly, to the best of our knowledge, little work has been devoted to combining RV literature with that on option pricing to construct *RV option pricing models*. Notable exceptions are the work of Stentoft

²The idea of RV measures goes back to the seminal work of Merton (1980), which shows that the integrated variance of a Brownian motion can be approximated by the sum of a large number of intra-day squared returns. This aspect allows us to exploit all the information contained in intra-day high-frequency data. This original intuition has been recently formalized and generalized in a series of papers that apply the quadratic variation theory to the class of L_2 semi-martingales; See, e.g., Andersen, Bollerslev, Diebold and Labys (2001), (2003) and Barndorff-Nielsen and Shephard (2001), (2002a), (2002b), (2005), and Comte and Renault (1998) .

(2008) and Christoffersen et al. (2010a).³ In Stentoft (2008) an Inverse Gaussian model of a 30-minute returns RV measure is used to price options on some individual stocks. However, this work does not provide a formal change of measure for the RV process but instead only considers the case when the risk neutral and physical dynamics of RV are the same (i.e. when the volatility risk is not priced). In a concurrent paper, Christoffersen et al. (2010a) generalize the GARCH option pricing approach by extending the Heston and Nandi (2000) GARCH model to include RV measures. However, they mainly focus on the RV contribution to short-medium term option pricing.

The aim of this paper is to use the RV as a proxy for the unobservable underlying volatility and to cast its dynamics into a model that is able to: (i) Reproduce the persistence observed in the data ($\mathbb{P}$ -dynamics), (ii) exploit the superior forecasting performance of high-frequency volatility estimates, and (iii) develop an explicit change of measure which preserves those features under $\mathbb{Q}$ for option pricing purposes. We propose to model the conditional mean of volatility by the Heterogeneous Auto-Regressive (HAR) multi-components model (see Corsi (2009)). The HAR specification can be considered an acceptable compromise between parameter parsimony and multi-factors specification. Despite the fact that the HAR model does not formally belong to the class of long memory processes, it is able to produce the same memory persistence observed in financial data. The HAR model provides only the first conditional moment of the RV. In order to specify the whole transition density and complete the probabilistic description of the RV process, we assume that the conditional distribution of the HAR is a non-central gamma. The resulting model belongs to the family of Auto-Regressive Gamma (ARG) processes, a class of discrete-time affine processes introduced by Gouriéroux and Jasiak (2006). Due to the combination of HAR and ARG processes, we label our new model the Heterogeneous Auto-Regressive Gamma (HARG). The HARG features both long-memory and an affine structure. The latter is particularly attractive for option pricing since, as with many affine processes, the HARG has a fully analytic Laplace transform; we thus derive semi-closed-form option pricing formulas in the paper. Moreover, in order to capture the asymmetric shape of the IV smile for S&P 500 options, we introduce a version of the HARG which includes the leverage effect. We label this the HARGL process. The HARGL is extremely flexible for option pricing purposes, capturing the asymmetric shape of the IV surface, but it loses the affine structure; option prices are then computed through Monte Carlo simulation which is straightforward for the HARGL model.

In addition to the more flexible model specification, the application of RV as a proxy for the unobservable volatility simplifies the model estimation. Filtering procedures are no longer required, and the HARGL model

³In a different direction, Bandi et al. (2008) use option prices to construct a metrics for evaluating the forecasting performance of different high-frequency volatility estimators.

can be directly estimated on the RV constructed from the available high-frequency data. Both the rich and fast-changing dynamics inherent in the RV measures and the simple long-memory structure of the proposed model contribute to improvements in pricing performance. These features allow our model to rapidly adapt to market changes and, at the same time, retain the ability to reproduce a realistic IV term structure under different market conditions (e.g., under different volatility regimes).

In terms of Root Mean Square Error on IV ($RMSE_{IV}$), the overall improvement of the HARGL over the Heston and Nandi (2000) GARCH and the Component GARCH of Christoffersen et al. (2008b) is approximately 24% and 10%, respectively. The results stem from a more accurate pricing of at-the-money (ATM) options, as reflected by even stronger results in terms of RMSE on prices ($RMSE_P$) (38% and 15% respectively). This is due, on the one hand, to the more flexible change of measure which allows our model to better match the level of the IV surface, removing any unconditional bias for ATM options, and, on the other hand, to the fast-adapting nature of the RV measure, which improves the pricing of short-term options. Considering the restricted specifications of the HARGL model, we are also able to disentangle the contribution of the different model ingredients to the overall pricing performance. Specifically, we isolate the importance of long-memory from the contribution of the RV measure: The long-memory structure of the HARGL leads to a 20% improvement in $RMSE_P$ with respect to the discretized version of the widely used CIR model (i.e. the ARG model).

The paper is organized as follows: Section 2 defines the HARGL model for log-returns and RV, under both the historical and risk neutral probability measures. Section 3 describes how to estimate the model and analyzes its dynamic features. In Section 4 we then present the HARGL option pricing performances and compare them with several option pricing benchmark models widely applied in the financial literature. Section 5 summarizes the results.

2 The model

2.1 Dynamics under Physical Probability

2.1.1 Log-returns dynamics

A well-established result in financial econometrics literature is that the marginal distribution of daily log-returns is not Gaussian but typically features fat tails (leptokurtic distribution). This fact has motivated the use of heavy-tailed distributions in several financial models. In spite of this consideration, Clark (1973) and Ane and Geman (2000) theoretically argue that, for an underlying continuous-time diffusion process, the standard Gaussian distribution can be recovered rescaling the log-returns by an appropriate measure of

the market activity. The basic intuition is that the log-returns process is a Brownian motion with a random time. Re-scaling the log-returns by an appropriate activity measure is equivalent to performing a time-change that restores the standard Brownian motion in calendar time. As such a measure of market activity, we propose to use the integrated variance ($\mathcal{IV}$). Then, we assume the following conditional dynamics for the log-returns:

$$\ln\left(\frac{S_{t+1}}{S_t}\right) := y_{t+1} = \mu_{t+1} + \sqrt{\mathcal{IV}_{t+1}}\epsilon_{t+1}, \quad (1)$$

$$\epsilon_{t+1}|\mathcal{IV}_{t+1} \sim N(0, 1).$$

In our notation, S_{t+1} , y_{t+1} , and $\mathcal{IV}_{t+1}$ are the price, the log-returns, and the integrated variance at time $t + 1$, respectively. For the drift of the log-returns under the physical measure, we propose the following specification:

$$\mu_{t+1} = r + \gamma\mathcal{IV}_{t+1}, \quad (2)$$

where r represents the risk-free rate minus the dividend yield and $\gamma := \tilde{\gamma} - 1/2$. The term $-1/2$ in γ is a convexity adjustment introduced such that the conditional expectation of returns becomes $\mathbb{E}[\exp y_{t+1}|\mathcal{IV}_{t+1}] = \exp(r + \tilde{\gamma}\mathcal{IV}_{t+1})$ and $\tilde{\gamma}$ can be interpreted as the price risk for volatility. We observe that our specification introduces a contemporaneous effect of $\mathcal{IV}_{t+1}$ on y_{t+1} . Specifically, the functional form we are proposing implies a stochastic drift, changing with the daily $\mathcal{IV}$. This feature has an interesting probabilistic implication, since our model can be embedded into the class of normal variance-mean mixture as in Barndorff-Nielsen et al. (1982), with $y_{t+1}|\mathcal{IV}_{t+1} \sim N\left(r + \left(\tilde{\gamma} - \frac{1}{2}\right)\mathcal{IV}_{t+1}, \mathcal{IV}_{t+1}\right)$.

As is customary in the RV literature, we estimate the unobservable $\mathcal{IV}$ by the corresponding continuous component of daily RV (details on the RV measure employed in the implementation of the model are given in Section 3). This choice has an empirical justification. Andersen, Bollerslev, Diebold and Labys (2000), (2001), (2003), Andersen, Bollerslev, Diebold and Ebens (2001), and Andersen et al. (2007), indeed showed that, when daily returns are standardized by the corresponding daily RV, the resulting distribution is nearly Gaussian. Also, for our S&P 500 data, the marginal distribution of log-returns standardized by the RV becomes quite close to Gaussian. This feature can be clearly seen from the density plots of Figure 1. Besides the graphical evidence, we also note that the values of the kurtosis are 7.32 and 3.11, for actual and for re-scaled log-returns, respectively.

[Figure 1 should be here]

2.1.2 Realized Volatility dynamics

Under the physical measure, the model is completed by specifying the dynamics of the RV process. To capture the well-documented feature of strong persistence in volatility, we follow Corsi (2009), and we model the conditional mean of the RV (given its past values) using the conditional expected value of an HAR (Heterogeneous AutoRegressive) process. The HAR model for RV is a multi-component volatility model specified as a sum of different volatility components defined over different time horizons. Specifically, the structure of the HAR model allows us to separate short-, medium-, and long-term volatility components. This feature has considerable option pricing implications as documented by Bollerslev and Mikkelsen (1996) and Comte et al. (2003). In addition, Adrian and Rosenberg (2007) show that a multi-component volatility model substantially improves the cross-sectional pricing of volatility risk. Finally, in order to take into account the asymmetry in the smile, we here extend the original HAR model by including a daily leverage effect.⁴

The HAR model specifies only the first conditional moment of RV. For option pricing purposes, we need the specification of the whole transition probability density or, equivalently, the specification of the conditional characteristic function. Therefore, we model RV as an Auto Regressive Gamma process (see Gouriéroux and Jasiak (2006)) with p -lags, i.e. ARG(p). In our model we set $p = 22$. Consequently, $RV_{t+1}|\mathcal{F}_t$ features a non-central Gamma distribution $\Gamma(\delta, \beta'(\mathbf{RV}_t, L_t), c)$ with shape and scale parameters δ and c , respectively, and location given by:

$$\beta'(\mathbf{RV}_t, L_t) = \beta_1 RV_t + \frac{\beta_2}{4} \left(\sum_{i=1}^4 RV_{t-i} \right) + \frac{\beta_3}{17} \left(\sum_{i=5}^{21} RV_{t-i} \right) + \beta_4 L_t, \quad (3)$$

where $\beta \in \mathbb{R}^4$ (row vector); $\mathbf{RV}_t = (RV_t, (1/4) \sum_{i=1}^4 RV_{t-i}, (1/17) \sum_{i=5}^{21} RV_{t-i})$ is a column vector in $\mathbb{R}^3$ and L_t represents the leverage effect $L_t = I_{(y_t < 0)} RV_t$ (where $I_{(y_t < 0)}$ takes value 1 if the log-return at date t is negative and takes value 0 otherwise).

Similar to Corsi (2009), the specification in Eq. (3) collects the lagged terms of an ARG(22) process in three different non-overlapping factors, defined as RV_t (short-term volatility factor), $RV_t^w := \sum_{i=1}^4 RV_{t-i}/4$ (medium-term volatility factor), and $RV_t^m := \sum_{i=5}^{21} RV_{t-i}/17$ (long-term volatility factor). Although different from the standard HAR parametrization, the parametrization in Eq. (3) does not imply any loss of information compared to the original one from Corsi (2009), since it relies only on a different rearrange-

⁴A different HAR model, featuring both heterogeneous leverage and jumps, has been recently proposed by Corsi and Renò (2009). We notice that the inclusion of jumps in volatility is not a crucial theoretical issue in our model setting, since the RV can be properly applied to model and estimate the jumps component. Modeling RV as a discrete-time process with jumps represents the first possible extension of our model.

ment of the terms. We thus term this model as the Heterogeneous Auto Regressive Gamma with Leverage (HARGL). Thanks to the strong analytical tractability of the HARGL specification, we can write down in closed-form the one-step-ahead Conditional Moment Generating Function (CMGF) under $\mathbb{P}$. In particular, from computations similar to those in Gagliardini et al. (2005), the CMGF of an HARGL process is:

$$\varphi_{ARG}^{\mathbb{P}}(\eta) := \mathbb{E}(\exp(-\eta RV_{t+1}) | (\mathbf{R}\mathbf{V}_t, L_t)) = \exp\left(-\frac{c\eta}{1+c\eta} (\beta'(\mathbf{R}\mathbf{V}_t, L_t) - \delta \ln(1+c\eta))\right), \quad (4)$$

where $\eta \in \mathbb{R}$ and $\beta'(\mathbf{R}\mathbf{V}_t, L_t)$ as in (3).

Setting $\beta_4 \equiv 0$, we preserve the long-memory property but lose the leverage effect. We label this model HARG. Furthermore, restricting β_2 and β_3 to zero, we obtain the simple ARG model as in Gouriou and Jasiak (2006). Both the last two restricted models belong to the class of affine processes. Thus, from Eq. (4), we can derive in closed-form the multi-step-ahead CMGF for the HARG/ARG processes. Analytical calculations are provided in the technical Appendix B. Due to the presence of the leverage effect, analogous calculations are not available for the multi-step-ahead CMGF of the HARGL⁵.

We remark that the need for the CMGF of RV under the physical measure is twofold. First, from the probabilistic point of view, the CMGF completely characterizes the distributional features of the RV (e.g. it uniquely defines its conditional moments and its transition density). Second, the CMGF of RV is important for option pricing purposes, as it is the necessary tool to describe the joint behavior of the log-returns process and RV needed in the change of probability measure.

2.1.3 Joint Conditional Moment Generating Function

Given the setup outlined in Sections 2.1.1 and 2.1.2, our model specification is:

$$\begin{aligned} y_{t+1} | RV_{t+1} &\sim N\left(r + \left(\tilde{\gamma} - \frac{1}{2}\right) RV_{t+1}, RV_{t+1}\right), \\ RV_{t+1} | \mathcal{F}_t &\sim \Gamma(\delta, \beta'(\mathbf{R}\mathbf{V}_t, L_t), c), \\ \beta'(\mathbf{R}\mathbf{V}_t, L_t) &= \beta_1 RV_t + \frac{\beta_2}{4} \left(\sum_{i=1}^4 RV_{t-i}\right) + \frac{\beta_3}{17} \left(\sum_{i=5}^{21} RV_{t-i}\right) + \beta_4 L_t. \end{aligned} \quad (5)$$

In order to complete the probabilistic description of the log-returns and RV dynamics, in this section we study the joint process $K'_{t+1} := (y_{t+1}, RV_{t+1})$. This is a bi-dimensional, real-valued process of log-returns and RV whose state space is $\mathbb{R} \times \mathbb{R}^+$. For the sake of notation, let $\mathcal{F}_t := \sigma(y_t, RV_t)$ indicate the σ -algebra containing the information about (y_t, RV_t) available at time t . Thanks to our model setup, we can easily obtain a closed-form expression for the CMGF of K'_{t+1} . This is an important result, since the joint CMGF provides with a complete characterization of the joint conditional (namely, given $\mathcal{F}_t$) transition probability

⁵The HARGL is, however, easy to simulate, as shown is Gouriou and Jasiak (2006).

that can be applied to derive an explicit one-to-one mapping between the parameters of (y_{t+1}, RV_{t+1}) under the measures $\mathbb{P}$ and $\mathbb{Q}$. Indeed, to capture the dynamics of K'_{t+1} under the martingale measure $\mathbb{Q}$, we simply need to estimate the parameters under $\mathbb{P}$ (using high-frequency historical data) and then apply the set of transformations described in Section 2.2. Proposition 1 defines the closed-form expression for the CMGF of K'_{t+1} .

Proposition 1. *If $RV_{t+1}|\mathcal{F}_t \sim \Gamma(\delta, \beta'(\mathbf{RV}_t, L_t), c)$, then the CMGF of $K'_{t+1} := (y_{t+1}, RV_{t+1}) \in \mathbb{R} \times \mathbb{R}^+$ in $\alpha' = (\alpha_1, \alpha_2)$ under the physical measure ($\mathbb{P}$) is, for $v \in \mathbb{R}$:*

$$\varphi_K^{\mathbb{P}}(v) = \exp(-b(v) - a(v)\beta(\mathbf{RV}_t, L_t)), \quad (6)$$

with $v = \alpha_2 + \gamma\alpha_1 - \alpha_1^2/2$ and the terms $b(v)$ and $a(v)$ given by

$$b(v) = \delta \ln(1 + cv) \quad (7)$$

and

$$a(v) = \frac{cv}{1 + cv}. \quad (8)$$

Proof. See Appendix A. □

2.2 Risk-neutral dynamics

The risk-neutral dynamics of both the log-returns and the RV process are obtained following the direct approach of Bertholon et al. (2008). We adopt a straightforward extension of standard discrete-time exponential affine SDF for the time $(t, t+1)$, as in Gagliardini et al. (2005) and Gouriéroux and Monfort (2007). Due to the long-memory dynamics of the RV described in Section 2.1.2, we introduce a SDF involving the $t+1$ log-returns and the three relevant lag components of the RV (short-, medium-, and long-term volatility components). More precisely, assuming $r_t = 0$ for computational convenience⁶, we specify the following SDF:

$$M_{t,t+1} = \exp(-\nu_0 - \nu_1 RV_{t+1} - \nu_2 RV_t^d - \nu_3 RV_t^w - \nu_4 RV_t^m - \nu_5 y_{t+1} - \nu_6 L_t). \quad (9)$$

Under specific restrictions on the vector $\nu \in \mathbb{R}^7$ which contains the elements $(\nu_0, \dots, \nu_6)$ (see Appendix A, proof A.2 for details), we state:

Proposition 2. *Under the model specifications in Eq.(5), the SDF in (9) is compatible with the no-arbitrage conditions, provided that suitable parameter-restrictions are satisfied. The parameter ν_1 remains a free parameter.*

⁶Clearly, all our results in the following propositions apply for $r_t \neq 0$, as well.

Proof. See Appendix A. □

The last Proposition shows that the SDF in (9) complies with the no-arbitrage conditions. Furthermore, we remark that Proposition 2 defines all the parameters in the SDF, but ν_1 . This is due to the fact that we are considering an incomplete market, and it is a crucial point of our model. In the estimation of the ν vector, ν_1 is the unique parameter to be calibrated. All the other parameters can be explicitly computed in closed-form, once ν_1 has been calibrated ⁷.

Moreover, thanks to the SDF specification in Eq. (9), it is possible to write down in closed-form the dynamics of RV under the martingale measure. To this end, we rely on the results in Proposition 1 and specify the CMGF under $\mathbb{Q}$ of the joint process K_{t+1} . We then have:

Proposition 3. *Under the R.N. probability measure $\mathbb{Q}$, the RV is still a HARGL, having parameters*

$$\beta^* = \frac{\beta}{(1 + c\lambda)^2}, \quad (10)$$

$$\delta^* = \delta, \quad (11)$$

$$c^* = \frac{c}{1 + c\lambda}, \quad (12)$$

with $\lambda = \nu_1 + \frac{\gamma^2}{2} - \frac{1}{8}$.

Proof. See Appendix A. □

The last Proposition provides in Eq. (10)-(12) the explicit formulas for the one-to-one mapping between the parameters of the RV under $\mathbb{Q}$ and $\mathbb{P}$. The availability of such formulas is a consequence of the affine specification of the SDF and of the high analytical tractability of the HARGL process.

From the previous results, we can finally conclude:

Corollary 4. *Under $\mathbb{Q}$, the log-returns follow a discrete-time stochastic volatility model, with dynamics as in Eq.(1), with risk premium $\gamma^* = -1/2$. The RV is a HARGL process, featuring a transition density given by a non-central gamma, with parameters β^*, δ^*, c^* .*

3 Dynamic properties under $\mathbb{P}$ and $\mathbb{Q}$

In this section we analyze the dynamic properties of the proposed HARGL model under the historical ($\mathbb{P}$) and martingale ($\mathbb{Q}$) measures. We compare the features of our model with those of several competitor

⁷The specific component of the vector ν that has to be calibrated does not need to be ν_1 but simply depends on the way the system of no-arbitrage restrictions is solved. We report the results in term of ν_1 just for convenience.

models. Specifically, we consider the Heston and Nandi (2000) GARCH(1,1) model, the Two Components GARCH(1,1) model by Christoffersen et al. (2008b), and the two restricted models: HARG and ARG.

3.1 Competitor models

The HARGL is a discrete-time model relying on historical observations. Thus, the first natural competitors come from the class of GARCH-type option pricing models. In particular, the first GARCH-type model we consider here is the widely applied GARCH(1,1) option pricing model, proposed by Heston and Nandi (2000) (GARCH hereafter). The Heston and Nandi model is an asymmetric GARCH where the log-returns under $\mathbb{P}$ are modeled by

$$\begin{aligned} y_{t+1} &= r + \lambda h_{t+1} + \sqrt{h_{t+1}} z_{t+1}, \\ h_{t+1} &= w + b h_t + a(z_t - c\sqrt{h_t})^2. \end{aligned} \quad (13)$$

with $z_{t+1} \sim N(0,1)$. In the conditional variance h_{t+1} , the parameter c captures the negative relation between shocks in the returns and volatility. The GARCH represents a first attempt to model the non-constant volatility process. Nevertheless, it is well-known that the GARCH fails in capturing the strong persistence and the volatility of the volatility process (see, e.g., Christoffersen et al. (2008b)). To overcome this problem the two-component GARCH (in the following, CGARCH) has been recently introduced. The second GARCH-type model we consider as a benchmark is then the CGARCH model, whose conditional variance equation is given by

$$\begin{aligned} h_{t+1} &= q_{t+1} + b_s(h_t - q_t) + a_s((z_t - c_s\sqrt{h_t})^2 - (1 + c_s^2 q_t)), \\ q_{t+1} &= \omega + b_l q_t + a_l((z_t^2 - 1) - 2c_l\sqrt{h_t} z_t), \end{aligned} \quad (14)$$

where $(h_t - q_t)$ and q_t represent the short- and long-run persistent components, respectively. As the component model of Engle and Lee (1999), the CGARCH model is the combination of two variance factors which give rise to a GARCH process with a more persistent dynamics than the standard one. Christoffersen et al. (2008b) show that the CGARCH is successful in option pricing, since the long memory plays a crucial role for the price of medium/long maturity ATM options. Thus, a comparison between the also highly persistent HARG(L) and the CGARCH models is instructive in gauging the improvements due to both the stochastic volatility model specification and to the use of RV.

Additional insights into the features of the HARGL model can be obtained by evaluating the gains due to the leverage effect and to the multi-factor specification (netting out the effect of the RV). To this end, a comparison between the HARGL and the HARG illustrates the importance of the leverage L_t , while a

comparison between the HARG and the ARG (i.e. the HARG with $\beta_2 \equiv \beta_3 \equiv 0$), which is the exact discrete-time version of the CIR process, illustrates the importance of multi-factor structure in stochastic volatility option pricing models.

3.2 Estimates

One of the main advantages of our methodology is related to the estimation of the parameters characterizing the HARGL volatility process.

For the family of RV option pricing models (i.e., ARG, HARG and HARGL), we employ the RV computed from tick-by-tick data for the S&P 500 Futures. Precisely, we estimate the continuous component of the RV using the tick-by-tick Two-Scales RV estimator proposed by Zhang et al. (2005) (with a fast scale of two ticks and a slower one of 20 ticks) and purified from the jump component by the Threshold Bipower variation method recently introduced in Corsi et al. (2009). To minimize the probability that a true jump is not identified by the jump test, we employ a conservative significance level of 90%. The RV measures the integrated variance during the trading period, i.e. from open to close. As a result, it neglects the contribution coming from overnight returns. To overcome this problem, we rescale our RV estimator to match the unconditional mean of the squared daily (i.e., close-to-close) returns. For all models, the sample period goes from January 1, 1990 to December 31, 2004.

Thanks to the use of RV as a proxy for the unobservable volatility, we can simply estimate the parameters of the HARGL process using Pseudo Maximum Likelihood⁸ on observable historical data. For the model specified in Eq.(5), the conditional mean and conditional variance are given by

$$\mathbb{E}_t(RV_{t+1}) = c\delta + \beta'(\mathbf{RV}_t, L_t), \quad (15)$$

$$\mathbb{V}_t(RV_{t+1}) = c^2\delta + 2c\beta'(\mathbf{RV}_t, L_t). \quad (16)$$

To estimate the market price of risk $\tilde{\gamma}$ in the log-returns equation, we apply the following simple regression model:

$$y_{t+1} = r_{t+1} + \left(\tilde{\gamma} - \frac{1}{2}\right) RV_{t+1} + \sqrt{RV_{t+1}}\epsilon_{t+1}. \quad (17)$$

⁸See Gouriéroux and Jasiak (2006) for a related discussion. In our implementation, we use the OLS estimates of $c\beta'$ as starting values for the PMLE maximization algorithm. Other estimation methods are the Simulated Maximum Likelihood (SML) or the Simulated Method of Moments. Those methods achieve a better efficiency, but they are computationally much more demanding.

This equation can be re-written as

$$\begin{aligned}\frac{y_{t+1} - r_{t+1} + \frac{1}{2}RV_{t+1}}{\sqrt{RV_{t+1}}} &= \tilde{\gamma}\sqrt{RV_{t+1}} + \epsilon_{t+1}, \\ \tilde{y}_{t+1} &= \tilde{\gamma}\sqrt{RV_{t+1}} + \epsilon_{t+1}.\end{aligned}\tag{18}$$

Eq.(18) defines a regression model for the response variable $\tilde{y}_{t+1}$, where the explanatory variable is $\sqrt{RV_{t+1}}$. The FED Fund rate provides us with a proxy for the risk-free rate (r_{t+1}). Following Stentoft (2008), estimation and testing for $\tilde{\gamma}$ can both be achieved by customary methods. To mitigate the impact of anomalous observations, we run a robust OLS regression. The estimated value of $\tilde{\gamma} = 0.51$, with a highly significant t-statistics of about 26. For the sake of comparability, all other competitor models are estimated using Pseudo Maximum Likelihood, as well, and the relevant moments are easily obtained from Eq.(13) and Eq.(14).

In Table 1, we show the estimated parameters, their standard deviations, and the value of the likelihood function for the HARGL and GARCH families, respectively.

[Table 1 should be here]

The impact of past lags on the present value of RV is given by the partial autocorrelation coefficients, $c\beta'$. According to our estimates (considering also the leverage effect L_t), the sensitivity of RV_t on the conditional mean of RV_{t+1} is $c(\beta_1 + \beta_4/2) = 0.494$, whereas the sensitivity of RV_t^w and RV_t^m are $c\beta_2 = 0.250$ and $c\beta_3 = 0.161$, respectively. We notice that, in line with the literature (see Corsi (2009)), the RV coefficients are all significant and show a decreasing impact of the past lags on the present value of the RV. Moreover, comparing the log-likelihood of the three RV models, we notice that the inclusion of both multiple factors (HARG) and leverage component (HARGL) improves upon the value of the likelihood⁹.

In order to compute the parameters of the HARG(L) under the martingale measure, we need to calibrate the free parameter ν_1 . This can be done, for instance, by setting ν_1 such that the unconditional volatility matches the average implied volatility of one-year maturity ATM options,¹⁰ which is 20.53% (see Section 4.1 for a detailed description of our data set). After the calibration of ν_1 , the other parameters under the martingale measure can be directly obtained using the mapping given in Proposition 3 and Corollary 4. Since both the HARG and ARG models are nested in our HARGL, the same change of measure as in Proposition 3 and Corollary 4 can be applied for each. For both the GARCH and CGARCH models, no

⁹The log-likelihood of the RV models and that of the GARCH-type models are clearly not comparable, an incompatibility arising from two different sets of data (RV and daily returns).

¹⁰Other possible calibrations may be employed such as matching the average IV for short- or medium-maturity options, but they are computationally more demanding while yielding similar results.

calibration is possible, and the change of measure is performed according to the methodology illustrated by Christoffersen et al. (2008b).

3.3 Moment term structure and conditional dynamics

We here analyze the dynamic properties of the HARG(L) model and its ability to replicate the observed features of log-returns and volatility processes. More precisely, we study the term structure of variance, since it yields important information about the ability of a given model to both (i) price options across different maturities and (ii) track the dynamics of the volatility surface over time (i.e. the implied volatility term structure). To assess the ability to replicate the persistence of the conditional variance process, we compute the variance term structure for the GARCH, CGARCH, HARGL, HARG and, ARG models. To compute the variance term structure, we apply the following measure (as in Christoffersen et al. (2008b)):

$$h_{t+1:t+T} \equiv \frac{1}{T} \sum_{k=1}^T E_t[h_{t+k}]. \quad (19)$$

That is, for a given horizon T , $h_{t+1:t+T}$ is the average of the model multi-step-ahead forecasts of the conditional variance from $t+1$ to $t+T$. For each of the different model specifications, we consider both a high- and a low-volatility regime as the conditioning state.

For the GARCH model we have,

$$h_{t+1:t+T}/\sigma^2 = 1 + \frac{1 - \tilde{b}^T}{1 - \tilde{b}} \frac{m - 1}{T},$$

with $\tilde{b} = b + ac^2$; we choose $m = 2$ in the case of a high-volatility regime and $m = 0.5$ for the low-volatility case.

For the CGARCH,

$$h_{t+1:t+T}/\sigma^2 = 1 + \frac{1 - \rho^T}{1 - \rho} \frac{m_1 - 1}{T} + \frac{1 - \tilde{b}_s^T}{1 - \tilde{b}_s} \frac{m_2 - m_1}{T},$$

with $\tilde{b}_s = b_s + a_s c_s^2$ and ρ corresponding to the persistence of the short- and long-run volatility components, respectively. Here $m_1 = 1.75$ and $m_2 = 2$ for high volatility and $m_1 = 0.75$ and $m_2 = 0.5$ for low volatility.

For the case of the HARG and HARGL,

$$h_{t+1:t+T}/\sigma^2 = 1 + \mathbf{F}_{(1,:)}^T \frac{\mathbf{V} - \sigma^2}{\sigma^2},$$

where $\mathbf{F}_{(1,:)}^T$ is a row vector in $\mathbb{R}^{22}$, representing the first-row of the matrix $\mathbf{F}^T$, which is the T -th power of

the (22×22) -matrix:

$$\mathbf{F} = \begin{pmatrix} \beta_1 + \beta_4/2 & \beta_2/4 & \cdots & \beta_2/4 & \beta_3/17 & \cdots & \beta_3/17 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{pmatrix}.$$

In computing the first coefficient of the matrix¹¹ $\mathbf{F}$, we follow Engle and Gallo (2006), relying on the approximation: $\mathbb{E}[L_{t+h}|\mathcal{F}_t] = \mathbb{E}[I_{(y_{t+h}<0)}|\mathcal{F}_t] \mathbb{E}[RV_{t+h}|\mathcal{F}_t]$ with $\mathbb{E}[I_{(y_{t+h}<0)}|\mathcal{F}_t] = \mathbb{E}[I_{(y_{t+h}<0)}] \approx 1/2$. Thanks to this approximation, we can consider $\beta_4/2$ to be the expected leverage coefficient. The vector $\mathbf{V} \in \mathbb{R}^{22}$ is defined as

$$\mathbf{V} = [m_d\sigma^2, m_w\sigma^2, \dots, m_w\sigma^2, m_m\sigma^2, \dots, m_m\sigma^2]',$$

where we choose $m_d = 2.15$, $m_w = 2$, $m_m = 1.5$, $m_d = 0.3$, $m_w = 0.5$, and $m_m = 0.9$ for the high and low volatility regimes, respectively. For the restricted HARG and ARG models, similar equations are obtained constraining the appropriate parameters to zero. For every multi-step-ahead forecast of the conditional variance $h_{t+1:t+T}/\sigma^2$, the quantity σ^2 represents the long-run volatility implied by the specific model at hand.

Figure 2 shows the term structure of the five models (normalized by their unconditional variances) under both $\mathbb{P}$ and $\mathbb{Q}$ and for different volatility regimes (high and low volatility).

[Figure 2 should be here]

As expected, the GARCH term structure has a fast decay at an exponential rate under both measures. This is at odds with the empirical high persistence of conditional volatility and thus suggests that difficulties will arise in pricing options with medium/long maturities. This lack of memory is cleared up by the CGARCH model. As evident in Figure 2 (left panels), the CGARCH indeed succeeds in significantly increasing the persistence. This feature is visible both under $\mathbb{P}$ (top panel) and $\mathbb{Q}$ (bottom panel) and for different volatility regimes.

The right-hand panels show the variance term structure for the HARG, ARG and HARGL models. We notice that, both under $\mathbb{P}$ (top panel) and $\mathbb{Q}$ (bottom panel), the persistence of the estimated HARGL model is similar to that of the CGARCH. This implies that, similar to the CGARCH, the HARGL is able to closely reproduce the long-memory observed in the volatility of financial data. Analogous considerations also

¹¹Details on $\mathbf{F}$ can be found in (Hamilton, 1994, pp. 7-9)

hold true for the HARG model. This means that the HARG(L) specification can be used as a parsimonious representation of a multi-factors volatility model. As for the GARCH in contrast, the ARG fails to reproduce this feature, since it has a fast decay towards the long run mean, under both $\mathbb{P}$ and $\mathbb{Q}$.

4 Option pricing: performance assessment

4.1 Data description

In this section, we describe the data employed in our empirical analysis. We use European options, written on the S&P 500 index. The observations for the option prices range from January 1, 1996 to December 31, 2004, and the data are downloaded from OptionMetrics. Following Barone-Adesi et al. (2008), options with time to maturity less than 10 days or more than 360 days, implied volatility larger than 70%, or prices less than 0.05 dollars are discarded. Moreover, we consider only out-of-the-money (OTM) put and call options for each Wednesday which tend to be more liquid. This procedure yields a total number of 31,365 observations. The numbers of put and call options are approximately the same, since we have about 51% put and 49% call options.

To perform our analysis, we split the options into different categories, classifying them according to either time to maturity or moneyness. In particular, in the remainder of this paper, we will use as a measure of moneyness the following quantity:

$$m = \frac{\ln(K/S_t)}{\sigma\sqrt{\tau}},$$

where σ is the average Implied Volatility (IV) over the whole sample and τ represents the maturity. This measure can be roughly interpreted as the distance, in terms of number of standard deviations, between the log-strike and the log-spot price, assuming Gaussian returns (i.e. in the ideal Black-Scholes model).

In our empirical application, we will consider a range of moneyness, m , which results in a range of options much wider than the one commonly used in other works. In fact, we noticed that the commonly made choice of moneyness (with S/K typically between 0.93 and 1.03) implies an overly narrow range of options, improperly discarding a large portion of liquid options (see Figure 3). In our classification, a put option is said to be Deep OTM (DOTM hereafter) if $m \leq -2$ and OTM if $-2 < m \leq 0$. A call option is said to be DOTM if $m \geq 1$ and OTM if $0 < m \leq 1$. According to maturity τ , we classify options as short maturity ($\tau < 60$ days), medium maturity (τ between 60 – 160 days), and long maturity ($\tau > 160$ days).

[Figure 3 should be here]

[Table 2 should be here]

Looking at the IV values reported in Table 2, we notice that for a given maturity, the data shows strong volatility smile for the short maturity options which tends to become a volatility smirk as maturity increases. Moreover, moving along different maturities and fixing the moneyness m , we observe the volatility term structure.

[Figure 4 should be here]

To provide a graphical representation of this phenomenon, we show in Figure 4 the average volatility surface (left panel) and the average volatility curves (right panel). As long as the time to maturity τ increases from 20 days to 1 year, both pictures show that the smile tends to vanish, while the smirk remains strong at one year maturity. This implies that the risk-neutral distribution of the log-returns is far from Gaussian, even after one year. This suggests that a persistent, either time-varying or stochastic volatility model needs to be applied to address these features. Finally, looking at the right panel, we notice that the IV of long-run (one year maturity) ATM options is around 20.53% (we use this value to calibrate the free parameter ν_1 , as described in section 3.2).

4.2 Option pricing method

In Section 2.1.3, we derive the one-step-ahead Laplace transform of the joint process K'_{t+1} , see Eq. (6). For any given maturity, the computation of the multi-step-ahead conditional Laplace transform of the HARG model can be obtained in an analogous way, solving a system of recursive equations and thus obtaining a semi-closed-form expression for the option price (see Technical Appendix B for details). For long maturities, the solving of this system of equations is computationally demanding. Therefore, we prefer to simplify and homogenize the pricing procedure, applying a straightforward Monte Carlo simulation method to all three RV option pricing models. In fact, one additional feature of our model is that both the sample paths of the log-returns and of the RV can be easily simulated¹². Thus, we obtain the option prices using the following four steps: (i) Estimation of the HARG(L) model under the physical measure $\mathbb{P}$; (ii) Unconditional calibration of the parameter ν_1 (by matching the average implied volatility of one-year maturity ATM options); (iii) Mapping of the parameters of the model estimated under $\mathbb{P}$ into the parameters under $\mathbb{Q}$ to specify the dynamics of the RV and the log-returns under the martingale measure $\mathbb{Q}$ using Eq. (10)-(12); (iv) For every t , simulation both the RV and log-returns $\mathbb{Q}$ -dynamics so that, for each maturity τ and strike K , we can compute the prices for call OTM options at time t as the average, $(1/L) \sum_{l=1}^L \max(S_\tau^{(l)} - K, 0)$. In the

¹²Moreover, the current availability of multi-processor, multi-core workstations along with, for example, the MatlabTM Parallel Computing Toolbox,TM dramatically reduces the time required for intensive simulating tasks.

previous formula, L represents the total number of Monte Carlo simulations. In our numerical analysis, we set $L = 20,000$. The simulated asset price at time τ in the l -th sample path is indicated by $S_\tau^{(l)}$. An analogous formula holds for put options.

To evaluate the performance of our model, we compute the Root Mean Square Error (RMSE) on prices ($RMSE_P$) and on the percentage IV ($RMSE_{IV}$) for the HARGL model and for competitors. $RMSE_P$ emphasizes the importance of ATM options which are more expensive, while the $RMSE_{IV}$ tends to put more weight on DOTM options (see Broadie et al. (2007) for a discussion on the $RMSE_{IV}$ properties and Christoffersen and Jacobs (2004) for one on the importance of the loss function in option pricing). Thanks to this comparison, we are able to assess absolute and relative performances of the different option pricing models.

4.3 Option pricing results

4.3.1 Static properties

In Table 3, we show the overall performance of our model in both absolute terms and relative to the different benchmark pricing models. We notice that, overall, the HARGL outperforms all competitors, both in terms of RMSE on price and on IV. The improvements on the benchmark models range from about 10% to more than 30%. The ranking is in accordance with that suggested by the theory: The CGARCH improves upon the simple GARCH, while the HARGL has smaller pricing errors than the HARG, which, in turn, outperforms the ARG specification. Specifically, the fact that both the HARG and the HARGL have a better $RMSE_P$ than the ARG highlights the favorable pricing impact of a more persistent volatility dynamics.

[Table 3 about here]

To gain a deeper understanding of the pricing performance, Table 4 and Table 5 report the results both in terms of $RMSE_{IV}$ and $RMSE_P$, disaggregated for different maturities and moneyness. In both tables, in Panel A we report the RMSE in absolute terms for the HARGL model, while in the remaining panels we compare the relative RMSE with respect to the other models. A value less than one highlights a RMSE of our HARGL model smaller than that of the corresponding benchmark, hence indicating an outperforming of the proposed model over the reference one.

[Table 4 about here]

[Table 5 about here]

In absolute terms, the HARGL performs well under both loss functions even though the $RMSE_{IV}$ signals some degree of under-pricing for DOTM put options. This is to be expected as it is a common feature of stochastic volatility option pricing models without jumps. The general improvement of the HARGL with respect to the GARCH model is remarkable across moneyness and maturity. Particularly for ATM options, the GARCH RMSE is almost two times larger. This gap is mitigated in the case of the CGARCH (Panel C) even if a consistent mispricing remains. A comparison between the HARG and the ARG models (Panel E) confirms the importance of the heterogeneous multi-component specification in considerably helping to price options over different maturities. Thanks to the fast-adapting nature of the RV measure, the ARG model does a reasonably good job in pricing very short maturity options. However, it altogether fails to correctly price the ATM term structure and the OTM put options, lacking adequate volatility persistence and asymmetry. In Panel D we notice that for ATM options, the HARG model performs comparably with the HARGL, since the two models share the same degree of persistence in the volatility term structure. However, as expected, the HARGL has a much smaller RMSE on DOTM and OTM put options due to the presence of the leverage component. It is evident from this analysis that both ingredients, volatility persistence and leverage effect, are necessary to accurately price options across different maturities and strikes. This is the reason why the HARGL consistently shows the best option pricing performance.

Finally, we analyze the performance of the option pricing models under different volatility regimes. To perform this analysis, we divide the sample period into three sub-samples: low and declining, medium and increasing, and high-volatility. The three different regimes have been identified using the volatility levels given by the VIX index, as shown in Figure 5. Figure 6 compares the average IV of near ATM options ($-0.1 < m < 0.1$) observed in the data with the IV obtained by the CGARCH, ARG and HARGL models. We analyze the three volatility regimes separately. The HARGL performs very well in all three volatility regimes; quite often the IV of the HARGL is extremely close to the IV observed in the data. In the low-volatility regime the pricing of the HARGL is almost perfect for short and medium maturities, while it tends to overestimate the IV at longer maturities. A similar pattern is present in the medium regime, but with a better agreement with the market data for long maturity options. In the high-volatility regime all the considered models underestimate the observed implied volatilities. Nevertheless, for all the considered maturities, the IV of the HARGL is the closest to the market data.

[Figure 5 about here]

[Figure 6 about here]

4.3.2 Dynamic properties

In order to compare the ability of the different models to dynamically track the evolution of ATM implied volatility over time, Figure 7 analyzes the dynamics of the ATM option bias for the different models. We notice that the GARCH-type models show a significant and persistent bias. The fact that GARCH-type models do not allow any parameter calibration certainly contributes to generating this bias. In fact, the absence of calibration clearly leads to a more rigid structure which yields a systematic mispricing of ATM options. In contrast, the three models in our RV option pricing approach allow for a more flexible change of measure. A convenient consequence of this additional flexibility is that no apparent unconditional bias in ATM options emerges. We notice that the HARG and HARGL models show markedly similar tracking performances for the evolution of ATM implied volatility, since the leverage effect has a negligible impact on the pricing of ATM options.

[Figure 7 about here]

Additional insights into the models' ability to track the dynamics of the volatility surface are gained by studying the evolution of the level and term structure implied by the CGARCH, ARG and HARGL models. The three panels in Figure 8 describe these dynamical features. Tracking the level (i.e., the average implied volatility of ATM options) is crucial for capturing the overall dynamics of the volatility surface. Comparing the level of the actual volatility surface with that of the CGARCH, we notice that the CGARCH model tends to reproduce the empirical level dynamics with some delay. Incidentally, we notice that the delay is smaller in the first part of the sample (until observation 50), which largely coincides with the observation period in Christoffersen et al. (2008b). However, it becomes more pronounced in the rest of the sample. A more reactive dynamics can be obtained using stochastic volatility models based on RV. This is the case with the baseline ARG. Although the ARG is quite capable of capturing the very short-term volatility dynamics, it is evident from the middle panel of Figure 8, that the ARG is totally incapable of reproducing the level dynamics of the volatility surface. This is due to the small persistence implied by the model (see Figure 2), which yields a very poor description of the data behavior. In contrast to the previous models, the HARGL is able to closely track the level of the empirical data (see bottom panel in Figure 8). This important feature is mainly due to: (i) The high persistence of the process and (ii) the much faster adapting nature of the RV measure compared to the conditional volatility filtered by GARCH-type models. Similar behavior is observed in Figure 9 for the dynamics of the term-structure slope (i.e. the difference between the average ATM long maturity options and level) where the superior tracking performance of the HARGL model is again apparent (see the bottom panel).

[Figure 8 should be here]

[Figure 9 should be here]

Summarizing, the proposed HARGL model seems, in general, better able to closely reproduce the IV level and dynamics, improving upon the already acceptable performance of the benchmark option pricing volatility models. The ability of the HARGL to generate a realistic degree of persistence leads to an accurate modeling of the term structure of the IV surface. Moreover, the fast-adapting properties of RV greatly help in closely tracking the evolution of the IV surface over time.

5 Conclusion

In this paper, we develop a discrete-time stochastic volatility option pricing model that exploits the historical information contained in the high-frequency data. Using the RV as a proxy for the unobservable returns volatility, we propose a long-memory process with a leverage effect: the HARGL process. Our model can be considered a reduced form, multi-factors model, since it is characterized by three volatility components (or frequencies): short-, medium-, and long-horizon. The proposed model, coupled with an exponential affine SDF, leads to tractable risk-neutral dynamics both for log-returns and volatility. Making the latent volatility observable (through the RV), the HARGL model can be easily estimated simply by using observed historical data. This is a clear advantage with respect to other stochastic volatility models, which rely on time-consuming filtering procedures. The parameters under $\mathbb{Q}$ are obtained by explicit formulas in which only one parameter must be calibrated. The extensive empirical analysis of the S&P 500 index options shows that two ingredients are crucial for option pricing performance: (i) The use of RV, which provides an accurate and fast adapting proxy for the unobserved volatility and (ii) the high persistence generated by the volatility model specification. Thanks to both these features, the HARGL is able to better reproduce the $\mathbb{Q}$ -dynamics, and hence, it considerably outperforms competing GARCH-type and stochastic volatility option pricing models (ARG and HARG). Specifically, the HARGL model improves upon the overall $RMSEP$ by about 38% over the Heston and Nandi (2000) GARCH, about 15% over the CGARCH of Christoffersen et al. (2008b), and about 20% over the discretized version of the CIR model (the ARG model).

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Parameter	ARG	HARG	HARGL	Parameter	GARGH	Parameter	CGARCH
$\tilde{\gamma}$		0.51 (0.0196)		λ	1.060 (1.585)	λ	2.939 (1.615)
c	54.93 (2.26)	43.81 (1.67)	42.38 (1.56)	ω	5.05e-19 (7.01e-09)	b_s	0.670 (0.064)
δ	1.182 (0.092)	0.496 (0.065)	0.581 (0.064)	b	0.881 (0.009)	a_s	1.49e-06 (7.62e-07)
β_1	0.014 (0.001)	0.012 (0.0008)	0.0098 (0.0018)	a	2.82e-06 (2.51e-07)	c_s	425.59 (208.49)
β_2	-	0.0051 (0.0021)	0.0059 (0.0024)	c	178.65 (15.20)	ω	1.27e-06 (2.06e-07)
β_3	-	0.0035 (0.0019)	0.0038 (0.0019)			b_l	0.986 (0.002)
β_4	-	-	0.0037 (0.0015)			a_l	2.45e-06 (2.96e-07)
						c_l	87.82 (15.53)
Log-likelihood	-26849.11	-26520.58	-26441.78	Log-likelihood	12428.14		12473.11
Persistence	0.760	0.919	0.908	Persistence	0.971		0.995

Table 1: Pseudo maximum likelihood estimates, robust standard errors, and models performance. The historical data for the ARG, HARG, and HARGL models are given by the daily RV computed on tick-by-tick data for the S&P500 Futures (see Section 3). For all three models, the estimation period ranges from January 1, 1990 to December 31, 2004.

Moneyness		Maturity							
		Less then 20		20 to 60		60 to 160		More then 160	
		Mean	Std	Mean	Std	Mean	Std	Mean	Std
$m \leq -2$	Put price \$	0.8963	1.0137	1.9934	2.0987	3.4019	3.4034	4.8004	4.7038
	IV	0.3262	0.0581	0.3360	0.0601	0.3403	0.0554	0.3283	0.0525
	Observations	2099		4019		3689		1563	
$-2 \leq m \leq -1$	Put price \$	2.8232	2.4050	6.3000	4.8417	11.5455	8.0134	16.6953	11.1661
	IV	0.2484	0.0521	0.2667	0.0540	0.2755	0.0485	0.2778	0.0469
	Observations	1700		4136		4075		2838	
$-1 \leq m \leq -0.2$	Put price \$	8.5810	5.3205	16.4179	9.5220	27.8586	15.6746	42.9110	20.7949
	IV	0.2093	0.0522	0.2205	0.0508	0.2275	0.0454	0.2354	0.0433
	Observations	1864		5103		5143		4160	
$-0.2 \leq m \leq 0.2$	Put price \$	15.3810	6.4545	28.0130	11.6261	47.1174	19.2829	70.7042	24.1143
	IV	0.1860	0.0515	0.1936	0.0474	0.1994	0.0419	0.2066	0.0410
	Observations	1045		3241		2952		2379	
$0.2 \leq m \leq 1$	Put price \$	6.1199	4.8675	11.8042	8.7999	21.3376	14.4224	31.8592	20.2136
	IV	0.1710	0.0478	0.1759	0.0461	0.1808	0.0387	0.1848	0.0361
	Observations	1977		4759		4740		3998	
$1 \leq m$	Put price \$	0.7098	1.0856	1.2420	1.9245	1.7657	2.5783	3.3532	3.9473
	IV	0.2199	0.0613	0.2160	0.0548	0.1966	0.0437	0.1844	0.0345
	Observations	2637		4982		5105		2554	

Table 2: Database description.

Means and standard deviations of prices and implied volatilities of S&P 500 index out-of-the-money options on each Wednesday from January 1, 1996 to December 31, 2004 (80,758 observations) sorted by moneyness/-maturity categories. IV is the Black-Scholes implied volatility. Moneyness is defined as $m = \ln(K/F)/\sigma\sqrt{\tau}$. σ is the average implied volatility equal to 0.22. Maturity is measured in calendar days.

Models	Performance measures	
	$RMSE_{IV}$	$RMSE_p$
HARGL	6.7391	6.8585
HARGL/GARCH	0.7603	0.6186
HARGL/CGARCH	0.8936	0.8459
HARGL/ARG	0.9079	0.8263
HARG/ARG	1.0717	0.9164
HARGL/HARG	0.8472	0.9017

Table 3: Global option pricing performance on S&P500 out-of-the-money options from January 1, 1996 to December 31, 2004.

We use the pseudo maximum likelihood parameter estimates from Table 1. First row: percentage implied volatility root mean squared error ($RMSE_{IV}$) and percentage price root mean squared error ($RMSE_p$) of the HARGL model. Second and subsequent rows: $RMSE_{IV}$ and $RMSE_p$ of the benchmark models relative to the HARGL.

Moneyness	Maturity			
	$\tau \leq 20$	$20 < \tau \leq 60$	$60 < \tau \leq 160$	$160 < \tau$
Panel A	HARGL Implied Volatility RMSE			
$m \leq -2$	16.181	14.001	11.561	9.701
$-2 \leq m \leq -1$	6.649	6.949	6.855	6.578
$-1 \leq m \leq -0.2$	3.933	4.146	4.122	4.121
$-0.2 \leq m \leq 0.2$	2.633	2.752	2.919	3.104
$0.2 \leq m \leq 1$	2.787	2.732	3.145	3.504
$1 \leq m$	4.259	3.978	4.035	4.346
Panel B	HARGL/GARCH Implied Volatility RMSE			
$m \leq -2$	0.937	1.073	0.879	0.760
$-2 \leq m \leq -1$	0.743	0.767	0.709	0.658
$-1 \leq m \leq -0.2$	0.621	0.633	0.584	0.540
$-0.2 \leq m \leq 0.2$	0.532	0.514	0.519	0.506
$0.2 \leq m \leq 1$	0.681	0.586	0.647	0.708
$1 \leq m$	0.413	0.415	0.519	0.706
Panel C	HARGL/CGARCH Implied Volatility RMSE			
$m \leq -2$	0.988	1.106	0.954	0.882
$-2 \leq m \leq -1$	0.801	0.838	0.831	0.829
$-1 \leq m \leq -0.2$	0.701	0.748	0.749	0.754
$-0.2 \leq m \leq 0.2$	0.640	0.681	0.750	0.782
$0.2 \leq m \leq 1$	0.870	0.905	1.130	1.284
$1 \leq m$	0.497	0.752	1.115	1.547

Moneyness	Maturity			
	$\tau \leq 20$	$20 < \tau \leq 60$	$60 < \tau \leq 160$	$160 < \tau$
Panel D	HARGL/ARG Implied Volatility RMSE			
$m \leq -2$	1.222	0.896	0.663	0.633
$-2 < m \leq -1$	0.957	0.878	0.816	0.778
$-1 < m \leq -0.2$	0.826	0.847	0.810	0.787
$-0.2 < m \leq 0.2$	0.657	0.675	0.721	0.775
$0.2 < m \leq 1$	0.619	0.606	0.743	0.869
$1 < m$	1.107	1.070	1.037	1.079
Panel E	HARG/ARG Implied Volatility RMSE			
$m \leq -2$	1.417	1.121	0.880	0.833
$-2 < m \leq -1$	1.095	1.035	0.997	0.978
$-1 < m \leq -0.2$	0.956	1.015	0.999	0.997
$-0.2 < m \leq 0.2$	0.708	0.751	0.779	0.836
$0.2 < m \leq 1$	0.604	0.579	0.680	0.760
$1 < m$	1.032	1.054	1.047	1.049
Panel F	HARGL/HARG Implied Volatility RMSE			
$m \leq -2$	0.863	0.800	0.754	0.760
$-2 < m \leq -1$	0.874	0.848	0.819	0.796
$-1 < m \leq -0.2$	0.864	0.835	0.810	0.789
$-0.2 < m \leq 0.2$	0.928	0.898	0.925	0.927
$0.2 < m \leq 1$	1.025	1.046	1.093	1.143
$1 < m$	1.073	1.016	0.990	1.028

Table 4: Option pricing performance on S&P500 out-of-the-money options from January 1, 1996 to December 31, 2004.

We use the pseudo maximum likelihood parameter estimates from Table 1. Panel A: percentage implied volatility root mean squared error ($RMSE_{IV}$) of the HARGL model sorted by moneyness and maturity. Panels B to F: $RMSE_{IV}$ of the benchmark models relative to the HARGL sorted by moneyness and maturity.

Moneyness	Maturity			
	$\tau \leq 20$	$20 < \tau \leq 60$	$60 < \tau \leq 160$	$160 < \tau$
Panel A	HARGL Price RMSE			
$m \leq -2$	1.986	3.324	4.639	5.819
$-2 < m \leq -1$	3.858	6.551	9.564	11.878
$-1 < m \leq -0.2$	4.483	7.179	10.174	12.675
$-0.2 < m \leq 0.2$	3.668	5.589	8.253	11.353
$0.2 < m \leq 1$	2.951	4.562	7.555	11.383
$1 < m$	1.463	2.392	3.778	5.857
Panel B	HARGL/GARCH Price RMSE			
$m \leq -2$	0.914	0.923	0.911	0.898
$-2 < m \leq -1$	0.794	0.799	0.769	0.732
$-1 < m \leq -0.2$	0.632	0.627	0.582	0.531
$-0.2 < m \leq 0.2$	0.513	0.490	0.482	0.489
$0.2 < m \leq 1$	0.651	0.559	0.624	0.719
$1 < m$	1.047	1.149	1.304	1.411
Panel C	HARGL/CGARCH Price RMSE			
$m \leq -2$	0.928	0.943	0.944	0.944
$-2 < m \leq -1$	0.843	0.866	0.868	0.864
$-1 < m \leq -0.2$	0.723	0.753	0.747	0.740
$-0.2 < m \leq 0.2$	0.638	0.665	0.715	0.786
$0.2 < m \leq 1$	0.851	0.850	1.067	1.299
$1 < m$	1.342	1.901	2.391	2.628
Moneyness	Maturity			
	$\tau \leq 20$	$20 < \tau \leq 60$	$60 < \tau \leq 160$	$160 < \tau$
Panel D	HARGL/ARG Price RMSE			
$m \leq -2$	0.932	0.925	0.914	0.906
$-2 < m \leq -1$	0.870	0.869	0.855	0.835
$-1 < m \leq -0.2$	0.823	0.851	0.829	0.794
$-0.2 < m \leq 0.2$	0.691	0.717	0.753	0.799
$0.2 < m \leq 1$	0.666	0.636	0.777	0.911
$1 < m$	0.939	0.971	1.065	1.122
Panel E	HARG/ARG Price RMSE			
$m \leq -2$	0.979	0.974	0.973	0.974
$-2 < m \leq -1$	0.963	0.979	0.986	0.987
$-1 < m \leq -0.2$	0.944	1.008	1.020	1.009
$-0.2 < m \leq 0.2$	0.755	0.806	0.833	0.839
$0.2 < m \leq 1$	0.649	0.603	0.695	0.765
$1 < m$	0.979	0.975	1.038	1.022
Panel F	HARGL/HARG Price RMSE			
$m \leq -2$	0.952	0.949	0.939	0.930
$-2 < m \leq -1$	0.904	0.888	0.867	0.846
$-1 < m \leq -0.2$	0.871	0.844	0.813	0.787
$-0.2 < m \leq 0.2$	0.915	0.890	0.905	0.952
$0.2 < m \leq 1$	1.027	1.055	1.118	1.191
$1 < m$	0.959	0.997	1.026	1.098

Table 5: Option pricing performance on S&P500 out-of-the-money options from January 1, 1996 to December 31, 2004.

We use the pseudo maximum likelihood parameter estimates from Table 1. Panel A: percentage price root mean squared error ($RMSE_p$) of the HARGL model sorted by moneyness and maturity. Panels B to F: $RMSE_p$ of the benchmark models relative to the HARGL sorted by moneyness and maturity.

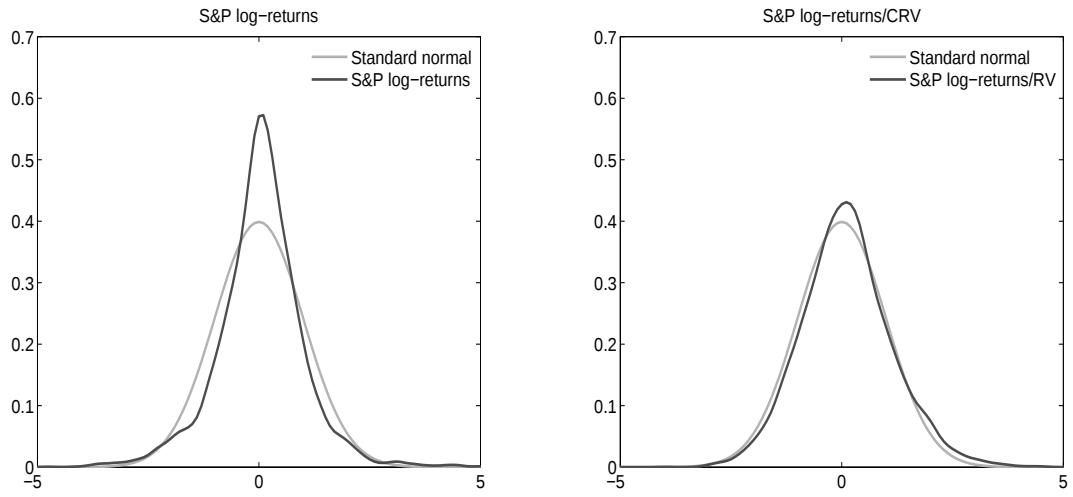


Figure 1: Log-returns distribution.

Comparison of the S&P 500 index log-returns distribution under different re-scaling measures. Left panel: Standard Normal distribution (grey line) and log-returns rescaled by the sample standard deviation (black line). Right Panel: Standard Normal distribution (grey line) and log-returns divided by contemporaneous realized volatility (black line).

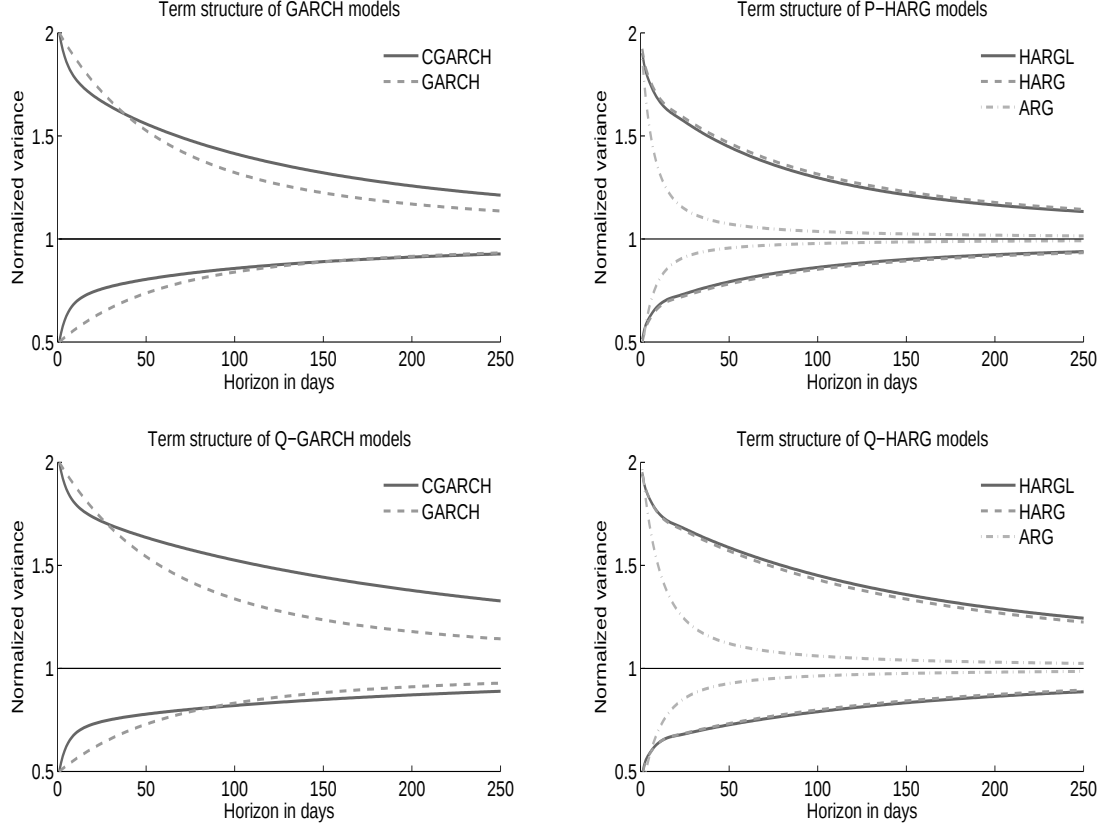


Figure 2: Term structure of variance.

Plot of the term structure of variance in high and low-volatility regimes for the GARCH and CGARCH (left panels), ARG, HARG, and HARGL models (right panels) normalized by their unconditional variance (σ^2) for 250 days. We plot the term structure both under the physical (top panels) and martingale measure (bottom panels). We use the pseudo maximum likelihood parameter estimates from Table 1. The GARCH-type models have initial values of $h_{t+1} = 2\sigma$ and $h_{t+1} = 0.5\sigma$. The CGARCH model has initial values of $h_{t+1} = 2\sigma$, $q_{t+1} = 1.75\sigma$ and $h_{t+1} = 0.5\sigma$, $q_{t+1} = 0.75\sigma$. The ARG, HARG, and HARGL models have initial values of $RV_d = 2.15\sigma$, $RV_w = 2\sigma$, $RV_m = 1.5\sigma$ and $RV_d = 0.3\sigma$, $RV_w = 0.5\sigma$, $RV_m = 0.9\sigma$, respectively.

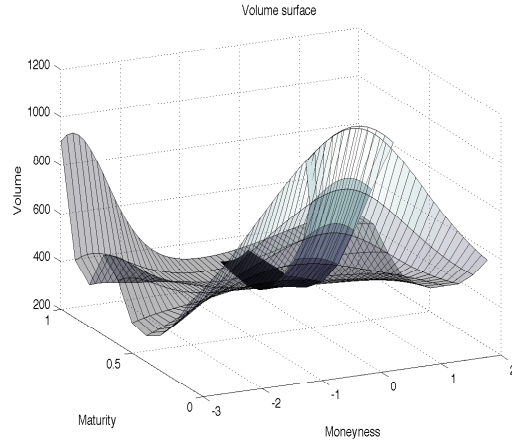


Figure 3: Volume surface.

Plot of a nonparametric smoothing of the volume surface in the moneyness and maturity dimension. Gaussian kernels are used with default bandwidth. The dark surface refers to values of S/K between 0.93 and 1.03 (the slightly difference in the two surfaces comes from the different default bandwidths).

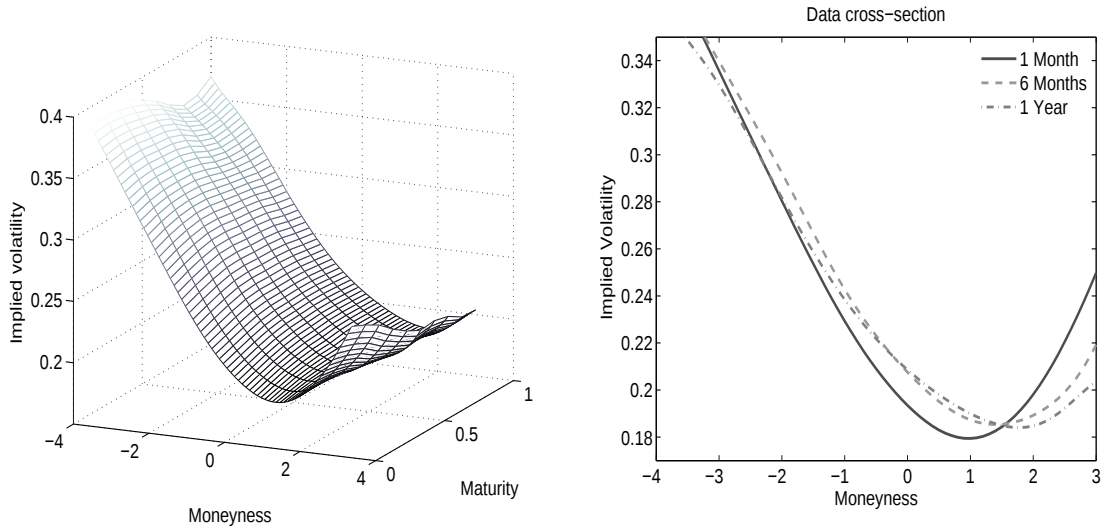


Figure 4: Volatility smile-smirk in S&P 500 index options. Left panel: implied volatility surface obtained via nonparametric smoothing of daily closing implied volatility quotes from January 1, 1996 to December 31, 2004 (80,758 observations). Independent Gaussian kernels with default bandwidth are used. Moneyness d is defined as $d = \ln(K/F)/\sigma\sqrt{\tau}$, where $\sigma = 24.38$ percent is the average implied volatility. Right panel: implied volatility curves obtained by slicing the implied volatility surface at maturities of 1 month (solid line), 6 months (dashed line), and 1 year (dotted line).

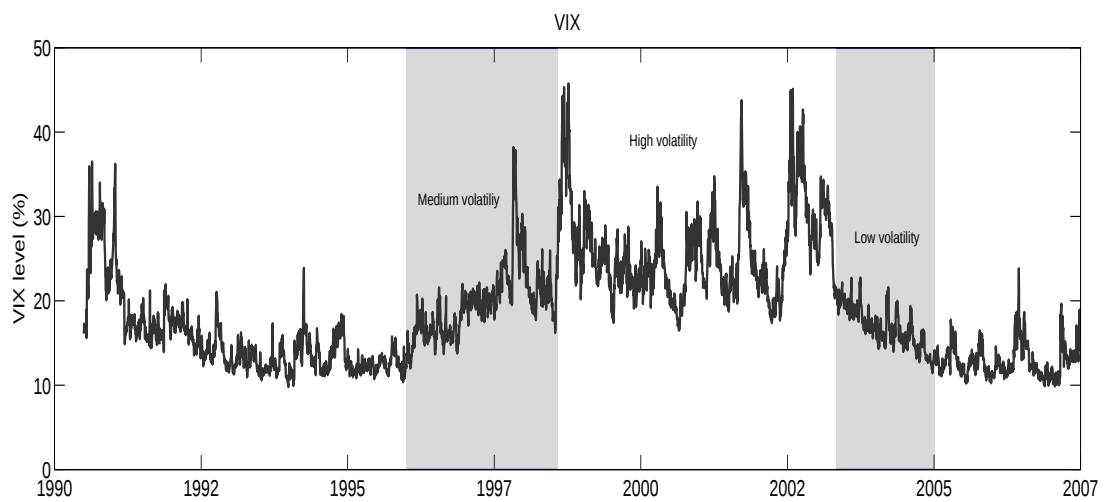


Figure 5: Volatility regimes.

Plot of the BBOE volatility index (VIX) from January 1, 1996 to December 31, 2004. We identify three different volatility regimes: medium volatility from January 1, 1996 to August 1, 1998, high volatility from August 2, 1998 to May 1, 2003, and low volatility from May 2, 2003 to December 31, 2004.

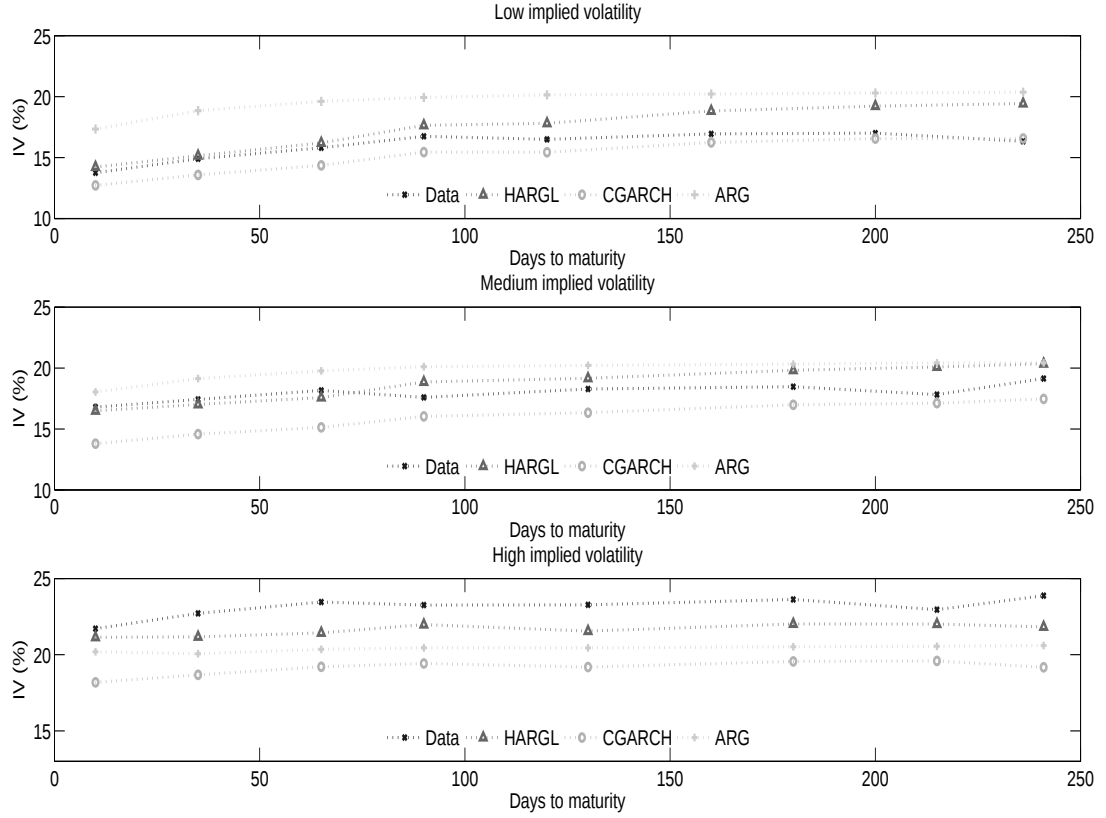


Figure 6: Implied volatility term structure for at-the-money options.

We use options with moneyness between -0.1 and 0.1. A dot represents the market-implied volatility, a triangle, the HARG model, a circle, the CGARCH model, and a cross, the ARG model. The top, mid and bottom panel correspond to the low, medium and high-volatility regimes as detected in Figure 5. The parameter estimates are taken from Table 1.

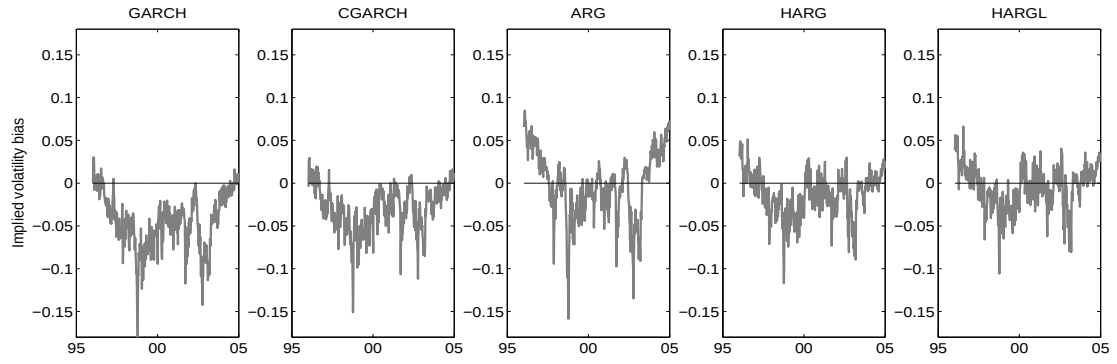


Figure 7: Weekly at-the-money options implied volatility bias.

Plot of the average difference between the model and the market-implied volatility for options with money-ness between -0.1 and 0.1. The parameter estimates are taken from Table 1

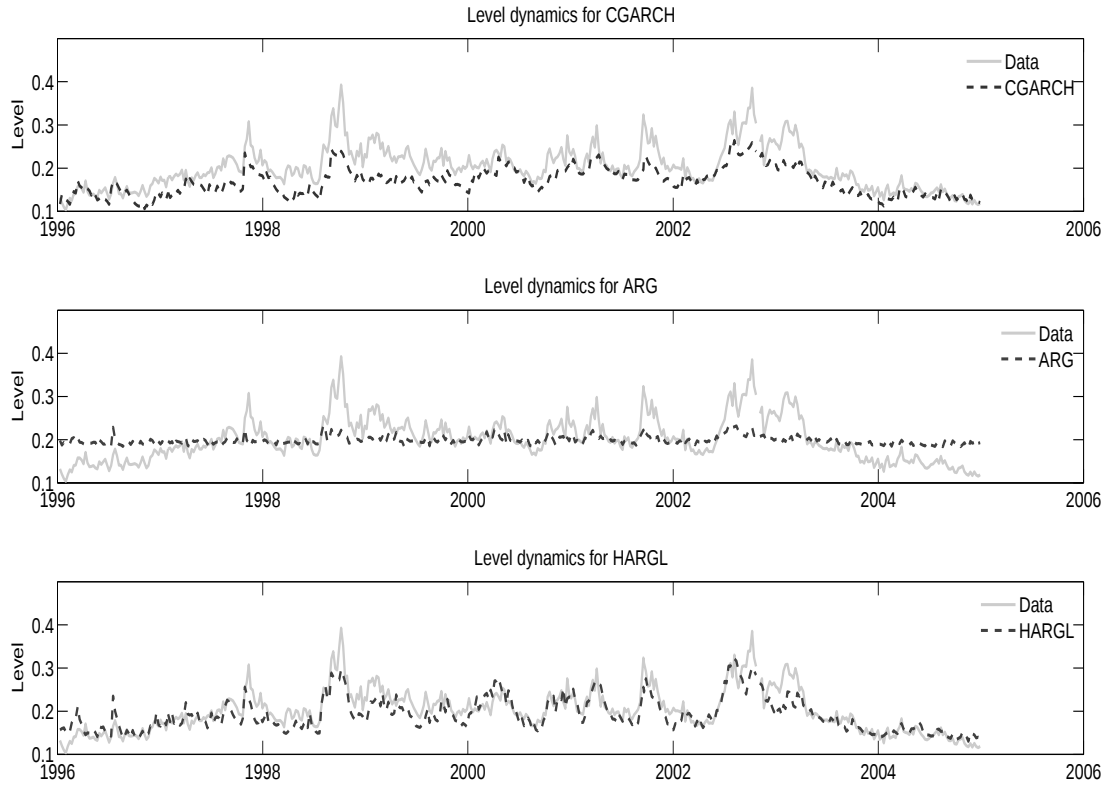


Figure 8: Level Dynamic option pricing performance from January 1, 1996 to December 31, 2004. Level is the average implied volatility of at-the-money options (moneyness between -0.1 and 0.1) with short maturity (between 20 and 80 days). In each panel, the light line represents the data, the dotted black line, the model. The top panel illustrates the performance of the CGARCH, the middle panel refers to the ARG model, and the bottom panel refers to the HARGL. The parameter estimates are taken from Table 1.

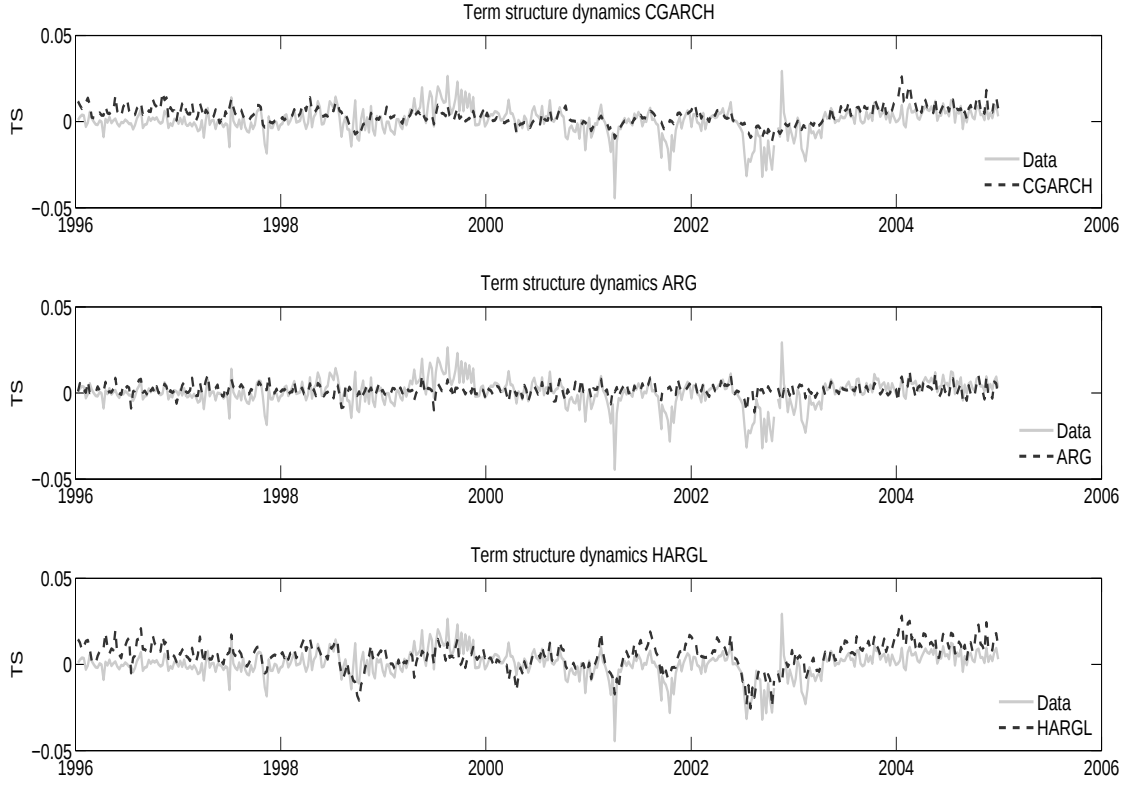


Figure 9: Term Structure Dynamic option pricing performance from January 1, 1996 to December 31, 2004. Term structure represents the slope of the implied volatility surface and is given by the difference between the average at-the-money long maturity (between 160 and 360 days) options and level. In each panel, the light line represents the data and the dotted black line represents the model. The top panel illustrates the performance of the CGARCH, the middle panel refers to the ARG model, and the bottom panel refers to the HARGL. The parameter estimates are taken from Table 1.

A Technical Appendix: Proofs

A.1 Proof of Proposition 1

Proof. Let us compute

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{P}} [\exp(-\alpha' K_{t+1})] &= \mathbb{E}_t^{\mathbb{P}} [\exp(-\alpha_1 y_{t+1} - \alpha_2 R V_{t+1})] \\
&= \mathbb{E}_t^{\mathbb{P}} \left[\exp(-\alpha_1 \sqrt{R V_{t+1}} \varepsilon_{t+1} - (\alpha_2 + \gamma \alpha_1) R V_{t+1}) \right] \\
&= \mathbb{E}_t^{\mathbb{P}} \left[\exp \left(- \left(\alpha_2 + \gamma \alpha_1 - \frac{1}{2} \alpha_1^2 \right) R V_{t+1} \right) \right] \\
&= \exp \left[-b \left(\alpha_2 + \gamma \alpha_1 - \frac{1}{2} \alpha_1^2 \right) - a \left(\alpha_2 + \gamma \alpha_1 - \frac{1}{2} \alpha_1^2 \right) \beta'(\mathbf{R} V_t, L_t) \right] \quad (20) \\
&= \varphi_K^{\mathbb{P}}(v), \quad (21)
\end{aligned}$$

with $v := \alpha_2 + \gamma \alpha_1 - \frac{1}{2} \alpha_1^2$. □

A.2 Proof of Proposition 2

Proof. For the sake of simplicity, we assume a zero expected instantaneous rate of return ($r = 0$). The no-arbitrage restrictions are:

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{P}}(M_{t,t+1}) &= 1 \\
\mathbb{E}_t^{\mathbb{P}}(M_{t,t+1} \exp(y_{t+1})) &= 1
\end{aligned}$$

$$\mathbb{E}_t^{\mathbb{P}}(\exp(-\nu_0 - \nu_1 R V_{t+1} - \nu_2 R V_t^d - \nu_3 R V_{t-1}^w - \nu_4 R V_{t-5}^m - \nu_5 y_{t+1} - \nu_6 L_t)) = 1 \quad (22)$$

$$\mathbb{E}_t^{\mathbb{P}}(\exp(-\nu_0 - \nu_1 R V_{t+1} - \nu_2 R V_t^d - \nu_3 R V_{t-1}^w - \nu_4 R V_{t-5}^m - (\nu_5 - 1)y_{t+1} - \nu_6 L_t)) = 1 \quad (23)$$

Using the moment generating function of y_{t+1} , the LHS of the first restriction (Eq. (22)) becomes:

$$\begin{aligned}
&\mathbb{E}_t^{\mathbb{P}} \{ \exp(-\nu_0 - \nu_1 R V_{t+1} - \nu_2 R V_t^d - \nu_3 R V_{t-1}^w - \nu_4 R V_{t-5}^m - \nu_5 y_{t+1} - \nu_6 L_t) \} \\
&= \mathbb{E}_t^{\mathbb{P}} \{ \exp(-\nu_0 - \nu_1 R V_{t+1} - \nu_2 R V_t^d - \nu_3 R V_{t-1}^w - \nu_4 R V_{t-5}^m - \nu_6 L_t) \\
&\quad \times \mathbb{E}_t^{\mathbb{P}} [\exp(-\nu_5 y_{t+1}) | R V_{t+1}] \} \\
&= \mathbb{E}_t^{\mathbb{P}} \left\{ \exp \left(-\nu_0 - \nu_1 R V_{t+1} - \nu_2 R V_t^d - \nu_3 R V_{t-1}^w - \nu_4 R V_{t-5}^m - \nu_5 \gamma R V_{t+1} - \nu_6 L_t \right) \right. \\
&\quad \left. \times \exp \left(\frac{R V_{t+1} \nu_5^2}{2} \right) \right\} \\
&= \mathbb{E}_t^{\mathbb{P}} \left\{ \exp \left(-\nu_0 - \left(\nu_1 + \nu_5 \gamma - \frac{\nu_5^2}{2} \right) R V_{t+1} - \nu_2 R V_t^d - \nu_3 R V_{t-1}^w - \nu_4 R V_{t-5}^m - \nu_6 L_t \right) \right\}.
\end{aligned}$$

We observe that the last equation is equivalent to:

$$\begin{aligned} & \mathbb{E}_t^{\mathbb{P}} \left\{ \exp \left(\nu_0 - \left(\nu_1 + \nu_5 \gamma - \frac{\nu_5^2}{2} \right) RV_{t+1} - \nu_2 RV_t^d - \nu_3 RV_t^w - \nu_4 RV_t^m - \nu_6 L_t \right) \right\} \\ &= \exp \left(-\nu_0 - \nu_2 RV_t^d - \nu_3 RV_t^w - \nu_4 RV_t^m - \nu_6 L_t \right) \varphi_{ARG}^{\mathbb{P}}(u), \end{aligned}$$

where $\varphi_{ARG}(u)$ is given in (4) and $u := \nu_1 + \nu_5 \gamma - \nu_5^2/2$. More specifically, we have

$$\begin{aligned} & \exp \left(-\nu_0 - \nu_2 RV_t^d - \nu_3 RV_t^w - \nu_4 RV_t^m - \nu_6 L_t \right) \exp \left(-a(u, \rho) \beta'(\mathbf{R}\mathbf{V}_t, L_t) - b(u) \right) \\ &= \exp \left(-\nu_0 - \nu_2 RV_t^d - \nu_3 RV_t^w - \nu_4 RV_t^m - \nu_6 L_t \right) \exp \left(-b(u) \right) \\ & \quad \times \exp \left(-\frac{cu}{1+cu} \left(\beta_1 \underbrace{RV_t^d}_{RV_t^d} + \beta_2 \underbrace{\left(\sum_{i=1}^4 RV_{t-i} \right)/4}_{RV_t^w} + \beta_3 \underbrace{\left(\sum_{i=5}^{21} RV_{t-i} \right)/17}_{RV_t^m} + \beta_4 L_t \right) \right), \end{aligned}$$

where $b(u)$ is given in (7).

Now, let us consider also the LHS of the equation (23) (applying the law of iterated expectations, as we have done for the LHS of equation (22)). We have

$$\begin{aligned} & \mathbb{E}_t^{\mathbb{P}} \left\{ \exp \left(-\nu_0 - \nu_1 RV_{t+1} - \nu_2 RV_t^d - \nu_3 RV_t^w - \nu_4 RV_t^m - \nu_6 L_t \right) \right. \\ & \quad \left. \times \mathbb{E}_t^{\mathbb{P}} \left[\exp \left(-(\nu_5 - 1) y_{t+1} \right) | RV_{t+1} \right] \right\} \\ &= \mathbb{E}_t^{\mathbb{P}} \left\{ \exp \left(-\nu_0 - \nu_1 RV_{t+1} - \nu_2 RV_t^d - \nu_3 RV_t^w - \nu_4 RV_t^m - (\nu_5 - 1) \gamma RV_{t+1} - \nu_6 L_t \right) \right. \\ & \quad \left. \times \exp \left(\frac{RV_{t+1}(\nu_5 - 1)^2}{2} \right) \right\} \\ &= \mathbb{E}_t^{\mathbb{P}} \left\{ \exp \left(-\nu_0 - \left(\nu_1 + (\nu_5 - 1) \gamma - \frac{(\nu_5 - 1)^2}{2} \right) RV_{t+1} - \nu_2 RV_t^d - \nu_3 RV_t^w - \nu_4 RV_t^m - \nu_6 L_t \right) \right\} \\ &= \exp \left(-\nu_0 - \nu_2 RV_t^d - \nu_3 RV_t^w - \nu_4 RV_t^m - \nu_6 L_t \right) \varphi_{ARG}^{\mathbb{P}}(\tilde{u}) \end{aligned}$$

where $\tilde{u} := \left(\nu_1 + (\nu_5 - 1) \gamma - \frac{(\nu_5 - 1)^2}{2} \right)$. The last equation then becomes:

$$\begin{aligned} & \exp \left(-\nu_0 - \nu_2 RV_t^d - \nu_3 RV_t^w - \nu_4 RV_t^m - \nu_6 L_t \right) \exp \left(-a(\tilde{u}, \rho) \beta'(\mathbf{R}\mathbf{V}_t, L_t) - b(\tilde{u}) \right) \\ &= \exp \left(-\nu_0 - \nu_2 RV_t^d - \nu_3 RV_t^w - \nu_4 RV_t^m - \nu_6 L_t \right) \exp \left(-b(\tilde{u}) \right) \\ & \quad \times \exp \left(-\frac{c\tilde{u}}{1+c\tilde{u}} \left(\beta_1 \underbrace{RV_t^d}_{RV_t^d} + \beta_2 \underbrace{\left(\sum_{i=1}^4 RV_{t-i} \right)/4}_{RV_t^w} + \beta_3 \underbrace{\left(\sum_{i=5}^{21} RV_{t-i} \right)/17}_{RV_t^m} + \beta_4 L_t \right) \right), \end{aligned}$$

where $b(\tilde{u})$ is given in (7).

Therefore, in order to satisfy both of the no-arbitrage conditions in (22) and (23), the following system has to be satisfied:

$$\begin{cases} -\nu_0 - \nu_2 RV_t^d - \nu_3 RV_t^w - \nu_4 RV_t^m - \nu_6 L_t - a(u, \rho) \beta'(\mathbf{R}\mathbf{V}_t, L_t) - b(u) = 0 \\ -\nu_0 - \nu_2 RV_t^d - \nu_3 RV_t^w - \nu_4 RV_t^m - \nu_6 L_t - a(\tilde{u}, \rho) \beta'(\mathbf{R}\mathbf{V}_t, L_t) - b(\tilde{u}) = 0 \end{cases},$$

or, equivalently,

$$\begin{cases} -\nu_0 - \nu_2 RV_t^d - \nu_3 RV_t^w - \nu_4 RV_t^m - \nu_6 L_t - \frac{cu}{1+cu} (\beta_1 RV_t^d + \beta_2 RV_t^w + \beta_3 RV_t^m + \beta_4 L_t) - b(u) = 0 \\ -\nu_0 - \nu_2 RV_t^d - \nu_3 RV_t^w - \nu_4 RV_t^m - \nu_6 L_t - \frac{c\tilde{u}}{1+c\tilde{u}} (\beta_1 RV_t^d + \beta_2 RV_t^w + \beta_3 RV_t^m + \beta_4 L_t) - b(\tilde{u}) = 0 \end{cases}. \quad (24)$$

The system in (24) implies (considering (8) and both the expressions for u and $\tilde{u}$) the following system of equations:

$$\begin{cases} \nu_0 + b\left(\nu_1 + \nu_5 \gamma - \frac{\nu_5^2}{2}\right) = 0 \\ \nu_0 + b\left(\nu_1 + (\nu_5 - 1)\gamma - \frac{(\nu_5 - 1)^2}{2}\right) = 0 \\ \nu_2 + \frac{c\left(\nu_1 + \nu_5 \gamma - \frac{\nu_5^2}{2}\right)}{1 + \left(\nu_1 + \nu_5 \gamma - \frac{\nu_5^2}{2}\right)c} \beta_1 = 0 \\ \nu_3 + \frac{c\left(\nu_1 + \nu_5 \gamma - \frac{\nu_5^2}{2}\right)}{1 + \left(\nu_1 + \nu_5 \gamma - \frac{\nu_5^2}{2}\right)c} \beta_2 = 0 \\ \nu_4 + \frac{c\left(\nu_1 + \nu_5 \gamma - \frac{\nu_5^2}{2}\right)}{1 + c\left(\nu_1 + \nu_5 \gamma - \frac{\nu_5^2}{2}\right)c} \beta_3 = 0 \\ \nu_6 + \frac{c\left(\nu_1 + \nu_5 \gamma - \frac{\nu_5^2}{2}\right)}{1 + c\left(\nu_1 + \nu_5 \gamma - \frac{\nu_5^2}{2}\right)c} \beta_4 = 0 \end{cases}.$$

To solve it, we first set

$$\begin{aligned} \nu_5 \gamma - \frac{\nu_5^2}{2} &= (\nu_5 - 1)\gamma - \frac{(\nu_5 - 1)^2}{2} \\ \nu_5 &= \gamma + \frac{1}{2}. \end{aligned}$$

The other solutions are:

$$\begin{aligned} \nu_0 &= -\delta \log \left(1 + c \left(\frac{1}{2} \gamma^2 + \nu_1 - \frac{1}{8} \right) \right), \\ \nu_2 &= -\frac{4c\beta_1 \gamma^2 - c\beta_1 + 8c\nu_1 \beta_1}{4c\gamma^2 - c + 8c\nu_1 + 8} \\ &= -c\beta_1 \frac{\gamma^2/2 + \nu_1 - 1/8}{c(\gamma^2/2 - 1/8 - \nu_1) + 1} \\ \nu_3 &= -c\beta_2 \frac{\gamma^2/2 + \nu_1 - 1/8}{c(\gamma^2/2 - 1/8 - \nu_1) + 1}, \end{aligned}$$

$$\begin{aligned}\nu_4 &= -c\beta_3 \frac{\gamma^2/2 + \nu_1 - 1/8}{c(\gamma^2/2 - 1/8 - \nu_1) + 1}, \\ \nu_6 &= -c\beta_4 \frac{\gamma^2/2 + \nu_1 - 1/8}{c(\gamma^2/2 - 1/8 - \nu_1) + 1}.\end{aligned}$$

□

A.3 Proof of Proposition 3

Proof. Let us compute

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}} [\exp(-\alpha' K_{t+1})] &= \mathbb{E}_t^{\mathbb{Q}} [\exp(-\alpha_1 y_{t+1} - \alpha_2 RV_{t+1})] \\ &= \mathbb{E}_t^{\mathbb{P}} [M_{t,t+1} \exp(-\alpha_1 y_{t+1} - \alpha_2 RV_{t+1})] \\ &= \mathbb{E}_t^{\mathbb{P}} \left[\exp(-\nu_0 - \nu_2 RV_t^d - \nu_3 RV_t^w - \nu_4 RV_t^m - y_{t+1}(\nu_5 + \alpha_1) - RV_{t+1}(\nu_1 + \alpha_2) - \nu_6 L_t) \right] \\ &= \mathbb{E}_t^{\mathbb{P}} (\exp(-\nu_0 - \nu_2 RV_t^d - \nu_3 RV_t^w - \nu_4 RV_t^m - (\gamma RV_{t+1} + \sqrt{RV_{t+1}} \epsilon_{t+1})(\nu_5 + \alpha_1)) \\ &\quad \times \exp(-RV_{t+1}(\nu_1 + \alpha_2) - \nu_6 L_t)) \\ &= \mathbb{E}_t^{\mathbb{P}} (\exp(-\nu_0 - \nu_2 RV_t^d - \nu_3 RV_t^w - \nu_4 RV_t^m + \frac{(\nu_5 + \alpha_1)^2}{2} RV_{t+1}) \\ &\quad \times (-RV_{t+1}((\nu_5 + \alpha_1)\gamma + (\nu_1 + \alpha_2)) - \nu_6 L_t)) \\ &= \mathbb{E}_t^{\mathbb{P}} \left[\exp \left(-\nu_0 - \nu_2 RV_t^d - \nu_3 RV_t^w - \nu_4 RV_t^m - \varpi RV_{t+1} - \nu_6 L_t \right) \right] \\ &= \exp \left(-\nu_0 - \nu_2 RV_t^d - \nu_3 RV_t^w - \nu_4 RV_t^m - \nu_6 L_t \right) \mathbb{E}_t^{\mathbb{P}} [\exp(-\varpi RV_{t+1})] \\ &= \exp \left(-\nu_0 - \nu_2 RV_t^d - \nu_3 RV_t^w - \nu_4 RV_t^m - \nu_6 L_t \right) \varphi_{ARG}^{\mathbb{P}}(\varpi) \\ &= \exp \left(-\nu_0 - \nu_2 RV_t^d - \nu_3 RV_t^w - \nu_4 RV_t^m - \nu_6 L_t \right) \exp(-a(\varpi)(\beta'(\mathbf{R}\mathbf{V}_t, L_t)) - b(\varpi)) \\ &= \exp(-\nu_0 - b(\varpi)) \exp \left(-\nu_2 RV_t^d - \nu_3 RV_t^w - \nu_4 RV_t^m - \nu_6 L_t \right) \\ &\quad \times \exp \left(-a(\varpi) \left(\beta_1 RV_t^d + \beta_2 RV_t^w + \beta_3 RV_t^m + \beta_4 L_t \right) \right) \\ &= \exp(-\nu_0 - b(\varpi)) \\ &\quad \times \exp \left(RV_t^d(-a(\varpi)\beta_1 - \nu_2) + RV_t^w(-a(\varpi)\beta_2 - \nu_3) + RV_t^m(-a(\varpi)\beta_3 - \nu_4) + L_t(-a(\varpi)\beta_4 - \nu_6) \right)\end{aligned}$$

with $\varpi := -\frac{(\nu_5 + \alpha_1)^2}{2} + (\nu_5 + \alpha_1)\gamma + (\nu_1 + \alpha_2)$. Now, considering the non-arbitrage conditions in Proposition 2, ϖ becomes

$$\begin{aligned}\varpi &= -\frac{(\nu_5 + \alpha_1)^2}{2} + (\nu_5 + \alpha_1)\gamma + (\nu_1 + \alpha_2) = \alpha_1(\gamma - \nu_5) + \alpha_2 - \frac{1}{2}\alpha_1^2 + \gamma\nu_5 + \nu_1 - \frac{\nu_5^2}{2} \\ &= -\frac{1}{2}\alpha_1 + \alpha_2 - \frac{1}{2}\alpha_1^2 + \lambda\end{aligned}$$

where $\lambda = \nu_1 + \frac{\gamma^2}{2} - \frac{1}{8}$, $\nu_0 = -b(\lambda)$, and we also have

$$\nu_2 = -a(\lambda)\beta_1, \nu_3 = -a(\lambda)\beta_2, \nu_4 = -a(\lambda)\beta_3 \text{ and } \nu_6 = -a(\lambda)\beta_4$$

Thus, we get

$$\mathbb{E}_t^{\mathbb{Q}} [\exp(-\alpha' K_{t+1})] = \exp \left[-a^* \left(-\frac{1}{2}\alpha_1 + \alpha_2 - \frac{1}{2}\alpha_1^2 \right) (\beta(\mathbf{R}\mathbf{V}_t, L_t)) - b^* \left(-\frac{1}{2}\alpha_1 + \alpha_2 - \frac{1}{2}\alpha_1^2 \right) \right],$$

in which the vector $a^*(\zeta)\beta^*$ is such that

$$a^*(\zeta)\beta = a(\zeta + \lambda)\beta - a(\lambda)\beta = \frac{c^*\beta\zeta}{1 + c^*\zeta},$$

$$b^*(\zeta) = b(\zeta + \lambda) - b(\lambda) = \delta^* \ln(1 + c^*\zeta),$$

where

$$\begin{aligned} \beta^* &= \frac{\beta}{(1 + c\lambda)^2}, \\ \delta^* &= \delta, \\ c^* &= \frac{c}{1 + c\lambda}. \end{aligned}$$

□

B Technical Appendix: Multi-step-ahead conditional Fourier transform of log-returns

In this technical appendix we compute the multi-steps-ahead Fourier transform for the ARG(p) process. This tool is needed for our option pricing purposes, since it determines the multi-period Fourier transform of the log-returns. Here, we show the calculation for a generic ARG(p) process. The calculation for the HARG can be easily recovered by aggregating the ARG(p) terms, as in the HARG formulation of the conditional mean (see Eq.(3)). We derive the multi-period Fourier transform for the log-returns under the $\mathbb{Q}$, using the parameters $a^*(\cdot)$ and $b^*(\cdot)$ as found in Proposition 3 and Corollary 4. The Fourier transform under the physical measure has the same expression, but with $a(\cdot)$ and $b(\cdot)$ as found in Proposition 1.

In our setting, the price for a call option with maturity h and strike k can be written as

$$c_t(h, k) = B(t, t+h) E^{\mathbb{Q}} \left[(\exp^{R_{t,h}} - k)^+ \middle| \mathbf{R}\mathbf{V}_t \right] = E^{\mathbb{Q}} \left[\left(\exp^{\tilde{R}_{t,h}} - \tilde{k} \right)^+ \middle| \mathbf{R}\mathbf{V}_t \right],$$

with $\tilde{R}_{t,h} = \sum_{i=1}^h \tilde{r}_{t+i}$ representing the cumulated excess return of the underlying asset between t and $t+h$ and $\tilde{k} = B(t, t+h)k$ representing the discounted moneyness of the option. Following Carr and Madan (1999), we introduce the variable $s := \log(\tilde{k})$, and we define the rescaled pay-off

$$\phi(s) := \exp^{\alpha s} E^{\mathbb{Q}} \left[\left(\exp^{\tilde{R}_{t,h}} - \exp^s \right)^+ \middle| \mathbf{R}\mathbf{V}_t \right],$$

for $s \in \mathbb{R}$ and for $\alpha > 0$.

Then, we can show the following:

Lemma 5. *Under the martingale measure $\mathbb{Q}$, the Fourier transform of $\phi(s)$ is given by*

$$\hat{\phi}(u) = \int \exp^{-ius} \phi(s) ds = \frac{\Phi(iu - \alpha - 1)}{\alpha^2 + \alpha - u^2 - iu(2\alpha + 1)} \quad (25)$$

with $i^2 = -1$, $u \in \mathbb{R}$ and where

$$\Phi(z) = E^{\mathbb{Q}} \left[\left(\exp^{-\sum_{i=1}^h z r_{t+i}} \right) \middle| \mathbf{RV}_t \right], \quad (26)$$

with $z \in \mathbb{R}$ and r_{t+i} standing for the return from t to $t+1$.

Proof.

$$\begin{aligned} \hat{\phi}(u) &= \int_{\mathbb{R}} \exp^{-ius} \phi(s) ds \\ &= \int_{\mathbb{R}} \exp^{-ius} \exp^{\alpha s} E^{\mathbb{Q}} \left[\left(\exp^{\tilde{R}_{t,h}} - \exp^s \right)^+ \middle| \mathbf{RV}_t \right] ds \\ &= \int_{\mathbb{R}} \exp^{-i(us-\alpha s)} E^{\mathbb{Q}} \left[\left(\exp^{\tilde{R}_{t,h}} - \exp^s \right)^+ \middle| \mathbf{RV}_t \right] ds \\ &= E^{\mathbb{Q}} \left[\exp^{\tilde{R}_{t,h}} \int_{-\infty}^{\tilde{R}_{t,h}} \exp^{-i(us-\alpha s)} ds + \int_{-\infty}^{\tilde{R}_{t,h}} \exp^{-i(u-\alpha-1)s} ds \middle| \mathbf{RV}_t \right] \\ &= E^{\mathbb{Q}} \left[-\frac{1}{iu-1} \exp^{-(iu-\alpha-1)\tilde{R}_{t,h}} + \frac{1}{iu-1-\alpha} \exp^{-(iu-\alpha-1)\tilde{R}_{t,h}} \middle| \mathbf{RV}_t \right] \\ &= \frac{\Phi(iu - \alpha - 1)}{\alpha^2 + \alpha - u^2 - iu(2\alpha + 1)}. \end{aligned} \quad (27)$$

□

To go further in our derivation of the multi-period joint Fourier transform, we need the expression for $\Phi(\cdot)$ in (27). To derive its expression under the martingale measure, we start from the return at time t , conditional on $\mathbf{RV}_t$. This quantity is given by $\tilde{r}_t = -\frac{1}{2}RV_t + \sqrt{RV_t}\varepsilon_t$, with $\varepsilon \sim N(0,1)$ and RV_t is an ARG(p) process, with parameters ρ^*, δ^*, c^* (see Proposition 2). Therefore, for $\mathbf{z} = (z, \dots, z)$, we have

$$\begin{aligned} \Phi(\mathbf{z}) &= E^{\mathbb{Q}} \left[\left(\exp^{-z \sum_{i=1}^h \tilde{r}_{t+i}} \right) \middle| \mathbf{RV}_t \right] \\ &= E^{\mathbb{Q}} \left[\left(\exp^{\frac{z}{2} \sum_{i=1}^h RV_{t+i} - z \sum_{i=1}^h RV_{t+i} \varepsilon_{t+i}} \right) \middle| \mathbf{RV}_t \right] \\ &= E^{\mathbb{Q}} \left[\left(\exp^{-\left(-\frac{z+z^2}{2}\right) \sum_{i=1}^h RV_{t+i}} \right) \middle| \mathbf{RV}_t \right], \end{aligned}$$

where the function $\Phi(z)$ is the Fourier transform of the ARG(p) process. We notice that in Eq. (28) we compute this function by premultiplying each return by the same real number z . Clearly, we can

generalize this formula and compute the function $\Phi(\mathbf{v})$ under the martingale measure for a generic vector $\mathbf{v} = (v_{t+1}, \dots, v_{t+h}) \in \mathbb{R}^h$. Then, the result in Eq. (28) follows as a special case. Therefore, we need a closed-form expression for

$$\Phi(\mathbf{v}) = E^{\mathbb{Q}} \left[\left(\exp^{-\sum_{i=1}^h v_{t+i} RV_{t+i}} \right) \middle| \mathbf{RV}_t \right]. \quad (28)$$

The expression of $\Phi(\mathbf{v})$ is provided in the following:

Lemma 6. *The Fourier transform $\Phi(\mathbf{v})$, for $\mathbf{v} = (v_{t+1}, \dots, v_{t+h}) \in \mathbb{R}^h$, for $h > 1$ and $h \in \mathbb{N}$, is given by*

$$\Phi(\mathbf{v}) = \exp^{-B_0 - \mathbf{A}_0' \mathbf{RV}_t}. \quad (29)$$

The coefficients B_0 and $\mathbf{A}_0 \in \mathbb{R}^p$ satisfy the following system of recursive equations for $m = 1, \dots, \max(j, p)$ and for every given step $j = h - 1, \dots, 1, 0$,

$$B_j = B_{j+1} + b^*(C_{j+1}(RV_{t+j})) \quad (30)$$

$$A_{m,j} = a_m^*(C_{j+1}(RV_{t+j})) + A(m+1, j+1) \quad (31)$$

with final conditions $B_h = 0$ and $A_{m,h} = v_{t+h-m+1}$, for every m . In our notation,

$$b^*(x) = \delta^* \log(1 + c^* x)$$

and

$$a_m^*(x) := \begin{cases} \frac{c^* x}{1+c^* x} \beta_m^* & \text{if } m \leq p \\ 0 & \text{if } m > p, \end{cases}$$

and $C_{j+1}(RV_{t+j+1})$ is the operator giving the coefficient of the RV_{t+j+1} at the $(j+1)$ th step (or equivalently, it is the first element of the matrix A at the $(j+1)$ th step).

Proof. The proof follows straight from Darolles et al. (2006) and Gouriéroux and Jasiak (2006). Specifically, we first need to rewrite the ARG(p) as a p -dimensional vector of ARG(1) as in Proposition 2 of Darolles et al. (2006). Then, we rewrite the Fourier transform for this vector as in formulas (8)-(9) of Gouriéroux and Jasiak (2006). Finally, the recursive formulas in Eq. (30) and Eq. (31) follow from Proposition 5 in Darolles et al. (2006). $\square$