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TETRAHEDRAL FORMS IN MONOIDAL CATEGORIES AND 3-MANIFOLD INVARIANTS

NATHAN GEER, RINAT KASHAEV, AND VLADIMIR TURAEV

ABSTRACT. We introduce systems of objects and operators in linear monoidal categories called $\widehat{\Psi}$-systems. A $\widehat{\Psi}$-system satisfying several additional assumptions gives rise to a topological invariant of triples (a closed oriented 3-manifold M, a principal bundle over M, a link in M). This construction generalizes the quantum dilogarithmic invariant of links appearing in the original formulation of the volume conjecture. We conjecture that all quantum groups at odd roots of unity give rise to $\widehat{\Psi}$-systems and we verify this conjecture in the case of the Borel subalgebra of quantum $\mathfrak{sl}_2$.

INTRODUCTION

One of the fundamental achievements of quantum topology was a discovery of deep connections between monoidal categories and 3-dimensional manifolds. It was first observed by O. Viro and V. Turaev that the category of representations of the quantum group $U_q(\mathfrak{sl}_2)$ gives rise to a topological invariant of 3-manifolds. The invariant is obtained as a state sum on a triangulation of a 3-manifold; the key ingredients of the state sum are the $6j$-symbols. This construction was generalized to other categories by several authors including J. Barrett, B. Westbury, A. Ocneanu, S. Gelfand, D. Kazhdan and others. Their results may be summarized by saying that every spherical fusion category gives rise to a state sum 3-manifold invariant. Similar methods apply to links in 3-manifolds and to 3-manifolds endowed with principal fiber bundles. A related but somewhat different line of development was initiated by Kashaev [8]. He defined a state sum invariant of links in 3-manifolds using "charged" versions of the $6j$-symbols associated with certain representations of the Borel subalgebra of $U_q(\mathfrak{sl}_2)$. The work of Kashaev was further extended by S. Baseilhac and R. Benedetti, see [1], [2].

The aim of this paper is to analyze categorical foundations of the Kashaev-Baseilhac-Benedetti theory. The key new notions in our approach are the ones of Ψ-systems and $\widehat{\Psi}$-systems in linear monoidal categories. The Ψ-systems provide a general framework for $6j$-symbols. Roughly speaking, a Ψ-system is a family of simple objects of the category $\{V_i\}_{i \in I}$ closed under duality and such that for "almost all" $i, j \in I$, the identity endomorphism of $V_i \otimes V_j$ splits as a sum of certain compositions $\{V_i \otimes V_j \to V_k \to V_i \otimes V_j\}_{k \in I}$ (see Section 1). Examples

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of Ψ-systems can be derived from quantum groups at roots of unity or, more generally, from Cayley-Hamilton Hopf algebras, see Section 11. Every fusion category has a Ψ-system formed by arbitrary representatives of the isomorphism classes of all simple objects. A Ψ-system in a linear monoidal category gives rise to a vector space H (the space of multiplicities), a linear form T on $H^{\otimes 4}$ (the tetrahedral evaluation form), and two automorphisms A, B (obtained by taking adjoints of morphisms). The vector space H has a natural symmetric bilinear form which allows us to consider the transposes A^*, B^* of A, B. We use T to define $6j$-symbols and we use A, B, A^*, B^* to formulate the tetrahedral symmetry of the $6j$-symbols. We also develop a T-calculus for endomorphisms of H which allows us to speak of equality/commutation of operators "up to composition with T". These definitions and results occupy Sections 1–5.

To define 3-manifold invariants we need to fix square roots of the operators

$$L = A^*A, \; R = B^*B, \; C = (AB)^3 \in \mathrm{End}(H).$$

A Ψ-system endowed with such square roots $L^{\frac{1}{2}}$, $R^{\frac{1}{2}}$, $C^{\frac{1}{2}}$ satisfying appropriate relations is said to be a $\widehat{\Psi}$-system. The $\widehat{\Psi}$-systems provide a general framework for so-called "charged" $6j$-symbols depending on two additional integer of half-integer parameters. The advantage of the charged $6j$-symbols lies in the simpler tetrahedral symmetry. This material occupies Sections 6–8.

We need two assumptions on a $\widehat{\Psi}$-system to produce a 3-manifold invariant. The first assumption says essentially that the operators $L^{\frac{1}{2}}$ and $R^{\frac{1}{2}}$ commute up to composition with T and multiplication by a certain scalar $\widetilde{q}$. The second assumption introduces additional data: a group G and a family of finite subsets $\{I_g\}_{g \in G}$ of I satisfying certain conditions. We use this data to define a numerical topological invariant of any tuple (a closed connected oriented 3-manifold M, a non-empty link $L \subset M$, a conjugacy class of homomorphisms $\pi_1(M) \to G$, an element of $H^1(M; \mathbb{Z}/2\mathbb{Z})$), see Sections 9, 10. The invariant in question is defined as a state sum on a Hamiltonian triangulation of (M, L). To encode the Hamiltonian path L into the state sum, we use the integral charges on H-triangulations first introduced in [8]. The theory of charges subsequently has been developed in [2]. It is a natural extension of the theory of angle structures due to W. Neumann, see, for example, [14, 15]. The key ingredients of our state sum are the charged $6j$-symbols. The resulting invariant is well-defined up to multiplication by integer powers of $\widetilde{q}$.

We conjecture that the Ψ-systems associated with quantum groups and their Borel subalgebras at odd roots of unity extend to $\widehat{\Psi}$-systems satisfying all our requirements. We verify this conjecture in the case of the Borel subalgebra of $U_q(\mathfrak{sl}_2)$, see Sections 11, 12. Geer and Patureau-Mirand [6] verify the conjecture for all quantum groups associated to simple Lie algebras and prove that the usual modular categories arising from quantum groups have $\widehat{\Psi}$-systems satisfying all our

requirements. The conjecture is open for Borel subalgebras of quantum groups other than $U_q(\mathfrak{sl}_2)$. We expect that the associated invariants are closely related with the invariants constructed in [7, 12]. In the case of the example of Section 12 with the trivial homomorphism $\pi_1(M) \to G$, this construction coincides with the one of Kashaev [8]. The latter invariant enters the volume conjecture [10], and for links in S^3, it is a specialization of the colored Jones polynomial [13]. Precise relationships of our invariants with the Baseilhac–Benedetti 3-manifold invariants are yet unclear.

1. Ψ-systems in monoidal categories

1.1. **Monoidal Ab-categories.** A *monoidal (tensor) category* $\mathcal{C}$ is a category equipped with a covariant bifunctor $\boxtimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the tensor product, an associativity constraint, a unit object $\mathbb{I}$, and left and right unit constraints such that the Triangle and Pentagon Axioms hold. When the associativity constraint and the left and right unit constraints are all identities, the category $\mathcal{C}$ is a *strict* monoidal (tensor) category. By MacLane's coherence theorem, any monoidal category is equivalent to a strict monoidal category. To simplify the exposition, we formulate further definitions only for strict monoidal categories; the reader will easily extend them to arbitrary monoidal categories.

A monoidal category $\mathcal{C}$ is said to be an *Ab-category* if for any objects V, W of $\mathcal{C}$, the set of morphisms $\mathrm{Hom}(V, W)$ is an additive abelian group and the composition and tensor product of morphisms are bilinear. Composition of morphisms induces a commutative ring structure on the abelian group $\mathsf{k} = \mathrm{End}(\mathbb{I})$. The resulting ring is called the *ground ring* of $\mathcal{C}$. For any objects V, W of $\mathcal{C}$ the abelian group $\mathrm{Hom}(V, W)$ becomes a left k-module via $kf = k \boxtimes f$ for $k \in \mathsf{k}$ and $f \in \mathrm{Hom}(V, W)$. We assume that the tensor multiplication of morphisms in $\mathcal{C}$ is k-bilinear.

An object V of $\mathcal{C}$ is *simple* if $\mathrm{End}(V) = \mathsf{k} \, \mathrm{Id}_V$. For any simple object V and $f \in \mathrm{End}(V)$, there is a unique $k \in \mathsf{k}$ such that $f = k \, \mathrm{Id}_V$. This k is denoted $\langle f \rangle$.

Fix from now on a monoidal Ab-category $\mathcal{C}$ whose ground ring k is a field. We shall use the symbol $\otimes$ for the tensor product of k-vector spaces over k and the symbol $\boxtimes$ for the tensor product in $\mathcal{C}$.

1.2. **Ψ-systems.** A *Ψ-system*[1] in $\mathcal{C}$ consists of

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[1]This name is inspired by the logo of the Indiana University, where the definition has been finalized.

(1) a distinguished set of simple objects $\{V_i\}_{i \in I}$ such that $\operatorname{Hom}(V_i, V_j) = 0$ for all $i \neq j$;

(2) an involution $I \to I$, $i \mapsto i^*$;

(3) two families of morphisms $\{b_i \colon \mathbb{I} \to V_i \boxtimes V_{i^*}\}_{i \in I}$ and $\{d_i \colon V_i \boxtimes V_{i^*} \to \mathbb{I}\}_{i \in I}$, such that for all $i \in I$,

$$(\operatorname{Id}_{V_i} \boxtimes d_{i^*})(b_i \boxtimes \operatorname{Id}_{V_i}) = \operatorname{Id}_{V_i} \quad \text{and} \quad (d_i \boxtimes \operatorname{Id}_{V_i})(\operatorname{Id}_{V_i} \boxtimes b_{i^*}) = \operatorname{Id}_{V_i}. \tag{1}$$

To formulate the fourth (and the last) requirement on the Ψ-systems, set

$$H_k^{ij} = \operatorname{Hom}(V_k, V_i \boxtimes V_j) \quad \text{and} \quad H_{ij}^k = \operatorname{Hom}(V_i \boxtimes V_j, V_k).$$

for any $i, j, k \in I$. We require that

(4) for any $i, j \in I$ such that $H_k^{ij} \neq 0$ for some $k \in I$, the identity morphism $\operatorname{Id}_{V_i \boxtimes V_j}$ is in the image of the linear map

$$\bigoplus_{k \in I} H_k^{ij} \otimes H_{ij}^k \to \operatorname{End}(V_i \boxtimes V_j), \quad x \otimes y \mapsto x \circ y. \tag{2}$$

We fix from now on a Ψ-system in $\mathcal{C}$ and keep the notation introduced above.

Lemma 1. *For any $i, j, k \in I$, the linear spaces H_k^{ij} and H_{ij}^k are finite dimensional, and the bilinear pairing*

$$H_{ij}^k \otimes H_k^{ij} \to \mathsf{k}, \quad x \otimes y \mapsto \langle x \circ y \rangle, \tag{3}$$

is non-degenerate. In particular, $\dim H_{ij}^k = \dim H_k^{ij}$.

Lemma 2. *For any $i, j \in I$, there are only finitely many $k \in I$ such that $H_k^{ij} \neq 0$.*

These lemmas will be proven in the next subsection after a little preparation.

1.3. **The operators A and B.** Consider the vector space $H = \hat{H} \oplus \check{H}$, where

$$\hat{H} = \bigoplus_{i,j,k \in I} H_{ij}^k \quad \text{and} \quad \check{H} = \bigoplus_{i,j,k \in I} H_k^{ij}.$$

Let

$$\pi_{ij}^k \colon H \to H_{ij}^k, \quad \pi_k^{ij} \colon H \to H_k^{ij}, \quad \hat{\pi} \colon H \to \hat{H}, \quad \check{\pi} \colon H \to \check{H}$$

be the obvious projections. We define linear maps $A, B \colon H \to H$ by

$$Ax = \sum_{i,j,k \in I} ((\operatorname{Id}_{V_{i^*}} \boxtimes \pi_{ij}^k x)(b_{i^*} \boxtimes \operatorname{Id}_{V_j}) + (d_{i^*} \boxtimes \operatorname{Id}_{V_j})(\operatorname{Id}_{V_{i^*}} \boxtimes \pi_k^{ij} x)),$$

$$Bx = \sum_{i,j,k \in I} ((\pi_{ij}^k x \boxtimes \operatorname{Id}_{V_{j^*}})(\operatorname{Id}_{V_i} \boxtimes b_j) + (\operatorname{Id}_{V_i} \boxtimes d_j)(\pi_k^{ij} x \boxtimes \operatorname{Id}_{V_{j^*}})).$$

For each $x \in H$, there are only finitely many non-zero terms in these sums, since x has only finitely many non-zero components $\pi_{ij}^k x$ and $\pi_k^{ij} x$. We can represent the definitions of A and B in the following graphical form:

where we use the graphical notation

Lemma 3. *The operators A and B are involutive and satisfy the "exchange" relations:*

$$\pi_k^{ij} A = A\pi_{i^*k}^{j}, \quad \pi_k^{ij} B = B\pi_{kj^*}^{i}, \quad \pi_{ij}^k A = A\pi_j^{i^*k}, \quad \pi_{ij}^k B = B\pi_i^{kj^*}. \qquad (4)$$

Proof. The exchange relations easily follow from the inclusions

$$A(H_{ij}^k) \subset H_{j}^{i^*k}, \quad A(H_k^{ij}) \subset H_{i^*k}^{j}, \quad B(H_{ij}^k) \subset H_i^{kj^*}, \quad B(H_k^{ij}) \subset H_{kj^*}^{i}. \qquad (5)$$

For any $x \in H$,

and

Thus, $A^2 = 1$. A similar calculation shows that $B^2 = 1$. $\qquad\square$

Proof of Lemma 1. Assume first that $H_k^{ij} \neq 0$. By the basic condition,

$$\mathrm{Id}_{V_i \boxtimes V_j} = \sum_{l \in X} \sum_{\alpha \in R_l} e_{l\alpha} e^{l\alpha}, \qquad (6)$$

where X is a finite subset of I and for all $l \in X$, we have a finite set of indices R_l, linearly independent vectors $\{e_{l\alpha}\}_{\alpha \in R_l}$ in H_l^{ij} and certain vectors $e^{l\alpha}$ in H_{ij}^l. For any $x \in H_k^{ij}$,

$$x = \mathrm{Id}_{V_i \boxtimes V_j} x = \sum_{l \in X} \sum_{\alpha \in R_l} e_{l\alpha} e^{l\alpha} x = \sum_{l \in X} \sum_{\alpha \in R_l} e_{l\alpha} \langle e^{l\alpha} x \rangle \delta_{k,l} = \sum_{\alpha \in R_k} e_{k\alpha} \langle e^{k\alpha} x \rangle, \quad (7)$$

where $\delta_{k,l}$ is the Kronecker delta. Thus, the vectors $e_{k\alpha}$ with $\alpha \in R_k$ generate H_k^{ij}. Since these vectors are linearly independent, they form a (finite) basis of H_k^{ij}. Similarly, for any $y \in H_{ij}^k$,

$$y = y \, \mathrm{Id}_{V_i \boxtimes V_j} = \sum_{\alpha \in R_k} \langle y e_{k\alpha} \rangle e^{k\alpha}. \qquad (8)$$

Therefore the vectors $e^{k\alpha}$ with $\alpha \in R_k$, generate H_{ij}^k. For all $\alpha, \beta \in R_k$, Formula (7) with $x = e_{k\beta}$ implies that $\langle e^{k\alpha} e_{k\beta} \rangle = \delta_{\alpha,\beta}$. Hence $\{e^{k\alpha}\}_{\alpha \in R_k}$ is a basis of H_{ij}^k dual to the basis $\{e_{k\alpha}\}_{\alpha \in R_k}$ of H_k^{ij} with respect to the pairing (3). Therefore, this pairing is non-degenerate.

It remains to show that $H_k^{ij} = 0$ implies $H_{ij}^k = 0$. Indeed, if $H_{ij}^k \neq 0$, then $H_j^{i*k} = A(H_{ij}^k) \neq 0$. By the preceding argument, $A(H_k^{ij}) = H_{i*k}^j \neq 0$. Hence $H_k^{ij} \neq 0$. $\qquad\square$

Proof of Lemma 2. If $H_k^{ij} \neq 0$, then by Formula (7), k belongs to the finite set X appearing in (6). $\qquad\square$

1.4. **Transposition of operators.** We provide the vector space $H = \hat{H} \oplus \check{H}$ with the symmetric bilinear pairing $\langle \, , \rangle$ by

$$\langle x, y \rangle = \sum_{i,j,k \in I} \left(\langle \pi_{ij}^k x \, \pi_k^{ij} y \rangle + \langle \pi_{ij}^k y \, \pi_k^{ij} x \rangle \right) \in \mathsf{k} \qquad (9)$$

for any $x, y \in H$. Note that $\langle \hat{H}, \hat{H} \rangle = \langle \check{H}, \check{H} \rangle = 0$.

A *transpose* of $f \in \mathrm{End}(H)$ is a map $f^* \in \mathrm{End}(H)$ such that $\langle fx, y \rangle = \langle x, f^*y \rangle$ for all $x, y \in H$. Lemma 1 implies that if a transpose f^* of f exists, then it is unique and $(f^*)^* = f$.

Lemma 4. *The canonical projections have transposes computed as follows:*

$$\check{\pi}^* = \hat{\pi} \quad \text{and} \quad (\pi_k^{ij})^* = \pi_{ij}^k.$$

Proof.

$$\langle x, \hat{\pi}y \rangle = \langle \check{\pi}x, \hat{\pi}y \rangle = \langle \check{\pi}x, y \rangle,$$
$$\langle x, \pi_{ij}^k y \rangle = \langle \pi_k^{ij} x, \pi_{ij}^k y \rangle = \langle \pi_k^{ij} x, y \rangle.$$

$\square$

Lemma 5. *The transposes of the operators A and B exist and*

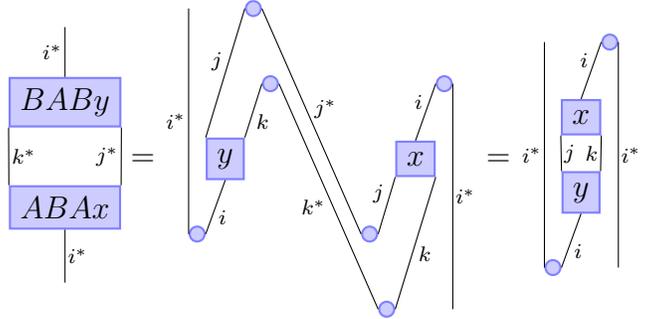$$A^* B^* A^* = BAB. \tag{10}$$

Proof. The existence of A^* and B^* follows from Lemma 1 and the inclusions (5). Note that

$$A^*(H_{ij}^k) \subset H_j^{i*k}, \quad A^*(H_k^{ij}) \subset H_{i*k}^j, \quad B^*(H_{ij}^k) \subset H_i^{kj*}, \quad B^*(H_k^{ij}) \subset H_{kj*}^i. \tag{11}$$

To prove (10), observe that for any $x \in \hat{H}$ and $y \in \check{H}$,

$$\langle x, y \rangle = \langle BABy, ABAx \rangle. \tag{12}$$

Here is a graphical proof of this formula for $x \in H_{jk}^i$ and $y \in H_i^{jk}$ with $i, j, k \in I$.

Now we can prove (10). Applying (12) to $x = x_1$ and $y = BABx_2$ with $x_1, x_2 \in \hat{H}$, we obtain

$$\langle x_1, BABx_2 \rangle = \langle (BAB)^2 x_2, ABAx_1 \rangle = \langle x_2, ABAx_1 \rangle = \langle ABAx_1, x_2 \rangle.$$

Applying (12) to $x = ABAy_1$ and $y = y_2$ with $y_1, y_2 \in \check{H}$, we obtain

$$\langle y_1, BABy_2 \rangle = \langle BABy_2, y_1 \rangle = \langle ABAy_1, y_2 \rangle.$$

Hence $BAB = (ABA)^* = A^*B^*A^*$. $\square$

2. The tetrahedral forms

2.1. **Operations on tensor powers.** We recall the usual notation for operations on the tensor powers of a vector space. Given a k-vector space V and an integer $n \geq 2$, the symbol $V^{\otimes n}$ denotes the tensor product of n copies of V over k. Let $\mathbb{S}_n$ be the symmetric group on $n \geq 2$ letters. Recall the standard action $\mathbb{S}_n \to \operatorname{Aut}(V^{\otimes n})$, $\sigma \mapsto P_\sigma$. By definition, for distinct $i, j \in \{1, \ldots, n\}$, the flip

$P_{(ij)}$ permutes the i-th and the j-th tensor factors keeping the other tensor factors. For $f \in \mathrm{End}(V)$ and $i = 1, \ldots, n$, set

$$f_i = \mathrm{id}^{\otimes(i-1)} \otimes f \otimes \mathrm{id}^{\otimes(n-i)} \in \mathrm{End}(V^{\otimes n}).$$

Note the exchange relations $P_\sigma f_i = f_{\sigma(i)} P_\sigma$ for any $\sigma \in \mathbb{S}_n$ and the commutativity relation $f_i g_j = g_j f_i$ for any $f, g \in \mathrm{End}(V)$ and $i \neq j$.

Given $F \in \mathrm{End}(V \otimes_{\mathsf{k}} V)$, we define for any $i, j \in \{1, \ldots, n\}$ with $i \neq j$ an endomorphism F_{ij} of $V^{\otimes n}$ as follows. If $i < j$, then F_{ij} acts as F on the i-th and j-th tensor factors of $V^{\otimes n}$ keeping the other tensor factors. If $i > j$, then $F_{ij} = P_{(ij)} F_{ji} P_{(ij)}$.

2.2. **The forms T and $\bar{T}$.** Recall the vector space $H = \hat{H} \oplus \check{H}$ from Section 1.3. We define two linear forms $T, \bar{T} \colon H^{\otimes 4} \to \mathsf{k}$ by the following diagrammatic formulae: for any $u, v, x, y \in H$,

The indices i, j, k, l, m, n in both formulas run over all elements of I. For any given $u, v, x, y \in H$, only a finite number of terms in these formulas may be non-zero.

Lemma 6. *We have*

$$TP_{(4321)} = \bar{T} A_1^* A_3, \tag{13a}$$

$$TP_{(23)} = \bar{T} A_2 B_3, \tag{13b}$$

$$TP_{(1234)} = \bar{T} B_2 B_4^*. \tag{13c}$$

Proof. Since A^* is an involution, (13a) is equivalent to the identity

$$T(u \otimes v \otimes x \otimes A^* y) = \bar{T}(y \otimes u \otimes Av \otimes x), \quad u, v, x, y \in H,$$

which is a direct consequence of the identity

$$\left\langle \;\; \right\rangle = \langle \xi, A^*y \rangle = \langle A\xi, y \rangle = \langle y, A\xi \rangle = \left\langle \;\; \right\rangle, \quad \xi = \;\; \in H_{in}^m.$$

The other two identities are verified in a similar manner. $\qquad\square$

The formulas

$$P_{(12)} = P_{(4321)}P_{(23)}P_{(1234)}, \quad P_{(34)} = P_{(1234)}P_{(23)}P_{(4321)},$$

allow us to compute the action of the permutations $P_{(12)}$ and $P_{(34)}$ on T:

$$TP_{(12)} = \bar{T}P_{(23)}P_{(1234)}A_4^*A_1 = TP_{(1234)}(BA)_1 A_2 A_4^* = \bar{T}(BA)_1(BA)_2(AB)_4^*, \quad (14)$$

$$TP_{(34)} = \bar{T}P_{(23)}P_{(4321)}B_4 B_1^* = TP_{(4321)}B_3(AB)_4 B_1^* = \bar{T}(BA)_1^*(AB)_3(AB)_4. \quad (15)$$

The action of the permutations on $\bar{T}$ can be easily determined from the involutivity of A, B, $P_{(12)}$, $P_{(23)}$, $P_{(34)}$. The resulting formulae can be obtained from those for T via the substitutions $T \leftrightarrow \bar{T}$ and $A \leftrightarrow B$.

2.3. **The adjoint operators.** We define the operators $\check{H}^{\otimes 2} \to \check{H}^{\otimes 2}$ adjoint to T and $\bar{T}$. For all $i, j, k \in I$ pick dual bases $(e_{k\alpha}^{ij})_\alpha$ and $(e_{ij}^{k\alpha})_\alpha$ in the multiplicity spaces H_k^{ij} and H_{ij}^k, respectively. For the vectors of these bases, we shall use the graphical notation

$$e_{k\alpha}^{ij} = \;\; \qquad \text{and} \qquad e_{ij}^{k\alpha} = \;\; .$$

Let $\tau, \bar{\tau} \in \operatorname{End}(\check{H}^{\otimes 2})$ be the operators defined by the graphical formulae

$$\tau(x \otimes y) = \sum_{i,\dots,n \in I} \sum_\alpha \; \boxed{\text{(diagram)}} \; \otimes \; \boxed{\text{(diagram)}}, \qquad \bar{\tau}(x \otimes y) = \sum_{i,\dots,n \in I} \sum_\alpha \; \boxed{\text{(diagram)}} \; \otimes \; \boxed{\text{(diagram)}}$$

where i, j, k, l, m, n run over all elements of I. By Lemma 2, for any $x, y \in \check{H}$, there are only finitely many terms in the expansions for $\tau(x \otimes y)$ and $\bar{\tau}(x \otimes y)$.

The operators τ and $\bar{\tau}$ do not depend on the choice of the bases in the multiplicity spaces. Indeed, these operators are adjoint to $T, \bar{T}$ in the sense that

$$\langle\langle u \otimes v, \tau(x \otimes y)\rangle\rangle = T(u \otimes v \otimes x \otimes y)$$

and

$$\langle\langle u \otimes v, \bar{\tau}(x \otimes y)\rangle\rangle = \bar{T}(u \otimes v \otimes x \otimes y)$$

for all $u, v \in \hat{H}, x, y \in \check{H}$. Here the bilinear pairing

$$\langle\langle \cdot, \cdot \rangle\rangle : (\hat{H} \otimes \hat{H}) \times (\check{H} \otimes \check{H}) \to \mathsf{k}$$

is defined by $\langle\langle u \otimes v, x \otimes y\rangle\rangle = \langle u, x \rangle \langle v, y \rangle$ where $\langle \cdot, \cdot \rangle$ is the symmetric bilinear form on $H = \hat{H} \oplus \check{H}$ introduced in Section 1.4.

2.4. **The Pentagon and Inversion identities.** To formulate the properties of τ and $\bar{\tau}$, we need further notation. For any $i, j \in I$ set

$$g_{i,j} = \begin{cases} 1 & \text{if there is } k \in I \text{ such that } H_k^{ij} \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

We define two endomorphisms $^\bullet\pi$ and $\pi^\bullet$ of $\check{H}^{\otimes 2}$ by

$$^\bullet\pi = \sum_{i,j,k,l,m \in I} g_{i,j} \, \pi_m^{il} \otimes \pi_l^{jk} \quad \text{and} \quad \pi^\bullet = \sum_{i,j,k,l,m \in I} g_{j,l} \, \pi_k^{ij} \otimes \pi_m^{kl}.$$

Clearly, $^\bullet\pi$ and $\pi^\bullet$ are commuting projectors onto certain subspaces of $\check{H}^{\otimes 2}$.

Lemma 7. *The operators τ and $\bar{\tau}$ satisfy*

(i) *the pentagon identity in* $\operatorname{End}(\check{H}^{\otimes 3})$:

$$\tau_{23}\tau_{13}\tau_{12} = \tau_{12}\tau_{23}(^\bullet\pi)_{21},$$

(ii) *the inversion relations in* $\operatorname{End}(\check{H}^{\otimes 2})$:

$$\tau_{21}\bar{\tau} = \pi^\bullet \quad \text{and} \quad \bar{\tau}\tau_{21} = {}^\bullet\pi,$$

where $\tau_{21} = P_{(12)}\tau P_{(12)}$ and $\bar{\tau}_{21} = P_{(12)}\bar{\tau} P_{(12)}$.

Proof. (**i**) For $x, y, z \in \check{H}$,

$$\tau_{23}\tau_{13}\tau_{12}(x \otimes y \otimes z)$$

$$= \sum_{i,\dots,n,\iota} \tau_{23}\tau_{13}\left(x \otimes \iota \otimes z \right) = \sum_{i,\dots,q,\iota,\kappa} \tau_{23}\left(x \otimes \iota \otimes \kappa \right)$$

$$= \sum_{i,\dots,r,\iota,\kappa,\lambda} x \otimes \iota \otimes \lambda = \sum_{i,\dots,r,\iota,\dots,\mu} \left\langle \lambda \atop \kappa \right\rangle x \otimes \mu \otimes \lambda$$

$$= \sum_{i,\dots,r,\lambda,\mu} g_{i,j}\, x \otimes \mu \otimes \lambda = \sum_{i,\dots,r,\lambda} g_{i,j}\,\tau_{12}\left(x \otimes y \otimes \lambda \right)$$

$$= \sum_{i,\dots,n} g_{i,j} \tau_{12} \tau_{23} \big(\boxed{x} \otimes \boxed{y} \otimes z \big) = \tau_{12} \tau_{23} ({}^{\bullet}\pi)_{21} (x \otimes y \otimes z).$$

All these equalities follow directly from the definitions except the fifth equality. The sum on its left-hand side is preserved if we insert an additional factor $g_{o,k}$. Indeed, if the triangular block with the vertices labeled μ, λ, ι contributes non-zero, then necessarily $g_{o,k} = 1$. This allows us to sum up over all p, κ and then over all k, ι to obtain the fifth equality.

(**ii**) For $x, y \in \check{H}$,

$$\tau_{21} \bar{\tau}(x \otimes y) = \sum_{i,\dots,n,\lambda} \tau_{21} \big(\boxed{x} \otimes \lambda \big) = \sum_{i,\dots,o,\lambda,\mu} \mu \otimes \lambda$$

$$= \sum_{i,\dots,m,\mu} g_{j,l} \, \mu \otimes \boxed{x} = \sum_{i,\dots,m} g_{j,l} \, \boxed{x} \otimes \boxed{y} = \pi^{\bullet}(x \otimes y).$$

The second inversion relation is proved similarly.   □

3. The $6j$-symbols

3.1. **Notation.** For any $i, j, k \in I$, the non-degenerate pairing $H_{ij}^k \otimes H_k^{ij} \to \mathsf{k}$ defined in Lemma 1 will be denoted $*_{ij}^k$. Composing this pairing with the flip $H_k^{ij} \otimes H_{ij}^k \to H_{ij}^k \otimes H_k^{ij}$, we obtain a non-degenerate pairing $H_k^{ij} \otimes H_{ij}^k \to \mathsf{k}$

denoted $*_k^{ij}$. We shall use these pairings to identify the dual of H_{ij}^k with H_k^{ij} and the dual of H_k^{ij} with H_{ij}^k. The pairings $*_{ij}^k$ and $*_k^{ij}$ induce the tensor contractions

$$U \otimes H_{ij}^k \otimes V \otimes H_k^{ij} \otimes W \to U \otimes V \otimes W,$$

$$U \otimes H_k^{ij} \otimes V \otimes H_{ij}^k \otimes W \to U \otimes V \otimes W,$$

where U, V, W are arbitrary k-vector spaces. These tensor contractions will be denoted by the same symbols $*_{ij}^k$ and $*_k^{ij}$ respectively.

3.2. **Definition of $6j$-symbols.** For any $i, j, k, l, m, n \in I$, the restriction of $T : H^{\otimes 4} \to \mathsf{k}$ to the tensor product

$$H_{kl}^m \otimes H_{ij}^k \otimes H_n^{jl} \otimes H_m^{in} \subset \hat{H} \otimes \hat{H} \otimes \check{H} \otimes \check{H} \subset H^{\otimes 4}$$

gives a vector in the k-vector space

$$\operatorname{Hom}(H_{kl}^m \otimes H_{ij}^k \otimes H_n^{jl} \otimes H_m^{in}, \mathsf{k}) = H_m^{kl} \otimes H_k^{ij} \otimes H_{jl}^n \otimes H_{in}^m. \qquad (16)$$

This vector is denoted

$$\left\{ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right\} \qquad (17)$$

and called the *positive $6j$-symbol* determined by the tuple i, j, k, l, m, n. In graphical notation, the $6j$-symbol (17) is the summand in the definition of T in Section 2.2 corresponding to the tuple $i, j, k, l, m, n \in I$. Thus, for any $u, v, x, y \in H$,

$$T(u \otimes v \otimes x \otimes y)$$

$$= \sum_{i,j,k,l,m,n \in I} *_{kl}^m *_{ij}^k *_n^{jl} *_m^{in} \left(\pi_{kl}^m(u) \otimes \pi_{ij}^k(v) \otimes \pi_n^{jl}(x) \otimes \pi_m^{in}(y) \otimes \left\{ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right\} \right).$$

The adjoint operator $\tau \in \operatorname{End}(\check{H}^{\otimes 2})$ expands as follows: for any $x, y \in \check{H}$,

$$\tau(x \otimes y) = \sum_{i,j,k,l,m,n \in I} *_n^{jl} *_m^{in} \left(\pi_n^{jl}(x) \otimes \pi_m^{in}(y) \otimes \left\{ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right\} \right).$$

Similarly restricting $\bar{T}$ to the tensor product $H_{in}^m \otimes H_{jl}^n \otimes H_k^{ij} \otimes H_m^{kl}$, we obtain the *negative $6j$-symbol*

$$\left\{ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right\}^- \in H_m^{in} \otimes H_n^{jl} \otimes H_{ij}^k \otimes H_{kl}^m. \qquad (18)$$

For any $u, v, x, y \in H$,

$$\bar{T}(u \otimes v \otimes x \otimes y)$$

$$= \sum_{i,j,k,l,m,n \in I} *_{in}^m *_{jl}^n *_k^{ij} *_m^{kl} \left(\pi_{in}^m(u) \otimes \pi_{jl}^n(v) \otimes \pi_k^{ij}(x) \otimes \pi_m^{kl}(y) \otimes \left\{ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right\}^- \right).$$

The adjoint operator $\bar{\tau} \in \text{End}(\check{H}^{\otimes 2})$ expands as follows: for any $x, y \in \check{H}$,

$$\bar{\tau}(x \otimes y) = \sum_{i,j,k,l,m,n \in I} *_k^{ij} *_m^{kl} \left(\pi_k^{ij}(x) \otimes \pi_m^{kl}(y) \otimes \left\{ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right\}^{-} \right).$$

3.3. **Identities.** The properties of the forms T and $\bar{T}$ established in Section 2 can be rewritten in terms of the $6j$-symbols. Formula (13a) yields

$$\left\{ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right\} = P_{(4321)} A_1 A_3^* \left(\left\{ \begin{array}{ccc} i^* & k & j \\ l & n & m \end{array} \right\}^{-} \right),$$

where A_1 is induced by the restriction of A to $H_n^{i^*m}$ and A_3^* is induced by the restriction of A^* to $H_{i^*k}^j$. Formula (13b) yields

$$\left\{ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right\} = P_{(23)} A_2^* B_3^* \left(\left\{ \begin{array}{ccc} k & j^* & i \\ n & m & l \end{array} \right\}^{-} \right),$$

where A_2^* is induced by the restriction of A^* to $H_l^{j^*n}$ and B_3^* is induced by the restriction of B^* to $H_{kj^*}^i$. Formula (13c) yields

$$\left\{ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right\} = P_{(1234)} B_2^* B_4 \left(\left\{ \begin{array}{ccc} i & n & m \\ l^* & k & j \end{array} \right\}^{-} \right),$$

where B_2^* is induced by the restriction of B^* to $H_j^{nl^*}$ and B_4 is induced by the restriction of B_4 to $H_{ml^*}^k$.

Note for the record that Formula (14) yields

$$\left\{ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right\} = P_{(12)} (BA)_1^* (BA)_2^* (AB)_4 \left(\left\{ \begin{array}{ccc} j & l & n \\ m^* & i^* & k^* \end{array} \right\}^{-} \right),$$

where the operator $(BA)_1^*$ is induced by the restriction of $(BA)^* = A^* B^*$ to $H_{i^*}^{jk^*}$; the operator $(BA)_2^*$ is induced by the restriction of $(BA)^*$ to $H_{k^*}^{lm^*}$, and $(AB)_4$ is induced by the restriction of AB to $H_{nm^*}^{i^*}$. Formula (15) yields

$$\left\{ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right\} = P_{(34)} (BA)_1 (AB)_3^* (AB)_4^* \left(\left\{ \begin{array}{ccc} m^* & i & n^* \\ j & l^* & k \end{array} \right\}^{-} \right),$$

where the operator $(BA)_1$ is induced by the restriction of BA to $H_{l^*}^{m^*k}$; the operator $(AB)_3^*$ is induced by the restriction of $(AB)^*$ to $H_{m^*i}^{n^*}$, and $(AB)_4^*$ is induced by the restriction of $(AB)^*$ to $H_{n^*j}^{l^*}$.

The pentagon identity yields that for any $j_0, j_1, \ldots, j_8 \in I$,

$$\sum_{j \in I} *_{j_7}^{jj_4} *_{j}^{j_2 j_3} *_{j_6}^{j_1 j} \left(\left\{ \begin{array}{ccc} j_1 & j_2 & j_5 \\ j_3 & j_6 & j \end{array} \right\} \otimes \left\{ \begin{array}{ccc} j_1 & j & j_6 \\ j_4 & j_0 & j_7 \end{array} \right\} \otimes \left\{ \begin{array}{ccc} j_2 & j_3 & j \\ j_4 & j_7 & j_8 \end{array} \right\} \right)$$

$$= g_{j_2,j_3} P_{(135642)} *_{j_0}^{j_5 j_8} \left(\left\{ \begin{matrix} j_1 & j_2 & j_5 \\ j_8 & j_0 & j_7 \end{matrix} \right\} \otimes \left\{ \begin{matrix} j_5 & j_3 & j_6 \\ j_4 & j_0 & j_8 \end{matrix} \right\} \right).$$

Here both sides lie in the k-vector space

$$H_{j_6}^{j_5 j_3} \otimes H_{j_5}^{j_1 j_2} \otimes H_{j_0}^{j_6 j_4} \otimes H_{j_1 j_7}^{j_0} \otimes H_{j_8}^{j_3 j_4} \otimes H_{j_2 j_8}^{j_7}. \tag{19}$$

To rewrite the inversion relations in terms of the $6j$-symbols, observe that the transpose of the pairing $*_{ij}^k : H_{ij}^k \otimes H_k^{ij} \to \mathsf{k}$ is a homomorphism $\mathsf{k} \to H_k^{ij} \otimes H_{ij}^k$. The image of the unit $1 \in \mathsf{k}$ under this homomorphism is denoted by δ_k^{ij}. In the notation of Section 2.3, we have $\delta_k^{ij} = \sum_\alpha e_{k\alpha}^{ij} \otimes e_{ij}^{k\alpha}$. The relation $\tau_{21}\bar{\tau} = \pi^\bullet$ may be rewritten as the identity

$$\sum_{n \in I} *_{jl}^n *_{in}^m \left\{ \begin{matrix} i & j & k \\ l & m & n \end{matrix} \right\} \otimes \overline{\left\{ \begin{matrix} i & j & k' \\ l & m & n \end{matrix} \right\}} = \delta_{k'}^k \, g_{j,l} \, P_{(432)}(\delta_m^{kl} \otimes \delta_k^{ij})$$

for all $i, j, k, k', l, m \in I$. The relation $\bar{\tau}\tau_{21} = {}^\bullet\pi$ may be rewritten as the identity

$$\sum_{k \in I} *_{ij}^k *_{kl}^m \overline{\left\{ \begin{matrix} i & j & k \\ l & m & n' \end{matrix} \right\}} \otimes \left\{ \begin{matrix} i & j & k \\ l & m & n \end{matrix} \right\} = \delta_{n'}^n \, g_{i,j} \, P_{(432)}(\delta_m^{in} \otimes \delta_n^{jl})$$

for all $i, j, l, m, n, n' \in I$.

Remark 8. *As an exercise, the reader may prove that $\tau_{12}\tau_{23}({}^\bullet\pi)_{21} = (\pi^\bullet)_{32}\tau_{12}\tau_{23}$. This formula does not give non-trivial identities between the $6j$-symbols.*

4. The T-calculus

4.1. **T-equalities.** We say that two endomorphisms a, b of H are *T-equal* and write $a \stackrel{T}{=} b$ if $Ta_i = Tb_i$ for all $i = 1, 2, 3, 4$. It is clear that the T-equality is an equivalence relation. If $a \stackrel{T}{=} b$, then $ac \stackrel{T}{=} bc$ for all $c \in \text{End}(H)$.

Though the definition of the T-equality involves four conditions, two of them may be eliminated as is clear from the following lemma.

Lemma 9. *For any $a, b \in \text{End}(H)$, we have $Ta_1 = Tb_1 \iff Ta_2 = Tb_2$ and $Ta_3 = Tb_3 \iff Ta_4 = Tb_4$.*

Proof. Formulas (13) and the identity $P_{(431)} = P_{(4321)}P_{(23)}$ imply that

$$TP_{(431)} = TA_1^*(BA)_2 A_3. \tag{20}$$

Multiplying on the right by $A_1^*(AB)_2 A_3 P_{(134)} = P_{(134)} A_1(AB)_2 A_4^*$, we obtain

$$TP_{(134)} = TA_1(AB)_2 A_4^*. \tag{21}$$

Similar arguments prove that

$$TP_{(124)} = TB_2(AB)_3 B_4^*, \tag{22}$$

$$TP_{(421)} = TB_1^*(BA)_3 B_4. \tag{23}$$

For each $i = 1, 2, 3, 4$ one of the Equations (20)–(23) has the form

$$TP_\sigma = TX_kY_lZ_m, \tag{24}$$

where

$$\sigma \in \mathbb{S}_4, \quad X, Y, Z \in \{A, A^*, B, B^*, AB, BA\}, \quad \{k, l, m\} = \{1, 2, 3, 4\} \setminus \{i\},$$

and the set $\{i, \sigma(i)\}$ is either $\{1, 2\}$ or $\{3, 4\}$. Set $j = \sigma(i)$ and observe that

$$Ta_i = TP_\sigma(X_kY_lZ_m)^{-1}a_i = TP_\sigma a_i(X_kY_lZ_m)^{-1} = Ta_jP_\sigma(X_kY_lZ_m)^{-1}.$$

Also, $Tb_i = Tb_jP_\sigma(X_kY_lZ_m)^{-1}$. Hence, $Ta_i = Tb_i$ if and only if $Ta_j = Tb_j$.   □

Corollary 10. *The T-equality $a \overset{T}{=} b$ holds if and only if $Ta_i = Tb_i$ for some $i \in \{1, 2\}$ and for some $i \in \{3, 4\}$.*

4.2. **T-scalars.** An endomorphism t of H is a *T-scalar* if t has a transpose t^* and

$$Tt_1 = Tt_2 = Tt_3^* = Tt_4^* \quad \text{and} \quad Tt_1^* = Tt_2^* = Tt_3 = Tt_4. \tag{25}$$

For example, all scalar automorphisms of H are T-scalars. A more interesting example of a T-scalar will be given in Lemma 14 below. If t is a T-scalar, then the adjoint operator $\tau : \check{H}^{\otimes 2} \to \check{H}^{\otimes 2}$ introduced in Section 2.3 satisfies

$$t_1\tau = t_2\tau = \tau t_1 = \tau t_2 \quad \text{and} \quad t_1^*\tau = t_2^*\tau = \tau t_1^* = \tau t_2^*. \tag{26}$$

If $t \in \text{End}(H)$ is a T-scalar, then so is t^*. If a T-scalar t is invertible in $\text{End}(H)$, then t^{-1} is a T-scalar. Indeed, the equality $Tt_1 = Tt_2$ implies that $Tt_1^{-1} = Tt_2^{-1}$ and similarly for all the other required equalities.

The product of any two T-scalars $t, u \in \text{End}(H)$ is a T-scalar. Indeed, for any $r \in \{1, 2\}$ and $s \in \{3, 4\}$,

$$T(tu)_r = Tt_ru_r = Tt_s^*u_r = Tu_rt_s^* = Tu_s^*t_s^* = T(tu)_s^*$$

and similarly $T(tu)_r^* = T(tu)_s$. Thus, the T-scalars form a subalgebra of the k-algebra $\text{End}(H)$ invariant under the involution $a \mapsto a^*$.

If t is a T-scalar, then $a \overset{T}{=} b \Longrightarrow ta \overset{T}{=} tb$ for any $a, b \in \text{End}(H)$. Indeed,

$$T(ta)_1 = Tt_1a_1 = Tt_2a_1 = Ta_1t_2 = Tb_1t_2 = Tt_2b_1 = Tt_1b_1 = T(tb)_1$$

and similarly, $T(ta)_3 = T(tb)_3$.

We call an invertible endomorphism t of H *unitary* if $t^* = t^{-1}$. More generally, an invertible endomorphism t of H is *T-unitary* if $t^* \overset{T}{=} t^{-1}$. For a T-unitary $t \in \text{End}(H)$, Equations (25) simplify to $Tt_rt_s = T$ for all $r \in \{1, 2\}$ and $s \in \{3, 4\}$.

4.3. T-commutation relations. We now show that T-scalars T-commute with every product of an even number of operators A, B, A^*, B^*. To give a more precise statement, we define a group F by the presentation

$$F = \langle a, b, a^*, b^* \mid a^2 = b^2 = (a^*)^2 = (b^*)^2 = 1 \rangle. \tag{27}$$

Consider the group homomorphism $F \to \mathbb{Z}/2\mathbb{Z}$ carrying the generators a, b, a^*, b^* to $1 \,(\mathrm{mod}\, 2)$. Elements of F belonging to the kernel of this homomorphism are said to be *even*; all other elements of F are said to be *odd*. In other words, an element of F is even if it expands as a product of an even number of generators and odd otherwise. The group F acts on H by $a \mapsto A$, $b \mapsto B$, $a^* \mapsto A^*$, $b^* \mapsto B^*$. The endomorphism of H determined by $g \in F$ is denoted $\underline{g}$.

Lemma 11. *For any T-scalar $t \in \mathrm{End}(H)$ any $g \in F$, we have $\underline{g}t \overset{T}{=} t\underline{g}$ if g is even and $\underline{g}t \overset{T}{=} t^*\underline{g}$ if g is odd.*

Proof. Fix a T-scalar $t \in \mathrm{End}(H)$. For $g \in F$, set $t^g = t$ if g is even and $t^g = t^*$ if g is odd. We need to prove that $\underline{g}t \overset{T}{=} t^g\underline{g}$ for all $g \in F$.

For $i = 1, 2, 3, 4$, set

$$\Delta^i = \{g \in F \mid T(\underline{g}t)_i = T(t^g\underline{g})_i\} \subset F.$$

By the previous lemma, $\Delta^1 = \Delta^2$ and $\Delta^3 = \Delta^4$. We claim that for any generator $c \in \{a, b, a^*, b^*\}$, we have $c\,\Delta^1 \subset \Delta^3$ and $c\,\Delta^3 \subset \Delta^1$. This will imply that the set $\Delta^1 \cap \Delta^3 \subset F$ is closed under left multiplication by the generators of F. Since this set contains the neutral element of F, we have $\Delta^1 \cap \Delta^3 = F$. In other words, $\Delta^i = F$ for all $i = 1, 2, 3, 4$. This means that $\underline{g}t \overset{T}{=} t^g\underline{g}$ for all $g \in F$.

To prove our claim, consider again Equality (24). Pick any $g \in \Delta^{\sigma(k)}$. Then

$$T(X\underline{g}t)_k = TP_\sigma(X_kY_lZ_m)^{-1}(X\underline{g}t)_k = TP_\sigma(Y_lZ_m)^{-1}(\underline{g}t)_k$$

$$= T(\underline{g}t)_{\sigma(k)}P_\sigma(Y_lZ_m)^{-1} = T(t^g\underline{g})_{\sigma(k)}P_\sigma(Y_lZ_m)^{-1} = Tt^g_{\sigma(k)}\underline{g}_{\sigma(k)}P_\sigma(Y_lZ_m)^{-1},$$

where the fourth equality follows from the inclusion $g \in \Delta^{\sigma(k)}$. Since $t \in \mathrm{End}\,H$ is a T-scalar, $Tt^g_{\sigma(k)} = Tt'_{\sigma(i)}$ for some $t' \in \{t, t^*\}$. Then

$$Tt^g_{\sigma(k)}\underline{g}_{\sigma(k)}P_\sigma(Y_lZ_m)^{-1} = Tt'_{\sigma(i)}\underline{g}_{\sigma(k)}P_\sigma(Y_lZ_m)^{-1} = TP_\sigma(Y_lZ_m)^{-1}t'_i\underline{g}_k = Tt'_i(X\underline{g})_k$$

where the last equality follows from (24). Since t is a T-scalar, $Tt'_i = Tt''_k$ for some $t'' \in \{t, t^*\}$. Recall that $X = \underline{x}$ for some $x \in \{a, a^*, b, b^*, ab, ba\}$. A case by case analysis shows that $t'' = t^{xg}$. Combining the formulas above, we obtain that $T(\underline{xg}t)_k = T(X\underline{g}t)_k = T(t^{xg}\underline{xg})_k$, i.e., $xg \in \Delta^k$. Thus, $x\Delta^{\sigma(k)} \subset \Delta^k$. Applying this inclusion to all forms (20)–(23) of (24) and to all possible choices of k, we obtain $c\,\Delta^1 \subset \Delta^3$ and $c\,\Delta^3 \subset \Delta^1$ for all $c \in \{a, b, a^*, b^*\}$. $\qquad\square$

5. Further properties of T

To study the form T we introduce the operators

$$L = A^*A, \ R = B^*B, \ C = (AB)^3 \in \text{End}(H).$$

We shall study these operators and show that the commutator of L and R is a T-scalar. Though this fact will not be directly used in the sequel, the properties of the operators L, R, C will lead us in the next section to a notion of a $\widehat{\Psi}$-system.

An operator $f \in \text{End}(H)$ such that $f^* = f$ is called *symmetric*. An operator $f \in \text{End}(H)$ such that $f(H_k^{ij}) \subset H_k^{ij}$ and $f(H_{ij}^k) \subset H_{ij}^k$ for all $i, j, k \in I$ is called *grading-preserving*.

Lemma 12. *The operators L, R, C are invertible, symmetric, and grading-preserving. They satisfy the following identities:*

$$ACA = BCB = C^{-1}, \tag{28}$$

$$LCL^{-1} = RCR^{-1} = C, \tag{29}$$

$$ALA = L^{-1}, \quad BRB = R^{-1}, \tag{30}$$

$$ARA = L^{-1}RC^{-1}, \quad BLB = R^{-1}LC. \tag{31}$$

Proof. That L, R, C are invertible follows from the fact that A and B are invertible. The inverses of these operators are computed by $L^{-1} = AA^*$, $R^{-1} = BB^*$, and $C^{-1} = (BA)^3$. The operators L and R are manifestly symmetric. We have

$$C = (AB)^3 = ABABAB = (ABA)(ABA)^*.$$

Therefore $C^* = C$. That L, R, C are grading-preserving follows from (5) and (11).

The identities (28)–(31) are checked as follows:

$$ACA = A(AB)^3 A = A^2(BA)^3 = (BA)^3 = C^{-1},$$

and similarly for BCB;

$$LCL^{-1} = A^*ACAA^* = A^*C^{-1}A^* = (AC^{-1}A)^* = C^* = C,$$

and similarly for RCR^{-1};

$$ALA = AA^*AA = AA^* = L^{-1},$$

and similarly for BRB;

$$L^{-1}RC^{-1} = AA^*B^*A(BA)^2 = AA^*B^*(BAB)^*BA = AB^*BA = ARA,$$

and similarly for $R^{-1}LC$. □

Lemma 13. *The following identities hold:*

$$TC_1C_2 = TC_3C_4, \tag{32a}$$

$$TR_1L_2 = TR_3L_4, \tag{32b}$$

$$TR_1R_2 = TC_3R_4, \tag{32c}$$

$$TL_1 = TC_2L_3L_4, \tag{32d}$$

Proof. The proof is based on a study of the action of the standard generators of $\mathbb{S}_4$ on T using the formulas of Section 2.2. We have

$$TP_{(12)}P_{(23)} = \bar{T}P_{(23)}(BA)_1(BA)_3(AB)_4^* = T(BA)_1B_2(ABA)_3(AB)_4^*.$$

The Coxeter relation $(P_{(12)}P_{(23)})^3 = 1$ yields Equation (32a):

$$T = T(P_{(12)}P_{(23)})^3 = T(P_{(12)}P_{(23)})^2(BA)_2B_3(ABA)_1(AB)_4^*$$
$$= TP_{(12)}P_{(23)}(BAB)_3(A(BA)^2)_2(BA)_1^2(AB)_4^{*2}$$
$$= T(BA)_1^3(BA)_2^3(AB)_3^3(AB)_4^{*3} = TC_1^{-1}C_2^{-1}C_3C_4.$$

The identity $P_{(12)}P_{(23)}P_{(34)}P_{(4321)} = 1$ implies that

$$T = TP_{(12)}P_{(23)}P_{(34)}P_{(4321)} = \bar{T}P_{(23)}P_{(34)}P_{(4321)}(BA)_2(BA)_1(AB)_4^*$$
$$= TP_{(34)}P_{(4321)}B_3(BA)_2(ABA)_1(AB)_4^*$$
$$= \bar{T}P_{(4321)}(A^*B^*BA)_2B_3((AB)^2A)_1(ABB^*A^*)_4$$
$$= T(B^*(AB)^2A)_1(A^*B^*BA)_2(ABB^*A^*)_4 = T(RC^{-1})_1(RC^{-1})_2(R^{-1}C)_4$$

This and (32a) yields (32c). The identity $P_{(34)}P_{(23)}P_{(12)}P_{(1234)} = 1$ implies that

$$T = TP_{(34)}P_{(23)}P_{(12)}P_{(1234)} = \bar{T}P_{(23)}P_{(12)}P_{(1234)}(BA)_1^*(AB)_4(AB)_3$$
$$= TP_{(12)}P_{(1234)}A_2(BA)_1^*(BAB)_4(AB)_3$$
$$= \bar{T}P_{(1234)}(BAA^*B^*)_1A_2((BA)^2B)_4(B^*A^*AB)_3$$
$$= T(BAA^*B^*)_1(B^*A^*AB)_3(A^*(BA)^2B)_4 = T(L^{-1}C^{-1})_1(LC)_3(LC)_4.$$

This and (32a) yields (32d). Finally,

$$TP_{(12)}P_{(34)} = \bar{T}P_{(34)}(BA)_1(BA)_2(AB)_3^*$$
$$= T(B^*A^*BA)_1(BA)_2(BAB^*A^*)_3(BA)_4.$$

The relation $(P_{(12)}P_{(34)})^2 = 1$ gives

$$T = T(P_{(12)}P_{(34)})^2 = TP_{(12)}P_{(34)}(B^*A^*BA)_2(BA)_1(BAB^*A^*)_4(BA)_3$$
$$= T(B^*A^*(BA)^2)_1(BAB^*A^*BA)_2(BAB^*A^*BA)_3((BA)^2B^*A^*)_4$$
$$= T(A^*A)_1(BAA^*B)_2(BAA^*B)_3(BB^*)_4 = TL_1(C^{-1}L^{-1}R)_2(C^{-1}L^{-1}R)_3R_4^{-1}.$$

This formula can be rewritten in the following equivalent form

$$T(R^{-1}L)_2C_3R_4 = TL_1C_2^{-1}(L^{-1}R)_3$$

which reduces to (32b) after using (32c) on the left hand side, and (32d) on the right hand side. $\qquad\square$

Lemma 14. $Q = LRL^{-1}R^{-1} \in \text{End}(H)$ *is a T-unitary T-scalar.*

Proof. Applying consecutively (32d) and (32b) in alternating order, we obtain

$$TQ_1 = T(RL^{-1}R^{-1})_1C_2L_3L_4 = T(L^{-1}R^{-1})_1(L^{-1}C)_2(RL)_3L_4^2$$
$$= TR_1^{-1}L_2^{-1}(L^{-1}RL)_3L_4 = TR_3^{-1}(L^{-1}RL)_3 = TQ_3^*.$$

Similar transformations using (32d) and (32c) yield $TQ_1 = TQ_4^*$. Analogously, using (32b) and (32c), we obtain $TQ_2 = TQ_4^*$. This verifies the first three of Equalities (25). The other three equalities are checked similarly.

Since Q is a T-scalar, so is $Q^* = R^{-1}L^{-1}RL$. We have

$$Q^{-1} = RLR^{-1}L^{-1} = LRQ^*R^{-1}L^{-1} \overset{T}{=} Q^*LRR^{-1}L^{-1} = Q^*,$$

where the T-equality follows from Lemma 11 applied to the T-scalar Q^*. $\qquad\square$

Remark 15. *It is clear that Q is grading-preserving. For any $i, j, k, l, m, n \in I$, the restrictions of Q to the corresponding multiplicity spaces induce the endomorphisms Q_1, Q_2, Q_3, Q_4 of the vector space (16). Lemma 14 implies that for any $r \in \{1, 2\}$ and $s \in \{3, 4\}$, the composition Q_rQ_s preserves the $6j$-symbol (17).*

6. $\widehat{\Psi}$-SYSTEMS

6.1. **The operators $C^{\frac{1}{2}}$, $R^{\frac{1}{2}}$, and $L^{\frac{1}{2}}$.** Recall the symmetric, grading-preserving, invertible operators $C, R, L \in \text{End}(H)$. Suppose that we have symmetric, grading-preserving, invertible operators $C^{\frac{1}{2}}, R^{\frac{1}{2}} \in \text{End}(H)$ such that

$$(C^{\frac{1}{2}})^2 = C, \quad AC^{\frac{1}{2}}A = BC^{\frac{1}{2}}B = C^{-\frac{1}{2}}, \tag{33a}$$

$$(R^{\frac{1}{2}})^2 = R, \quad BR^{\frac{1}{2}}B = R^{-\frac{1}{2}}, \quad R^{\frac{1}{2}}C^{\frac{1}{2}} = C^{\frac{1}{2}}R^{\frac{1}{2}} \tag{33b}$$

where by definition $C^{-\frac{1}{2}} = (C^{\frac{1}{2}})^{-1}$ and $R^{-\frac{1}{2}} = (R^{\frac{1}{2}})^{-1}$. Set

$$L^{\frac{1}{2}} = BAR^{-\frac{1}{2}}AB \in \text{End}(H) \quad \text{and} \quad L^{-\frac{1}{2}} = (L^{\frac{1}{2}})^{-1} = BAR^{\frac{1}{2}}AB \in \text{End}(H). \tag{34}$$

The properties of $L^{\frac{1}{2}}$ are summarized in the following lemma.

Lemma 16. *The operator $L^{\frac{1}{2}}$ is symmetric, grading-preserving, and*

$$(L^{\frac{1}{2}})^2 = L, \quad AL^{\frac{1}{2}}A = L^{-\frac{1}{2}}, \quad L^{\frac{1}{2}}C^{\frac{1}{2}} = C^{\frac{1}{2}}L^{\frac{1}{2}}, \tag{35}$$

Proof. We have

$$(L^{\frac{1}{2}})^2 = BAR^{-1}AB = BABB^*AB = A^*B^*A^*B^*AB = A^*ABAAB = A^*A = L,$$

$$AL^{\frac{1}{2}}A = ABAR^{-\frac{1}{2}}ABA = (AB)^2 R^{\frac{1}{2}}(BA)^2$$
$$= BACR^{\frac{1}{2}}C^{-1}AB = BAR^{\frac{1}{2}}AB = L^{-\frac{1}{2}},$$

$$(L^{\frac{1}{2}})^* = B^*A^*R^{-\frac{1}{2}}A^*B^* = B^*A^*B^*R^{\frac{1}{2}}B^*A^*B^* = ABAR^{\frac{1}{2}}ABA = AL^{-\frac{1}{2}}A = L^{\frac{1}{2}}.$$

That $L^{\frac{1}{2}}$ is grading-preserving and commutes with $C^{\frac{1}{2}}$ follows from the definitions.
$\square$

6.2. $\widehat{\Psi}$**-systems in** $\mathcal{C}$. A $\widehat{\Psi}$-*system* in $\mathcal{C}$ is a Ψ-system in $\mathcal{C}$ together with a choice of invertible, symmetric, grading-preserving operators $C^{\frac{1}{2}}, R^{\frac{1}{2}} \in \text{End}(H)$ satisfying Equalities (33) as well as the identities

$$TC_1^{\frac{1}{2}}C_2^{\frac{1}{2}} = TC_3^{\frac{1}{2}}C_4^{\frac{1}{2}}, \tag{36a}$$

$$TR_1^{\frac{1}{2}}L_2^{\frac{1}{2}} = TR_3^{\frac{1}{2}}L_4^{\frac{1}{2}}, \tag{36b}$$

$$TR_1^{\frac{1}{2}}R_2^{\frac{1}{2}} = TC_3^{\frac{1}{2}}R_4^{\frac{1}{2}}, \tag{36c}$$

$$TL_1^{\frac{1}{2}} = TC_2^{\frac{1}{2}}L_3^{\frac{1}{2}}L_4^{\frac{1}{2}}. \tag{36d}$$

where $L^{\frac{1}{2}}$ is defined by (34).

Generally speaking, a Ψ-system may not allow operators $C^{\frac{1}{2}}, R^{\frac{1}{2}} \in \text{End}(H)$ as above.

Equations (36c) and (36d) above are not independent. In fact, any one of them may be omitted.

We suppose from now on that we do have a $\widehat{\Psi}$-system.

6.3. **Commutation relations.** We establish commutation relation analogous to those in Lemma 11. Consider the group

$$\widehat{F} = \langle a, b, a^*, b^*, c, r \mid a^2 = b^2 = (a^*)^2 = (b^*)^2 = 1 \rangle.$$

Consider the group homomorphism $\widehat{F} \to \mathbb{Z}/2\mathbb{Z}$ carrying a, b, a^*, b^* to $1 \, (\text{mod} \, 2)$ and carrying c, r to 0. Elements of $\widehat{F}$ belonging to the kernel of this homomorphism are said to be *even*; all other elements of $\widehat{F}$ are said to be *odd*. The group $\widehat{F}$ acts on H by

$$a \mapsto A, \; b \mapsto B, \; a^* \mapsto A^*, \; b^* \mapsto B^*, \; c \mapsto C^{\frac{1}{2}}, \; r \mapsto R^{\frac{1}{2}}.$$

The endomorphism of H determined by $g \in \widehat{F}$ is denoted $\underline{g}$.

Lemma 17. *Let* $t \in \text{End}(H)$ *be a T-scalar that T-commutes with $C^{\frac{1}{2}}$ in the sense that $tC^{\frac{1}{2}} \stackrel{T}{=} C^{\frac{1}{2}}t$. For $g \in \widehat{F}$, set $t^g = t$ if g is even and $t^g = t^*$ if g is odd. Then $\underline{g}t \stackrel{T}{=} t^g\underline{g}$ for all $g \in \widehat{F}$.*

Proof. Observe that t also T-commutes with $C^{-\frac{1}{2}}$ and t^* T-commutes with both $C^{\frac{1}{2}}$ and $C^{-\frac{1}{2}}$. Indeed, let $\{i,j\}$ be the set $\{1,2\}$ or the set $\{3,4\}$. Set $\{k,l\} = \{1,2,3,4\} \setminus \{i,j\}$. Formula (36a) implies that

$$T(tC^{-\frac{1}{2}})_i = Tt_iC_i^{-\frac{1}{2}} = Tt_jC_i^{-\frac{1}{2}} = TC_i^{-\frac{1}{2}}t_j = TC_j^{\frac{1}{2}}C_k^{-\frac{1}{2}}C_l^{-\frac{1}{2}}t_j$$
$$= Tt_jC_j^{\frac{1}{2}}C_k^{-\frac{1}{2}}C_l^{-\frac{1}{2}} = Tt_iC_j^{\frac{1}{2}}C_k^{-\frac{1}{2}}C_l^{-\frac{1}{2}} = TC_j^{\frac{1}{2}}C_k^{-\frac{1}{2}}C_l^{-\frac{1}{2}}t_i = T(C^{-\frac{1}{2}}t)_i,$$

$$T(t^*C^{\frac{1}{2}})_i = Tt_i^*C_i^{\frac{1}{2}} = Tt_kC_i^{\frac{1}{2}} = TC_i^{\frac{1}{2}}t_k = TC_j^{-\frac{1}{2}}C_k^{\frac{1}{2}}C_l^{\frac{1}{2}}t_k$$
$$= Tt_kC_j^{-\frac{1}{2}}C_k^{\frac{1}{2}}C_l^{\frac{1}{2}} = Tt_i^*C_j^{-\frac{1}{2}}C_k^{\frac{1}{2}}C_l^{\frac{1}{2}} = TC_j^{-\frac{1}{2}}C_k^{\frac{1}{2}}C_l^{\frac{1}{2}}t_i^* = T(C^{\frac{1}{2}}t^*)_i.$$

For $i \in \{1,2,3,4\}$, set

$$\widehat{\Delta}_i = \{g \in \widehat{F}|\ T(\underline{g}t)_i = T(t^g\underline{g})_i\} \subset \widehat{F}.$$

By Lemma 9, we have $\widehat{\Delta}_1 = \widehat{\Delta}_2$ and $\widehat{\Delta}_3 = \widehat{\Delta}_4$. Set $\widehat{\Delta} = \widehat{\Delta}_1 \cap \widehat{\Delta}_3 \subset \widehat{F}$. Clearly, $1 \in \widehat{\Delta}$. It is enough to show that $\widehat{\Delta} = \widehat{F}$.

Pick any index $i \in \{1,2,3,4\}$ and let $j,k,l \in \{1,2,3,4\}$ be such that either $\{i,j\} = \{1,2\}$ or $\{i,j\} = \{3,4\}$, and $\{k,l\} = \{1,2,3,4\} \setminus \{i,j\}$. For $X = \underline{c} = C^{\frac{1}{2}}$ and any $g \in \widehat{\Delta}$, Formula (36a) implies that

$$T(\underline{cg}t)_i = T(X\underline{g}t)_i = T(\underline{g}t)_iX_j^{-1}X_kX_l = T(t^g\underline{g})_iX_j^{-1}X_kX_l = Tt_j^g X_j^{-1}X_kX_l\underline{g}_i$$

(the last two equalities follow from the assumptions $g \in \widehat{\Delta}$ and t is a T-scalar, respectively). Since t, t^* both T-commute with $C^{\pm\frac{1}{2}}$, we similarly have

$$Tt_j^g X_j^{-1}X_kX_l\underline{g}_i = TX_j^{-1}X_kX_l\underline{g}_i t_j^g = T(X\underline{g})_i t_j^g$$
$$= Tt_j^g(X\underline{g})_i = T(t^g X\underline{g})_i = T(t^{cg}\underline{cg})_i.$$

Thus, $T(\underline{cg}t)_i = T(t^{cg}\underline{cg})_i$ for all i so that $cg \in \widehat{\Delta}$. This shows the inclusion $c\widehat{\Delta} \subset \widehat{\Delta}$. The inclusion $c^{-1}\widehat{\Delta} \subset \widehat{\Delta}$ is proved similarly.

Next, pick indices i,j,k such that $\{i,j,k\} = \{1,2,4\}$. Set $X = \underline{r} = R^{\frac{1}{2}}$ and $Y = C^{\frac{1}{2}}$. For any $g \in \widehat{\Delta}$, Formula (36c) implies that for some $\varepsilon = \pm 1$ and $X', X'' \in \{X, X^{-1}\}$,

$$T(\underline{rg}t)_i = TX_i(\underline{g}t)_i = TY_3^\varepsilon X_j'X_k''(\underline{g}t)_i = T(t^g\underline{g})_iY_3^\varepsilon X_j'X_k'' = Tt_3^g\underline{g}_iY_3^\varepsilon X_j'X_k'',$$

where $t' \in \{t, t^*\}$ is such that $Tt_i^g = Tt_3'$. Since $t'Y \overset{T}{=} Yt'$,

$$Tt_3'\underline{g}_iY_3^\varepsilon X_j'X_k'' = TY_3^\varepsilon X_j'X_k''\underline{g}_it_3' = T(X\underline{g})_it_3' = Tt_3'(X\underline{g})_i = T(t^g X\underline{g})_i = T(t^{rg}\underline{rg})_i.$$

Thus, $rg \in \widehat{\Delta}$ and so $r\widehat{\Delta} \subset \widehat{\Delta}$. Similarly, $r^{-1}\widehat{\Delta} \subset \widehat{\Delta}$. The rest of the proof is as in Lemma 11. □

6.4. Stable T-equivalence. We say that two operators $a, b \in \mathrm{End}(H)$ are *stably T-equal* and write $a \stackrel{sT}{=} b$ if $\underline{f}a \stackrel{T}{=} \underline{f}b$ for all $f \in \widehat{F}$. Obviously, if $a \stackrel{sT}{=} b$, then $a \stackrel{T}{=} b$ and $\underline{g}a \stackrel{sT}{=} \underline{g}b$ for all $g \in \widehat{F}$.

Corollary 18. *For any $g \in \widehat{F}$ and any T-scalar t that T-commutes with $C^{\frac{1}{2}}$,*

$$\underline{g}\, t \stackrel{sT}{=} t^g \underline{g}. \tag{37}$$

Indeed, by Lemma 17, for all $f \in \widehat{F}$,

$$\underline{f}\,\underline{g}\, t = \underline{fg}\, t \stackrel{T}{=} t^{fg} \underline{fg} = (t^g)^f \underline{f}\,\underline{g} \stackrel{T}{=} \underline{f}t^g\underline{g}.$$

Corollary 18 and the evenness of the elements r and $bar^{-1}ab$ of $\widehat{F}$ imply that if a T-scalar t commutes with $C^{\frac{1}{2}}$, then t stably T-commutes with $R^{\frac{1}{2}}$ and $L^{\frac{1}{2}}$ in the sense that $R^{\frac{1}{2}}t \stackrel{sT}{=} tR^{\frac{1}{2}}$ and $L^{\frac{1}{2}}t \stackrel{sT}{=} tL^{\frac{1}{2}}$.

6.5. The T-scalar q. The following T-scalar will play a key role in the sequel.

Lemma 19. *The grading-preserving operator*

$$q = R^{\frac{1}{2}}AR^{-\frac{1}{2}}AL^{-\frac{1}{2}}C^{-\frac{1}{2}} = R^{\frac{1}{2}}BL^{\frac{1}{2}}BL^{-\frac{1}{2}}C^{-\frac{1}{2}}$$

is a unitary T-scalar commuting with $C^{\frac{1}{2}}$.

Proof. That q is grading-preserving is obvious because $R^{\frac{1}{2}}$, $C^{\frac{1}{2}}$, and $L^{\frac{1}{2}}$ are grading-preserving and A is involutive. Since $R^{\frac{1}{2}}, L^{\frac{1}{2}}$, and $AR^{-\frac{1}{2}}A$ commute with $C^{\frac{1}{2}}$, the operator q also commutes with $C^{\frac{1}{2}}$. To prove the remaining claims, set

$$D = AR^{-\frac{1}{2}}A = BL^{\frac{1}{2}}B. \tag{38}$$

It is clear that $DC^{\frac{1}{2}} = C^{\frac{1}{2}}D$. Note also that

$$D^* = LDL^{-1}, \tag{39a}$$

$$D^2 = R^{-1}LC = CR^{-1}L. \tag{39b}$$

Indeed,

$$D^* = A^*R^{-\frac{1}{2}}A^* = A^*ABL^{\frac{1}{2}}BAA^* = LBL^{\frac{1}{2}}BL^{-1} = LDL^{-1}$$

and

$$D^2 = BLB = BA^*AB = BB^*A^*A^*B^*A^*AB$$
$$= BB^*A^*AABABAB = R^{-1}LC.$$

We claim that the operator D satisfies the following identities

$$TD_1 = TR_2^{\frac{1}{2}}L_3^{\frac{1}{2}}D_4^*, \tag{40a}$$

$$TC_1^{\frac{1}{2}}D_2 = TR_3^{\frac{1}{2}}D_4^*, \tag{40b}$$

$$TL_2^{\frac{1}{2}}D_3C_4^{\frac{1}{2}} = TD_1^*. \tag{40c}$$

Formula (40a) is proved as follows:

$$TD_1 = TP_{(134)}A_4^*(BA)_2(R^{-\frac{1}{2}}A)_1 = TL_4^{\frac{1}{2}}R_1^{-\frac{1}{2}}L_2^{-\frac{1}{2}}P_{(134)}A_4^*(BA)_2A_1$$
$$= TP_{(134)}L_3^{\frac{1}{2}}(R^{-\frac{1}{2}}A^*)_4(L^{-\frac{1}{2}}BA)_2A_1 = TR_2^{\frac{1}{2}}L_3^{\frac{1}{2}}D_4^*,$$

where in the first, second, etc. equalities we use respectively: the definition of D and (21); the action of the permutation group $\mathbb{S}_4$ and (36b); the action of $\mathbb{S}_4$; the definitions of $L^{\frac{1}{2}}$, D and (21). The proof of (40b) is similar:

$$TD_2 = TP_{(124)}B_4^*(BA)_3(L^{\frac{1}{2}}B)_2 = TL_1^{\frac{1}{2}}C_2^{-\frac{1}{2}}L_3^{-\frac{1}{2}}P_{(124)}B_4^*(BA)_3B_2$$
$$= TP_{(124)}L_4^{\frac{1}{2}}C_1^{-\frac{1}{2}}L_3^{-\frac{1}{2}}B_4^*(BA)_3B_2 = TC_1^{-\frac{1}{2}}R_3^{\frac{1}{2}}D_4^*,$$

where we use consecutively: the definition of D and (22); the action of $\mathbb{S}_4$ and (36d); the action of $\mathbb{S}_4$; the definitions of $L^{\frac{1}{2}}$, D and (22). Finally, we prove (40c):

$$TD_3 = TP_{(431)}A_1^*(AB)_2(R^{-\frac{1}{2}}A)_3 = TR_2^{\frac{1}{2}}C_3^{-\frac{1}{2}}R_4^{-\frac{1}{2}}P_{(431)}A_1^*(AB)_2A_3$$
$$= TP_{(431)}R_2^{\frac{1}{2}}C_4^{-\frac{1}{2}}R_1^{-\frac{1}{2}}A_1^*(AB)_2A_3 = TD_1^*L_2^{-\frac{1}{2}}C_4^{-\frac{1}{2}}.$$

Here we use: the definition of D and (20); the action of $\mathbb{S}_4$ and (36c); the action of $\mathbb{S}_4$; the definitions of $L^{\frac{1}{2}}$, D and (20).

The definition of q may be rewritten as

$$q = R^{\frac{1}{2}}DL^{-\frac{1}{2}}C^{-\frac{1}{2}}. \tag{41}$$

We can now prove the unitarity of q:

$$q^* = C^{-\frac{1}{2}}L^{-\frac{1}{2}}D^*R^{\frac{1}{2}} = C^{-\frac{1}{2}}L^{\frac{1}{2}}DL^{-1}R^{\frac{1}{2}}$$

$$= C^{-\frac{1}{2}}L^{\frac{1}{2}}D^{-1}D^2L^{-1}R^{\frac{1}{2}} = C^{-\frac{1}{2}}L^{\frac{1}{2}}D^{-1}CR^{-\frac{1}{2}} = C^{\frac{1}{2}}L^{\frac{1}{2}}D^{-1}R^{-\frac{1}{2}} = q^{-1},$$

where we use the relations (39). Now,

$$Tq_1 = TR_2^{-\frac{1}{2}}C_3^{\frac{1}{2}}R_4^{\frac{1}{2}}(DL^{-\frac{1}{2}}C^{-\frac{1}{2}})_1 = TL_3^{\frac{1}{2}}D_4^*C_3^{\frac{1}{2}}R_4^{\frac{1}{2}}(L^{-\frac{1}{2}}C^{-\frac{1}{2}})_1$$
$$= TC_2^{-\frac{1}{2}}L_4^{-\frac{1}{2}}D_4^*C_3^{\frac{1}{2}}R_4^{\frac{1}{2}}C_1^{-\frac{1}{2}} = TC_1^{-\frac{1}{2}}C_2^{-\frac{1}{2}}C_3^{\frac{1}{2}}(qC^{\frac{1}{2}})_4^* = Tq_4^*,$$

where we use consecutively: Formulas (41) and (36c); Formula (40a); Formula (36d); Formula (41); Formula (36a). Similarly,

$$Tq_2 = TR_1^{-\frac{1}{2}}C_3^{\frac{1}{2}}R_4^{\frac{1}{2}}(DL^{-\frac{1}{2}}C^{-\frac{1}{2}})_2 = TC_1^{-\frac{1}{2}}R_3^{\frac{1}{2}}D_4^*R_1^{-\frac{1}{2}}C_3^{\frac{1}{2}}R_4^{\frac{1}{2}}(L^{-\frac{1}{2}}C^{-\frac{1}{2}})_2$$
$$= TC_1^{-\frac{1}{2}}L_4^{-\frac{1}{2}}D_4^*C_3^{\frac{1}{2}}R_4^{\frac{1}{2}}C_2^{-\frac{1}{2}} = TC_1^{-\frac{1}{2}}C_2^{-\frac{1}{2}}C_3^{\frac{1}{2}}(qC^{\frac{1}{2}})_4^* = Tq_4^*,$$

where we use consecutively: Formulas (41) and (36c); Formula (40b); Formula (36b); Formula (41); Formula (36a). Similarly,

$$Tq_3 = TR_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_4^{-\frac{1}{2}} (DL^{-\frac{1}{2}} C^{-\frac{1}{2}})_3 = TD_1^* C_4^{-\frac{1}{2}} R_1^{\frac{1}{2}} L_4^{-\frac{1}{2}} (L^{-\frac{1}{2}} C^{-\frac{1}{2}})_3$$

$$= TL_1^{-\frac{1}{2}} C_2^{\frac{1}{2}} D_1^* C_4^{-\frac{1}{2}} R_1^{\frac{1}{2}} C_3^{-\frac{1}{2}} = T(qC^{\frac{1}{2}})_1^* C_2^{\frac{1}{2}} C_3^{-\frac{1}{2}} C_4^{-\frac{1}{2}} = Tq_1^*,$$

where we use: Formulas (41) and (36b); Formula (40c); Formula (36d); Formula (41); Formula (36a). Together with the unitarity of q these identities imply that q is a T-scalar. □

Lemma 20. *For all $a, b \in \frac{1}{2}\mathbb{Z}$,*

$$L^a R^b \stackrel{sT}{=} q^{8ab} R^b L^a. \tag{42}$$

For all $a, b \in \frac{1}{2}\mathbb{Z}$ and $c \in \mathbb{Z}$,

$$(L^a R^b)^c \stackrel{sT}{=} q^{4abc(1-c)} L^{ac} R^{bc}. \tag{43}$$

Proof. Formulas (41) and (39b) imply that

$$qL^{\frac{1}{2}} R^{-\frac{1}{2}} q = R^{-\frac{1}{2}} L^{\frac{1}{2}}.$$

This and (37) yield (42) for $a = b = \frac{1}{2}$:

$$L^{\frac{1}{2}} R^{\frac{1}{2}} = R^{\frac{1}{2}} R^{-\frac{1}{2}} L^{\frac{1}{2}} R^{\frac{1}{2}} = R^{\frac{1}{2}} q L^{\frac{1}{2}} R^{-\frac{1}{2}} q R^{\frac{1}{2}} \stackrel{sT}{=} q^2 R^{\frac{1}{2}} L^{\frac{1}{2}} R^{-\frac{1}{2}} R^{\frac{1}{2}} = q^2 R^{\frac{1}{2}} L^{\frac{1}{2}}.$$

Assuming (42) for some $a \in \frac{1}{2}\mathbb{Z}$ and $b = \frac{1}{2}$, we obtain

$$L^{a+\frac{1}{2}} R^{\frac{1}{2}} = L^a L^{\frac{1}{2}} R^{\frac{1}{2}} \stackrel{sT}{=} L^a q^2 R^{\frac{1}{2}} L^{\frac{1}{2}}$$

$$\stackrel{sT}{=} q^2 L^a R^{\frac{1}{2}} L^{\frac{1}{2}} \stackrel{sT}{=} q^2 q^{4a} R^{\frac{1}{2}} L^a L^{\frac{1}{2}} = q^{4(a+\frac{1}{2})} R^{\frac{1}{2}} L^{a+\frac{1}{2}}.$$

This proves (42) for all positive $a \in \frac{1}{2}\mathbb{Z}$ and $b = \frac{1}{2}$. Similarly, assuming (42) for some positive $a, b \in \frac{1}{2}\mathbb{Z}$, we obtain

$$L^a R^{b+\frac{1}{2}} = L^a R^{\frac{1}{2}} R^b \stackrel{sT}{=} q^{4a} R^{\frac{1}{2}} L^a R^b \stackrel{sT}{=} q^{4a} R^{\frac{1}{2}} q^{8ab} R^b L^a$$

$$\stackrel{sT}{=} q^{4a} q^{8ab} R^{\frac{1}{2}} R^b L^a = q^{8a(b+\frac{1}{2})} R^{b+\frac{1}{2}} L^a.$$

This proves (42) for all positive $a, b \in \frac{1}{2}\mathbb{Z}$. Moreover, for such a, b,

$$L^{-a} R^b = L^{-a} R^b L^a L^{-a} \stackrel{sT}{=} L^{-a} q^{-8ab} L^a R^b L^{-a} \stackrel{sT}{=} q^{-8ab} L^{-a} L^a R^b L^{-a} = q^{-8ab} R^b L^{-a},$$

$$L^a R^{-b} = R^{-b} R^b L^a R^{-b} \stackrel{sT}{=} R^{-b} q^{-8ab} L^a R^b R^{-b} \stackrel{sT}{=} q^{-8ab} R^{-b} L^a R^b R^{-b} = q^{-8ab} R^{-b} L^a,$$

and

$$L^{-a} R^{-b} = R^{-b} R^b L^{-a} R^{-b} \stackrel{sT}{=} R^{-b} q^{8ab} L^{-a} R^b R^{-b} \stackrel{sT}{=} q^{8ab} R^{-b} L^{-a},$$

This proves (42) for all non-zero a, b. For $a = 0$ or $b = 0$, Formula (42) is obvious.

The case $c = 0$ of (43) is obvious. Assuming (43) for some $c \in \mathbb{Z}$, we obtain

$$(L^a R^b)^{c+1} = (L^a R^b)^c L^a R^b \overset{sT}{=} q^{4abc(1-c)} L^{ac} R^{bc} L^a R^b \overset{sT}{=} q^{4abc(1-c)} L^{ac} q^{-8abc} L^a R^{bc} R^b$$

$$\overset{sT}{=} q^{-4abc(c+1)} L^{ac} L^a R^{bc} R^b = q^{4ab(c+1)(1-(1+c))} L^{a(c+1)} R^{b(c+1)}.$$

This implies (43) for all $c \geq 0$. The proof for negative c is similar.   □

Remark 21. *Applying Formula 42 to $a = b = 1$, we obtain $q^8 \overset{sT}{=} Q$, where $Q = LRL^{-1}R^{-1}$ is the operator studied in Lemma 14.*

7. CHARGED T-FORMS

7.1. **Definition.** For any $a, c \in \frac{1}{2}\mathbb{Z}$, we define the "charged" T-forms

$$T(a, c) = T q_1^{4ac} R_1^c R_2^{-a} L_3^{-a} R_3^{-c} \colon H^{\otimes 4} \to \mathsf{k}$$

and

$$\bar{T}(a, c) = \bar{T} q_1^{-4ac} L_2^{-a} R_2^{-c} R_3^{-a} R_4^c \colon H^{\otimes 4} \to \mathsf{k}.$$

Lemma 22. *Set $\mathsf{A} = AL^{-\frac{1}{2}}, \mathsf{B} = BR^{-\frac{1}{2}} \in \mathrm{End}(H)$. Then A, B are symmetric involutions and for any $a, b, c \in \frac{1}{2}\mathbb{Z}$ such that $a + b + c = \frac{1}{2}$,*

$$T(a, c) P_{(4321)} = \bar{T}(a, b) q_1^{-2a} \mathsf{A}_1 \mathsf{A}_3, \tag{44a}$$

$$T(a, c) P_{(23)} = \bar{T}(b, c) q_1^{2c} \mathsf{A}_2 \mathsf{B}_3, \tag{44b}$$

$$T(a, c) P_{(1234)} = \bar{T}(a, b) q_1^{-2a} \mathsf{B}_2 \mathsf{B}_4. \tag{44c}$$

Proof. The equalities $\mathsf{A}^2 = \mathsf{B}^2 = 1$ directly follow from the the involutivity of the operators A, B and the formulas $AL^{\frac{1}{2}}A = L^{-\frac{1}{2}}$ and $BR^{\frac{1}{2}}B = R^{-\frac{1}{2}}$. We have

$$\mathsf{A}^* = (L^{-\frac{1}{2}})^* A^* = L^{-\frac{1}{2}} A^* = L^{-\frac{1}{2}} AAA^* = AL^{\frac{1}{2}} AA^* = AL^{\frac{1}{2}} L^{-1} = AL^{-\frac{1}{2}} = \mathsf{A}.$$

Similarly, $\mathsf{B}^* = \mathsf{B}$. To prove (44), observe that

$$AL^{-b} R^{-c} \mathsf{A} = \mathsf{A} L^{\frac{1}{2}-b} R^{-c} \mathsf{A} = L^{b-\frac{1}{2}} \mathsf{A} R^{-c} \mathsf{A} = L^{b-\frac{1}{2}} \left(\mathsf{A} R^{-\frac{1}{2}} \mathsf{A} \right)^{2c}$$

$$= L^{b-\frac{1}{2}} \left(L^{\frac{1}{2}} R^{-\frac{1}{2}} C^{\frac{1}{2}} q \right)^{2c} \overset{sT}{=} q^{2c} C^c L^{b-\frac{1}{2}} \left(L^{\frac{1}{2}} R^{-\frac{1}{2}} \right)^{2c}$$

$$\overset{sT}{=} q^{2c} C^c L^{b-\frac{1}{2}} q^{-2c(1-2c)} L^c R^{-c} \overset{sT}{=} q^{4c^2} C^c L^{b+c-\frac{1}{2}} R^{-c}.$$

Here the first, second, etc. equalities follow respectively from: the definition of A; the formula $\mathsf{A} L^{\frac{1}{2}} = L^{-\frac{1}{2}} \mathsf{A}$; the involutivity of A; the definition of q; the fact that $R^{\frac{1}{2}}$ and $L^{\frac{1}{2}}$ commute with $C^{\frac{1}{2}}$ and stably T-commute with q; Formula (43); the fact that $L^{\frac{1}{2}}$ stably T-commutes with q. A similar computation shows that

$$BL^{-a} R^{-b} \mathsf{B} \overset{sT}{=} q^{4a^2-4a} C^{-a} L^{-a} R^{a+b-\frac{1}{2}}. \tag{45}$$

Formula (44b) is equivalent to the formula $\bar{T}(b,c)q_1^{2c}\mathsf{A}_2\mathsf{B}_3 P_{(23)} = T(a,c)$ which we now prove:

$$\bar{T}(b,c)q_1^{2c}\mathsf{A}_2\mathsf{B}_3 P_{(23)} = \bar{T}q_1^{2c(1-2b)}L_2^{-b}R_2^{-c}R_3^{-b}R_4^c\mathsf{A}_2\mathsf{B}_3 P_{(23)}$$

$$= TP_{(23)}A_2 B_3 q_1^{2c(1-2b)}L_2^{-b}R_2^{-c}R_3^{-b}R_4^c\mathsf{A}_2\mathsf{B}_3 P_{(23)}$$

$$= Tq_1^{2c(1-2b)}(BR^{-b}\mathsf{B})_2(AL^{-b}R^{-c}\mathsf{A})_3 R_4^c$$

$$= Tq_1^{2c(1-2b)}q_3^{4c^2}R_2^{b-\frac{1}{2}}(C^c L^{-a}R^{-c})_3 R_4^c$$

$$= Tq_1^{4ac}R_2^{b-\frac{1}{2}}(C^c L^{-a}R^{-c})_3 R_4^c$$

$$= T(R^c q^{4ac})_1 R_2^{-a}(L^{-a}R^{-c})_3 = T(q^{4ac}R^c)_1 R_2^{-a}(L^{-a}R^{-c})_3 = T(a,c).$$

Here we use consecutively: the definition of $\bar{T}(b,c)$; Formula (13b); the action of $\mathbb{S}_4$; the equalities $BR^{-b}\mathsf{B} = R^{b-\frac{1}{2}}$ and $AL^{-b}R^{-c}\mathsf{A} \overset{T}{=} q^{4c^2}C^c L^{-a}R^{-c}$ established above; the equality $Tq_3 = Tq_1^{-1}$; Formula (36c); the fact that R stably T-commutes (and therefore T-commutes) with q; the definition of $T(a,c)$.

The proof of (44c) is similar:

$$\bar{T}(a,b)q_1^{-2a}\mathsf{B}_2\mathsf{B}_4 P_{(4321)} = TP_{(1234)}B_2 B_4^* q_1^{-2a(2b+1)}L_2^{-a}R_2^{-b}R_3^{-a}R_4^b\mathsf{B}_2\mathsf{B}_4 P_{(4321)}$$

$$= T(B^*R^b\mathsf{B})_1 q_2^{-2a(2b+1)}(BL^{-a}R^{-b}\mathsf{B})_3 R_4^{-a} = TR_1^{\frac{1}{2}-b}q_2^{-2a(2b+1)}(BL^{-a}R^{-b}\mathsf{B})_3 R_4^{-a}$$

$$= TR_1^{\frac{1}{2}-b}q_2^{4ac}(C^{-a}L^{-a}R^{-c})_3 R_4^{-a} = TR_1^c(R^{-a}q^{4ac})_2(L^{-a}R^{-c})_3 = T(a,c).$$

Here we use consecutively: the definition of $\bar{T}(a,b)$ and Formula (13c); the action of $\mathbb{S}_4$; Formula $B^*R^b\mathsf{B} = R^{\frac{1}{2}-b}$ which follows from the definitions of R, B and the equality $BR^{\frac{1}{2}}B = R^{-\frac{1}{2}}$; Formula (45) and the equality $Tq_3 = Tq_2^{-1}$; Formula (36c); the formulas $Rq \overset{T}{=} qR$, $Tq_2 = Tq_1$ and the definition of $T(a,c)$.

Finally, we prove (44a):

$$\bar{T}(a,b)q_1^{-2a}\mathsf{A}_1\mathsf{A}_3 P_{(1234)} = TP_{(4321)}A_1^* A_3 q_1^{-2a(2b+1)}L_2^{-a}R_2^{-b}R_3^{-a}R_4^b\mathsf{A}_1\mathsf{A}_3 P_{(1234)}$$

$$= T(L^{-a}R^{-b})_1(AR^{-a}\mathsf{A})_2 R_3^b(A^*q^{-2a(2b+1)}\mathsf{A})_4$$

$$= T(L^{-a}R^{-b})_1(q^{4a^2}C^a L^{a-\frac{1}{2}}R^{-a})_2 R_3^b(q^{2a(2b+1)}L^{\frac{1}{2}})_4$$

$$= TR_1^{-b}(q^{-4ac}L^{a-\frac{1}{2}}R^{-a})_2(R^b L^{-a})_3 L_4^{\frac{1}{2}-a}$$

$$= T(q^{-4ac}R^c)_1 R_2^{-a}(R^{-c}L^{-a})_3 = T(a,c),$$

where we use (among others) Formulas (13a), (36d), (36b), and (42). $\qquad\square$

Remark 23. *Though we shall not need it in the sequel, note that the involutions* $\mathsf{A}, \mathsf{B}\colon H \to H$ *introduced in Lemma 22 satisfy the relations* $(\mathsf{B}\mathsf{A})^3 = q^2$ *and*

$$(\mathsf{A}C^{\frac{1}{2}})^2 = (\mathsf{B}C^{\frac{1}{2}})^2 = (\mathsf{A}q)^2 = (\mathsf{B}q)^2 = (\mathsf{A}L^{\frac{1}{2}})^2 = (\mathsf{B}R^{\frac{1}{2}})^2 = 1.$$

7.2. **The charged Pentagon and Inversion identities.** For any $a, c \in \frac{1}{2}\mathbb{Z}$, we define the "charged" operators $\tau(a,c), \bar{\tau}(a,c) : \check{H}^{\otimes 2} \to \check{H}^{\otimes 2}$ by

$$\tau(a,c) = q_1^{-4ac} R_1^c R_2^{-a} \tau L_1^{-a} R_1^{-c} \quad \text{and} \quad \bar{\tau}(a,c) = q_1^{4ac} R_2^{-c} L_2^{-a} \bar{\tau} R_1^{-a} R_2^c,$$

where τ and $\bar{\tau}$ are the endomorphisms of $\check{H}^{\otimes 2}$ introduced in Section 2.3. The operator $\tau(a,c)$ is adjoint to $T(a,c)$: for any $u, v \in \hat{H}, x, y \in \check{H}$,

$$\langle\langle u \otimes v, \tau(a,c)(x \otimes y)\rangle\rangle = \langle\langle u \otimes v, (q^{-4ac} R^c \otimes R^{-a})\tau(L^{-a}R^{-c}(x) \otimes y)\rangle\rangle$$

$$= \langle\langle q^{4ac} R^c(u) \otimes R^{-a}(v), \tau(L^{-a}R^{-c}(x) \otimes y)\rangle\rangle$$

$$= T(q^{4ac} R^c(u) \otimes R^{-a}(v) \otimes L^{-a}R^{-c}(x) \otimes y) = T(a,c)(u \otimes v \otimes x \otimes y),$$

where we use the pairing $\langle\langle \cdot, \cdot \rangle\rangle$ introduced in Section 2.3, the unitarity of q, and the symmetry of $L^{\frac{1}{2}}$, $R^{\frac{1}{2}}$. Similarly, $\bar{\tau}(a,c)$ is adjoint to $\bar{T}(a,c)$.

Lemma 24. *The charged operators τ, $\bar{\tau}$ satisfy the following identities.*

(i) *The charged pentagon identity:*

$$\tau_{23}(a_0, c_0)\, \tau_{13}(a_2, c_2)\, \tau_{12}(a_4, c_4) = \tau_{12}(a_3, c_3)\, \tau_{23}(a_1, c_1)\, ({}^\bullet\pi)_{21}$$

for any $a_0, a_1, a_2, a_3, a_4, c_0, c_1, c_2, c_3, c_4 \in \frac{1}{2}\mathbb{Z}$ such that

$$a_1 = a_0 + a_2, \ a_3 = a_2 + a_4, \ c_1 = c_0 + a_4, \ c_3 = a_0 + c_4, \ c_2 = c_1 + c_3. \qquad (46)$$

(ii) *The charged inversion identities: for any $a, c \in \frac{1}{2}\mathbb{Z}$,*

$$\tau_{21}(a,c)\, \bar{\tau}(-a,-c) = \pi^\bullet \quad \text{and} \quad \bar{\tau}(-a,-c)\, \tau_{21}(a,c) = {}^\bullet\pi.$$

Proof. We first rewrite Formulas (36) in terms of τ and $\bar{\tau}$:

$$C_1^{\frac{1}{2}} C_2^{\frac{1}{2}} \tau = \tau C_1^{\frac{1}{2}} C_2^{\frac{1}{2}}, \ R_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \tau = \tau R_1^{\frac{1}{2}} L_2^{\frac{1}{2}}, \ R_1^{\frac{1}{2}} R_2^{\frac{1}{2}} \tau = \tau C_1^{\frac{1}{2}} R_2^{\frac{1}{2}}, \ L_1^{\frac{1}{2}} C_2^{-\frac{1}{2}} \tau = \tau L_1^{\frac{1}{2}} L_2^{\frac{1}{2}}.$$

In the following computations, the underlined expressions are transformed via one of these four equalities. We have

$$q_1^{4(a_1 c_1 + a_3 c_3)} \tau_{12}(a_3, c_3)\, \tau_{23}(a_1, c_1)\, ({}^\bullet\pi)_{21}$$

$$= R_1^{c_3} R_2^{-a_3} \tau_{12} L_1^{-a_3} R_1^{-c_3} R_2^{c_1} R_3^{-a_1} \tau_{23} L_2^{-a_1} R_2^{-c_1} ({}^\bullet\pi)_{21}$$

$$= R_1^{c_3} R_2^{-a_3} R_3^{-a_1} \underline{\tau_{12} R_2^{c_1}} \tau_{23} L_1^{-a_3} R_1^{-c_3} L_2^{-a_1} R_2^{-c_1} ({}^\bullet\pi)_{21}$$

$$= R_1^{c_3} R_2^{-a_3} R_3^{-a_1} R_1^{c_1} R_2^{c_1} \tau_{12} C_1^{-c_1} \tau_{23} L_1^{-a_3} R_1^{-c_3} L_2^{-a_1} R_2^{-c_1} ({}^\bullet\pi)_{21}$$

$$= R_1^{c_2} R_2^{c_0 - a_2} R_3^{-a_1} \tau_{12} \tau_{23} ({}^\bullet\pi)_{21} C_1^{-c_1} L_1^{-a_3} R_1^{-c_3} L_2^{-a_1} R_2^{-c_1},$$

where the last equality follows from the definition of $^\bullet\pi$ and the fact that $R^{\frac{1}{2}}$, $L^{\frac{1}{2}}$, $C^{\frac{1}{2}}$ are grading-preserving operators. We also have

$$q_1^{4(a_0c_0+a_2c_2+a_4c_4)}\,\tau_{23}(a_0,c_0)\,\tau_{13}(a_2,c_2)\,\tau_{12}(a_4,c_4)$$

$$= R_2^{c_0}R_3^{-a_0}\tau_{23}L_2^{-a_0}R_2^{-c_0}R_1^{c_2}R_3^{-a_2}\tau_{13}L_1^{-a_2}R_1^{-c_2}R_1^{c_4}R_2^{-a_4}\tau_{12}L_1^{-a_4}R_1^{-c_4}$$

$$= R_1^{c_2}R_2^{c_0}R_3^{-a_0}\underline{\tau_{23}R_3^{-a_2}}\tau_{13}L_1^{-a_2}R_1^{-a_0-c_1}L_2^{-a_0}R_2^{-c_1}\tau_{12}L_1^{-a_4}R_1^{-c_4}$$

$$= R_1^{c_2}R_2^{c_0-a_2}R_3^{-a_0-a_2}\tau_{23}C_2^{a_2}\tau_{13}L_1^{-a_2}R_1^{-a_0}L_2^{-a_0}\underline{R_1^{-c_1}R_2^{-c_1}\tau_{12}}L_1^{-a_4}R_1^{-c_4}$$

$$= R_1^{c_2}R_2^{c_0-a_2}R_3^{-a_1}\tau_{23}\tau_{13}L_1^{-a_2}C_2^{a_2}\underline{R_1^{-a_0}L_2^{-a_0}\tau_{12}}C_1^{-c_1}R_2^{-c_1}L_1^{-a_4}R_1^{-c_4}$$

$$= R_1^{c_2}R_2^{c_0-a_2}R_3^{-a_1}\tau_{23}\tau_{13}\underline{L_1^{-a_2}C_2^{a_2}\tau_{12}}R_1^{-a_0}L_2^{-a_0}C_1^{-c_1}L_1^{-a_4}R_1^{-c_4}R_2^{-c_1}$$

$$= R_1^{c_2}R_2^{c_0-a_2}R_3^{-a_1}\tau_{23}\tau_{13}\tau_{12}C_1^{-c_1}L_1^{-a_2}R_1^{-a_0}L_1^{-a_4}R_1^{-c_4}L_2^{-a_1}R_2^{-c_1}$$

$$= q_1^{-8a_0a_4}R_1^{c_2}R_2^{c_0-a_2}R_3^{-a_1}\tau_{23}\tau_{13}\tau_{12}C_1^{-c_1}L_1^{-a_3}R_1^{-c_3}L_2^{-a_1}R_2^{-c_1}.$$

Comparing the obtained expressions and using that $\tau_{12}\tau_{23}(^\bullet\pi)_{21}=\tau_{23}\tau_{13}\tau_{12}$, we conclude that the charged pentagon equality follows from the formula

$$a_0c_0 + a_2c_2 + a_4c_4 + 2a_0a_4 = a_1c_1 + a_3c_3.$$

This formula is verified as follows:

$$a_0c_0 + a_2c_2 + a_4c_4 + 2a_0a_4 = a_0(c_0+a_4) + a_2c_2 + a_4(a_0+c_4)$$

$$= a_0c_1 + a_2(c_1+c_3) + a_4c_3 = (a_0+a_2)c_1 + (a_2+a_4)c_3 = a_1c_1 + a_3c_3.$$

We now prove the first inversion identity:

$$\tau_{21}(a,c)\,\bar{\tau}(-a,-c) = q_1^{-4ac}R_2^cR_1^{-a}\tau_{21}L_2^{-a}R_2^{-c}q_1^{4ac}R_2^cL_2^a\bar{\tau}R_1^aR_2^{-c}$$

$$= R_2^cR_1^{-a}\tau_{21}\bar{\tau}R_1^aR_2^{-c} = R_2^cR_1^{-a}\pi^\bullet R_1^aR_2^{-c} = \pi^\bullet,$$

where the second equality follows from (26) since q is a T-scalar. The equality $R_2^cR_1^{-a}\pi^\bullet R_1^aR_2^{-c} = \pi^\bullet$ follows from the fact that $\pi^\bullet$ is the projector on a direct sum of multiplicity spaces. The second inversion identity is proved similarly. $\square$

8. Charged $6j$-symbols

Let $i, j, k, l, m, n \in I$ and $a, c \in \frac{1}{2}\mathbb{Z}$. Replacing T by $T(a,c)$ in the definition of the positive $6j$-symbol in Section 3.2, we obtain the *charged positive $6j$-symbol*

$$\left\{\begin{matrix} i & j & k \\ l & m & n \end{matrix}\right\}(a,c) \in H_m^{kl} \otimes H_k^{ij} \otimes H_{jl}^n \otimes H_{in}^m. \tag{47}$$

Replacing $\bar{T}$ by $\bar{T}(a,c)$ in the definition of the negative $6j$-symbol in Section 3.2, we obtain the *charged negative $6j$-symbol*

$$\left\{\begin{matrix} i & j & k \\ l & m & n \end{matrix}\right\}^{-}(a,c) \in H_m^{in} \otimes H_n^{jl} \otimes H_{ij}^k \otimes H_{kl}^m. \tag{48}$$

The formulas of Section 3.2 computing $T, \bar{T}, \tau, \bar{\tau}$ in terms of the $6j$-symbols extend to the present setting by inserting (a, c) after each occurrence of $T, \bar{T}, \tau, \bar{\tau}$ and after each $6j$-symbol.

The properties of the charged T-forms established in Section 7 can be rewritten in terms of the charged $6j$-symbols. For any $a, b, c \in \frac{1}{2}\mathbb{Z}$ such that $a + b + c = \frac{1}{2}$, Formula (44a) yields

$$\left\{ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right\} (a, c) = P_{(4321)} q_1^{2a} \mathsf{A}_1 \mathsf{A}_3 \left(\left\{ \begin{array}{ccc} i^* & k & j \\ l & n & m \end{array} \right\}^- (a, b) \right), \qquad (49)$$

where A_1 is induced by the restriction of A to $H_n^{i^*m}$ and A_3 is induced by the restriction of A to $H_{i^*k}^j$. Formula (44b) yields

$$\left\{ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right\} (a, c) = P_{(23)} q_1^{-2c} \mathsf{A}_2 \mathsf{B}_3 \left(\left\{ \begin{array}{ccc} k & j^* & i \\ n & m & l \end{array} \right\}^- (b, c) \right), \qquad (50)$$

where A_2 is induced by the restriction of A to $H_l^{j^*n}$ and B_3 is induced by the restriction of B to $H_{kj^*}^i$. Finally, Formula (44c) yields

$$\left\{ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right\} (a, c) = P_{(1234)} q_1^{2a} \mathsf{B}_2 \mathsf{B}_4 \left(\left\{ \begin{array}{ccc} i & n & m \\ l^* & k & j \end{array} \right\}^- (a, b) \right), \qquad (51)$$

where B_2 is induced by the restriction of B to $H_j^{nl^*}$ and B_4 is induced by the restriction of B to $H_{ml^*}^k$.

The charged pentagon identity yields that for any $j_0, j_1, \ldots, j_8 \in I$ and any $a_0, a_1, a_2, a_3, a_4, c_0, c_1, c_2, c_3, c_4 \in \frac{1}{2}\mathbb{Z}$ satisfying (46),

$$\sum_{j \in I} *_{j_7}^{jj_4} *_{j}^{j_2 j_3} *_{j_6}^{j_1 j} \left(\left\{ \begin{array}{ccc} j_1 & j_2 & j_5 \\ j_3 & j_6 & j \end{array} \right\} (a_0, c_0) \otimes \left\{ \begin{array}{ccc} j_1 & j & j_6 \\ j_4 & j_0 & j_7 \end{array} \right\} (a_2, c_2) \right.$$

$$\otimes \left\{ \begin{array}{ccc} j_2 & j_3 & j \\ j_4 & j_7 & j_8 \end{array} \right\} (a_4, c_4) \right)$$

$$= g_{j_2, j_3} P_{(135642)} *_{j_0}^{j_5 j_8} \left(\left\{ \begin{array}{ccc} j_1 & j_2 & j_5 \\ j_8 & j_0 & j_7 \end{array} \right\} (a_1, c_1) \otimes \left\{ \begin{array}{ccc} j_5 & j_3 & j_6 \\ j_4 & j_0 & j_8 \end{array} \right\} (a_3, c_3) \right). \qquad (52)$$

Here both sides lie in the k-vector space (19).

The first inversion relations gives that for all $i, j, k, k', l, m \in I$ and $a, c \in \frac{1}{2}\mathbb{Z}$,

$$\sum_{n \in I} *_{jl}^n *_{in}^m \left(\left\{ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right\} (a, c) \otimes \left\{ \begin{array}{ccc} i & j & k' \\ l & m & n \end{array} \right\}^- (-a, -c) \right)$$

$$= \delta_{k'}^k \, g_{j,l} \, P_{(432)} (\delta_m^{kl} \otimes \delta_k^{ij}).$$

The second inversion relations gives that for all $i, j, l, m, n, n' \in I$ and $a, c \in \frac{1}{2}\mathbb{Z}$,

$$\sum_{k \in I} *_{ij}^{k} *_{kl}^{m} \left(\left\{ \begin{array}{ccc} i & j & k \\ l & m & n' \end{array} \right\}^{-} (-a, -c) \otimes \left\{ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right\} (a, c) \right)$$

$$= \delta_{n'}^{n} \, g_{i,j} \, P_{(432)}(\delta_{m}^{in} \otimes \delta_{n}^{jl}).$$

9. Three-manifold invariants

In this section, following the ideas of the paper [8], we associate to a $\widehat{\Psi}$-system an invariant of an oriented compact three-manifold together with a non-empty link.

9.1. **Topological preliminaries.** Throughout this subsection, the symbol M denotes a closed connected orientable 3-manifold. Following [2], by a *quasi-regular triangulation* of M we mean a decomposition of M as a union of embedded tetrahedra such that the intersection of any two tetrahedra is a union (possibly, empty) of several of their vertices, edges, and (2-dimensional) faces. Quasi-regular triangulations differ from the usual triangulations in that they may have tetrahedra meeting along several vertices, edges, and faces. Note that each edge of a quasi-regular triangulation has two distinct endpoints.

A *Hamiltonian link* in a quasi-regular triangulation $\mathcal{T}$ of M is a set $\mathcal{L}$ of unoriented edges of $\mathcal{T}$ such that every vertex of $\mathcal{T}$ belongs to exactly two elements of $\mathcal{L}$. Then the union of the edges of $\mathcal{T}$ belonging to $\mathcal{L}$ is a link L in M. We call the pair $(\mathcal{T}, \mathcal{L})$ an *H-triangulation* of (M, L).

Proposition 25 ([2], Proposition 4.20). *For any non-empty link L in M, the pair (M, L) admits an H-triangulation.*

H-triangulations of (M, L) can be related by elementary moves of two types, the *H-bubble moves* and the *H-Pachner* $2 \leftrightarrow 3$ *moves*. The *positive H-bubble move* on an H-triangulation $(\mathcal{T}, \mathcal{L})$ starts with a choice of a face $F = v_1 v_2 v_3$ of $\mathcal{T}$ such that at least one of its edges, say $v_1 v_3$, is in $\mathcal{L}$. Consider two tetrahedra of $\mathcal{T}$ meeting along F. We unglue these tetrahedra along F and insert a 3-ball between the resulting two copies of F. We triangulate this 3-ball by adding a vertex v_4 at its center and three edges connecting v_4 to v_1, v_2, and v_3. The edge $v_1 v_3$ is removed from $\mathcal{L}$ and replaced by the edges $v_1 v_4$ and $v_3 v_4$. This move can be visualized as the transformation
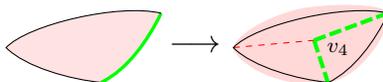
where the bold (green) edges belong to $\mathcal{L}$. The inverse move is the *negative H-bubble move*. The *positive H-Pachner* $2 \leftrightarrow 3$ *move* can be visualized as the

transformation
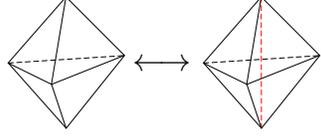
This move preserves the set $\mathcal{L}$. The inverse move is the *negative H-Pachner move*; it is allowed only when the edge common to the three tetrahedra on the right is not in $\mathcal{L}$.

Proposition 26 ([2], Proposition 4.23). *Let L be a non-empty link in M. Any two H-triangulations of (M, L) can be related by a finite sequence of H-bubble moves and H-Pachner moves in the class of H-triangulations of (M, L).*

Integral charges on H-triangulations first have been introduced in [8] and the corresponding theory subsequently has been developed in [2]. This theory is a natural extension of the theory of angle structures, see for example [14, 15], into the framework of arbitrary triangulated three-manifolds.

By a *charge* on a tetrahedron T, we mean a $\frac{1}{2}\mathbb{Z}$-valued function c on the set of edges of T such that $c(e) = c(e')$ for any opposite edges e, e' and $c(e_1) + c(e_2) + c(e_3) = 1/2$ for any edges e_1, e_2, e_3 of T forming the boundary of a face. Consider now an H-triangulation $(\mathcal{T}, \mathcal{L})$ of (M, L) as above. Let $E(\mathcal{T})$ be the set of edges of $\mathcal{T}$ and let $\widehat{E}(\mathcal{T})$ be the set of pairs (a tetrahedron T of $\mathcal{T}$, an edge of T). Let $\epsilon_{\mathcal{T}} : \widehat{E}(\mathcal{T}) \to E(\mathcal{T})$ be the obvious projection. For any edge e of $\mathcal{T}$, the set $\epsilon_{\mathcal{T}}^{-1}(e)$ has n elements, where n is the number of tetrahedra of $\mathcal{T}$ adjacent to e.

Definition 27. *A* charge *on $(\mathcal{T}, \mathcal{L})$ is a map $c : \widehat{E}(\mathcal{T}) \to \frac{1}{2}\mathbb{Z}$ such that*

  (1) *the restriction of c to any tetrahedron T of $\mathcal{T}$ is a charge on T,*
  (2) *for each edge e of $\mathcal{T}$ not belonging to $\mathcal{L}$ we have $\sum_{e' \in \epsilon_{\mathcal{T}}^{-1}(e)} c(e') = 1$,*
  (3) *for each edge e of $\mathcal{T}$ belonging to $\mathcal{L}$ we have $\sum_{e' \in \epsilon_{\mathcal{T}}^{-1}(e)} c(e') = 0$.*

Each charge c on $(\mathcal{T}, \mathcal{L})$ determines a cohomology class $[c] \in H^1(M; \mathbb{Z}/2\mathbb{Z})$ as follows. Let s be a simple closed curve in M which lies in general position with respect to $\mathcal{T}$ and such that s never leaves a tetrahedron T of $\mathcal{T}$ through the same 2-face by which it entered. Thus each time s passes through T, it determines a unique edge e belonging to both the entering and departing faces. The sum of the residues $2\,c|_T(e)(mod\,2) \in \mathbb{Z}/2\mathbb{Z}$ over all passages of s through tetrahedra of $\mathcal{T}$ depends only on the homology class of s and is the value of $[c]$ on s. According to [3], for any non-empty link $L \subset M$ and any $\zeta \in H^1(M; \mathbb{Z}/2\mathbb{Z})$, each H-triangulation of (M, L) has a charge representing ζ.

Lemma 28. *Let $(\mathcal{T}, \mathcal{L})$ and $(\mathcal{T}', \mathcal{L}')$ be H-triangulations of (M, L) such that $(\mathcal{T}', \mathcal{L}')$ is obtained from $(\mathcal{T}, \mathcal{L})$ by an H-Pachner move or an H-bubble move. Let c be a charge on $(\mathcal{T}, \mathcal{L})$. Then there exists a charge c' on $(\mathcal{T}', \mathcal{L}')$ such that c'*

equals c on all pairs (a tetrahedron T of $\mathcal{T}$ not involved in the move, an edge of T) and for any common edge e of $\mathcal{T}$ and $\mathcal{T}'$,

$$\sum_{a \in \epsilon_{\mathcal{T}}^{-1}(e)} c(a) = \sum_{a' \in \epsilon_{\mathcal{T}'}^{-1}(e)} c'(a'). \tag{53}$$

Moreover, $[c] = [c']$.

Proof. A straightforward calculation, cf. [2], Lemma 4.10. $\qquad\square$

The charge c' in this lemma is unique if the move $(\mathcal{T}, \mathcal{L}) \mapsto (\mathcal{T}', \mathcal{L}')$ is negative. In this case we say that c' is *induced* by c. If the move $(\mathcal{T}, \mathcal{L}) \mapsto (\mathcal{T}', \mathcal{L}')$ is positive, then c' is not unique, see Lemma 4.12 of [2].

9.2. **The algebraic data.** We describe the algebraic data needed to define our 3-manifold invariant. Let $\mathcal{C}$ be a monoidal Ab-category whose ground ring k is a field. Fix a $\widehat{\Psi}$-system in $\mathcal{C}$ with distinguished simple objects $\{V_i\}_{i \in I}$. Fix a family $\{I_g\}_{g \in G}$ of finite subsets of the set I numerated by elements of a group G and satisfying the following conditions:

(1) for any $g \in G$, if $i \in I_g$, then $i^* \in I_{g^{-1}}$;
(2) for any $i_1 \in I_{g_1}, i_2 \in I_{g_2}, k \in I \setminus I_{g_1 g_2}$ with $g_1, g_2 \in G$, we have $H_k^{i_1 i_2} = 0$;
(3) if $i_1 \in I_{g_1}, i_2 \in I_{g_2}$ with $g_1, g_2 \in G$, then either $I_{g_1 g_2} = \emptyset$ or there is $k \in I_{g_1 g_2}$ such that $H_k^{i_1 i_2} \neq 0$;
(4) for any finite family $\{g_r \in G\}_r$, there is $g \in G$ such that $I_{g g_r} \neq \emptyset$ for all r;
(5) we are given a map $\mathsf{b} : I \to \mathsf{k}$ such that $\mathsf{b}(i) = \mathsf{b}(i^*)$ for all $i \in I$, and for any $g_1, g_2 \in G$, $k \in I_{g_1 g_2}$ such that $I_{g_1} \neq \emptyset$ and $I_{g_2} \neq \emptyset$,

$$\sum_{i_1 \in I_{g_1}, i_2 \in I_{g_2}} \mathsf{b}(i_1)\, \mathsf{b}(i_2) \dim(H_k^{i_1 i_2}) = \mathsf{b}(k).$$

9.3. **G-colorings and state sums.** Fix algebraic data as in Section 9.2. Let M be a closed connected orientable 3-manifold and $\mathcal{T}$ a quasi-regular triangulation of M as in Section 9.1. A *G-coloring* of $\mathcal{T}$ is a map Φ from the set of oriented edges of $\mathcal{T}$ to G such that

(1) $\Phi(-e) = \Phi(e)^{-1}$ for any oriented edge e of $\mathcal{T}$, where $-e$ is e with opposite orientation and
(2) if e_1, e_2, e_3 are ordered edges of a face of $\mathcal{T}$ endowed with orientation induced by the order, then $\Phi(e_1)\, \Phi(e_2)\, \Phi(e_3) = 1$.

A *G-gauge of $\mathcal{T}$* is a map from the set of vertices of $\mathcal{T}$ to G. The G-gauges of $\mathcal{T}$ form a multiplicative group which acts on the set of G-colorings of $\mathcal{T}$ as follows. If δ is a G-gauge of $\mathcal{T}$ and Φ is a G-coloring of $\mathcal{T}$, then the G-coloring $\delta\Phi$ is given by

$$(\delta\Phi)(e) = \delta(v_e^-)\, \Phi(e)\, \delta(v_e^+)^{-1},$$

where v_e^- (resp. v_e^+) is the initial (resp. terminal) vertex of an oriented edge e.

Let $\mathcal{M}(M, G)$ be the set of conjugacy classes of group homomorphisms from the fundamental group of M to G. The elements of $\mathcal{M}(M, G)$ bijectively correspond to the G-colorings of $\mathcal{T}$ considered up to gauge transformations. Indeed, for a vertex x_0 of $\mathcal{T}$, each G-coloring Φ of $\mathcal{T}$ determines a homomorphism $\pi_1(M, x_0) \to G$. To compute this homomorphism on an element of $\pi_1(M, x_0)$, one represents this element by a loop based at x_0 and formed by a sequence of oriented edges of $\mathcal{T}$; then one takes the product of the values of Φ on these edges. Let $[\Phi] \in \mathcal{M}(M, G)$ be the conjugacy class of this homomorphism. We say that Φ represents $[\Phi]$. The assignment $\Phi \mapsto [\Phi]$ establishes the bijective correspondence mentioned above.

A *state* of a G-coloring Φ of $\mathcal{T}$ is a map φ assigning to every oriented edge e of $\mathcal{T}$ an element $\varphi(e)$ of $I_{\Phi(e)}$ such that $\varphi(-e) = \varphi(e)^*$ for all e. The set of all states of Φ is denoted $\mathrm{St}(\Phi)$. The identity $\mathsf{b}(\varphi(e)) = \mathsf{b}(\varphi(-e))$ allows us to use the notation $\mathsf{b}(\varphi(e))$ for non-oriented edges. It is easy to see that $\mathrm{St}(\Phi) \neq \emptyset$ if and only if $I_{\Phi(e)} \neq \emptyset$ for all oriented edges e of $\mathcal{T}$. In this case we say that Φ is *admissible*.

Let now L be a non-empty link in M and $(\mathcal{T}, \mathcal{L})$ be an H-triangulation of (M, L) with charge c. From this data, we derive a certain partition function (state sum) as follows. Fix a total order on the set of vertices of $\mathcal{T}$. Consider a tetrahedron T of $\mathcal{T}$ with vertices v_1, v_2, v_3, v_4 in increasing order. We say that T is *right oriented* if the tangent vectors $v_1 v_2, v_1 v_3, v_1 v_4$ form a positive basis in the tangent space of M; otherwise T is *left oriented*. For an admissible G-coloring Φ of $\mathcal{T}$ and a state $\varphi \in \mathrm{St}(\Phi)$, set

$$i = \varphi(\overrightarrow{v_1 v_2}), j = \varphi(\overrightarrow{v_2 v_3}), k = \varphi(\overrightarrow{v_1 v_3}), l = \varphi(\overrightarrow{v_3 v_4}), m = \varphi(\overrightarrow{v_1 v_4}), n = \varphi(\overrightarrow{v_2 v_4}),$$

where $\overrightarrow{v_i v_j}$ is the oriented edge of T going from v_i to v_j. Set

$$|T|_\varphi = \begin{cases} \left\{ \begin{matrix} i & j & k \\ l & m & n \end{matrix} \right\} (c(v_1 v_2), c(v_2 v_3)) \text{ if } T \text{ is right oriented,} \\ \left\{ \begin{matrix} i & j & k \\ l & m & n \end{matrix} \right\}^- (c(v_1 v_2), c(v_2 v_3)) \text{ if } T \text{ is left oriented.} \end{cases}$$

The $6j$-symbol $|T|_\varphi$ belongs to the tensor product of 4 multiplicity modules associated to the faces of T. Specifically,

$$|T|_\varphi \in \begin{cases} H_m^{kl} \otimes H_k^{ij} \otimes H_{jl}^n \otimes H_{in}^m & \text{if } T \text{ is right oriented,} \\ H_m^{in} \otimes H_n^{jl} \otimes H_{ij}^k \otimes H_{kl}^m & \text{if } T \text{ is left oriented.} \end{cases}$$

Note that any face of $\mathcal{T}$ belongs to exactly two tetrahedra of $\mathcal{T}$, and the associated multiplicity modules are dual to each other, see Lemma 1. These dualities allow us to contract $\otimes_T |T|_\varphi$ into a scalar. Denote by cntr the tensor product of all these tensor contractions determined by the faces of $\mathcal{T}$. Set

$$\mathsf{K}(\mathcal{T}, \mathcal{L}, \Phi, c) = \sum_{\varphi \in \mathrm{St}(\Phi)} \left(\prod_{e \in \mathcal{L}} \mathsf{b}(\varphi(e)) \right) \mathrm{cntr} \left(\bigotimes_T |T|_\varphi \right) \in \mathsf{k},$$

where T runs over all tetrahedra of $\mathcal{T}$. To compute $\mathsf{K}(\mathcal{T}, \mathcal{L}, \Phi, c)$ we may need to order the faces of $\mathcal{T}$, but the result does not depend on this order.

Theorem 29. *Suppose that there exists a scalar $\widetilde{q} \in \mathsf{k}$ such that q is T-equal to the operator*

$$\widetilde{q} \operatorname{Id}_{\hat{H}} \oplus \widetilde{q}^{-1} \operatorname{Id}_{\check{H}} \in \operatorname{End}(H).$$

Then, up to multiplication by integer powers of $\widetilde{q}$, the state sum $\mathsf{K}(\mathcal{T}, \mathcal{L}, \Phi, c)$ depends only on the isotopy class of L in M, the conjugacy class $[\Phi] \in \mathcal{M}(M, G)$, and the cohomology class $[c] \in H^1(M; \mathbb{Z}/2\mathbb{Z})$ (and does not depend on the choice of c in its cohomology class, the admissible representative Φ of $[\Phi]$, the H-triangulation $\mathcal{T}$ of (M, L), and the ordering of the vertices of $\mathcal{T}$).

A proof of this theorem will be given in Section 10.

Lemma 30. *Any element of the set $\mathcal{M}(M, G)$ can be represented by an admissible G-coloring on an arbitrary quasi-regular triangulation $\mathcal{T}$ of M.*

Proof. Take any G-coloring Φ of $\mathcal{T}$ representing the given element of $\mathcal{M}(M, G)$. We say that a vertex of $\mathcal{T}$ is *bad* for Φ if there is an oriented edge e in $\mathcal{T}$ incident to this vertex and such that $I_{\Phi(e)} = \emptyset$. It is clear that Φ is admissible if and only if Φ has no bad vertices. We show how to modify Φ in the class $[\Phi]$ to reduce the number of bad vertices. Observe first that each pair (a vertex v of $\mathcal{T}$, an element g of G) determines a G-gauge $\delta^{v,g}$ whose value on any vertex u of $\mathcal{T}$ is defined by

$$\delta^{v,g}(u) = \begin{cases} g & \text{if } u = v, \\ 1 & \text{else.} \end{cases} \tag{54}$$

Let v be a bad vertex for Φ. Pick any $g \in G$ such that $I_{g\Phi(e)} \neq \emptyset$ for all edges e of $\mathcal{T}$ adjacent to v and oriented away from v. The G-coloring $\delta^{v,g}\Phi$ takes values in the set $\{h \in G \mid I_h \neq \emptyset\}$ on all edges of $\mathcal{T}$ incident to v and takes the same values as Φ on all edges of $\mathcal{T}$ not incident to v. Here we use the fact that the edges of $\mathcal{T}$ are not loops which is ensured by the quasi-regularity of $\mathcal{T}$. The transformation $\Phi \mapsto \delta^{v,g}\Phi$ decreases the number of bad vertices. Repeating this argument, we find a G-coloring without bad vertices in the class $[\Phi]$. $\qquad\square$

We represent any $h \in \mathcal{M}(M, G)$ by an admissible G-coloring Φ of $\mathcal{T}$ and any $\zeta \in H^1(M; \mathbb{Z}/2\mathbb{Z})$ by a charge c and set

$$\mathsf{K}(M, L, h, \zeta) = \mathsf{K}(\mathcal{T}, \mathcal{L}, \Phi, c) \in \mathsf{k}.$$

By Theorem 29, $\mathsf{K}(M, L, h, \zeta)$ is a topological invariant of the tuple (M, L, h, c).

10. Proof of Theorem 29

Throughout this section we keep the assumptions of Theorem 29.

Lemma 31. *Up to multiplication by integer powers of $\widetilde{q}$, $\mathsf{K}(\mathcal{T}, \mathcal{L}, \Phi, c)$ does not depend on the ordering of the vertices of $\mathcal{T}$.*

Proof. Consider the natural action of the symmetric group on the orderings of the vertices of $\mathcal{T}$. As the symmetric group is generated by simple transpositions $(r, r + 1)$, it is enough to consider the action of one such transposition on an ordering. If the vertices labelled by r and $r + 1$ do not span an edge of $\mathcal{T}$, then the new state sum is identical to the old one. Suppose that an edge, e, of $\mathcal{T}$ connects the vertices labelled by r and $r + 1$. Let P be the set of all labels p such that the vertices labeled by $r, r + 1$, and p form a face of $\mathcal{T}$. This face, denoted f_p, belongs to two adjacent tetrahedra of $\mathcal{T}$ containing e and determines two dual multiplicity spaces.

For a tetrahedron T of $\mathcal{T}$, consider the transformation of the $6j$-symbol $|T|_\varphi$ under the permutation $(r, r + 1)$. If T does not contain e, then $|T|_\varphi$ does not change. We claim that for $T \supset e$, the $6j$-symbol $|T|_\varphi$ is multiplied by an integer power of $\widetilde{q}$ independent of φ and composed with the tensor product of operators acting on the multiplicity spaces corresponding to f_p, where p runs over the 2-element set $\{p \in P \,|\, f_p \subset T\}$. Here the operator corresponding to p in this set is A if $p > r$ and B if $p < r$. Since A and B are involutive and self-dual, the effect of this transformation after the tensor contraction cntr will be multiplication by an integer power of $\widetilde{q}$ independent of φ. This will imply the lemma.

The claim above follows from Equations (49)-(51). Indeed, let $r, r + 1, p, p'$ be the labels of the vertices of T. Suppose for concreteness that $p < r$ and $r + 1 < p'$ (the other cases are similar). Then the left (resp. right) hand side of Formula (50) with $a = c(T, v_p v_r), b = c(T, v_p v_{r+1}), c = c(T, v_r v_{r+1})$ computes $|T|_\varphi$ before (resp. after) the permutation of r and $r + 1$. The operators A_2 and B_3 in (50) act on the multiplicity spaces corresponding to the faces $f_{p'}$ and f_p, respectively. Therefore Formula (50) implies our claim. $\qquad\square$

Lemma 32. *Let $(\mathcal{T}, \mathcal{L})$, $(\mathcal{T}', \mathcal{L}')$ be H-triangulations of (M, L) such that $(\mathcal{T}', \mathcal{L}')$ is obtained from $(\mathcal{T}, \mathcal{L})$ by a negative H-Pachner move or a negative H-bubble move. Then any admissible G-coloring Φ on $\mathcal{T}$ restricts to an admissible G-coloring Φ' of $\mathcal{T}'$. For any charge c on $(\mathcal{T}, \mathcal{L})$, we have (up to multiplication by powers of $\widetilde{q}$)*

$$\mathsf{K}(\mathcal{T}, \mathcal{L}, \Phi, c) = \mathsf{K}(\mathcal{T}', \mathcal{L}', \Phi', c'), \tag{55}$$

where c' is the charge on $(\mathcal{T}', \mathcal{L}')$ induced by c.

Proof. The values of Φ' form a subset of the set of values of Φ; therefore the admissibility of Φ implies the admissibility of Φ'.

The rest of the proof follows the lines of [16, Section VII.2.3] via translating the geometric moves into algebraic identities. First, we prove (55) for a negative H-bubble move. Let v_1, v_2, v_3, v_4 be the vertices of $\mathcal{T}$ and $F = v_1 v_2 v_3$ the face of $\mathcal{T}'$ as in the description of the bubble move in Section 9.1 (see Figure 1). Since our state sums do not depend on the ordering of the vertices, we assume that v_4 is the last in the order of the vertices of $\mathcal{T}$ and the order of the vertices of $\mathcal{T}'$ is induced by the order of the vertices of $\mathcal{T}$. We can also assume that $v_1 v_4, v_3 v_4 \in \mathcal{L}$ and

$v_1 v_3 \in \mathcal{L}'$. Let T_r (resp. T_l) be the right oriented (resp. left oriented) tetrahedron of $\mathcal{T}$ disappearing under the move. Let $a, b \in \frac{1}{2}\mathbb{Z}$ be the c-charges of the edges $v_1 v_2$, $v_2 v_3$ of T_r respectively. The properties of a charge imply that the charges of the edges $v_1 v_2$, $v_2 v_3$ of T_l are $-a$ and $-b$, respectively.

Fix a state $\varphi' \in \mathrm{St}(\Phi')$ and let $S \subset \mathrm{St}(\Phi)$ be the set of all states of Φ extending φ'. It is enough to show that the term $\mathsf{K}_{\varphi'}$ of $\mathsf{K}(\mathcal{T}', \mathcal{L}', \Phi', c')$ associated to φ' is equal to the sum $\sum_{\varphi \in S} \mathsf{K}_\varphi$ of the terms of $\mathsf{K}(\mathcal{T}, \mathcal{L}, \Phi, c)$ associated to all $\varphi \in S$. Set $i = \varphi'(\overrightarrow{v_1 v_2})$, $j = \varphi'(\overrightarrow{v_2 v_3})$, and $k = \varphi'(\overrightarrow{v_1 v_3})$. For any distinct indices $p, q \in \{1, 2, 3, 4\}$, set $I_{pq} = I_{\Phi(\overrightarrow{v_p v_q})} \subset I$. The admissibility of Φ implies that $I_{pq} \neq \emptyset$ for all p, q. Clearly, $i \in I_{12}, j \in I_{23}, k \in I_{13}$. A state $\varphi \in S$ is determined by the labels

$$ l = \varphi(\overrightarrow{v_3 v_4}) \in I_{34}, \quad m = \varphi(\overrightarrow{v_1 v_4}) \in I_{14}, \quad \text{and} \quad n = \varphi(\overrightarrow{v_2 v_4}) \in I_{24}. $$

We have

$$ |T_r|_\varphi = \left\{ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right\} (a, b) \quad \text{and} \quad |T_l|_\varphi = \left\{ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right\}^{-} (-a, -b). $$

It is convenient to write $|T_r|_{lmn}^{ijk}$ for $|T_r|_\varphi$ and $|T_l|_{lmn}^{ijk}$ for $|T_l|_\varphi$.

Denote by $*_f$ the tensor contraction determined by a face f. We have $\mathsf{K}_{\varphi'} = *_F(\mathsf{b}(k)X)$, where X is the term of the state sum determined by φ' before contraction $*_F$ and multiplication by $\mathsf{b}(k)$. Let F_r and F_l be the faces of T_r and T_l,
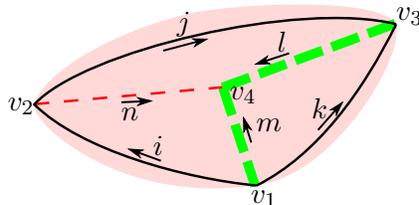

FIGURE 1. $T_l \cup T_r$ colored by $\varphi \in S$

respectively, with vertices v_1, v_2, v_3. We have

$$\sum_{\varphi \in S} \mathsf{K}_\varphi = *_{F_r} *_{F_l} \left(X \otimes \sum_{l \in I_{34}, m \in I_{14}, n \in I_{24}} \mathsf{b}(l)\, \mathsf{b}(m) *_m^{kl} *_{jl}^n *_{in}^m \left(|T_r|_{lmn}^{ijk} \otimes |T_l|_{lmn}^{ijk} \right) \right)$$

$$= *_{F_r} *_{F_l} \left(X \otimes \sum_{l \in I_{34}, m \in I_{14}} \mathsf{b}(l)\, \mathsf{b}(m) *_m^{kl} \left(\sum_{n \in I_{24}} *_{jl}^n *_{in}^m |T_r|_{lmn}^{ijk} \otimes |T_l|_{lmn}^{ijk} \right) \right)$$

$$= *_{F_r} *_{F_l} \left(X \otimes \sum_{l \in I_{34}, m \in I_{14}} \mathsf{b}(l)\, \mathsf{b}(m) *_m^{kl} \left(g_{j,l} \left(\delta_m^{kl} \otimes \delta_k^{ij} \right) \right) \right)$$

$$= *_{F_r} *_{F_l} \left(X \otimes \sum_{l \in I_{34}, m \in I_{14}} \mathsf{b}(l)\, \mathsf{b}(m)\, g_{j,l} \dim(H_m^{kl})\, \delta_k^{ij} \right),$$

where the third equality follows from the first inversion relation and Condition (2) in Section 9.2. The existence of the admissible coloring Φ and Condition (3) of Section 9.2 imply that in the latter expression $g_{j,l} = 1$ for all $l \in I_{34}$. Therefore this expression is equal to

$$*_{F_r} *_{F_l} \left(X \otimes \sum_{l \in I_{34}, m \in I_{14}} \mathsf{b}(l)\, \mathsf{b}(m^*) \dim(H_{k^*}^{lm^*})\, \delta_k^{ij} \right)$$

$$= *_{F_r} *_{F_l} \left(X \otimes \sum_{l \in I_{34}, m' \in I_{41}} \mathsf{b}(l)\, \mathsf{b}(m') \dim(H_{k^*}^{lm'})\, \delta_k^{ij} \right)$$

$$= *_{F_r} *_{F_l} \left(X \otimes \mathsf{b}(k^*)\, \delta_k^{ij} \right) = *_{F_r} *_{F_l} \left(X \otimes \mathsf{b}(k)\, \delta_k^{ij} \right) = *_F (\mathsf{b}(k) X) = \mathsf{K}_{\varphi'},$$

where the second equality is ensured by Condition (5) in Section 9.2. This proves the lemma for the H-bubble moves. Similarly, the H-Pachner move translates into the charged pentagon identity (52), see Figure 2. $\qquad\square$
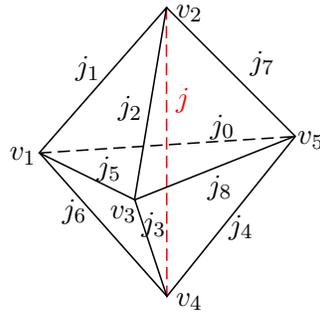

FIGURE 2. Labeling for H-Pachner move

Lemma 33. *Let v be a vertex of $\mathcal{T}$ and let $g \in G$ be such that $I_g \neq \emptyset$. If Φ and $\delta^{v,g}\Phi$ are admissible G-colorings of $\mathcal{T}$, then $\mathsf{K}(\mathcal{T}, \mathcal{L}, \Phi, c) = \mathsf{K}(\mathcal{T}, \mathcal{L}, \delta^{v,g}\Phi, c)$, where $\delta^{v,g}$ is the G-gauge of $\mathcal{T}$ defined in (54).*

Proof. A similar claim in a simpler setting (no charges and G is abelian) was established in [7], Lemma 27. The proof there relies on Lemma 26 of the same paper. Replacing Lemma 26 by Lemma 32 above and making the appropriate adjustments, we easily adapt the argument in [7] to the present setting. $\square$

Lemma 34. *If admissible G-colorings Φ and Φ' of $\mathcal{T}$ represent the same element of $\mathcal{M}(M, G)$, then $\mathsf{K}(\mathcal{T}, \mathcal{L}, \Phi, c) = \mathsf{K}(\mathcal{T}, \mathcal{L}, \Phi', c)$ for any charge c.*

Proof. Since $[\Phi] = [\Phi']$, there are pairs $(v_i, g_i) \in \{\text{vertices of } \mathcal{T}\} \times G$ such that

$$\Phi' = \delta^{v_n, g_n} \delta^{v_{n-1}, g_{n-1}} \cdots \delta^{v_1, g_1} \Phi.$$

Note that the gauges $\delta^{v,g}$ and $\delta^{v',g'}$ commute for all $g, g' \in G$ provided the vertices v, v' are distinct. Using this property and the identity $\delta^{v,g}\delta^{v,g'} = \delta^{v,gg'}$, we can ensure that all the vertices v_i in the expansion of Φ' are pairwise distinct.

We prove the lemma by induction on n. If $n = 0$, then $\Phi' = \Phi$ and there is nothing to prove. For $n \geq 1$, pick any $g \in G$ such that the sets I_g, $I_{gg_1^{-1}}$, $I_{g\Phi(e)}$, and $I_{gg_1^{-1}\Phi'(e)}$ are non-empty for all oriented edges e of $\mathcal{T}$ outgoing from v_1. Then the colorings $\delta^{v_1, g}\Phi$ and $\delta^{v_1, gg_1^{-1}}\Phi'$ are admissible. Clearly,

$$\delta^{v_1, gg_1^{-1}}\Phi' = \delta^{v_n, g_n} \delta^{v_{n-1}, g_{n-1}} \cdots \delta^{v_2, g_2} \delta^{v_1, g}\Phi.$$

Lemma 33 and the induction assumption imply that

$$
\begin{aligned}
\mathsf{K}(\mathcal{T}, \mathcal{L}, \Phi, c) &= \mathsf{K}(\mathcal{T}, \mathcal{L}, \delta^{v_1, g}\Phi, c) \\
&= \mathsf{K}(\mathcal{T}, \mathcal{L}, \delta^{v_n, g_n} \delta^{v_{n-1}, g_{n-1}} \cdots \delta^{v_2, g_2} \delta^{v_1, g}\Phi, c) \\
&= \mathsf{K}(\mathcal{T}, \mathcal{L}, \delta^{v_1, gg_1^{-1}}\Phi', c) = \mathsf{K}(\mathcal{T}, \mathcal{L}, \Phi', c).
\end{aligned}
$$

$\square$

Lemma 34 implies that $\mathsf{K}(\mathcal{T}, \mathcal{L}, \Phi, c)$ depends only on the element of $\mathcal{M}(M, G)$ represented by Φ. We represent any $h \in \mathcal{M}(M, G)$ by an admissible G-coloring Φ of $\mathcal{T}$ and set $\mathsf{K}(\mathcal{T}, \mathcal{L}, h, c) = \mathsf{K}(\mathcal{T}, \mathcal{L}, \Phi, c)$. The scalar $\mathsf{K}(\mathcal{T}, \mathcal{L}, h, c)$ is invariant under negative H-Pachner/H-bubble moves. More precisely, Lemma 32 implies that under the assumptions of this lemma, for any $h \in \mathcal{M}(M, G)$, we have (up to multiplication by powers of $\widetilde{q}$)

$$\mathsf{K}(\mathcal{T}, \mathcal{L}, h, c) = \mathsf{K}(\mathcal{T}', \mathcal{L}', h, c'). \tag{56}$$

Lemma 35. *For any $h \in \mathcal{M}(M, G)$, the scalar $\mathsf{K}(\mathcal{T}, \mathcal{L}, h, c)$ does not depend on the choice of the charge c in its cohomology class.*

Proof. The lemma is proved in two steps: first, any two charges are connected by a finite sequence of local modifications and second, the state sum is shown to be preserved under these modifications. Recall from Section 9.1 the set $\widehat{E}(\mathcal{T})$ of pairs (a tetrahedron T of $\mathcal{T}$, an edge of T). Let $*$ be the involution on $\widehat{E}(\mathcal{T})$ carrying a pair (T, e) to the pair $*(T, e) = (T, *e)$ where $*e$ is the edge of T opposite to e. Fix from now on an order on the set of vertices of $\mathcal{T}$.

For each edge e of $\mathcal{T}$ we define a map $d(e) : \widehat{E}(\mathcal{T}) \to \{-1/2, 0, 1/2\}$ as follows. Let v be the vertex of e which is largest in the ordering of vertices. Let $T_0, T_1, T_2, ..., T_n = T_0$ be the cyclically ordered tetrahedra of $\mathcal{T}$ adjacent to e. We choose the cyclic order so that the induced orientation in the plane transversal to e followed by the orientation of e towards v determines the given orientation of M. For $i = 1, \ldots, n$, denote by e_i the only edge of the 2-face $T_{i-1} \cap T_i$ which is distinct from e and incident to v. For any $a \in \widehat{E}(\mathcal{T})$, set

$$d(e)(a) = \begin{cases} 1/2 & \text{if } a = (T_{i-1}, e_i) \text{ or } a = *(T_{i-1}, e_i) \text{ for some } i \in \{1, ..., n\}, \\ -1/2 & \text{if } a = (T_i, e_i) \text{ or } a = *(T_i, e_i) \text{ for some } i \in \{1, ..., n\}, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that for any family of integers $\{\lambda_e\}_e$ numerated by the edges e of $\mathcal{T}$ the sum $c + \sum_e \lambda_e d(e)$ is a charge of $(\mathcal{T}, \mathcal{L})$ and $[c + \sum_e \lambda_e d(e)] = [c]$. The following is due to Baseilhac [1], see also Neumann [14] and [2], Proposition 4.8:

Fact. For any pair of charges c, c' of $(\mathcal{T}, \mathcal{L})$ with $[c] = [c']$, there is a family of integers $\{\lambda_e\}_e$ numerated by the edges e of $\mathcal{T}$ such that $c' = c + \sum_e \lambda_e d(e)$.

Therefore to prove the lemma, it is enough to show that for any edge e of $\mathcal{T}$,

$$\mathsf{K}(\mathcal{T}, \mathcal{L}, h, c) = \mathsf{K}(\mathcal{T}, \mathcal{L}, h, c + d(e)). \tag{57}$$

We show how to reduce the case $e \in \mathcal{L}$ to the case $e \notin \mathcal{L}$. Suppose that $e \in \mathcal{L}$. Pick a face F of $\mathcal{T}$ containing e. We apply the H-bubble move to F producing a new H-triangulation $(\mathcal{T}_b, \mathcal{L}_b)$ such that e viewed as an edge of $\mathcal{T}_b$ does not belong to $\mathcal{L}_b$. Pick a charge c_b on $\mathcal{T}_b$ inducing the charge c on $\mathcal{T}$. A direct calculation shows that the charge $c_b + d(e)$ on $\mathcal{T}_b$ induces the charge $c + d(e)$ on $\mathcal{T}$. By (56), $\mathsf{K}(\mathcal{T}, \mathcal{L}, h, c) = \mathsf{K}(\mathcal{T}_b, \mathcal{L}_b, h, c_b)$ and $\mathsf{K}(\mathcal{T}, \mathcal{L}, h, c + d(e)) = \mathsf{K}(\mathcal{T}_b, \mathcal{L}_b, h, c_b + d(e))$. Therefore, it is enough to prove (57) in the case $e \notin \mathcal{L}$.

Suppose from now on that $e \notin \mathcal{L}$. We first reduce the proof of (57) to the case where e is contained in at least 3 tetrahedra of $\mathcal{T}$ and $\mathcal{T}$ has a vertex such that there is precisely one face of $\mathcal{T}$ containing e and this vertex. Let V be the set of vertices of $\mathcal{T}$. Pick a face of $\mathcal{T}$ not containing e and having at least one side in $\mathcal{L}$. We apply to this face the H-bubble move producing a new H-triangulation $(\mathcal{T}', \mathcal{L}')$ whose set of vertices is the union of V with a 1-point set $\{v_0\}$. Pick a charge c' on $\mathcal{T}'$ inducing the charge c on $\mathcal{T}$. We can find a sequence $T_0, T_1, ..., T_m$ of distinct tetrahedra of $\mathcal{T}'$ with $m \geq 1$ such that: $v_0 \in T_0$ and $T_0 \cap T_1$ is the face

of T_0 opposite to v_0; $T_i \cap T_{i+1}$ contains a common face of T_i and T_{i+1} for all i; T_i does not contain e for all $i < n$ and $e \subset T_m$. Generally speaking, the intersection $T_i \cap T_{i+1}$ may contain more than one face; we pick any face in this intersection and denote it $T_i \cap' T_{i+1}$. We apply to $\mathcal{T}'$ a sequence of m positive H-Pachner moves. The first move replaces the pair T_0, T_1 by 3 tetrahedra and adds an edge connecting v_0 to the vertex of T_1 opposite to $T_0 \cap' T_1 = T_0 \cap T_1$. One of these new 3 tetrahedra, t, contains the face $T_1 \cap' T_2$. The second move replaces the pair t, T_2 by 3 tetrahedra and adds an edge connecting v_0 to the vertex of T_2 opposite to $T_1 \cap' T_2$. Continuing in this way m times, we transform $(\mathcal{T}', \mathcal{L}')$ into a new H-triangulation $(\mathcal{T}'', \mathcal{L}'')$ having a (unique) face that contains both v_0 and e. The triangulation $\mathcal{T}''$ and all the intermediate triangulations are quasi-regular because the newly added edges always connect v_0 to another vertex (belonging to V). Our transformations preserve the set $V \cup \{v_0\}$ of the vertices of the triangulation and lift to the charges (though non-uniquely). If the charge c' of $\mathcal{T}'$ is transformed into a charge c_k at the k-th step, then $c_k + d(e)$ is a transformation of $c' + d(e)$ (this is obvious for $k < m$ and is verified by a direct computation for $k = m$). Set $c'' = c_m$ and observe that as above,

$$\mathsf{K}(\mathcal{T}, \mathcal{L}, h, c) = \mathsf{K}(\mathcal{T}', \mathcal{L}', h, c') = \mathsf{K}(\mathcal{T}'', \mathcal{L}'', h, c'')$$

and

$$\mathsf{K}(\mathcal{T}, \mathcal{L}, h, c + d(e)) = \mathsf{K}(\mathcal{T}', \mathcal{L}', h, c' + d(e)) = \mathsf{K}(\mathcal{T}'', \mathcal{L}'', h, c'' + d(e)),$$

where on the right hand side we view e as an edge of $\mathcal{T}''$. Note that e is contained in at least 3 tetrahedra of $\mathcal{T}''$ because at the $(m-1)$-st step the edge e is contained in T_m and in at least one other tetrahedron of the triangulation, and the m-th move above creates three tetrahedra of which two contain e. Moreover, there is precisely one face of $\mathcal{T}''$ containing e and the vertex v_0.

Let A_1, A_2 be the vertices of e and $t_1, t_2, ..., t_n$ with $n \geq 3$ be the cyclically ordered tetrahedra of $\mathcal{T}''$ adjacent to e. Let $B_1 = v_0, B_2, ..., B_n$ be the vertices of $\mathcal{T}''$ (possibly with repetitions) such that A_1, A_2, B_i, B_{i+1} are the vertices of t_i for all i (here $B_{n+1} = B_1$). Clearly, $B_i \neq B_1$ for all $i \neq 1$. If $n > 3$, then we apply to $\mathcal{T}''$ a positive H-Pachner move replacing t_1, t_2 by 3 tetrahedra and adding an edge connecting $B_1 = v_0$ to B_3. This produces a quasi-regular triangulation $(\mathcal{T}''', \mathcal{L}''')$ of (M, L) with the same properties as $\mathcal{T}''$ but having $n - 1$ tetrahedra adjacent to e. As above, $\mathsf{K}(\mathcal{T}'', \mathcal{L}'', h, c'') = \mathsf{K}(\mathcal{T}''', \mathcal{L}''', h, c''')$ and

$$\mathsf{K}(\mathcal{T}'', \mathcal{L}'', h, c'' + d(e)) = \mathsf{K}(\mathcal{T}''', \mathcal{L}''', h, c''' + d(e))$$

for a certain charge c''' on $\mathcal{T}'''$. Proceeding by induction, we reduce ourselves to the case $n = 3$. In this case, the edge e may be eliminated by a negative H-Pachner move so that the equality (57) follows from (56). $\square$

Proof of Theorem 29. Theorem 29 follows from the results of this section and Proposition 26. $\square$

11. Cayley–Hamilton Hopf algebras

In this section, we recall some of the results of [5] and use them to construct Ψ-systems in categories. We assume that the ground field k is algebraically closed and is of characteristic 0.

11.1. Cayley–Hamilton algebras.

Definition 36. *An* algebra with trace *is an (associative) algebra $\mathcal{R}$ over k with a k-linear map $t\colon \mathcal{R} \to \mathcal{R}$ such that for all $a, b \in \mathcal{R}$,*

$$t(a)b = bt(a), \quad t(ab) = t(ba), \quad and \quad t(a)\,t(b) = t(t(a)b).$$

The image $t(\mathcal{R})$ of t is a subalgebra of $\mathcal{R}$ called the *trace subalgebra*. Note that $t(\mathcal{R})$ is contained in the center Z of $\mathcal{R}$.

In the polynomial algebra $\mathsf{k}[x_1, \ldots, x_n]$ define the elementary symmetric functions $e_i(x_1, \ldots, x_n)$:

$$\prod_{i=1}^{n}(x - x_i) = x^n + \sum_{i=1}^{n} e_i(x_1, \ldots, x_n)x^{n-i},$$

and the Newton functions

$$\psi_k(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^k, \quad 1 \le k \le n,$$

which are well known to be related

$$e_i(x_1, \ldots, x_n) = P_i(\psi_1(x_1, \ldots, x_n), \ldots, \psi_i(x_1, \ldots, x_n))$$

for certain uniquely defined polynomials $P_i(y_1, \ldots, y_i)$.

Definition 37. *An algebra with trace $(\mathcal{R}, t)$ is an nth-Cayley–Hamilton algebra, if $t(1) = n$ and*

$$a^n + \sum_{i=1}^{n} P_i(t(a), t(a^2), \ldots, t(a^i))a^{n-i} = 0$$

for any $a \in \mathcal{R}$.

A prototypical example of an nth-Cayley–Hamilton algebra is the matrix algebra $M_n(\mathsf{k})$ of $n \times n$ matrices over k with the usual trace (with values in $\mathsf{k} = \mathsf{k}\,\mathrm{Id} \subset M_n(\mathsf{k})$).

Let $(\mathcal{R}, t)$ be a finitely generated n^{th}-Cayley–Hamilton algebra with trace subalgebra $A = t(\mathcal{R})$. In the rest of this section we assume that:

(1) $\mathcal{R}$ is prime (that is the product of any two non-zero ideals is non-zero),
(2) $\mathcal{R}$ is a finite A-module,
(3) the center Z of $\mathcal{R}$ is integrally closed,
(4) A is a finitely generated algebra over k.

The *reduced trace* of $\mathcal{R}$ is defined by the formula $t_{\mathcal{R}/A} = \frac{m}{n}t$, where $m \geq 1$ is the minimal divisor of n such that $\mathcal{R}$ is an mth-Cayley–Hamilton algebra with trace $\frac{m}{n}t$. Then $m = [\mathcal{R} : A] = \dim_A \mathcal{R}$ and there exists a reduced trace $t_{\mathcal{R}/Z} : \mathcal{R} \to Z$ such that $t_{\mathcal{R}/A} = t_{Z/A} \circ t_{\mathcal{R}/Z}$. Note that $[\mathcal{R} : A] = [\mathcal{R} : Z][Z : A]$.

By an n-dimensional representation of $(\mathcal{R}, t)$ we mean an algebra homomorphism $\phi : \mathcal{R} \to M_n(\mathsf{k})$ which is compatible with traces in the sense that $t(a) = Tr(\phi(a))$ for all $a \in \mathcal{R}$, where Tr is the standard trace on $M_n(\mathsf{k})$. Let $V(A)$ be the affine algebraic variety associated to A, which can be identified with the maximal spectrum of A or with the set of homomorphisms $A \to \mathsf{k}$. By Theorem 3.1 of [5], the (closed) points of $V(A)$ parametrize semi-simple representations of $(\mathcal{R}, t_{\mathcal{R}/A})$ of dimension $m = [\mathcal{R} : A]$. We can similarly use the points of $V(A)$ to parametrize semi-simple representations of $(Z, t_{Z/A})$ of dimension $p = [Z : A]$. Since $\mathcal{R}$ is a finite A-module, Z also is a finite A-module. Then Z is a finitely generated algebra over k and its associated affine variety $V(Z)$ parametrizes semi-simple representations of $(\mathcal{R}, t_{\mathcal{R}/Z})$ of dimension $[\mathcal{R} : Z]$. Observe finally that the inclusion $A \subset Z$ defines a morphism of algebraic varieties $\pi \colon V(Z) \to V(A)$ of degree p.

Given a point $x \in V(A)$, denote by N_x the corresponding m-dimensional semi-simple representation of $\mathcal{R}$. Given a point $P \in V(Z)$, denote by M_P the corresponding $[\mathcal{R} : Z]$-dimensional semi-simple representation of $\mathcal{R}$. For $x \in V(A)$, the fiber $\pi^{-1}(x)$ (with multiplicities) is a cycle $\sum h_i P_i$ of degree $\sum h_i = p$, where $P_i \in V(Z)$ and $h_i \geq 1$. One has the following equality of $\mathcal{R}$-modules:

$$N_x = \oplus_i h_i M_{P_i}.$$

The Zariski open subset of $V(A)$ consisting of the points x such that $\pi^{-1}(x)$ consists of p distinct points is called the *unramified locus* of $\mathcal{R}$. For x in the unramified locus, any $P_i \in \pi^{-1}(x)$ corresponds to an irreducible representation M_{P_i}.

11.2. Cayley–Hamilton Hopf algebras.

Definition 38. *An nth-Cayley–Hamilton Hopf algebra is a Hopf algebra which is also an nth-Cayley–Hamilton algebra such that the trace subalgebra is a Hopf subalgebra.*

Assume now that $\mathcal{R}$ is a Cayley–Hamilton Hopf algebra satisfying the assumptions (1)-(4) of the last subsection. The co-multiplication on A defines an associative binary operation on the variety $G = V(A)$ while the antipode defines the inverse operation, so G becomes an algebraic group. One has the following decomposition formula for $x, y \in G$ (Proposition 5.15 of [5]):

$$N_x \otimes N_y = mN_{xy}$$

where $m = [\mathcal{R} : A]$. A pair $x, y \in G$ is *generic* if x, y and xy lie in the unramified locus. Then for each point $P \in V(Z)$ lying in the fiber of x, y or xy the corresponding representation M_P is irreducible. For $P \in \pi^{-1}(x)$ and $Q \in \pi^{-1}(y)$, one

has the following Clebsch–Gordan decomposition (Theorem 5.16 of [5]):

$$M_P \otimes M_Q \simeq \bigoplus_{O \in \pi^{-1}(xy)} M_O^{\oplus h_O^{P,Q}} \quad \text{and} \quad \sum_O h_O^{P,Q} = [\mathcal{R} : Z] \qquad (58)$$

for some non-negative integers $h_O^{P,Q}$. Also $\sum_{P,Q} h_O^{P,Q} = m$ for all $O \in \pi^{-1}(xy)$. Note that generic pairs (x,y) form a Zariski open subvariety in $G \times G$.

For $x \in G$ we define a set I_x as follows: if x is in the unramified locus, then $I_x = \pi^{-1}(x)$, otherwise $I_x = \emptyset$. Set $I = \bigcup_{x \in G} I_x$ and consider the family $\{M_P\}_{P \in I}$ of irreducible representations discussed above.

Theorem 39. *Let $\mathcal{C}$ be the monoidal Ab-category of $\mathcal{R}$-modules of finite dimension over k. Then $\mathcal{C}$ has a Ψ-system with distinguished simple objects $\{M_P\}_{P \in I}$.*

Proof. Any $P \in I$ belongs to I_x for a unique point x of the unramified locus. Using the antipode $S : \mathcal{R} \to \mathcal{R}$, we associate to the irreducible representation M_P the dual representation M_P^* together with the evaluation morphism of $\mathcal{R}$-modules $M_P^* \otimes M_P \to \mathsf{k}$ and the coevaluation morphism $\mathsf{k} \to M_P \otimes M_P^*$ determined by $1 \mapsto \sum v_i \otimes v_i^*$ where $\{v_i\}$ is a basis of M_P and $\{v_i^*\}$ is the dual basis of M_P^*. The representation M_P^* is isomorphic to M_{P^*} where $P^* = P \circ S \in \pi^{-1}(x^{-1})$ and $x^{-1} = x \circ S$. Thus, we obtain an involution $I \to I$, $P \mapsto P^*$ and $\mathcal{R}$-module morphisms $d_{P^*} : M_{P^*} \otimes M_P \to \mathsf{k}$, $b_P : \mathsf{k} \to M_P \otimes M_{P^*}$ satisfying Equation (1). Equation (2) follows from (58). Thus, $\{M_P\}_{P \in I}$ is a Ψ-system. $\qquad \square$

Theorem 40. *Let $\mathsf{b} : I \to \mathsf{k}$ be the constant function taking the value $\frac{1}{m}$. The triple (G, I, b) satisfies Conditions 1-5 of Subsection 9.2 (where instead of $\widehat{\Psi}$-systems we should speak of Ψ-systems).*

Proof. Condition 1 follows from the fact that if x is in the unramified locus then so is $x^{-1} = x \circ S$. Conditions 2 and 3 follow from (58). Moreover, (58) implies that if $x_1, x_2, x_1 x_2$ are in the unramified locus and $P \in I_{x_1 x_2}$, then

$$\sum_{P_1 \in I_{x_1}, P_2 \in I_{x_2}} \mathsf{b}(P_1)\, \mathsf{b}(P_2) \dim(H_P^{P_1 P_2}) = \frac{1}{m^2} \sum_{P_1 \in I_{x_1}, P_2 \in I_{x_2}} \dim(H_P^{P_1 P_2}) = \frac{m}{m^2} = \mathsf{b}(P).$$

This implies Condition 5. Condition 4 holds since the unramified locus is a Zariski open subset of $G = V(A)$. $\qquad \square$

Remark 41. *A Cayley–Hamilton Hopf algebra $\mathcal{R}$ is* sovereign *if $\mathcal{R}$ contains a group-like element ϕ such that $S^2(x) = \phi^{-1} x \phi$ for all $x \in \mathcal{R}$. If $\mathcal{R}$ is sovereign then the results of [4] imply that the category $\mathcal{C}$ of Theorem 39 is sovereign (aka. pivotal). In this case, the right duality comes from the sovereign structure on $\mathcal{C}$. Then the general theory of sovereign categories implies that the operator $C = (AB)^3 \in \mathrm{End}(H)$ is the identity.*

11.3. **Examples from quantum groups at roots of unity.** Let $\mathfrak{g}$ be a simple Lie algebra of rank n over $\mathsf{k} = \mathbb{C}$ with the root system Δ. Fix simple roots $\alpha_1, \ldots, \alpha_n \in \Delta^+$ and denote by $(a_{ij})_{i,j=1}^n$ the corresponding Cartan matrix. Denote by d_i the length of the i-th simple root.

For an odd positive integer N denote by ϖ a primitive complex root of 1 of order N (in the case of type G_2 we require that $N \notin 3\mathbb{Z}$).

Consider the quantized universal enveloping algebra $\mathcal{U}_\varpi = U_\varpi(\mathfrak{g})$. It is an associative unital algebra over $\mathbb{C}$ generated by K_μ, where μ runs over the weight lattice of $\mathfrak{g}$ and E_i, F_i, $i = 1, \ldots, n$ with the defining relations:

$$K_\mu K_\nu = K_{\mu+\nu}, \quad K_0 = 1,$$

$$K_\mu E_i = \varpi^{\alpha_i(\mu)} E_i K_\mu, \quad K_\mu F_i = \varpi^{-\alpha_i(\mu)} F_i K_\mu,$$

$$E_i F_j - F_j E_i = \delta_{ij}(K_{\alpha_i} - K_{\alpha_i}^{-1})/(\varpi_i - \varpi_i^{-1}),$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{\varpi_i} E_i^{1-a_{ij}-k} E_j E_i^k = 0, \quad i \neq j,$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{\varpi_i} F_i^{1-a_{ij}-k} F_j F_i^k = 0, \quad i \neq j,$$

where $\varpi_i = \varpi^{d_i}$,

$$\begin{bmatrix} m \\ k \end{bmatrix}_\varpi = \frac{[m]_\varpi!}{[m-k]_\varpi![k]_\varpi!}, \quad [m]_\varpi! = [m]_\varpi[m-1]_\varpi \cdots [2]_\varpi[1]_\varpi, \quad [m]_\varpi = \frac{\varpi^m - \varpi^{-m}}{\varpi - \varpi^{-1}}.$$

The formulas

$$\Delta(K_\mu) = K_\mu \otimes K_\mu,$$

$$\Delta(E_i) = E_i \otimes 1 + K_{\alpha_i} \otimes E_i,$$

$$\Delta(F_i) = 1 \otimes F_i + F_i \otimes K_{-\alpha_i},$$

define a homomorphism of algebras $\Delta \colon \mathcal{U}_\varpi \to \mathcal{U}_\varpi \otimes \mathcal{U}_\varpi$. There are unique counit and antipode turning $\mathcal{U}_\varpi$ into a Hopf algebra with comultiplication Δ. We denote by $\mathcal{U}_\varpi^\pm$ the subalgebras of $\mathcal{U}_\varpi$ generated by $\{E_i\}_i$ and $\{F_i\}_i$ respectively. The subalgebra generated by $\{K_\mu\}_\mu$ will be denoted by $\mathcal{U}_\varpi^0$. We also consider Hopf subalgebras $\mathcal{B}_\varpi^\pm = \mathcal{U}_\varpi^0 \otimes \mathcal{U}_\varpi^\pm$. It is known that the subalgebras $Z_0^\pm \subset \mathcal{B}_\varpi^\pm$ generated by E_α^N, $K_{\alpha_i}^N$ (respectively F_α^N, $K_{\alpha_i}^N$) and the subalgebra $Z_0 \subset \mathcal{U}_\varpi$ generated by E_α^N, F_α^N, $K_{\alpha_i}^N$ are central Hopf subalgebras. Moreover, $\mathcal{B}_\varpi^\pm$ and $\mathcal{U}_\varpi$ are Cayley–Hamilton Hopf algebras with trace subalgebras $Z_0^\pm$ and Z_0 respectively, see [5]. In all these three situations, Theorem 39 produces a monoidal Ab-category $\mathcal{C}$ with a Ψ-system.

Conjecture 42. *This Ψ-system can be extended to a $\widehat{\Psi}$-system in $\mathcal{C}$ such that there exists a scalar $\widetilde{q} \in \mathbb{C}$ for which the operator $q \in \mathrm{End}(H)$ of Lemma 19 is T-equal to $\widetilde{q}\,\mathrm{Id}_{\widehat{H}} \oplus \widetilde{q}^{-1}\,\mathrm{Id}_{\check{H}}$.*

If this conjecture is true then Theorem 29 implies that the state sum arising from $\mathcal{C}$ with this $\widehat{\Psi}$-system and the algebraic data of Theorem 40 is a topological invariant of the triple (a closed connected oriented 3-manifold M, a non-empty link in M, a conjugacy class of homomorphisms $\pi_1(M) \to G$). In the next section, we verify Conjecture 42 for the Borel subalgebra of $U_\varpi(\mathfrak{sl}_2)$ for any primitive complex root of unity ϖ of odd order N. The corresponding topological invariant generalizes the one constructed in [8] which in the case of links in the 3-sphere coincides with the N-colored Jones polynomial evaluated at ϖ.

12. The Borel subalgebra of $U_\varpi(\mathfrak{sl}_2)$

12.1. **The Ψ-system.** As above, fix a positive integer N and a primitive N-th root of unity ϖ. In what follows, $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$. Consider the Hopf algebra B_ϖ defined by the following presentation

$$\mathbb{C}\langle a^{\pm 1}, b \,|\, ab = \varpi ba, \ \Delta(a) = a \otimes a, \ \Delta(b) = a \otimes b + b \otimes 1\rangle.$$

Following [8, 9], we consider the cyclic representations of B_ϖ, i.e. the representations carrying b to an invertible operator.

Let $G = \mathbb{R} \times \mathbb{R}_{>0}$ be the upper half plane with the group structure given by $(x, y)(u, v) = (x + yu, yv)$. As a topological space, the set $I = G \setminus (\{0\} \times \mathbb{R}_{>0})$ has two connected components $I_\pm = \{(x, y) \in G \,|\, \pm x > 0\}$. We fix $\epsilon \in \mathbb{C}$ such that $\epsilon^N = -1$. In particular, in the case of odd N, we assume that $\epsilon = -1$. We define the N-th root function $\sqrt[N]{x}$ on real numbers x by the condition that it is positive real for positive real x and $\sqrt[N]{x} = \epsilon \sqrt[N]{-x}$ for negative x. Define two maps

$$u \colon G \to \mathbb{R}_{>0}, \quad v \colon G \to \mathbb{R}_{>0} \sqcup \epsilon \mathbb{R}_{>0}$$

$$u(g) = u_g = \sqrt[N]{y}, \quad v(g) = v_g = \sqrt[N]{x}, \quad g = (x, y) \in G.$$

We have the following properties

$$u_{gh} = u_g u_h, \quad u_{g^{-1}} = \frac{1}{u_g},$$

and

$$v_{g^{-1}} = \epsilon_g \frac{v_g}{u_g}, \quad \epsilon_g = \epsilon^{\pm 1}, \quad g \in I_\pm.$$

To any $g \in I$, we associate a B_ϖ-module V_g which is an N-dimensional vector space with a distinguished basis $\{w_i\}_{i \in \mathbb{Z}_N}$, and the (left) B_ϖ-module structure is given by the formulae:

$$aw_i = u_g \varpi^i w_i, \quad bw_i = v_g w_{i+1}, \quad i \in \mathbb{Z}_N.$$

Note that the distinguished basis permits to identify V_g with $\mathbb{C}^N$.

In what follows, we need the following function

$$\Phi_{g,m} = (-\epsilon_g)^m \varpi^{m(m-1)/2}, \quad \overline{\Phi}_{g,m} = \frac{1}{\Phi_{g,m}}, \quad m \in \mathbb{Z}_N.$$

Proposition 43. *In the category of B_ϖ-modules, the set of objects $\{V_g\}_{g \in I}$ with the involution $g^* = g^{-1}$ is a Ψ-system, where the duality morphisms*

$$d_g \colon V_g \otimes V_{g^*} \to \mathbb{C}, \quad b_g \colon \mathbb{C} \to V_g \otimes V_{g^*}$$

are given by the formulae

$$d_g(w_i \otimes w_j) = \begin{cases} \Phi_{g,i} & \text{if } i+j=0; \\ 0 & \text{otherwise,} \end{cases} \quad b_g(1) = \sum_{i \in \mathbb{Z}_N} \overline{\Phi}_{g^*,-i} w_i \otimes w_{-i},$$

and the multiplicity spaces $H_h^{f,g}$ are such that $\dim(H_h^{f,g})$ is N if $h = fg$ and is zero otherwise.

Proof. It is straightforward to verify that d_g and b_g are morphisms of the category of B_ϖ-modules. The dimensions of the multiplicity spaces were calculated in [8, 9]. $\qquad\square$

12.2. **Calculation of the operators A, B, L, and R.** Let us call a pair of elements $g, h \in I$ *admissible* if $gh \in I$. For an operator E satisfying the equation $E^N = -1$ and an admissible pair (g, h), we associate an operator valued function $\Psi_{g,h}(E)$ as a solution of the functional equation

$$\frac{\Psi_{g,h}(\varpi E)}{\Psi_{g,h}(E)} = \frac{v_g - u_g v_h E}{v_{gh}}.$$

More precisely, we choose numerical coefficients $\psi_{g,h,m}$, $m \in \mathbb{Z}_N$, such that

$$\Psi_{g,h}(E) = \sum_{m \in \mathbb{Z}_N} \psi_{g,h,m} (\epsilon E)^m.$$

The above functional equation translates to the following difference equation

$$\frac{\psi_{g,h,m}}{\psi_{g,h,m-1}} = \frac{\epsilon^{-1} u_g v_h}{v_g - v_{gh} \varpi^m}.$$

We fix a unique solution of the latter equation normalized so that $\psi_{g,h,0} = 1$. Using the notation of [11], we have

$$\psi_{g,h,m} = w(v_{gh}, u_g v_h/\epsilon, v_g | m), \quad \Psi_{g,h}(E) = f\left(v_{gh}/v_g, 0 \,| E u_g v_h/v_g\right),$$

where

$$w(x, y, z | m) = \frac{(y/z)^m}{(\varpi x/z; \varpi)_m}, \quad x^N + y^N = z^N,$$

$$f(x, y | z) = \sum_{m \in \mathbb{Z}_N} \frac{(\varpi y; \varpi)_m}{(\varpi x; \varpi)_m} z^m, \quad 1 - x^N = (1 - y^N) z^N,$$

and

$$(x; \varpi)_m = \prod_{j=0}^{m-1} (1 - x \varpi^j).$$

For two operators U and V satisfying the conditions $U^N = V^N = 1$, $UV = VU$, we also define

$$L(U, V) = \frac{1}{N} \sum_{i,j \in \mathbb{Z}_N} \varpi^{ij} U^i V^j.$$

In what follows, the standard basis $\{e_i\}$ of $\mathbb{C}^N$ will be indexed by elements of $\mathbb{Z}_N$. Define operators $X, Y \in \mathrm{Aut}(\mathbb{C}^N)$,

$$X e_i = \varpi^i e_i, \quad Y e_i = e_{i+1}, \quad i \in \mathbb{Z}_N.$$

For $g \in I$, let $\pi_g \colon B_\varpi \to \mathrm{End}(V_g)$ be the algebra homomorphism corresponding to the B_ϖ-module structure of V_g.

Lemma 44. *For any admissible pair (g, h), the operator valued function*

$$S_{g,h} = \Psi_{g,h}(-Y^{-1}X \otimes Y)L(Y \otimes 1, 1 \otimes X)$$

takes its values in the set of invertible matrices and satisfies the equation

$$(\pi_g \otimes \pi_h)(\Delta(x)) = S_{g,h}(\pi_{gh}(x) \otimes \mathrm{id}_{\mathbb{C}^N})S_{g,h}^{-1}, \quad \forall x \in B_\varpi.$$

Proof. A straightforward computation. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

The operator $S_{g,h}$, considered as a linear map $S_{g,h} \colon V_{gh} \otimes \mathbb{C}^N \to V_g \otimes V_h$, permits to identify the multiplicity spaces $H_{gh}^{g,h}$ and $H_{g,h}^{gh}$ with $\mathbb{C}^N$ and $(\mathbb{C}^N)^*$, respectively, through the formulae

$$fv = S_{g,h}(v \otimes f), \quad f \in H_{gh}^{g,h}, \ v \in V_{gh},$$

$$f(v \otimes w) = (\mathrm{id}_{V_{gh}} \otimes f)(S_{g,h}^{-1}(v \otimes w)), \quad v \in V_g, \ w \in V_h, \ f \in H_{g,h}^{gh}.$$

In what follows, we shall use the notation e_i and e_i^* for the dual bases in $H_{gh}^{g,h}$ and $H_{g,h}^{gh}$, respectively, which correspond to the standard dual bases in $\mathbb{C}^N$ and $(\mathbb{C}^N)^*$.

Lemma 45. *The action of the operators A, A^*, B, and B^* is given by*

$$A e_i = \Psi_{g^*,gh}(\epsilon_g \varpi) \sum_{j \in \mathbb{Z}_N} \Phi_{g^*,i-j} e_j^*, \quad A^* e_i = \Psi_{g,h}(\varpi/\epsilon_g) \sum_{j \in \mathbb{Z}_N} \Phi_{g,j-i} e_j^*,$$

$$e_i \in H_{gh}^{g,h}, \quad e_j^* \in H_{g^*,gh}^h,$$

$$A e_i^* = \frac{1}{N\Psi_{g,h}(\varpi/\epsilon_g)} \sum_{j \in \mathbb{Z}_N} \overline{\Phi}_{g,j-i} e_j, \quad A^* e_i^* = \frac{1}{N\Psi_{g^*,gh}(\epsilon_g \varpi)} \sum_{j \in \mathbb{Z}_N} \overline{\Phi}_{g^*,i-j} e_j,$$

$$e_i^* \in H_{g,h}^{gh}, \quad e_j \in H_h^{g^*,gh},$$

$$B e_i = \frac{1}{\nu(v_{gh}/v_g)} \Phi_{h,i} e_{-i}^*, \quad B^* e_i = \frac{1}{\nu(v_g/v_{gh})} \Phi_{h^*,-i} e_{-i}^*, \quad e_i \in H_{gh}^{g,h}, \quad e_{-i}^* \in H_{gh,h^*}^g,$$

$$B e_i^* = \nu(v_g/v_{gh})\overline{\Phi}_{h^*,-i} e_{-i}, \quad B^* e_i^* = \nu(v_{gh}/v_g)\overline{\Phi}_{h,i} e_{-i}, \quad e_i^* \in H_{g,h}^{gh}, \quad e_{-i} \in H_g^{gh,h^*},$$

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where

$$\nu(x) = \frac{1 - x^N}{N(1 - x)}.$$

Proof. From Lemma 44 it follows that for any $x \in V_h$, $y \in H_h^{g^*, gh}$, $z \in H_{gh}^{g,h}$,

$$(d_{g^*} \otimes \mathrm{id}_{V_h})(\mathrm{id}_{V_{g^*}} \otimes S_{g,h})(S_{g^*,gh} \otimes \mathrm{id}_{H_{gh}^{g,h}})(x \otimes y \otimes z) = x \langle y, Az \rangle.$$

Starting from this identity, and using the results of Appendices A and C of [11], a straightforward calculation yields the formulas above for the action of A.

Similarly, for any $x \in V_g$, $y \in H_g^{gh,h^*}$, $z \in H_{gh}^{g,h}$ the identity

$$(\mathrm{id}_{V_g} \otimes d_h)(S_{g,h}P \otimes \mathrm{id}_{V_{h^*}})(\mathrm{id}_{H_{gh}^{g,h}} \otimes S_{gh,h^*})(z \otimes x \otimes y) = x \langle y, Bz \rangle.$$

gives rise to the action of the operator B. $\qquad\square$

Lemma 46. *The action of the operators $L = A^*A$ and $R = B^*B$ is given by*

$$Le_i = \left(\frac{u_g v_h}{v_{gh}} \right)^{N-1} e_{i-1}, \quad e_i \in H_{gh}^{g,h},$$

$$Le_i^* = \left(\frac{u_g v_h}{v_{gh}} \right)^{N-1} e_{i+1}^*, \quad e_i^* \in H_{g,h}^{gh},$$

$$Re_i = \varpi^{-i} \left(\frac{v_g}{v_{gh}} \right)^{N-1} e_i, \quad e_i \in H_{gh}^{g,h},$$

$$Re_i^* = \varpi^{-i} \left(\frac{v_g}{v_{gh}} \right)^{N-1} e_i^*, \quad e_i^* \in H_{g,h}^{gh}.$$

Proof. The case of R is straightforward. In the case of L we use the equality $z^{N-1} f(x, 0 | z\varpi) = x^{N-1} f(z, 0 | x\varpi)$ for the function $f(x, y | z)$ of [11]. $\qquad\square$

12.3. The $\widehat{\Psi}$-system for odd N. Assume from now on that N is odd.

Proposition 47. *The Ψ-system of Proposition 43 extends to a $\widehat{\Psi}$-system where $C^{\frac{1}{2}} = \mathrm{Id}_H$ and $R^{\frac{1}{2}}$ is given by*

$$R^{\frac{1}{2}} e_i = \varpi^{-\left(\frac{N+1}{2} \right)i} \left(\frac{v_g}{v_{gh}} \right)^{\frac{N-1}{2}} e_i, \quad e_i \in H_{gh}^{g,h},$$

$$R^{\frac{1}{2}} e_i^* = \varpi^{-\left(\frac{N+1}{2} \right)i} \left(\frac{v_g}{v_{gh}} \right)^{\frac{N-1}{2}} e_i^*, \quad e_i^* \in H_{g,h}^{gh}.$$

Proof. Since N is odd, we can set $\epsilon = -1$, and for any $g \in I$, the coordinates u_g and v_g are real numbers. In particular, as the operator $r = R^N$ has a positive spectrum, we define $\sqrt{r}$ as the unique positive operator such that $(\sqrt{r})^2 = r$. Define

$$R^{\frac{1}{2}} = (\sqrt{r})^{-1} R^{\frac{N+1}{2}}.$$

Let us calculate $R^{\frac{1}{2}} e_i$ for $e_i \in H_{g,h}^{gh}$. Write $R = R_0 R_1$ as the product of commuting operators R_0 and R_1 where $R_0 = (v_g/v_{gh})^{N-1}$ and $R_1 e_j = \varpi^{-j} e_j$ for $e_j \in H_{g,h}^{gh}$. Then we have $R_1^N = \mathrm{Id}_H$ and $r = R^N = R_0^N R_1^N = R_0^N$ so

$$(\sqrt{r})^{-1} R^{\frac{N+1}{2}} e_i = (R_0^N)^{-1/2} R_0^{(N+1)/2} R_1^{(N+1)/2} e_i = (v_g/v_{gh})^{\frac{N-1}{2}} \varpi^{-\left(\frac{N+1}{2}\right)i} e_i.$$

The computation of $R^{\frac{1}{2}} e_i^*$ is similar.

Next consider the operator

$$C = (AB)^3 = A(BAB)AB = AA^*B^*A^*AB = L^{-1}B^*LB.$$

From Lemmas 45 and 46 it is easy to see that $L^{-1}B^*LB = \mathrm{Id}_H$. Thus, we can define $C^{\frac{1}{2}} = \mathrm{Id}_H$. Then it is easy to see that $C^{\frac{1}{2}}, R^{\frac{1}{2}}$ satisfy Equations (33) and (36) since C and R satisfy analogous formulas without square roots. $\square$

Lemma 48. *The operator* $L^{\frac{1}{2}} = BAR^{-\frac{1}{2}}AB$ *is given by*

$$L^{\frac{1}{2}} e_i = \left(\frac{u_g v_h}{v_{gh}}\right)^{\frac{N-1}{2}} e_{i-(N+1)/2}, \quad e_i \in H_{gh}^{g,h},$$

$$L^{\frac{1}{2}} e_i^* = \left(\frac{u_g v_h}{v_{gh}}\right)^{\frac{N-1}{2}} e_{i+(N+1)/2}^*, \quad e_i^* \in H_{g,h}^{gh}.$$

Proof. Write $L = L_0 L_1$ as the product of commuting operators L_0 and L_1 where $L_0|_{H_{gh}^{g,h} \oplus H_{g,h}^{gh}} = \left(\frac{u_g v_h}{v_{gh}}\right)^{N-1}$ and L_1 is a translation operator such that $L_1^N = \mathrm{Id}_H$. We will show that $L^{\frac{1}{2}} = L_0^{1/2} L_1^{(N+1)/2}$.

Let $r' = L^N = BAr^{-1}AB$ where $r = R^N$. Then

$$L^{\frac{1}{2}} = BA\sqrt{r}R^{-\frac{N+1}{2}}AB = (\sqrt{r'})^{-1}BAR^{-\frac{N+1}{2}}AB = (\sqrt{r'})^{-1}L^{\frac{N+1}{2}}.$$

Since $r' = L^N = L_0^N$ we have

$$L^{\frac{1}{2}} = \sqrt{r'}^{-1} L^{\frac{N+1}{2}} = (L_0^N)^{-\frac{1}{2}} L_0^{\frac{N+1}{2}} L_1^{\frac{N+1}{2}} = L_0^{\frac{1}{2}} L_1^{\frac{N+1}{2}}$$

and the formulas of the lemma follow. $\square$

Lemma 49. *The operator* $q = R^{\frac{1}{2}} BL^{\frac{1}{2}} BL^{-\frac{1}{2}} C^{-\frac{1}{2}}$ *has the following form*

$$q = (-1)^{(N-1)/2}(\varpi^{-a} \mathrm{Id}_{\hat{H}} \oplus \varpi^a \mathrm{Id}_{\check{H}}) \in \mathrm{End}(H)$$

where $a = \frac{N^2-1}{8}$.

Proof. A direct computation using Proposition 47 and Lemma 48 shows that

$$qe_i = (-1)^{(N-1)/2} \varpi^{\left(\frac{N^2-1}{8}\right)} e_i$$

for $e_i \in H_{gh}^{g,h}$. Similarly, $qe_i^* = (-1)^{(N-1)/2} \varpi^{-\left(\frac{N^2-1}{8}\right)} e_i^*$ for $e_i^* \in H_{g,h}^{gh}$. $\square$

For $g \in G$ set $I_g = \{g\} \subset I$, if $g \in I$ and $I_g = \emptyset$, otherwise. Let $\mathsf{b} : I \to \mathbb{C}$ be the constant function taking the value $\frac{1}{N}$. Then the triple (G, I, b) satisfies Conditions 1-5 of Subsection 9.2. We summarize the results of this section in the following theorem.

Theorem 50. *For any odd N, the category of B_ϖ-modules has a $\widehat{\Psi}$-system such that the algebraic data (G, I, b) satisfies Conditions 1-5 of Subsection 9.2 and $q = (-1)^{(N-1)/2}(\varpi^{-a} \operatorname{Id}_{\hat{H}} \oplus \varpi^a \operatorname{Id}_{\check{H}})$ where $a = \frac{N^2-1}{8}$.*

Thus, the category of B_ϖ-modules gives rise to a topological invariant as in Theorem 29 and as was mentioned above this invariant generalizes that of [8].

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