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Stokes Phenomenon, dynamical r-matrices and Poisson Geometry

THÈSE

présentée à la Faculté des Sciences de l'Université de Genève
pour obtenir le grade de Docteur ès sciences, mention mathématiques

par

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de

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intitulée :

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La Faculté des sciences, sur le préavis de Monsieur A. ALEKSEEV, professeur ordinaire et directeur de thèse (Section de mathématiques), Monsieur P. SEVERA, professeur (Section de mathématiques), Monsieur V. FOCK, professeur (Département de mathématiques, Université de Strasbourg, France) et Monsieur V. TOLEDANO LAREDO, professeur (Department of mathematics, Northeastern University, Massachusetts, United States), autorise l'impression de la présente thèse, sans exprimer d'opinion sur les propositions qui y sont énoncées.

Genève, le 20 mai 2016

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Le Doyen

N.B. - La thèse doit porter la déclaration précédente et remplir les conditions énumérées dans les "Informations relatives aux thèses de doctorat à l'Université de Genève".

Résumé

Dans cette thèse, nous étudions la géométrie de Poisson des espaces des modules des connexions plates et méromorphes sur des surfaces de Riemann, ces dernières étant liées au phénomène de Stokes. Notre objectif est de comprendre certains nouveaux résultats dans ce domaine du point de vue de la physique mathématique. Ceci est motivé par 1) la comparaison de deux approches différentes du théorème de linéarisation de Ginzburg-Weinsten [49]: l'approche de Boalch [16] utilisant le phénomène de Stokes et l'approche de Enriquez-Etingof-Marshall [32] basée sur la théorie des r-matrices dynamiques et les groupes quantiques; 2) le travaille recent de Toledano Laredo [78], où les phoenomènes de Stokes à valeurs dans $U(\mathfrak{g})$ ont été utilisés pour donner une construction trascendente du groupe quantique de Drinfeld-Jimbo.

Les résultats principaux de cette thèse sont les suivants:

• La construction d'une transformation de jauge entre la r-matrice classique standard et la r-matrice dynamique de Alekseev-Meinrenken en utilisant les données de monodromie (matrice de connexion) d'un certain problème irrégulier de Riemann-Hilbert. Cette transformation de jauge a été utilisée par Enriquez-Etingof-Marshall [32] pour construire la linéarisation de Ginzburg-Weinstein. Sur la base de ce résultat, une relation étonnante entre le phénomène de Stokes et le twist de Drinfeld est donnée. Parmi les autres sous-produits on trouve une généralisation d'une version du voisinage symplectique du théorème de linéarisation de Ginzburg-Weinstein, ainsi qu'une nouvelle description du double symplectique de Lu-Weinstein [64].

• Une extension de l'analogue quantique des matrices de Stokes, et sa relation avec l'équation de Yang-Baxter. Ceci est basé sur le travail de Toledano Laredo [78], dans lequel il a utilisé le phénomène de Stokes à valeurs dans $U\mathfrak{g}$ pour construire une torsion annihilant l'associateur KZ, et donc donner une construction transcendantale canonique du groupe quantique de Drinfeld-Jimbo. L'aspect de la géométrie de Poisson du phénomène de Stokes dans [16] est démontré être la limite classique de la construction dans [78], et amène ainsi une interprétation de la théorie des groupes quantiques. Dans le même esprit, une description symplectique des équations de déformation isomonodromique de Jimbo, Miwa et Ueno [55] [18] est calculée comme

la limite classique de l'équation de Casimir [77].

• À la fin, nous regardons l'espace des modules symplectiques de connexions plates sur les surfaces de Riemann. Nous utilisons une r-matrice dynamique généralisée, induite par la procédure de fixation de jauge, pour donner une nouvelle description de dimension finie de la structure symplectique d'Atiyah-Bott sur l'espace des modules. Avec cela, nous trouvons le groupoïde Poisson de symétrie de l'espace des modules. Nous donnons aussi une étude systématique de la théorie des r-matrices dynamiques classiques généralisées.

Abstract

In this thesis, we study the Poisson geometry of moduli spaces of flat and meromorphic connections over Riemann surfaces, the latter involves the Stokes phenomenon. The aim is to understand some new achievements in this direction from the perspective of mathematical physics. This is motivated by 1) the comparison of two different approaches to the Ginzburg-Weinsten linearization theorem [49], i.e., the approach of Boalch [16] using the Stokes phenomenon and the approach of Enriquez-Etingof-Marshall [32] based on the theory of dynamical r-matrices and quantum groups; 2) the recent work of Toledano Laredo [78], where $U(\mathfrak{g})$–valued Stokes phenomenon was used to give a canonical transcendental construction of the Drinfeld–Jimbo quantum group.

This thesis consists of three main results:

• A construction of a gauge transformation between the standard classical r-matrix and the Alekseev-Meinrenken dynamical r-matrix, using the monodromy data (connection matrix) of a certain irregular Riemann-Hilbert problem. This gauge transformation was used by Enriquez-Etingof-Marshall [32] to construct Ginzburg-Weinstein linearization. Based on this result, a surprising relation between the Stokes phenomenon and the Drinfeld twist is given. Further byproducts include a generalization of a symplectic neighborhood version of the Ginzburg-Weinstein linearization theorem as well as a new description of the Lu-Weinstein symplectic double [64].

• An extension to the quantum analogue of the Stokes matrices, and its relation with the Yang-Baxter equation. This is based on Toledano Laredo's work [78], where he used $U\mathfrak{g}$-valued Stokes phenomenon to construct a Drinfeld twist killing the KZ associator. The Poisson geometry aspect of the Stokes phenomenon in [16] is proven to be the classical limit of the construction in [78], and thus carries an interpretation from the theory of quantum groups. In the same spirit, a symplectic description of the isomonodromic deformation equations of Jimbo, Miwa and Ueno [55][18] is derived as the classical limit of the Casimir equation [77].

• In the end, we look at Atiyah-Bott symplectic moduli spaces of flat connections over Riemann surfaces. We use a generalized dynamical r-matrix, induced by the

gauge fixing procedure, to give a new finite dimensional description of the Atiyah-Bott symplectic structure on the moduli space. Using this, we find a Poisson groupoid symmetry of the moduli space. We also give a systematic investigation of the theory of generalized classical dynamical r-matrices.

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In the end, I would like to express my love to my father Guanying Xu and my mother Yali Zu. There are no words that can express my thanks for them. I would like to dedicate this dissertation to them.

Contents

Chapter 1

Introduction

Since Atiyah and Bott introduced canonical symplectic structures on the moduli spaces of flat connections over Riemann surface in [10], a lot of attention has been paid to the moduli spaces by mathematicians and physicists due to their rich mathematical structures and their links with a variety of topics. From the physics perspective, a major motivation for their study is their role in Chern-Simons theory. An independent mathematical motivation for investing in moduli spaces of flat connections arises from Poisson geometry.

The extension to the meromorphic case of the Atiyah-Bott symplectic moduli spaces was due to Boalch [17]. The biggest difference between holomorphic connections and meromorphic connections is that there are local moduli at the poles of general meromorphic connections. In other words, it is not sufficient to only consider the complement of the polar divisor and take the corresponding monodromy. The extra data, in terms of which one can describe the local moduli of meromorphic connections, has been studied in the theory of differential equations and is known as Stokes matrices. Roughly speaking, given a meromorphic connection, in $\mathrm{gl}(n, \mathbb{C})$ case we can think of a matrix-valued first order ordinary differential equation with arbitrary order poles, the solution will generally have exponential behaviour at the poles. However, on different sectors at each pole, this exponential behaviour varies, and the Stokes matrices encode the change in asymptotic behaviour of solutions on different sectors. See e.g. [13][14][81] for more details. Thus the monodromy data of meromorphic connections should incorporate the Stokes matrices at each pole, besides the fundamental group representation.

Later on, more study of the geometry of moduli spaces of meromorphic connections has been done by many others. To begin an introduction to the thesis, we next present a very brief review of some of these works. We will discuss them in more details in the following chapters.

• In [17], natural symplectic structures (generalising the Atiyah-Bott approach) were described on moduli spaces of meromorphic connections. Explicitly, the extended moduli space (see Definition 2.6 of [17]) of meromorphic connections on a holomorphic vector bundle V over $\mathbb{P}^1$ with poles on an effective divisor $D = \sum_{i=1}^m k_i(a_i)$ and a fixed irregular type at each a_i was proven to be isomorphic to the symplectic quotient of the form $\widetilde{O}_1 \times \cdots \times \widetilde{O}_m /\!\!/ G$, where $\widetilde{O}_i$ is an extended orbit with natural symplectic structure associated to the irregular type at a_i. In [18], it was shown that these results extend to any complex reductive group G by introducing G-valued Stokes data for meromorphic connections on principal G-bundles.

• In [19], a finite dimensional construction of the natural symplectic structures on the spaces of monodromy/Stokes data of meromorphic connections over Riemann surfaces is given, by using the quasi-Hamiltonian geometry introduced by Alekseev–Malkin–Meinrenken [2]. Explicitly, a family of new examples of complex quasi-Hamiltonian G-spaces $\widetilde{\mathcal{C}}$ with G-valued moment maps was introduced as generalization of the conjugacy class example in [2]. It was further shown that given the divisor $D = \sum_{i=1}^m k_i(a_i)$, the symplectic spaces of monodromy data for meromorphic connections on V with poles on D and fixed irregular types are isomorphic to the quasi-Hamiltonian quotient spaces $\widetilde{\mathcal{C}}_1 \circledast \cdots \widetilde{\mathcal{C}}_m /\!\!/ G$, where $\widetilde{\mathcal{C}}_i$ is the space of monodromy data at a_i and $\circledast$ denotes the fusion product between quasi Hamiltonian G-manifolds.

The main result of [17] [19] leads to that the irregular Riemann-Hilbert correspondence

$$\nu : (\widetilde{O}_1 \times \cdots \widetilde{O}_m) /\!\!/ \hookrightarrow (\widetilde{\mathcal{C}}_1 \times \cdots \times \widetilde{\mathcal{C}}_m) /\!\!/ G$$

associating monodromy/Stokes data to a meromorphic connection on V is a symplectic map. In a particular case, this statement is equivalent to the main theorem in [16], where the Stokes map is proven to give rise to a Ginzburg-Weinstein linearization. This statement is closely related to the study of Poisson Lie groups.

To get more feeling about the objects we will work with, we set the stage by reviewing some related notions.

The notion of quantum group was introduced by Drinfeld [24]. Since then, a lot of attention has been paid into this field, due to its rich mathematical structure and its links with a variety of topics from geometry, algebra and analysis. The idea behind quantum groups is that we consider the deformation of a group algebra (a universal enveloping algebra) within the category of Hopf algebras, that are not required to be either commutative or cocommutative. Then we think of the deformed object as an algebra of functions on a "noncommutative space", in the spirit of the noncommutative geometry. Thus in Drinfeld's approach, quantum groups arise as Hopf algebras depending on an auxiliary parameter $\hbar$, which become universal enveloping algebras of a certain Lie algebra when $\hbar = 0$. While the notion of quantum enveloping algebras is very general, the most famous example is the Drinfeld-Jimbo quantum group $U_\hbar(\mathfrak{g})$ [24]. Just as groups often appear as symmetries, quantum groups act on many other mathematical objects, for example quantum planes and quantum Grassmannians.

In a recent paper of Toledano Laredo [78], $U(\mathfrak{g})$–valued Stokes phenomenon were used to give a canonical transcendental construction of the Drinfeld–Jimbo quantum group. We will come back to this point later. At this moment, let us recall the classical counterpart of a quantum group.

The notion of Poisson Lie groups arose as the classical limit of the Drinfeld's quantum groups. A Poisson–Lie group is a Lie group G equipped with a Poisson bracket for which the group multiplication $G \times G \to G$ is a Poisson map, where the manifold $G \times G$ has been given the structure of a product Poisson manifold. The most important example of Poisson Lie groups is the standard Poisson structure on a semisimple Lie group [64][75].

Poisson Lie groups also give a geometric interpretation of the classical Yang-Baxter equation. The classical Yang-Baxter equation (CYBE) plays a key role in the theory of integrable systems. The classical dynamical Yang-Baxter equation (CDYBE) is a differential equation analogue to CYBE and is introduced by Felder in [40] as the consistency condition for the differential Knizhnik-Zamolodchikov-Bernard equations for correlation functions in conformal field theory on tori. It was shown by Etingof and Varchenko [37] that dynamical r-matrices correspond to Poisson Lie groupoids (a notion introduced by Weinstein [82]) in much the same way as classical r-matrices correspond to Poisson Lie groups.

In the study of non-commutative Weil algebra [3], Alekseev and Meinrenken introduced a particular dynamical r-matrix r_{AM}, which is an important special case of classical dynamical r-matrices ([40], [37]). Let $\mathfrak{g}$ be a complex reductive Lie algebra and $t \in S^2(\mathfrak{g})^2$ the element corresponding to an invariant inner product on $\mathfrak{g}$, then r_{AM}, as a map from $\mathfrak{g}^*$ to $\mathfrak{g} \wedge \mathfrak{g}$, is defined by

$$r_{\mathrm{AM}}(x) := (\mathrm{id} \otimes \phi(\mathrm{ad}_{x^\vee}))(t), \ \forall x \in \mathfrak{g}^*, \tag{1.1}$$

where $x^\vee = (x \otimes \mathrm{id})(\mathrm{t})$ and $\phi(z) := -\frac{1}{z} + \frac{1}{2}\mathrm{cotanh}\frac{z}{2}$, $z \in \mathbb{C} \setminus 2\pi i \mathbb{Z}^*$. Remarkably, this r-matrix came to light naturally in two different applications, i.e., in the context of equivariant cohomology [3] and in the description of a Poisson structure on the chiral WZNW phase space compatible with classical G-symmetry [11].

Among the properties of Poisson Lie groups, the existence of the dual and of the double of a Poisson Lie group is the most important. The Ginzburg-Weinstein linearlization theorem [49] states that for any compact Lie group K with its standard Poisson structure, the dual Poisson Lie group K^* is Poisson diffeomorphic to the dual of the Lie algebra, with the canonical Poisson structure. This result was later reproved and generalized in [4], [5], [16], [32]. Among these works, new insights to the linearization theorem and its connections with many other fields were developed.

Ginzburg-Weinstein linearization via the Stokes phenomenon. Let G be a complex reductive Lie algebra with $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{t} \subset \mathfrak{g}$ a Cartan subalgebra. Let G be equipped with the standard Poisson structure, and G^* the corresponding dual Lie group. We consider the meromorphic connection on a holomorphic principal G-bundle P over the unit disc $D \subset \mathbb{C}$, which has the form (by choosing a trivialization of P)

$$\nabla = d - (\frac{A_0}{z^2} + \frac{B}{z})dz \tag{1.2}$$

where $A_0 \in \mathfrak{t}_{\mathrm{reg}}$ and $B \in \mathfrak{g}$ an arbitrary element. It follows from [17] that the resulting moduli space is isomorphic to G^*: actually the elements of G^* are the Stokes matrices of such connections. The natural Poisson structure on this moduli space from the Atiyah-Bott construction is proven to coincide with the dual Poisson Lie structure on G^*. Thus the statement of irregular Riemann-Hilbert correspondence becomes

Theorem 1.0.1. [16] *For each choice of A_0, the Stokes map*

$$\nu_{A_0} : \mathfrak{g}^* \to G^*$$

taking an element $B \in \mathfrak{g} \cong \mathfrak{g}^*$ to the Stokes data of the corresponding connection (4.3)*, is a local Poisson isomorphism, relating the linear Poisson structure on $\mathfrak{g}^*$ and the dual (non-linear) Poisson structure on G^*.*

Ginzburg-Weinstein linearization via dynamical r-matrices. In [32], Enriquez, Etingof and Marshall constructed formal Poisson isomorphisms between the formal Poisson manifolds $\mathfrak{g}^*$ and G^* (the dual Poisson Lie group). Here g^* is equipped with its Kostant-Kirillov-Souriau structure, and G^* with its Poisson-Lie structure given by the standard classical r-matrix r. Their result relies on constructing a formal map $g : \mathfrak{g}^* \to G$ satisfying the following gauge transformation equation (as identity of formal maps $\mathfrak{g}^* \to \wedge^2(\mathfrak{g})$)

$$g_1^{-1}d_2(g_1) - g_2^{-1}d_1(g_2) + (\otimes^2\mathrm{Ad}_g)^{-1}r_0 + \langle \mathrm{id} \otimes \mathrm{id} \otimes x, [g_1^{-1}d_3(g_1), g_2^{-1}d_3(g_2)]\rangle = r_{\mathrm{AM}},$$

Here $r_0 := \frac{1}{2}(r - r^{2,1})$, $g_1^{-1}d_2(g)(x) = \sum_i g^{-1}\partial_{\varepsilon^i}g(x) \otimes e_i$ is viewed as a formal function $\mathfrak{g}^* \to \mathfrak{g}^{\otimes 2}$, (ε^i), (e_i) are dual bases of $\mathfrak{g}^*$ and $\mathfrak{g}$, $g_i^{-1}d_j(g_i) = (g_1^{-1}d_2(g_1))^{i,j}$ and $\partial_\xi g(x) = (\frac{d}{d\varepsilon})|_{\varepsilon=0}g(x + \varepsilon\xi)$.

Theorem 1.0.2. [32] *Let* $g(x) \in \mathrm{Map}(\mathfrak{g}^*, G)$ *be a solution of the above equation. Then there exists a unique formal Poisson isomorphism* $g^*(x) : \mathfrak{g}^* \to G^*$*, defined by the identity*

$$g(x)e^{x^\vee}g(x)^{-1} = L(g^*(x))R(g^*(x))^{-1}.$$

Here $L, R : G^* \to G$ *are the formal group morphisms corresponding to the Lie algebra morphisms* $L, R : \mathfrak{g}^* \to \mathfrak{g}$ *given by* $L(x) := (x \otimes id)(r)$*,* $R(x) := -(id \otimes x)(r^{21})$*.*

Note that this theorem may be viewed as a generalization of the formal version of the Ginzburg-Weinstein linearization theorem [49]. Two constructions of solutions of (1.3) are given: the first one uses the theory of the classical Yang-Baxter equation and gauge transformations; the second one relies on the theory of quantization of Lie bialgebras. We will present the latter in more details as follows.

Ginzburg-Weinstein linearization via the Drinfeld twists. Let us take an admissible Drinfeld associator Φ, and $(U(\mathfrak{g}), m, \Delta, \Phi)$ be the corresponding quasi-Hopf algebra. (See e.g. [25]) For the associator Φ, there exists a Drinfeld twist killing Φ (see [25][34]), and according to [33], this twist can be made admissible by a suitable gauge transformation. The resulting twist $J \in U(\mathfrak{g})^{\hat{\otimes}2}[\![\hbar]\!]$ satisfies $J = 1 - \hbar\frac{r}{2} + \circ(\hbar)$, $\hbar\log(J) \in U(\mathfrak{g})\hat{\otimes}U(\hbar\mathfrak{g})[\![\hbar]\!]$, $(\varepsilon \otimes \mathrm{id})(J) = (\mathrm{id} \otimes \varepsilon)(J) = 1$ (ε is the counit), and

$$\Phi = (J^{2,3}J^{1,23})^{-1}J^{1,2}J^{12,3}. \tag{1.3}$$

Let us identify the second component $U(\mathfrak{g})$ of this tensor J with $\mathbb{C}[\mathfrak{g}^*]$ via the symmetrization (PBW) isomorphism $S^{\cdot}(\mathfrak{g}) \to U(\mathfrak{g})$. We use this identification view J as a function from $\mathfrak{g}^*$ to $U(\mathfrak{g})[\![\hbar]\!]$, denoted by $J(x)$. Then the admissibility of J guarantees the reduction module $\hbar$ of $J(\hbar^{-1}x)$, i.e., $J(\hbar^{-1}x)|_{\hbar=0}$ is a well-defined map.

Theorem 1.0.3. [32] *The reduction modulo* $\hbar$ *of* $J(x)$*, denoted by* $g(x) = J(x)|_{\hbar=0}$*, belongs to* $\exp(\mathfrak{g} \otimes \hat{S}(\mathfrak{g}))_{>0}$ *(thus a formal map from* $\mathfrak{g}^*$ *to* $\exp(\mathfrak{g})$*) and satisfies the equation* $r_0^{g(x)} = r_{\mathrm{AM}}$*.*

Therefore, the classical limit of an admissible Drinfeld twist gives rise to a Ginzburg-Weinstein linearization.

Motivations.

Apart from the desire to learn more mathematical fields, the motivation for studying the Poisson geometry of meromorphic connections arose from the following two aspects.

(1) Firstly, the three different subjects, i.e., Stokes phenomenon, dynamical r-matrices and Drinfeld twists naturally appear in the study of linearization problem of Poisson Lie groups. It is a natural attempt to make connections among them.

(2) At first glance, the appearance of Stokes phenomenon in the study of Poisson Lie groups is very surprising. Thus the understanding of the deep connection between Stokes phenomenon and Poisson geometry may be a formidable task. However, our results in [85] (based on the first motivation) suggest some new ways to this task. On the other hand, in a recent paper of Toledano Laredo [78], the Stokes phenomenon of the dynamical KZ equations (introduced by Felder, Markov, Tarasov and Varchenko in [41]) was used to construct Drinfeld twists. Recall that Poisson Lie groups can be viewed as the classical limit of quantum groups. Thus the point we wish to make is that the role of Poisson Lie groups as moduli space of meromorphic connections is not occasional, and it carries an interpretation from the theory of quantum groups.

Summary of results.

Irregular Riemann-Hilbert correspondence, Drinfeld twists and Alekseev-Meinrenken r-matrices. In [85], we construct explicit solutions of (1.3) via the monodromy (connection matrix) for a certain irregular Riemann-Hilbert problem. This allows us to understand the geometric meaning of equation (1.3) and clarify its relation with certain irregular Riemann-Hilbert correspondence. To be precise, let us consider the meromorphic connection ∇ that appeared in (1.2). Then one can take the monodromy of ∇ from 0 to ∞, known as the connection matrix of ∇, which is computed as the ratio between canonical fundamental solutions of ∇ at one chosen Stokes sector at 0 and at ∞. Thus we get a map, denoted by C, associating $B \in \mathfrak{g}^*$ to the connection matrix $C(B)$ of $\nabla = d - (\frac{A_0}{z^2} + \frac{B}{z})dz$. Here we assume that $\mathfrak{g}$ is a complex reductive Lie algebra.

Theorem 1.0.4. *[85] The rescaled connection matrix $C_{2\pi i} \in \mathrm{Map}(\mathfrak{g}^*, G)$, defined by $C_{2\pi i}(x) := C(\frac{1}{2\pi i}x)$ for all $x \in \mathfrak{g}^*$, is a solution of equation* (1.3) *(provided $r_0 \in \mathfrak{g} \wedge \mathfrak{g}$ is the skew-symmetric part of the standard classical r-matrix for $\mathfrak{g}$).*

Having proved that the connection matrix satisfies the gauge transformation equation, we further discuss its relation with Drinfeld twist. In [32], the gauge transformation equation was interpreted as the classical limit of a vertex-IRF transformation equation (see [33]) between a dynamical twist quantization $J_d(x) \in \mathrm{Map}(\mathfrak{g}^*, U(\mathfrak{g})^{\hat{\otimes}2}[\![\hbar]\!])$ of r_{AM} and a constant twist quantization $J_c \in U(\mathfrak{g})^{\hat{\otimes}2}[\![\hbar]\!]$ of r_0 associated to an admissible associator Φ. As a result, the quasi-classical limit of each vertex-IRF transformation $\rho \in \mathrm{Map}(\mathfrak{g}^*, U(\mathfrak{g})[\![\hbar]\!])$ which maps $J(x)$ to J_c gives rise to a solution of (1.3). According to [31][32], the renormalization of an admissible Drinfeld twist $J \in U(\mathfrak{g})^{\otimes 2}$ (killing the associator Φ) gives rise to such a vertex-IRF transformation, where J is regarded as an element in $(U(\mathfrak{g})\hat{\otimes}\hat{S}(\mathfrak{g}))[\![\hbar]\!]$, a formal map from $\mathfrak{g}^*$ to $U(\mathfrak{g})[\![\hbar]\!]$, by identifying the second component $U(\mathfrak{g})$ with $\hat{S}(\mathfrak{g})$ via symmetrization (PBW) isomorphism (see section 4). In particular, an admissible Drinfeld twist provides us with a solution of (1.3). On the other hand, for a semisimple Lie algebra $\mathfrak{g}$, the inverse is also true, i.e., for any $g(x) \in \mathrm{Map}(\mathfrak{g}^*, G)$ satisfying (1.3), there exists an admissible Drinfeld twist $J \in U(\mathfrak{g})^{\hat{\otimes} 2}[\![\hbar]\!]$ whose renormalized quasi-classical limit is $g(x)$. In particular, with the help of Theorem 1.0.4, we have

Theorem 1.0.5. *For each (rescaled) connection matrix $C_{2\pi i} \in \mathrm{Map}(\mathfrak{g}^*, G)$, regarded as an element in $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ by taking Taylor expansion at 0 and identifying the $S^{\cdot}(\mathfrak{g})$ component with $U(\mathfrak{g})$ via PBW isomorphism, there exists a Drinfeld twist killing the associator Φ whose renormalized quasi-classical limit is $C_{2\pi i}$.*

In particular, if Φ is the Knizhnik-Zamolodchikov (KZ) associator Φ_{KZ}, then the connection matrix (the KZ associator) can be seen as the monodromy from 0 to ∞ (1 to ∞) of the differential equation with one order two pole at 0 and one simple pole at ∞ (three simple poles at 0, 1, ∞). The confluence of two simple poles in KZ_3 system may be related to the fact that certain Drinfeld twist kills the KZ associator. The relation between Stokes phenomenon and the theory of quantum groups will be discussed more in our next result.

Stokes phenomenon, Poisson Lie groups and quantum groups. We have seen that G–valued Stokes phenomena were used in [16] and [85], to give a canonical, analytic linearisation of the Poisson–Lie group structure on G^* and its symplectic neighbourhood respectively. On the other hand, in a recent paper of Toledano Laredo [78], the Stokes phenomenon of the dynamical KZ equations (introduced by Felder, Markov, Tarasov and Varchenko in [41]) was used to construct Drinfeld twists. In the joint paper with Toledano Laredo [79], we observe that the quantization problem (quantization of Poisson Lie groups to Quantum groups) is analogous to the deformation of certain irregular Riemann-Hilbert problem (meromorphic ODE). The main contributions are

- we prove that the quantum Stokes matrix for the dynamical KZ equations satisfies Yang-Baxter equation, therefore is a quantum R-matrix.
- we show that the classical limit of the dynamical KZ equation is the meromorphic differential equation (1.2). Along the way, the constructions in [16] and [85] can be obtained as semiclassical limits of the one in [78]. Thus we give an interpretation of the appearance of Poisson geometry in the study of Stokes phenomenon from the perspective of quantization of Lie bialgebras.

• The dependence of quantum Stokes matrices on regular elements $A_0 \in \mathfrak{t}_{\rm reg}$ is described by a differential gauge transformation equation. Hamiltonian description of isomonodromic deformation equations [55][18] is then proven to be the classical limit of this equation.

Generalized classical dynamical r-matrices and moduli space of flat connections. Having studied the Poisson geometry aspect of the moduli space of meromorphic connections, we come back to Atiyah-Bott symplectic moduli spaces of flat connections over Riemann surface. Several simple finite dimensional descriptions of this symplectic structure are then given by many authors using different approaches. See e.g. [2][8][45][50][51][56]. One possibility is to obtain the moduli space of flat G-connections on a surface $\Sigma_{g,n}$ of genus g with n punctures by (quasi-)Poisson reduction from an enlarged ambient G^{n+2g}. In Fock-Rosly's approach [47], the Poisson structure on G^{n+2g} is described using a classical r-matrix. In Alekseev-Malkin-Meinrenken's approach via Lie group valued moment maps [2], the moduli space is obtained by a reduction of a canonical quasi-Poisson structure on G^{n+2g}.

In the meantime, besides the applications of the classical dynamical Yang-Baxter equation (CDYBE) we mentioned before, it is also proven to be closely connected with the theory of homogeneous Poisson spaces [25], Dirac structures and Lie bialgebroids [67]. See e.g., [64], [62] and references therein. Inspired by the study of Lie bialgebroids, the notion of generalized classical dynamical Yang-Baxter equations was introduced by Liu and Xu [62], in which the base manifold underlying the CDYBE can be a general Poisson manifold. Despite its importance, this subject suffered from the lack of examples for a long time.

Some recent works indicate the possible connection between these two very different subjects of dynamical Yang-Baxter equations and moduli spaces of flat connections. From the viewpoint of Hamiltonian formalism, the moduli spaces of flat connections can be viewed as constrained Hamiltonian systems. In [72], Meusburger and Schönfeld proved that Dirac gauge fixing for the moduli space of flat $\mathrm{ISO}(2,1)$-connections on a Riemann surface gives rise to generalized classical dynamical r-matrices.

In [86], we deepen the connection between these two subjects by giving a systematic investigation of the theory of generalized classical dynamical r-matrices:

• we explain how generalized dynamical r-matrices can be obtained by (quasi-)Poisson reduction. New examples of Poisson structures, Poisson G-spaces and Poisson groupoid actions naturally appear in this setting. For example, associated to a classical dynamical r-matrix, we construct a natural Poisson manifold carrying simultaneously a Hamiltonian action and a Poisson action, whose Hamiltonian reduction gives rise to a homogeneous Poisson space;

• we concretely analyze the dynamical r-matrices arising from the reduction of the canonical quasi-Poisson manifold $G \circledast G$ (see [2]). We also introduce the notion of gauge transformations for generalized dynamical r-matrices. As an application, we use these dynamical r-matrices to give a new finite dimensional description of the Atiyah-Bott symplectic structure on the moduli space. Using this description, we find a Poisson groupoid symmetry of the moduli space.

• two examples are given, one of them was previously studied by Meusburger-Schönfeld in the framework of the ISO$(2,1)$-Chern-Simons theory of (2+1)-dimensional gravity.

The thesis consists of the following parts. We start with two introductory chapters. In the first one, we recall some basic notions in Poisson and quasi-Hamiltonian geometry. In the second one, we recall the symplectic moduli space of meromorphic connections, and the irregular Riemann-Hilbert correspondence. The core part of the thesis is Chapters 4-6, which are from the following publications:

Chapter 4: X. Xu, *Irregular Riemann-Hilbert correspondence, Alekseev-Meinrenken dynamical r-matrices and Drinfeld twists.* Preprint arxiv:1507.07149.

Chapter 5: V. Toledano Laredo and X. Xu, *Stokes phenomenon, Poisson Lie groups and quantum groups.*

Chapter 6: X. Xu, *Generalized classical dynamical r-matrices and moduli spaces of flat connections over surfaces*, Commun. Math. Phys. 341, 523-542 (2016).

Another publications of mine that are independent of this thesis are:

X. Xu, Twisted Courant algebroids and coisotropic Cartan geometries, *J. Geom. Phys.* 82 (2014), 124–131.

N. Ikeda and X. Xu, *Canonical functions, differential graded symplectic pairs in supergeometry, and Alexandrov-Kontsevich-Schwartz-Zaboronsky sigma models with boundaries, J. Math. Phys*, 55 (2014).

Z. Liu, Y. Sheng and X. Xu, *The Pontryagin class for pre-Courant algebroids,* Journal of Geometry and Physics (2016).

H. Lang, Y. Sheng and X. Xu, *Nonabelian Omni-Lie algebras, Banach Center Publications,* Polish Acad. Sci., Warsaw (2016).

H. Lang, Y. Sheng and X. Xu, *Strong homotopy Lie algebras, homotopy Poisson manifolds and Courant algebroids*. Preprint arXiv:1312.4609,

N. Ikeda and X. Xu, *Current Algebras from DG symplectic Pairs in Supergeometry*. Preprint arXiv:1308.0100.

Chapter 2

Background Material: Poisson Geometry

In this chapter we recall the background material we will use. It includes some basic notations and definitions regarding Poisson geometry and quasi-Hamiltonian geometry. The main references used are [3][38][63][80].

2.1 Poisson manifolds

Definition 2.1.1. *A Poisson bracket on a smooth manifold M is an $\mathbb{R}$-bilinear map $\{\cdot,\cdot\}$ on the algebra $C^\infty(M)$ of smooth functions on M satisfying the following conditions:*

(i) Skew-symmetry: $\{f,g\} = -\{g,f\}$.

(ii) Jacobi identity: $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$.

(iii) Lebniz's rule: $\{fg,h\} = f\{g,h\} + g\{f,h\}$.

The pair $(M,\{\cdot,\cdot\})$, a manifold equipped with a Poisson bracket, is called a Poisson manifold.

Note that the first two conditions make $\{\cdot,\cdot\}$ a Lie algebra structure on $C^\infty(M)$, and the third guarantees that the operation $\{f,\cdot\} : C^\infty(M) \to C^\infty(M)$, for each $f \in C^\infty(M)$, is a derivation of the commutative product on $C^\infty(M)$, i.e., is a vector field. It follows that given a Poisson manifold $(M,\{\cdot\cdot\})$, we can define a bivector field $\pi \in \mathfrak{X}^2(M)$ by

$$\pi(df,dg) = \{f,g\}.$$

Conversely, given any bivector field π on M, we can define a bilinear skew-symmetric bracket $\{\cdot,\cdot\}$ (that automatically obeys Leibniz's rule) by the formula $\{f,g\} = \pi(df \wedge dg)$. However, in general, this bracket will not satisfy the Jacobi identity. The extra property, that ensuring that the bivector field defines a Poisson bracket, is characterized by the non-linear partial differential equation $[\pi,\pi] = 0$, where

$$[\cdot,\cdot] : \mathfrak{X}^p(M) \times \mathfrak{X}^q(M) \to \mathfrak{X}^{p+q-1}(M)$$

denotes the Schouten–Nijenhuis bracket (see e.g. [42]) on multivector fields. None of these descriptions of a Poisson structure is always the most efficient one, thus we will be switching between the bracket and the bivector field points of view.

Example 2.1.2. *Suppose we are given a linear coordinates $(q^1,...,q^n,p_1,...,p_n)$ on $\mathbb{R}^{2n}$. Then the formula*

$$\{f,g\} := \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} \tag{2.1}$$

defines a Poisson bracket (Here we use the Einstein summation convention). It is a direct check that this Poisson bracket is also characterized by the bivector

$$\pi := \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i}$$

Example 2.1.3. *Associated to any $n \times n$ skew symmetric matrix $A = (a_{ij})$, there is a quadratic Poisson bracket on $\mathbb{R}^n$ defined by the formula*

$$\{f,g\}_A := a_{ij}x^i x^j \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}.$$

Example 2.1.4. *Let $\mathfrak{g}$ be any finite dimensional Lie algebra. Let $f \in C^\infty(\mathfrak{g}^*)$ be any smooth function, and $x \in \mathfrak{g}^*$. Because $\mathfrak{g}$ is a vector space, the differential df_x of f at $x \in \mathfrak{g}^*$ can be seen as an element of $\mathfrak{g}$. One can check that the following binary operation on $C^\infty(\mathfrak{g}^*)$ gives rise to a Poisson bracket*

$$\{f,g\}(x) := \langle [df_x, dg_x], x\rangle.$$

Now suppose we are given two Poisson manifolds $(M_1, \{\cdot,\cdot\}_1)$ and $(M_2, \{\cdot,\cdot\}_2)$, a map $\Phi : M_1 \to M_2$ is called a Poisson map if the pull-back preserves the Poisson bracket, i.e.,

$$\{f \circ \Phi, g \circ \Phi\}_1 = \{f,g\}_2 \circ \Phi, \ \forall f,g \in C^\infty(M_2).$$

Using Poisson bivector fields, a Poisson map can be described as follows. Let (M,π_M) and (N,π_N) be Poisson manifolds and $\Phi : M \to N$ a smooth map. Then Φ is a Poisson map if and only if $\Phi_*(\pi_M) = \pi_N$.

Example 2.1.5. *Given any Lie algebra homomorphism $\Psi : \eta \to \mathfrak{g}$, the transpose map $\Psi^* : \mathfrak{g}^* \to \eta^*$ is a Poisson map, where $\mathfrak{g}^*$ and η^* are equipped with the linear Poisson bracket induced from the Lie algebra structure.*

Example 2.1.6. *The Cartesian product $(M \times N, \pi_M \times \pi_N)$ of two Poisson manifolds (M, π_M) and (N, π_N) is again a Poisson manifold, and the canonical projections $\mathrm{Pr}_M : M \times N \to M$ and $\mathrm{Pr}_N : M \times N \to N$ are Poisson maps.*

Besides enabling us to describe a Poisson bracket by a bivector field, the Leibniz rule for a Poisson bracket leads to the following definition.

Definition 2.1.7. *Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold. Then Hamiltonian vector field of a function $H \in C^\infty(M)$ is the vector field $X_H \mathfrak{X}$ defined by*

$$X_H(f) := \{H, f\}, \ \forall f \in C^\infty(M).$$

One immediate property about the Hamiltonian vector fields is as follows.

Proposition 2.1.8. *The assignment $C^\infty(M) \to \mathfrak{X}$; $f \mapsto X_f$ is a Lie algebra morphism. That is for any $f, g \in C^\infty(M)$,*

$$X_{\{f,g\}} = [X_f, X_g].$$

A relevant notion is the Casimir functions for any Poisson manifold (M, π). A function $f \in C^\infty(M)$ is called a Casimir function if the associated Hamiltonian vector field $X_f = 0$. These notions are originated from the study of classical mechanics. See e.g. [9] for a thorough discussion in this direction.

2.1.1 Symplectic leaves

Let $\pi \in \mathfrak{X}^2(M)$ be a bivector field. Then π induces a smooth bundle map $\pi^\sharp : T^*M \to TM$ which is defined by (on the sections)

$$\pi^\sharp : \Omega^1(M) \to \mathfrak{X}(M); \ \pi^\sharp(\alpha) \mapsto i_\alpha \pi, \ \forall \alpha \in \Omega^1(M).$$

The bivector field $\pi \in \mathfrak{X}^2(M)$ is called non-degenerate (at $x \in M$) if the map $\pi^\sharp : T^*M \to TM$ is an isomorphism (at $x \in M$). The inverse of this isomorphism $\pi^\sharp$ determines a map $(\pi^\sharp)^{-1} : TM \to T^*M$.

Similarly, a two form $\omega \in \Omega^2(M)$ is called non-degenerate if the following map is an isomorphism for any $x \in M$

$$\omega^\flat : T_xM \to T_x^*M; \ \omega^\flat(X) := i_X \omega_x, \ \forall X \in \mathfrak{X}(M).$$

Definition 2.1.9. *A symplectic form/structure ω on a manifold M is a closed non-degenerate differential two-form. Then pair (M, ω) is called a symplectic manifold.*

It is a direct check that there is a one-to-one correspondence between non-degenerate bivector fields π and non-degenerate two-forms ω such that

$$\omega^{\flat} = (\pi^{\sharp})^{-1} \longleftrightarrow \pi^{\sharp} = (\omega^{\flat})^{-1},$$

and under this one-to-one correspondence, a non-degenerate Poisson bivector field π becomes a non-degenerate closed two form ω and vice versa. Thus we have

Proposition 2.1.10. *There is a one-to-one correspondence between non-degenerate Poisson structures and symplectic structures on a manifold M.*

From this perspective, the notion of Poisson manifolds is a natural generalization of the notion of symplectic manifolds. The point we want to make in the following is that a Poisson manifold is naturally partitioned into regularly immersed symplectic manifolds.

Recall that a bivector field π induces a map $\pi^{\sharp} : T^*M \to TM$ which is generally degenerate. The rank of π at a point $x \in M$ is defined as the rank of the induced linear mapping $\pi^{\sharp}_x$. Then a point $x \in M$ is called regular for a Poisson structure π on M if the rank of π is constant on an open neighborhood of $x \in M$. Otherwise, it is called singular. Regular points form an open dense subset. If all the points are regular, we call the Poisson structure itself regular.

An integral submanifold for the (singular) distribution $\pi^{\sharp}(T^*M)$ is a path connected submanifold S satisfying

$$T_x S = \pi^{\sharp}(T^*_x M), \ \forall x \in S.$$

On each intergral submanifold S, there is a natural symplectic form $\omega_S \in \Omega^2(S)$ determined by

$$\omega_S(X_f, X_g)(x) = -\{f, g\}(x), \ \forall f, g \in C^{\infty}(M), \ \forall x \in S.$$

Here we use the fact that the image of $\pi^{\sharp} : T^*M \to TM$ consists of the values $X_f(x)$ of all Hamiltonian vector fields evaluated at x.

Definition 2.1.11. *Maximal integral submanifolds of π are called the symplectic leaves of the Poisson manifold (M, π).*

Because the space of regular points and its complement are saturated by symplectic leaves, symplectic leaves may be either regular or singular. Thus a Poisson manifold is partitioned into its symplectic leaves.

2.2 Poisson Lie groups

Most of the material in this section is based on [63].

A Poisson Lie group is a Lie group G equipped with a Poisson bracket such that the group multiplication $G \times G \to G$ is a Poisson map. Here the manifold $G \times G$ is given the structure of a product Poisson manifold. Explicitly, a Poisson bivector π on G gives rise to a Poisson Lie structure if and only if the following identity holds:

$$\pi(gg') = L_{g*}(\pi(g')) + R_{g'*}(\pi(g)).$$

Note that we always have $\pi(e) = 0$, thus a Poisson Lie structure on G is never symplectic.

The Lie algebra $\mathfrak{g}$ of a Poisson Lie group G has an extra structure induced from the Poisson structure. In brief, the Lie group structure gives the Lie bracket on $\mathfrak{g}$ as usual, and the differential of the Poisson bivector induces a linear map

$$d_e\pi : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g},$$

which is called the linearization of π at $e \in G$. Thus associated to a Poisson Lie group, there is a Lie algebra $\mathfrak{g}$ and a linearization $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$. To characterise this pair, we need the following notion.

Definition 2.2.1. *A Lie bialgebra is a pair $(\mathfrak{g}, \delta)$ of a Lie algebra $\mathfrak{g}$ and a linear map $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$, such that the following cocyle relation is satisfied:*

(i) the dual map $\delta^* : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$ defines a Lie bracket on $\mathfrak{g}^*$;

(ii) $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ is an 1-cocycle on $\mathfrak{g}$, i.e.,

$$\delta([X, Y]) = (\mathrm{ad}_X \otimes 1 + 1 \otimes \mathrm{ad}_X)\, \delta(Y) - (\mathrm{ad}_Y \otimes 1 + 1 \otimes \mathrm{ad}_Y)\, \delta(X).$$

Then we have

Theorem 2.2.2. [24] *Let (G, π) be a Poisson Lie group, then the linearization $d_e\pi$ of π at $e \in G$ defines a Lie bialgebra structure $(\mathfrak{g}, d_e\pi)$ on $\mathfrak{g}$, called the tangent Lie bialgebra to (G, π). Conversely, if G is connected and simply connected, then every Lie bialgebra $(\mathfrak{g}, \delta)$ on $\mathfrak{g}$ defines a unique Poisson Lie structure π on G such that $(\mathfrak{g}, \delta)$ is the tangent Lie bialgebra to (G, π).*

Proof. See e.g. [63] for a proof and a thorough discussion in this direction. ∎

The name "bialgebra" stands for the following facts: if $(\mathfrak{g}, \delta)$ is a Lie bialgebra, then the map dual to δ, $\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$, gives rise to a structure of Lie algebra on the dual vector space $\mathfrak{g}^*$. Furthermore, the definition of Lie bialgebras is symmetric, i.e., if $(\mathfrak{g}, \delta)$ is a Lie algebra, then $(\mathfrak{g}^*, \delta^*)$ is also a Lie bialgebra, called the dual Lie bialgebra. It naturally leads to the following definition.

Definition 2.2.3. *Let (G, π) be a Poisson Lie group with tangent Lie bialgebra $(\mathfrak{g}, \delta)$, and let G^* be the connected and simply-connected Lie group with the Lie algebra $\mathfrak{g}^*$ induced by δ. Then G^* with the unique Poisson Lie structure, such that $(\mathfrak{g}^*, \delta^*)$ as its tangent Lie bialgebra, is called the dual Poisson Lie group of (G, π).*

Example 2.2.4. *Let us take a Lie group $G = \mathrm{SL}(2, \mathbb{R})$ with the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. Let*

$$e_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_3 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

be a basis of $\mathfrak{g}$. We equip G with a bivector field $\pi(g) = 2(r_g(e_2 \wedge e_3) - l_g(e_2 \wedge e_3))$, where r_g and l_g denote the right and left translations in G by g respectively. Then one can check that (G, π) is a Poisson Lie group, whose dual Poisson lie group can be identified with the Lie group

$$\mathrm{SB}(2, \mathbb{C}) = \{\begin{pmatrix} a & b+ic \\ 0 & a^{-1} \end{pmatrix}, \mid a > 0, b, c \in \mathbb{R}\}.$$

By Theorem 2.2.2, more examples of Poisson Lie groups can be obtained by constructing Lie bialgebras. One main source of the Lie bialgebras structure on a Lie algebra $\mathfrak{g}$ are from classical r-matrices.

Definition 2.2.5. *An element $r \in \mathfrak{g} \otimes \mathfrak{g}$ is called a classical r-matrix if $r + r^{2,1} \in S^2(\mathfrak{g})^{\mathfrak{g}}$ and r satisfies the classical Yang-Baxter equation:*

$$[r^{1,2}, r^{1,3}] + [r^{1,2}, r^{2,3}] + [r^{1,3}, r^{2,3}] = 0. \tag{2.2}$$

A straightforward computation leads to the following proposition.

Proposition 2.2.6. *A classical r-matrix r on $\mathfrak{g}$ induces a Lie bialgebra $(\mathfrak{g}, \delta)$, where the linear map $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ is given by*

$$\delta(x) = dr(x) := [x \otimes 1 + 1 \otimes x, r], \ \forall x \in \mathfrak{g}. \tag{2.3}$$

We denote by $r_0 := \frac{1}{2}(r - r^{2,1})$ the skew-symmetric part of a classical r-matrix r. The following proposition is straightforward.

Proposition 2.2.7. (see [64]) Associated to any classical r-matrix $r \in \mathfrak{g} \otimes \mathfrak{g}$, there is a Poisson Lie structure π on G taking the form

$$\pi(g) = r_g(r_0) - l_g(r_0), \ \forall g \in G,$$

where r_g and l_g are respectively the right and left multiplication given by $g \in G$. Furthermore, the tangent Lie bialgebra to (G, π) is $(\mathfrak{g}, dr)$.

The propositions listed above can be illuminated in the following example. Let $\mathfrak{g}$ be a semisimple Lie algebra. Let $\mathfrak{t} \subset \mathfrak{g}$ be a Cartan subalgebra, and $\Delta \in \mathfrak{t}^*$ the root system. Let E_α denote the one-dimensional root subspace with respect to $\alpha \in \Delta$. Then it is well-known that associated to a polarization $\Delta = \Delta_+ \cup \Delta_-$ of Δ, there is a classical r-matrix defined by

$$r := \frac{1}{2}t + \frac{1}{2}\sum_{\alpha\in\Sigma_+} E_\alpha \wedge E_{-\alpha}, \tag{2.4}$$

where $t \in S^2(\mathfrak{g})^{\mathfrak{g}}$ is the Casimir element. We call r the standard classical r-matrix, and denote by $\pi_r \in \mathfrak{X}^2(G)$ the corresponding Poisson Lie structure on G. Thus by Proposition 2.2.7 (G, π_r) is a Poisson Lie group, and its dual Poisson Lie group is given in the following. First, the standard classical r-matrix determines a Lie bialgebra on $\mathfrak{g}$, and the induced dual Lie algebra structure on $\mathfrak{g}^*$ can be described as follows. Let $T \subset G$ be a maximal torus with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$. Let $B_\pm$ denote a pair of opposite Borel subgroups corresponding to the choice of positive roots Δ_+. Using the explicit expression in (2.3) and a straightforward calculation we find the simply connected dual Lie group associated to $(\mathfrak{g}, r)$ is

$$G^* = U_- \times U_+ \times \mathfrak{t},$$

where $U_\pm$ are the unipotent parts of the Borel subgroups $B_\pm$ and $\mathfrak{t}$ is viewed as an abelian group under addition. This dual Poisson Lie group G^* will play an important role in this thesis.

A somewhat not obvious fact about G^* is the description of the Poisson bivector field on it: let us consider the natural embedding of G^* to G

$$a : G^* \to G;\ a(u_-, u_+, \lambda) = u_-^{-1}u_+e^\lambda,\ \forall(u_-, u_+, \lambda) \in G^*.$$

Then the dual Poisson Lie structure π_{G^*} on G^*, such that $(\mathfrak{g}^*, (dr)^*)$ as its tangent Lie bialgebra, is determined by

Proposition 2.2.8. [75] *The image of the Poisson bivector field π_{G^*} on G^* under the embedding $a : G^* \to G$ coincides with the Poisson bivector field π_G on G*

$$\pi_G = \frac{1}{2}\sum_{a\in I} R_a \wedge L_a + \frac{1}{2}r_0^{ab}(L_a + R_a) \wedge (L_b + R_b),$$

where $\{e_a\}_{a\in I}$ is an orthogonal basis of $\mathfrak{g}$, R_a, L_a are right and left invariant vectors generated by e_a, and r_0 is the skew-symmetric part of $r = r^{ab}e_a \otimes e_b$.

We will next discuss the group action within the Poisson category.

Definition 2.2.9. *An action $\rho : G \times M \to M$ of a Poisson Lie group G on a Poisson manifold P is called a Poisson action if ρ is a Poisson map, where $G \times M$ is equipped with the product Poisson structure.*

Notice that the left (right) action of a Poisson Lie group G on itself is a Poisson action. Given the action ρ, let $\rho_g : M \to M;\ \rho_g(m) := g \cdot m$ and $\rho_m : G \to M;\ \rho_m(g) := g \cdot m$ for any $g \in G$, $m \in M$ be the two natural induced maps. Let π and π_M be the Poisson vector fields on G and M respectively. Then ρ is a Poisson action if and only if

$$\pi_{g \cdot m} = \rho_g \pi_m(m) + \rho_m \pi(g),\ \forall g \in G,\ m \in M.$$

When the Poisson structure π is zero, a Poisson action on M is just an action by Poisson automorphisms. However, in general, a Poisson action by G does not necessarily preserve the Poisson structures on M. Thus some notions involving group actions, like moment maps, should be different from their counterparts in the study of symplectic geometry.

Let $\rho : G \times M \to M$ be a left Poisson action of a Poisson Lie group (G, π_G) on a Poisson manifold (M, π_M). Let (G^*, π_{G^*}) be the dual Poisson Lie group of G. Denote by the same symbol $\rho : \mathfrak{g} \to \mathfrak{X}(M)$ the infinitesimal group action, and denote by $\theta \in \Omega^1(M) \otimes \mathfrak{g}$ the left invariant Cartan one-form on G.

Definition 2.2.10. [63] *A smooth map $\mu : M \to G^*$ is called a moment map for the Poisson action ρ if*

$$\rho_{\mathfrak{g}} = \pi_M^\sharp(\mu^*(\theta)).$$

Here $\rho_{\mathfrak{g}} \in \mathfrak{X}(M) \otimes \mathfrak{g}$ is defined by $\rho_{\mathfrak{g}} := \sum \rho(e_a) \otimes e_a$ in terms of an orthogonal basis $\{e_a\}$ of $\mathfrak{g}$.

If G has the zero Poisson structure, and M has a non-degenerate Poisson (therefore symplectic) structure, then the above definition reduces to the usual definition of the moment map in symplectic geometry. See e.g. [52].

Example 2.2.11. *Let G be a Poisson Lie group, and G^* a dual Poisson Lie group. Define maps*

$$\lambda : \mathfrak{g}^* \to \mathfrak{X}(G);\ x \mapsto \pi_G^\sharp(x^l),\ \ \rho : \mathfrak{g}^* \to \mathfrak{X}(G);\ x \mapsto -\pi_G^\sharp(x^r),$$

where x^l (x^r) is the left (right) invariant one-form on G generated by $x \in \mathfrak{g}^*$. One can check that ρ (λ) is a Lie algebra (anti-)morphism. The integration of the Lie algebra (anti-)morphism ρ (λ), if exists, gives rise to an action of G^* on G, called the right (left) dressing transformation. For both the left and right dressing transformations of G^* on G, the identity map of G is a moment map.

The usual symplectic reduction procedure can be carried out for Poisson actions with moment maps. See e.g. [63] for more details.

2.3 Quasi-Hamiltonian spaces

Let G be a Lie group with the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\theta, \bar{\theta}$ denote the left and right invariant $\mathfrak{g}$-valued Cartan one-forms on G respectively, and let ψ denote the canonical three-form of G, i.e., $\psi := \frac{1}{6}\langle\theta, [\theta, \theta]\rangle$.

Definition 2.3.1. ([2]) A quasi-Hamiltonian G-space is a G-manifold M with a G-equivariant map $\mu : M \to G$ (where G acts on itself by conjugation), and a G-invariant two-form $\omega \in \Omega^2(M)$ such that

(i) $d\omega = \mu^*(\psi)$, where ψ is the canonical three-form on G;

(ii) $\omega(v_X, \cdot) = \frac{1}{2}\mu^*(\theta + \bar{\theta}, X) \in \Omega^1(M)$, for all $X \in \mathfrak{g}$, where v_X is the fundamental vector field $(v_X)_m = -\frac{d}{dt}(e^{tX} \cdot m)|_{t=0}$.

(iii) the kernel of ω at each point $m \in M$ is

$$\mathrm{ker}\omega_m = \{(v_X)_m \mid X \in \mathfrak{g} \textit{ such that } hXh^{-1} = -X, \textit{ where } h := \mu(m) \in G\} \quad (2.5)$$

The most important example of a quasi-Hamiltonian G-space is as follows.

Example 2.3.2. *Suppose $\mathcal{C} \subset G$ is a conjugacy class with the conjugation action of G. Then $\mathcal{C}$ is a quasi-Hamiltonian G-space with the moment map μ given by the inclusion map, and two-form ω defined by*

$$\omega_h(v_X, v_Y) = \frac{1}{2}(\langle X, \mathrm{Ad}_h Y\rangle - \langle Y, \mathrm{Ad}_h X\rangle), \quad (2.6)$$

for any $X, Y \in \mathfrak{g}$ and v_X, v_Y the fundamental vector field with respect to the conjugation action of G.

Theorem 2.3.3. ([2]) *Suppose M is a quasi-Hamiltonian $(G\times H)$-space with moment map $(\mu, \mu_H) : M \to G \times H$. If the quotient μ^{-1}/G of the inverse image $\mu^{-1}(1)$ of the identity under the first moment map is a manifold, then the restriction of ω to $\mu^{-1}(1)$ descends to the reduced space $M/\!\!/G := \mu^{-1}/G$ and makes it into a quasi-Hamiltonian H-space. In particular, if H is abelian, then $M/\!\!/G$ is a symplectic manifold.*

We can introduce the following monoid structure in the category of quasi-Hamiltonian G-spaces.

Definition 2.3.4. ([2]) *Let M_1 and M_2 be quasi-Hamiltonian G-spaces with moment map μ_1 and μ_2 respectively. Their fusion product $M_1 \circledast M_2$ is defined to be the quasi-Hamiltonian G-space $M_1 \times M_2$, where G acts diagonally, with two-form*

$$\widetilde{\omega} = \omega_1 + \omega_2 - \frac{1}{2}(\mu_1^*\theta, \mu_2^*\bar{\theta}) \tag{2.7}$$

and moment map

$$\widetilde{\mu} = \mu_1 \cdot \mu_2 : \ M \to G. \tag{2.8}$$

One of the main applications of quasi-Hamiltonian geometry is to give a finite dimensional description of the Atiyah-Bott symplectic moduli space [10]. We will discuss this in the next chapter.

The axioms in the definition of a quasi-Hamiltonian G-space are motivated in terms of Hamiltonian loop group manifolds. Let us finish this chapter by recalling this in brief. See [2] for more details.

Let G be a compact Lie group, and $LG = \mathrm{Map}(S^1, G)$ the loop group consisting of continuous maps and pointwise group multiplication. Its Lie algebra is the space of maps $L\mathfrak{g} = \Omega^0(S^1, \mathfrak{g})$, while the dual space $L\mathfrak{g}^*$ of $L\mathfrak{g}$ is the space of one forms $L\mathfrak{g}^* = \Omega^1(S^1, \mathfrak{g})$. The natural pairing between $L\mathfrak{g}$ and $L\mathfrak{g}^*$ is given by

$$\langle A, \xi \rangle = \oint_{S^1} (A, \xi).$$

Definition 2.3.5. *A Hamiltonian LG-space is a Banach manifold N together with an LG action, an invariant closed two form $\omega_L \in \Omega^2(N)$, and an equivariant map $\Phi : N \to L\mathfrak{g}^*$ such that*

(i) The induced map $\omega_L^\flat : T_xN \to T_x^*N$ is injective.

(ii) The map Φ is a moment map for the LG action, i.e.,

$$i_{v_\xi}\omega_L = d \oint_{S^1} (\Phi, \xi).$$

Now Let us choose a local coordinate on S^1, and let the based loop group $\Omega G \subset LG$ be the kernel of the evaluation map $LG \to G;\ g \mapsto g(0)$. Then the action of ΩG on $L\mathfrak{g}^*$ is free, and by equivalence of Φ, its action on N is also free. Assume further that the moment map Φ is proper, then the quotient $M := N/\Omega G$ is a smooth finite-dimensional manifold. Since $G = LG/\Omega G$, the LG action on N defines a G action on M.

By identifying $L\mathfrak{g}^*$ with the space of connections on the trivial bundle $S^1 \times G$, we can define a holonomy map

$$\mathrm{Hol} : L\mathfrak{g}^* \to G;\ A \mapsto g,$$

where g is the holonomy of A around S^1 based at 0 and in a positive sense. In particular, this map is equivariant with respect to the gauge action of G (constant loop in LG) on $L\mathfrak{g}^*$ and the adjoint action of G on itself. Thus the LG equivariant moment map $\Phi : N \to L\mathfrak{g}^*$ descends to a G equivariant map $u : M \to G$, through the holonomy map $\mathrm{Hol} : L\mathfrak{g}^* \to G$. In the end, we get a pair (M, μ) of a quotient manifold $M = N/\Omega G$ and a G equivariant map $\mu : M \to G$ out of any Hamiltonian LG-space N with proper moment map. The quotient $M = N/\Omega G$ inherits a natural two form which can be described as follows. Let us choose the local coordinate s on S^1 and define a two form on $L\mathfrak{g}^*$: $\bar{\omega} = \frac{1}{2}\int_0^1 ds(\mathrm{Hol}_s^*\bar{\theta}, \frac{\partial}{\partial s}\mathrm{Hol}_s^*\bar{\theta})$.

Theorem 2.3.6. [2] *Let (N, ω_L, Φ) be a Hamiltonian LG-space with proper moment map, then the two form $\omega_L + \Phi^*(\bar{\omega})$ is the pull-back $\mathrm{Hol}^*\omega$ of a unique two form ω on $M = N/\Omega G$, and (M, ω, μ) is a quasi-Hamiltonian G-space, called the holonomy manifold of N. Conversely, given a quasi-Hamiltonian G-space (M, ω, μ) there is a unique Hamiltonian LG-space (N, ω_L, Φ) such that M is its holonomy manifold.*

We have seen that the idea of quasi-Hamiltonian geometry is to replace the infinite dimensional Hamiltonian LG-space N with the finite dimensional space M with μ as a moment map. There are many aspects of the Hamiltonian geometry that carry over the quasi-Hamiltonian geometry, with suitable modifications. For a select few developments in this direction, see [70][71]. Also see [69] for good lecture notes on quasi-Hamiltonian geometry.

Chapter 3

Background Material: Moduli Spaces of Meromorphic Connections over Riemann Surfaces

In this chapter, we will recall some basic definitions in the study of meromorphic connections. In particular, we recall symplectic moduli spaces of meromorphic connections on a trivial holomorphic vector bundle, the corresponding symplectic spaces of monodromy data and the irregular Riemann-Hilbert correspondence between them. We mainly follow the papers [17][18][19] of Boalch.

3.1 Moduli spaces of flat connections over Riemann surfaces

For simplicity, let us explain the picture in the case $G = \mathrm{GL}_n(\mathbb{C})$. Let us consider the following meromorphic connection on the trivial holomorphic vector bundle of rank n over $\mathbb{P}^1$ (the Riemann sphere)

$$\nabla := d - (\frac{A_1}{z-a_1} + \cdots + \frac{A_m}{z-a_m})dz,$$

where $A_1, ..., A_m \in \mathrm{GL}_n\mathbb{C}$ are m matrices and $a_1, ..., a_m$ are distinct complex numbers. Let us assume the only simple poles of ∇ are $\{a_i\}$, which imposes the condition $A_1 + \cdots + A_m = 0$. In particular, it is a flat holomorphic connection on $\mathbb{P}^1 \setminus (D_1 \cup \cdots \cup D_m)$, where D_i is a small open disc around a_i for each i. We also say ∇ is a flat connection over $\mathbb{P}^1$ with m punctures. Therefore, taking the monodromy of ∇ gives rise to a representation of $\pi_1(\mathbb{P}^1 \setminus \{a_i\})$, the fundamental group of the punctured sphere.

Explicitly, this monodromy map associates the representations of the fundamental group (monodromy around each pole) to the connections ∇:

$$\{(A_1, \cdots, A_m) \mid A_1 + \cdots + A_m = 0\} \to \{(M_1, \cdots, M_m) \mid M_1 \cdots M_m = 1\}. \quad (3.1)$$

The symplectic geometry of the monodromy map starts with the assignment of natural symplectic structures to both the set of connections and the set of representations of the fundamental groups. To this end, we restrict the matrices A_i to be in fixed adjoint orbits, which implies that the monodromy of ∇ around a_i will be inside the conjugacy class $\mathcal{C}_i$ containing $e^{2\pi i A_i}$. Then each (co)adjoint orbit (identified with the trace) has the natural Kostant-Kirillov-Souriau symplectic structure. Thus the symplectic manifold we consider is the product of m generic coadjoint orbits O_i.

Note that (1) removing the dependence of the choice of a basepoint amounts to quotient on both sides of (3.1) by the diagonal conjugation $\mathrm{GL}_n(\mathbb{C})$ action; (2) the condition $A_1 + \cdots + A_m = 0$ can be expressed as the vanishing of the moment map for the diagonal conjugation G action on $O_1 \times \cdots \times O_m$. Therefore, the monodromy map becomes

$$\mu : O_1 \times \cdots \times O_m /\!\!/ G \to \mathrm{Hom}_{\mathcal{C}}(\pi_1, G)/G.$$

Here $\mathcal{C}$ denotes the choice of the set of conjugacy classes.

The symplectic geometry of this representation space is due to Atiyah and Bott [10]. Their method starts with all smooth connections on $\mathbb{P}^1$ with punctures $\{a_i\}$, and interprets the representation space as an infinite dimensional symplectic quotient. The above finite dimensional description of these moduli spaces can be carried over to a higher genus Riemann surface: let $\Sigma_{g,n}$ be an oriented surface of genus g with n punctures and $\{\mathcal{C}_i\}_{i=1,\ldots,n}$ a set of conjugacy classes of G. Then the moduli space of flat G-connections on $\Sigma_{g,n}$ is given by the character variety, i.e., the space of group homomorphisms $h : \pi_1(\Sigma_{g,n}) \to G$ that map the homotopy equivalence class of a loop around the i-th puncture to the associated conjugacy class $\mathcal{C}_i \subset G$. Two such group homomorphisms describe gauge-equivalent connections if and only if they are related by conjugation with an element of G. This implies that the moduli space of flat G-connections on $\Sigma_{g,n}$ is given by

$$\mathrm{Hom}_{\mathcal{C}}(\pi_1(\Sigma_{g,n}, G))/G = \{h \in \mathrm{Hom}(\pi_1(\Sigma_{g,n}), G) \mid h(m_i) \in \mathcal{C}_i\}/G,$$

where G acts by conjugation, $\mathcal{C}$ denotes the choice of the set of conjugacy classes, and $m_i \in \pi_1(\Sigma_{g,n})$ corresponds to the loop around i-th puncture.

Several simple finite dimensional descriptions of this symplectic structure are then given by many authors using different approaches. See e.g. [2][8][45][50][51][56]. For example, the Alekseev-Malkin-Meinrenken approach gives rise to such a description based on the quasi-Hamiltonian geometry.

Theorem 3.1.1. [6] *Consider the quasi-Hamiltonian G-manifold*

$$P_{g,n} = \mathcal{C}_1 \circledast \ldots \circledast \mathcal{C}_n \circledast D(G) \circledast \ldots \circledast D(G),$$

where $\mathcal{C}_1,\ldots,\mathcal{C}_n$ (conjugacy classes of G) and $D(G)$ carry quasi-Hamiltonian structures defined in Chapter 2, and $\circledast$ denotes the fusion product of quasi-Hamiltonian G-spaces. Then the quasi-Hamiltonian reductions $P_{g,n}/\!\!/G$ of $P_{g,n}$ are isomorphic to the moduli spaces of flat G-connections on $\Sigma_{g,n}$ with the Atiyah-Bott symplectic form.

3.2 Moduli spaces of meromorphic connections

The above picture can be generalised to meromorphic connections with higher order poles in a similar way, such that the corresponding irregular Riemann-Hilbert map is symplectic. We will recall this construction in the rest of this chapter. Let $D = \sum_{i=1}^{m} k_i(a_i) > 0$ be an effective divisor on $\mathbb{P}^1$ and V a rank n holomorphic vector bundle.

Definition 3.2.1. *A meromorphic connection ∇ on V with poles on D is a map $\nabla : V \to V \otimes K(D)$ from the sheaf of holomorphic sections of V to the sheaf of sections of $V \otimes K(D)$, satisfying the Leibniz rule: $\nabla(fv) = (df) \otimes v + f\nabla v$, where v is a local section of V, f is a local holomorphic function and K is the sheaf of holomorphic one-forms on $\mathbb{P}^1$.*

Let us choose a local coordinate z on $\mathbb{P}^1$ vanishing at a_i, and a local trivialisation of V. Then any meromorphic connection ∇ takes the form of $\nabla = d - A$, where

$$A = \frac{A_{k_i}}{z^{k_i}}dz + \cdots \frac{A_1}{z}dz + A_0 dz + \cdots, \tag{3.2}$$

and $A_j \in \mathrm{End}(\mathbb{C}^n)$, $j \leq k_i$. We restrict to generic meromorphic connections ∇ whose leading coefficient A_{k_i} at each a_i is diagonalizable with distinct eigenvalues (for $k_i \geq 2$), or diagonalizable with distinct eigenvalues mod $\mathbb{Z}$ (for $k_i = 1$).

Definition 3.2.2. [17] *A compatible framing at a_i of a vector bundle V with generic connection ∇ is an isomorphism $g_0 : V_{a_i} \to \mathbb{C}^n$ between the fibre V_{a_i} and $\mathbb{C}^n$ such that the leading coefficient of ∇ is diagonal in any local trivialisation of V extending g_0.*

Let us fix at each point a_i a germ $d - {}^i A^0$ of a diagonal generic meromorphic connection. Let $\nabla = d - A$ be a meromorphic connection and z_i a local coordinate vanishing at a_i, then (∇, V) with compatible framing g_0 at a_i has irregular type ${}^i A^0$ if there is some formal bundle automorphism $g \in GL_n[\![z_i]\!]$ with $g(a_i) = g_0$ such that $gAg^{-1} + dg \cdot g^{-1} = {}^i A^0 + \frac{{}^i\Lambda}{z_i} dz_i$ for some diagonal matrix ${}^i\Lambda$. Let $\mathbf{a}$ denote the choice of the effective divisor D and all the germs ${}^i A^0$.

Definition 3.2.3. ([17]) *The extended moduli space $\widetilde{\mathcal{M}^*}(\mathbf{a})$ is the set of isomorphism classes of triples $(V, \nabla, \mathbf{g})$ consisting of a generic connection ∇ with poles on D on a trivial holomorphic vector bundle V over $\mathbb{P}^1$ with compatible framing $\mathbf{g} = (g_0)$ such that $(V, \nabla, \mathbf{g})$ has irregular type ${}^i A^0$ at each a_i.*

3.2.1 Symplectic moduli space of meromorphic connections

Next let us recall (from [17] Section 2) the building blocks $\widetilde{O}$ of the moduli space $\widetilde{\mathcal{M}^*}(\mathbf{a})$. Fix an integer $k \neq 2$. We define $G_k := G(\mathbb{C}[\xi]/\xi^k)$ as the group of $(k-1)$-jets of bundle automorphisms, where ξ is an indeterminate. We denote by $\mathfrak{g}_k = \mathrm{Lie(G_k)}$ its Lie algebra, which contains elements of the form $X = X_0 + X_1\xi + \cdots + X_{k-1}\xi^{k-1}$ with $X_i \in \mathfrak{g}$.

Let us denote by B_k the subgroup of G_k of elements having constant term 1. Then the Lie group G_k is the semi-direct product $G \ltimes B_k$, where G acts on B_k by conjugation. The Lie algebra of G_k accordingly decomposes as a vector space direct sum, and we have (by taking the dual): $\mathfrak{g}_k^* = \mathfrak{b}_k^* \oplus \mathfrak{g}^*$. Elements of $\mathfrak{g}_k^*$ will be written as

$$A = A_0 \frac{d\xi}{\xi^k} + \cdots + A_{k-1} \frac{d\xi}{\xi} \tag{3.3}$$

via the pairing with $\mathfrak{g}_k$ given by $\langle A, X \rangle := \mathrm{Res}_0(A, X) = \sum_{i+j=k-1} (A_i, X_j)$. In this way $\mathfrak{b}_k^*$ is identified with the set of A having zero residue and $\mathfrak{g}^*$ with those having only a residue term (zero irregular part). Let $\pi_{\mathrm{res}} : \mathfrak{g}_k^* \to \mathfrak{g}^*$ and $\pi_{\mathrm{irr}} : \mathfrak{g}_k^* \to \mathfrak{b}_k^*$ denote the corresponding projections.

Now choose an element $A^0 = A_0^0 \frac{dz}{z^k} + \cdots + A_{k-2}^0 \frac{dz}{z^2}$ of $\mathfrak{b}_k^*$ with $A_i^0 \in \mathfrak{t}$ and with regular leading coefficient $A_0^0 \in \mathfrak{t}_{\mathrm{reg}}$. Let $O_{A^0} \subset \mathfrak{b}_k^*$ denote the B_k coadjoint orbit containing A^0.

Definition 3.2.4. ([17]) The extended orbit $\widetilde{O} \subset G \times \mathfrak{g}_k^*$ associated to O_{A^0} is

$$\widetilde{O} := \{(g_0, A) \in G \times \mathfrak{g}_k^* \mid \pi_{irr}(g_0 A g_0^{-1}) \in O_{A^0}\} \tag{3.4}$$

where $\pi_{irr} : \mathfrak{g}_k^* \to \mathfrak{b}_k^*$ is the natural projection removing the residue.

We have a natural symplectic form on $\widetilde{O}$ such that is naturally a Hamiltonian G-manifold. Any tangents v_1, v_2 to $\widetilde{O} \in G \times \mathfrak{g}_k^*$ at (g_0, A) are of the form

$$v_i = (X_i(0), [A, X_i] + g_0^{-1} R_i g_0) \in \mathfrak{g} \oplus \mathfrak{g}_k^* \tag{3.5}$$

for some $X_1, X_2 \in \mathfrak{g}_k$ and $R_1, R_2 \in \mathfrak{t}^*$ (where $\mathfrak{g} \cong T_{g_0} G$ via left multiplication), and the symplectic structure on $\widetilde{O}$ is given by

$$\omega_{\widetilde{O}}(v_1, v_2) = \langle R_1, \mathrm{Ad}_{g_0} X_2 \rangle - \langle R_2, \mathrm{Ad}_{g_0} X_1 \rangle + \langle A, [X_1, X_2] \rangle. \tag{3.6}$$

Proposition 3.2.5. ([17]) The G action $h \cdot (g_0, A) := (g_0 h^{-1}, hAh^{-1})$ on $(\widetilde{O}, \omega_{\widetilde{O}})$ is a Hamiltonian action with the moment map $\mu_G : \widetilde{O} \to \mathfrak{g}^*$, $\mu(g_0, A) = \pi_{res}(A)$.

In the simple pole case $k = 1$ we define

$$\widetilde{O} := \{(h, x) \in G \times \mathfrak{g}^* \mid \mathrm{Ad}_h x \in \mathfrak{t}'\} \subset G \times \mathfrak{g}^*. \tag{3.7}$$

The spaces $\widetilde{O}$ enable one to construct global symplectic moduli spaces of meromorphic connections on trivial G-bundles over $\mathbb{P}^1$ as symplectic quotients of the form $\widetilde{O}_1 \times \cdots \times \widetilde{O}_m /\!\!/ G$ (the Hamiltonian reduction of the direct product of m Hamiltonian G-spaces).

Proposition 3.2.6. ([17]) $\widetilde{\mathcal{M}^*}(\mathbf{a})$ *is isomorphic to the symplectic quotient*

$$\widetilde{\mathcal{M}^*}(\mathbf{a}) \cong \widetilde{O}_1 \times \cdots \times \widetilde{O}_m /\!\!/ G \tag{3.8}$$

where $\widetilde{O}_i \subset G \times \mathfrak{g}_{k_i}^*$ is the extended coadjoint orbit associated to $O_{A^i} \subset \mathfrak{b}_k^*$, the B_k coadjoint orbit containing the diagonal element A^i which arises from the irregular part of ${}^i A^0$ at a_i.

3.2.2 symplectic spaces of monodromy data

In this subsection, we explain what is the monodromy data (Stokes matrices, connection matrices and formal monodromy) of a meromorphic connection. Then we define the space of monodromy data with the Atiyah-Bott symplectic structure (generalized to meromorphic connections setting). We mainly follow [16] and [19]. Let us first take a meromorphic connection with an order two pole as an example to motivate the general construction.

Monodromy data of meromorphic connections with one degree two pole. Let V be a rank n trivial holomorphic vector bundle on $\mathbb{P}^1$. Let $A_0 \in \mathrm{GL}(n, \mathbb{C})$ be a diagonal matrix with distinct diagonal elements and $B \in \mathrm{gl}(n, \mathbb{C})$ an arbitrary matrix. Choose coordinate $\{z\}$ to identify $\mathbb{P}^1$ with $\mathbb{C} \cup \infty$ and a trivialization of V. We consider the following meromorphic connection on V

$$\nabla := d - (\frac{A_0}{z^2} + \frac{B}{z})dz, \tag{3.9}$$

which has an order two pole at origin and (if $B \neq 0$) a first order pole at ∞. Next we fill in the details of the definition of the monodromy data of ∇.

Definition 3.2.7. *The Stokes rays of the connection ∇ are the rays $\mathbb{R}_{>0}\zeta$, where ζ ranges over the non-zero eigenvalues of ad_{A_0}.*

We choose an initial sector Sect_0 at 0 bounded by two adjacent Stokes rays and a branch of $\log(z)$ on Sect_0. Then we label the Stokes rays $d_1, d_2, ..., d_{2l}$ going in a positive sense and starting on the positive edge of Sect_0. Set $\mathrm{Sect_i = Sect(d_i, d_{i+1})}$ for the open sector swept out by rays moving from d_i to d_{i+1} in a positive sense. (Indices are taken modulo $2l$, so $\mathrm{Sect}_0 = \mathrm{Sect(d_{2l}, d_1)}$.

The following basic result is well-known for $G = \mathrm{GL}(n, \mathbb{C})$ and A regular and was extended in [18] to the case of complex reductive groups. It was further extended to the general simple $A \in \mathfrak{t}$ in [21].

Theorem 3.2.8. (see e.g. [14][18][21][66][68]) *On each sector Sect_i, there is a unique holomorphic functions F_i such that*

$$\nabla F_i = \frac{dF_i}{dz} - (\frac{A_0}{z^2} + \frac{B}{z})F_i = 0,$$

and functions F_i can be analytically continued to the i-th 'supersector' $\widehat{\mathrm{Sect}_i} := \mathrm{Sect}(d_i - \frac{\pi}{2}, d_i + \frac{\pi}{2})$ and then

$$F_i \cdot e^{\frac{A_0}{z}} \mapsto 1, \ as\ z \mapsto 0 \ \in \widehat{\mathrm{Sect}_i}.$$

On the other hand, there is a unique fundamental solutions

$$\chi := Hz^B \textit{ on a neighbourhood of } \infty \textit{ slit along } d_1,$$

where $H : \mathbb{P}^1 \setminus \{0\} \to \mathrm{GL}(n, \mathbb{C})$ is a holomorphic map such that $H(\infty) = 1$ (See e.g. [81] for the existence and uniqueness of H).

One more data we need to describe the monodromy data of ∇ is a permutation matrix.

Definition 3.2.9. *The permutation matrix* $P \in \mathrm{GL}(n, \mathbb{C})$ *associated to the choice of* Sect_0 *is defined by* $(P)_{ij} = \delta_{\pi(i)j}$ *where* π *is the permutation of* $\{1, ..., n\}$ *corresponding to the dominance ordering of* $\{e^{q_1}, ..., e^{q_n}\}$ *along the direction* γ *bisecting the sector* $\mathrm{Sect}(d_1, d_l)$*:*

$$\pi(i) \le \pi(j) \quad \Longleftrightarrow \quad e^{q_i}/e^{q_j} \to 0 \text{ as } z \to 0 \text{ along } \gamma.$$

The monodromy data of ∇ is then composed of the quadruple (S_-, S_+, Λ, C), where

(i) the Stokes matrices (S_-, S_+) are determined by

$$F_l = F_0 P S_- P^{-1}, \text{when } F_l \text{ is continued in a positive sense to } \mathrm{Sect}_0,$$
$$F_0 = F_l P S_+ P^{-1} e^{2\pi i \delta(B)}, \text{ when } F_l \text{ is continued in a positive sense to } \mathrm{Sect}_0;$$

(ii) the formal monodromy Λ is defined by $\Lambda := P^{-1} e^{2\pi i \delta(B)} P$;

(iii) the connection matrix $C \in \mathrm{GL}(n, \mathbb{C})$ is determined by the identity $\chi = F_0 \cdot C$ in the domain of definition of F_0, where χ is extended along a path in Sect_0.

Let U_+ (U_-) be the set of strictly upper (lower) triangular matrices. It follows from the asymptotic behavior of F_0 and F_l at 0 that

Proposition 3.2.10. (see [14]). $S_+ \in U_+$ *and* $S_- \in U_-$.

It motivates us to define the monodromy manifold as

$$\widetilde{\mathcal{M}}(A_0) := G \times U_- \times U_+ \times T.$$

Then the monodromy map associates a point $(C, S_-, S_+, \Lambda) \in \widetilde{\mathcal{M}}(A_0)$ to any connection ∇ in (3.9). This monodromy manifold is equipped with the Atiyah-Bott symplectic structure (generalized to meromorphic connections setting), which will be described in the following via quasi-Hamiltonian geometry.

Local moduli of meromorphic connections with an arbitrary degree pole. We have seen that for the local moduli of meromorphic connections with one degree two pole, it is not sufficient to only consider the complement of the polar divisor and take the corresponding monodromy. The extra data, Stokes matrices are part of the monodromy data. In general, the local moduli of meromorphic connections with an arbitrary order pole can be described in a similar way. They play the role of the building blocks of the monodromy data of meromorphic connection on the trivial vector bundle over $\mathbb{P}^1$, with fixed irregular types at a Divisor D. One should compare them to the building blocks $\widetilde{\mathcal{O}}$ in (3.2.5).

Let us now recall these building blocks of the spaces of monodromy data, and the quasi-Hamiltonian description of the symplectic structure on these monodromy/Stokes data. Let $\theta, \bar{\theta}$ denote the left and right invariant $\mathfrak{g}$-valued Cartan one-forms on G respectively. Let ψ denote the canonical three-form of G, i.e., $\psi := \frac{1}{6}\langle\theta, [\theta, \theta]\rangle$.Let T be a maximal torus of G with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$ and let $B_\pm$ denote a pair of opposite Borel subgroups with $B_+ \cap B_- = T$. Let us consider the family of complex manifolds (see [19] for the geometrical origins of these spaces where their infinite-dimensional counterparts are described)

$$\widetilde{\mathcal{C}} := \{(C, \mathbf{d}, \mathbf{e}, \Lambda) \in G \times (B_- \times B_+)^{k-1} \times t \mid \delta(d_j)^{-1} = e^{\frac{\pi i \Lambda}{k-1}} = \delta(e_j) \ for \ all \ j\},$$

parameterised by an integer $k \geq 2$, where $\mathbf{b} = (d_1, ..., d_{k-1})$, $\mathbf{e} = (e_1, ..., e_{k-1})$ with $d_{even}, e_{odd} \in B_+$ and $d_{odd}, e_{even} \in B_-$, and $\delta : B_+ \to T$ is the homomorphism with kernel $U_\pm$.

Proposition 3.2.11. ([19]) The manifold $\widetilde{\mathcal{C}}$ is a complex quasi-Hamiltonian $G \times T$-space with the G-action

$$(g, t) \cdot (C, \mathbf{d}, \mathbf{e}, \Lambda) = (tCg^{-1}, td_1t^{-1}, ..., td_{k-1}t^{-1}, te_1t^{-1}, ..., te_{k-1}t^{-1}, \Lambda) \in \widetilde{\mathcal{C}}, \quad (3.10)$$

and the moment map $(\mu, e^{-2\pi i \Lambda}) : \widetilde{\mathcal{C}} \to G \times T$ where

$$\mu : \widetilde{\mathcal{C}} \to G, \quad (C, \mathbf{d}, \mathbf{e}, \Lambda) \mapsto C^{-1}d_1^{-1} \cdots d_{k-1}^{-1}e_{k-1} \cdots e_1C, \quad (3.11)$$

and two-form

$$\omega = \frac{1}{2}(\bar{\mathcal{D}}, \bar{\mathcal{E}}) + \frac{1}{2}\sum_{j=1}^{k-1}(\mathcal{D}_j, \mathcal{D}_{j-1}) - (\mathcal{E}_j, \mathcal{E}_{j-1}) \quad (3.12)$$

where $\bar{\mathcal{D}} = D^*\bar{\theta}$, $\bar{\mathcal{E}} = E^*\bar{\theta}$, $\mathcal{D}_j = D_j^*\theta$, $\mathcal{E}_j = E_j^*\theta \in \Omega^1(\widetilde{\mathcal{C}}, \mathfrak{g})$ for maps D_j, $E_j : \widetilde{\mathcal{C}} \to G$ defined by $D_i(C, , , \Lambda) = d_i \cdots d_1C$, $E_i = e_i \cdots e_1C$, $D := D_{k-1}$, $E := E_{k-1}$, $E_0 = D_0 := C$.

For example, in the order two pole case $k = 2$, we recover the monodromy manifold defined before

$$\widetilde{\mathcal{C}}_{k=2} \cong G \times U_- \times U_+ \times T, \quad \mu = C^{-1}b_-^{-1}b_+C,$$
$$\omega = \frac{1}{2}(D^*\bar{\theta}, E^*\bar{\theta}) + \frac{1}{2}(D^*\theta, C^*\theta) - \frac{1}{2}(E^*\theta, C^*\theta),$$

where $D = b_- C, E = b_+ C$. In general, the quotient $\widetilde{\mathcal{C}}/G$ has an induced Poisson structure [2]. For $k = 2$ this Poisson structure coincides with standard Poisson structure on the dual Poisson Lie group $G^* = U_- \times U_+ \times \mathfrak{t}$ (see Chapter 2).

In first order pole case $k = 1$ let us define $\widetilde{\mathcal{C}}_{k=1} := G \times e^{\mathfrak{t}^*} \subset \widetilde{\mathcal{C}}_{k=2} \cong G \times G^*$. The two-form and moment map in this case are the restriction of the two-form and moment map (4.54) of $\widetilde{\mathcal{C}}_{k=2}$ to $\widetilde{\mathcal{C}}_{k=1}$.

Symplectic moduli space of meromorphic connections over Riemann sphere. The next step is to build the spaces of monodromy data of meromorphic connections with arbitrary poles out of the local moduli spaces defined as above. This can be done by using the quasi-Hamiltonian fusion procedure in [2], which amounts to gluing two surfaces with one boundary component into two of the holes of a three-holed sphere.

Given a divisor $D = \sum_{i=1}^m k_i(a_i)$ having each $k_i \geq 1$ at a_i on $\mathbb{P}^1$, let $\widetilde{\mathcal{M}}(\mathbf{a})$ denote the corresponding monodromy manifold for compatibly meromorphic connections $(V, \nabla, \mathbf{g})$ with irregular type $\mathbf{a}$. The extension of the Atiyah-Bott symplectic structure to the case of singular $\mathbb{C}^\infty$-connections induces natural symplectic structure on $\widetilde{\mathcal{M}}(\mathbf{a})$ (see [17]). On the other hand, the infinite dimensional description of the Atiyah-Bott symplectic structure leads to certain Hamiltonian loop group manifolds, and in each local moduli $\widetilde{\mathcal{C}}$ in Proposition 3.2.11 is the corresponding quasi-Hamiltonian space. See the end of Chapter 2 for the equivalence between Hamiltonian loop group manifolds and quasi-Hamiltonian spaces.

Therefore, Proposition 3.2.11 and the quasi-Hamiltonian fusion procedure enable us to give explicit description of the symplectic manifold $\widetilde{\mathcal{M}}(\mathbf{a})$.

Proposition 3.2.12 (Lemma 3.1 [19])**.** *The symplectic space $\widetilde{\mathcal{M}}(\mathbf{a})$ is isomorphic to the quasi-Hamiltonian quotient $\widetilde{\mathcal{C}}_1 \circledast \cdots \circledast \widetilde{\mathcal{C}}_m /\!\!/ G$, where $\circledast$ denotes the fusion product of two quasi-Hamiltonian G-manifolds.*

3.2.3 Irregular Riemann-Hilbert correspondence

Let $\mathbf{a}$ be the data of a divisor $D = \sum k_i(a_i)$ and connection germs $d - {}^i A^0$ at each a_i. The irregular Riemann-Hilbert map, which depends on a choice of tentacles (see Definition 3.9 in [17]), is a map ν from the global symplectic moduli space of meromorphic connections $\widetilde{\mathcal{M}}^*(\mathbf{a}) \cong (\widetilde{O}_1 \times \cdots \widetilde{O}_m)/\!\!/G$ to the symplectic space of monodromy data $\widetilde{\mathcal{M}}(\mathbf{a}) \cong (\widetilde{\mathcal{C}}_1 \times \cdots \times \widetilde{\mathcal{C}}_m)/\!\!/G$. In brief, the map arises as follows. Let $(V, \nabla, \mathbf{g})$ be a compatibly framed meromorphic connection on a holomorphic vector bundle V with irregular type $\mathbf{a}$. The irregular type $\mathbf{a}$ canonically determines some directions at a_i (Stokes rays) for each i. We then consider the Stokes sectors at each a_i bounded by these directions (and having some small fixed radius). The key fact is that, similar to the discussion in subsection 3.2.2, the framings $\mathbf{g}$ (and a choice of branch of logarithm at each pole) determine, in a canonical way, a choice of basis of solutions of the connection ∇ on each Stokes sector at each pole. Now along any path in the punctured sphere $\mathbb{P}^1 \setminus \{a_1, ..., a_m\}$ between two such sectors, we can extend the two corresponding bases of solutions and obtain a constant n by n matrix relating these two bases. The monodromy data of $(V, \nabla, \mathbf{g})$ is simply the set of all such constant matrices, plus the exponents of formal monodromy, thus it corresponds to a point in the space of monodromy data $\widetilde{\mathcal{C}}_1 \times \cdots \times \widetilde{\mathcal{C}}_m$. Note that $(V, \nabla, \mathbf{g})$ can be seen as a point in $\widetilde{O_1} \times \cdots \widetilde{O_m}$. Such a map taking the monodromy data of meromorphic connections is G-equivariant and descends to give ν. The main result of [17] leads to:

Theorem 3.2.13. ([17]) The irregular Riemann-Hilbert map

$$\nu : (\widetilde{O}_1 \times \cdots \widetilde{O}_m) /\!\!/ G \hookrightarrow (\widetilde{\mathcal{C}}_1 \circledast \cdots \circledast \widetilde{\mathcal{C}}_m) /\!\!/ G \tag{3.13}$$

associating monodromy/Stokes data to a meromorphic connection on the trivial G-bundle over $\mathbb{P}^1$ is a symplectic map (provided the symplectic structure on the right-hand side is divided by $2\pi i$).

The two by two matrix case. In the following, we will work out the Stokes matrices and irregular Riemann-Hilbert maps in two by two matrix case. We mainly follow [14].

Let us consider the meromorphic connection on the trivial rank 2 holomorphic vector bundle of the following form (under a chosen trivialization)

$$\frac{dF}{dz} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} + \frac{1}{z}\begin{pmatrix} t_1 & b_2 \\ b_1 & t_2 \end{pmatrix} F, \tag{3.14}$$

where $F(z)$ is valued in $\mathbb{C}^2$.

Remark 3.2.14. *In previous discussion we assume the irregular singularity is at* 0*, this meromorphic connection has one degree two pole at* ∞*. However, every notion and property given in previous subsections carry over to this case in an obvious way.*

Let $\hat{F} := \hat{\rho} z^{\Lambda'} e^{\Lambda z}$ denote the formal fundamental solution matrix, where $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $\Lambda' = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$, and $\hat{\rho} = \mathrm{Id} + \sum_1^\infty \rho_m z^{-m}$ is a formal series. Suppose the ordering of $\lambda_1,\ \lambda_2$ is such that $\lambda_2 - \lambda_1 = \rho e^{i\theta}$, $-\frac{\pi}{2} < \theta \leq \frac{\pi}{2}$. Then by definition, $\frac{3\pi}{2} - \theta$ and $-\frac{\pi}{2} - \theta$ determine the Stokes' rays of equation (3.14), and the corresponding Stokes sectors are

$$\mathrm{Sect}_0 : -\frac{\pi}{2} - \theta < \arg(z) < \frac{3\pi}{2} - \theta; \quad \mathrm{Sect}_1 : \frac{\pi}{2} - \theta < \arg(z) < \frac{5\pi}{2} - \theta.$$

Next, we will find the canonical fundamental solutions in each sector. Let us define α, β by

$$\alpha+\beta=t_2-t_1,\ \ \alpha\beta=-b_1b_2,$$

and denote $F_1(a,c;z)$ by Kummer's function

$$F_1(a,c;z)=\sum_{n=0}^{\infty}\frac{(a)_nz^n}{n!(c)_n},$$

where $(a)_0=1,\ (a)_n=a(a+1)\cdots(a+n-1)$ and c is not a negative integer or 0. Using the identities for Kummer functions (see e.g. [15]), one can verify that

$$Y(z)=\begin{pmatrix} b_2e^{\lambda_1 z}(\xi)^{t_1+\beta}F_1(\beta,\beta-\alpha+1;-\xi) & b_2e^{\lambda_1 z}(\xi)^{t_1+\alpha}F_1(\alpha,\alpha-\beta+1;-\xi) \\ \beta e^{\lambda_2 z}(\xi)^{t_1+\beta}F_1(-\alpha,\beta-\alpha+1;\xi) & \alpha e^{\lambda_2 z}(\xi)^{t_1+\alpha}F_1(-\beta,\alpha-\beta+1;\xi)\end{pmatrix},$$

where $\xi:=z(\lambda_1-\lambda_2)$, is a fundamental solution matrix for (3.14) provided F_1 is well-defined, i.e., $b_2\neq 0$ and $\alpha-\beta$ is not an integer. In the following, we make this assumption. Then consistent with $\xi=z(\lambda_1-\lambda_2)$, we define $\arg(\xi)=\arg(z)+\theta-\pi$. It follows that

$$-\frac{3\pi}{2}<\arg(z)<\frac{\pi}{2},\ \ for\ z\in\mathrm{Sect}_0,$$
$$-\frac{\pi}{2}<\arg(z)<\frac{3\pi}{2},\ \ for\ z\in\mathrm{Sect}_1.$$

In the expression of $\xi^\lambda=z^\lambda(\lambda_2-\lambda_1)^\lambda e^{-i\pi\lambda}$ in $Y(z)$, $(\lambda_2-\lambda_1)^\lambda$ is defined as the principal value and z^λ is defined according to $z\in\mathrm{Sect}_0$ or Sect_1. Thus we use the notation $Y_0(z\in\mathrm{Sect}_0)$ and $Y_1(z\in\mathrm{Sect}_1)$ to identify the function $Y(z)$ for these choices of ξ^λ.

It then follows from the asymptotic expansion formulas for Kummer functions (see, e.g., [73])

$$Y_0(z)\sim\hat{\rho}(z)z^\Lambda e^{\Lambda z}DU_0=\hat{F}(z)DU_0,\ \ z\to\infty,\ \ z\in\mathrm{Sect}_0,$$
$$Y_1(z)\sim\hat{\rho}(z)z^\Lambda e^{\Lambda z}DU_1=\hat{F}(z)DU_1,\ \ z\to\infty,\ \ z\in\mathrm{Sect}_1,$$

where

$$D=\begin{pmatrix} b_2(\lambda_2-\lambda_1)^{t_1}e^{-i\pi t_1} & 0 \\ 0 & (\lambda_2-\lambda_1)^{t_2}e^{-i\pi t_2}\end{pmatrix},$$

and

$$U_0=\begin{pmatrix} \frac{\Gamma(\beta-\alpha+1)}{\Gamma(1-\alpha)} & \frac{\Gamma(\alpha-\beta+1)}{\Gamma(1-\beta)} \\ \frac{e^{\pi i\alpha}\Gamma(\beta-\alpha+1)}{\Gamma(\beta)} & \frac{e^{\pi i\beta}\Gamma(\alpha-\beta+1)}{\Gamma(\alpha)}\end{pmatrix},\ U_1=\begin{pmatrix} \frac{\Gamma(\beta-\alpha+1)}{\Gamma(1-\alpha)} & \frac{\Gamma(\alpha-\beta+1)}{\Gamma(1-\beta)} \\ \frac{e^{-\pi i\alpha}\Gamma(\beta-\alpha+1)}{\Gamma(\beta)} & \frac{e^{-\pi i\beta}\Gamma(\alpha-\beta+1)}{\Gamma(\alpha)}\end{pmatrix}.$$

Due to the uniqueness of the fundamental solution matrices with the asymptotic expansion $\hat{\rho} z^{\Lambda'} e^{\Lambda z}$ as $z \to \infty$ in Sect_0 and Sect_1, we have that $F_i(z) := Y_i(z) U_i^{-1} D^{-1}$, for $z \in \mathrm{Sect}_i$, is the canonical fundamental solution matrix with $\hat{F}(z)$ as the asymptotic expansion in Sect_i.

Therefore by the definition of Stokes matrices (see subsection 3.2.2), for $z \in \mathrm{Sect}_{1,2} = \{z \mid \frac{3\pi}{2} - \theta < \arg(z) < \frac{\pi}{2} - \theta\}$, we have that $S_- = DU_1U_0^{-1}D^{-1}$. Similar, for $z \in \mathrm{Sect}_{2,1} = \{z \mid \frac{\pi}{2} - \theta < \arg(z) < \frac{-\pi}{2} - \theta\}$, we have that $S_+ = DU_0PU_1^{-1}D^{-1}e^{-2\pi i \Lambda'}$, where $P = \begin{pmatrix} e^{2\pi i(t_1+\beta)} & 0 \\ 0 & e^{2\pi i(t_1+\alpha)} \end{pmatrix}$. The Stokes matrices (S_-, S_+) are explicitly given by Γ functions

$$S_- = \begin{pmatrix} 1 & 0 \\ \frac{2\pi i b_1 (\lambda_2-\lambda_1)^{t_2-t_1} e^{\pi i(t_1-t_2)}}{\Gamma(1+\alpha)\Gamma(1+\beta)} & 1 \end{pmatrix}, \quad S_+ = \begin{pmatrix} 1 & \frac{2\pi i b_2(\lambda_2-\lambda_1)^{t_1-t_2}}{\Gamma(1-\alpha)\Gamma(1-\beta)} \\ 0 & 1 \end{pmatrix}.$$

Note that (t_1, t_2, b_1, b_2) are coordinates on $\mathrm{GL}(2,\mathbb{C})$. Let us define the coordinate (r_1, r_2, s_-, s_+) on the space $U_- \times U_+ \times \mathfrak{t}$ of monodromy data:

$$\begin{pmatrix} 1 & 0 \\ s_- & 1 \end{pmatrix}, \begin{pmatrix} 1 & s_+ \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}.$$

Thus the above discussion gives rise to

Proposition 3.2.15. *For any fixed* $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$*, the irregular Riemann-Hilbert map is given by*

$$\nu : \mathrm{GL}(2,\mathbb{C}) \to U_- \times U_+ \times \mathfrak{t}; \; (t_1, t_2, b_1, b_2) \mapsto (e^{2\pi i t_1}, e^{2\pi i t_2}, s_-, s_+),$$

where $s_- = \frac{2\pi i b_1(\lambda_2-\lambda_1)^{t_2-t_1} e^{\pi i(t_1-t_2)}}{\Gamma(1+\alpha)\Gamma(1+\beta)}$ *and* $s_+ = \frac{2\pi i b_2(\lambda_2-\lambda_1)^{t_1-t_2}}{\Gamma(1-\alpha)\Gamma(1-\beta)}$.

Using some standard identities for Γ functions, one can check that in this $\mathfrak{g} = \mathrm{GL}(2,\mathbb{C})$ case, the irregular Riemann-Hilbert map ν pushes forward the canonical linear Poisson structure on $\mathfrak{g}^*$ to the dual Poisson Lie structure on $G^* = U_- \times U_+ \times \mathfrak{t}$. This is an special case of Theorem 3.2.13. From this example, one can see the complexity of irregular Riemann-Hilbert maps.

Chapter 4

Irregular Riemann-Hilbert correspondence, Alekseev-Meinrenken dynamical r-matrices and Drinfeld twists

Xiaomeng Xu

Abstract: In 2004, Enriquez-Etingof-Marshall suggested a new approach to the Ginzburg-Weinstein linearization theorem. This approach is based on solving a system of PDEs for a gauge transformation between the standard classical r-matrix and the Alekseev-Meinrenken dynamical r-matrix. In this paper, we explain that this gauge transformation can be constructed as a monodromy (connection matrix) for a certain irregular Riemann-Hilbert problem. This further indicates a surprising relation between the connection matrix and Drinfeld twist. Our construction is based on earlier works by Boalch. As byproducts, we get a symplectic neighborhood version of the Ginzburg-Weinstein linearization theorem as well as a new description of the Lu-Weinstein symplectic double.

[0] *Keyword: Irregular Riemann-Hilbert correspondence, Alekseev-Meinrenken dynamical r-matrix, Drinfeld twist, Ginzburg-Weinstein linearization, Lu-Weinstein symplectic double*

[0] *MSC*: 53D17, 34M40, 17B37.

1 Introduction

In the study of non-commutative Weil algebra [3], Alekseev and Meinrenken introduced a particular dynamical r-matrix r_{AM}, which is an important special case of classical dynamical r-matrices ([40], [37]). Let $\mathfrak{g}$ be a complex reductive Lie algebra and $t \in S^2(\mathfrak{g})^2$ the element corresponding to an invariant inner product on $\mathfrak{g}$, then r_{AM}, as a map from $\mathfrak{g}^*$ to $\mathfrak{g} \wedge \mathfrak{g}$, is defined by

$$r_{\mathrm{AM}}(x) := (\mathrm{id} \otimes \phi(\mathrm{ad}_{x^\vee}))(t), \ \forall x \in \mathfrak{g}^*, \tag{4.1}$$

where $x^\vee = (x \otimes \mathrm{id})(\mathrm{t})$ and $\phi(z) := -\frac{1}{z} + \frac{1}{2}\mathrm{cotanh}\frac{z}{2}$, $z \in \mathbb{C} \setminus 2\pi i\mathbb{Z}^*$. Remarkably, this r-matrix came to light naturally in two different applications, i.e., in the context of equivariant cohomology [3] and in the description of a Poisson structure on the chiral WZNW phase space compatible with classical G-symmetry [11].

Let $r \in \mathfrak{g} \otimes \mathfrak{g}$ be a classical r-matrix such that $r + r^{2,1} = t$ (thus $(\mathfrak{g}, r)$ is a quasitriangular Lie bialgebra). In [32], Enriquez, Etingof and Marshall constructed formal Poisson isomorphisms between the formal Poisson manifolds $\mathfrak{g}^*$ and G^* (the dual Poisson Lie group). Here $\mathfrak{g}^*$ is equipped with its Kostant-Kirillov-Souriau structure, and G^* with its Poisson-Lie structure given by r. Their result relies on constructing a formal map $g : \mathfrak{g}^* \to G$ satisfying the following gauge transformation equation (as identity of formal maps $\mathfrak{g}^* \to \wedge^2(\mathfrak{g})$)

$$g_1^{-1}d_2(g_1) - g_2^{-1}d_1(g_2) + (\otimes^2\mathrm{Ad}_g)^{-1}r_0 + \langle \mathrm{id} \otimes \mathrm{id} \otimes x, [g_1^{-1}d_3(g_1), g_2^{-1}d_3(g_2)]\rangle = r_{\mathrm{AM}} \tag{4.2}$$

Here $r_0 := \frac{1}{2}(r - r^{2,1})$, $g_1^{-1}d_2(g)(x) = \sum_i g^{-1}\partial_{\varepsilon^i}g(x) \otimes e_i$ is viewed as a formal function $\mathfrak{g}^* \to \mathfrak{g}^{\otimes 2}$, (ε^i), (e_i) are dual bases of $\mathfrak{g}^*$ and $\mathfrak{g}$, $g_i^{-1}d_j(g_i) = (g_1^{-1}d_2(g_1))^{i,j}$ and $\partial_\xi g(x) = (\frac{d}{d\varepsilon})|_{\varepsilon=0}g(x+\varepsilon\xi)$. Two constructions of solutions of (4.2) are given: the first one uses the theory of the classical Yang-Baxter equation and gauge transformations; the second one relies on the theory of quantization of Lie bialgebras. The result in [32] may be viewed as a generalization of the formal version of [49], in which Ginzburg and Weinstein proved the existence of a Poisson diffeomorphism between the real Poisson manifolds k^* and K^*, where K is a compact Lie group and k is its Lie algebra. Different approaches to similar results in the subject of linearization of Poisson structures can be found in [1] and [16].

The main purpose of the present paper is to give an explicit solution of the above equation when r is a standard classical r-matrix. This allows us to understand the geometric meaning of equation (4.2) and clarify its relation with irregular Riemann-Hilbert correspondence. The solutions will be constructed as the monodromy of certain differential equations with irregular types. To be precise, for the case $G = \mathrm{GL}(n, \mathbb{C})$ with $\mathfrak{g} := \mathrm{Lie}(G) = \mathrm{gl}(n, \mathbb{C})$ and $r_0 \in \mathfrak{g} \wedge \mathfrak{g}$ the skew-symmetric part of the standard classical r-matrix, let us consider the meromorphic connection on V over the unit disc $D \subset \mathbb{C}$ which has the form (by choosing a trivialization of V)

$$\nabla = d - (\frac{A_0}{z^2} + \frac{B}{z})dz \tag{4.3}$$

where $A_0 \in \mathrm{gl}(n, \mathbb{C})$ is a diagonal matrix with distinct diagonal elements and $B \in \mathrm{gl}(n, \mathbb{C})$ an arbitrary matrix. One can take the monodromy of ∇ from 0 to ∞, known as the connection matrix $C(B)$ of ∇, which is computed as the ratio of canonical fundamental solutions of ∇ at one chosen Stokes sector Sect_0 at 0 and ∞ (see section 3). Thus we get a map, also denoted by C, associating $B \in \mathfrak{g}^*$ to the connection matrix $C(B)$ of $\nabla = d - (\frac{A_0}{z^2} + \frac{B}{z})dz$.

Theorem 1.1. *The rescaled connection matrix* $C_{2\pi i} \in \mathrm{Map}(\mathfrak{g}^*, G)$*, defined by* $C_{2\pi i}(x) := C(\frac{1}{2\pi i}x)$ *for all* $x \in \mathfrak{g}^*$*, is a solution of equation* (4.2) *(provided* $r_0 \in \mathfrak{g} \wedge \mathfrak{g}$ (4.2) *is the skew-symmetric part of the standard classical* r*-matrix for* $\mathrm{GL}(n, \mathbb{C})$*).*

The meromorphic connections ∇ taking the form of (4.3) were previously studied by Boalch. In particular, link between the connections ∇ and the dual Poisson Lie group G^* was discovered in [16], where the Poisson manifold G^* is proven to be a space of Stokes data, and local analytic isomorphisms $\mathfrak{g}^*$ to G^* in a neighbourhood of 0 were constructed. Furthermore, the connection matrix C was used by Boalch to construct the Duistermaat twist [30].

Having proved the connection matrix satisfies the gauge transformation equation, we can further discuss its relation with Drinfeld twist. This is based on a series work of Enriquez, Etingof and others. In [32], the gauge transformation equation was interpreted as the classical limit of a vertex-IRF transformation equation (see [33]) between a dynamical twist quantization $J_d(x) \in \mathrm{Map}(\mathfrak{g}^*, U(\mathfrak{g})^{\hat{\otimes} 2}[\![\hbar]\!])$ of r_{AM} and a constant twist quantization $J_c \in U(\mathfrak{g})^{\hat{\otimes} 2}[\![\hbar]\!]$ of r_0 associated to an admissible associator Φ. As a result, the quasi-classical limit of each vertex-IRF transformation $\rho \in \mathrm{Map}(\mathfrak{g}^*, U(\mathfrak{g})[\![\hbar]\!])$ which maps $J(x)$ to J_c gives rise to a solution of (4.2). According to [31][32], the renormalization of an admissible Drinfeld twist $J \in U(\mathfrak{g})^{\otimes 2}$ (killing the associator Φ) gives rise to such a vertex-IRF transformation, where J is regarded as an element in $(U(\mathfrak{g})\hat{\otimes}\hat{S}(\mathfrak{g}))[\![\hbar]\!]$, a formal map from $\mathfrak{g}^*$ to $U(\mathfrak{g})[\![\hbar]\!]$, by identifying the second component $U(\mathfrak{g})$ with $\hat{S}(\mathfrak{g})$ via symmetrization (PBW) isomorphism (see section 4). Thus in particular, an admissible Drinfeld twist provides us with a solution of (4.2). On the other hand, for semisimple Lie algebra $\mathfrak{g}$, the inverse is also true, i.e., for any $g(x) \in \mathrm{Map}(\mathfrak{g}^*, G)$ satisfying (4.2), there exists an admissible Drinfeld twist $J \in U(\mathfrak{g})^{\hat{\otimes} 2}[\![\hbar]\!]$ whose renormalized quasi-classical limit is $g(x)$. In particular, with the help of Theorem 1.1, we have

Theorem 1.2. *For each (rescaled) connection matrix $C_{2\pi i} \in \mathrm{Map}(\mathfrak{g}^*, G)$, regarded as an element in $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ by taking Taylor expansion at* 0 *and identifying the $S^{\cdot}(\mathfrak{g})$ component with $U(\mathfrak{g})$ via PBW isomorphism, there exists a Drinfeld twist killing the associator Φ whose renormalized quasi-classical limit is $C_{2\pi i}$.*

In particular, if Φ is the KZ associator Φ_{KZ}, then the connection matrix (the KZ associator) can be seen as the monodromy from 0 to ∞ (1 to ∞) of the differential equation with one order two pole at 0 and one simple pole ∞ (three simple poles at 0, 1, ∞). Naively it seems that the confluence of two simple poles at 0 and 1 in the KZ case turns the monodromy representing KZ associator to the monodromy representing the connection matrix. Thus one may expect an explanation of the above theorem from this perspective. It also indicates that the confluence of two simple poles in KZ equation may be related to the fact that certain Drinfeld twist kills the KZ associator.

In the second part of this paper, we clarify the relation between the gauge transformation equation (4.2) and certain irregular Riemann-Hilbert correspondence. This is motivated and based on Boalch's early works, e.g. [17] [18] [19] [20], on the study of the geometry of moduli spaces of meromorphic connections on a trivial principal G(a complex reductive Lie group)-bundle over Riemann surfaces with divisors. We next present a brief review of these works. In [17], natural symplectic structures were found and described on such moduli spaces both explicitly and from an infinite dimensional viewpoint (generalising the Atiyah-Bott approach). Explicitly, the extended moduli space (see Definition 2.6 of [17]) of meromorphic connections on a holomorphic vector bundle V over $\mathbb{P}^1$ with poles on an effective divisor $D = \sum_{i=1}^m k_i(a_i)$ and a fixed irregular type at each a_i was proven to be isomorphic to the symplectic quotient of the form $\widetilde{O}_1 \times \cdots \times \widetilde{O}_m /\!\!/ G$, where $\widetilde{O}_i$ is an extended orbit with natural symplectic structure associated to the irregular type at a_i. In [19], a family of new examples of complex quasi-Hamiltonian G-spaces $\widetilde{\mathcal{C}}$ with G-valued moment maps was introduced as generalization of the conjugacy class example of Alekseev–Malkin–Meinrenken [2]. It was further shown that given the divisor $D = \sum_{i=1}^m k_i(a_i)$, the symplectic spaces of monodromy data for meromorphic connections on V with poles on D and fixed irregular types is isomorphic to the quasi-Hamiltonian quotient space $\widetilde{\mathcal{C}}_1 \circledast \cdots \widetilde{\mathcal{C}}_m /\!\!/ G$, where $\widetilde{\mathcal{C}}_i$ is the space of monodromy data at a_i and $\circledast$ denotes the fusion product between quasi Hamiltonian G-manifolds [2]. In the simple pole case, it recovers the quasi-Hamiltonian description of moduli spaces of flat connections in [2]. The main result of [17] [19] leads to that the irregular Riemann-Hilbert correspondence

$$\nu : (\widetilde{O}_1 \times \cdots \widetilde{O}_m) /\!\!/ \hookrightarrow (\widetilde{\mathcal{C}}_1 \times \cdots \times \widetilde{\mathcal{C}}_m) /\!\!/ G \tag{4.4}$$

associating monodromy/Stokes data to a meromorphic connection on V is a symplectic map. In [18], it was shown that these results extend to any complex reductive group G by introducing G-valued Stokes data for meromorphic connections on principal G-bundles.

In [16], Boalch studied a T-reduction version of the irregular Riemann-Hilbert correspondence in the case of the meromorphic connections on V with one simple pole and one order two pole. The key feature of this case is that the correspondence gives rise to a Poisson map from the dual of the Lie algebra $\mathfrak{g}^*$ to the dual Poisson Lie group G^* associated to the standard classical r-matrix on $\mathfrak{g}$. To have more details, one can define Stokes matrices (see e.g [14]) of the meromorphic connection ∇ in (4.3), a pair of lower and upper triangular matrices $(S_-(B), S_+(B))$, as the ratio of canonical fundamental solutions of ∇ on two chosen Stokes sectors. Then the main result of [16] shows that for each choice of diagonal matrix A_0, the irregular Riemann-Hilbert map $S : \mathfrak{g}^* \to G^*$ relating $B \in \mathfrak{g}^*$ to the Stokes data $(S_-(B), S_+(B)) \in G^*$, is a Poisson map. Note that the connection matrix uniquely determines the Stokes matrices via the important monodromy relation (from the fact that a simple positive loop around 0 is also a simple negative loop around ∞)

$$C(B)e^{2\pi i B^\vee}C(B)^{-1} = S_-S_+e^{2\pi i\delta(B)}, \tag{4.5}$$

where $\delta(B)$ is the diagonal part of B.

To prove our main Theorem 1.1, the first step is to find a symplectic geometric interpretation of equation (4.2), which turns to be a new geometric framework generalizing the Ginzburg-Weinstein linearization. For this purpose, we consider a symplectic slice Σ of T^*G and its Poisson Lie analogue, a symplectic submanifold Σ' of the Lu-Weinstein symplectic double Γ (locally isomorphic to $G \times G^*$) [64]. (See section 2.1 for more details). Then associated to any map $g \in \mathrm{Map}(\mathfrak{g}^*, G)$, we define a local diffeomorphism $F_g : (\Sigma, \omega) \to (\Sigma', \omega')$. Then a symplectic geometric interpretation of the gauge transformation equation is as follows.

Theorem 1.3. *F_g is a local symplectic isomorphism from (Σ, ω) to (Σ', ω') if and only if $g \in \mathrm{Map}(\mathfrak{g}^*, G)$ satisfies equation (4.2).*

With the help of the above theorem, we only need to prove the expected symplectic geometry property of the connection matrix C. This is immediate as long as we consider the irregular Riemann-Hilbert correspondence in the setting of the extended moduli space (see Definition 2.6 in [17]) of meromorphic connections with one simple pole and one order two pole. Actually, following the discussion above, the corresponding irregular Riemann-Hilbert map is

$$\nu : (\widetilde{O}_1 \times \widetilde{O}_2)/\!\!/G \hookrightarrow (\widetilde{\mathcal{C}}_1 \times \widetilde{\mathcal{C}}_2)/\!\!/G. \tag{4.6}$$

On the other hand, the Hamiltonian and quasi-Hamiltonian quotient $(\widetilde{O}_1 \times \widetilde{O}_2)/\!\!/G$ and $(\widetilde{\mathcal{C}}_1 \times \widetilde{\mathcal{C}}_2)/\!\!/G$ are isomorphic to Σ and Σ' respectively. We thus obtain a symplectic map $\nu : \Sigma \to \Sigma'$. Next, following the construction of the irregular Riemann-Hilbert map, we prove that ν can be chosen in such a way that for any $(h, \lambda) \in \Sigma \subset T^*G \cong G \times \mathfrak{g}^*$ (via left multiplication)

$$\nu(h, \lambda) = F_C(h, \lambda), \tag{4.7}$$

where $C(B)$ is the connection matrix of ∇ in (4.3). Therefore, combining with Theorem 2.2, we prove that the connection matrix $C \in \mathrm{Map}(\mathfrak{g}^*, G)$ satisfies the gauge transformation equation (4.2). This clarifies the relation between the gauge transformation of dynamical r-matrices and certain irregular Riemann-Hilbert problem. As a byproduct, we give a new description of Lu-Weinstein symplectic groupoid via Alekseev-Meinrenken r-matrix. We also clarify the meaning of the gauge transformation equation in the framework of generalized classical dynamical r-matrix.

The organisation of this paper is as follows. The next section gives the background material and a geometric description of the equation (4.2). Section 3 defines the connection matrix $C(B)$ of the meromorphic connection ∇ in 4.3 and states that $C : \mathfrak{g}^* \to G$ gives rise to a solution of (4.2), i.e., a gauge transformation from r_0 to r_{AM}. Section 4 discusses the quantum version, i.e., the vertex-IRF transformation equation and formulates a surprising relation between connection matrices and Drinfeld twists. Section 5 gives the background material on the moduli space of meromorphic connections over surfaces and irregular Riemann-Hilbert correspondence in this setting. At the second part of Section 5, we study in details one special case of this correspondence and show that how it gives rise to the equivariant geometric description of the equation (4.2). Section 6 describes Lu-Weinstein symplectic groupoid via Alekseev-Meinrenken r-matrix. Next Section 7 studies the Poisson structure on one symplectic slice Σ' of Lu-Weinstein symplectic double and gives a proof of the main theorem in Section (2.1).

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2 Gauge transformation between the standard r-matrix and the Alekseev-Meinrenken r-matrix

Throughout this section, let $\mathfrak{g}$ be a complex reductive Lie algebra and $t \in S^2(\mathfrak{g})^{\mathfrak{g}}$ the element corresponding to an invariant inner product on $\mathfrak{g}$.

First recall that an element $r \in \mathfrak{g}\otimes\mathfrak{g}$ is called a classical r-matrix if $r+r^{2,1} \in S^2(\mathfrak{g})^{\mathfrak{g}}$ and r satisfies the classical Yang-Baxter equation:

$$[r^{1,2}, r^{1,3}] + [r^{1,2}, r^{2,3}] + [r^{1,3}, r^{2,3}] = 0. \tag{4.8}$$

Throughout this paper, we will denote by $r_0 := \frac{1}{2}(r - r^{2,1})$ the skew-symmetric part of a classical r-matrix r.

A dynamical analog of a classical r-matrix is as follows. Let $\eta \subset \mathfrak{g}$ be a Lie subalgebra. Then a classical dynamical r-matrix is an η-equivariant map $r : \eta^* \to \mathfrak{g}\otimes\mathfrak{g}$ such that $r + r^{2,1} \in S^2(\mathfrak{g})^{\mathfrak{g}}$ and r satisfies the dynamical Yang-Baxter equation (CDYBE):

$$\mathrm{Alt}(dr) + [r^{1,2}, r^{1,3}] + [r^{1,2}, r^{2,3}] + [r^{1,3}, r^{2,3}] = 0, \tag{4.9}$$

where $\mathrm{Alt}(dr(x)) \in \wedge^3\mathfrak{g}$ is the skew-symmetrization of $dr(x) \in \eta \otimes \mathfrak{g} \otimes \mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ for all $x \in \eta^*$.

In the distinguished special case $\eta = \mathfrak{g}$, the Alekseev-Meinrenken dynamical r-matrix $r_{\mathrm{AM}} : \mathfrak{g}^* \to \mathfrak{g} \otimes \mathfrak{g}$ is defined by

$$r_{\mathrm{AM}}(x) = (\mathrm{id} \otimes \phi(\mathrm{ad}_{x^\vee}))(t), \ \forall x \in \mathfrak{g}^*, \tag{4.10}$$

where $x^\vee = (x \otimes \mathrm{id})(t)$ and $\phi(z) := -\frac{1}{z} + \frac{1}{2}\mathrm{cotanh}\frac{z}{2}$, $z \in \mathbb{C} \setminus 2\pi i\mathbb{Z}^*$. Taking the Taylor expansion of ϕ at 0, we see that $\phi(z) = \frac{z}{12} + \circ(z^2)$, thus $\phi(\mathrm{ad_x})$ is well-defined. One can check that $r_{\mathrm{AM}} + \frac{t}{2}$ is a classical dynamical r-matrix (The maximal domain of definition of $\phi(\mathrm{ad}_x)$ contains all $x \in \mathfrak{g}^*$ for which the eigenvalues of $\mathrm{ad_x}$ lie in $\mathbb{C} \setminus 2\pi i\mathbb{Z}^*$).

Denote by G the formal group with Lie algebra $\mathfrak{g}$ and by $\mathrm{Map}_0(\mathfrak{g}^*, G)$ the space of formal maps $g : \mathfrak{g}^* \to G$ such that $g(0) = 1$, i.e., the space of maps of the form e^m, where $m \in \mathfrak{g} \otimes \hat{S}(\mathfrak{g})_{\geq 0}$ ($\hat{S}(\mathfrak{g})$ is the degree completion of the symmetric algebra $S(\mathfrak{g})$). The following theorem states that existence of formal solution of gauge transformation equation (4.2)

Theorem 2.1. [32] *Let r be a classical r-matrix with $r + r^{2,1} = t$. Then there exists a formal map $g \in \mathrm{Map}_0(\mathfrak{g}^*, G)$, such that*

$$g_1^{-1}d_2(g_1) - g_2^{-1}d_1(g_2) + (\otimes^2 \mathrm{Ad}_g)^{-1} r_0 + \langle \mathrm{id} \otimes \mathrm{id} \otimes x, [g_1^{-1}d_3(g_1), g_2^{-1}d_3(g_2)] \rangle = r_{\mathrm{AM}}. \quad (4.11)$$

Here $r_0 := \frac{1}{2}(r - r^{2,1})$, $g_1^{-1}d_2(g)(x) = \sum_i g^{-1}\partial_{\varepsilon^i} g(x) \otimes e_i$ is viewed as a formal function $\mathfrak{g}^* \to \mathfrak{g}^{\otimes 2}$, (ε^i), (e_i) are dual bases of $\mathfrak{g}^*$ and $\mathfrak{g}$, $g_i^{-1}d_j(g_i) = (g_1^{-1}d_2(g_1))^{i,j}$ and $\partial_\xi g(x) = (\frac{d}{d\varepsilon})|_{\varepsilon=0} g(x + \varepsilon\xi)$.

We will denote by $r_0^g \in \mathrm{Map}(\mathfrak{g}^*, \mathfrak{g} \wedge \mathfrak{g})$ the left hand side of equation (4.11). Because of the $\mathfrak{g}$-invariance of t, one checks that $r_0^g = r_{\mathrm{AM}}$ if and only if $r^g = r_{\mathrm{AM}} + \frac{t}{2}$. In [32], this equation is proven to be the classical limit of vertex-IRF transformation between certain dynamical twists(see section 4) and the authors give two constructions of the formal solutions of equation (4.11) based on formal calculation and quantization of Lie bialgebras respectively. In the following two sections, we will give a geometric interpretation and a construction of explicit solutions of equation (4.11), where instead of the formal setting, we will work on a local theory.

2.1 Geometric construction

Following the same convention from last section, and let $\mathfrak{t} \subset \mathfrak{g}$ be a maximal abelian subalgebra and $\mathfrak{t}'$ the complement of the affine root hyperplanes: $\mathfrak{t}' := \{\Lambda \in \mathfrak{t} \mid \alpha(\Lambda) \notin \mathbb{Z}\}$. In the following, $\mathfrak{t}'$ is regarded as a subspace of $\mathfrak{g}^*$ via the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ induced by inner product. Let Σ be a cross-section of $T^*G \cong G \times \mathfrak{g}^*$ (identification via left multiplication), defined by

$$\Sigma := \{(h, \lambda) \in G \times \mathfrak{g}^* \mid \lambda \in \mathfrak{t}'\}. \quad (4.12)$$

Then one can check that Σ is a symplectic submanifold of T^*G with the canonical symplectic structure (see [52] Theorem 26.7). The induced symplectic structure ω on Σ is given for any tangents $v_1 = (X_1, R_1), v_2 = (X_2, R_2) \in \mathfrak{g} \times \mathfrak{g}^*$, where $R_1, R_2 \in \mathfrak{t}^*$, at $(h, \lambda) \in \Sigma$ by

$$\omega(v_1, v_2) = \langle R_1, X_2 \rangle - \langle R_2, X_1 \rangle + \langle \lambda, [X_1, X_2] \rangle. \quad (4.13)$$

Now let $r \in \mathfrak{g} \otimes \mathfrak{g}$ be a classical r-matrix with $r + r^{2,1} = t$. Let G^* be the simply connected dual Poisson Lie group associated to the quasitriangular Lie biaglebra $(\mathfrak{g}, r)$ and D the double Lie group with Lie algebra $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^*$ which is locally diffeomorphic to $G \times G^*$ (see e.g [63]). A natural symplectic structure on D is given by the following bivector,

$$\pi_D = \frac{1}{2}(r_d \pi_0 + l_d \pi_0), \quad (4.14)$$

where $\pi_0 \in \mathfrak{d} \wedge \mathfrak{d}$ such that $\pi_0(\xi_1+X_1, \xi_2+X_2) = \langle X_1, \xi_2\rangle - \langle X_2, \xi_1\rangle$ for $\xi_1+X_1, \xi_2+X_2 \in \mathfrak{d}^* \cong \mathfrak{g}^* \oplus \mathfrak{g}$.

Following [64], the Lu-Weinstein double symplectic groupoid, associated to the Lie bialgebra $(\mathfrak{g}, r)$, is the set

$$\Gamma := \{(h, h^*, u, u^*) \mid h, u \in G, h^*, u^* \in G^*, hh^* = u^*u \in D\} \tag{4.15}$$

with a unique Poisson structure π_Γ such that the local diffeomorphism $(\Gamma, \pi_\Gamma) \to (D, \pi_D)$: $(h, h^*, u, u^*) \mapsto hh^*$ is Poisson. We define a submanifold Σ' of Γ, as a Poisson Lie analogue of Σ, by

$$\Sigma' := \{(h, h^*, u, u^*) \in \Gamma \mid h^* \in e^{\mathfrak{t}'} \subset G^*\} \tag{4.16}$$

(e denotes the exponential map with respect to the Lie algebra g^*). In section 7, we will prove that Σ' is a symplectic submanifold of (Γ, π_Γ). Now let us take this fact and denote the induced symplectic structure on Σ' by ω'. On the other hand, the map

$$\Sigma' \to G \times e^{\mathfrak{t}'};\ (h, e^\lambda, u, u^*) \mapsto (h, e^\lambda) \tag{4.17}$$

expresses Σ' as a cover of a dense subset of $G \times e^{\mathfrak{t}'} \subset G \times G^*$. Thus associated to any $g \in \mathrm{Map}_0(\mathfrak{g}^*, G)$, we have a local diffeomorphism (defined on a dense subset of Σ) $F_g : \Sigma \to \Sigma'$ defined by

$$F_g(h, \lambda) := (g(\mathrm{Ad}_h \lambda)h, e^\lambda, u, u^*),\ \forall (h, \lambda) \in \Sigma, \tag{4.18}$$

where $u \in G, u^* \in G^*$ are determined by the identity $he^\lambda = u^*u$. Note that F_g is well-defined for elements $(h, \lambda) \in \Sigma$ such that $h\lambda$ in the double Lie group D is sufficiently near the unit, and this is enough for our purpose.

Theorem 2.2. *F_g is a local symplectic isomorphism from (Σ, ω) to (Σ', ω') if and only if $g \in \mathrm{Map}(\mathfrak{g}^*, \mathrm{G})$ satisfies the gauge transformation equation (4.11), $r_0^g = r_{\mathrm{AM}}$.*

Proof. See section 7.

The case when r is a standard r-matrix. Let $T \subset G$ be a maximal torus with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$. Let $B_\pm$ denote a pair of opposite Borel subgroups with $B_+ \cap B_- = T$. For the choice of positive roots Σ_+ corresponding to Borel subgroup B_+, we take the standard r-matrix given by

$$r := \frac{1}{2} t + \frac{1}{2} \sum_{\alpha \in \Sigma_+} E_\alpha \wedge E_{-\alpha}, \tag{4.19}$$

where $t \in S^2(\mathfrak{g})^{\mathfrak{g}}$ is the Casimir element. In this case, the simply connected dual Poisson Lie group associated to $(\mathfrak{g}, r)$ is

$$G^* = \{(b_-, b_+, \Lambda) \in B_- \times B_+ \times \mathfrak{t} \mid \delta(b_-)\delta(b_+) = 1, \delta(b_+) = \exp(\pi \mathrm{i}\Lambda)\}, \quad (4.20)$$

where $\delta : \mathfrak{g} \to \mathfrak{t}$ takes the diagonal part. Thus Σ' is a submanifold of the double

$$\Gamma : \{h, (b_-, b_+, \Lambda_b), u, (c_-, c_+, \Lambda_c) \mid hb_\pm = c_\pm u\} \subset (G \times G^*)^2, \quad (4.21)$$

defined by

$$\Sigma' := \{(h, (e^{-\pi i\Lambda}, e^{\pi i\Lambda}, \Lambda), u, (c_+, c_-, \Lambda_c)) \in \Gamma \mid he^{\pm\pi i\Lambda} = c_\pm u, \Lambda \in \mathfrak{t}'\}, \quad (4.22)$$

where $\mathfrak{t}' \subset \mathfrak{t}$ is the complement of the affine root hyperplanes. To simplify notation, we will write $e^{2\pi i\lambda}$ instead of $(e^{\pi i\lambda^\vee}, e^{-\pi i\lambda^\vee}, \lambda^\vee) \in G^*$, where $\lambda \in \mathfrak{g}^*$, $\lambda^\vee = (\lambda \otimes \mathrm{id})(t) \in \mathfrak{g}$ and $e^{2\pi i\lambda}$ $(e^{\pi i\lambda^\vee})$ is respect to the exponential map of the Lie algebra $\mathfrak{g}^*$ $(\mathfrak{g})$.

Now given any $g \in \mathrm{Map}(\mathfrak{g}^*, G)$, let us consider a local diffeomorphism $F_g' : \Sigma \to \Sigma'$ which will be more directly involved in the following discussion,

$$F_g'(h, \lambda) := (g(2\pi i\mathrm{Ad}_h\lambda)h, e^{2\pi i\lambda}, u, u^*), \ \forall (h, \lambda) \in \Sigma \quad (4.23)$$

where $u \in G, u^* \in G^*$ are uniquely determined by the identity $he^\lambda = u^*u$. It is obvious that the map $(\Sigma, \omega) \to (\Sigma, \frac{1}{2\pi i}\omega)$, $(h, \lambda) \mapsto (h, 2\pi i\lambda)$ is symplectic. Therefore, as a corollary of Theorem 2.2, we have

Corollary 2.3. *The map $F_{g_{2\pi i}}$ is a local symplectic isomorphism from (Σ, ω) to (Σ', ω') (provided the symplectic structure on the right-hand side is divided by $2\pi i$) if and only if g satisfies the gauge transformation equation (4.11), $r_0^g = r_{\mathrm{AM}}$.*

3 A gauge transformation between r_{AM} and the standard r-matrix from connection matrix

In this section, we will construct an explicit solution of the gauge transformation equation (4.11) for the case $G = \mathrm{GL}(n, \mathbb{C})$ and r is the standard r-matrix. However, our result does not depend on the choice of $\mathrm{GL}(n, \mathbb{C})$, and so extends to any connected complex reductive group G with the choice of a maximal tours and the corresponding r-matrix given in (4.19) as long as we use the G-valued Stokes data for meromorphic connections on a principal G-bundle (see [18]) and consider the irregular Riemann-Hilbert correspondence in this setting.

Let V be a rank n trivial holomorphic vector bundle on $\mathbb{P}^1$. Let $A_0 \in \mathrm{GL}(n, \mathbb{C})$ be a diagonal matrix with distinct diagonal elements and $B \in \mathrm{gl}(n, \mathbb{C})$ an arbitrary matrix. Choose coordinate $\{z\}$ to identify $\mathbb{P}^1$ with $\mathbb{C} \cup \infty$ and a trivialization of V. We consider the following meromorphic connection on V which has an order 2 pole at origin and (if $B \neq 0$) a first order pole at ∞,

$$\nabla := d - (\frac{A_0}{z^2} + \frac{B}{z})dz. \tag{4.24}$$

Associated to the connection ∇, one can define a connection/monodromy matrix $C \in \mathrm{GL}(n, \mathbb{C})$ as the "monodromy" of ∇ from 0 to ∞. Our aim in this section is to fill in the details of the definition of C.

Definition 3.1. *Let us assume $-A_0/z = diag(q_1, ..., q_n)$. The anti-Stokes directions at 0 on the complex plane are the directions between pairs of eigenvalues of A_0 (when plotted in the z-plane).*

We choose an initial sector Sect_0 at 0 bounded by two adjacent anti-Stokes directions and a branch of $\log(z)$ on Sect_0. Then we label the anti-Stokes directions $d_1, d_2, ..., d_{2l}$ going in a positive sense and starting on the positive edge of Sect_0. Set $\mathrm{Sect_i} = \mathrm{Sect(d_i, d_{i+1})}$ for the open sector swept out by rays moving from d_i to d_{i+1} in a positive sense. (Indices are taken modulo $2l$, so $\mathrm{Sect}_0 = \mathrm{Sect(d_{2l}, d_1)}$.

Given a map $f : \mathbb{C} \to \mathrm{GL}(n, \mathbb{C})$, we denote the gauge action of f on any meromorphic connection $d - A$ by square bracket:

$$f[d - A] = d - fAf^{-1} - df \cdot f^{-1}. \tag{4.25}$$

Proposition 3.2. (see [14]) *There is a unique $\hat{F} \in \mathrm{GL}_n(\mathbb{C}[\![z]\!])$ with $\hat{F}(0) = 1$ such that*

$$\hat{F}[\frac{A_0}{z^2}dz + \frac{\delta(B)}{z}da] = \frac{A_0}{z^2}dz + \frac{B}{z}dz$$

as formal series, where $\delta(B)$ is the diagonal part of B.

The radius of convergence of the series $\hat{F}$ is generally zero. Thus $\hat{F}z^{\delta(B)}e^{-\frac{A_0}{z}}$ is just a formal fundamental solution for ∇. Now the point is that on each Sect_i, there is a canonical way to choose one holomorphic isomorphism between ∇^0 and ∇, where $\nabla^0 := d - (\frac{A_0}{z^2} + \frac{\delta(B)}{z})dz$.

Theorem 3.3. (see e.g. [14][66][68]) *On each sector Sect_i, there is a unique invertible $n \times n$ matrix of holomorphic functions F_i such that $F_i[\nabla^0] = \nabla$, and the matrix of functions F_i can be analytically continued to the i-th 'supersector' $\widehat{\mathrm{Sect}}_i := \mathrm{Sect}(d_i - \frac{\pi}{2}, d_i + \frac{\pi}{2})$ and then F_i is asymptotic to $\hat{F}$ at 0 within $\widehat{\mathrm{Sect}}_i$.*

Definition 3.4. *The canonical fundamental solution of* ∇ *on* $\mathrm{Sect_i}$ *is*

$$\Phi_i := F_i z^{\delta(B)} e^{-\frac{A_0}{z}}, \tag{4.26}$$

where (by convention) the branch of $\log(z)$ *chosen on* Sect_0 *is extended to the other sectors in a negative sense.*

Now let us consider the fundamental solutions of ∇:

$$\Phi := \Phi_0 \text{ on } \mathrm{Sect}_0, \tag{4.27}$$

$$\chi := Hz^B \text{ on a neighbourhood of } \infty \text{ slit along } d_1, \tag{4.28}$$

where $H : \mathbb{P}^1 \setminus \{0\} \to \mathrm{GL}(n, \mathbb{C})$ is a holomorphic map such that $H[d - \frac{B}{z}dz] = \nabla$ and $H(\infty) = Id$, here the square bracket denotes the gauge action. (See e.g. [81] for the existence, uniqueness of H.) Finally, for any $B \in \mathrm{gl}(n, \mathbb{C})$ we obtain a transition matrix $C'(B)$ relating these two fundamental solutions, i.e., if χ is extended along a path in Sect_0 then $\chi = \Phi \cdot C'$ in the domain of definition of Φ. C' is a constant invertible matrix because both bases extend uniquely (as solutions of ∇) along the path and they are both ∇-horizontal bases. Further, associated to the diagonal matrix A_0 and the choice of Sect_0, there is a permutation matrix which will be used to "normalize" the matrix $C'(B)$.

Definition 3.5. *The permutation matrix* $P \in \mathrm{GL}(n, \mathbb{C})$ *associated to the choice of* Sect_0 *is defined by* $(P)_{ij} = \delta_{\pi(i)j}$ *where* π *is the permutation of* $\{1, ..., n\}$ *corresponding to the dominance ordering of* $\{e^{q_1}, ..., e^{q_n}\}$ *along the direction* γ *bisecting the sector* $\mathrm{Sect}(d_1, d_l)$*:*

$$\pi(i) \le \pi(j) \quad \Longleftrightarrow \quad e^{q_i}/e^{q_j} \to 0 \text{ as } z \to 0 \text{ along } \gamma.$$

Definition 3.6. *The connection matrix* $C(B)$ *of* $\nabla = d - (\frac{A_0}{z^2} + \frac{B}{z})$ *associated to the choice of* Sect_0 *and the branch of* $\log(z)$*, is defined as* $C(B) := P^{-1}C'(B)$*.*

Thus we obtain a map C (depends on the choice of A_0) from $\mathfrak{g}^*$ to $\mathrm{GL}(n, \mathbb{C})$ which associating any $B \in \mathfrak{g}^*$ to the connection matrix $C(B)$ of ∇. Now we can introduce our main theorem.

Theorem 3.7. *The map* $C_{2\pi i} \in \mathrm{Map}(\mathfrak{g}^*, G)$*, defined by* $C_{2\pi i}(x) := C(\frac{1}{2\pi i}x)$ *for all* $x \in \mathfrak{g}^*$*, is a solution of the gauge transformation equation* (4.11) *(provided* r_0 *in* (4.11) *is the skew-symmetric part of the standard r-matrix associated to* $\mathrm{GL}(n, \mathbb{C})$*).*

A proof will be given in Section 5. The idea is as follows. Following Theorem 2.2, to prove $r_0^C = r_{\mathrm{AM}}$, we only need to verify its symplectic geometric counterpart. This will be realized as certain irregular Riemann-Hilbert correspondence in section 5.

4 Vertex-IRF transformations and Drinfeld twists

In this section, we will recall the notion of vertex-IRF transformations between dynamical twists [35]. Following [32], equation (4.11) is the classical limit of a vertex-IRF transformation equation between a constant twist J_c and a dynamical twist $J_d(x)$ which are the twist quantizations of r_0 and r_{AM} respectively. In particular, a Drinfeld twist which kills an admissible associator gives rise to such a vertex-IRF transformation. Further discussions will lead to a conjectural relation between connection matrices and Drinfeld twists.

Definition 4.1. *Let $\Phi = 1 + \frac{[t^{12}, t^{23}]}{24}\hbar^2 + O(\hbar^3) \in U(\mathfrak{g})^{\otimes 3}[\![\hbar]\!]$ be such that Φ is $\mathfrak{g}$-invariant and satisfies the pentagon equation and the counit axiom. Then a function $J_d : \mathfrak{g}^* \to U(\mathfrak{g})^{\otimes 2}[\![\hbar]\!]$ is called a dynamical twist associated to Φ if $J_d(x) = 1 + O(\hbar)$ is $\mathfrak{g}$-invariant and*

$$J_d^{12,3} J_d^{1,2}(x + \hbar h^{(3)}) = \Phi^{-1} J_d^{1,23}(x) J_d^{2,3}(x), \tag{4.29}$$

where for $J_d^{1,2}(x + \hbar h^3)$ we use the dynamical convention, i.e.,

$$J_d^{1,2}(x + \hbar^3) = \sum_{N \geq 0} \frac{\hbar^N}{N!} \sum_{i_1, \dots, i_N}^{n} (\partial_{\xi^{i_1}} \cdots \partial_{\xi^{i_N}} J_d)(x) \otimes (e_{i_1} \cdots e_{i_N}) \tag{4.30}$$

where $n = \dim(\mathfrak{g})$, and $\{e_i\}_{i=1,\dots,n}$, $\{\xi^i\}_{i=1,\dots,n}$ are dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$.

Assume $(\Phi, J_d(x))$ satisfies the conditions in Definition 4.1. Let $j(x) := (\frac{J_d(x)-1}{\hbar}) \mod \hbar$, and $r(x) := j(x) - j(x)^{2,1}$. Then following [31], $r(x) + \frac{t}{2}$ is a solution of the CDYBE (4.9), i.e., a classical dynamical r-matrix. In this case, $J_d(x)$ is called a dynamical twist quantization of $r(x)$.

In particular, a constant twist $J_c \in U(\mathfrak{g})^{\otimes 2}[\![\hbar]\!]$ is such that

$$J_c^{12,3} J_c^{1,2} = \Phi^{-1} J_c^{1,23} J_c^{2,3}. \tag{4.31}$$

Similarly, we say $J_c = 1 + \hbar\frac{r}{2} + \circ(\hbar^2)$ is a twist quantization of $r_0 := \frac{1}{2}(r - r^{2,1})$.

Set $U' := U(\hbar\mathfrak{g})[\![\hbar]\!]$, the subalgebra generated by $\hbar x$, $\forall x \in \mathfrak{g}$. Note that $U'/\hbar U' = \hat{S}(\mathfrak{g})$. An associator $\Phi \in U(\mathfrak{g})^{\widehat{\otimes} 3}[\![\hbar]\!]$ is called admissible (see [33]) if

$$\Phi \in 1 + \frac{\hbar^2}{24}[t^{1,2}, t^{2,3}] + O(\hbar^3), \qquad \hbar\log(\Phi) \in (U'(\mathfrak{g}))^{\widehat{\otimes} 3}.$$

Given an admissible associator $\Phi \in U(\mathfrak{g})^{\otimes 3}[\![\hbar]\!]$, we identify the third component $U(\mathfrak{g})$ of this tensor cube with $\mathbb{C}[\mathfrak{g}^*]$ via the symmetrization (PBW) isomorphism $S^\cdot(\mathfrak{g}) \to U(\mathfrak{g})$ and use this identification view Φ^{-1} as a function from $\mathfrak{g}^*$ to $U(\mathfrak{g})^{\otimes 2}[\![\hbar]\!]$, denoted by $\Phi^{-1}(x)$. Then we have $\Phi^{-1}(\hbar^{-1}x)$ is a well-defined element in $U(\mathfrak{g})^{\otimes 2}\hat{\otimes}\mathbb{C}[\mathfrak{g}^*]$. Following [33], any universal Lie associator gives rise to an admissible associator.

Theorem 4.2. [31] Assume that Φ is the image in $U(\mathfrak{g})^{\otimes 3}[\![\hbar]\!]$ of a universal Lie associator. Let $J_d(x) := \Phi^{-1}(\hbar^{-1}x)$, where Φ^{-1} is regarded as an element of $(U(\mathfrak{g})^{\otimes 2}\otimes \mathbb{C}[\mathfrak{g}^*])[\![\hbar]\!]$. Then

(1). $J_d(x)$ is a formal dynamical twist. More precisely, $J_d(x) = 1+\hbar j(x)+O(\hbar^2) \in (U(\mathfrak{g})^{\otimes 2}\hat{\otimes}\hat{S}(\mathfrak{g}))[\![\hbar]\!]$, is a series in nonnegative powers of $\hbar$ and satisfies the dynamical twist equation.

(2). $J_d(x)$ is a twist quantization of the Alekseev-Meinrenken dynamical r-matrix, that is $r_{\mathrm{AM}} = j(x) - j(x)^{2,1}$.

(3). If Φ_{KZ} is the Knizhnik-Zamolodchikov associator, then $J_d(x)$ is holomorphic on an open set and extends meromorphically to the whole $\mathfrak{g}^*$.

Definition 4.3. [35] *Let $J_d(x) : \mathfrak{g}^* \longrightarrow U(\mathfrak{g})^{\otimes 2}[\![\hbar]\!]$ be a function with invertible values and $\rho : \mathfrak{g}^* \longrightarrow U(\mathfrak{g})[\![\hbar]\!]$ a function with invertible values such that $\varepsilon(\rho(\lambda)) = 1$ (ε is the counit). Set*

$$J_d^{\rho}(x) = \Delta(\rho(x))J_d(x)\rho^1(x - \hbar h^{(2)})^{-1}\rho^2(x)^{-1}, \tag{4.32}$$

and call ρ a vertex-IRF transformation from $J_d(x)$ to $J_d^{\rho}(x)$, where for $\rho^{-1}(x - \hbar h^{(2)})$ we use the dynamical convention.

Now let us take an admissible associator Φ. Let J_c ($J_d(x)$) be a (dynamical) twist quantization of r_0 (r_{AM}). Let $\rho(x) \in (U(\mathfrak{g}) \otimes \hat{S}\cdot\mathfrak{g})[\![\hbar]\!]$ be a formal vertex-IRF transformation which maps the $\mathfrak{g}$-invariant but dynamical twist $J_d(x)$ to the constant but non-invariant twist J_c. This is to say

$$J_c = \Delta(\rho(x))J_d(x)\rho^1(x - h^{(2)})^{-1}\rho^2(x)^{-1}. \tag{4.33}$$

Then by comparing the coefficients of equation (4.33) up to the first order of $\hbar$, we have

Proposition 4.4. [32] *The reduction modulo $\hbar$ of $\rho(x)$, denoted by $g(x) = \rho(x)|_{\hbar=0}$, belongs to $\exp(\mathfrak{g} \otimes \hat{S}(\mathfrak{g}))_{>0}$ (thus a formal map from $\mathfrak{g}^*$ to $\exp(\mathfrak{g})$) and satisfies the equation $r_0^{g(x)} = r_{\mathrm{AM}}$.*

Let $J_d(x) = \Phi(\hbar^{-1}x)$ be the dynamical twist in Theorem 4.2, then the IRF-transformations satisfying (4.33) are constructed in [32] as follows. For the admissible associator Φ, there exists a twist killing Φ (see [25][34]), and according to [33], this twist can be made admissible by a suitable gauge transformation. The resulting twist $J \in U(\mathfrak{g})^{\hat{\otimes} 2}[\![\hbar]\!]$ satisfies $J = 1-\hbar\frac{r}{2}+\circ(\hbar)$, $\hbar\log(J) \in U'^{\widehat{\otimes} 2}$, $(\varepsilon\otimes\mathrm{id})(J) = (\mathrm{id}\otimes\varepsilon)(J) = 1$, and

$$\Phi = (J^{2,3}J^{1,23})^{-1}J^{1,2}J^{12,3}. \tag{4.34}$$

Let us now identity the second component $U(\mathfrak{g})$ of J with $\mathbb{C}(\mathfrak{g}^*)$ via PBW isomorphism $S^{\cdot}(\mathfrak{g}) \cong U(\mathfrak{g})$, and regard J as a formal function from $\mathfrak{g}^*$ to $U(\mathfrak{g})[\![\hbar]\!]$, denoted by $J(x)$. Let $\rho(x) := J(\hbar^{-1}x) \in \mathrm{Map}(\mathfrak{g}^*, U(\mathfrak{g})[\![\hbar]\!])$ denote the corresponding renormalization by sending $x \in \mathfrak{g}^*$ to $\hbar^{-1}x$. Then if we identify the third component $U(\mathfrak{g})$ of the tensor cube with $\mathbb{C}(\mathfrak{g}^*)$ in equation (4.33) and renormalize the resulting formal maps from $\mathfrak{g}^*$ to $U(\mathfrak{g})^{\hat{\otimes}2}$ by sending $x \in \mathfrak{g}^*$ to $\hbar x$, the equation (4.34) becomes

$$J^{-1} = \Delta(\rho(x))J_d(x)\rho^1(x - h^{(2)})^{-1}\rho^2(x)^{-1} \tag{4.35}$$

(Here $J_d(x) := \Phi^{-1}(\hbar^{-1}x)$ is the dynamical twist in Theorem 4.2). One checks that $J_c := J^{-1}$ satisfies (4.31) (thus a constant twist). Therefore, the admissible Drinfeld twist J gives rise to a vertex-IRF transformation between $J_d(x)$ and $J_c = J^{-1}$.

Following Proposition 4.4 and the above discussion, we know that the renormalized quasi-classical limit $g(x) \in \mathrm{Map}(\mathfrak{g}^*, G)$ of an admissible Drinfeld twist J satisfies equation (4.11). Now for the case $\mathfrak{g}$ is semisimple, we prove that the inverse is also true, i.e., given any solution $g(x)$ of (4.11), there exists an admissible Drinfeld twist J whose quasi-classical limit is $g(x)$.

Note that $\mathrm{Map}_0(\mathfrak{g}^*, G)$ has a group structure, defined by $(g_1 * g_2)(x) = g_2(\mathrm{Ad}^*_{g_1(x)}x)g_1(x)$. Following [32], we have a subgroup $\mathrm{Map}_0{}^{ham}(\mathfrak{g}^*, G)$ whose elements $g(x)$ are such that (use the same convention in (4.11))

$$g_1^{-1}d_2(g_1) - g_2^{-1}d_1(g_2) + \langle \mathrm{id} \otimes \mathrm{id} \otimes x, [g_1^{-1}d_3(g_1), g_2^{-1}d_3(g_2)]\rangle = 0. \tag{4.36}$$

Furthermore, the right action of $\mathrm{Map}_0(\mathfrak{g}^*, G)$ on itself restricts to an action of $\mathrm{Map}_0^{ham}(\mathfrak{g}^*, G)$ on the space of solutions of equation (4.11), i.e., for $\alpha \in \mathrm{Map}_0^{ham}(\mathfrak{g}^*, G)$, $(\alpha * g)(x) = g(\mathrm{Ad}^*_{\alpha(x)}x)\alpha(x)$ is a solution of (4.11) if $g(x)$ is.

The infinitesimal of the above action is as follows. For each $a = w \otimes f \in \wedge^n(\mathfrak{g}) \otimes \hat{S}(\mathfrak{g})$, we define $da := \sum_i w \otimes e_i \otimes \frac{d}{d\varepsilon}a(x + \varepsilon\xi^i)$ and if $v \in \wedge^{n-1}(\mathfrak{g}) \otimes \mathfrak{g}$, set $\mathrm{Alt}(v \otimes f) := (v + v^{2,\ldots,n,1} + \cdots + v^{n,1,\ldots,n-1}) \otimes f$. Then it is direct to check that $\mathrm{Map}_0^{ham}(\mathfrak{g}^*, G)$ is a prounipotent Lie group with Lie algebra $\{\alpha \in \mathfrak{g} \otimes \hat{S}(\mathfrak{g})_{\geq 1} \mid \mathrm{Alt}(d\alpha) = 0\}$. This Lie algebra is isomorphic to $(\hat{S}(\mathfrak{g})_{>1}, \{-,-\})$ under the map $d : f \to df$, $f \in \hat{S}(\mathfrak{g})_{>1}$. Then the infinitesimal action of $\mathrm{Map}_0^{ham}(\mathfrak{g}^*, G)$ on the space of all $g(x) \in \mathrm{Map}_0(\mathfrak{g}, G)$ satisfying (4.11) is described as follows: the Lie algebra $(\hat{S}(\mathfrak{g})_{>1}, \{-,-\})$ acts by vector fields on the space of solutions by

$$g^{-1}\delta_f(g) = \langle \mathrm{id} \otimes \mathrm{id} \otimes x, [d_3(f_2), g_{12}^{-1}d_3 g_{12}]\rangle - d_1(f_2) \in \mathfrak{g} \otimes \hat{S}(\mathfrak{g})_{\geq 0}. \tag{4.37}$$

Next let $U'_0 := \mathrm{Ker}(\varepsilon) \cap U'$. Then $V := \{u \in \hbar^{-1}U'_0 \subset U(\mathfrak{g})[\![\hbar]\!]\} \mid u = O(\hbar)\}$ is a Lie subalgebra for the commutator. One checks that $e^u * J := (e^u)^1(e^u)^2 J(\Delta(e^u))^{-1}$ is a solution of (4.34) if J is. Thus V acts on the set of admissible Drinfeld twists by $\delta_u(J) = u^1 J + u^2 J - J u^{12}$, $u \in V$. Note that $V/\hbar V = (\hat{S}(\mathfrak{g})_{>1}, \{-,-\})$. The reduction modulo $\hbar$ of the Lie algebra V action on a twist is described as follows. The Lie algebra $(\hat{S}(\mathfrak{g})_{>1}, \{-,-\})$ acts on the set of solutions of (4.11) by $\delta_f(g) = \{1 \otimes f, g\} - g \cdot df$, $f \in \hat{S}(\mathfrak{g})_{>1}$, which coincides with the action (4.37). Therefore, this action is the infinitesimal of the right action of $\mathrm{Map}(\mathfrak{g}^*, G)$ on the set of solutions of (4.11).

Recall that for an admissible Drinfeld twist J, regarded as a formal function $J(x) : \mathfrak{g}^* \to U(\mathfrak{g})[\![\hbar]\!]$ (via PBW), the renormalized quasi-classical limit of J is $g(x) := J(\hbar^{-1}x)|_{\hbar=0} \in \mathrm{Map}(\mathfrak{g}^*, U(\mathfrak{g})[\![\hbar]\!])$ ($g(x)$ is actually in $\mathrm{Map}_0(\mathfrak{g}^*, G)$ by Proposition 4.4).

Proposition 4.5. *Given any $g \in \mathrm{Map}_0(\mathfrak{g}^*, G)$ satisfying $r_0^g = r_{\mathrm{AM}}$, there exists an admissible Drinfeld twist J whose renormalized quasi-classical limit is $g(x)$, and the identity $r_0^g = r_{\mathrm{AM}}$ is the classical limit of the identity $\Phi = (J^{2,3}J^{1,23})^{-1}J^{1,2}J^{12,3}$.*

Proof. Let J' be an admissible Drinfeld twist with $g'(x)$ as its renormalization classical limit (thus a solution of (4.11)). Following [32], $\mathrm{Map}_0^{ham}(\mathfrak{g}^*, G)$ acts simply and transitively on the space of solutions of (4.11). Therefore, there exists an $\alpha \in \mathrm{Map}_0^{ham}(\mathfrak{g}^*, G)$ such that $g(x) = g' * \alpha$. Assume $a \in \hat{S}(\mathfrak{g})_{>1}$ (Lie algebra of $\mathrm{Map}_0^{ham}(\mathfrak{g}^*, G)$) be such that $e^a = \alpha$. Let us take $u \in V \subset U(\mathfrak{g})[\![\hbar]\!]$ whose (renormalized) reduction modulo $\hbar$ is a. We thus have that $J := e^u * J'$ is also an admissible twist. Let us regard J as a formal function from $\mathfrak{g}^*$ to $U(\mathfrak{g})[\![\hbar]\!]$ by identifying the second component $U(\mathfrak{g})$ with $\hat{S}(\mathfrak{g})$, then we can check that $g(x) = J(\hbar^{-1}x)|_{\hbar=0}$. ■

In particular, given any connection matrix C, $C_{2\pi i} \in \mathrm{Map}_0(\mathfrak{g}^*, G)$ is a solution of (4.11) (see section 3), thus can be quantized. From the above discussion, it means that if we regard $C_{2\pi i}$ as an element in $U(\mathfrak{g})^{\otimes 2}$ by taking the Taylor expansion at 0 and identifying $\hat{S}(\mathfrak{g})$ with $U(\mathfrak{g})$, then there exists an admissible Drinfeld twist $J \in U(\mathfrak{g})^{\otimes 2}[\![\hbar]\!]$ satisfying (4.34) whose renormalized quasi-classical limit is $C_{2\pi i}$.

Theorem 4.6. *Assume Φ is the image in $U(\mathfrak{g})^{\hat{\otimes} 3}$ of a universal Lie associator. Then for any connection matrix $C \in \mathrm{Map}_0(\mathfrak{g}^*, G)$, there exists an admissible Drinfeld twist J killing the associator Φ whose renormalized quasi-classical limit is $C_{2\pi i}$.*

Remark 4.7. *In particular, we can take Φ to be the KZ associator. Then the connection matrix (the KZ associator) can be seen as the monodromy from* 0 *to* ∞ *(*1 *to* ∞*) of the differential equation with one order two pole at* 0 *and one simple pole* ∞ *(three simple poles at* 0, 1, ∞*). Naively it seems that the confluence of two simple poles at* 0 *and* 1 *in the KZ case turns the monodromy representing KZ associator to the monodromy representing connection matrix. Thus one may expect an explanation of the above theorem from this perspective. It also indicates that the* *identity* $\Phi_{KZ} = (J^{2,3}J^{1,23})^{-1}J^{1,2}J^{12,3}$ *may be related to the confluence of two simple poles.*

5 Irregular Riemann-Hilbert correspondence

In this section, we will recall symplectic moduli spaces of meromorphic connections on a trivial holomorphic vector bundle, the corresponding symplectic spaces of monodromy data and the irregular Riemann-Hilbert correspondence between them. We mainly follow the papers [17][18][19] of Boalch, in which these symplectic spaces are found and described both explicitly and from an infinite dimensional viewpoint (generalising the Atiyah-Bott approach). After that, we will consider the case of the meromorphic connection with one simple pole and one order two pole, and show that the irregular Riemann-Hilbert correspondence in this case gives rise to a gauge transformation between r_0 and r_{AM}.

5.1 Moduli spaces of meromorphic connections and the spaces of monodromy data

Let $D = \sum_{i=1}^{m} k_i(a_i) > 0$ be an effective divisor on $\mathbb{P}^1$ and V a rank n holomorphic vector bundle.

Definition 5.1. *A meromorphic connection* ∇ *on* V *with poles on* D *is a map* $\nabla : V \to V \otimes K(D)$ *from the sheaf of holomorphic sections of* V *to the sheaf of sections of* $V \otimes K(D)$*, satisfying the Leibniz rule:* $\nabla(fv) = (df) \otimes v + f\nabla v$*, where* v *is a local section of* V*,* f *is a local holomorphic function and* K *is the sheaf of holomorphic one-forms on* $\mathbb{P}^1$*.*

Let z be a local coordinate on $\mathbb{P}^1$ vanishing at a_i then in terms of a local trivialisation of V, any meromorphic connection ∇ takes the form of $\nabla = d - A$, where

$$A = \frac{A_{k_i}}{z^{k_i}}dz + \cdots \frac{A_1}{z}dz + A_0 dz + \cdots \tag{4.38}$$

is a matrix of meromorphic one-forms and $A_j \in \text{End}(\mathbb{C}^n)$, $j \leq k_i$. ∇ is called generic if at each a_i the leading coefficient A_{k_i} is diagonalizable with distinct eigenvalues (for $k_i \geq 2$), or diagonalizable with distinct eigenvalues mod $\mathbb{Z}$ (for $k_i = 1$).

Definition 5.2. [17] *A compatible framing at* a_i *of a vector bundle* V *with generic connection* ∇ *is an isomorphism* $g_0 : V_{a_i} \to \mathbb{C}^n$ *between the fibre* V_{a_i} *and* $\mathbb{C}^n$ *such that the leading coefficient of* ∇ *is diagonal in any local trivialisation of* V *extending* g_0*.*

At each point a_i choose a germ $d - {}^iA^0$ of a diagonal generic meromorphic connection in some trivialisation of V. Let $\nabla = d - A$ in some local trivialisation and z_i a local coordinate vanishing at a_i, then (∇, V) with compatible framing g_0 at a_i has irregular type ${}^iA^0$ if there is some formal bundle automorphism $g \in GL_n[[z_i]]$ with $g(a_i) = g_0$ such that $gAg^{-1} + dg \cdot g^{-1} = {}^iA^0 + \frac{{}^i\Lambda}{z_i} dz_i$ for some diagonal matrix ${}^i\Lambda$. Let $\mathbf{a}$ denote the choice of the effective divisor D and all the germs ${}^iA^0$.

Definition 5.3. ([17]) *The extended moduli space $\widetilde{\mathcal{M}^*}(\mathbf{a})$ is the set of isomorphism classes of triples $(V, \nabla, \mathbf{g})$ consisting of a generic connection ∇ with poles on D on a trivial holomorphic vector bundle V over $\mathbb{P}^1$ with compatible framing $\mathbf{g} = (g_0)$ such that $(V, \nabla, \mathbf{g})$ has irregular type ${}^iA^0$ at each a_i.*

Next let us recall (from [17] Section 2) the building blocks $\widetilde{O}$ of the moduli space $\widetilde{\mathcal{M}^*}(\mathbf{a})$. Fix an integer $k \neq 2$. Let $G_k := G(\mathcal{C}[z]/z^k)$ be the group of $(k-1)$-jets of bundle automorphisms, and let $\mathfrak{g}_k = \mathrm{Lie(G_k)}$ be its Lie algebra, which contains elements of the form $X = X_0 + X_1 z + \cdots + X_{k-1} z^{k-1}$ with $X_i \in \mathfrak{g}$. Let B_k be the subgroup of G_k of elements having constant term 1. The group G_k is the semi-direct product $G \ltimes B_k$ (where G acts on B_k by conjugation). Correspondingly the Lie algebra of G_k decomposes as a vector space direct sum and dualising we have: $\mathfrak{g}_k^* = \mathfrak{b}_k^* \oplus \mathfrak{g}^*$. Elements of $\mathfrak{g}_k^*$ will be written as

$$A = A_0 \frac{dz}{z^k} + \cdots + A_{k-1} \frac{dz}{z} \tag{4.39}$$

via the pairing with $\mathfrak{g}_k$ given by $\langle A, X \rangle := \mathrm{Res}_0(A, X) = \sum_{i+j=k-1} (A_i, X_j)$. In this way $\mathfrak{b}_k^*$ is identified with the set of A having zero residue and $\mathfrak{g}^*$ with those having only a residue term (zero irregular part). Let $\pi_{\mathrm{res}} : \mathfrak{g}_k^* \to \mathfrak{g}^*$ and $\pi_{\mathrm{irr}} : \mathfrak{g}_k^* \to \mathfrak{b}_k^*$ denote the corresponding projections.

Now choose an element $A^0 = A_0^0 \frac{dz}{z^k} + \cdots + A_{k-2}^0 \frac{dz}{z^2}$ of $\mathfrak{b}_k^*$ with $A_i^0 \in \mathfrak{t}$ and with regular leading coefficient $A_0^0 \in \mathfrak{t}_{\mathrm{reg}}$. Let $O_{A^0} \subset \mathfrak{b}_k^*$ denote the B_k coadjoint orbit containing A^0.

Definition 5.4. ([17]) The extended orbit $\widetilde{O} \subset G \times \mathfrak{g}_k^*$ associated to O_{A^0} is

$$\widetilde{O} := \{(g_0, A) \in G \times \mathfrak{g}_k^* \mid \pi_{irr}(g_0 A g_0^{-1}) \in O_{A^0}\} \tag{4.40}$$

where $\pi_{irr} : \mathfrak{g}_k^* \to \mathfrak{b}_k^*$ is the natural projection removing the residue.

$\widetilde{O}$ is naturally a Hamiltonian G-manifold. Any tangents v_1, v_2 to $\widetilde{O} \in G \times \mathfrak{g}_k^*$ at (g_0, A) are of the form

$$v_i = (X_i(0), [A, X_i] + g_0^{-1} R_i g_0) \in \mathfrak{g} \oplus \mathfrak{g}_k^* \tag{4.41}$$

for some $X_1, X_2 \in \mathfrak{g}_k$ and $R_1, R_2 \in \mathfrak{t}^*$ (where $\mathfrak{g} \cong T_{g_0}G$ via left multiplication), and the symplectic structure on $\widetilde{O}$ is given by

$$\omega_{\widetilde{O}}(v_1, v_2) = \langle R_1, \mathrm{Ad}_{g_0} X_2 \rangle - \langle R_2, \mathrm{Ad}_{g_0} X_1 \rangle + \langle A, [X_1, X_2] \rangle. \tag{4.42}$$

Proposition 5.5. ([17]) The G action $h \cdot (g_0, A) := (g_0 h^{-1}, hAh^{-1})$ on $(\widetilde{O}, \omega_{\widetilde{O}})$ is Hamiltonian with moment map $\mu_G : \widetilde{O} \to \mathfrak{g}^*$, $\mu(g_0, A) = \pi_{res}(A)$.

In the simple pole case $k = 1$ we define

$$\widetilde{O} := \{(h, x) \in G \times \mathfrak{g}^* \mid \mathrm{Ad}_h x \in \mathfrak{t}'\} \subset G \times \mathfrak{g}^*. \tag{4.43}$$

One checks that the map $\widetilde{O} \to \Sigma$, $(h, x) \mapsto (h, \mathrm{Ad}_h x)$ is an isomorphic of $\widetilde{O}$ to the symplectic slice Σ defined in section 3.

The spaces $\widetilde{O}$ enable one to construct global symplectic moduli spaces of meromorphic connections on trivial G-bundles over $\mathbb{P}^1$ as symplectic quotients of the form $\widetilde{O}_1 \times \cdots \times \widetilde{O}_m /\!\!/ G$ (the Hamiltonian reduction of the direct product of m Hamiltonian G-spaces).

Proposition 5.6. ([17]) $\widetilde{\mathcal{M}^*}(\mathbf{a})$ *is isomorphic to the symplectic quotient*

$$\widetilde{\mathcal{M}^*}(\mathbf{a}) \cong \widetilde{O}_1 \times \cdots \times \widetilde{O}_m /\!\!/ G \tag{4.44}$$

where $\widetilde{O}_i \subset G \times \mathfrak{g}^*_{k_i}$ *is the extended coadjoint orbit associated to* $O_{A^i} \subset \mathfrak{b}^*_k$*, the* B_k *coadjoint orbit containing the diagonal element* A^i *which arises from the irregular part of* ${}^iA^0$ *at* a_i*.*

Quasi-Hamiltonian G-spaces and symplectic spaces of monodromy data. Next, let us recall the quasi-Hamiltonian description of the symplectic structure on the space of monodromy/Stokes data. Let G be a Lie group with the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\theta, \bar{\theta}$ denote the left and right invariant $\mathfrak{g}$-valued Cartan one-forms on G respectively, and let ψ denote the canonical three-form of G, i.e., $\psi := \frac{1}{6}\langle \theta, [\theta, \theta] \rangle$.

Definition 5.7. ([2]) A quasi-Hamiltonian G-space is a G-manifold M with a G-equivariant map $\mu : M \to G$ (where G acts on itself by conjugation), and a G-invariant two-form $\omega \in \Omega^2(M)$ such that

(i) $d\omega = \mu^*(\psi)$, where ψ is the canonical three-form on G;

(ii) $\omega(v_X, \cdot) = \frac{1}{2}\mu^*(\theta + \bar{\theta}, X) \in \Omega^1(M)$, for all $X \in \mathfrak{g}$, where v_X is the fundamental vector field $(v_X)_m = -\frac{d}{dt}(e^{tX} \cdot m)|_{t=0}$.

(iii) the kernel of ω at each point $m \in M$ is

$$\ker\omega_m = \{(v_X)_m \mid X \in \mathfrak{g} \text{ such that } hXh^{-1} = -X, \text{ where } h := \mu(m) \in G\} \quad (4.45)$$

The most important example of a quasi-Hamiltonian G-space is as follows.

Example 5.8. *Suppose $\mathcal{C} \subset G$ is a conjugacy class with the conjugation action of G. Then $\mathcal{C}$ is a quasi-Hamiltonian G-space with the moment map μ given by the inclusion map, and two-form ω defined by*

$$\omega_h(v_X, v_Y) = \frac{1}{2}(\langle X, \mathrm{Ad}_h Y\rangle - \langle Y, \mathrm{Ad}_h X\rangle), \quad (4.46)$$

for any $X, Y \in \mathfrak{g}$ and v_X, v_Y the fundamental vector field with respect to the conjugation action of G.

Theorem 5.9. ([2]) *Suppose M is a quasi-Hamiltonian $(G \times H)$-space with moment map $(\mu, \mu_H) : M \to G \times H$. If the quotient μ^{-1}/G of the inverse image $\mu^{-1}(1)$ of the identity under the first moment map is a manifold, then the restriction of ω to $\mu^{-1}(1)$ descends to the reduced space $M/\!\!/G := \mu^{-1}/G$ and makes it into a quasi-Hamiltonian H-space. In particular, if H is abelian, then $M/\!\!/G$ is a symplectic manifold.*

We can introduce the following monoid structure in the category of quasi-Hamiltonian G-spaces.

Definition 5.10. ([2]) *Let M_1 and M_2 be quasi-Hamiltonian G-spaces with moment map μ_1 and μ_2 respectively. Their fusion product $M_1 \circledast M_2$ is defined to be the quasi-Hamiltonian G-space $M_1 \times M_2$, where G acts diagonally, with two-form*

$$\widetilde{\omega} = \omega_1 + \omega_2 - \frac{1}{2}(\mu_1^*\theta, \mu_2^*\bar{\theta}) \quad (4.47)$$

and moment map

$$\widetilde{\mu} = \mu_1 \cdot \mu_2 : \ M \to G. \quad (4.48)$$

The quasi-Hamiltonian spaces from conjugacy classes can be seen as the building blocks of moduli spaces of flat connections on trivial G-bundles over $\mathbb{P}^1$. Indeed, let Σ_m be a sphere with m boundary components, the quasi-Hamiltonian reduction

$$\mathcal{C}_1 \circledast \cdots \circledast \mathcal{C}_m /\!\!/ G \quad (4.49)$$

of the fusion product of m conjugacy classes $\mathcal{C}_i$ is isomorphic to the moduli space of flat connections over Σ_m with the Atiyah-Bott symplectic form.

Now let us recall the building blocks of the spaces of monodromy data. They are the monodromy manifolds corresponding to higher degree poles, and conjugacy classes can be seen as the monodromy manifolds in first order pole cases. Let T be a maximal torus of G with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$ and $B_\pm$ denote a pair of opposite Borel subgroups with $B_+ \cap B_- = T$. Let us consider the family of complex manifolds (see [19] for the geometrical origins of these spaces where their infinite-dimensional counterparts are described)

$$\widetilde{\mathcal{C}} := \{(C, \mathbf{d}, \mathbf{e}, \Lambda) \in G \times (B_- \times B_+)^{k-1} \times t \mid \delta(d_j)^{-1} = e^{\frac{\pi i \Lambda}{k-1}} = \delta(e_j) \ for\ all\ j\}, \tag{4.50}$$

parameterised by an integer $k \geq 2$, where $\mathbf{b} = (d_1, ..., d_{k-1})$, $\mathbf{e} = (e_1, ..., e_{k-1})$ with $d_{even}, e_{odd} \in B_+$ and $d_{odd}, e_{even} \in B_-$ and $\delta : B_+ \to T$ is the homomorphism with kernel $U_\pm$.

Proposition 5.11. ([19]) The manifold $\widetilde{\mathcal{C}}$ is a complex quasi-Hamiltonian $G \times T$-space with action

$$(g, t) \cdot (C, \mathbf{d}, \mathbf{e}, \Lambda) = (tCg^{-1}, td_1t^{-1}, ..., td_{k-1}t^{-1}, te_1t^{-1}, ..., te_{k-1}t^{-1}, \Lambda) \in \widetilde{\mathcal{C}}, \tag{4.51}$$

and the moment map $(\mu, e^{-2\pi i \Lambda}) : \widetilde{\mathcal{C}} \to G \times T$ where

$$\mu : \widetilde{\mathcal{C}} \to G, \quad (C, \mathbf{d}, \mathbf{e}, \Lambda) \mapsto C^{-1} d_1^{-1} \cdots d_{k-1}^{-1} e_{k-1} \cdots e_1 C, \tag{4.52}$$

and two-form

$$\omega = \frac{1}{2}(\bar{\mathcal{D}}, \bar{\mathcal{E}}) + \frac{1}{2}\sum_{j=1}^{k-1}(\mathcal{D}_j, \mathcal{D}_{j-1}) - (\mathcal{E}_j, \mathcal{E}_{j-1}) \tag{4.53}$$

where $\bar{\mathcal{D}} = D^*\bar{\theta}$, $\bar{\mathcal{E}} = E^*\bar{\theta}$, $\mathcal{D}_j = D_j^*\theta$, $\mathcal{E}_j = E_j^*\theta \in \Omega^1(\widetilde{\mathcal{C}}, \mathfrak{g})$ for maps D_j, $E_j : \widetilde{\mathcal{C}} \to G$ defined by $D_i(C, , , \Lambda) = d_i \cdots d_1 C$, $E_i = e_i \cdots e_1 C$, $D := D_{k-1}$, $E := E_{k-1}$, $E_0 = D_0 := C$.

For example, in the order two pole case $k = 2$,

$$\widetilde{\mathcal{C}}_{k=2} \cong G \times G^*, \quad \mu = C^{-1} b_-^{-1} b_+ C, \quad \omega = \frac{1}{2}(D^*\bar{\theta}, E^*\bar{\theta}) + \frac{1}{2}(D^*\theta, C^*\theta) - \frac{1}{2}(E^*\theta, C^*\theta) \tag{4.54}$$

where $D = b_- C, E = b_+ C$. In general the quotient $\widetilde{\mathcal{C}}/G$ has an induced Poisson structure [2] and for $k = 2$ this coincides with standard Poisson structure on G^*.

In first order pole case $k = 1$ we define $\widetilde{\mathcal{C}}_{k=1} := \{(h, (e^{-\pi i \lambda}, e^{\pi i \lambda}, \lambda)) \mid h \in G, \lambda \in \mathfrak{t}'\}$ which is a submanifold of $\widetilde{\mathcal{C}}_{k=2} \cong G \times G^*$. The 2 form and moment map in this case are the restriction of the 2-form and moment map (4.54) of $\widetilde{\mathcal{C}}_{k=2}$ to $\widetilde{\mathcal{C}}_{k=1}$.

Given a divisor $D = \sum_{i=1}^m k_i(a_i)$ having each $k_i \geq 1$ at a_i on $\mathbb{P}^1$, the above proposition enables one to construct the symplectic space $\widetilde{\mathcal{M}}(\mathbf{a})$ of monodromy data for compatibly meromorphic connections $(V, \nabla, \mathbf{g})$ with irregular type $\mathbf{a}$.

Proposition 5.12 (Lemma 3.1 [19]). *The symplectic space $\widetilde{\mathcal{M}}(\mathbf{a})$ is isomorphic to the quasi-Hamiltonian quotient $\widetilde{\mathcal{C}}_1 \circledast \cdots \circledast \widetilde{\mathcal{C}}_m /\!\!/ G$, where $\circledast$ denotes the fusion product of two quasi-Hamiltonian G-manifolds.*

The extension of the Atiyah-Bott symplectic structure to the case of singular $\mathbb{C}^\infty$-connections given in [17] leads to certain Hamiltonian loop group manifolds and $\widetilde{\mathcal{C}}$ is the corresponding quasi-Hamiltonian space.

5.2 Irregular Riemann-Hilbert correspondence

Let $\mathbf{a}$ be the data of a divisor $D = \sum k_i(a_i)$ and connection germs $d - {}^iA^0$ at each a_i. The irregular Riemann-Hilbert map, which depends on a choice of tentacles (see Definition 3.9 in [17]), is a map ν from the global symplectic moduli space of meromorphic connections $\widetilde{\mathcal{M}^*}(\mathbf{a}) \cong (\widetilde{O}_1 \times \cdots \widetilde{O}_m) /\!\!/ G$ to the symplectic space of monodromy data $\widetilde{\mathcal{M}}(\mathbf{a}) \cong (\widetilde{\mathcal{C}}_1 \times \cdots \times \widetilde{\mathcal{C}}_m) /\!\!/ G$. In brief, the map arises as follows. Let $(V, \nabla, \mathbf{g})$ be a compatibly framed meromorphic connection on a holomorphic vector bundle V with irregular type $\mathbf{a}$. The irregular type $\mathbf{a}$ canonically determines some directions at a_i (Stokes rays) for each i. We then consider the Stokes sectors at each a_i bounded by these directions (and having some small fixed radius). The key fact is that, similar to the discussion in subsection 3.2.2, the framings $\mathbf{g}$ (and a choice of branch of logarithm at each pole) determine, in a canonical way, a choice of basis of solutions of the connection ∇ on each Stokes sector at each pole. Now along any path in the punctured sphere $\mathbb{P}^1 \setminus \{a_1, ..., a_m\}$ between two such sectors, we can extend the two corresponding bases of solutions and obtain a constant n by n matrix relating these two bases. The monodromy data of $(V, \nabla, \mathbf{g})$ is simply the set of all such constant matrices, plus the exponents of formal monodromy, thus it corresponds to a point in the space of monodromy data $\widetilde{\mathcal{C}}_1 \times \cdots \times \widetilde{\mathcal{C}}_m$. Note that $(V, \nabla, \mathbf{g})$ can be seen as a point in $\widetilde{O_1} \times \cdots \widetilde{O_m}$. Such a map taking the monodromy data of meromorphic connections is G-equivariant and descends to give ν. The main result of [17] leads to:

Theorem 5.13. ([17]) The irregular Riemann-Hilbert map

$$\nu : (\widetilde{O}_1 \times \cdots \widetilde{O}_m) /\!\!/ G \hookrightarrow (\widetilde{\mathcal{C}}_1 \circledast \cdots \circledast \widetilde{\mathcal{C}}_m) /\!\!/ G \tag{4.55}$$

associating monodromy/Stokes data to a meromorphic connection on the trivial G-bundle over $\mathbb{P}^1$ is a symplectic map (provided the symplectic structure on the right-hand side is divided by $2\pi i$).

We will analyze the case with one pole of order one and one pole of order two and show that the irregular Riemann-Hilbert map ν gives rise to a local symplectic isomorphism from (Σ, ω) to (Σ', ω'). Furthermore, given a choice of tantacles, the corresponding map ν can be expressed explicitly by the connection matrix $C \in \mathrm{Map}(\mathfrak{g}^*, G)$ defined in section 3. Thus with the help of Theorem 2.2, one can show that C is a solution of the equation (4.11), i.e., $r_0^C = r_{\mathrm{AM}}$.

Proposition 5.14. *Let $\widetilde{O}_1$ and $\widetilde{O}_2$ be two copies of $\widetilde{O}$ with $k = 1$ and $k = 2$ respectively. Then the Hamiltonian quotient $\widetilde{O}_1 \times \widetilde{O}_2 /\!/ G$ is symplectic isomorphic to (Σ, ω).*

Proof. By definition, $\widetilde{\mathcal{O}}_1 = \{(g_1, B_1) \in G \times \mathfrak{g}^* \mid g_1 B_1 g_1^{-1} \in \mathfrak{t}'\}$ and $\widetilde{\mathcal{O}}_2 = \{(g_2, A, B_2) \in G \times \mathfrak{g}^* \times \mathfrak{g}^* \mid \mathrm{Ad}_{g_2} A = A_0\}$, where A_0 is a fixed diagonal matrix with distinct elements. Because A is determined by g_2, $\widetilde{\mathcal{O}}_2$ is naturally isomorphic to $G \times \mathfrak{g}^*$ by sending (g_2, A, B_2) to (g_2, B_2). Note that the moment map is

$$\mu : \widetilde{O}_1 \times \widetilde{O}_2 \longrightarrow \mathfrak{g}^*; \ (g_1, B_1, g_2, B_2) \mapsto B_1 + B_2. \tag{4.56}$$

The submanifold $\mu^{-1}(0)$ is defined by $\mu^{-1}(0) := \{(g_1, B_1, g_2, -B_1) \in (G \times \mathfrak{g}^*)^2 \mid \mathrm{Ad}_{g_1} B_1 \in \mathfrak{t}'\}$. We have a subjective map

$$\iota : \mu^{-1}(0) \longrightarrow \Sigma; \ (g_1, B_1, g_2, -B_1) \mapsto (g_2 g_1^{-1}, -\mathrm{Ad}^*_{g_1} B_1) \tag{4.57}$$

whose fibres are the G orbits. Thus it induces an isomorphism from $\widetilde{O}_1 \times \widetilde{O}_2 /\!/ G$ to (Σ, ω). To verify this is actually a symplectic isomorphism, let us take two tangents v_1, v_2 to $\mu^{-1}(0)$ which at each point $(g_1, B_1, g_2, -B_1)$ take the forms $v_i = (0, \mathrm{Ad}_{g_1^{-1}} R_i, \mathrm{Ad}_{g_2^{-1}} X_i, -\mathrm{Ad}_{g_1^{-1}} R_i)$ for some $X_i \in \mathfrak{g}, R_i \in \mathfrak{t}^*$ and $i = 1, 2$ ($\mathfrak{g} \cong T_{g_2} G$ via left multiplication).

Let $\omega_{\mu^{-1}(0)}$ be the restriction of the (direct sum) symplectic structure $\omega_{\widetilde{O}_1 \times \widetilde{O}_2}$ on $\mu^{-1}(0)$. Following the formula (4.42), we have that at $(g_1, B_1, g_2, -B_1)$,

$$\begin{aligned} \omega_{\mu^{-1}(0)}(v_1, v_2) &= \omega_{\widetilde{O}_1}((0, \mathrm{Ad}_{g_1^{-1}} R_1), (0, \mathrm{Ad}_{g_1^{-1}} R_2)) \\ &\quad + \omega_{\widetilde{O}_2}((\mathrm{Ad}_{g_2^{-1}} X_1, -\mathrm{Ad}_{g_1^{-1}} R_1), (\mathrm{Ad}_{g_2^{-1}} X_2, -\mathrm{Ad}_{g_1^{-1}} R_2)) \\ &= \langle R_2, \mathrm{Ad}_{g_1 g_2^{-1}} X_1 \rangle - \langle R_1, \mathrm{Ad}_{g_1 g_2^{-1}} X_2 \rangle - \langle B_1, \mathrm{Ad}_{g_2^{-1}}([X_1, X_2]) \rangle \end{aligned} \tag{4.58}$$

On the other hand, a direct computation gives $\iota_*(v_i) = (\mathrm{Ad}_{g_1 g_2^{-1}} X_i, -R_i)$ at $(g_2 g_1^{-1}, Ad_{g_1} B_1)$, here $\mathfrak{g} \cong T_{g_2 g_1^{-1}} G$ via left multiplication. Formula (4.13) makes it transparent that at $(g_2 g_1^{-1}, -\mathrm{Ad}^*_{g_1} B_1) \in \Sigma$,

$$\begin{aligned} \omega(\iota_*(v_1), \iota_*(v_2)) &= \omega((\mathrm{Ad}_{g_2 g_1^{-1}} X_1, -R_1), (\mathrm{Ad}_{g_2 g_1^{-1}} X_2, -R_2)) \\ &= \langle R_2, \mathrm{Ad}_{g_1 g_2^{-1}} X_1 \rangle - \langle R_1, \mathrm{Ad}_{g_1 g_2^{-1}} X_2 \rangle - \langle B_1, Ad_{g_2^{-1}}([X_1, X_2]) \rangle \end{aligned} \tag{4.59}$$

Therefore, we have that $\iota^*\omega = \omega_{\mu^{-1}(0)}$, i.e., ι induces a symplectic isomorphism between $\widetilde{O}_1 \times \widetilde{O}_2 /\!\!/ G$ and (Σ, ω). ■

As for the Poisson Lie counterpart, we have

Proposition 5.15. *Let $\widetilde{\mathcal{C}}_1$ and $\widetilde{\mathcal{C}}_2$ be two copies of $\widetilde{\mathcal{C}}$ with $k=1$ and $k=2$ respectively. Then the quasi-Hamiltonian reduction of the fusion of $\widetilde{\mathcal{C}}_1$ and $\widetilde{\mathcal{C}}_2$ is isomorphic as a symplectic manifold to the symplectic submanifold (Σ', ω') of the double Γ.*

Proof. We assume that the Borels chosen at the first pole are opposite to those chosen at the second (which we may since isomonodromy will give symplectic isomorphisms with the spaces arising from any other choice of Borels intersecting in T). Thus we have,

$$\widetilde{\mathcal{C}}_1 = \{(h, e_-^{2\pi i\lambda}) \mid h \in G, \lambda \in \mathfrak{t}'\}, \qquad \widetilde{\mathcal{C}}_2 = \{(C, (b_-, b_+, \Lambda)) \mid \delta(b_\pm) = e^{\pm\pi i\Lambda}\}, \quad (4.60)$$

here $e_-^{2\pi i\lambda} = (e^{\pi i\lambda^\vee}, e^{-\pi i\lambda^\vee}, \lambda^\vee)$ (exponential map of $\mathfrak{g}^*$ with the opposite Borels chosen). The moment map on $\widetilde{\mathcal{C}}_1 \circledast \widetilde{\mathcal{C}}_2$ is $\mu = h^{-1}e^{-2\pi i\lambda^\vee}hC^{-1}b_-^{-1}b_+C$. Therefore the condition $\mu = 1$ becomes $Ce^{2\pi i \mathrm{Ad}_{h^{-1}}\lambda^\vee}C^{-1} = b_-^{-1}b_+$, where $B := \mathrm{Ad}_{h^{-1}}(\lambda^\vee)$. Recall that Σ' is a submanifold of Lu-Weinstein symplectic double Γ,

$$\Sigma' := \{(g_1, e_-^{2\pi i\lambda}, g_2, (b_-, b_+, \Lambda)) \in \Gamma \mid \delta(b_\pm) = e^{\pm\pi i\Lambda},\ g_1 e^{\pm\pi i\lambda^\vee} = b_\pm g_2\}. \quad (4.61)$$

We have a surjective map from $\mu^{-1}(1) = \{(h, e_-^{2\pi i\lambda}, C, (b_+, b_-, \Lambda)) \mid e^{2\pi i\mathrm{Ad}_{h^{-1}}\lambda^\vee} = Cb_-^{-1}b_+C^{-1}\}$ to Σ',

$$(h, e_-^{2\pi i\lambda}, C, (b_+, b_-, \Lambda)) \to (Ch^{-1}, e^{-2\pi i\lambda}, u, (b_-, b_+, \Lambda)) \quad (4.62)$$

whose fibres are precisely the G orbits, where $u := b_+^{-1}Ch^{-1}e^{\pi i\lambda} \in G$. Therefore, it induces an isomorphism from $\widetilde{\mathcal{C}}_1 \circledast \widetilde{\mathcal{C}}_2 /\!\!/ G$ to Σ'. An explicit formula for the symplectic structure on Σ' can be computed by using Theorem 3 of [4]. On the other hand we have an explicit formula for the symplectic structure on $\widetilde{\mathcal{C}}_1 \circledast \widetilde{\mathcal{C}}_2 /\!\!/ G$. A straightforward calculation shows these explicit formulae on each side agree. ■

To specify an irregular Riemann-Hilbert map, we have to make a choice of tentacles (see [17]). We introduce coordinate z to identify $\mathbb{P}^1$ with $\mathbb{C} \cap \infty$ and assume the divisor D has one pole of order two at $a_2 := 0$ and one pole of order one at $a_1 := \infty$. Then we consider the meromorphic connections ∇ on the trivial rank n holomorphic vector bundle V over $\mathbb{P}^1$ with compatible framings $\mathbf{g}$ such that $(V, \nabla, \mathbf{g})$ have an irregular type $-\frac{A_0}{z}$ at 0, where A_0 is a diagonal matrix with distinct elements. Let us take a prior Sect_0 between two anti-Stokes rays (only depend on A_0) at 0, and make a choice of tentacles as follows.

(i) A choice of a point p_2 in Sect_0 at 0 and a point p_1 in Sect_0 near ∞.

(ii) A lift $\hat{p_i}$ of each p_i to the universal cover of a punctured disc $D_i \backslash \{a_i\}$ containing p_i for $i = 1, 2$.

(iii) A base point p_0 which coincides with p_1.

(iv) A contractible path $\gamma : [0, 1] \to \mathbb{P}^1 \setminus \{0, \infty\}$ in the punctured sphere, from p_0 to p_1.

Note that the chosen point $\hat{p_2}$ determines a branch of $\log z$ on Sect_0. According to section 3, let $C \in \mathrm{Map}(\mathfrak{g}^*, G)$ be the connection matrix associated to the diagonal matrix A_0, the choice of Sect_0 and the branch of $\log z$. Then we have

Proposition 5.16. *For the above choice of tentacles, the corresponding irregular Riemann-Hilbert map* $\nu : (\widetilde{O}_1 \times \widetilde{O}_2)/\!\!/G \cong \Sigma \to (\widetilde{\mathcal{C}}_1 \times \widetilde{\mathcal{C}}_2)/\!\!/G \cong \Sigma'$ *is given by*

$$\nu(h, \lambda) = (C(\mathrm{Ad}^*_h \lambda)h, e^{2\pi i\lambda}, u, u^*), \ \forall (h, \lambda) \in \Sigma, \tag{4.63}$$

for certain $u \in G, u^* \in G^*$ *satisfying* $C(\mathrm{Ad}^*_h \lambda)he^{2\pi i\lambda} = u^* u$.

Proof. Let $(V, \nabla, \mathbf{g} = (g_1, g_2))$ be a compatibly framed meromorphic connection with irregular type $-\frac{A_0}{z}$ at a_2, where A_0 is a diagonal matrix with distinct elements. Let us choose a trivialization of V, then the compatible framing $\{g_i\}_{i=1,2}$ are represented by constant matrices (also denoted by $\{g_i\}_{i=1,2}$). We assume Φ_0, Φ_1 and Φ_2 are canonical fundamental solutions (see Definition 3.6 in [17]) of ∇ on a neighbourhood of $p_0 = p_1$ and p_2 with respect to the compatible framing 1, g_1 and g_2 respectively. Then the monodromy data of $(V, \nabla, \mathbf{g})$, corresponds to a point

$$(C_1, e_-^{2\pi i\lambda}, C_2, (b_-, b_+, \Lambda)) \in \widetilde{\mathcal{C}}_1 \times \widetilde{\mathcal{C}}_2 \cong G \times e_-^{\mathfrak{t}'} \times G \times G^*, \tag{4.64}$$

($e_-^{2\pi i\lambda} = (e^{\pi i\lambda^\vee}, e^{-\pi i\lambda^\vee}, \lambda^\vee)$ is the exponential map of $\mathfrak{g}^*$ with the opposite Borels chosen) is the set of constant $n \times n$ matrix C_i (the ratio of the canonical solutions Φ_i at p_i with Φ_0 at p_0 for $i = 1, 2$), as well as the Stokes data (b_-, b_+) at a_2 and the formal monodromy at a_1, a_2. They can be described as follows.

• along the path γ in the punctured sphere $\mathbb{P}^1 \setminus \{0, \infty\}$, we can extend the two corresponding bases of solutions Φ_0 and Φ_2, then $\Phi_2 P_1 C_2 = \Phi_0 \in G$, where P_1 is the permutation matrix (see section 3) depends on the choice of Sect_0 at a_2;

• p_0, p_1 can be seen as connected by an identity path, thus $\Phi_1 C_1 = \Phi_0$. Therefore C_1 is equal to g_1, the ratio of the frame chosen at p_0 and p_1;

• b_-, b_+ at a_2 can be determined by the monodromy relation from the fact that a simple positive loop around 0 is also a simple negative loop around ∞.

Explicitly, we assume $(V, \nabla, \mathbf{g})$ represents a point $(g_1, -B', g_2, A, B') \in \mu^{-1}(0) \subset \widetilde{O}_1 \times \widetilde{O}_2$, where μ is the moment map (see Proposition 5.14). To obtain the corresponding monodromy, we need to find the canonical fundamental solutions $\{\Phi_i\}_{i=0,2}$ of the meromorphic connection $\nabla = d - (\frac{A}{z^2} + \frac{B'}{z})dz$. By the assumption that ∇ has the irregular type $-\frac{A_0}{z}$, we get $g_2[\nabla] = d - (\frac{A_0}{z^2} + \frac{B}{z})dz$, where $B := \mathrm{Ad}_{g_2} B'$. According to section 3, let Φ_0 and H be the canonical fundamental solutions of $d - (\frac{A_0}{z^2} + \frac{B}{z})dz$ on a neighbourhood of $a_2 = 0$ and $a_0 = \infty$ respectively. Then one checks that $g_2^{-1} H g_2$ and $g_2^{-1}\Phi$ are the fundamental solutions of ∇ on a neighbourhood of p_0 and p_2 with respect to the local trivialisation of V given by 1 and g_2 respectively. From the uniqueness, we have that $\Phi_0 = g_2^{-1} H g_2$ and $\Phi_2 = g_2^{-1}\Phi$. Now by the definition of connection matrix, $C(B)$ (C_2) is the ratio of the two solutions Φ (Φ_0) and H (Φ_2), i.e., $HPC_2 = \Phi$ $(\Phi_2 P C_2 = \Phi_0)$, here P is the permutation matrix depends on A_0 and Sect_0 (see section 3). Then it is easy to see that $C_2 = C(B) g_2$. On the other hand, the formal monodromy $e^{2\pi i \lambda} = e^{2\pi i \mathrm{Ad}_{g_1} B'^{\vee}}$. Therefore, the corresponding map ν' which associates any $(V, \nabla, \mathbf{g})$ to its monodromy data is given by

$$\nu'(g_1, -B', g_2, B') = (g_1, e_-^{-2\pi i \mathrm{Ad}_{g_1} B'}, C(B) g_2, (b_-, b_+, \Lambda)) \in \mu'^{-1}(1) \subset \widetilde{\mathcal{C}}_1 \times \widetilde{\mathcal{C}}_2, \quad (4.65)$$

where $b_- \in B_-$, $b_+ \in B_+$ satisfy the identity (the moment map condition)

$$C(B) e^{2\pi i B} C(B)^{-1} = b_-^{-1} b_+. \quad (4.66)$$

This map $\nu' : \mu^{-1}(0) \subset \widetilde{O}_1 \times \widetilde{O}_2 \to \mu'^{-1}(0) \subset \widetilde{\mathcal{C}}_1 \times \widetilde{\mathcal{C}}_2$ is G-equivariant and descends to give the irregular Riemann-Hilbert map $\nu : \Sigma \to \Sigma'$ which takes the form

$$\nu(g_2 g_1^{-1}, \mathrm{Ad}_{g_1} B') = (C(B) g_2 g_1^{-1}, e^{2\pi i \mathrm{Ad}_{g_1} B'}, u, (b_-, b_+, \Lambda)), \quad (4.67)$$

here $u := b_+^{-1} C(B) g_2 e^{\pi i B'^{\vee}} g_1$ and we use the isomorphisms $\mu^{-1}(0)/G \cong \Sigma$ and $\mu'^{-1}(1)/G \cong \Sigma'$ constructed in Propostion 5.14 and 5.15 respectively. It indicates that the map ν is given by

$$\nu(h, \lambda) = (C(\mathrm{Ad}_h^* \lambda) h, e^{2\pi i \lambda}, u, u^*), \quad (4.68)$$

for any $(h, \lambda) \in \Sigma$ and certain $u \in G, u^* \in G^*$ satisfying $C(\mathrm{Ad}_h^* \lambda) h e^{2\pi i \lambda} = u^* u$. ■

The proof of Theorem 3.7. Following Proposition 5.16, the irregular Riemann-Hilbert map ν coincides with the local diffeomorphic map $F'_{C_{2\pi i}} : \Sigma \to \Sigma'$ defined in Section 2, where $C_{2\pi i}(x) = C(\frac{1}{2\pi i} x)$, for all $x \in \mathfrak{g}^*$. Thus, according to Corollary 2.3, $\nu : (\Sigma, \omega) \to (\Sigma', \omega')$ is a symplectic map provided the symplectic structure on the right-hand side is divided by $2\pi i$ if and only if $C_{2\pi i}$ is a solution of the gauge transformation equation (4.11). However, the former is guaranteed by Theorem

5.13. As a result, we get a proof of Theorem 3.7, i.e., $C_{2\pi i}$ is a solution of the gauge transformation equation (4.11).

The T-reduction version. Let T be the set of diagonal matrices. The irregular Riemann-Hilbert map $\nu = F'_{C_{2\pi i}} : \Sigma \to \Sigma'$ is equivariant with respect to the T-actions on Σ and Σ' given by

$$a \cdot (h, \lambda) = (ha, \lambda), \qquad a \cdot (h, e^{\lambda}, u, u^*) = (ha, e^{\lambda}, u, u^*), \tag{4.69}$$

for any $a \in T$. Thus we can consider a T reduction version of F'_C. Define two maps $P : \Sigma \to \mathfrak{g}^*$, $P' : \Sigma' \to G^*$ whose fibres are the T orbit as follows

$$P(h, \lambda) = \mathrm{Ad}_h^* \lambda,\ \forall (h, \lambda) \in \Sigma, \qquad P'(h, e^{\lambda}, u, u^*) = d_h e^{\lambda},\ \forall (h, e^{\lambda}, u, u^*) \in \Sigma'. \tag{4.70}$$

Here d denotes the left dressing transformation of G on G^*. Then there exists a unique map $S_C : \mathfrak{g}^* \to G^*$ such that the following diagram commutes:

$$\begin{array}{ccc} \Sigma & \xrightarrow{F'_{C_{2\pi i}}} & \Sigma' \\ {\scriptstyle P_1}\big\downarrow & & {\scriptstyle P_2}\big\downarrow \\ \mathfrak{g}^* & \xrightarrow{S_C} & G^* \end{array}$$

This map S_C naturally appears in the theory of Stokes matrix. Actually, given the meromorphic connection $\nabla = d - (\frac{A_0}{z^2} + \frac{B}{z})dz$ on a trivial holomorphic vector bundle, the Stokes matrices of ∇ are $S_+ = e^{-\pi i \delta(B)} b_+(B) e^{2\pi i \delta(B)}$ and $S_- = b_-^{-1}(B) e^{-\pi i \delta(B)}$ (see e.g [16]). Thus the Stokes matrices are uniquely determined by the connection matrix $C(B)$ of ∇ through the monodromy relation (from the fact that a simple positive loop around 0 is also a simple negative loop around ∞)

$$C(B) e^{2\pi i B^{\vee}} C(B)^{-1} = S_- S_+ e^{2\pi i \delta(B)}. \tag{4.71}$$

The main result of [16] shows that the monodromy map $S_C : \mathfrak{g}^* \to G^*$ relating $B \in \mathfrak{g}^*$ to the monodromy data $(b_-(B), b_+(B)) \in G^*$ at 0 of ∇, is a Poisson map. Here we use an extended moduli space version (see Definition 2.6 in [17]) of the one used in [16] to get a generalization, a symplectic neighborhood version, of the linearization of Poisson structures in [49]. Geometrically, from the above discussion one can view the Poisson monodromy map $S_C : \mathfrak{g}^* \to G^*$ as a T-reduction version of the symplectic map $F'_{C_{2\pi i}} : \Sigma \to \Sigma'$.

6 Lu-Weinstein symplectic groupoids via Alekseev-Meinrenken r-matrices

As an application of the above construction, we describe the Lu-Weinstein symplectic groupoids via Alekseev-Meinrenken r-matrices.

Following the construction of section 2.1 in [86], for any r-matrix with the skew-symmetric part r_0,

$$\pi_{\mathrm{AM}}(h,x) := \pi_{\mathrm{KKS}}(x) + l_h(\theta) + l_h(r_{AM}(x)) - r_h(r_0), \tag{4.72}$$

defines a symplectic structure on $G \times \mathfrak{g}^*$, where $\theta := \frac{\partial}{\partial x^a} \wedge e_a \in \Gamma(\wedge^2(T\mathfrak{g}^* \oplus \mathfrak{g}))$ for a base $\{e_a\}$ of $\mathfrak{g}$ and the corresponding coordinates $\{x^a\}$ on $\mathfrak{g}^*$. We will prove that $(G \times \mathfrak{g}^*, \pi_{AM})$ is a natural symplectic groupoid and is locally symplectic isomorphic to Lu-Weinstein double sympelctic groupoid with respect to $(\mathfrak{g}, r)$.

To do this, let us consider the Semenov-Tian-Shansky (STS) Poisson tensor on $\mathfrak{g}^*$ defined by

$$\pi_{\mathrm{STS}}(x)(df,dg) = \langle df(x) \otimes dg(x), \mathrm{ad}_x \otimes \frac{1}{2}\mathrm{ad}_x \mathrm{coth}(\frac{1}{2}\mathrm{ad}_x)(t) - \otimes^2 \mathrm{ad}_x(r_0)\rangle, \tag{4.73}$$

for any $f, g \in C^\infty(\mathfrak{g}^*)$. We denote by L, R the group morphisms corresponding to the Lie algebra morphisms $L, R : \mathfrak{g}^* \to \mathfrak{g}$

$$L(x) := (x \otimes \mathrm{id})(r), \qquad R(x) := -(x \otimes \mathrm{id})(r^{2,1}) \qquad \forall\, x \in \mathfrak{g}^*. \tag{4.74}$$

Let (G^*, π_{G^*}) be the simply connected Poisson Lie group associated to the quasitriangular Lie bialgebra $(\mathfrak{g}, r)$.

Proposition 6.1. [38] The map $I : (\mathfrak{g}^*, \pi_{STS}) \to (G^*, \pi_{G^*})$ determined by $e^{x^\vee} = L(I(x))^{-1}R(I(x))$ for any $x \in \mathfrak{g}^*$, is a Poisson map.

Actually, the STS Poisson structure on $\mathfrak{g}^*$ is completely determined by the above proposition. Now let us take any solution $g \in \mathrm{Map}(\mathfrak{g}^*, G)$ of the gauge transformation equation (4.11).

Theorem 6.2. $(G \times \mathfrak{g}^*, \pi_{\mathrm{AM}})$ *is a Poisson(therefore symplectic) groupoid over* $\mathfrak{g}^*$ *with the structure maps given by*

$$\begin{aligned} &\alpha(h,x) \to x \in \mathfrak{g}^*,\ \beta(h,x) \to Ad^*_h x \in \mathfrak{g}^*,\ \forall h \in G, x \in \mathfrak{g}^*,\\ &\varepsilon : \mathfrak{g}^* \to G \times \mathfrak{g}^* :\ x \mapsto (g(x), Ad^*_{g(x)}x) \in G \times \mathfrak{g}^*,\\ &m : \mathcal{G}_2 := \{((h_1,x),(h_2,y)) \in \mathcal{G} \times \mathcal{G} \mid \beta(h_1) = \alpha(h_2)\} \to \mathcal{G} :\\ &(h_1,x),(h_2, y = Ad^*_{h_1}x) \mapsto Ad^*_{h_1 h_2}x,\ \forall (h_1,h_2) \in \mathcal{G}_2. \end{aligned} \tag{4.75}$$

Furthermore, it is the integration of the STS Poisson structure on $\mathfrak{g}^*$.

A straightforward proof can be obtained verifying the following equivalent conditions one by one.

Lemma 6.3. ([83]) ($\mathcal{G} \Longrightarrow M, \pi$) is a Poisson groupoid if and only if all the following hold.

(1) For all $(x, y) \in \mathcal{G}_2$,

$$\pi(xy) = R_Y\pi(x) + L_X\pi(y) - R_Y L_X \pi(m), \tag{4.76}$$

where $m = \beta(x) = \alpha(y)$ and X, Y are (local) bisections through x and y respectively.

(2) M is a coisotropic submanifold of $\mathcal{G}$,

(3) For all $x \in \mathcal{G}$, $\alpha_*\pi(x)$ and $\beta_*\pi(x)$ only depend on the base points $\alpha(x)$ and $\beta(y)$ respectively.

(4) For all $f, g \in C^\infty(M)$, one has $\{\alpha^* f, \beta^* g\} = 0$,

(5) The vector field $X_{\beta^* f}$ is left invariant for all $f \in C^\infty(M)$.

Theorem 6.4. *The map $v : G \times \mathfrak{g}^* \to G \times G^*$, $v(h, x) = (g(x)h, g^*(x))$ for all $(h, x) \in G \times \mathfrak{g}^*$, gives a local symplectic isomorphism from $(G \times \mathfrak{g}^*, \pi_{\mathrm{AM}})$ to Lu-Weinstein symplectic double Γ, where $g^* : \mathfrak{g}^* \to G^*$ is the unique local isomorphism defined by the identity*

$$g(x) e^{x^\vee} g(x)^{-1} = L(g^*(x)) R(g^*(x))^{-1}. \tag{4.77}$$

Actually, under the transformation $F : G \times \mathfrak{g}^* \to G \times \mathfrak{g}^*$ given by $F(h, x) = (hg^{-1}(x), Ad^*_{g(x)} x)$, then the Poisson tensor π_{AM} becomes

$$F_*(\pi_{\mathrm{AM}})(h, x) = \pi_{\mathrm{STS}}(x) + l_h(\theta^g(x)) + l_h(r_0) - r_h(r_0), \ \forall (h, x) \in G \times \mathfrak{g}^* \tag{4.78}$$

where $\theta^g \in \Gamma(\wedge^2(T\mathfrak{g}^* \oplus \mathfrak{g}))$ is the gauge transformation of θ under $g \in \mathrm{Map}(\mathfrak{g}^*, G)$ (see [86] for more details about a generalized dynamical r-matrix and its gauge transformations). According to [86], we have that $(\pi_{\mathrm{STS}}, \theta^g, r_0)$ is a gauge transformation of the dynamical r-matrix $(\pi_{\mathrm{KKS}}, \theta, r_{\mathrm{AM}})$ under $g \in \mathrm{Map}(\mathfrak{g}^*, G)$, thus also a generalized classical dynamical r-matrix. It therefore verifies that equation (4.11) is indeed a gauge transformation equation in the theory of generalized dynamical r-matrices.

7 Proof of Theorem 2.2

In this section, we will study in details the symplectic submanifold Σ' of Lu-Weinstein symplectic double groupoid Γ and then give a proof of Theorem 2.2. According Section 2, associated to a quasitriangular Lie bialgebra $(\mathfrak{g}, r)$, (Γ, π_Γ) is the set

$$\Gamma = \{(h, h^*, u, u^*) \mid h, u \in G, h^*, u^* \in G^*, hh^* = u^* u\} \tag{4.79}$$

with the unique Poisson structure π_Γ such that the local diffeomorphism $(\Gamma, \pi_\Gamma) \to (D, \pi_D)$: $(h, h^*, u, u^*) \mapsto hh^*$ is Poisson (D is the double of $(\mathfrak{g}, r)$)). Then the submanifold Σ' takes the form

$$\Sigma' = \{(h, h^*, u, u^*) \in \Gamma \mid h^* \in e^{\mathfrak{t}'} \subset G^*\}. \tag{4.80}$$

Proposition 7.1. *Σ' is a symplectic submanifold of the Lu-Weinstein symplectic double (Γ, π_Γ).*

Proof. An explicit formula for the restriction of symplectic 2-form on $\Sigma' \in \Gamma$ can be computed by using Theorem 3 of [3]. One checks directly that it is symplectic. ∎

Thus Σ' inherits a symplectic structure. We denote by π' the corresponding Poisson tensor. Note that the inclusion map $(\Sigma', \pi') \hookrightarrow (\Gamma, \pi_\Gamma)$ and the dressing transformation map $(\Gamma, \pi_\Gamma) \to (G^*, \pi_{G^*})$; $(h, h^*, u, u^*) \mapsto d_h(h^*)$ are Poisson, so is their composition. Thus we have

Proposition 7.2. *The map*

$$P' : (\Sigma', \pi') \to (G^*, \pi_{G^*});\ (h, e^\lambda, u, u^*) \mapsto d_h e^\lambda \tag{4.81}$$

is a Poisson map.

To simplify the notation, we take a local model of (Σ', π') as follows. Note that we have the local diffeomorphism

$$\Sigma' \to G \times e^{\mathfrak{t}'};\ (h, e^\lambda, u, u^*) \mapsto (h, e^\lambda). \tag{4.82}$$

To simplify the notation, we will take $G \times e^{\mathfrak{t}'}$ as a local model of (Σ', π) with the induced Poisson tensor, denoted also by π'. Generally, π' is only defined on a dense subset of $G \times e^{\mathfrak{t}'}$, however this is enough for our purpose.

Let T act on $G \times e^{\mathfrak{t}'}$ by $t \cdot (h, e^\lambda) = (ht, e^\lambda)$. The fibres of the map $P' : G \times \mathfrak{t}' \to G^*$, $(h, e^\lambda) \mapsto d_h(e^\lambda)$ are precisely the T-orbits. Thus a general 1-form on $G \times e^{\mathfrak{t}'}$ takes the form $P'^*(\beta) + \hat{\eta}$, where $\beta \in \Omega^1(G^*)$, $\eta \in \mathfrak{t}^* \subset \mathfrak{g}^*$ (via inner product) and at each point (h, e^λ), $\hat{\eta} := (l_{h^{-1}} \circ r_{e^{-\lambda}})^* \eta$.

Proposition 7.3. *At each point (h, e^λ), π' is given for any forms $\phi_1 := P'^*(\beta_1) + \hat{\eta}_1$, $\phi_2 := P'^*(\beta_2) + \hat{\eta}_2$ by*

$$\begin{aligned} \pi'(h, e^\lambda)(\phi_1, \phi_2) &= \pi_{G^*}(d_h e^\lambda)(\beta_1, \beta_2) + \langle X_1, \eta_2 \rangle - \langle X_2, \eta_1 \rangle \\ &+ (l_{h^{-1}} \pi_G(h))(\eta_1, \xi_2) - (l_{h^{-1}} \pi_G(h))(\eta_2, \xi_1) + (l_{h^{-1}} \pi_G(h))(\eta_1, \eta_2) \end{aligned} \tag{4.83}$$

where $\xi_i + X_i \in \mathfrak{g}^* \otimes \mathfrak{g}$ is the pull back of $P'^*(\beta_i)$ under $l_{h^{-1}} \circ r_{e^{-\lambda}}$ for $i = 1, 2$.

Proof. Following [63], if $m \in D$ (the double Lie group) can be factored as $m = hu$ for some $h \in G$ and $u \in G^*$ (locally it is always the case), then explicit formula for π_D is given by

$$\begin{aligned} & ((l_{h^{-1}} \circ r_{u^{-1}})(\pi_D))(m)(\xi_1 + X_1, \xi_2 + X_2) \\ = & \langle X_1, \xi_2 \rangle - \langle X_2, \xi_1 \rangle + (l_{h^{-1}} \pi_G(h))(\xi_1, \xi_2) + (r_{u^{-1}} \pi_{G^*}(u))(X_1, X_2) \end{aligned} \quad (4.84)$$

for $\xi_1 + X_1, \xi_2 + X_2 \in \mathfrak{g}^* \oplus \mathfrak{g}$.

On one hand, Proposition 7.2 gives that

$$\pi'(P'^*(\beta_1), P'^*(\beta_2)) = \pi_{G^*}(\beta_1, \beta_2), \quad (4.85)$$

for any $\beta_1, \beta_2 \in \Omega^1(G^*)$.

On the other hand, let us consider the one form taking the form of $\hat{\eta} := l^*_{h^{-1}}(r^*_{e^{-\lambda}}\eta)$, $\eta \in \mathfrak{t}^* \subset \mathfrak{g}^*$. From the expression of π_D, we see that $\pi^\sharp_D(he^\lambda)(\hat{\eta})$ is tangent to $G \times e^{\mathfrak{t}'}$ at (h, e^λ). Thus

$$\begin{aligned} \pi'(\hat{\eta}_1 + P'^*(\beta_1), \hat{\eta}_2) &= \pi_D(\hat{\eta}_1 + d^*(\beta_1), \hat{\eta}_2)|_{G \times e^{\mathfrak{t}'}} \\ &= \langle X_1, \eta_2 \rangle + l_{h^{-1}} \pi_G(\xi_1, \eta_2) + l_{h^{-1}} \pi_G(\eta_1, \eta_2) \end{aligned} \quad (4.86)$$

where $\xi_1 + X_1 \in \mathfrak{g}^* \otimes \mathfrak{g}$ is the pull back of $P'^*(\beta_1)$ under $l_{h^{-1}} \circ r_{e^{-\lambda}}$. The above two identities indicate the expression (4.83) of π'. ■

In the following, we will give a description of the Poisson space $(G \times e^{\mathfrak{t}'}, \pi')$ by using r-matrices. Let us define a bivector field on $G \times \mathfrak{t}'$ which at each point $(h, \lambda) \in G \times \mathfrak{t}'$ takes the form

$$\pi_r(h, \lambda) = l_h(t_i) \wedge \frac{\partial}{\partial t^i} + l_h((\mathrm{id} \otimes \mathrm{ad}^{-1}_{\lambda^\vee})(t)) + l_h(r_{\mathrm{AM}}(\lambda)) - r_h(r_0) \quad (4.87)$$

where $t \in S^2(\mathfrak{g})^{\mathfrak{g}}$ is the Casimir element, $\{t_i\}$ is a basis of $\mathfrak{t}$ and $\{t^i\}$ the corresponding coordinates on $\mathfrak{t}^*$ and at any point $x \in \mathfrak{g}$, $\mathrm{ad}^{-1}_x : \mathfrak{g} \to \mathfrak{g}$ is the trivial extension of the map $\mathrm{ad}^{-1}_x : \mathfrak{g}^\perp_x \to \mathfrak{g}^\perp_p \subset \mathfrak{g}$ corresponding to the decomposition $\mathfrak{g} = \mathfrak{g}_x \oplus \mathfrak{g}^\perp_x$. Here $\mathfrak{g}_x$ is the isotropic subalgebra of $\mathfrak{g}$ at x and $\mathfrak{g}^\perp_x$ its complement with respect to the inner product. By using the dynamical Yang-Baxter equation of r_{AM}, one can show that π_r is a Poisson tensor.

Proposition 7.4. *The image of π_r under the diffeomorphism $\Psi : G \times \mathfrak{t}' \to G \times e^{\mathfrak{t}'}$, $(h, \lambda) \mapsto (h, e^\lambda)$ coincides with π'.*

Proof. Recall that (from the discussion above Proposition 7.3) a general 1-form on $G \times e^{\mathfrak{t}'}$ takes the form $P'^*(\beta) + \hat{\eta}$, where $\beta \in \Omega^(G^*)$ and $\eta \in \mathfrak{t}^* \subset \mathfrak{g}^*$. We will prove that $\Psi_*(\pi_r)(\hat{\eta},\cdot) = \pi'(\hat{\eta},\cdot)$ and $\Psi_*(\pi_r)(P'^*(\beta),\cdot) = \pi'(P'^*(\beta),\cdot)$ respectively. First note that at each point (h,λ) (by the equivariance of r_{AM})

$$l_h(r_{\mathrm{AM}}(\lambda) + (\mathrm{id}\otimes \mathrm{ad}^{-1}_{\lambda^\vee})(t)) = r_h(r_{\mathrm{AM}}(x) + (\mathrm{id}\otimes \mathrm{ad}^{-1}_{x^\vee})(t)) = r_h((\mathrm{id}\otimes \coth(\frac{1}{2}\mathrm{ad}_{x^\vee})(t)) \tag{4.88}$$

where $x = \mathrm{Ad}^*_h\lambda \in \mathfrak{g}^*$. By the definition of the map P, a direct calculation gives that

$$\pi_r(h,\lambda)(P^*(\alpha_1), P^*(\alpha_2)) = (\mathrm{ad}^*_{x^\vee}\otimes \frac{1}{2}\mathrm{ad}^*_{x^\vee}\coth(\frac{1}{2}\mathrm{ad}^*_{x^\vee})(t) - \otimes^2\mathrm{ad}^*_{x^\vee}(r_0)(\alpha_1,\alpha_2), \tag{4.89}$$

where $(h,\lambda)\in G\times\mathfrak{t}'$ and $x = \mathrm{Ad}^*_h\lambda$. In other words,

$$\pi_r(h,\lambda)(P^*(\alpha_1), P^*(\alpha_2)) = \pi_{\mathrm{STS}}(x)(\alpha_1,\alpha_2). \tag{4.90}$$

On the other hand, we have the following commutative diagram

$$\begin{array}{ccc} G\times\mathfrak{t}' & \xrightarrow{\Psi} & G\times e^{\mathfrak{t}'} \\ {\scriptstyle P}\downarrow & & {\scriptstyle P'}\downarrow \\ \mathfrak{g}^* & \xrightarrow{I} & G^* \end{array},$$

where $I:(\mathfrak{g}^*,\pi_{\mathrm{STS}})\to(G^*,\pi_{G^*})$ is the local Poisson isomorphism defined in Section 6. Thus $P^*(I^*(\beta_i)) = \Psi^*(P'^*(\beta_i))$ for any $\beta_i\in\Omega^1(G^*)$, $i=1,2$. Therefore,

$$\begin{aligned} \Psi_*\pi_r(P'^*(\beta_1),P'^*(\beta_2)) &= \pi_r(P^*(I^*(\beta_1)),P^*(I^*\beta_2)) = \pi_{\mathrm{STS}}(I^*(\beta_1),I^*(\beta_2)), \\ \pi'(P'^*(\beta_1),P'^*(\beta_2)) &= \pi_{G^*}(\beta_1,\beta_2). \end{aligned} \tag{4.91}$$

Combining with Proposition 6.1 which says $\pi_r(I^*(\beta_1),I^*(\beta_2)) = \pi_{G^*}(\beta_1,\beta_2)$, we have that

$$\Psi_*\pi_r(P'^*(\beta_1),P'^*(\beta_2)) = \pi'(P'^*(\beta_1),P'^*(\beta_2)). \tag{4.92}$$

For the remaining part, by the definition of the diffeomorphism Ψ and the expression of $\pi_G = l_h(r_0) - r_h(r_0)$, one can easily get that

$$\Psi_*(\pi_r)(P'^*(\beta_1)+\hat{\eta}_1,\hat{\eta}) = \langle X_1,\eta\rangle + l_{h^{-1}}\pi_G(\xi_1,\eta) + l_{h^{-1}}\pi_G(\eta_1,\eta), \tag{4.93}$$

where $\xi_1 + X_1\in\mathfrak{g}^*\otimes\mathfrak{g}$ is the pull back of $P'^*(\beta_1)$ under $l_{h^{-1}}\circ r_{e^{-\lambda}}$ and $\eta,\eta_1\in\mathfrak{t}'$. By comparing with the expression of π', we have that $\Psi_*(\pi_r)(\hat{\eta},\cdot) = \pi'(\hat{\eta},\cdot)$ for any $\eta\in\mathfrak{t}^*$.

Eventually, we prove that $\Psi_*(\pi_r)(P'^*(\beta)+\hat{\eta},\cdot) = \pi'(P'^*(\beta)+\hat{\eta},\cdot)$ for any $\beta\in\Omega^1(G^*)$, $\eta\in\mathfrak{t}^*\subset\mathfrak{g}^*$. That is, the image of π_r under the diffeomorphism

Ψ coincides with π'. ■

In other words, we have a local symplectic isomorphism (denoted by same symbol) $\Psi : (G \times \mathfrak{t}', \pi_r) \to \Sigma'$, $(h, \lambda) \mapsto (h, e^\lambda, u, u^*)$, where $u \in G, u^* \in G^*$ are unique determined by the identity $he^\lambda = u^* u$.

For the Poisson tensor π corresponding to the symplectic form ω on $G \times \mathfrak{t}'$, we have

Proposition 7.5. *The Poisson tensor π takes the form*

$$\pi(h, \lambda) = l_h(t_j) \wedge \frac{\partial}{\partial t^j} + l_h(\mathrm{id} \otimes (\mathrm{ad}_{\lambda^\vee}^{-1})(t)) \tag{4.94}$$

where $\{t_j\}$ is a basis of $\mathfrak{t}$ and $\{t^j\}$ the corresponding coordinates on $\mathfrak{t}^*$.

After these preliminary work, we can give a proof of Theorem 2.2. Following Proposition 7.4, $(G \times \mathfrak{t}^*_{\mathrm{reg}}, \pi_r)$ is locally isomorphic to (Σ', π'). Therefore we can take $(G \times \mathfrak{t}', \pi_r)$ as a local model of (Σ', π') and then the map defined by (4.18) becomes $F_g : \Sigma = G \times \mathfrak{t}' \to G \times \mathfrak{t}'$, $(h, \lambda) \mapsto (g(x)h, \lambda)$, where $x = \mathrm{Ad}_h^* \lambda$. Theorem 2.2 is thus equivalent to that

Theorem 7.6. *$F_g : (G \times \mathfrak{t}', \pi) \to (G \times \mathfrak{t}', \pi_r)$ is a Poisson map if and only if $g \in \mathrm{Map}(\mathfrak{g}^*, G)$ satisfies the gauge transformation equation (4.11), $r_0^g = r_{\mathrm{AM}}$.*

Proof. We only need to show that $F_{g*}\pi = \pi_r$ is equivalent to the equation $r_0^g = r_{\mathrm{AM}}$. By comparing the expressions of π and π_r, we have

$$\pi_r(h, \lambda) = \pi(h, \lambda) + l_h(r_{\mathrm{AM}}(\lambda) - \otimes^2 \mathrm{Ad}_{h^{-1}}(r_0)). \tag{4.95}$$

At any point $(h, \lambda) \in G \times \mathfrak{t}'$, $F_g(h, \lambda) = (g(x)h, \lambda)$ where $x := \mathrm{Ad}_h^* \lambda \in \mathfrak{g}^*$. We take $\{e^i\}$, $\{e_i\}$ as dual bases of $\mathfrak{g}^*$, $\mathfrak{g}$ and $\{t^j\}$, $\{t_j\}$ dual bases of $\mathfrak{t}^*$ and $\mathfrak{t}$. A straightforward calculation gives that at each point $(g(x)h, \lambda) \in G \times \mathfrak{t}^*_{\mathrm{reg}}$

$$\begin{aligned} F_{g*}(l_h(e_i)) &= l_{gh}(e_i) + l_{gh}(h^{-1} g^{-1} \frac{\partial g}{\partial X^i} h), & (4.96)\\ F_{g*}(\frac{\partial}{\partial t_j}) &= \frac{\partial}{\partial t_j} + l_{gh}(h^{-1} g^{-1} \frac{\partial g}{\partial T^j} h) & (4.97) \end{aligned}$$

where $X^i := [\mathrm{Ad}_h e_i, x]$, $T^j := \mathrm{Ad}_h^* t^j$ are tangent vectors at $x = \mathrm{Ad}_h^* \lambda$. Note that $T^m \in \mathfrak{g}_x$ (the isotropic subalgebra at x) and X^i span the tangent space $T_x \mathfrak{g}^*$ and thus the above formulas involve all the possible derivative of $g \in \mathrm{Map}(\mathfrak{g}^*, G)$. A direct computation shows that at each point $(g(x)h, \lambda) \in G \times \mathfrak{t}'$ (here $x = \mathrm{Ad}_h^* \lambda \in \mathfrak{g}^*$)

$$F_{g*}(\pi)(g(x)h, \lambda) = \pi(g(x)h, \lambda) + l_{gh}(\otimes^2 \mathrm{Ad}_{h^{-1}} U(x)), \tag{4.98}$$

where $U(x) \in \mathfrak{g} \wedge \mathfrak{g}$ is defined, by using the notation in Theorem 2.1, as

$$U(x) = g_1^{-1} d_2(g_1) - g_2^{-1} d_1(g_2) + \langle \mathrm{id} \otimes \mathrm{id} \otimes x, [g_1^{-1} d_3(g_1), g_2^{-1} d_3(g_2)] \rangle. \quad (4.99)$$

Thus by comparing with the expression of π_r,

$$\pi_r(g(x)h, \lambda) = \pi(g(x)h, \lambda) + l_{gh}(r_{AM}(\lambda) - \otimes^2 \mathrm{Ad}_{(gh)^{-1}} r_0), \quad (4.100)$$

we obtain that $F_{g*}(\pi) = \pi_r$ at point $(g(x)h, \lambda) \in G \times \mathfrak{t}'$ if and only if

$$r_{AM}(\lambda) = \otimes^2 \mathrm{Ad}_{(gh)^{-1}} r_0 + \otimes^2 Ad_{h^{-1}} U(x). \quad (4.101)$$

Note that $x = \mathrm{Ad}_h^* \lambda$, by the equivariance of r_{AM}, we have $\otimes^2 \mathrm{Ad}_h r_{AM}(\lambda) = r_{AM}(x)$. Thus the above formula is exactly the gauge transformation equation $r_0^g = r_{\mathrm{AM}}$. ■

Chapter 5

Stokes phenomena, Poisson–Lie groups and quantum groups

1 Introduction

This chapter is part of the joint work with Toledano Laredo [79].

Let $\mathfrak{g}$ be a complex, semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, and $b_\pm \subset \mathfrak{g}$ a pair of opposite Borel subalgebras intersecting along $\mathfrak{h}$. Let G be the simply-connected Poisson-Lie group corresponding to $(\mathfrak{g}, r)$, and $G* = B_- \times_H B_+$ its dual. We have seen from last chapter that G–valued Stokes phenomenon were used in [16] and [85], to give a canonical, analytic linearisation of the Poisson–Lie group structure on G^* and its symplectic neighbourhood in respectively. On the other hand, in a recent paper of Toledano Laredo [78], the Stokes phenomenon of the dynamical KZ equations (introduced by Felder, Markov, Tarasov and Varchenko in [41]) was used to construct a Drinfeld twist killing the KZ associator, and therefore give an explicit transcendental construction of the Drinfeld-Jimbo quantum group $U_h(\mathfrak{g})$. Because a Poisson-Lie group can be viewed as a classical limit of a quantum group, thus the Stokes phenomenon used in [16] and [78] should relate to each other in a similar way.

In this chapter, we show that the quantization problem (quantization of Poisson Lie groups to Quantum groups) can be studied in the frame of the deformation of certain irregular Riemann-Hilbert problem (meromorphic ODE). The main contributions are

• we prove that the quantum Stokes matrix for the dynamical KZ equations satisfies Yang-Baxter equation, therefore is a quantum R-matrix.

• we observe that the classical limit of the dynamical KZ equation is the meromorphic differential equation (1.2). Along the way, we show that the (quantum)

Stokes matrices relating the fusion operators of $\mathfrak{g}$ constructed in [78] have as classical limit the Stokes matrices map $\mathfrak{g}^* \to G^*$ constructed in [16]. In a similar way, we show that the semiclassical limit of the differential twist(s) constructed in [78] is the connection map(s) $\mathfrak{g}^* \to G$ studied in [85]. We give an interpretation of the appearance of Poisson geometry in the study of Stokes phenomenon from the perspective of quantization of Lie bialgebras.

• We prove that the dependence of quantum Stokes matrices on regular elements $A_0 \in \mathfrak{t}_{\mathrm{reg}}$ is described by a differential equation (a quantum isomonodromic deformation equation). Hamiltonian description of isomonodromic deformation equations [55][18] is then proven to be the classical limit of this equation.

2 Stokes phenomena and Poisson–Lie groups

2.1 G–valued Stokes phenomena

Let G be a complex, reductive group, $H \subset G$ a maximal torus, and $\mathfrak{h} \subset \mathfrak{g}$ the Lie algebras of H and G respectively. Let $\Phi \subset \mathfrak{h}^*$ be the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$, and $\mathfrak{h}_{\mathrm{reg}} = \mathfrak{h} \setminus \bigcup_{\alpha\in\Phi} \mathrm{Ker}\,\alpha$ the set of regular elements in $\mathfrak{h}$.

Let $\mathcal{P}$ be the holomorphically trivial, principal G–bundle on $\mathbb{P}^1$, and consider the meromorphic connection on $\mathcal{P}$ given by

$$\nabla = d - \left(\frac{A_0}{z^2} + \frac{B}{z}\right) dz. \tag{5.1}$$

where $A_0, B \in \mathfrak{g}$. We assume henceforth that $A_0 \in \mathfrak{h}_{\mathrm{reg}}$. The *Stokes rays* of the connection ∇ are the rays $\mathbb{R}_{>0} \cdot \alpha(A_0) \subset \mathbb{C}^*$, $\alpha \in \Phi$. The *Stokes sectors* are the open regions of $\mathbb{C}^*$ bounded by them. A ray r is called *admissible* if it is not a Stokes ray.

To each admissible ray r, there is a canonical fundamental solution Υ_r of ∇ with prescribed asymptotics in the half–plane

$$\mathbb{H}_r = \left\{ue^{\iota\phi} |\, u \in r, \phi \in (-\pi/2, \pi/2)\right\}$$

Specifically, the following result is proved in [14] for $G = GL_n(\mathbb{C})$, in [18] for G reductive, and in [22] for an arbitrary affine algebraic group.[1] Denote by $[B]$ the projection of B onto $\mathfrak{h}$ corresponding to the root space decomposition $\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$.

Theorem 2.1. *Let $r = \mathbb{R}e^{\iota\theta}$ be an admissible ray. Then, there is a unique holomorphic function $H_r : \mathbb{H}_r \to G$ such that*

[1]We use the formulation of [22], which does not rely on formal power series solutions.

1. *H_r tends to 1 as $z \to 0$ in any closed sector of $\mathbb{H}_r$ of the form*

$$|\arg(z \cdot e^{-\iota\theta})| \leq \frac{\pi}{2} - \delta, \qquad \delta > 0$$

2. *for any determination of $\log(z)$, the function*

$$\Upsilon_r = H_r \cdot e^{-A_0/z} \cdot z^{[B]}$$

satisfies $\nabla\Upsilon_r = 0$.

The canonical solutions Υ_r are locally constant with respect to the choice of r, so long as r does not cross a Stokes ray. Indeed, if r, r' are two admissible rays such that $r \neq -r'$, the corresponding solutions are related by $\Upsilon_{r'} = \Upsilon_r \cdot S$ on $\mathbb{H}_r \cap \mathbb{H}_{r'}$, where S is an element of G. The asymptotic behaviour of $\Upsilon_r, \Upsilon_{r'}$ implies that

$$z^{[B]} \cdot e^{-A_0/z} \cdot S \cdot e^{A_0/z} \cdot z^{-[B]} \to 1 \quad \text{as} \quad z \to 0 \text{ in } \mathbb{H}_r \cap \mathbb{H}_{r'} \tag{5.2}$$

and the result is a consequence of the following [18].

Proposition 2.2. *If $S \in G$ is such that (5.2) holds, then $S = \exp(X)$, where X lies in the nilpotent subalgebra*

$$\bigoplus_{\substack{\alpha\in\Phi: \\ \alpha(A_0)\in\Sigma(r,r')}} \mathfrak{g}_\alpha \subset \mathfrak{g}$$

where $\Sigma(r, r') \subset \mathbb{C}^*$ is the closed convex sector bounded by r and r'.

It follows in particular that Υ_r only depends on the Stokes sector Σ containing r, and will henceforth be denoted by Υ_Σ.

2.2 Stokes matrices and linearisation of G^*

Fix a Stokes sector Σ with boundary rays ℓ, ℓ', listed in counterclockwise order, and choose the determination of $\log(z)$ with a cut along ℓ. The *Stokes matrices* of ∇ relative to Σ are the elements $S_\pm$ of G determined by

$$\Upsilon_\Sigma = \Upsilon_{-\Sigma} \cdot S_+ \qquad \text{and} \qquad \Upsilon_{-\Sigma} \cdot e^{-2\pi\iota[B]} = \Upsilon_\Sigma \cdot S_-$$

where the first (resp. second) identity is understood to hold in $-\Sigma$ (resp. Σ) after Υ_Σ (resp. $\Upsilon_{-\Sigma}$) has been analytically continued counterclockwise.

The Stokes sector Σ determines a partition of the root system of $\mathfrak{g}$ as follows. Let $\mathcal{L}_+$ (resp. $\mathcal{L}_-$) be the collection of Stokes rays which one crosses when going from Σ to $-\Sigma$ in the counterclockwise (resp. clockwise) direction. Then $\Phi = \Phi_+ \sqcup \Phi_-$, where

$$\Phi_\pm = \{\alpha \in \Phi |\, \alpha(A_0) \in \ell, \ell \in \mathcal{L}_\pm\} = -\Phi_\mp$$

It follows from Proposition 2.2 that the Stokes matrices S_+, S_- lie in N_+, N_- respectively, where $N_\pm \subset G$ are the unipotent subgroups with Lie algebra $\mathfrak{n}_\pm = \bigoplus_{\alpha\in\Phi_\pm} \mathfrak{g}_\alpha$.

Let $B_\pm = N_\pm \rtimes H \subset G$ be the opposite Borel subgroups with Lie algebras $\mathfrak{b}_\pm = \mathfrak{n}_\pm \rtimes \mathfrak{h}$. Consider the Stokes map $\mathcal{S} : \mathfrak{g}^* \to B_- \times_H B_+$ given by $B \to (e^{-\iota\pi[B]} \cdot S_-^{-1}, e^{\iota\pi[B]} \cdot S_+)$. Endow $\mathfrak{g}^*$ with its standard Kirillov–Kostant–Souriau Poisson structure given by

$$\{f, g\}(x) = \langle [df, dg](x), x\rangle$$

where $df(x), dg(x) \in T_x^*\mathfrak{g}^* = \mathfrak{g}$ are the differentials of f, g and $[\cdot,\cdot]$ is the Lie bracket on $\mathfrak{g}$. Endow $G^* = B_- \times_H B_+$ with the dual Poisson Lie structure (see e.g. Chapter 2 or Chapter 4).

The following remarkable result is due to Boalch [16, 18].

Theorem 2.3. *The Stokes map is a local analytic Poisson isomorphism.*

In particular, $\mathcal{S}$ gives a Ginzburg-Weinstein linearisation of the Poisson-Lie structure on G^*.

2.3 Quasi–triangular Poisson–Lie groups

Let $(\mathfrak{p}, r)$ be a finite–dimensional quasitriangular Lie bialgebra over a field k of characteristic zero. Thus, $\mathfrak{p}$ is a Lie algebra, $r \in \mathfrak{p} \otimes \mathfrak{p}$ satisfies the classical Yang–Baxter equations (CYBE)

$$[r_{12}, r_{23} + r_{13}] + [r_{13}, r_{23}] = 0$$

and is such that $t = r + r^{21}$ is $\mathfrak{p}$–invariant. In particular, $\mathfrak{p}$ is a Lie bialgebra, with cobracket $\delta : \mathfrak{p} \to \mathfrak{p} \wedge \mathfrak{p}$ given by $\delta(x) = [x \otimes 1 + 1 \otimes x, r]$. Assume that r is non–degenerate, that is that the map $^\vee : \mathfrak{p}^* \to \mathfrak{p}$, $\lambda^\vee = \lambda \otimes \mathrm{id}(t)$ is a bijection.

Let $\mathfrak{p}^*$ be the dual Lie bialgebra to $\mathfrak{p}$, and P, P^* the formal Poisson–Lie groups with Lie algebras $\mathfrak{p}, \mathfrak{p}^*$. By definition, a formal map $\mathfrak{p}^* \to L$, where $L = \exp(\mathfrak{l})$ is a formal group, is a map of the form e^x, where $x \in \mathsf{k}[[\mathfrak{p}^*]]_+ \otimes \mathfrak{l}$ is a formal $\mathfrak{l}$–valued function on $\mathfrak{p}^*$ such that $x(0) = 0$.

Enriquez–Etingof–Marshall obtained a linearisation of P^* from any formal map $g : \mathfrak{p}^* \to P$ satisfying the differential equation

$$(g^{-1}dg)_{12} - (g^{-1}dg)_{21} + \mathrm{Ad}(g^{\otimes 2})^{-1} r_0 + \mathrm{id}^{\otimes 2} \otimes \lambda\, [(g^{-1}dg)_{13}, (g^{-1}dg)_{23}] = r_{\mathrm{AM}} \quad (5.3)$$

as an identity of maps $\mathfrak{p}^* \to \wedge^2\mathfrak{p}$. Here, the derivative $g^{-1}dg$ is thought of as a map

$$\mathfrak{p}^* \ni \lambda \to \mathrm{Hom}_{\mathsf{k}}(T_\lambda \mathfrak{p}^*, T_1 P) = \mathfrak{p} \otimes \mathfrak{p}$$

given by $g^{-1}dg(\lambda) = \sum_i g^{-1}\partial_{e^i} g(\lambda) \otimes e_i$, where $\{e^i\}, \{e_i\}$ are dual bases of $\mathfrak{p}^*, \mathfrak{p}$, r_0 is the antisymmetrisation $\frac{1}{2}(r - r_{21})$, and $r_{\rm AM} : \mathfrak{p}^* \to \wedge^2\mathfrak{p}$ is the Alekseev–Meinrenken dynamical r–matrix given by

$$r_{\rm AM}(\lambda) = \mathrm{id} \otimes \varphi(\mathrm{ad}\,\lambda^\vee)(t)$$

where $\varphi(z) = -1/z + 1/2\coth(z/2)$.

To state the main result of [32], note that the CYBE imply that the maps $\ell, \rho : \mathfrak{p}^* \to \mathfrak{p}$ given by

$$\ell(\lambda) = \lambda \otimes \mathrm{id}(r) \qquad \text{and} \qquad \rho(\lambda) = -\,\mathrm{id} \otimes \lambda(r)$$

are Lie algebra homomorphisms. We denote the corresponding morphisms of formal groups $P^* \to P$ by L, R respectively. Since $(\ell - \rho)\lambda = \lambda \otimes \mathrm{id}(t)$, the non–degeneracy of r implies that the map $P^* \to P$, $g^* \to L(g^*) \cdot R(g^*)^{-1}$ is an isomorphism of formal manifolds.

Theorem 2.4 ([32]). *Let $g : \mathfrak{p}^* \to P$ be a formal map satisfying the partial differential equation* (5.3)*. If r is non–degenerate, the map $g^* : \mathfrak{p}^* \to P^*$ uniquely defined by*

$$g(\lambda) \cdot e^{\lambda^\vee} \cdot g(\lambda)^{-1} = L(g^*(\lambda)) \cdot R(g^*(\lambda))^{-1}$$

is a formal isomorphism of Poisson manifolds

2.4 Connection matrices and dynamical gauge transformations

Retain the notation of 2.1–2.2. In particular, G is a complex, reductive group and ∇ the connection 5.1 determined by $A_0, B \in \mathfrak{g}$, with $A_0 \in \mathfrak{h}_{\rm reg}$. ∇ is said to be *non–resonant* at $z = \infty$ if the eigenvalues of $\mathrm{ad}(B)$ are not positive integers. The following is well–know (see, *e.g.*, [81] for $G = GL_n(\mathbb{C})$).

Lemma 2.5. *If ∇ is non–resonant, there is a unique holomorphic function $H_\infty : \mathbb{P}^1 \setminus \{0\} \to G$ such that $H_\infty(\infty) = 1$ and, for any determination of* $\log(z)$*, the function $\Upsilon_\infty = H_\infty \cdot z^B$ is a solution of $\nabla\Upsilon_\infty = 0$.*

For any Stokes sector Σ of ∇, define the *connection matrix* $C_\Sigma \in G$ by

$$\Upsilon_\infty = \Upsilon_\Sigma \cdot C_\Sigma$$

where the identity is understood to hold in $\mathbb{H}_r$, and the underlying determination of $\log(z)$ is chosen with a cut on the clockwise most edge of Σ. The connection matrix C_Σ is related to the Stokes factors $S_\pm$ by the following monodromy relation.

Lemma 2.6. *The following holds*

$$C_\Sigma \cdot e^{2\pi\iota B} \cdot C_\Sigma^{-1} = S_- \cdot e^{2\pi\iota[B]} \cdot S_+$$

Proof. By definition of $S_\pm$, the monodromy of Υ_Σ around a positive loop γ_0 around 0 based at a point $z_0 \in \Sigma$ is the right–hand side of the stated identity. On the other hand, the monodromy of Υ_∞ around γ_0 is $e^{2\pi\iota B}$. Since $\Upsilon_\Sigma = \Upsilon_\infty \cdot C_\Sigma^{-1}$, the former monodromy is conjugate to the latter by C_Σ.

Let $\mathfrak{g}_{\mathrm{nr}} \subset \mathfrak{g}$ be the dense open subset consisting of elements B such that the eigenvalues of $\mathrm{ad}(B)$ do not contain positive integer multiples of $2\pi\iota$. Consider the map $C : \mathfrak{g}_{\mathrm{nr}} \to G$ given by mapping B to the connection matrix of $\nabla = d - (A_0/z^2 + B/2\pi\iota z)$. Identify $\mathfrak{g} \cong \mathfrak{g}^*$ and let $\mathfrak{g}^*_{\mathrm{nr}} \subset \mathfrak{g}^*$ the subset corresponding to $\mathfrak{g}_{\mathrm{nr}}$.

Theorem 2.7 ([85])**.** *The map $C : \mathfrak{g}^*_{\mathrm{nr}} \to G$ satisfies the partial differential equation* (5.3)*, together with the initial condition $C(0) = 1$.*

Combining Theorems 2.7 and 2.4 gives rise to a Poisson map $\mathfrak{g}^*_{\mathrm{nr}} \to G^*$, $B \to (b_-, b_+)$, where $b_\pm \in B_\pm$ are uniquely determined by $C(B) \cdot e^B \cdot C(B)^{-1} = b_-^{-1} \cdot b_+$. By the monodromy relation of Lemma 2.6, this implies that $b_- = e^{-[B]/2} \cdot S_-^{-1}$ and $b_+ = e^{[B]/2} \cdot S_+$. This gives another proof of Boalch's theorem 2.3.

3 Stokes phenomena and quantum groups

This section is an exposition of [78]. Throughout the paper, $\mathsf{h}, \hbar$ are two formal parameters related by $\hbar = 2\pi\iota\mathsf{h}$.

3.1 Filtered algebras

Let $A = \bigcup_{n\geq 0} A_n$ be a filtered $\mathbb{C}$–algebra over $\mathbb{C}$ with $A_0 = \mathbb{C}$. Given a sequence $o = \{o_k\}_{k\in\mathbb{N}}$ of non–negative integers, define $A[\![\hbar]\!]^o \subset A[\![\hbar]\!]$ by

$$A[\![\hbar]\!]^o = \{\sum_{k\geq 0} a_k\hbar^k |\, a_k \in A_{o_k}\}$$

Note that:

1. $A[\![\hbar]\!]^o$ is a (closed) $\mathbb{C}[\![\hbar]\!]$–submodule of $A[\![\hbar]\!]$ if o is increasing.[2]

[2] $A[\![\hbar]\!]^o$ is then the $\hbar$–adic completion of the Rees algebra of A corresponding to the filtration $A_{o_0} \subset A_{o_1} \subset \cdots$

2. $A[\![\hbar]\!]^o$ is a subalgebra of $A[\![\hbar]\!]$ if o is subadditive, that is such that $o_k + o_l \leq o_{k+l}$ for any $k, l \in \mathbb{N}$. This implies in particular that o is increasing, and that $o_0 = 0$.

If the subspaces $A_k \subset A$ are finite–dimensional, so are the quotients

$$A[\![\hbar]\!]^o/(\hbar^{p+1}A[\![\hbar]\!] \cap A[\![\hbar]\!]^o) \cong A_{o_0} \oplus \hbar A_{o_1} \oplus \cdots \oplus \hbar^p A_{o_p}$$

Assuming this, we shall say that a map $F : X \to A[\![\hbar]\!]^o$, where X is a topological space (resp. a smooth or complex manifold), is continuous (resp. smooth or holomorphic) if each of its truncations $F_p : X \to A[\![\hbar]\!]^o/(\hbar^{p+1}A[\![\hbar]\!] \cap A[\![\hbar]\!]^o)$ are.

We shall mainly be interested in the case $A = U\mathfrak{g}^{\otimes n}$ endowed with the standard order filtration given by $\deg(x^{(i)}) = 1$ for $x \in \mathfrak{g}$, where

$$x^{(i)} = 1^{\otimes(i-1)} \otimes x \otimes 1^{\otimes(n-i)}$$

The sequence o will be chosen subadditive, and such that $o_1 \geq 2$ in order for $\hbar\Omega_{ij}, \hbar\Delta^{(n)}(\mathcal{K}_\alpha) \in A[\![\hbar]\!]^o$. Note that $\mathfrak{g} \cap U\mathfrak{g}[\![\hbar]\!]^o = \{0\}$ since $o_0 = 0$, but that the adjoint action of $\mathfrak{g}$ on $U\mathfrak{g}^{\otimes n}$ induces one on by derivations on $U\mathfrak{g}^{\otimes n}[\![\hbar]\!]^o$. Note also that $U\mathfrak{g}[\![\hbar]\!]^o$ is **not** a Hopf algebra, since $\Delta : U\mathfrak{g}[\![\hbar]\!]^o \to U\mathfrak{g}^{\otimes 2}[\![\hbar]\!]^o \supsetneq (U\mathfrak{g}[\![\hbar]\!]^o)^{\otimes 2}$.

Let $\mathcal{A}$ be a $\mathbb{C}[\![\hbar]\!]$–module and consider the natural map $\imath : \mathcal{A} \to \varprojlim \mathcal{A}/\hbar^n\mathcal{A}$. Recall that $\mathcal{A}$ is *separated* if $\imath$ is injective, and *complete* if it is surjective. By definition, $\mathcal{A}$ is *topologically free* if it is separated, complete and torsion–free.

Consider now the map

$$\imath : \mathrm{End}_{\mathbb{C}[\![\hbar]\!]}(\mathcal{A}) \to \varprojlim \mathrm{End}_{\mathbb{C}[\![\hbar]\!]}(\mathcal{A}/\hbar^n\mathcal{A})$$

Lemma 3.1. *Assume that $\mathcal{A}$ is separated. Then,*

1. *$\imath$ is injective.*
2. *If $\mathcal{A}$ is complete, $\imath$ is surjective.*

Proof. (1) If $T \in \mathrm{End}_{\mathbb{C}[\![\hbar]\!]}$ is such that $\imath T = 0$, then $T(\mathcal{A}) \subset \bigcap_n \hbar^n\mathcal{A} = 0$. (2) Let $\{T_n\} \in \lim_n \mathrm{End}_{\mathbb{C}[\![\hbar]\!]}(\mathcal{A}/\hbar^n\mathcal{A})$. For any $a \in \mathcal{A}$, the sequence $\{T_n a\}$ lies in $\lim_n \mathcal{A}/\hbar^n\mathcal{A}$ and is therefore the image of a unique element $a' \in \mathcal{A}$. The assignment $a \to Ta = a'$ is easily seen to define an element of $\mathrm{End}_{\mathbb{C}[\![\hbar]\!]}(\mathcal{A})$ which projects to each of the T_n.

Corollary 3.2. *If $\mathcal{A}$ is topologically free, the map $\imath : \mathrm{End}_{\mathbb{C}[\![\hbar]\!]}(\mathcal{A}) \to \varprojlim \mathrm{End}_{\mathbb{C}[\![\hbar]\!]}(\mathcal{A}/\hbar^n\mathcal{A})$ is an isomorphism.*

3.2 The dynamical KZ equations

In what follows, we set $\mathcal{A} = U\mathfrak{g}^{\otimes 2}[\![\hbar]\!]^o$ and $\mathcal{E} = \mathrm{End}_{\mathbb{C}[\![\hbar]\!]}(\mathcal{A})$. The DKZ connection is the $\mathcal{E}$–valued connection on $\mathbb{C}$ given by

$$\nabla_{\mathrm{DKZ}} = \left(d - \left(\mathsf{h}\frac{\Omega}{z} + \mathrm{ad}\,\mu^{(1)}\right) dz\right)$$

3.3 Fundamental solution at $z = 0$

Proposition 3.3. [78]

1. *For any $\mu \in \mathfrak{h}$, there is a unique holomorphic function $H_0 : \mathbb{C} \to \mathcal{A}$ such that $H_0(0, \mu) = 1$ and, for any determination of $\log(z)$, the $\mathcal{E}$–valued function*

$$\Upsilon_0(z, \mu) = e^{z\,\mathrm{ad}\,\mu^{(1)}} \cdot H_0(z, \mu) \cdot z^{\mathsf{h}\Omega}$$

   satisfies $\nabla_{\mathrm{DKZ}}\Upsilon_0 = 0$.

2. *H_0 and Υ_0 are holomorphic functions of μ, and Υ_0 satisfies*

$$\left(d_{\mathfrak{h}} - \frac{\mathsf{h}}{2}\sum_{\alpha\in\Phi_+}\frac{d\alpha}{\alpha}\Delta(\mathcal{K}_\alpha) - z\,\mathrm{ad}(d\mu^{(1)})\right)\Upsilon_0 = \Upsilon_0\left(d_{\mathfrak{h}} - \frac{\mathsf{h}}{2}\sum_{\alpha\in\Phi_+}\frac{d\alpha}{\alpha}\Delta(\mathcal{K}_\alpha)\right)$$

3.4 Fundamental solutions at $z = \infty$

Let $\mathbb{H}_\pm = \{z \in \mathbb{C} |\ \mathrm{Im}(z) \gtrless 0\}$.

Theorem 3.4. [78]

1. *For any $\mu \in \mathfrak{h}^{\mathbb{R}}_{\mathrm{reg}}$, there are unique holomorphic functions $H_\pm : \mathbb{H}_\pm \to \mathcal{A}$ such that $H_\pm(z, \mu)$ tends to 1 as $z \to \infty$ in any sector of the form $|\arg(z)| \in (\delta, \pi-\delta)$, $\delta > 0$ and, for any determination of $\log(z)$, the $\mathcal{E}$–valued function*

$$\Upsilon_\pm(z, \mu) = H_\pm(z, \mu) \cdot z^{\mathsf{h}\Omega_0} \cdot e^{z\,\mathrm{ad}\,\mu^{(1)}}$$

   satisfies $\nabla_{\mathrm{DKZ}}\Upsilon_\pm = 0$.

2. *$H_\pm$ and $\Upsilon_\pm$ are smooth functions of μ, and $\Upsilon_\pm$ satisfies*

$$\left(d_{\mathfrak{h}} - \frac{\mathsf{h}}{2}\sum_{\alpha\in\Phi_+}\frac{d\alpha}{\alpha}\Delta(\mathcal{K}_\alpha) - z\,\mathrm{ad}(d\mu^{(1)})\right)\Upsilon_\pm = \Upsilon_\pm\left(d_{\mathfrak{h}} - \frac{\mathsf{h}}{2}\sum_{\alpha\in\Phi_+}\frac{d\alpha}{\alpha}({\mathcal{K}_\alpha}^{(1)} + {\mathcal{K}_\alpha}^{(2)})\right)$$

3.5 $\mathbb{Z}_2$–equivariance

Let $\mathcal{U} \subset \mathbb{C}$ be a subset. For any function $F : \mathcal{U} \to \mathcal{A}$ (resp. $F : \mathcal{U} \to \mathcal{E}$), define $F^\vee : -\mathcal{U} \to \mathcal{A}$ (resp. $F^\vee : -\mathcal{U} \to \mathcal{E}$) by $F^\vee(z) = F(-z)^{21}$ (resp. $F^\vee(z) = (1\,2) \cdot F(-z) \cdot (1\,2)$). If F is a local solution of the dynamical KZ equations with values in $\mathcal{A}^\mathfrak{h}$ (resp. $\mathcal{E}^\mathfrak{h}$), then so is $F^\vee$.

Lemma 3.5. *The following holds*

1. *For $z \in \mathbb{H}_\pm$,*
$$\Upsilon_0^\vee(z) = \Upsilon_0(z) \cdot e^{\mp\pi\iota\mathsf{h}\Omega}$$

2. *For $z \in \mathbb{H}_\mp$,*
$$\Upsilon_\pm^\vee(z) = \Upsilon_\mp(z) \cdot e^{\pm\pi\iota\mathsf{h}\Omega_0}$$

Proof. (1) The uniqueness of the holomorphic part H_0 of Υ_0 implies that $H_0^\vee = H_0$. It follows that $\Upsilon_0^\vee(z) = H_0(z) \cdot (-z)^{\mathsf{h}\Omega} = \Upsilon_0(z) \cdot e^{\mp\iota\pi\mathsf{h}\Omega_0}$ since $\log(-z) = \log(z) \mp \iota\pi$, depending on whether $\operatorname{Im} z \gtrless 0$. (2) follows similarly from the fact that $H_\pm^\vee = H_\mp$ on $\mathbb{H}_\mp$. ■

3.6 Differential twist

Fix henceforth the standard determination of $\log(z)$ with a cut along the negative real axis, and let $\Upsilon_0, \Upsilon_\pm$ be the corresponding fundamental solutions of the dynamical KZ equations given in 3.3 and 3.4 respectively. We shall consider Υ_0 and $\Upsilon_\pm$ as (single–valued) holomorphic functions on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

Let $\mathcal{C} \subset \mathfrak{h}^\mathbb{R}_{\text{reg}}$ be the fundamental Weyl chamber.

Definition 3.6. [78] *The differential twist is the holomorphic function $J_\pm : \mathcal{C} \to U\mathfrak{g}^{\otimes 2}[\![\hbar]\!]^o$ defined by*
$$J_\pm = \Upsilon_0(z)^{-1} \cdot \Upsilon_\pm(z)$$
where $z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

Theorem 3.7. [78]

1. *$J_\pm$ kills the KZ associator $\Phi_{\text{KZ}} \in U\mathfrak{g}^{\otimes 3}[\![\hbar]\!]^o$, that is*
$$\Phi_{\text{KZ}} \cdot \Delta \otimes \operatorname{id}(J_\pm) \cdot J_\pm \otimes 1 = \operatorname{id} \otimes \Delta(J_\pm) \cdot 1 \otimes J_\pm$$

2. *Modulo $\hbar^2$,*
$$J_\pm = 1^{\otimes 2} + \frac{\hbar}{2}\left(\mp\Omega_- + \frac{1}{\pi\iota}\sum_{\alpha\in\Phi_+}(\Omega_\alpha + \Omega_{-\alpha})(\log\alpha + \gamma)\right)$$

where $\Omega_\alpha = x_\alpha \otimes x_{-\alpha}$, $\Omega_\pm = \sum_{\alpha\in\Phi_+} \Omega_{\pm\alpha}$, and $\gamma = \lim_n(\sum_{k=1}^n 1/k - \log(n))$ is the Euler–Mascheroni constant.

In particular, the antisymmetrisation of $J_\pm$ is equal to

$$\pm\frac{\hbar}{4}(\Omega_+ - \Omega_-) = \pm\frac{\hbar}{2}\sum_{\alpha\in\Phi_+} x_\alpha \wedge x_{-\alpha} \tag{5.4}$$

3. *$J_\pm$ satisfies*

$$dJ_\pm = \frac{\mathsf{h}}{2}\sum_{\alpha\in\Phi_+}\frac{d\alpha}{\alpha}\left(\Delta(\mathcal{K}_\alpha)J_\pm - J_\pm({\mathcal{K}_\alpha}^{(1)} + {\mathcal{K}_\alpha}^{(2)})\right)$$

Proposition 3.8. *The following holds*

$$J_- = R_{\text{KZ}} \cdot J_+^{21} \cdot (R^0_{\text{KZ}})^{-1}$$

where $R_{\text{KZ}} = e^{\hbar\Omega/2}$ and $R^0_{\text{KZ}} = e^{\hbar\Omega_0/2}$.

Proof. By definition, $J_+^{21} = (\Upsilon_0^\vee)^{-1} \cdot \Upsilon_+^\vee$, where the right–hand side is evaluated for $\operatorname{Im} z < 0$. By Lemma 3.5, this is equal to $e^{-\pi\iota\mathsf{h}\Omega} \cdot \Upsilon_0^{-1} \cdot \Upsilon_- \cdot e^{\pi\iota\mathsf{h}\Omega_0}$.

3.7 Quantisation of $(\mathfrak{g}, r)$

Let

$$r = \Omega_+ + \frac{1}{2}\Omega_0 = \sum_{\alpha\in\Phi_+} x_\alpha \otimes x_{-\alpha} + \frac{1}{2}\Omega_0$$

be the Drinfeld–Sklyanin r–matrix corresponding to the triangular decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$, and $(\mathfrak{g}, r)$ the corresponding quasitriangular Lie bialgebra.

Set $R_{\text{KZ}} = e^{\hbar\Omega/2}$, and let

$$(U\mathfrak{g}[\![\hbar]\!], \Delta_0, R_{\text{KZ}}, \Phi_{\text{KZ}})$$

be the quasitriangular quasi–Hopf algebra structure on $U\mathfrak{g}$ underlying the monodromy of the KZ equations [27], where Δ_0 is the standard cocommutative coproduct on $U\mathfrak{g}$. The differential twist $J_\pm$ allows to twist this structure, and yields a quasitriangular Hopf algebra $(U\mathfrak{g}[\![\hbar]\!], \Delta_\pm, R_\pm)$, where[3]

$$\Delta_\pm(x) = J_\pm^{-1} \cdot \Delta_0(x) \cdot J_\pm \qquad \text{and} \qquad R_\pm = (J_\pm^{-1})^{21} \cdot R_{\text{KZ}} \cdot J_\pm$$

[3]Note that $\Delta_\pm$ and $R_\pm$ depend on the additional choice of $\mu \in \mathcal{C}$. Specifically, if $\mu_0, \mu_1 \in \mathcal{C}$, $p : [0,1] \to \mathcal{C}$ is a path with $p(0) = \mu_0, p(1) = \mu_1$, and $a_p \in U\mathfrak{g}[\![\hbar]\!]_0$ is the holonomy of the Casimir connection along p, then

$$\Delta_\pm(x)(\mu_1) = a_p^{\otimes 2} \cdot \Delta_\pm(a_p^{-1} x a_p)(\mu_0) \cdot (a_p^{\otimes 2})^{-1} \qquad \text{and} \qquad R_\pm(\mu_1) = a_p^{\otimes 2} \cdot R_\pm(\mu_0) \cdot (a_p^{\otimes 2})^{-1}$$

In particular, the quasitriangular Hopf algebras corresponding to different values of $\mu \in \mathcal{C}$ are all isomorphic.

Theorem 3.9.

1. *$(U\mathfrak{g}[\![\hbar]\!], \Delta_+, R_+)$ is a quantisation of $(\mathfrak{g}, r)$.*

2. *$(U\mathfrak{g}[\![\hbar]\!], \Delta_-, R_-)$ is a quantisation of $(\mathfrak{g}, r^{21})$.*

3. *Each of $(U\mathfrak{g}[\![\hbar]\!], \Delta_\pm, R_\pm)$ is isomorphic to the Drinfeld–Jimbo quantum group corresponding to $\mathfrak{g}$.*

Proof. (1)–(2) By (5.4), the coefficient of $\hbar$ in $R_\pm$ is $\frac{1}{2}(\Omega \pm \Omega_+ \mp \Omega_-)$, which is equal to r for R_+ and r^{21} for R_-.

(3) This follows, for example, from Drinfeld's uniqueness of the quantisation of $(\mathfrak{g}, r)$ [24] given that the Chevalley involution of $\mathfrak{g}$ clearly lifts to $(U\mathfrak{g}[\![\hbar]\!], \Delta_\pm, R_\pm)$.

Remark 3.10. *It follows from (4) of Theorem 3.7 that*

$$R_- = R^0_{\mathrm{KZ}} \cdot R^{21}_+ \cdot (R^0_{\mathrm{KZ}})^{-1} \tag{5.5}$$

4 The R–matrix as a quantum Stokes matrix

4.1 Quantum Stokes matrices

Recall that $\mathbb{H}_\pm = \{z \in \mathbb{C} | \operatorname{Im}(z) \gtrless 0\}$. Define the quantum Stokes matrices $S_{\hbar\pm} \in U\mathfrak{g}^{\otimes 2}[\![\hbar]\!]^o$ by

$$\Upsilon_+ = \Upsilon_- \cdot S_{\hbar+} \qquad \text{and} \qquad \Upsilon_- \cdot e^{\hbar\Omega_0} = \Upsilon_+ \cdot S_{\hbar-}$$

where the first identity is understood to hold in $\mathbb{H}_-$ after Υ_+ has been continued across the ray $\mathbb{R}_{\geq 0}$, and the second in $\mathbb{H}_+$ after Υ_- has been continued across $\mathbb{R}_{\leq 0}$. Here we use $S_{\hbar\pm}$ to distinguish from the classical Stokes matrices $S_\pm$.

Proposition 4.1. *The following holds*

1. $S_{\hbar-} = e^{-\iota\pi\mathsf{h}\Omega_0} \cdot S^{21}_{\hbar+} \cdot e^{\iota\pi\mathsf{h}\Omega_0}$.

2. $J_+^{-1} \cdot e^{2\pi\iota\mathsf{h}\Omega} \cdot J_+ = S^{-1}_{\hbar+} \cdot e^{2\pi\iota\mathsf{h}\Omega_0} \cdot S^{-1}_{\hbar-}$

3. *As functions of $\mu \in \mathcal{C}$, the quantum Stokes matrices $S_{\hbar\pm}$ satisfy*

$$d_{\mathfrak{h}} S_{\hbar\pm} = \frac{\mathsf{h}}{2} \sum_{\alpha\in\Phi_+} \frac{d\alpha}{\alpha} \left[\mathcal{K}_\alpha{}^{(1)} + \mathcal{K}_\alpha{}^{(2)}, S_{\hbar\pm}\right]$$

Proof. (1) Let f be a holomorphic function on $\mathbb{H}_\pm$, and denote by $\mathcal{P}_\pm(f)$ the analytic continuation of f to $\mathbb{H}_\mp$ across the half–axis $\mathbb{R}_{\gtrless 0}$. By Lemma 3.5, and the definition of $S_{\hbar -}$,

$$\mathcal{P}_-(\Upsilon_+^\vee) = \mathcal{P}_-(\Upsilon_-) \cdot e^{\iota\pi\mathsf{h}\Omega_0} = \Upsilon_+ \cdot S_{\hbar -} \cdot e^{-\iota\pi\mathsf{h}\Omega_0}$$

On the other hand, if $\imath : \mathbb{C} \to \mathbb{C}$ is the inversion $z \to -z$,

$$\begin{aligned}\mathcal{P}_-(\Upsilon_+^\vee) &= (1\,2) \cdot \mathcal{P}_-(\Upsilon_+ \circ \imath) \cdot (1\,2) = (1\,2) \cdot \mathcal{P}_+(\Upsilon_+) \circ \imath \cdot (1\,2)\\ &= (1\,2) \cdot \Upsilon_- \circ \imath \cdot S_{\hbar +} \cdot (1\,2) = \Upsilon_-^\vee \cdot S_{\hbar +}^{21}\\ &= \Upsilon_+ \cdot e^{-\iota\pi\mathsf{h}\Omega_0} \cdot S_{\hbar +}^{21}\end{aligned}$$

where the last identity uses Lemma 3.5.

(2) By construction, the monodromy of the fundamental solution Υ_0 around a positively oriented loop γ_0 around 0 is $e^{2\pi\iota\mathsf{h}\Omega}$. Let now γ_∞ be a clockwise loop around ∞ based at $x_0 \in \mathbb{H}_+$. Since such a loop crosses the negative real axis before the positive one, the monodromy of Υ_+ around γ_+ is $S_{\hbar +}^{-1} \cdot e^{2\pi\iota\mathsf{h}\Omega_0} \cdot S_{\hbar -}^{-1}$. The result now follows from the fact that γ_∞ is homotopic to γ_0, and $\Upsilon_+ = \Upsilon_0 \cdot J_+$.

(3) follows from the PDE satisfies by Υ_0 and $\Upsilon_\pm$. ∎

4.2 The R–matrix as a quantum Stokes matrix

Now we state a surprising relation between the quantum Stokes phenomenon and the Yang-Baxter equation, which says that the quantum Stokes matrix, after correction by formal monodromy, gives rise to quantum R-matrices (solution of Yang-Baxter equations).

Theorem 4.2. *The following holds*

$$R_+ = e^{\pi\iota\mathsf{h}\Omega_0} \cdot S_-^{-1} \qquad \text{and} \qquad R_- = e^{\pi\iota\mathsf{h}\Omega_0} \cdot S_+^{-1}$$

Proof. By definition of S_+, $\Upsilon_+ = \Upsilon_- \cdot S_+$, when both $\Upsilon_\pm$ are considered as single–valued functions on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. On the other hand, by definition of $J_\pm$,

$$\Upsilon_+ = \Upsilon_0 \cdot J_+ = \Upsilon_- \cdot J_-^{-1} \cdot J_+$$

Using Proposition 3.8 therefore yields

$$S_+ = e^{\iota\pi\mathsf{h}\Omega_0} \cdot (J_+^{-1})^{21} \cdot e^{-\iota\pi\mathsf{h}\Omega} \cdot J_+ = e^{\iota\pi\mathsf{h}\Omega_0} \cdot (R_+^{-1})^{21}$$

where the last equality uses the fact that $R_{\mathrm{KZ}} = \exp(\pi\iota\mathsf{h}\Omega) = R_{\mathrm{KZ}}^{21}$. The first stated identity now follows from (1) of Proposition 4.1. The second one follows from the first and (5.5). ∎

5 Semiclassical limits

In this section, we prove that the remarkable result in [16] can be viewed as the classical limit of the construction in [78]. Thus the quantization problem (quantization of Poisson Lie groups to Quantum groups) can be studied in the frame of the deformation of certain irregular Riemann-Hilbert problem (meromorphic ODE).

5.1 The algebra $\mathfrak{U}$

Set $\mathfrak{U} = U\mathfrak{g}[\![\hbar]\!]$ and, for any $n \geq 1$, $\mathfrak{U}^{\otimes n} = \mathfrak{U}\widehat{\otimes}\cdots\widehat{\otimes}\mathfrak{U} = U\mathfrak{g}^{\otimes n}[\![\hbar]\!]$, where $\widehat{\otimes}$ is the completed tensor product of $\mathbb{C}[\![\hbar]\!]$–modules. Let $\eta : \mathbb{C}[\![\hbar]\!] \to \mathfrak{U}$ and $\epsilon : \mathfrak{U} \to \mathbb{C}[\![\hbar]\!]$ be the unit and counit of $\mathfrak{U}$. $\mathfrak{U}$ splits as $\mathrm{Ker}(\epsilon) \oplus \mathbb{C}[\![\hbar]\!] \cdot 1$, with projection onto the first summand given by $\pi = \mathrm{id} - \eta \circ \epsilon$. Let $\Delta^{(n)} : \mathfrak{U} \to \mathfrak{U}^{\otimes n}$ be the iterated coproduct recursively defined by $\Delta^{(0)} = \epsilon$, $\Delta^{(1)} = \mathrm{id}$, and $\Delta^{(n)} = \Delta \otimes \mathrm{id}^{\otimes(n-2)} \circ \Delta^{(n-1)}$ for $n \geq 2$.

Define $\mathfrak{U}' \subset \mathfrak{U}$ by [24, 48]

$$\mathfrak{U}' = \{x \in \mathfrak{U} |\, \pi^{\otimes n} \circ \Delta^{(n)}(x) \in \hbar^n \mathfrak{U}^{\otimes n},\, n \geq 1\}$$

The following is well–known

Lemma 5.1.

1. *$\mathfrak{U}' = U(\hbar\mathfrak{g}[\![\hbar]\!])$. That is, $x = \sum_{n \geq 0} \hbar^n x_n$, $x_n \in U\mathfrak{g}$, lies in $\mathfrak{U}'$ is, and only if the filtration order of x_n in $U\mathfrak{g}$ is less than or equal to n.*

2. *$\mathfrak{U}'$ is a flat deformation of the completed symmetric algebra $\widehat{S}\mathfrak{g} = \prod_{n \geq 0} S^n\mathfrak{g}$.*

Proof. (1) follows from the easily proved formula

$$\pi^{\otimes n} \circ \Delta^{(n)}(x_1 \cdots x_k) = \sum_{\substack{I_1 \sqcup \cdots \sqcup I_n = \{1,\ldots,k\} \\ |I_i| \neq 0}} x_{I_1} \otimes \cdots \otimes x_{I_n}$$

where $x_1, \ldots, x_k \in \mathfrak{g}$ and, for any $I = \{i_1, \ldots, i_m\}$, with $i_1 < \cdots < i_m$, $x_I = x_{i_1} \cdots x_{i_m}$.

(2) follows from (1) and the fact that any topologically free $\mathbb{C}[\![\hbar]\!]$–module $\mathfrak{r}A$ is a flat deformation of $\mathfrak{r}A/\hbar\mathfrak{r}A$.

5.2 Semiclassical limit

If $A \in \mathfrak{U}^{\otimes n}\widehat{\otimes}\mathfrak{U}'$, the *semiclassical limit* of A, denoted by $\mathrm{scl}\,(A)$ is the image of A in

$$\mathfrak{U}^{\otimes n}\widehat{\otimes}\mathfrak{U}'/\hbar(\mathfrak{U}^{\otimes n}\widehat{\otimes}\mathfrak{U}') = U\mathfrak{g}^{\otimes n} \otimes \widehat{S}\mathfrak{g}$$

Given that $\widehat{S}\mathfrak{g} = \mathbb{C}[[\mathfrak{g}^*]]$, we will regard $\mathrm{scl}\,(A)$ as formal function on $\mathfrak{g}^*$ with values in $U\mathfrak{g}^{\otimes n}$.

Note that the subalgebras $U\mathfrak{g}^{\otimes n}[\![\hbar]\!]^o \subset \mathfrak{U}^{\otimes n}$ defined in 3.1 is contained in $\mathfrak{U}^{\otimes(n-1)} \otimes \mathfrak{U}'$. In particular, the semiclassical limit of an element of $U\mathfrak{g}^{\otimes n}[\![\hbar]\!]^o$ is well–defined.

5.3 Semiclassical limit of the DKZ equations

Proposition 5.2.

1. *Let Υ be a solution of the dynamical KZ equations with values in $U\mathfrak{g}^{\otimes 2}[\![\hbar]\!]^o \subset \mathfrak{U} \otimes \mathfrak{U}'$. Then, $F = \mathrm{scl}\,(\Upsilon)$ is a solution of*

$$\frac{dF}{dz} = \left(\frac{B}{z} + \mathrm{ad}\,\mu\right) F$$

where $B : \mathfrak{g}^* \to \mathfrak{g}$ is the linear isomorphism given by $\lambda \to \lambda \otimes \mathrm{id}(\Omega)$, thought of as a formal function $\mathfrak{g}^* \to U\mathfrak{g}$

2. *If $\Upsilon_0 = H_0 \cdot z^{\mathsf{h}\Omega}$ is the canonical solution of the DKZ equations given by Proposition 3.3, then $F_0 = \mathrm{scl}\,(\Upsilon_0)$ is of the form $h_0 \cdot z^B$, where h_0 is an entire function of z such that $h_0(0) = 1$.*

3. *If $\Upsilon_\pm = H_\pm \cdot z^{\mathsf{h}\Omega_0} \cdot e^{z\,\mathrm{ad}\,\mu^{(1)}}$ is the canonical solution of the DKZ equations given by Proposition 3.3, then $F_\pm = \mathrm{scl}\,(\Upsilon_\pm)$ is of the form $h_\pm \cdot z^{[B]} \cdot e^{z\,\mathrm{ad}\,\mu}$ where $h_\pm$ is a holomorphic function on $\mathbb{H}_\pm$ such that $\mathbb{H}_\pm \to 1$ as $z \to \infty$ in any closed sector of the form $|\arg(z)| \in [\delta, \pi - \delta]$, $\delta > 0$, and $[B] : \mathfrak{g}^* \to \mathfrak{h}$ is the composition of B with the projection $\mathfrak{g} \to \mathfrak{h}$.*

Proof. (1) follows from the fact that, under the identification $\mathfrak{U}\widehat{\otimes}\mathfrak{U}'/\hbar(\mathfrak{U}\widehat{\otimes}\mathfrak{U}') = U\mathfrak{g} \otimes \mathbb{C}[[\mathfrak{g}^*]]$, $\mathrm{scl}\,(\mathsf{h}\Omega)$ corresponds to the identification $\mathfrak{g}^* \to \mathfrak{g}$.

(2)–(3) are direct consequences of (1).

5.4 Semiclassical limit of the differential twist

Let $J_\pm = \Upsilon_0^{-1} \cdot \Upsilon_\pm$ be the differential twist defined in 3.6.

Theorem 5.3.

1. *The semiclassical limit $C = \mathrm{scl}\,(J_\pm^{-1})$ is the connection matrix map $C : \mathfrak{g}^* \to G$ defined in 2.4 after the change of variable $z \to 1/z$.*

2. *C satisfies the EEM partial differential equation* (5.3).

Proof. (1) Follows from Proposition 5.2 and the uniqueness of canonical solutions of the connection (5.1). (2) follows from the fact that, by Theorem 3.7, $J_\pm$ kills the KZ associator and Lemma 3.4 of [32] according to which the semiclassical limit of any (admissible) associator killing Φ_{KZ} gives rise to a solution of (5.3).

5.5 Semiclassical limit of the quantum Stokes matrices

The following result is a direct consequence of Proposition 5.2.

Theorem 5.4. *The semiclassical limit of the Stokes matrices $S_{\hbar\pm}$ of the dynamical KZ equation are the Stokes matrices $S_\pm$ of the connection* (5.1)*, thought of as functions $\mathfrak{g}^* \to B_\pm$.*

It follows that the Stokes map $\mathfrak{g}^* \to G^*$ associated to the connection (5.1) is a Poisson map. See e.g. Section 3.2.2 in [32]. Thus we obtain an interpretation of the appearance of Poisson Lie groups in Stokes phenomenon from a new perspective.

6 Quantum and classical isomonodromy equations

6.1 Gauge action on quantum Stokes matrices

Let $U_0' := \mathrm{Ker}(\varepsilon) \cap U(\mathfrak{g})[\![\hbar]\!]^\circ$ and let $V := \{u_\hbar \in \hbar^{-1}U_0' \subset U(\mathfrak{g})[\![\hbar]\!]\} \mid u_\hbar = O(\hbar)\}$ be the Lie subalgebra for the commutator. The reduction module $\hbar$ of the Lie algebra V is $V/\hbar V = (\hat{S}(\mathfrak{g})_{>1}, \{-,-\})$.

Let $A_0 \in \mathcal{C} \subset \mathfrak{h}_{\mathrm{reg}}^{\mathbb{R}}$, and $S_{\hbar\pm}$ be the associated quantum Stokes matrices defined as before. Given any $u_\hbar \in V$ and $u \in \hat{S}(\mathfrak{g})$ its reduction module $\hbar$, the gauge action of $u_\hbar \in V$ on the quantum Stokes matrices is given by $e^{u_\hbar} * S_{\hbar\pm} := (e^{u_\hbar})^{\otimes 2} S_{\hbar\pm} (e^{u_\hbar})^{\otimes 2^{-1}}$, and its infinitesimal action is given by

$$\delta_{u_\hbar}(S_{\hbar\pm}) = [u_\hbar^{(1)} + u_\hbar^{(2)}, S_{\hbar\pm}] \tag{5.6}$$

The reduction modulo $\hbar$ of this action can be described as follows: let $S_\pm \in \exp(\mathfrak{g})\hat{\otimes}\hat{S}(\mathfrak{g})$ (Stokes matrices) be the semiclassical limit of $S_{\hbar\pm}$. The Lie algebra $(\hat{S}(\mathfrak{g}), \{\cdot,\cdot\})$ acts on $S_\pm \in \exp(\mathfrak{g})\hat{\otimes}\hat{S}(\mathfrak{g})$ by $\delta_u(S_\pm) = \{1 \otimes u, S_\pm\}$.

Thus if we view $S_\pm \in \exp(\mathfrak{g})\hat{\otimes}\hat{S}(\mathfrak{g})$ as a map, the infinitesimal action of u on $S_\pm \in \mathrm{Map}_0(\mathfrak{g}^*, G)$ is given by

$$\delta_u(S_\pm)(x) = (S_\pm)_*(H_u(x)), \tag{5.7}$$

where $u \in \hat{S}(\mathfrak{g})$ and H_u is the Hamiltonian vector field on $\mathfrak{g}^*$ generated by u, i.e., $H_u = \{u, \cdot\}$.

6.2 Isomonodromic deformation equation.

By Theorem 4.1 as functions of $\mu \in \mathcal{C}$, the quantum Stokes matrices $S_{\hbar\pm}$ satisfy

$$d_{\mathfrak{h}} S_{\hbar\pm} = \frac{\mathsf{h}}{2} \sum_{\alpha\in\Phi_+} \frac{d\alpha}{\alpha} \left[\mathcal{K}_\alpha{}^{(1)} + \mathcal{K}_\alpha{}^{(2)}, S_{\hbar\pm} \right]. \tag{5.8}$$

Let $\omega_\hbar := \frac{\hbar}{2} \sum_{\alpha\in\Phi_+} \mathcal{K}_\alpha \frac{d\alpha}{\alpha} \in U(\mathfrak{g})[\![\hbar]\!] \otimes \Omega^1(\mathcal{C})$. Then one checks that $\omega_\hbar \in V \otimes \Omega^1(\mathcal{C})$. The above equation is an infinitesimal gauge equation on $S_\pm$ under the action of $\omega_\hbar$. ("time-dependent or $\mathcal{C} \subset \mathfrak{h}_{\mathrm{reg}}^{\mathbb{R}}$-dependent" gauge action).

Let $\omega \in \Omega^1(\mathcal{C}) \otimes \hat{S}(\mathfrak{g})$ be the reduction module $\hbar$ of the Casimir operator $\omega_\hbar$ (to be more precise, ω is a one-form on $\mathcal{C}$ whose coefficients are quadratic polynomials on $\mathfrak{g}^*$, and by definition, under the PBW isomorphism, ω coincides with $\frac{1}{2} \sum_{\alpha\in\Phi_+} \mathcal{K}_\alpha \frac{d\alpha}{\alpha}$.). By taking the Hamiltonian vector field generated by $\hat{S}(\mathfrak{g})$, we define $H_\omega \in \Omega^1(\mathcal{C}) \otimes \mathfrak{X}(\mathfrak{g}^*)$. Then it follows from the disscussion above, in particular equation (5.7), the reduction module $\hbar$ of equation (5.8) gives rise to

$$d_{\mathfrak{h}} S_\pm(x) + (S_\pm)_*(H_\omega(x)) = 0, \ \forall x \in \mathfrak{g}^*. \tag{5.9}$$

Here $S_\pm$ is a map from $\mathcal{C} \times \mathfrak{g}^* \to G$, and $H_\omega \in \Omega^1(\mathcal{C}) \otimes \mathfrak{X}(\mathfrak{g}^*)$.

This gives the isomonodromic deformation equation [55][18] as follows. Choose $A_t \in \mathcal{C}$ a one parameter family. Assume $x_t \in \mathfrak{g}^*$ is an isomonodromic flow, i.e., x_t is such that the Stokes matrices $s_{A_t\pm}(x_t)$ of the connection $\nabla_t = d - (\frac{A_t}{z^2} + \frac{x_t}{z})$ is fixed. Taking the derivative with respect to t, we get

$$\frac{dS_{A_t\pm}(x_0)}{dt}|_{t=0} + S_\pm(\dot{x}) = 0.$$

where $\dot{x} := \frac{dx_t}{dt}|_{t=0}$. From the arbitrarity of the one parameter family A_t in $\mathcal{C}$, we have the isomonodromic equation becomes $d_{\mathfrak{h}} S_\pm(x) + S_\pm(d_{\mathfrak{h}} x) = 0$. Comparing this with equation (5.9), we obtain

$$(S_\pm)_*(d_{\mathfrak{h}} x) = (S_\pm)_*(H_\omega(x)).$$

We eventually obtain

Theorem 6.1. [18] *The isomonodromic deformation equation takes the form* $d_{\mathfrak{h}} x = H_\omega(x)$.

This is a time-dependent Hamiltonian description of the isomonodromic deformation. Recall that by the definition of ω, under the PBW isomorphism, ω (one-form on $\mathcal{C}$ whose coefficients are quadratic polynomials on $\mathfrak{g}^*$) coincides with $\frac{1}{2} \sum_{\alpha\in\Phi_+} \mathcal{K}_\alpha \frac{d\alpha}{\alpha}$. The symplectic nature of the isomonodromic deformation equation is interpreted from the perspective of the gauge action of Casimir operator on quantum Stokes matrices.

Chapter 6

Generalized classical dynamical Yang-Baxter equations and moduli spaces of flat connections on surfaces

Xiaomeng Xu

Abstract: In this paper, we explain how generalized dynamical r-matrices can be obtained by (quasi-)Poisson reduction. New examples of Poisson structures, Poisson G-spaces and Poisson groupoid actions naturally appear in this setting. As an application, we use a generalized dynamical r-matrix, induced by the gauge fixing procedure, to give a new finite dimensional description of the Atiyah-Bott symplectic structure on the moduli space of flat connections on a surface. Using this, we find a Poisson groupoid symmetry of the moduli space.

1 Introduction

The classical Yang-Baxter equation (CYBE) plays a key role in the theory of integrable systems. A geometric interpretation of CYBE was given by Drinfeld and gave rise to the theory of Poisson-Lie groups. The classical dynamical Yang-Baxter equation (CDYBE) is a differential equation analogue to CYBE and introduced by Felder in [40] as the consistency condition for the differential Knizhnik-Zamolodchikov-Bernard equations for correlation functions in conformal field theory on tori. It was shown by Etingof and Varchenko [37] that dynamical r-matrices correspond to Poisson-Lie groupoids (a notion introduced by Weinstein [82]) in much the same way as r-matrices correspond to Poisson-Lie groups. In the meantime, the classical dynamical Yang-Baxter equation is proven to be closely connected with the theory of homogeneous Poisson spaces [26], Dirac structures and Lie bialgebroids [67], see [65], [62] and references therein. Inspired by the study of Lie bialgebroids, the notion of generalized classical dynamical Yang-Baxter equations was introduced by Liu and Xu [62] in which the base manifold underlying the CDYBE can be a general Poisson manifold. Despite its importance, this subject suffered from the lack of examples for a long time.

Since Atiyah and Bott introduced canonical symplectic structures on the moduli spaces of flat connections on Riemann surface in [10], a lot of attention has been paid to the moduli spaces by mathematicians and physicists due to their rich mathematical structures and their links with a variety of topics. From the physics perspective, a major motivation for their study is their role in Chern-Simons theory. An independent mathematical motivation for investing moduli spaces of flat connections arises from Poisson geometry. The Atiyah-Bott symplectic structure on the moduli of flat G-connections over oriented surface Σ admits several finite dimensional descriptions. The first such description appears in Goldman's study of symplectic structures on character varieties $\mathrm{Hom}(\pi_1(\Sigma), G)/G$, see [50]. Another possibility is to obtain the moduli space of flat G-connections on a surface $\Sigma_{g,n}$ of genus g with n punctures by (quasi-)Poisson reduction from an enlarged ambient G^{n+2g}. In the Fock-Rosly's approach [47], the Poisson structure on G^{n+2g} is described using a classical r-matrix. In the Alekseev-Malkin-Meinrenken's theory of Lie group valued moment maps [2], the moduli space is obtained by a reduction of a canonical quasi-Poisson tensor on G^{n+2g}.

These two subjects of dynamical Yang-Baxter equations and moduli spaces of flat connections appear to be very different. However, some recent works indicate the possible connection between them. From the viewpoint of Hamiltonian formalism, the moduli spaces of flat connections can be viewed as constrained Hamiltonian systems. Dirac gauge fixing for the moduli space of flat $\mathrm{ISO}(2,1)$-connections on a Riemann surface has been shown to give rise to generalized classical dynamical r-matrices in [72]. On the other hand, gauge fixing in the Poisson-Lie context has been shown to give rise to classical dynamical r-matrices in some cases [39].

In this paper, we deepen the connection between these two subjects by giving a systematic investigation of the theory of generalized classical dynamical r-matrices. We explain how generalized dynamical r-matrices can be obtained by (quasi-)Poisson reduction. Furthermore, we show that new examples of Poisson structures, Poisson G-spaces and Poisson groupoid actions naturally appear in this setting. As a result, associated to a classical dynamical r-matrix, there is naturally a Poisson manifold carrying simultaneously a Hamiltonian action and a Poisson action, whose Hamiltonian reduction gives a homogeneous Poisson space. After that, we take the canonical quasi-Poisson manifold $G \circledast G$ as an example and concretely analyze the dynamical r-matrices arising from the quasi-Poisson reduction of $G \circledast G$. We also introduce the notion of gauge transformations for generalized dynamical r-matrices. As an application, we use a dynamical r-matrix, induced by the gauge fixing procedure, to give a new finite dimensional description of the symplectic structure on the moduli space. Using this, we find a Poisson groupoid symmetry of the moduli space. We end up with two examples, one of them was previously studied by Meusburger-Schönfeld in the framework of the ISO$(2, 1)$-Chern-Simons theory of $(2 + 1)$-dimensional gravity.

Our paper is structured as follows. In section 2, we recall the definition of generalized classical dynamical Yang-Baxter equations and present some examples. After that, we give new examples of generalized dynamical r-matrices from (quasi-)Poisson reduction. We show that new Poisson structures, Poisson G-spaces and Poisson groupoid actions naturally appear in this setting. Moduli space dynamical r-matrices and gauge transformations for generalized dynamical r-matrices are introduced. In section 3, we use a generalized dynamical r-matrix induced by the gauge fixing procedure to give a new finite dimensional description of the Atiyah-Bott symplectic structure on the moduli space of flat connections on surfaces.

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2 Generalized classical dynamical Yang-Baxter equations

First we fix some notations. Let M be a manifold, then a Poisson bivector π on M gives rise to a Lie algebroid structure on T^*M, denoted by (T^*M, π), where the anchor map is $\pi^\sharp : T^*M \to TM$ and the Lie bracket is

$$[\alpha, \beta] = L_{\pi^\sharp\alpha} - L_{\pi^\sharp\beta} - d(\pi(\alpha, \beta)),\ \forall \alpha, \beta \in \Omega^1(M). \tag{6.1}$$

Let G be a Lie group with $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\{e_i\}_{i=1,..,n}$ be a basis of $\mathfrak{g}$. Given a tensor $\theta = \sum X_i \otimes e_i \in \Gamma(TM \otimes \mathfrak{g})$, a smooth map $r : M \to \mathfrak{g} \wedge \mathfrak{g}$ and a linear map $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$, we define the following operations

$$\delta\theta = \sum X_i \otimes \delta e_i, \qquad [r, \theta] = \sum X_i \otimes [r, e_i], \tag{6.2}$$

$$[\theta, \theta] = \sum [X_i, X_j] \otimes e_i \wedge e_j, \qquad \theta \wedge \theta = \sum X_i \wedge X_j \otimes [e_i, e_j]. \tag{6.3}$$

We denote by $\theta^\sharp : T^*M \to M \times \mathfrak{g}$ the morphism associated with $\theta \in \Gamma(TM \otimes \mathfrak{g})$. With these preparatory notations, we can now give the following definition.

Definition 2.1. [62] *For a Poisson manifold (M, π) and a Lie algebra $\mathfrak{g}$, assume that there exists a tensor $\theta \in \Gamma(TM \otimes \mathfrak{g})$ such that $\theta^\sharp : (T^*M, \pi) \to \mathfrak{g}$ is a Lie algebroid morphism. A function $r \in C^\infty(M, \wedge^2\mathfrak{g})$ is called a dynamical r-matrix coupled with the Poisson manifold (M, π) via θ if*

$$\frac{1}{2}[\theta, \theta] = [r, \theta] - \pi^\sharp(dr), \tag{6.4}$$

and the generalized DYBE is satisfied:

$$\mathrm{Alt}(\theta^\sharp dr) + \frac{1}{2}[r, r] = \Omega, \tag{6.5}$$

where $\mathrm{Alt}(\theta^\sharp dr(x))$ is the skew-symmetrization of $\theta^\sharp dr(x) \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ for all $x \in M$, and $\Omega \in (\wedge^3\mathfrak{g})^\mathfrak{g}$ is an invariant element and regarded as a constant section of $\wedge^3(M \times \mathfrak{g})$.

We call r a triangular dynamical r-matrix coupled with M via θ if the corresponding $\Omega = 0$ in (6.5). Throughout this paper, we will also use the triple $((M, \pi), \theta, r)$ to denote a generalized dynamical r-matrix.

Example 2.2. *Let η be a Lie subalgebra of $\mathfrak{g}$ and $M = \eta^*$ with the natural linear Poisson structure. Let $\theta^\sharp : T^*\eta^* \to \mathfrak{g}$ be the natural projection: $(\xi, v) \to v$, for all $(\xi, v) \in \eta^* \times \eta$. We can write $\theta = \sum_{i=1}^k \frac{\partial}{\partial x_i} \otimes e_i$ in a basis $\{e_1, ..., e_k\}$ of η and a dual basis $\{x_1, ..., x_k\}$ of η^*. One checks that $\theta^\sharp$ is a Lie algebroid morphism, and $[\theta, \theta] = 0$. Therefore, the condition $\frac{1}{2}[\theta, \theta] = [r, \theta] - \pi^*(dr)$ reduces to $[r, \theta] = \pi^*(dr)$, which says that the map $r : \eta^* \to \mathfrak{g} \wedge \mathfrak{g}$ is η-equivariant. In this case, equation $\mathrm{Alt}(\theta^\sharp dr) + \frac{1}{2}[r, r] = \Omega \in (\wedge^3\mathfrak{g})^\mathfrak{g}$ becomes the classical dynamical Yang-Baxter equation and a solution r is a classical dynamical r-matrix [40].*

Similarly, we introduce a notion of the generalized Poisson-Lie dynamical Yang-Baxter equation.

Definition 2.3. *For a Poisson manifold* (M,π) *and a Lie bialgebra* $(\mathfrak{g},\delta)$*, assume that there exists a tensor* $\theta\in\Gamma(TM\otimes\mathfrak{g})$ *such that* $\theta^\sharp:(T^*M,\pi)\to\mathfrak{g}$ *is a Lie algebroid morphism. A function* $r\in C^\infty(M,\wedge^2\mathfrak{g})$ *is called a Poisson-Lie dynamical r-matrix coupled with the Poisson manifold* (M,π) *via* θ *if*

(a) $\delta\theta+\frac{1}{2}[\theta,\theta]=[r,\theta]-\pi^\sharp(dr)$,

(b) the generalized Poisson-Lie DYBE is satisfied:

$$\mathrm{Alt}(\theta^\sharp dr)+\frac{1}{2}[r,r]+\delta r=\Omega\in(\wedge^3\mathfrak{g})^{\mathfrak{g}}. \tag{6.6}$$

Example 2.4. *Let* G *be a Poisson-Lie group and* (G^*,π) *the simply connected dual Poisson-Lie group. Let* $\mathfrak{g}=\mathrm{Lie}(G)$ *and* $\mathfrak{g}^*=\mathrm{Lie}(G^*)$*. To consider the generalized dynamical r-matrices on* (G^*,π)*, we take a natural section* θ *of* $TG^*\otimes\mathfrak{g}$*, which is induced by the isomorphism* $T^*G^*\to G^*\times\mathfrak{g}$*. Then one checks that* $\theta^\sharp:T^*G^*\to\mathfrak{g}$ *is a Lie algebroid morphism. A direct calculation shows that equation* (a) *and* (b) *in Definition 2.3 reduces to*

$$\mathrm{dress}^L_a(r)+[a\otimes 1+1\otimes a,r]=0,$$
$$[r,r]+\mathrm{Alt}(d^Lr)+\mathrm{Alt}((\delta\otimes\mathrm{id})(r))=\Omega,$$

for $r:G^*\to\mathfrak{g}\wedge\mathfrak{g}$ *and any* $a\in\mathfrak{g}$*, where* dress^L_a *denotes the left dressing vector field generated by* a *and* $d^Lr(g):=e_i\otimes\frac{d}{dt}|_{t=0}r(\mathrm{e}^{-te_i}g)$ *for each* $g\in G^*$ *and an orthonormal basis* $\{e_i\}$ *of* $\mathfrak{g}$*. Thus a map* $r:G^*\to\mathfrak{g}\wedge\mathfrak{g}$ *is a generalized dynamical r-matrix coupled with* G^* *via* θ *if and only if* r *is a Poisson-Lie dynamical r-matrix [31].*

2.1 Generalized classical dynamical r-matrices from (quasi-)Poisson reduction

In this subsection, we explain that generalized classical dynamical r-matrices naturally appear in the theory of (quasi-)Poisson reduction. First, let us recall the definition of quasi-Poisson G-manifolds.

We assume that $(\mathfrak{g},\langle\cdot,\cdot\rangle)$ is a quadratic Lie algebra, ϕ is the Cartan 3-tensor. In terms of an orthogonal basis $\{e_a\}$ of $\mathfrak{g}$, $\phi\in\wedge^3\mathfrak{g}$ is given by

$$\phi=\frac{1}{12}f_{abc}e_a\wedge e_b\wedge e_c, \tag{6.7}$$

where $f_{abc} = \langle e_a, [e_b, e_c]\rangle$ are the structure constants of $\mathfrak{g}$. Given a G-manifold M, the Lie algebra homomorphism $\rho : \mathfrak{g} \to TM$ can be extended to an equivariant map,

$$\rho : \wedge^\bullet \mathfrak{g} \to \wedge^\bullet TM,$$

preserving wedge products and Schouten brackets.

Definition 2.5. [6] *A quasi-Poisson manifold is a G-manifold M, equipped with an invariant bivector field $\pi \in \Gamma(\wedge^2 TM)$ such that*

$$[\pi, \pi] = \rho(\phi). \tag{6.8}$$

Example 2.6. *Let G be a Lie group and $\{e_a\}_{a\in I}$ be an orthogonal basis of its Lie algebra $\mathfrak{g}$. Define a bivector field on G by $\pi_G = \frac{1}{2}\sum_{a\in I} R_a \wedge L_a$, where R_a and L_a are right and left invariant vectors generated by e_a. Then (G, π_G) is a quasi-Poisson G-manifold, where G acts on itself by conjugation.*

Generally, the G-invariant functions on a quasi-Poisson manifold (M, π) form a Poisson algebra under the binary bracket induced by π. Thus we obtain a Poisson algebra on $C^\infty(M)^G$. Given M a G-manifold with G acting locally freely and $\rho : \mathfrak{g} \to TM$ the corresponding infinitesimal action, we will use the same symbol ρ to denote the following natural extension map:

$$\rho : \wedge^\bullet(TM \oplus \mathfrak{g}) \to \wedge^\bullet TM. \tag{6.9}$$

Throughout this paper, we denote the skew-symmetrization of any section $A \in \Gamma(\wedge^\bullet(TM) \otimes \wedge^\bullet \mathfrak{g})$ by $\hat{A} \in \Gamma(\wedge^\bullet(TM \oplus \mathfrak{g}))$. Thus, if $\theta = f^{ia}(x)\frac{\partial}{\partial x_i} \otimes e_a \in \Gamma(TM \otimes \mathfrak{g})$ in local coordinates $\{x_i\}$ of U and a basis $\{e_a\}$ of $\mathfrak{g}$, we have $\hat{\theta} = f^{ia}(x)\frac{\partial}{\partial x_i} \wedge e_a$.

Theorem 2.7. *Let $U \subset M$ be a cross-section of the G action and π_M be a bivector field on M. Then there exists a unique triple (π_U, θ, r), where $\pi_U \in \Gamma(\wedge^2 TU)$, $\theta \in \Gamma(TU \otimes \mathfrak{g})$ and $r : U \to \mathfrak{g} \wedge \mathfrak{g}$ such that*

$$\pi_M|_U = \pi_U - \rho(\hat{\theta})|_U + \rho(r)|_U. \tag{6.10}$$

Moreover,

(*a*) *if π_M is a G-invariant Poisson tensor on M, then (U, π_U) is a Poisson manifold and r is a triangular dynamical r-matrix coupled with U via θ.*

(*b*) *if π_M is a quasi-Poisson tensor on M, then (U, π_U) is a Poisson manifold and r is a dynamical r-matrix coupled with U via θ with respect to the Cartan 3-tensor, i.e., $\Omega = -\frac{1}{2}\phi$ in the generalized CDYBE.*

Proof. Because G acts locally freely and U is a cross-section, for any $x \in U$ there is a canonical splitting $T_xM = T_xU \oplus \rho_x(\mathfrak{g})$ of the sequence

$$0 \to \mathfrak{g} \to T_xM \to T_xU \to 0.$$

Thus, there exists unique $\pi_U \in \Gamma(\wedge^2 TU)$, $\theta \in \Gamma(TU \otimes \mathfrak{g})$ and $r : U \to \mathfrak{g} \wedge \mathfrak{g}$ such that

$$\pi_M|_U = \pi_U - \rho(\hat{\theta})|_U + \rho(r)|_U,$$

where π_U is tangent to U.

If π_M is a G-invariant Poisson tensor or quasi-Poisson tensor, it induces a Poisson bracket $\{\cdot, \cdot\}$ on U. From the expression (6.10) and the fact $\rho(e)f' = 0$ for any $e \in \mathfrak{g}$ and $f' \in C^\infty(M)^G$, we have that $\{f, g\} = \pi_U(df, dg)$ for any $f, g \in C^\infty(U)$. This is to say (U, π_U) is a Poisson manifold. The remaining thing is to check that the triple (π_U, θ, r) satisfies the required compatibility condition and the generalized CDYBE. This can be seen from the proofs of Theorem 2.8 and Theorem 2.12. ■

Theorem 2.7 suggests the following construction which generalizes the construction for ordinary classical dynamical r-matrices in [84]. Given a manifold M, $M \times G$ carries natural right and left G-actions defined respectively by $(x, p) \cdot g = (x, pg)$ and $g \cdot (x, p) = (x, gp)$ for all $x \in M$, $p, g \in G$. Let ρ^L denote the infinitesimal left G-actions. Then we have

Theorem 2.8. *Let (M, π_M) be a Poisson manifold and $\theta \in \Gamma(TM \otimes \mathfrak{g})$. Then any smooth function $r : M \to \mathfrak{g} \wedge \mathfrak{g}$ induces a left G-invariant bivector π_r on $M \times G$ which is given by*

$$\pi_r = \pi_M + \rho^L(\hat{\theta}) + \rho^L(r), \tag{6.11}$$

and

(a) π_r is a Poisson tensor iff r is a triangular generalized dynamical r-matrix.

(b) π_r is a quasi-Poisson tensor iff r is a generalized dynamical r-matrix with respect to the Cartan 3-tensor.

Proof. Note that the vector field on $M \times G$ has a natural bigrading: elements in TM have degree $(1, 0)$ while elements in TG have degree $(0, 1)$. It is simple to see that $[\pi_M, \pi_M]$ is of degree $(3, 0)$, $[\pi_M, \rho^L(\hat{\theta})]$ is of degree $(2, 1)$, $[\pi_M, \rho^L(r)]$ is of degree $(1, 2)$ and $[\rho^L(r), \rho^L(r)]$ is of degree $(0, 3)$. On the other hand, $[\rho^L(\hat{\theta}), \rho^L(r)]$ consists of elements of degree $(1, 2)$ and of $(0, 3)$ and $[\rho^L(\hat{\theta}), \rho^L(\hat{\theta})]$ consists of elements of degree $(2, 1)$ and of $(1, 2)$. For any $S \in \wedge^3(TM \oplus TG)$, let $S = \sum_{0 \le i,j \le 3} S^{(i,j)}$ be its decomposition with respect to this bigrading. The following equations can be verified by a direct computation:

$$[\rho^L(\hat{\theta}), \rho^L(\hat{\theta})]^{(1,2)} = \rho^L(\widehat{[\theta,\theta]}), \qquad [\rho^L(\hat{\theta}), \rho^L(\hat{\theta})]^{(2,1)} = 2\rho^L(\widehat{\theta\wedge\theta}), \tag{6.12}$$
$$[\rho^L(\hat{\theta}), \rho^L(r)]^{(0,3)} = \rho^L(\mathrm{Alt}(\theta^* dr)), \qquad [\pi_M, \rho^L(r)] = \rho^L(\pi_M^\sharp(dr)), \tag{6.13}$$
$$[\rho^L(\hat{\theta}), \rho^L(r)]^{(1,2)} = -\rho^L(\widehat{[r,\theta]}), \qquad [\pi_M, \rho^L(\hat{\theta})] = \rho^L(d_{\pi_M}\theta) \tag{6.14}$$

where the operations $[\theta,\theta]$, $\theta\wedge\theta$ and $[r,\theta]$ for $\theta \in \Gamma(TM\otimes\mathfrak{g})$ and $r \in C^\infty(M, \wedge^2\mathfrak{g})$ are defined as (6.2) and (6.3). Eventually, by using the facts $[\rho^L(r), \rho^L(r)] = \rho^L([r,r])$ and $[\pi_M, \pi_M] = 0$ we have

$$\begin{aligned}
[\pi_r, \pi_r] &= [\pi_M + \rho^L(\hat{\theta}) + \rho^L(r), \pi_M + \rho^L(\hat{\theta}) + \rho^L(r)] \\
&= [\rho^L(\hat{\theta}), \rho^L(\hat{\theta})] + 2[\pi_M, \rho^L(\hat{\theta})] + 2[\rho^L(\hat{\theta}), \rho^L(r)] + 2[\pi_M, \rho^L(r)] + [\rho^L(r), \rho^L(r)] \\
&= 2\rho^L(\mathrm{Alt}(\theta^* dr) + \frac{1}{2}[r,r]) + 2\rho^L(\widehat{[\theta,\theta]} - \widehat{[r,\theta]} + \widehat{\pi^*(dr)}) + 2\rho^L(\widehat{\theta\wedge\theta} + d_{\pi_M}\theta).
\end{aligned}$$

Note that the map $\theta^\sharp : T^*M \to \mathfrak{g}$ is a Lie algebroid morphism if and only if

$$\widehat{\theta\wedge\theta} + d_{\pi_M}\theta = 0.$$

Therefore we have that $[\pi_r, \pi_r] = 0$ iff r is a triangular generalized dynamical r-matrix and $[\pi_r, \pi_r] = \rho^R(\phi)$ iff r is a generalized dynamical r-matrix w.r.t $\Omega = -\frac{1}{2}\phi$. ■

Similarly, we have the following theorem.

Theorem 2.9. *Let (N, π_N) be a quasi-Poisson G-space and $\rho : \mathfrak{g} \to TN$ be the infinitesimal action. Then for any generalized dynamical r-matrix coupled with (M, π_M) via θ with respect to $\Omega = -\frac{1}{2}\phi$,*

$$\pi := \pi_M + \pi_N + \rho(\hat{\theta}) + \rho(r) \tag{6.15}$$

is a Poisson tensor on $M \times N$.

Proof. We need to prove $[\pi, \pi] = 0$. Note that $[\rho(r), \pi_N] = [\rho(\hat{\theta}), \pi_N] = 0$ because of the invariance of π_N. Thus we have

$$\begin{aligned}
[\pi, \pi] &= [\pi_M + \pi_N + \rho(\hat{\theta}) + \rho(r), \pi_M + \pi_N + \rho(\hat{\theta}) + \rho(r)] \\
&= [\pi_M, \pi_M] + [\pi_N, \pi_N] + \rho([r,r]) + 2[\rho(\hat{\theta}), \rho(r)] + 2[\pi_M, \rho(\hat{\theta})] + 2[\pi_M, r] \\
&= [\pi_N, \pi_N] + 2\rho(\mathrm{Alt}(\theta^* dr) + \frac{1}{2}[r,r]) \\
&= \rho(\phi) - \rho(\phi) = 0.
\end{aligned}$$

This finishes the proof. ■

Now we discuss the relation between generalized dynamical r-matrices and the reduction of the fusion product of two quasi-Poisson manifolds. Let M, N be two G-manifolds and ρ_M, ρ_N be the corresponding infinitesimal G action. We define a bivector field on $M \times N$ by

$$\Phi = \sum_{a \in I} \rho_M(e_a) \wedge \rho_N(e_a),$$

where $\{e_a\}_{a\in I}$ is an orthogonal basis of $\mathfrak{g}$.

Proposition 2.10. [6] *If (M, π_M) and (N, π_N) are two quasi-Poisson G-manifolds, then $\pi_M + \pi_N - \Phi$ gives a quasi-Poisson structure on $M \times N$ for the diagonal G-action. This quasi-Poisson manifold, denoted by $M \circledast N$, is called the fusion product of M and N.*

Example 2.11. *Let (G, π_G) be the quasi-Poisson G-manifold given in Example 2.6. By doing the fusion product with itself, we get a quasi-Poisson manifold $D(G) := G \circledast G$. Let R^i_a and L^i_a denote the right and left invariant vector fields on the i-th copy of $G \times G$ generated by e_a, then the quasi-Poisson tensor takes the form:*

$$\pi_{G^2} = \frac{1}{2}\sum_a (R^1_a \wedge L^1_a + R^2_a \wedge L^2_a + (L^1_a - R^1_a) \wedge (L^2_a - R^2_a)). \tag{6.16}$$

Let (M, π_M) and (N, π_N) be two quasi-Poisson G-manifolds. We assume G acts locally freely on M. Let $U \subset M$ be any cross-section. By Theorem 2.7, associated to U, there is a triple (π_U, θ, r) such that r is a generalized dynamical r-matrix coupled with (U, π_U) via θ. On the other hand, $U \times N$ is a cross-section of the diagonal G action on $M \times N$. So it inherits a Poisson structure π_{red} by the reduction from the quasi-Poisson structure on $M \circledast N$.

Theorem 2.12. *The Poisson tensor π_{red} on $U \times N$ takes the form of*

$$\pi_{\text{red}} = \pi_U + \rho_N(\hat{\theta}) + \rho_N(r) + \pi_N. \tag{6.17}$$

Before giving a proof, we show the following lemma which says that on the reduction level, fusion product and direct product give rise to the same Poisson structure.

Lemma 2.13. *Let M and N be two quasi-Poisson G-manifolds. Then for any diagonal G-invariant functions $f, g \in C^\infty(M \times N)^G$, $\Phi(df, dg) = 0$. Moreover, $\pi_M + \pi_N$ induces a Poisson algebra structure on $C^\infty(M \times N)^G$ which is the same as the Poisson algebra on $C^\infty(M \circledast N)^G$.*

Proof. Note that $\Phi = \sum \rho_M(e_a) \wedge \rho_N(e_a)$, and $(\rho_M(e_a) + \rho_N(e_a))f = 0$ for all $f \in C^\infty(M \times N)^G$. So $\Phi = -\sum \rho_N(e_a) \wedge \rho_N(e_a) = 0$ when restricts to G-invariant functions. ■

Proof of Theorem 2.12 For any $f, g \in C^\infty(U \times N)$, let $f', g' \in C^\infty(M \times N)^G$ be the diagonal G-invariant extension of f, g respectively. Then by Lemma 2.13, $\pi_{\text{red}}(df, dg) = (\pi_M + \pi_N)(df', dg')|_{U\times N}$. Following Theorem 2.7, we have

$$\pi_M|_U = \pi_U - \rho_M(\hat{\theta})|_U + \rho_M(r)|_U.$$

Together with the fact that $(\rho_M(e_a) + \rho_N(e_a))F = 0$ for any $F \in C^\infty(M \times N)^G$, we get

$$(\pi_M + \pi_N)(df', dg')|_{U\times N} = (\pi_U + \rho_N(\hat{\theta}) + \rho_N(r) + \pi_N)(df', dg')|_{U\times N}.$$

Note that $\pi_U + \rho_N(\hat{\theta}) + \rho_N(r) + \pi_N$ is tangent to $U \times N$ and $f'|_U = f$, $g'|_U = g$. Therefore,

$$(\pi_U + \rho_N(\hat{\theta}) + \rho_N(r) + \pi_N)(df', dg')|_{U\times N} = (\pi_U + \rho_N(\hat{\theta}) + \rho_N(r) + \pi_N)(df, dg).$$

This is to say $\pi_{\text{red}} = \pi_U + \rho_N(\hat{\theta}) + \rho_N(r) + \pi_N$. ■

2.2 Classical dynamical r-matrices and Poisson G-spaces

Let G be a complex semisimple Lie group with $\text{Lie}(G) = \mathfrak{g}$, $\kappa \in (S^2\mathfrak{g})^{\mathfrak{g}}$ the element corresponding to the Killing form on $\mathfrak{g}$ and $\Lambda = \frac{1}{2}\kappa + r_0$ any classical quasi-triangular r-matrix with skew-symmetric part r_0. Let ρ^R and ρ^L denote respectively the infinitesimal actions of the right and left translations. We have

$$\pi_G := \rho^L(r_0) - \rho^R(r_0) \tag{6.18}$$

defines a Lie Poisson structure on G.

Let H be a Lie subgroup of G with $\mathfrak{h} = \text{Lie}(H)$ and $r = \frac{1}{2}\kappa + A_r : \mathfrak{h}^* \to \mathfrak{g} \otimes \mathfrak{g}$ a classical dynamical r-matrix with the skew-symmetric part A_r, i.e., r is an H-equivariant map and satisfies the classical dynamical Yang-Baxter equation

$$\text{Alt}(\theta^\sharp(dA_r)) + \frac{1}{2}[A_r, A_r] = \frac{1}{4}[\kappa^{12}, \kappa^{23}] = -\frac{1}{2}\phi, \tag{6.19}$$

where ϕ is the Cartan 3-tensor, $\theta = \frac{\partial}{\partial x^i} \wedge e_i$ for a basis $\{e_i\}$ of $\mathfrak{h}$ and the corresponding coordinates $\{x^i\}$ on $\mathfrak{h}^*$.

On the other hand, we can think of r_0 as a dynamical r-matrix over a point. Therefore, similar to Theorem 2.9, we have

Proposition 2.14. *Associated to the dynamical r-matrix A_r, there is a Poisson structure π_r on $\mathfrak{h}^* \times G$ defined by*

$$\pi_r = \pi_{\mathrm{KKS}} + \rho^L(\theta) + \rho^L(A_r) - \rho^R(r_0), \tag{6.20}$$

where π_{KKS} is the Kirillov-Kostant-Souriau (KKS) Poisson structure on $\mathfrak{h}^*$.

Proof. Note that the Schouten brackets $[\pi_{KKS}, \rho^R(r_0)]$, $[\rho^L(\theta), \rho^R(r_0)]$ and $[\rho^L(A_r), \rho^R(r_0)]$ are zero, and $[\rho^R(r_0), \rho^R(r_0)] = -\rho^R[r_0, r_0] = \rho^R(\phi) = \rho^L(\phi)$. A straightforward calculation shows that $[\pi_r, \pi_r] = 0$. ■

Now let us consider the H action on $\mathfrak{h}^* \times G$ given by

$$h \cdot (x, g) = (\mathrm{Ad}_h^* x, gh), \ \forall h \in H, x \in \mathfrak{h}^*, g \in G. \tag{6.21}$$

Proposition 2.15. *The Poisson tensor π_r on $\mathfrak{h}^* \times G$ is H-invariant.*

Proof. In the expression of the Poisson tensor π_r on $\mathfrak{h}^* \times G$, the components π_{KKS}, $\rho^L(\theta)$ and $\rho^R(r_0)$ are obviously H-invariant. Therefore, π_r is H-invariant as long as $\rho^L(A_r)$ is H-invariant which can be seen from the fact that the map $A_r : \mathfrak{h}^* \to \mathfrak{g} \wedge \mathfrak{g}$ is H-equivariant. ■

Recall that an action of the Poisson-Lie group (G, π_G) on a Poisson manifold M is said to be Poisson if the action map $G \times M \to M$ is a Poisson map, where $G \times M$ is equipped with the product Poisson structure. In our case, $\mathfrak{h}^* \times G$ carries a natural left G-action, i.e.,

$$g_1 \cdot (x, g_2) = (x, g_1 g_2), \ \forall x \in \mathfrak{h}^*, g_1, g_2 \in G. \tag{6.22}$$

Proposition 2.16. *$(\mathfrak{h}^* \times G, \pi_r)$ is a Poisson (G, π_G)-space with respect to the left G-action.*

Proof. By definition, $G \times (\mathfrak{h}^* \times G) \to \mathfrak{h}^* \times G$ is a Poisson map if and only if for all $g_1 \in G$ and $a = (x, g_2) \in \mathfrak{h}^* \times G$,

$$\pi_r(g_1 \cdot a) = g_{1*}(\pi_r(a)) + a_* \pi_G(g_1), \tag{6.23}$$

where a in the last term denotes the map $a : G \to \mathfrak{h}^* \times G$, $a(g_1) := (x, g_1 g_2)$, for any $g_1 \in G$. Equation (6.23) can be obtained by the following calculation

$$\begin{aligned} \pi_r(g_1 \cdot a) &= \pi_r(x, g_1 g_2) \\ &= \pi_{KKS}(x, g_1 g_2) + L_{g_1 g_2}\theta(x) + L_{g_1 g_2} A_r(x) - R_{g_1 g_2} r_0 \\ &= L_{g_1}(\pi_{KKS}(x, g_2) + \theta(x, g_2)) + L_{g_1}(L_{g_2} A_r(x) - R_{g_2} r_0) + R_{g_2}(L_{g_1} r_0 - R_{g_1} r_0) \\ &= g_{1*}(\pi_r(a)) + a_*(\pi_G(g_1)), \end{aligned} \tag{6.24}$$

where $R_g(L_g)$ denotes the right(left) translation from the identity element to g. ■

By Poisson reduction, the H-invariant Poisson tensor π_r on $\mathfrak{h}^* \times G$ induces a Poisson structure π_s on $(\mathfrak{h}^* \times G)/H = G \times_H \mathfrak{h}^*$. Since the G-action and the H-action on $\mathfrak{h}^* \times G$ commute, $(\mathfrak{h}^* \times G)/H$ carries an induced left G action, $G \times (\mathfrak{h}^* \times G)/H \to (\mathfrak{h}^* \times G)/H$. Furthermore, we have

Theorem 2.17. *$((\mathfrak{h}^* \times G)/H, \pi_s)$ is a Poisson (G, π_G)-space.*

The above theorem is a consequence of the following general proposition.

Proposition 2.18. *Let (M, π_M) be a Poisson (G, π_G)-space. Suppose a Lie group H acts freely on M, commuting with the G action, such that π_M is H-invariant. Then the reduced Poisson manifold $(M/H, \pi_{red})$ is a Poisson (G, π_G)-space with respect to the induced G-action.*

Proof. By definition, for any $g \in G$ and $a \in \mathfrak{h}^* \times G$, we have

$$\pi_r(g \cdot a) = g_* \pi_r(a) + a_*(\pi_G(g)). \tag{6.25}$$

By quotienting the H action on the two sides, we see that the left action map of G on $(\mathfrak{h}^* \times G)/H$ is a Poisson map. ■

The result in this subsection gives a geometric interpretation of Lu's construction of Poisson homogeneous spaces from dynamical r-matrices in [65].

2.3 Generalized dynamical r-matrices associated with conjugacy classes

Given a conjugacy class $\mathcal{C}$ in G, let us identify the tangent space $T_g\mathcal{C}$ at g with $\mathfrak{g}_g^\perp$, where $\mathfrak{g}_g$ is the centralizer of g and $\mathfrak{g}_g^\perp$ its complement. The operator $\mathrm{Ad}_g - 1$ is invertible on $\mathfrak{g}_g^\perp$. Thus we get a linear operator

$$\frac{\mathrm{Ad}_g + 1}{\mathrm{Ad}_g - 1}|\mathfrak{g}_g^\perp := (\frac{\mathrm{Ad}_g + 1}{\mathrm{Ad}_g - 1}) \circ \mathrm{Pr}_{\mathfrak{g}_g^\perp} : \mathfrak{g} \to \mathfrak{g},$$

where $\mathrm{Pr}_{\mathfrak{g}_g^\perp}$ is the projection of $\mathfrak{g}$ on $\mathfrak{g}_g^\perp$. Let $\sum_{a \in I} R_a \wedge L_a$ be a bivector field on G (take the convention in Example 2.6). Then we have

Proposition 2.19 (Proposition 3.4, [6])**.** *$\sum_{a \in I} R_a \wedge L_a = \sum_{a,b \in I} \frac{1}{2}(\frac{\mathrm{Ad}_g+1}{\mathrm{Ad}_g-1}|\mathfrak{g}_g^\perp)_{ab} X_a \wedge X_b$ as bivector fields on G, where $\{e_a\}_{a \in I}$ is a basis of $\mathfrak{g}$ and $X_a = L_a - R_a$ for any $e_a \in \mathfrak{g}$.*

Let $(G \times G, \pi_{G^2})$ be the quasi-Poisson G-manifold given in Example 2.11. As a result of the above proposition, for any conjugacy classes $\mathcal{C}_1$, $\mathcal{C}_2$, $\pi_{G^2}|_{\mathcal{C}_1\times\mathcal{C}_2}$ is tangent to $\mathcal{C}_1 \times \mathcal{C}_2$, i.e., $(\mathcal{C}_1 \times \mathcal{C}_2, \pi_{G^2}|_{\mathcal{C}_1\times\mathcal{C}_2})$ is a quasi-Poisson manifold with respect to the diagonal conjugation G-action. Assume that the G-action is locally free on some open subset of $\mathcal{C}_1 \times \mathcal{C}_2$, which is ture for most interesting cases, for example when G is semisimple and $\mathcal{C}_1$, $\mathcal{C}_2$ are generic. We will choose a cross-section and study the associated generalized dynamical r-matrix. To do this, let $T \subset G$ be a maximal torus. Let $p \in \mathcal{C}_1 \cap T$ and G_p the centralizer of p with $\mathfrak{g}_p = \mathrm{Lie}(G_p)$. For any some open subset of $\mathcal{C}_2$ where the conjugation G_p-action is locally free, we then choose a cross-section $U \subset \mathcal{C}_2$. Thus $\{p\} \times U$ is a cross-section of the G-action on $\mathcal{C}_1 \times \mathcal{C}_2$. Following Theorem 2.7, associated to the choice of $\{p\} \times U$, there is a dynamical r-matrix $(\pi_{p\times U}, \theta, r)$. To write it down explicitly, we introduce a function $H \in C^\infty(U, \mathrm{End}(\mathfrak{g}))$ as follows. For each point $x \in U$, let $\mathfrak{g}'_x$ be the subspace of $\mathfrak{g}$ defined by

$$\mathfrak{g}'_x = \{e \in \mathfrak{g} \mid \frac{d}{dt}|_{t=0} \mathrm{Ad}_{\exp(te)} x \in T_x U\}.$$

Because U is a cross-section of the conjugation G_p action, we have a decomposition $\mathfrak{g} = \mathfrak{g}_p \oplus \mathfrak{g}'_x$. Then we define $H_x \in \mathrm{End}(\mathfrak{g})$ to be the projection of $\mathfrak{g}$ on $\mathfrak{g}'_x$. Furthermore, associated to each $e_a \in \mathfrak{g}$, we define a function $H(e_a) \in C^\infty(U, \mathfrak{g})$ and a vector field $H(X_a) \in \Gamma(TU)$ by

$$H(X_a)|_x := \frac{d}{dt}|_{t=0} \mathrm{Ad}_{\exp(tH_x(e_a))} x, \ \forall x \in U. \tag{6.26}$$

Theorem 2.20. *The generalized dynamical r-matrix $(\pi_{p\times U}, \theta, r)$ induced from the reduction of the quasi-Poisson tensor π_{G^2} on $\mathcal{C}_1 \times \mathcal{C}_2$ takes the form*

$$\pi_{p\times U} = \frac{1}{2}\sum_{a,b}((\frac{\mathrm{Ad}_p + 1}{\mathrm{Ad}_p - 1}|_{\mathfrak{g}_p^\perp})_{ab} + \frac{\mathrm{Ad}_x + 1}{\mathrm{Ad}_x - 1}|\mathfrak{g}_x^\perp)_{ab}) H(X_a) \wedge H(X_b), \tag{6.27}$$

$$\theta = \frac{1}{2}\sum_a H(X_a) \otimes e_a + \frac{1}{2}(\frac{\mathrm{Ad}_p + 1}{\mathrm{Ad}_p - 1}|_{\mathfrak{g}_p^\perp})_{ab} H(X_a) \otimes H(e_b) + \frac{1}{2}\sum_{a,b}(\frac{\mathrm{Ad}_x + 1}{\mathrm{Ad}_x - 1}|\mathfrak{g}_x^\perp)_{ab} H(X_a) \otimes (H(e_b) - e_b), \tag{6.28}$$

$$r = \frac{1}{2}\sum_a e_a \wedge H(e_a) + \frac{1}{2}(\frac{\mathrm{Ad}_p + 1}{\mathrm{Ad}_p - 1}|_{\mathfrak{g}_p^\perp})_{ab} H(e_a) \wedge H(e_b) + \frac{1}{2}\sum_{a,b}(\frac{\mathrm{Ad}_x + 1}{\mathrm{Ad}_x - 1}|\mathfrak{g}_x^\perp)_{ab}(e_a - H(e_a)) \wedge (e_b - H(e_b)). \tag{6.29}$$

Proof. Note that for the cross-section $p \times U$ of the G_p action on $\mathcal{C}_1 \times \mathcal{C}_2$, the triple $(\pi_{p\times U}, \theta, r)$ corresponds to the decomposition of $\pi_{G^2}|_{p\times U}$ with respect to TU and the complement $\rho(\mathfrak{g})|_U$ generated by the diagonal G-action, where $\rho(e_a) = X_a^1 + X_a^2$ for any $e_a \in \mathfrak{g}$. On the other hand, from Example 2.11 and Proposition 2.19, we have at any point $(p, x) \in \{p\} \times U$

$$\pi_{G^2} = \frac{1}{4}\sum_{a,b}(\frac{\mathrm{Ad}_p + 1}{\mathrm{Ad}_p - 1}|\mathfrak{g}_p^\perp)_{ab} X_a^1 \wedge X_b^1 + \frac{1}{4}\sum_{a,b}(\frac{\mathrm{Ad}_x + 1}{\mathrm{Ad}_x - 1}|\mathfrak{g}_x^\perp)_{ab} X_a^2 \wedge X_b^2 + \sum_a X_a^1 \wedge X_a^2 \quad (6.30)$$

With the help of this expression, we just need to compute the decomposition of $X_a^i \in \Gamma(T(\mathcal{C}_1 \times \mathcal{C}_2)|_U)$ along the two directions TU and $\rho(\mathfrak{g})|_U$. After a direct computation, we get the following decompositions:

$$X_a^1|_{p\times U} = -H(X_a) + \rho(H(e_a))|_{p\times U}, \quad (6.31)$$

$$X_a^2|_{p\times U} = H(X_a) + \rho(e_a - H(e_a))|_{p\times U}, \quad (6.32)$$

where $\rho(H(e_a)) \in \Gamma(T(\mathcal{C}_1 \times \mathcal{C}_2))$ is given by $\rho(H(e_a))|_{y,x} = (X^1_{H_x(e_a)} + X^2_{H_x(e_a)})|_{y,x}$ for all $(y, x) \in \mathcal{C}_1 \times \mathcal{C}_2$. To be precise, by the definition of H, $e_a - H(e_a) \in C^\infty(U, \mathfrak{g}_p)$ where $\mathfrak{g}_p$ is the Lie algebra of the isotropic group G_p, so we get $X_a^1 = X^1_{H(e_a)}$ when restricts to $p \times U$. Thus we have

$$-H(X_a) + \rho(H(e_a))|_{p\times U} = -H(X_a) + X^1_{H(e_a)}|_{p\times U} + H(X_a) = X_a^1|_{p\times U}.$$

A similar calculation gives the equation (6.32).

In the end, we get the expression of the triple $(\pi_{p\times U}, \theta, r)$ by plugging (6.31) and (6.32) in the expression of $\pi_{G^2}|_{p\times U}$. ■

We refer to the generalized dynamical r-matrices associated to two conjugacy classes in G as moduli space generalized dynamical r-matrices.

Let us take $G = \mathrm{SU}(2)$ for a concrete example. Let

$$e_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

be a basis of su(2) and $\mathcal{C} \subset \mathrm{SU}(2)$ the conjugacy class through p= $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Then $\mathcal{C}$ can be identified with the sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ and an element in $\mathcal{C}$ takes the form $\begin{pmatrix} ix & y + iz \\ -y + iz & -ix \end{pmatrix}$. The diagonal matrix $\begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{pmatrix}$ acts on $\mathcal{C} = S^2 \in \mathbb{R}^3$ by rotation with respect to x-axis, i.e., $e^{i\beta} \circ (x, y, z) = (x, e^{2i\beta}y, e^{2i\beta}z)$. For this S^1 action, we choose a simple cross-section

$$U := \{(x, y, z) \mid -1 < x < 1,\ y = 0,\ z > 0\} \subset \mathcal{C} = S^2 \in \mathbb{R}^3.$$

We parameterize U by introducing α such that $x = \sin\alpha$ and $z = \cos\alpha$. Then the isotropic subspace of $\mathfrak{g}$ at a point $\alpha \in U \subset \mathfrak{g}$ is defined by

$$\mathfrak{g}'_\alpha = \{A \in \mathfrak{g} \mid \frac{d}{dt}|_{t=0} \mathrm{e}^{tA}\alpha\mathrm{e}^{-tA} \in T_\alpha U\}.$$

A direct calculation gives

Proposition 2.21. *In terms of* e_1, e_2, e_3, $\mathfrak{g}'_\alpha = \mathrm{Span}\{e_1 + \tan\alpha e_2, e_3\}$.

It follows that the function $H \in C^\infty(U, \mathrm{End}(\mathfrak{g}))$ is given by

$$H(e_1) = 0,\ H(e_2) = \cot\alpha e_1 + e_2,\ H(e_3) = e_3.$$

The corresponding vector fields on U generated by adjoint action are given by

$$H(X_1) = 0,\ H(X_2) = 0,\ H(X_3) = \frac{\partial}{\partial\alpha}.$$

Another straightforward computation shows that $\mathfrak{g}_\alpha^\perp = \mathrm{Span}\{\tan\alpha e_1 - e_2, e_3\}$, where $\mathfrak{g}_\alpha$ is the Lie subalgebra of the stabilizer of $\mathfrak{g}$ at $\alpha \in U$. However, we have

$$(\mathrm{Ad}_\alpha + 1)e_3 = (\mathrm{Ad}_\alpha + 1)(\tan\alpha e_1 - e_2) = 0.$$

It indicates that $(\frac{\mathrm{Ad}_\alpha+1}{\mathrm{Ad}_\alpha-1}|\mathfrak{g}_\alpha^\perp) : \mathfrak{g}_\alpha^\perp \to \mathfrak{g}_\alpha^\perp$ is a zero transformation. Eventually, by Theorem 2.20, the dynamical r-matrix associated to the local section $p \times U$ of $(\mathcal{C} \times \mathcal{C})/G$ takes the form of

$$r = \tan\alpha e_1 \wedge e_2, \qquad \theta = \frac{\partial}{\partial\alpha} \otimes e_3. \tag{6.33}$$

2.4 Gauge transformations of generalized classical dynamical r-matrices

Let G be a Lie group and $\Theta = g^{-1}dg$ the Cartan one form. Let $r : U \to \mathfrak{g} \wedge \mathfrak{g}$ be a generalized dynamical r-matrix coupled with a Poisson manifold (U, π_U) via $\theta \in \Gamma(TU \otimes \mathfrak{g})$ w.r.t some Ω.

Definition 2.22. *We define the gauge transformation of a smooth map* $\sigma : U \to G$ *on* (π_U, θ, r) *by*

$$r^\sigma := \mathrm{Ad}_\sigma \otimes \mathrm{Ad}_\sigma(r + \widehat{\langle\sigma^*\Theta, \theta\rangle} + \langle\pi_U, \sigma^*\Theta \wedge \sigma^*\Theta\rangle), \tag{6.34}$$
$$\theta^\sigma := \mathrm{Ad}_\sigma(\theta + \langle\pi_U, \sigma^*\Theta\rangle), \tag{6.35}$$

where $\langle\cdot,\cdot\rangle$ *is the natural pairing between forms and (muti)vector fields, and* $\widehat{\langle\sigma^*\Theta, \theta\rangle} :$ $U \to \mathfrak{g} \wedge \mathfrak{g}$ *is the skew-symmetrization of* $\langle\sigma^*\Theta, \theta\rangle$.

Proposition 2.23. *r^σ is a generalized classical dynamical r-matrix coupled with U via θ^σ with respect to Ω.*

Proof. Following Theorem 2.8, given the dynamical r-matrix (π_U, θ, r), we can construct a right invariant bivector $\pi := \pi_U + \rho^L(\hat{\theta}) + \rho^L(r)$ on $U \times G$ such that $[\pi, \pi] = -2\rho(\Omega)$. We denote the graph of the map $\sigma : U \to G$ by $U' \subset U \times G$. Then π is a right G-invariant bivector fields and U' is a cross-section of the right G action. Following the argument in Theorem 2.7, associated to π and U' there exists a dynamical r-matrix $(\pi_{U'}, \theta_{U'}, r_{U'})$ w.r.t Ω. Let us take the isomorphism $F : U \to U'$; $F(x) = (x, \sigma(x)) \in U'$ for any $x \in U$. A straightforward calculation shows that

$$F_*\pi_U = \pi_{U'}, \ F^*\theta^\sharp_{U'} = \theta^{\sigma\sharp} \text{ and } r_{U'} \circ F = r^\sigma.$$

It means that if we identify U' with U by F, the triple $(\pi_{U'}, \theta_{U'}, r_{U'})$ becomes $(\pi_U, \theta^\sigma, r^\sigma)$. It finishes the proof. ■

The geometric meaning of gauge transformations of generalized dynamical r-matrices is illuminated in the proof of Proposition 2.23. Another interpretation is as follows. Let (M, π_M) be a (quasi-)Poisson G-manifold and (π_U, θ, r) be a dynamical r-matrix with respect to Ω. Given a gauge transformation $\sigma : U \to G$, we define a diffeomorphism from $U \times M$ to itself by

$$\sigma \cdot (x, p) = (x, \sigma(x) \cdot p), \ \forall p \in G. \tag{6.36}$$

Proposition 2.24. *Following Theorem 2.12, let $(U \times M, \pi_r)$ and $(U \times M, \pi_{r^\sigma})$ be the Poisson manifolds associated to (π_U, θ, r) and $(\pi_U, \theta^\sigma, r^\sigma)$ respectively. Then we have*

$$\{F \circ \sigma, G \circ \sigma\}_r = \{F, G\}_{r^\sigma} \circ \sigma,$$

for any $F, G \in C^\infty(U \times G)$.

2.5 Generalized classical dynamical r-matrices and Poisson groupoids

In this subsection, we discuss the geometric interpretation of the generalized CDYB equation. Recall that in [37], Etingof and Varchenko found a geometric interpretation of the CDYB equation that generalizes Drinfeld's interpretation of the CYB equation in terms of Poisson-Lie groups. Namely, they constructed a so called dynamical Poisson-Lie groupoid structure on the direct product manifold $\eta^* \times G \times \eta^*$, where η is a Lie subalgebra of $\mathfrak{g}$. The CYBE can be viewed as the special case of the generalized CYBE (see Example 2.2). An observation here is that $\eta^* \times G \times \eta^*$ is the Lie groupoid integrating the Lie algebroid $T\eta^* \oplus \mathfrak{g}$. Furthermore, the Poisson structure on $\eta^* \times G \times \eta^*$ induces a Lie bialgebroid structure on $T\eta^* \oplus \mathfrak{g}$. Similarly, in the case of the generalized dynamical r-matrix, we have the following theorem. Let M be a manifold, $\mathfrak{g}$ be a Lie algebra and $(TM \oplus \mathfrak{g}, [\cdot,\cdot]_L)$ be a Lie algebroid with the anchor map given by the projection to TM, the bracket given by

$$[X + A, Y + B]_L = [X, Y] + L_X B - L_Y A + [A, B]_{\mathfrak{g}}, \tag{6.37}$$

for all $X, Y \in \Gamma(TM)$ and $A, B \in \Gamma(M \times \mathfrak{g})$.

Theorem 2.25 (Theorem 4.5, [62]). *A solution r of the generalized DYBE coupled with (M, π) via θ induces a coboundary Lie bialgebroid structure $(TM \oplus \mathfrak{g}, d_*)$ where the differential $d_* : \Gamma(\wedge^\bullet(TM \oplus \mathfrak{g})) \to \Gamma(\wedge^{\bullet+1}(TM \oplus \mathfrak{g}))$ corresponding to the Lie algebroid structure on $T^*M \oplus \mathfrak{g}^*$ is of the form*

$$d_* = [\pi_M + \hat{\theta} + r, \cdot]_L,$$

where $[\cdot,\cdot]_L$ is the Schouten bracket on $\wedge^\bullet(TM \oplus \mathfrak{g})$.

Similarly, a solution of the generalized Poisson-Lie DYBE coupled with (M, π_M) via θ gives a Lie bialgebroid $(TM \oplus \mathfrak{g}, d_*)$, where the differential $d_* = \delta + [\pi_M + \hat{\theta} + r, \cdot]$. According to the theory of integration of Lie bialgebroids in [67], we have

Corollary 2.26. *Associated to a generalized classical dynamical r-matrix coupled with (M, π) via θ, there is a Poisson groupoid structure on $\mathcal{G} = M \times G \times M$ whose tangent Lie bialgebroid is $(TM \oplus \mathfrak{g}, d_*)$.*

Thus the Poisson groupoid $M \times G \times M$ gives a geometric interpretation of the generalized DYBE that generalizes Drinfeld's interpretation of the CYBE in terms of Poisson-Lie groups.

A smooth manifold M is called $\mathcal{G}$-space for a Lie groupoid $(\mathcal{G} \rightrightarrows P, s, t)$ if there are two smooth maps, the moment map and the action map, $J : M \to P$ and

$$\alpha : \mathcal{G} \times_J M = \{(x, m) \in \mathcal{G} \times M \mid t(x) = J(m)\} \to M$$

such that, writing $\alpha(x, m) = x \cdot m$, for all compatible $x, y \in \mathcal{G}$ and $m \in M$,

(i) $J(x \cdot m) = s(x)$;

(ii) $(x \cdot y) \cdot m = x \cdot (y \cdot m)$;

(iii) $J(m) \cdot m = m$.

Now suppose that M is a $\mathcal{G}$-space. The action of $\mathcal{G}$ on M is a Poisson action if its graph $\{(x, m, x \cdot m) \mid t(x) = J(m)\}$ is a coisotropic submanifold of $\mathcal{G} \times M \times \overline{M}$ [82]. Then M is called a Poisson $\mathcal{G}$-space.

Associated to a dynamical r-matrix r coupled with (M, π_M) via θ with respect to the Cartan 3-tensor and a quasi-Poisson G-space N, we have a Poisson groupoid $\mathcal{G} = M \times G \times M$ (Corollary 2.26) and a Poisson manifold $(M \times N, \pi = \pi_M + \rho_N(\hat{\theta}) + \rho_N(r) + \pi_N)$ (by Theorem 2.9). Furthermore, there is a natural action of the groupoid $\mathcal{G} \rightrightarrows M$ on $M \times N$,

$$(x, g, y) \cdot (y, p) = (x, g \cdot p)$$

for all $x, y \in M$, $g \in G$ and $p \in N$. This is a Lie groupoid action with respect to the moment map $J : M \times N \to M$ given by the natural projection. An observation here is that this action is a Poisson action.

Theorem 2.27. *The $\mathcal{G}$-space $(M \times N, \pi)$ is a Poisson $\mathcal{G}$-space.*

To prove this theorem, we need the following results.

Lemma 2.28 (Theorem 3.3, [53])**.** *Let $\mathcal{G}$ be a Poisson groupoid with its tangent Lie bialgebroid (A, A^*). Then a Poisson manifold (M, π) is a Poisson $\mathcal{G}-$space if and only if the vector bundle morphism from T^*M to A^*, the dual of the infinitesimal action map, is a Lie algebroid morphism.*

Lemma 2.29 (Theorem 3.1, [53])**.** *Let A^* be a Lie algebroid over P, and (M, π) a Poisson manifold. Then, for the cotangent Lie algebroid T^*M induced by the Poisson structure, a vector bundle morphism $\Phi : T^*M \to A^*$ over $J : M \to P$ is a Lie algebroid morphism if and only if the following two conditions hold:*

(i) $H_{J^*f} = -\Phi^*(d_*f), \ \forall f \in C^\infty(P)$;

(ii) $L_{\Phi^*(X)}\pi = -\Phi^*(d_*S), \ \forall S \in \Gamma(A)$,

where H_{J^*f} denotes the Hamiltonian vector field on M, which is defined by $H_g h = \pi(dg, dh)$ for all $g, h \in C^\infty(M)$. The differential d_* comes from the Lie algebroid structure on A^*.

Proof of Theorem 2.27 In our case, the Poisson manifold is $(M \times N, \pi = \pi_M + \rho_N(\hat{\theta}) + \rho_N(r) + \pi_N)$ and the Lie bialgebroid is $(TM \oplus \mathfrak{g}, d_* = [\pi_M + \hat{\theta} + r, \cdot])$. So by Lemma 2.28 and Lemma 2.29, we just need to prove $H_{J^*f} = -F(d_*f), \ \forall f \in C^\infty(M)$ and $L_{F(S)}\pi_M = -F(d_*S), \ \forall S \in \Gamma(TM \otimes \mathfrak{g})$, where the bundle map $F : TM \oplus \mathfrak{g} \to T(M \times N)$ is the infinitesimal action of $M \times G \times M$ on $M \times N$, explicitly given by $F(X + e) = X + \rho(e)$ for $X \in \Gamma(TM)$ and $e \in \mathfrak{g}$.

(1) Following the expression of the Poisson tensor π on $M \times N$ given in Theorem 2.9, we have that for all $f \in C^\infty(M)$,

$$H_{J^*f} = \pi_M^*(df) + \rho(\theta^*(df)),$$

where $J : M \times N \to M$ is the natural projection map. On the other hand, by the definition of the differential d_*,

$$d_*f = [\pi_M, f] + [\hat{\theta}, f] + [r, f].$$

Note that $[r, f] = 0$, $F([\pi_M, f]) = [\pi_M, f]$, and $F([\hat{\theta}, f]) = F(-\theta^*(df)) = -\rho(\theta^*(df))$. Therefore,

$$H_{J^*f} = -[\pi_M + \rho(\hat{\theta}) + \rho(r), f] = -F(d_*f).$$

(2) Set $S = X + e \in \Gamma(TM \oplus \mathfrak{g})$, then

$$d_*S = [\pi_M + \hat{\theta} + r, X + e] = [\pi_M + \hat{\theta} + r, X] + [\pi_M + \hat{\theta} + r, e].$$

By the definition of $F : \wedge^*(TM \oplus \mathfrak{g}) \to T(M \times N)$, we see that the map F and the Schouten-bracket $[\cdot, \cdot]_L$ on $\wedge^*(TM \otimes \mathfrak{g})$ commute. As a result,

$$F([\pi_M + \hat{\theta} + r, X]) = [F(\pi_M) + F(\hat{\theta}) + F(r), F(X)] = [\pi, X].$$

Similarly,

$$F([\pi_M + \hat{\theta} + r, e]) = [F(\pi_M) + F(\hat{\theta}) + F(r), F(e)] = [\pi, \rho(e)].$$

Eventually, we get $-F(d_*S) = L_{F(S)}\pi$. This finishes the proof. ■

3 Generalized dynamical r-matrices and moduli spaces of flat connections on surfaces

3.1 Poisson and quasi-Poisson structures on the moduli spaces of flat connections on surfaces

In [10], Atiyah and Bott introduced canonical symplectic structures on the moduli spaces of flat G-connections on oriented surfaces. A convenient finite dimensional description of these moduli spaces is as follows. Let $\Sigma_{g,n}$ be an oriented surface of genus g with n punctures and $\{\mathcal{C}_i\}_{i=1,\ldots,n}$ a set of conjugacy classes of G. Then the moduli space of flat G-connections on $\Sigma_{g,n}$ is given by the character variety, i.e., the space of group homomorphisms $h : \pi_1(\Sigma_{g,n}) \to G$ that map the homotopy equivalence class of a loop around the i-th puncture to the associated conjugacy class $\mathcal{C}_i \subset G$. Two such group homomorphisms describe gauge-equivalent connections if and only if they are related by conjugation with an element of G. This implies that the moduli space of flat G-connections on $\Sigma_{g,n}$ is given by

$$X_{G,\mathcal{C}}(\Sigma_{g,n}) = \mathrm{Hom}_{\mathcal{C}_1,\dots,\mathcal{C}_n}(\pi_1(\Sigma_{g,n}, G))/G = \{h \in \mathrm{Hom}(\pi_1(\Sigma_{g,n}), G) \mid h(m_i) \in \mathcal{C}_i\}/G,$$

where G acts by conjugation, $\mathcal{C}$ denotes the choice of the set of conjugacy classes, and $m_i \in \pi_1(\Sigma_{g,n})$ corresponds to the loop around i-th puncture. By characterising the group homomorphisms in terms of the images of the generators of $\pi_1(S_{g,n})$, it is the set

$$\{(M_1, ..., M_n, A_1, B_1, ..., A_g, B_g) \in G^{n+2g} | M_i \in \mathcal{C}_i, [B_g, A_g] \cdots [B_1, A_1] \cdot M_n \cdots M_1 = 1\}/G,$$

where the quotient is taken with respect to the diagonal action of G on G^{n+2g}. The smooth part of this space carries a natural symplectic structure [50]. In the Fock-Rosly approach [47], an explicit description of the symplectic structure is obtained by Poisson reduction of a Poisson structure on the enlarged ambient space G^{n+2g}. To write down the Poisson tensor on this enlarged space, let us introduce two natural operators ∇_R, $\nabla_L \in \Gamma(TG \otimes \mathfrak{g}^*)$ which are given for all $A \in \mathfrak{g}$, $p \in G$ by

$$\langle \nabla_R, A \rangle f(p) := \frac{d}{dt}|_{t=0} f(p\mathrm{e}^{-tA}), \tag{6.38}$$

$$\langle \nabla_L, A \rangle f(p) := \frac{d}{dt}|_{t=0} f(\mathrm{e}^{tA} p). \tag{6.39}$$

Then we define $2(n + 2g)$ covariant differential operators in the following way:

$$\begin{aligned} \nabla_{2i-1} &= \nabla_R^{M_i}, & \nabla_{2i} &= \nabla_L^{M_i} \; for \; i = 1, ..., n; \\ \nabla_{n+4i-3} &= \nabla_R^{A_i}, & \nabla_{n+4i-1} &= \nabla_L^{A_i} \; for \; i = 1, ..., g; \\ \nabla_{n+4i-2} &= \nabla_R^{B_i}, & \nabla_{n+4i-1} &= \nabla_L^{B_i} \; for \; i = 1, ..., g. \end{aligned} \tag{6.40}$$

Definition 3.1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. For any $r \in \mathfrak{g} \otimes \mathfrak{g}$, the corresponding the Fock-Rosly bivector $B_r^{n,g} \in \Gamma(\wedge^2(TG^{n+2g}))$ is defined by*

$$B_r^{n,g}(df, dh) := \frac{1}{2} \sum_i \langle r, \nabla_i f \wedge \nabla_i h \rangle + \sum_{i<j} \langle r, \nabla_i f \wedge \nabla_j h \rangle. \tag{6.41}$$

Theorem 3.2. [47] Let $\mathfrak{g}$ be a Lie algebra with a non-degenerate Ad-invariant symmetric bilinear form. If $r \in \mathfrak{g} \otimes \mathfrak{g}$ is a solution of the classical Yang-Baxter equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0, \tag{6.42}$$

then $B_r^{n,g}$ defines a Poisson structure on G^{n+2g}. Furthermore, when the symmetric part κ of r dual to the bilinear form on $\mathfrak{g}$, this Poisson structure induces the canonical symplectic structure on the moduli space of flat G-connections on $\Sigma_{g,n}$.

From the expression of $B_r^{n,g}$, we see that the Poisson bracket of two functions on G^{n+2g} depends only on the symmetric component of r if one of the two functions is invariant under the diagonal action of G on G^{n+2g}. Thus we can use the symmetric part κ of r and reduction procedure to describe the Poisson structure on the quotient space G^{n+2g}/G. Notice that the bivector $B_\kappa^{g,n}$ given in (6.41) is the part of $B_r^{g,n}$ which only depends on κ. It turns out that $B_\kappa^{g,n}$ coincides with the quasi-Poisson bivector on the fusion product $G \circledast ... \circledast G \circledast D(G) \circledast ... \circledast D(G)$ (n copies of G and g coies of $D(G)$, where G and $D(G)$ are the quasi-Poisson manifolds given in Example 2.6 and 2.11 respectively. If we restrict to a set of conjugacy classes $\{\mathcal{C}_i\}_{i=1,...,n}$, then it gives a way to describe the standard symplectic structure on $X_{G,\mathcal{C}}(\Sigma_{n,g}) = \{h \in \mathrm{Hom}(\pi_1(\Sigma_{g,n}, G)|h(m_i) \in \mathcal{C}_i\}/G$ by using quasi-Poisson geometry.

Theorem 3.3. [6] *Consider the quasi-Poisson manifold*

$$P_{g,n} = \mathcal{C}_1 \circledast ... \circledast \mathcal{C}_n \circledast D(G) \circledast ... \circledast D(G),$$

where $\mathcal{C}_1$,...,$\mathcal{C}_n$ are conjugacy classes of G. Then the quasi-Poisson reductions of $P_{g,n}$ are isomorphic to the moduli spaces of flat G-connections on $\Sigma_{g,n}$ with the Atiyah-Bott symplectic form.

3.2 GCDYB equations and moduli spaces of flat connections on surfaces

In this subsection, we will combine the discussion in previous sections and give our main result which describes the canonical symplectic structure on the moduli spaces of flat connections on surfaces by using generalized dynamical r-matrices.

Following Theorem 3.3, the symplectic structure on $X_{G,\mathcal{C}}(\Sigma_{g,n})$ is given by the reduction of the quasi-Poisson structure $B_\kappa^{g,n}$ on $P_{g,n}$ with respect to the simultaneous conjugation action of G.

Reduction with respect to two punctures

We assume that there are at least two punctures on the surface $\Sigma_{g,n}$, ie., $n \geq 2$. If we choose a local cross-section of the G action on $X_{G,\mathcal{C}}(\Sigma_{g,n})$, then the reduced Poisson structure on this section is viewed to be a local model of the Poisson structure on $X_{G,\mathcal{C}}(\Sigma_{g,n})$. We proceed the reduction in a "minimal" way, i.e., imposing gauge fixing conditions on the first two punctures as follows. First, we think of $P_{g,n}$ as the fusion product of $(\mathcal{C}_1 \times \mathcal{C}_2, \pi_{G^2})$ and $(P_{g,n-2}, B_\kappa^{g,n-2})$, i.e., $P_{g,n} = (\mathcal{C}_1 \circledast \mathcal{C}_2) \circledast P_{g,n-2}$, where $P_{g,n-2} := \mathcal{C}_3 \circledast ... \circledast \mathcal{C}_n \circledast D(G) \circledast ... \circledast D(G)$. Then let U be any local cross-section of the diagonal conjugation action of G on $\mathcal{C}_1 \times \mathcal{C}_2$ and (π_U, θ, r) the associated moduli space generalized dynamical r-matrix. Finally, we have that $U \times P_{g,n-2}$ is a local cross-section of the G action on $X_{G,\mathcal{C}}(\Sigma_{g,n})$, and by Theorem 2.9, the reduced Poisson structure on it is given by

$$\pi_{\text{red}} = \pi_U + \rho(\hat{\theta}) + \rho(r) + B_\kappa^{g,n-2},$$

where $\rho : \mathfrak{g} \to P_{g,n-2}$ is the infinitesimal action generated by simultaneous conjugation G action. Furthermore, a simple comparison shows that $\rho(r) = B_r^{g,n-2}$ as bivector fields on $U \times P_{g,n-2}$, where $B_r^{g,n-2}$ is the Fock-Rosly bivector field associated to $r : U \to \mathfrak{g} \wedge \mathfrak{g}$ given by (6.41) (depending on a parameter space U). As a result, $\rho(r) + B_\kappa^{g,n-2} = B_{r+\kappa}^{g,n-2}$, where r and κ can be seen as skew-symmetric and symmetric parts of an entire function $r + k \in C^\infty(U, \mathfrak{g} \otimes \mathfrak{g})$. Eventually, we obtain

Theorem 3.4. *The quasi-Poisson structure $B_\kappa^{g,n}$ on $P_{g,n}$ induces a Poisson bracket on $U \times \mathcal{C}_3 ... \times \mathcal{C}_n \times G^{2g}$, which is isomorphic to the Atiyah-Bott symplectic structure and takes the following form:*

(1) *For $f, g \in C^\infty(\mathcal{C}_3 ... \times \mathcal{C}_n \times G^{2g})$.*

$$\{f, g\} = B_{r+\kappa}^{n-2,g}(df, dg) \tag{6.43}$$

(2) *For $f \in C^\infty(\mathcal{C}_3 ... \times \mathcal{C}_n \times G^{2g})$ and $\phi, \varphi \in C^\infty(U)$:*

$$\{f, \phi\} = \rho(\hat{\theta})(df, d\phi) \tag{6.44}$$

$$\{\phi, \varphi\} = \pi_U(d\phi, d\varphi), \tag{6.45}$$

Note that the original Fock-Rosly bivector field $B_r^{n,g}$ on G^{n+2g} is associated to a classical r-matrix. Here we introduce a dynamical version of the Fock-Rosyly bivector field which is related to a dynamical r-matrix, and use it to give a new description of the Atiyah-Bott symplectic structure on the moduli space $X_{G,\mathcal{C}}(\Sigma_{g,n})$. One immediate consequence of this viewpoint is the following proposition due to Theorem 2.27. It indicates a Poisson groupoid symmetry of the moduli space $X_{G,\mathcal{C}}(\Sigma_{g,n})$.

Proposition 3.5. *Let $\mathcal{G} = U \times G \times U$ be the Poisson Lie groupoid associated to the moduli space dynamical r-matrix (π_U, r, θ), then $(U \times \mathcal{C}_3 ... \times \mathcal{C}_n \times G^{2g}, \{\cdot,\cdot\})$ carries a natural Poisson $\mathcal{G}$ action.*

Gauge fixing and classical dynamical r-matrices in $\mathrm{ISO}(2,1)$-Chern-Simons theory.

In [72], Meusburger and Schönfeld obtained classical dynamical r-matrices by considering gauge fixing in $\mathrm{ISO}(2,1)$-Chern-Simons theory. Now, we interpret these classical dynamical r-matrices as moduli space dynamical r-matrices corresponding to the special case $G = \mathrm{ISO}(2,1)$.

First, let us give the required notations. We denote by $e_0 = (1,0,0)$, $e_1 = (0,1,0)$, $e_2 = (0,0,1)$ the standard basis of $\mathbb{R}^3$. By ε_{abc} we denote the totally skew-symmetric tensor in three dimensions with the convention $\varepsilon_{012} = 1$. The indices of ε_{abc} are raised with the three-dimensional Minkowski metric $\eta = \mathrm{diag}(1,-1,-1)$.

The Poincaré group in 3-D is the semidirect product $\mathrm{ISO}(2,1) = \mathrm{SO}_+(2,1) \ltimes \mathbb{R}^3$ of the proper orthochronous Lorentz group $\mathrm{SO}_+(2,1)$ and the translation group $\mathbb{R}^3$. The elements of $\mathrm{ISO}(2,1)$ are parameterized as

$$(u, \mathbf{a}) = (u, 0) \cdot (1, -\mathbf{j}) = (u, -\mathrm{Ad}(u)\mathbf{j}) \textit{ with } u \in \mathrm{SO}_+(2,1),\ \mathbf{j}, \mathbf{a} \in \mathbb{R}^3.$$

The corresponding coordinate functions $\{j^a\}_{a=0,1,2}$ are given by

$$j^a : \mathrm{ISO}(2,1) \to \mathbb{R}, \quad (u, -\mathrm{Ad}(u)\mathbf{q}) \to q^a.$$

Let $\{J_a\}_{a=0,1,2}$ be a basis of $\mathfrak{so}(2,1)$ such that the Lie bracket takes the form $[J_a, J_b] = \varepsilon_{ab}{}^c J_c$. Hence a basis of the Lie algebra $\mathfrak{iso}(2,1)$ is given by $\{J_a\}_{a=0,1,2}$ together with a basis $\{P_a\}_{a=0,1,2}$ of the abelian Lie algebra $\mathbb{R}^3$.

The moduli space of flat G-connections $X_{G,\mathcal{C}}(\Sigma_{g,n})$ can be viewed as a constrained system in the sense of Dirac [23]. In this spirit, the moduli space is obtained from $P_{n,g} = \mathcal{C}_1 \times ...\mathcal{C}_n \times D(G) \times ... \times D(G)$ by imposing a group-valued constraint that arises from the defining relation of the fundamental group $\pi_1(\Sigma_{g,n})$. In the case of $G = \mathrm{ISO}(2,1)$, the group-valued constraint is a set of six first constraints in the Dirac gauge fixing formalism for the Fock-Rosly Poisson tensor on $P_{n,g}$. The associated gauge transformations which they generate via the Poisson bracket are given by the diagonal action of $\mathrm{ISO}(2,1)$ on $\mathrm{ISO}(2,1)^{n+2g}$.

A choice of gauge fixing conditions for the constraints is investigated in [72]. These gauge fixing conditions implement the quotient by $\mathrm{ISO}(2,1)$ and restrict the first two components of all points $(M_1, ..., B_g) \in \Sigma = C^{-1}(0)$ in such a way that M_1, M_2 are determined uniquely by two real parameters ψ and α given in terms of the components of the product $M_2 \cdot M_1 = (u_{12}, -Ad(u_{12})\mathbf{j}_{12})$ as

$$\psi = f(\mathrm{Tr}(u_{12})), \qquad \alpha = g(\mathrm{Tr}(u_{12}))\mathrm{Tr}(j_{12}^a J_a \cdot u_{12}) + h(\mathrm{Tr}(u_{12})), \tag{6.46}$$

where $f, g \in C^\infty(\mathbb{R})$ are arbitrary diffeomorphisms and $h \in C^\infty(R)$. This allows us to identify the constraint surface $\Sigma = C^{-1}(0)$ with a subset of $\mathbb{R}^2 \times \mathrm{ISO}(2,1)^{n-2+2g}$, where the $\mathbb{R}^2$ is parameterized by (ψ, α) and $\mathrm{ISO}(2,1)^{n-2+2g}$ by $(M_3, ..., B_g)$.

Let us pose the Dirac gauge fixing constraints in such form. By Theorem 4.5 in [72], there exist maps

$$\mathbf{q}_\psi,\ \mathbf{q}_\alpha,\ \mathbf{q}_\delta,\ m:\mathbb{R}^2\to\mathbb{R}^3,\ V:\mathbb{R}\to\mathrm{Mat}(3,\mathbb{R})$$

such that the associated Dirac bracket is given in terms of them. On the other hand, the Dirac gauge fixing is equivalent to choose a cross-section of the ISO(2, 1) action on $\mathcal{C}_1\times\mathcal{C}_2$, which is the locus of the constraint functions. Therefore by Theorem 3.4, associated to this cross-section, there is a moduli space dynamical r-matrix. It interprets the origin of the dynamical r-matrices found in [72], which are given in our framework by the following propostion.

Proposition 3.6. *The moduli space dynamical classical r-matrix (π,θ,r) corresponding to the Dirac gauge fixing procedure is given by*

$$\pi=0,\qquad \theta=q_\alpha^a\frac{\partial}{\partial\alpha}\otimes J_a+q_\psi^a\frac{\partial}{\partial\psi}\otimes P_a+q_\delta^a\frac{\partial}{\partial\alpha}\otimes P_a, \tag{6.47}$$

$$r=-V^{bc}(\psi)(P_b\otimes J^c-J^c\otimes P_b)+\varepsilon^{bcd}m_d(\psi,\alpha)P_b\otimes P_c. \tag{6.48}$$

Moreover, the induced Poisson bracket takes the following form:
for any $f,g\in C^\infty(\mathrm{ISO}(2,1)^{n-2+2g})$,

$$\{\alpha,\psi\}=0,\quad \{\alpha,f\}=\rho(\hat{\theta})(d\alpha,df),\quad \{f,g\}=B_{r+\kappa}^{n-2,g}(df,dg) \tag{6.49}$$

where $\kappa=P_a\otimes J^a$.

Given a map $\sigma:\mathbb{R}^2\to\mathrm{ISO}(2,1)$, let us consider the smooth map

$$\Phi^\sigma:\mathbb{R}^2\times\mathrm{ISO}(2,1)^{n-2+2g}\to\mathbb{R}^2\times\mathrm{ISO}(2,1)^{n-2+2g},$$
$$(\psi,\alpha,M_3,...,B_g)\mapsto(\psi,\alpha,\mathrm{Ad}_\sigma M_3,...,\mathrm{Ad}_\sigma B_g).$$

As a consequence of Proposition 2.24, we have

Corollary 3.7. *Let $\{\cdot,\cdot\}$ be the bracket given in (6.49) with respect to (θ,r). Then for all $F,G\in C^\infty(\mathbb{R}^2\times\mathrm{ISO}(2,1)^{n-2+2g})$,*

$$\{F\circ\Phi^\sigma,G\circ\Phi^\sigma\}=\{F,G\}^\sigma\circ\Phi^\sigma, \tag{6.50}$$

where $\{\cdot,\cdot\}_D^\sigma$ is the bracket given in (6.43) with respect to (θ^σ,r^σ), the gauge transformation of $\sigma:\mathbb{R}^2\to\mathrm{ISO}(2,1)$ on (θ,r).

Particularly, the map $\sigma = (g, -Ad(g)\mathbf{t})$ satisfying $\partial_\alpha g = \partial_\alpha^2 \mathbf{t} = 0$ is called dynamical Poincaré transformation in [72]. Dynamical r-matrices from different gauge fixing conditions subject to extra conditions given in [72] are related by dynamical Poincaré transformations.

A standard set of dynamical r-matrices from the Dirac gauge fixing in ISO$(2,1)$-Chern-Simons theory is given explicitly in [72]. This set of solutions corresponds to special gauge fixing condition which is motivated by its direct physical interpretation in the application to the Chern-Simons formulation of $(2+1)$-gravity.

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