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BACKREACTION OF LINEAR PERTURBATIONS AND DARK ENERGY^a

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A simple discussion on the backreaction of inhomogeneities in cosmology, focusing on the possibility that it could explain the present acceleration and solve the coincidence problem.

1 Introduction

The coincidence problem. Perhaps the most surprising observation in modern cosmology is that the expansion of the universe has apparently started to accelerate in the recent past, at a redshift of probably less than one. This phenomenon has usually been interpreted in the context of a homogeneous and isotropic model of the universe. The acceleration can then be accommodated only by modifying gravity or by introducing an energy component with negative pressure (or energy density), 'dark energy', the magnitude of which is fit to the observations.

These models suffer from the *coincidence problem*: they do not explain why the energy densities of matter and dark energy have become comparable only recently. If there is a dynamical explanation for the acceleration having started in the near past –instead of it being a coincidence– then presumably it is related to the dynamics we see in the universe. The most significant change in the universe at small redshifts is the formation of large scale structure, so it seems a natural possibility that the observed deviation from the natural prediction of the homogeneous and isotropic model could be related to the growth of inhomogeneities in the universe.

The fitting problem. The reasoning behind using a homogeneous and isotropic model is that the universe appears to be homogeneous and isotropic when averaged over sufficiently large

^aContribution to the proceedings of the 2004 Moriond workshop "Exploring the Universe", La Thuile, Italy, March 28-April 4. Based on ¹, where more details and references can be found.

scales. In the usual approach one *first* takes the average of the metric and the energy-momentum tensor, and then plugs these smooth quantities into the Einstein equation. However, physically one should first plug the inhomogeneous quantities into the Einstein equation and *then* take the average. Because the Einstein equation is non-linear, these two procedures are not equivalent:

$$\langle G_{\mu\nu}(g_{\alpha\beta}) \rangle \neq G_{\mu\nu}(\langle g_{\alpha\beta} \rangle) , \quad (1)$$

where $G_{\mu\nu}$ is the Einstein tensor, $g_{\alpha\beta}$ is the metric and $\langle \rangle$ stands for averaging. The content of (1) is that *the average behaviour of an inhomogeneous spacetime is not the same as the behaviour of the corresponding smooth spacetime*. Here “corresponding” means that the smooth and average quantities have the same initial conditions. In other words, the average properties of an inhomogeneous spacetime (energy density, expansion rate, ...) do not satisfy the Einstein equation.

This is the *fitting problem* discussed by George Ellis in 1983². Simply put, how does one find the average model which best fits the real inhomogeneous universe? The difference between the behaviour of average and smooth quantities is also known as *backreaction*.

The equations satisfied by the average quantities in a general inhomogeneous spacetime have been obtained³. However, these equations are not closed, which simply means that different spacetimes sharing the same initial average values evolve differently even as far as their averages are concerned – not a surprising result. We have taken (following⁴) a more modest approach, in which one assumes a smooth background and studies the effect of perturbations.

2 The backreaction calculation

The expansion rate. We consider a homogeneous and isotropic universe that is spatially flat and filled with a pressureless fluid. We assume that the fluid has scale-invariant adiabatic perturbations (with zero mean) and an amplitude given by the measurements of the CMB. In first order perturbation theory, the metric in the longitudinal gauge reads

$$ds^2 = -(1 + 2\Phi(\bar{x}))dt^2 + (1 - 2\Phi(\bar{x}))a(t)^2d\bar{x}^2 , \quad (2)$$

where $a = (t/t_0)^{2/3}$ and Φ is constant in time.

We want to find out the effect of the perturbations on the expansion rate measured by a comoving observer. Since the spacetime is inhomogeneous, the expansion rate is different for observers in different points, and to make the comparison to a homogeneous and isotropic universe, we should take an average. The expansion rate is a covariantly defined scalar quantity, so this operation is well defined, but the result depends on which hypersurface one chooses to take the average on^{4,5}. We *define* the average expansion rate to be the average of the expansion rates measured by comoving observers (weighed by the volume element) whose clocks show the same proper time τ (measured from the big bang). In other words, we average on the hypersurface of constant proper time τ of comoving observers.

The covariant definition of the local expansion rate is $\theta = u^\mu{}_{;\mu}$, where u^μ is the four-velocity of the matter fluid. From the metric (2), we find u^μ and from that we obtain the expansion rate θ and the proper time τ . Then we simply express θ in terms of τ and take the average. Note that this calculation involves no dynamics: we just take a given metric and calculate the observable of interest for that spacetime.

Naively, one might expect the average expansion rate to be

$$\langle \theta \rangle \simeq 3H_\tau \left(1 + \alpha_1 \langle \Phi \rangle + \alpha_2 \langle \Phi^2 \rangle + \alpha_3 \langle \Phi^3 \rangle \right) , \quad (3)$$

where $H_\tau = 2/(3\tau)$ is the background expansion rate in terms of the proper time, and α_i are some constants of order one. Here and in what follows we expand only to second order in Φ (note

that we use results from first-order perturbation theory, so that this is not a consistent second order calculation). Since Φ has zero mean, the linear term and its square vanish, and since the amplitude of Φ is 10^{-5} , the quadratic term is of the order of 10^{-10} – completely negligible.

When one actually calculates the expansion rate, the result is not (3), but instead

$$\begin{aligned} \langle \theta \rangle &\simeq 3H_\tau \left(1 + \beta_1 \frac{1}{(a_\tau H_\tau)^2} \langle \nabla^2 \Phi \rangle + \beta_2 \frac{1}{(a_\tau H_\tau)^4} \langle \nabla^2 \Phi \rangle^2 + \beta_3 \frac{1}{(a_\tau H_\tau)^2} \langle \partial_i \Phi \partial_i \Phi \rangle \right. \\ &\quad \left. + \beta_4 \frac{1}{(a_\tau H_\tau)^2} \langle \partial_i (\Phi \partial_i \Phi) \rangle + \beta_5 \frac{1}{(a_\tau H_\tau)^4} \langle \partial_i (\nabla^2 \Phi \partial_i \Phi) \rangle \right) \\ &\equiv 3H_\tau \left(1 + \lambda_1 a_\tau + \lambda_2 a_\tau^2 \right) , \end{aligned} \quad (4)$$

where β_i are given constants of order one, $a_\tau = (\tau/\tau_0)^{2/3}$, and λ_i are constants.

The plain powers of Φ that one would expect do not appear, and instead the backreaction is given by gradients of Φ . The structure of the backreaction terms in (4) is easy to understand: each gradient has to be accompanied by $1/a$ for reasons of covariance and by $1/H$ for reasons of dimensionality. Since θ is a scalar, gradients have to appear in pairs, and since the Einstein equation is second order, there are at most two gradients for each Φ . Since $1/(aH)^2 \propto a$, the backreaction grows relative to the background, and thus behaves like dark energy.

Backreaction as dark energy. For a homogeneous and isotropic spatially flat universe, the expansion law is (in this case $\theta = 3H$)

$$(\theta/3)^2 = \frac{1}{3M_{Pl}^2} \rho , \quad (5)$$

where M_{Pl} is the Planck mass and ρ is the energy density, which for matter behaves like $\rho \propto a^{-3}$.

In contrast to (5), the average of the inhomogeneous expansion law is, from (4),

$$\begin{aligned} \langle \theta/3 \rangle^2 &\simeq H_\tau^2 (1 + \lambda_1 a_\tau + \lambda_2 a_\tau^2)^2 \\ &\propto a_\tau^{-3} + 2\lambda_1 a_\tau^{-2} + (\lambda_1^2 + 2\lambda_2) a_\tau^{-1} + 2\lambda_1 \lambda_2 + \lambda_2^2 a_\tau , \end{aligned} \quad (6)$$

where in the second equality we have taken into account $H_\tau^2 \propto a_\tau^{-3}$.

An observer in a perturbed universe trying to fit the smooth model (5) to the observed expansion rate (6) would conclude that there is a mysterious energy component which is nowhere to be seen but which affects the expansion rate. Writing $\rho = \rho_m + \rho_{de}$ where ρ_m is the energy density of matter, and ρ_{de} is the apparent energy density of this 'dark energy', one finds from the usual relation $\rho_{de} \propto a^{-3(1+w)}$ the equations of state $w = -1/3, -2/3, -1$ and $-4/3$.

The equation of state is negative because the backreaction grows relative to the background for which $w = 0$. Note that there is nothing unnatural about having an equation of state which is more negative than -1 (as marginally indicated by observations^{6,7}). It does not imply violation of the weak energy condition, since the 'dark energy' is only a parametrisation of our ignorance of the real expansion law, and does not correspond to an actual energy component.

Backreaction has a negative equation of state that could produce acceleration, so the next question is whether the effect is significant. In other words: what are the values of λ_1 and λ_2 ? For perturbations in the linear regime, we have

$$\lambda_1 = \beta_1 \langle \partial_i \Phi \partial_i \Phi \rangle / H_0^2 \sim 10^{-5} , \quad \lambda_2 = \beta_2 \langle \partial_i (\nabla^2 \Phi \partial_i \Phi) \rangle / H_0^4 \sim \langle \delta_0^2 \rangle \sim 1 , \quad (7)$$

where H_0 is the Hubble parameter today and δ_0 is the density perturbation today. The constant λ_1 is easy to evaluate and is negligible, while λ_2 is more involved. We have used the relation $\nabla^2 \Phi = -3(aH)^2 \delta / 2$, valid in the linear regime of perturbations, which we have taken to end at $\langle \delta^2 \rangle = 1$. The quantity to be averaged can then (by definition) be of order one point by point. However, since it is a total derivative, the periodicity implied by the standard decomposition in terms of Fourier modes puts it artificially to zero. The contribution of similar terms at order Φ^4

could be non-vanishing, since the square of a total derivative is not a total derivative. If we would have taken the average of θ^2 instead of averaging θ and squaring it, we would have found

$$\langle(\theta/3)^2\rangle \simeq H_\tau^2 \left(1 + \frac{4}{81} \frac{1}{(aH)^4} \langle \nabla^2 \Phi \nabla^2 \Phi \rangle\right) = H_\tau^2 \left(1 + \frac{1}{9} \langle \delta^2 \rangle\right), \quad (8)$$

where all subdominant terms have been dropped. For comparison to the observations it is more correct to take the average first, but (8) shows that backreaction from linear perturbations can have a sizeable effect; taking the cut-off to be at $\langle \delta^2 \rangle = 1$, the correction is of the order of 10%.

The reason that the effect can be large even though the metric perturbation Φ is small is that the physics is not given just by the metric but by its first and second derivatives. The derivatives of the metric are dimensional quantities, so they are small or large only with respect to some scale. In the case of spatial derivatives in cosmology, the comparison scale turns out to be aH , the size of the horizon, compared to which the spatial gradients are large.

3 Conclusion

In summary, the backreaction of linear perturbations on the expansion rate is non-zero, boosted by powers of $k^2/(aH)^2$ from the naive expectation of powers of Φ , and naturally involves a negative equation of state. For periodic boundary conditions, it is also numerically negligible at second order in Φ .

In evaluating the backreaction of linear perturbations, we had to introduce a cut-off to obtain a finite result. A more realistic treatment would take non-linear perturbations and structure formation explicitly into account; note that the presence of δ relates backreaction directly to the growth of structure. This calculation has not been done, but one would expect the growth of the gravitational potential to increase the magnitude of backreaction and make its equation of state more negative. The backreaction of perturbations breaking away from the linear regime could thus possibly give 'dark energy' at the right time to solve the coincidence problem.

Whether or not backreaction turns out to be quantitatively important, the impact of inhomogeneities on the expansion rate should be properly evaluated to solve the fitting problem and make sure that we are using the right theoretical equations to explain increasingly precise cosmological observations.

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