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## Branch Groups

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## Branch Groups

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## Introduction

Branch groups were defined only recently although they make non-explicit appearances in the literature in the past, starting with the article of John Wilson [Wil71]. Moreover, such examples as infinitely iterated wreath products or the group of tree automorphisms  $\text{Aut}(\mathcal{T})$ , where  $\mathcal{T}$  is a regular rooted tree, go back to the work of Lev Kaloujnine, Bernard Neumann Philip Hall and others.

Branch groups were explicitly defined for the first time at the 1997 St-Andrews conference in Bath in a talk by the second author. Immediately, this sparked a great interest among group theorists, who started investigating numerous properties of branch groups (see [Gri00, BG00b, BG02, GW00]) as well as John Wilson's classification of just-infinite groups.

There are two new approaches to the definition of a branch group, given in [Gri00]. The first one is purely algebraic, defining branch groups as groups whose lattice of subnormal subgroups is similar to the structure of a spherically homogeneous rooted tree. The second one is based on a geometric point of view according to which branch groups are groups acting spherically transitively on a spherically homogeneous rooted tree and having structure of subnormal subgroups similar to the corresponding structure in the full group  $\text{Aut}(\mathcal{T})$  of automorphisms of the tree.

Until 1980 no examples of finitely generated branch groups were known and the first such examples were constructed in [Gri80]. These examples are usually referred to as the first and the second Grigorchuk groups, following Pierre de la Harpe ([Har00]). Other examples soon appeared in [Gri83, Gri84, Gri85a, GS83a, GS83b, GS84, Neu86] and these examples are the basic examples of branch groups, the study of which continues at the present time. Let us mention that the examples of Sergey Alëshin [Ale72] and Vitalii Sushchanskii [Suš79] that appeared earlier also belong to the class of finitely generated branch groups, but the methods used in the study the groups from [Ale72] and [Suš79] did not allow the discovery of the branch structure and this was done much later.

Already in [Gri80] the main features of a general method which works for almost any finitely generated branch group had appeared: one considers the stabilizer of a vertex on the first level and projects it on the corresponding subtree. Then either this projection is equal to the initial group and thus one gets a self-similarity property, or otherwise one gets a finite or infinite chain of branch groups related by some homomorphisms. One of the essential properties of this chain is that these homomorphisms satisfy a “Lipschitz” property of norm reduction, which lends itself to arguments using direct induction on length in the case of a self-similar group, or simultaneous induction on length for all groups in a chain.

Before we give more information of a historical character and briefly describe the main directions of investigation and the main results in the area, let us explain why the class of branch groups is important. There is a lot of evidence that this is indeed the case. For example

- (1) The class of branch groups is one of the classes into which the class of just-infinite groups naturally splits (just-infinite groups are groups whose proper quotients are finite).
- (2) The class contains groups with many extraordinary properties, like infinite finitely generated torsion groups, groups of intermediate growth, amenable but not elementary amenable groups, groups of finite width, etc.
- (3) Branch groups have many applications and are related to analysis, geometry, combinatorics, probability, computer science, etc.

- (4) They are relatively easy to handle and usually the proofs even of deep theorems are short and do not require special techniques. Therefore the branch groups constitute an easy-to-study class of groups, whose basic examples have already appeared in many textbooks and lecture notes, for example in [KM82, Bau93, Rob96, Har00].

This survey article deals almost exclusively with abstract branch groups. The theory of profinite branch groups is also being actively developed at present (see [GHZ00, Gri00, Wil00]), but we hardly touch on this subject.

The survey does not pretend to be complete. There are several topics that we did not include in the text due to the lack of space and time. Among them, we mention the results of Said Sidki from [Sid97] on thin algebras associated to Gupta-Sidki groups, the results on automorphisms of branch groups of Said Sidki from [Sid87a] and the recent results of Lavreniuk and Nekrashevich from [LN02], the results of Claas Röver on embeddings of Grigorchuk groups into finitely presented simple groups and on abstract commensurators from [Röv99, Röv00], the results of Vitalii Sushchanskiĭ on factorizations based on the use of torsion branch groups [Sus89, Sus94], the results of B. Fine, A. Gaglione, A. Myasnikov and D. Spellman from [FGMS01] on discriminating groups, etc.

### 0.1. Just-infinite groups

Let  $\mathcal{P}$  be any property which is preserved under homomorphic images (we call such a property an  $\mathcal{H}$ -property). Any infinite finitely generated group can be mapped onto a just-infinite group (see [Gri00, Har00]), so if there is an infinite finitely generated group with the  $\mathcal{H}$ -property  $\mathcal{P}$  then there is a just-infinite finitely generated group with the same property. Among the  $\mathcal{H}$ -properties let us mention the property of being a torsion group, not containing the free group  $F_2$  on two elements as a subgroup, having subexponential growth, being amenable, satisfying a given identity, having bounded generation, finite width, trivial space of pseudocharacters (for a relation to bounded cohomology see [Gri95]), only finite-index maximal subgroups,  $T$ -property of Kazhdan etc.

The branch just-infinite groups are precisely the just-infinite groups whose structure lattice of subnormal subgroups (with some identifications) is isomorphic to the lattice of closed and open subsets of a Cantor set. This is the approach of John Wilson from [Wil71].

In that paper, John Wilson split the class of just-infinite groups into two subclasses – the groups with finite and the groups with infinite structure lattice. The dichotomy of John Wilson can be reformulated (see [Gri00]) in the form of a trichotomy according to which any finitely generated just-infinite group is either a branch group or can easily be constructed from a simple group or from a hereditarily just-infinite group (i.e., a residually finite group all of whose subgroups of finite index are just-infinite).

Therefore the study of finitely generated just-infinite groups naturally splits into the study of branch groups, infinite simple groups and hereditarily just-infinite groups. Unfortunately, at the moment, none of these classes of groups are well understood, but we have several (classes of) examples.

There are several examples and constructions of finitely generated infinite simple groups, probably starting with the example of Graham Higman in [Hig51], followed by the finitely presented example of Richard Thompson, generalized by Graham Higman in [Hig74] (see also the survey [CFP96] and [Bro87]), the constructions of different monsters by Alexander Ol'shanskiĭ (see [OI'91]), as well as by Sergei Adyan and Igor Lysionok in [AL91], and more recently some finitely presented examples by Claas Röver in [Röv99]. The  $\mathcal{H}$ -properties that can be satisfied by such groups are, for instance, the Burnside identity  $x^p$ , for large prime  $p$ , and triviality of the space of pseudocharacters. The latter holds for the simple groups  $T$  and  $V$  of Richard Thompson (this follows from the results on finiteness of commutator length, see [GS87, Bro87]).

All known hereditarily just-infinite groups (like the projective groups  $\mathrm{PSL}(n, \mathbb{Z})$  for  $n > 2$ ) are linear (in the profinite case there are extra examples like Nottingham group), so by the alternative

of Tits they contain  $F_2$  as a subgroup and therefore cannot be amenable, of intermediate growth, torsion etc. However, they can have bounded generation: it is shown in [CK83] that this holds for  $SL(n, \mathbb{Z})$ ,  $n > 2$ , and therefore also for  $PSL(n, \mathbb{Z})$ .

It seems that there are fewer constraints in the class of branch groups and that they can have various  $\mathcal{H}$ -properties, some of which are listed below. It is conjectured that many of these properties do not hold for groups from the other two classes. On the other hand, branch groups cannot satisfy nontrivial identities (see [Leo97b] and [Wil00] where the proof is given for the just-infinite case).

## 0.2. Algorithmic aspects

Branch groups have good algorithmic properties. In the branch groups of G or GGS type (or more generally spinal type groups) the word problem is solvable by an universal branch algorithm described in [Gri84]. This algorithm is very fast and requires a minimal amount of memory.

The conjugacy problem was unsettled for a long time, and it was solved for the basic examples of branch groups just recently. The article [WZ97] solves the problem for regular branch  $p$ -groups, where  $p$  is an odd prime, and the argument uses the property of “conjugacy separability” as well as profinite group machinery. In [Leo98a] and [Roz98] a different approach was used, which also works in case  $p = 2$ . This ideas were developed in [GW00] in different directions. For instance, it was shown that, under certain conditions, the conjugacy problem is solvable for all subgroups of finite index in a given branch group (we mention here that the property of solvability of conjugacy problem, in contrast with the word problem, is not preserved when one passes to subgroups of finite index). Still, we are far from understanding if the conjugacy problem is solvable in all branch groups with solvable word problem.

The isomorphism problem was also considered in [Gri84] where it is proven that each of the uncountably many constructed groups  $G_\omega$  is isomorphic to at most countably many of them, thus showing that the construction gives uncountably many non-isomorphic examples. It would be very interesting to distinguish all these examples.

Branch groups are related to groups of finite automata. A brief account is given in Section 1.5 (see also [GNS00]). Every group generated by finite automata has a solvable word problem. It is unclear if every such group has solvable conjugacy problem. On the other hand, it seems that the isomorphism problem cannot be solved in this particular case. Indeed, according to the results in [KBS91], the freeness of a matrix group with integer entries cannot be determined, and the general linear group  $GL(n, \mathbb{Z})$  can be embedded in the group of automata defined over an alphabet on  $2^n$  letters as shown by A. Brunner and Said Sidki (see [BS98]).

## 0.3. Group presentations

In Chapter 4 we study presentations of branch groups by generators and relations. It seems probable that no branch group is finitely presented. However, the regular branch groups have nice recursive presentations called  $L$ -presentations. The first such presentation was found for the first Grigorchuk group by Igor Lysionok in [Lys85]. Shortly afterwards, Said Sidki devised a general method yielding recursive definitions of such groups, and applied it to the Gupta-Sidki group [Sid87b]. In [Gri98, Gri99] the idea and the result of Igor Lysionok were developed in different directions. In [Gri99] it was proven that the Igor Lysionok system of relations is minimal and the Schur multiplier of the group was computed: it is  $(\mathbb{Z}/2)^\infty$ . Thus the second homology group of the first Grigorchuk group is infinite dimensional. In [Gri99] it was indicated that the Gupta-Sidki  $p$ -groups also have finite  $L$ -presentations. On the other hand, it was shown in [Gri98] how  $L$ -presentations can be used to embed some branch groups into finitely presented groups using just one HNN extension. The important feature of this embedding is that it preserves the amenability. The first examples of finitely presented, amenable but not elementary amenable groups were constructed this way, thus providing new examples of good fundamental groups (in the terminology of Freedmann and Teichler [FT95]).

In [Bar] the notion of an  $L$ -presentation was slightly extended to the notion of an endomorphic presentation in a way that allowed to show that a finitely generated, fractal, regular branch group satisfying some natural extra conditions has a finite endomorphic presentation. A number of concrete  $L$  and endomorphic presentations of branch groups appear in the article along with general facts on such presentations.

As was stated, no known branch group has finite presentation. For the first Grigorchuk group this was already mentioned in [Gri80] with a sketch of a proof that was given completely in [Gri99]. In [Gri84] two other proofs were presented. Yet another approach in proving the absence of finite presentations is used by Narain Gupta in [Gup84]. More on the history of the presentation problem and related methods appears in Section 4.1.

#### 0.4. Burnside groups

The third part of the survey is devoted to the algebraic properties of branch groups in general and of the most important examples.

The first examples of branch groups appeared in [Gri80] as examples of infinite finitely generated torsion groups. Thus the branch groups are related to the Burnside Problem on torsion groups. This difficult problem has three branches: the Unbounded Burnside Problem, the Bounded Burnside Problem and the Restricted Burnside problem (see [Ady79, Kos90, and]) and, in one way or another, all of them are solved. However, there is still a series of unsolved problems in the neighborhood of the Burnside problem and which are very important to the theory of groups — to name one, “is there a finitely presented infinite torsion group?”. The first example of an infinite finitely generated torsion group, which provided a negative answer to the General Burnside Problem, was constructed by Golod in [Gol64] and it was based on the Golod-Shafarevich Theorem. The actual problem of constructing simple examples which do not require the use of such deep results as Golod-Shafarevich Theorem remained open until such examples appeared in [Gri80]. Soon, more examples appeared in [Gri83, Gri84, Gri85a, GS83a, GS83b, GS84] and more recently in [BŠ01, Gri00, Bar00a, Šun00]. The early examples are finitely generated infinite  $p$ -groups, for  $p$  a prime, and the latter papers contain interesting examples that are not  $p$ -groups.

We already mentioned the idea of using induction on word length, based on the fact that the projections on coordinates decrease the length. In conjunction, the idea of fixing larger and larger layers of the tree under taking powers was developing. The stabilization occurs in the first Grigorchuk group from [Gri80] after three steps, and for the second example from the same article it occurs after the second step of taking  $p$ -th powers. Using a slightly modified metric on the group [Bar98], the stabilization can be made to appear after just one step; this is extensively developed in [BŠ01]. Examples with strong stabilization properties for the standard word metric are constructed in [GS83a], where stabilization takes place after the first step. The notion of *depth* of an element, i.e., the number of decompositions one must perform to decrease the length down to 1 was introduced by Said Sidki in [Sid87a], and is very useful in some situations.

One of the important principles of modern group theory is to try to develop asymptotic methods related to growth, amenability and other asymptotic notions. In [Gri84] the torsion growth functions were introduced for finitely generated torsion groups and it was shown that some examples from [Gri80, Gri83, Gri84] have polynomial growth in this sense. These results were improved in many directions in [Lys98, Leo97a, Leo99, BŠ01] and some of them are described in Chapter 6.

Among the main consequences of the theory of Efim Zelmanov (see [Zel90, Zel91]) is that if a finitely generated torsion residually finite group has finite exponent (i.e., there exists  $n \neq 0$  such that  $g^n = 1$  for every element  $g$  in the group) then the group is finite. Although the results of Efim Zelmanov do not depend on the classification of finite simple groups, the above mentioned consequence does and it would be nice to produce a proof that is independent of the classification. A simple proof that finitely generated torsion branch groups always have infinite exponent is provided (Theorem 6.9).

The profinite completion of a group of finite exponent is a torsion profinite group. If it has a just-infinite quotient (and this is the case if the group  $G$  is a virtually pro- $p$ -group) then one gets a profinite just-infinite torsion group of bounded exponent. By Wilson's alternative such a group is either just-infinite branch, or hereditarily just-infinite; but by the results from [GHZ00] if it were branch it would have unbounded exponent, so the search for profinite groups of bounded exponent can be narrowed to hereditarily just-infinite groups. We believe such groups do not exist.

In Chapter 6 we give a simple proof of the fact that a finitely generated torsion branch group has infinite exponent. An interesting question is to describe the type of torsion growth that distinguishes the finite and infinite groups (it follows from the results of Efim Zelmanov that there exists a recursive, unbounded function  $z_k(n)$ , depending on the number of generators  $k$ , such that every torsion group on  $k$  generators whose torsion growth is bounded above by  $z_k(n)$  is finite). It seems likely that the problem can be reduced to the case of branch torsion groups.

In Chapter 6 we also analyze carefully the idea of the first example in [Gri80] which is not related to the stabilization, but rather uses a covering of a group by kernels of homomorphisms. The torsion groups (called  $G$  groups) that generalize the constructions of [Gri84, Gri85a] are investigated in greater detail in [BŠ01, Šun00]. We exhibit the construction and some interesting examples, based on existence of finite groups with certain required properties (Derek Holt's example, for instance).

Until recently, it was not known whether there exist torsion-free just-infinite branch groups. Such an example was constructed in [BG02].

## 0.5. Subgroups of branch groups

The study of the subgroup structure of any class of groups is an important part of the investigation. Branch groups have a rich and nice subgroup structure which has not yet been completely investigated. In the early works attention was paid to some particular subgroups of small index such as the stabilizers of the first few levels, the initial members of the lower central series, derived series, etc. A fundamental observation made by Narain Gupta and Said Sidki in [GS84] is that many GGS groups contain a normal subgroup  $K$  of finite index with the property that  $K$  contains *geometrically*  $K^m$  ( $m$  is the degree of the tree and "geometrically" means that the product  $K^m$  acts on the subtrees on the first level) as a subgroup of finite index. It so happens that all the main examples of branch groups have such a subgroup and this fact lies at the base of the definition of regular branch groups.

Rigid stabilizers and stabilizers of the first Grigorchuk group are described in [Roz90, BG02]. For the Gupta-Sidki  $p$ -groups this is done in [Sid87a]. The structure of normal subgroups of Peter Neumann's groups [Neu86] is very simple, since the normal subgroups coincide with the (rigid) stabilizers. For branch  $p$ -groups it is more difficult to obtain the structure of the lattice of normal subgroups (this is related to the fact that pro- $p$ -groups usually have rich structure of subgroups of finite index). The lattice of normal subgroups of the first Grigorchuk group was recently described by the first author in [Bar00c] (see also [CST01] where the normal subgroups are described up to the fourth level).

In the study of infinite finitely generated groups an important role is played by the maximal and weakly maximal subgroups (i.e., subgroups of infinite index maximal with respect to this property). It is strange that little attention was paid to the latter until recently. The main result of Ekaterina Pervova [Per00] claims that in the basic examples of branch groups every proper maximal subgroup has finite index. This is in contrast with lattices in semisimple Lie groups, as follows from results of Margulis and Soifer [MS81].

Important examples of weakly maximal subgroups are the parabolic subgroups, i.e., the stabilizers of infinite paths in the tree. The structure of parabolic subgroups is described in [BG02] for some particular examples. They are not finitely generated and have a tree-like structure. It would be interesting to obtain a complete description of weakly maximal subgroups in branch groups.

## 0.6. Lie algebras

Chapter 8 deals with central series and associated Lie algebras of branch groups. There is a canonical way, due to Wilhelm Magnus in which a central series corresponds to a graded Lie ring or Lie algebra. The most interesting central series are the lower central series and the series of dimension subgroups. It was proved in [Gri89] that the Cesàro averages of the ranks of the factors in the lower central series of the first Grigorchuk group (which are elementary 2-groups) are uniformly bounded and it was conjectured that the ranks themselves were uniformly bounded, i.e., that the first Grigorchuk group has finite width. An important step in proving this conjecture was made in [Roz96], and a complete proof appears in [BG00a], using ideas of Lev Kaloujnine [Kal48] and the notion of uniserial module. Moreover, a negative answer to a problem of Efim Zelmanov on the classification of just-infinite profinite groups of finite width is provided in [BG00a], a new example of a group of finite width was constructed and the structure of the Cayley graph of the associated Lie algebras was described. This is one of the few cases of a nontrivial computation of a Cayley (or Lie) graph of a graded Lie algebra.

The question of the finiteness of width of other basic branch groups (first of all the Gupta-Sidki groups) was open for a long time and recently answered negatively by the first author in [Bar00c]. These results are also presented in Chapter 8.

An important role in the study of profinite completions of branch group is played by the “congruence subgroup property” with respect to the sequence of stabilizers, meaning that every finite-index subgroup contains a level stabilizer  $\text{St}_G(n)$  for some  $n$ , and which holds for many branch groups. Nevertheless, there are branch groups without this property and the complete solution of the congruence subgroup property problem for the class of all branch groups is not completely resolved.

## 0.7. Growth

The fourth part of the paper deals with some geometric and analytic properties of branch groups. The main notion in asymptotic group theory is the notion of *growth* of a finitely generated group. The growth function  $\gamma(n)$  of a finitely generated group  $G$  with respect to a system of generators  $S$  counts the number of group elements of length at most  $n$ . The group’s type of growth — exponential, intermediate, polynomial — does not depend on the choice of  $S$ . One can easily construct an example of a group of polynomial growth of any given degree  $d$  (for instance  $\mathbb{Z}^d$ ) or a group of exponential growth (like  $F_2$ , the free group on two generators) but it is a highly non-trivial task to construct a group of intermediate growth. The question of existence of such groups of intermediate growth was posed by John Milnor [Mil68c] and solved fifteen years later in [Gri83, Gri84, Gri85a], where the second author shows that the first group in [Gri80] and all  $p$ -groups  $G_\omega$  in [Gri84, Gri85a] have intermediate growth; the estimates are of the form

$$e^{\sqrt{n}} \lesssim \gamma(n) \lesssim e^{n^\beta}, \quad (1)$$

for some  $\beta < 1$ , where for two functions  $f, g : \mathbb{N} \rightarrow \mathbb{N}$  we write  $f \lesssim g$  to mean that there exists a constant  $C$  with  $f(n) \leq g(Cn)$  for all  $n \in \mathbb{N}$ .

Milnor’s problem was therefore solved using branch groups. Up to the present time all known groups of intermediate growth are either branch groups or groups constructed using branch groups and we believe that all just-infinite groups of intermediate growth are branch groups.

By using branch groups, the second author showed in [Gri84] that there exist uncountable chains and anti-chains of intermediate growth functions of groups acting on trees.

The upper bound in (1) was improved in [Bar98], and a general improvement of the upper bounds for all groups  $G_\omega$  was given in [BŠ01] and [MP01].

One of the main remaining question on growth is whether there exists a group with growth precisely  $e^{\sqrt{n}}$ . It is known that if a group is residually nilpotent and its growth is strictly less than  $e^{\sqrt{n}}$ , then the group is virtually nilpotent and therefore has polynomial growth [Gri89, LM89]. Using arguments given in [Gri89] (see also [BG00a]), it follows that if a group of growth  $e^{\sqrt{n}}$

exists in the class of residually- $p$  groups, then it must have finite width. For some time, among all known examples of groups of intermediate growth only the first Grigorchuk group was known to have finite width, and the second author conjectured that this group has precisely this type of growth. However, his conjecture was infirmed by Yuriï Leonov [Leo00] and the first author [Bar01]; indeed the growth of the first Grigorchuk group is bounded below by  $e^{n^\alpha}$  for some  $\alpha > \frac{1}{2}$ .

The notion of growth can be defined for other algebraic and geometric objects as well: algebras, graphs, etc. A very interesting topic is the study of the growth of graded Lie algebras  $\mathcal{L}(G)$  associated to groups. In case of GGS groups some progress is achieved in [Bar00c], where the growth of  $\mathcal{L}(G)$  for the Gupta-Sidki 3-group and some other groups is computed; in particular, it is shown that the Gupta-Sidki group does not have finite width. A connection between the Lie algebra structure and the tree structure is used in the majoration of the growth of the associated Lie algebra by the growth of any homogeneous space  $G/P$ , where  $P$  is a parabolic subgroup, i.e., the stabilizer of an infinite path in a tree [Bar00c]. As metric spaces, these homogeneous spaces are equivalent to Schreier graphs. These graphs have an interesting structure: they are substitutional graphs, and have a fractal behavior in the case of many fractal branch groups. They have polynomial growth, usually of non-integral degree. These results are presented in Section 10.3.

One of the promising directions of research is the study of spectral properties of the above graphs. This question is linked to several famous problems of operator  $K$ -theory and theory of  $C^*$ -algebras. One of the first works in this direction is [BG00b], where it is shown that the spectrum of the discrete Laplace operator on such graphs can be a Cantor set, optionally with extra isolated points. The computation of these spectra is related to operator recursions that hold for the Laplace or Hecke-type operators associated to the dynamical system  $(G, \partial\mathcal{T}, \mu)$ , where  $(\partial\mathcal{T}, \mu)$  is the boundary of the tree endowed with the uniform measure. The main results are presented in Chapter 11.

Finally, there are a great number of open questions on branch groups. Some of them are listed in the final part of the paper; we hope that they will stimulate the development of the subject.

## 0.8. Acknowledgments

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## 0.9. Some notation

We include zero in the set of natural numbers  $\mathbb{N} = \{0, 1, 2, \dots\}$ . The set of positive integers is denoted by  $\mathbb{N}_+ = \{1, 2, \dots\}$ .

Expressions as  $((1, 2)(3, 4, 5))$  listing non-trivial cycles are used to describe permutations in the group of symmetries  $\text{Sym}(n)$  of  $n$  elements

We want all the group actions to be on the right. Thus we conjugate as follows,

$$g^h = h^{-1}gh,$$

and we denote

$$[g, h] = g^{-1}h^{-1}gh = g^{-1}g^h.$$

The commutator subgroup of  $G$  is denoted by  $[G, G]$  and the abelianization  $G/[G, G]$  by  $G^{ab}$ .

Let the group  $A$  act on the right on the group  $H$  through  $\alpha : A \rightarrow \text{Aut}(H)$ . We define the *semidirect product*  $G = H \rtimes_\alpha A$  as the group whose elements are the ordered pairs from the set  $H \times A$  and the operation is given by

$$(h, a)(g, b) = (hg^{(a^{-1})\alpha}, ab).$$

After the identification  $(h, 1) = h$  and  $(1, a) = a$  we see that  $G$  is a group containing  $H$  as a normal subgroup and  $A$  as its complement, i.e.,  $HA = G$  and  $H \cap A = 1$ . Moreover, the conjugation of the elements in  $H$  by the elements in  $A$  is given by the action  $\alpha$ , i.e.,  $h^a = h^{(a)\alpha}$ .

If we start with a group  $G$  that has a normal subgroup  $H$  with a complement  $A$  in  $G$ , we say that  $G$  is the *internal semidirect product* of  $H$  and  $A$ . Indeed,  $G = H \rtimes A$  where the action of  $A$  on  $H$  is through conjugation (note that  $hagb = hg^{a^{-1}}ab$  for  $h, g \in H$  and  $a, b \in A$ ).

Let  $G$  and  $A$  be groups acting on the set  $X$  and the finite set  $Y$ , respectively. We define the *permutational wreath product*  $G \wr_Y A$  that acts on the set  $Y \times X$  (note the change in the order) as follows: let  $A$  act on the direct power  $G^Y$  on the right by permuting the coordinates of  $G^Y$  by

$$(h^a)_y = h_{y^{a^{-1}}}, \quad (2)$$

for  $h \in G^Y$ ,  $a \in A$ ,  $y \in Y$ ; then define  $G \wr_Y A$  as the semidirect product

$$G \wr_Y A = G^Y \rtimes A$$

obtained through the action of  $A$  on  $G^Y$ ; finally let the wreath product act on the right on the set  $Y \times X$  by

$$(y, x)^{ha} = (y^a, x^{h_y}),$$

for  $y \in Y$ ,  $x \in X$ ,  $h \in G^Y$  and  $a \in A$ . Note that the equality (2) which represents the action of  $A$  on  $G^Y$ , also represents conjugation in the wreath product, exactly as we want, and that this wreath product is associative, modulo the necessary natural identifications.

All actions defined by now were right actions. However, we achieved this by introducing inversion at several crucial places, thus introducing left actions through the back door. Another possibility was to let the semidirect product of  $A$  and  $H$  as above be the group whose elements the ordered pairs in  $A \times H$  and define  $(a, h)(b, g) = (ab, h^{(b)\alpha}g)$ . This works well, but we choose not to do it.

We introduce here the basic notation of growth series. Growth series will be used in Chapters 8 and 10.

Let  $X$  be a set on which the group  $G$  acts, and fix a base point  $* \in X$  and a set  $S$  that generates  $G$  as a monoid. The *growth function* of  $X$  is

$$\gamma_{*,G}(n) = |\{x \in X \mid x = *^{s_1 \dots s_n} \text{ for some } s_i \in S\}|.$$

The *growth series* of  $X$  is

$$\text{growth}(X) = \sum_{n \geq 0} \gamma_{*,G}(n) \hbar^n.$$

Let  $V = \bigoplus_{n \geq 0} V_n$  be a graded vector space. The *Hilbert-Poincaré series* of  $V$  is the formal power series

$$\text{growth}(V) = \sum_{n \geq 0} \dim V_n \hbar^n.$$

A preorder  $\preceq$  is defined on the set of non-decreasing functions  $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  by  $f \preceq g$  if there exists a positive constant  $C$  such that  $f(n) \leq g(Cn)$ , for all  $n$  in  $\mathbb{R}_{\geq 0}$ . An equivalence relation  $\sim$  is defined by  $f \sim g$  if  $f \preceq g$  and  $g \preceq f$ .

Several branch groups are distinguished enough to be given separate notation. They are the first Grigorchuk group  $\mathfrak{G}$ , the Grigorchuk supergroup  $\tilde{\mathfrak{G}}$ , the Gupta-Sidki 3-group  $\overline{\Gamma}$ , the Fabrykowski-Gupta group  $\Gamma$  and the Bartholdi-Grigorchuk group  $\overline{\Gamma}$ . See Section 1.6 for the definitions.

## Part 1

# Basic Definitions and Examples

## Branch Groups and Spherically Homogeneous Trees

### 1.1. Algebraic definition of a branch group

We start with the main definition of the survey, namely the definition of a branch group. The definition is given in purely algebraic terms, emphasizing the subgroup structure of the groups. We give a geometric version of the definition in Section 1.3 in terms of actions on rooted trees. The two definitions are not equivalent and we will say something about the difference later.

**DEFINITION 1.1.** *Let  $G$  be a group. We say that  $G$  is a branch group if there exist two decreasing sequences of subgroups  $(L_i)_{i \in \mathbb{N}}$  and  $(H_i)_{i \in \mathbb{N}}$  and a sequence of integers  $(k_i)_{i \in \mathbb{N}}$  such that  $L_0 = H_0 = G$ ,  $k_0 = 1$ ,*

$$\bigcap_{i \in \mathbb{N}} H_i = 1$$

and, for each  $i$ ,

- (1)  $H_i$  is a normal subgroup of  $G$  of finite index.
- (2)  $H_i$  is a direct product of  $k_i$  copies of the subgroup  $L_i$ , i.e., there are subgroups  $L_i^{(1)}, \dots, L_i^{(k_i)}$  of  $G$  such that

$$H_i = L_i^{(1)} \times \dots \times L_i^{(k_i)} \tag{3}$$

and each of the factors is isomorphic to  $L_i$ .

- (3)  $k_i$  properly divides  $k_{i+1}$ , i.e.,  $m_{i+1} = k_{i+1}/k_i \geq 2$ , and the product decomposition (3) of  $H_{i+1}$  refines the product decomposition (3) of  $H_i$  in the sense that each factor  $L_i^{(j)}$  of  $H_i$  contains  $m_{i+1}$  of the factors of  $H_{i+1}$ , namely the factors  $L_{i+1}^{(\ell)}$  for  $\ell = (j-1)m_{i+1} + 1, \dots, jm_{i+1}$ .
- (4) conjugations by the elements in  $G$  transitively permute the factors in the product decomposition (3).

The definition implies that branch groups are infinite, but residually finite groups. Note that the subgroups  $L_i$  are not normal, but they are subnormal of defect 2.

**DEFINITION 1.2.** *Let  $G$  be a branch group. Keeping the notation from the previous definition, we call the sequence of pairs  $(L_i, H_i)_{i \in \mathbb{N}}$  a branch structure on  $G$ .*

The branch structure of a branch group is depicted in Figure 1.1. The branch structure on a

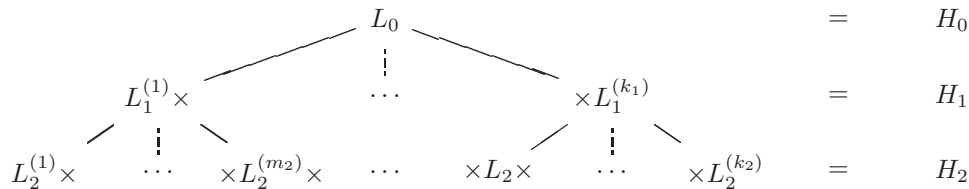


FIGURE 1.1. Branch structure of a branch group

group  $G$  is not unique, since any subsequence of pairs  $(L_{i_j}, H_{i_j})_{j=1}^\infty$  is also a branch structure on  $G$ .

One can quickly construct examples of branch groups, using infinitely iterated wreath products. For example, let  $p$  be a prime and  $\mathbb{Z}/p\mathbb{Z}$  act on the set  $Y = \{1, \dots, p\}$  by cyclic permutations. Define the permutational wreath product

$$G_n = \underbrace{((\mathbb{Z}/p\mathbb{Z} \wr_Y \dots) \wr_Y \mathbb{Z}/p\mathbb{Z}) \wr_Y \mathbb{Z}/p\mathbb{Z}}_n,$$

and let  $G$  be the inverse limit  $\varprojlim G_n$ , where the projections from  $G_n$  to  $G_{n-1}$  are just the natural restrictions. Since  $G = G \wr_Y \mathbb{Z}/p\mathbb{Z}$ ,  $G$  is a branch group.

Similarly, for  $m \geq 2$ , let  $\text{Sym}(m)$  be the group of permutations of  $Y = \{1, \dots, m\}$ , define  $G_n$  as the permutational wreath product

$$G_n = \underbrace{((\text{Sym}(m) \wr_Y \dots) \wr_Y \text{Sym}(m)) \wr_Y \text{Sym}(m)}_n,$$

and  $G$  as the inverse limit  $\varprojlim G_n$ . Since  $G = G \wr_Y \text{Sym}(m)$ ,  $G$  is a branch group.

In the next section we will look at the last group from a geometric point of view. We will also develop some terminology for groups acting on rooted trees that will be used for the second, more geometric, definition of branch groups.

## 1.2. Spherically homogeneous rooted trees

We will define the notion of a spherically homogeneous tree as a set of words ordered by the prefix relation and then make a connection to the graph-theoretical version of the same notion. We find it useful to live in both worlds and use their terminology and notation.

### 1.2.1. The trees. Let

$$\bar{m} = m_1, m_1, m_3, \dots$$

be a sequence of integers with  $m_i \geq 2$  and let

$$\bar{Y} = Y_1, Y_2, Y_3, \dots$$

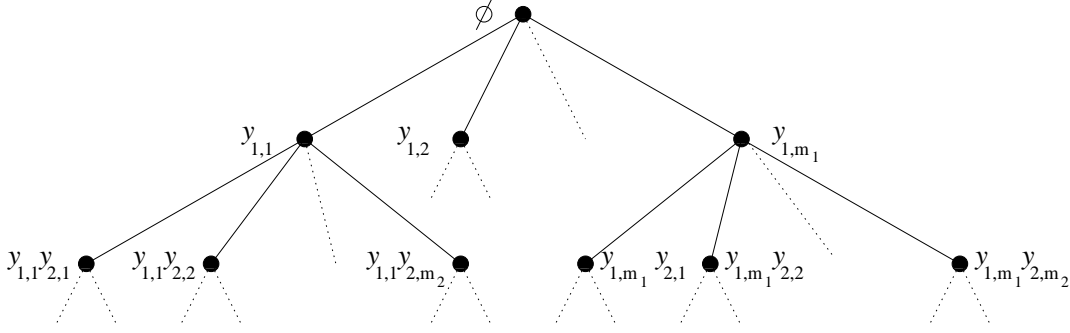
be a sequence of alphabets with  $|Y_i| = m_i$ . A *word of length  $n$*  over  $\bar{Y}$  is any sequence of letters of the form  $w = y_1 y_2 \dots y_n$  where  $y_i \in Y_i$  for all  $i$ . The unique word of length 0, the *empty word*, is denoted by  $\emptyset$ . The length of the word  $u$  is denoted by  $|u|$ . Denote the set of words over  $\bar{Y}$  by  $\bar{Y}^*$ . We introduce a partial order on the set of all words over  $\bar{Y}$  by the *prefix relation*  $\leq$ . Namely,  $u \leq v$  if  $u$  is an initial segment of the sequence  $v$ , i.e., if  $u = u_1 \dots u_n$ ,  $v = v_1 \dots v_k$ ,  $n \leq k$ , and  $u_i = v_i$ , for  $i \in \{1, \dots, n\}$ . The partially ordered set of words over  $\bar{Y}$ , denoted by  $\mathcal{T}^{(\bar{Y})}$ , is called the *spherically homogeneous tree* over  $\bar{Y}$ . The sequence  $\bar{m}$  is the *sequence of branching indices* of the tree  $\mathcal{T}^{(\bar{Y})}$ . If there is no room for confusion we denote  $\mathcal{T}^{(\bar{Y})}$  by  $\mathcal{T}$ . For the remainder of the section (and later on) we think of  $\bar{Y}$  as being fixed, and let  $Y_i = \{y_{i,1}, \dots, y_{i,m_i}\}$ , for  $i \in \mathbb{N}_+$ . In case all the sets  $Y_i$  are equal, say to  $Y$ , the tree  $\mathcal{T}^{(\bar{Y})}$  is said to be *regular*, and is denoted by  $\mathcal{T}^{(Y)}$ .

Let us give now the graph-theoretical interpretation of  $\mathcal{T}$  and thus justify our terminology. Every word over  $\bar{Y}$  represents a vertex in a rooted tree. Namely, the empty word  $\emptyset$  represents the *root*, the  $m_1$  one-letter words  $y_{1,1}, \dots, y_{1,m_1}$  represent the  $m_1$  children of the root, the  $m_2$  two-letter words  $y_{1,1}y_{2,1}, \dots, y_{1,1}y_{2,m_2}$  represent the  $m_2$  children of the vertex  $y_{1,1}$ , etc. More generally, if  $u$  is a word over  $\bar{Y}$ , then the words  $uy$ , for  $y$  in  $Y_{|u|+1}$ , of length  $|u| + 1$  represent the  $m_{|u|+1}$  *children* (or *successors*) of  $u$  (see Figure 1.2).

The graph structure of  $\mathcal{T}$  induces a distance function on the set of words by

$$d(u, v) = |u| + |v| - 2|u \wedge v|,$$

where  $u \wedge v$  is the longest common prefix of  $u$  and  $v$ . In particular, the words of length  $n$  represent the vertices that are at distance  $n$  to the root. Such vertices constitute the *level  $n$*  of the tree, denoted by  $\mathcal{L}_n^{(\bar{Y})}$  or, when  $\bar{Y}$  is assumed, just by  $\mathcal{L}_n$ . In the terminology of metric spaces, the vertices on the

FIGURE 1.2. The tree  $\mathcal{T}$ 

level  $n$  are precisely the elements of the *sphere of radius  $n$*  with center at the root. In the sequel, we will rarely make any distinction between a word  $u$  over  $\overline{Y}$ , the vertex represented by  $u$  and the unique path from the root to the vertex  $u$ .

**1.2.2. Tree automorphisms.** A permutation of the words over  $\overline{Y}$  that preserves the prefix relation is an *automorphism* of the tree  $\mathcal{T}$ . From the graph-theoretical point of view an automorphism of  $\mathcal{T}$  is just a graph automorphism that fixes the root. We denote the group of automorphisms of  $\mathcal{T}$  by  $\text{Aut}(\mathcal{T})$ . Clearly, the orbits of the action of  $\text{Aut}(\mathcal{T})$  on  $\mathcal{T}$  are precisely the levels of the tree. The fact that the automorphism group acts transitively on the spheres centered at the root is precisely the reason for which these trees are called spherically homogeneous.

Consider an automorphism  $f$  of  $\mathcal{T}$  and a word  $u$  over  $\overline{Y}$ . The image of  $u$  under  $f$  is denoted by  $u^f$ . For a letter  $y$  in  $Y_{|u|+1}$  we have  $(uy)^f = u^f y'$  where  $y'$  is a uniquely determined letter in  $Y_{|u|+1}$ . Clearly, the induced map  $y \mapsto y'$  is a permutation of  $Y_{|u|+1}$ , we denote this permutation by  $(u)f$  and we call it the *vertex permutation* of  $f$  at  $u$ . If we denote the image of  $y$  under  $(u)f$  by  $y^{(u)f}$ , we have

$$(uy)^f = u^f y^{(u)f}, \quad (4)$$

and this easily extends to

$$(y_1 y_2 \dots y_n)^f = y_1^{(\emptyset)f} y_2^{(y_1)f} \dots y_n^{(y_1 y_2 \dots y_{n-1})f}. \quad (5)$$

Any tuple  $((u)g)_{u \in \overline{Y}^*}$ , indexed by the words  $u$  over  $\overline{Y}$ , where the entry  $(u)g$  is a permutation of the alphabet  $Y_{|u|+1}$ , determines an automorphism  $g$  of  $\mathcal{T}$  given by

$$(y_1 y_2 \dots y_n)^g = y_1^{(\emptyset)g} y_2^{(y_1)g} \dots y_n^{(y_1 y_2 \dots y_{n-1})g}.$$

Therefore, we can think of an automorphism  $f$  of  $\mathcal{T}$  as the tuple of vertex permutations  $((u)f)_{u \in \overline{Y}^*}$  and we can represent the automorphism  $f$  on the tree  $\mathcal{T}$  by decorating each vertex  $u$  in  $\mathcal{T}$  by its permutation  $(u)f$  (see Figure 1.3). The decorated tree that represents  $f$  is called the *portrait* of  $f$ .

The portrait of  $f$  gives an intuitively clear picture of the action of  $f$  on  $\mathcal{T}$ , if we can imagine what happens when we perform all the vertex permutations at once. If only finitely many vertex permutations are non-trivial this is not difficult to do.

By using (4), we can easily see that

$$(u)fg = (u)f \circ (u^f)g \quad \text{and} \quad (u)f^{-1} = [(u^f)^{-1}]^{-1}, \quad (6)$$

for all words  $u$  over  $\overline{Y}$  and automorphisms  $f$  and  $g$  of  $\mathcal{T}$ .

We introduce the *shift operator*  $\sigma$  that acts on sequences as follows:

$$\sigma(s_1, s_2, s_3, \dots) = s_2, s_3, s_4, \dots$$

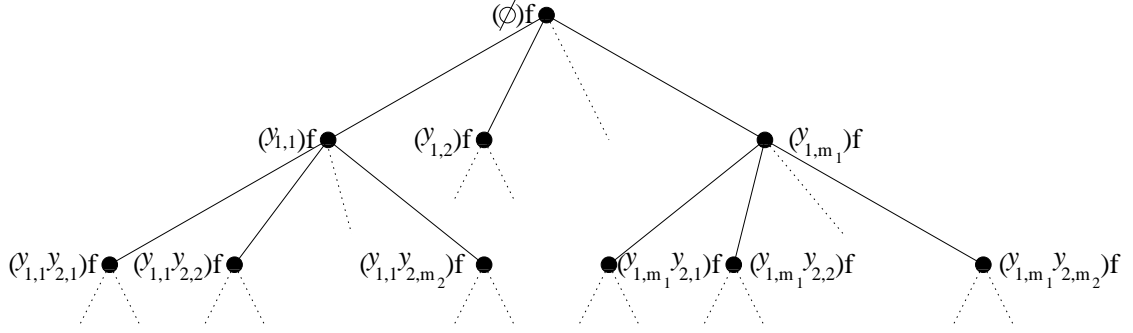


FIGURE 1.3. The automorphism  $f$  of  $\mathcal{T}$

Acting  $n$  times on the sequence of alphabets  $\overline{Y}$  gives the shifted sequence of alphabets

$$\sigma^n \overline{Y} = Y_{n+1}, Y_{n+2}, Y_{n+3}, \dots$$

which has the following shifted sequence of branching indices

$$\sigma^n \overline{m} = m_{n+1}, m_{n+2}, m_{n+3}, \dots,$$

and this new sequence of alphabets defines the spherically homogeneous tree  $\mathcal{T}^{(\sigma^n \overline{Y})}$ . Let  $u$  be a word over  $\overline{Y}$  of length  $n$  and denote by  $\mathcal{T}_u^{(\overline{Y})}$  the spherically homogeneous tree that consists of all words over  $\overline{Y}$  with prefix  $u$  ordered by the prefix relation. It is the subtree of  $\mathcal{T}^{(\overline{Y})}$  that is hanging below the vertex  $u$ . Clearly, the trees  $\mathcal{T}_u^{(\overline{Y})}$  and  $\mathcal{T}^{(\sigma^n \overline{Y})}$  are canonically isomorphic under the isomorphism  $\delta_u$  that deletes the prefix  $u$  from the words in  $\mathcal{T}_u^{(\overline{Y})}$ , and any two trees  $\mathcal{T}_u^{(\overline{Y})}$  and  $\mathcal{T}_v^{(\overline{Y})}$ , where  $u$  and  $v$  are words over  $\overline{Y}$  of the same length, are canonically isomorphic under the isomorphism that deletes the prefix  $u$  and replaces it by the prefix  $v$  (this isomorphism is just the composition  $\delta_u \delta_v^{-1}$ ). In order to avoid cumbersome notation we denote the tree  $\mathcal{T}^{(\sigma^n \overline{Y})}$  by  $\mathcal{T}_n$  and the tree  $\mathcal{T}_u^{(\overline{Y})}$  by  $\mathcal{T}_u$  when  $\overline{Y}$  is assumed to be fixed. The previous observations then say that  $\mathcal{T}_{|u|}$  and  $\mathcal{T}_u$  are canonically isomorphic (see Figure 1.4).

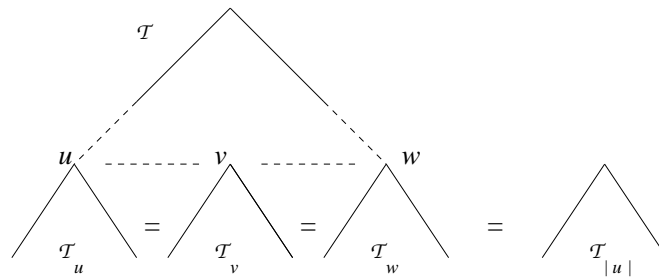


FIGURE 1.4. Canonically isomorphic subtrees and shifted trees

Let  $f$  be an automorphism of  $\mathcal{T}$  and  $u$  a word over  $Y$ . The *section* of  $f$  at  $u$  (other words in use are *component*, *projection* and *slice*), is the the automorphism  $f_u$  of  $\mathcal{T}_{|u|}$  defined by the vertex permutations

$$(w)f_u = (uw)f, \tag{7}$$

for all words  $w$  over  $\sigma^{|u|} \overline{Y}$ . Therefore,  $f_u$  uses the vertex permutations of  $f$  at and below the vertex  $u$  and assigns them to words over  $\sigma^{|u|} \overline{Y}$  in a natural way.

The set  $G_u = \{g_u | g \in G\}$  of sections at  $u$  of the elements in  $G$  is called the *section* of  $G$  at  $u$ . We mention that the section  $G_u$  is not necessarily a subgroup of  $G$  even if the tree is regular.

It is easy to show, using (5) and (7), that

$$(uv)^f = u^f v^{f_u} \quad (8)$$

for all automorphisms  $f$  of  $\mathcal{T}$ , words  $u$  over  $\bar{Y}$  and words  $v$  over  $\sigma^{|u|}\bar{Y}$ . Using (6), (7) and (8) we obtain the equalities

$$(fg)_u = f_u g_{u^f} \text{ and } (f^{-1})_u = (f_{u^{f^{-1}}})^{-1}, \quad (9)$$

that hold for all automorphisms  $f$  and  $g$  of  $\mathcal{T}$  and words  $u$  over  $\bar{Y}$ .

Before we move on, let us look at trees and automorphisms from another point of view. The set of infinite paths (rays) from the root  $\emptyset$  in  $\mathcal{T}$  is called the *boundary* of  $\mathcal{T}$  and is denoted by  $\partial\mathcal{T}$ . We define a metric  $d$  on  $\partial\mathcal{T}$  by

$$d(r, s) = \begin{cases} \frac{1}{2^{|r \wedge s|}} & \text{if } r \neq s \\ 0 & \text{if } r = s, \end{cases}$$

for all infinite rays  $r$  and  $s$  in  $\partial\mathcal{T}$ . Any automorphism  $f$  of  $\mathcal{T}$  defines an isometry  $\bar{f}$  of the space  $\partial\mathcal{T}$  given by

$$(y_1 y_2 y_3 \dots)^{\bar{f}} = y_1^{(\emptyset)f} y_2^{(y_1)f} y_3^{(y_1 y_2)f} \dots$$

Conversely, any isometry  $\bar{g}$  of  $\partial\mathcal{T}$  defines an automorphism  $g$  of  $\mathcal{T}$  as follows:  $u^g$  is the prefix of  $r^{\bar{g}}$  of length  $|u|$ , where  $r$  is any infinite path in  $\partial\mathcal{T}$  with prefix  $u$ . Therefore,  $\text{Aut}(\mathcal{T}) = \text{Isom}(\partial\mathcal{T})$ .

Note that the definition of the metric  $d$  above was very arbitrary. Given a strictly decreasing sequence  $\bar{d} = (d_i)_{i \in \mathbb{N}}$  of positive numbers with limit 0, we could define a metric on  $\partial\mathcal{T}$  by  $d(r, s) = d_{|r \wedge s|}$  if  $r \neq s$ , and it can be shown that the topology of the metric space  $\partial\mathcal{T}$  is independent of the choice of the sequence  $\bar{d}$ .

The metric space  $(\partial\mathcal{T}, d)$  is a universal model for ultrametric homogeneous spaces as is mentioned in [Gri00] and explained in more details in Proposition 6.2 in [GNS00].

**1.2.3. Level and rigid stabilizers.** We introduce the notions of (rigid) vertex and level stabilizers, as well as the congruence subgroup property.

**DEFINITION 1.3.** *Let  $G$  be a group of automorphisms of  $\mathcal{T}$ . The subgroup  $\text{St}_G(u)$  of  $G$ , called the vertex stabilizer of  $u$  in  $G$ , consists of those automorphisms in  $G$  that fix the vertex  $u$ .*

For any two automorphisms  $f$  and  $g$  in  $\text{St}_G(u)$ , by using (9), we have

$$(fg)_u = f_u g_{u^f} = f_u g_u,$$

so that the map

$$\varphi_u^G : \text{St}_G(u) \rightarrow \text{Aut}(\mathcal{T}_{|u|})$$

given by

$$(f)\varphi_u^G = f_u$$

is a homomorphism. We call this homomorphism the *section homomorphism* at  $u$ , and we usually avoid the superscript. We denote the image of the section homomorphism  $\varphi_u$  by  $U_u^G$ , or just by  $U_u$  when  $G$  is assumed, and call it the *upper companion* of  $G$  at  $u$ . Note that the upper companion of  $G$  at  $u$  is a subgroup of  $\text{Aut}(\mathcal{T}_{|u|})$ , and is not necessarily a subgroup of  $G$ , even in case of a regular tree. Nevertheless, in many important cases in which the tree  $\mathcal{T}$  is regular the upper companion groups are equal to  $G$  after the canonical identification of the original tree  $\mathcal{T}$  with its subtrees.

**DEFINITION 1.4.** *Let  $G$  be a group of automorphisms of a regular tree  $\mathcal{T}$ . The group  $G$  is fractal if for every vertex  $u$  the upper companion group  $U_u$  is equal to  $G$  (after the tree identifications  $\mathcal{T} = \mathcal{T}_{|u|}$ ).*

The vertex stabilizers lead to the notion of level stabilizers as follows:

DEFINITION 1.5. Let  $G$  be a group of automorphisms of  $\mathcal{T}$  and let  $\text{St}_G(\mathcal{L}_n)$ , called the  $n$ -th level stabilizer in  $G$ , denote the subgroup of  $G$  consisting of the automorphisms of  $\mathcal{T}$  that fix all the vertices on the level  $n$  (and up of course), i.e.,

$$\text{St}_G(\mathcal{L}_n) = \bigcap_{u \in \mathcal{L}_n} \text{St}_G(u).$$

The homomorphism

$$\psi_n^G : \text{St}_G(\mathcal{L}_n) \rightarrow \prod_{u \in \mathcal{L}_n} U_u \leq \prod_{u \in \mathcal{L}_n} \text{Aut}(\mathcal{T}_{|u|})$$

given by

$$(f)\psi_n^G = ((f)\varphi_u^G)_{u \in \mathcal{L}_n} = (f_u)_{u \in \mathcal{L}_n}$$

is an embedding, since the only automorphism that fixes all the vertices at level  $n$  and acts trivially on all the subtrees hanging below the level  $n$  is the trivial one. In case  $n = 1$  we almost always omit the index 1 in  $\psi_1$ , and we omit the superscript  $G$ , for all  $n$ . We will see in a moment that the level stabilizers of  $G$  have finite index in  $G$ . It follows that the same is true for the vertex stabilizers.

We note that the current literature contains several versions of definitions of fractal branch groups. In some of them the sufficient condition from Lemma 1.7 below is used as a definition. One can impose even stronger conditions.

DEFINITION 1.6. Let  $G$  be a group of automorphisms of a regular tree  $\mathcal{T}$ . The group  $G$  is strongly fractal if it is fractal and the embedding

$$\psi : \text{Stab}_G(\mathcal{L}_1) \rightarrow \prod_{i=1}^m G$$

is subdirect, i.e. surjective on each factor.

LEMMA 1.7. Let  $G$  be a group of automorphisms of  $\mathcal{T}^{(p)}$ ,  $p$  a prime, and let all vertex permutations of the automorphisms in  $G$  are powers of a fixed cyclic permutation of order  $p$ . Then,  $G$  is fractal if and only if  $G$  is strongly fractal.

DEFINITION 1.8. A group  $G$  of tree automorphisms satisfies the congruence subgroup property if every finite index subgroup of  $G$  contains a level stabilizer  $\text{St}_G(\mathcal{L}_n)$ , for some  $n$ .

We now move to the rigid version of stabilizers:

DEFINITION 1.9. The rigid vertex stabilizer of  $u$  in  $G$ , denoted by  $\text{Rst}_G(u)$ , is the subgroup of  $G$  that consists of those automorphisms of  $\mathcal{T}$  that fix all vertices not having  $u$  as a prefix.

The automorphisms in  $\text{Rst}_G(u)$  must also fix  $u$ , and the only vertex permutations that are possibly non-trivial are those corresponding to the vertices in  $\mathcal{T}_u$  (see Figure 1.5).

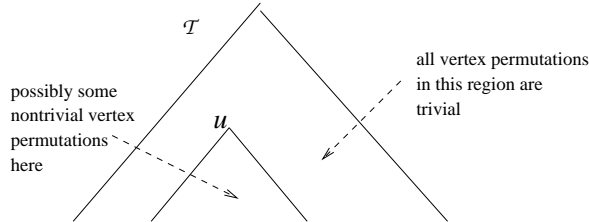


FIGURE 1.5. An automorphism in the rigid stabilizer of  $u$

The rigid stabilizer  $\text{Rst}_G(u)$  is also known as the *lower companion* of  $G$  at  $u$ , denoted by  $L_u^G$ , or by  $L_u$  when  $G$  is assumed. Clearly, the lower companion group at  $u$  can be embedded in the upper companion group which is contained in the corresponding section, i.e.

$$L_u \hookrightarrow U_u \subseteq G_u.$$

DEFINITION 1.10. *The subgroup of  $G$  generated by all the rigid stabilizers of vertices on the level  $n$  is the rigid  $n$ -th level stabilizer and it is denoted by  $\text{Rst}_G(\mathcal{L}_n)$ .*

Clearly, automorphisms in different rigid vertex stabilizers on the same level commute and

$$\text{Rst}_G(\mathcal{L}_n) = \prod_{u \in \mathcal{L}_n} \text{Rst}_G(u).$$

The level stabilizer  $\text{St}_G(\mathcal{L}_n)$  and the rigid level stabilizer  $\text{Rst}_G(\mathcal{L}_n)$  are normal in  $\text{Aut}(\mathcal{T})$ . Further, the following relations hold:

$$\prod_{u \in \mathcal{L}_n} L_u = \text{Rst}_G(\mathcal{L}_n) \leq \text{St}_G(\mathcal{L}_n) \xrightarrow{\psi} \prod_{u \in \mathcal{L}_n} U_u.$$

In contrast to the level stabilizers, the rigid level stabilizers may have infinite index, and may even be trivial.

Let us restrict our attention, for a moment, to the case when  $G$  is the full automorphism group  $\text{Aut}(\mathcal{T})$ . Clearly, every automorphism of  $\mathcal{T}_{|u|}$  is a section of an automorphism of  $\mathcal{T}$ , since any choice of vertex permutations at and below  $u$  is possible for automorphisms of the tree  $\mathcal{T}$  that fix  $u$ . Therefore,  $\text{Aut}(\mathcal{T})_u = \text{Aut}(\mathcal{T}_{|u|}) = \text{Aut}(\mathcal{T}_u)$ , i.e., the section is equal to the full automorphism group of the corresponding subtree. Moreover, the section groups are equal to the corresponding upper companion groups. It is also clear that the rigid stabilizer  $\text{Rst}_{\text{Aut}(\mathcal{T})}(u)$  is canonically isomorphic to  $\text{Aut}(\mathcal{T}_u)$ , that the rigid and the level stabilizer of the same level are equal, and  $\psi$  is an isomorphism.

Consider the subgroup  $\text{Aut}_f(\mathcal{T})$  of automorphisms that have only finitely many non-trivial vertex permutations. The automorphisms in this group are called *finitary*. The group of finitary automorphisms is the union of the chain of subgroups  $\text{Aut}_{[n]}(\mathcal{T})$  for  $n \in \mathbb{N}$ , where  $\text{Aut}_{[n]}(\mathcal{T})$  denotes the group of tree automorphisms whose vertex permutations at level  $n$  and below are trivial. The group  $\text{Aut}_{[n]}(\mathcal{T})$  is canonically isomorphic to the automorphism group  $\text{Aut}(\mathcal{T}_{[n]})$  of the finite tree  $\mathcal{T}_{[n]}$  that consists of the vertices of  $\mathcal{T}$  represented by words no longer than  $n$  (level  $n$  and above). The group  $\text{Aut}(\mathcal{T}_{[n]})$  is isomorphic to the iterated permutational wreath product

$$\text{Aut}(\mathcal{T}_{[n]}) \cong ((\dots (\text{Sym}(Y_n) \wr \text{Sym}(Y_{n-1})) \wr \dots) \wr \text{Sym}(Y_1)),$$

and its cardinality is  $m_1!(m_2!)^{m_1}(m_3!)^{m_1 m_2} \dots (m_n!)^{m_1 m_2 \dots m_{n-1}}$ . Also, the equality

$$\text{Aut}(\mathcal{T}) = \text{St}_{\text{Aut}(\mathcal{T})}(\mathcal{L}_n) \rtimes \text{Aut}_{[n]}(\mathcal{T}),$$

holds. As the intersection of all level stabilizers is trivial we see that  $\text{Aut}(\mathcal{T})$  is residually finite and, as a corollary, every subgroup of  $\text{Aut}(\mathcal{T})$  is residually finite.

We organize some of the remarks we already made in the following diagram:

$$\begin{array}{ccccccc} \text{Aut}(\mathcal{T}) & & \text{Aut}(\mathcal{T}) & & \text{Aut}(\mathcal{T}) & & \\ \parallel & & \parallel & & \parallel & & \\ \text{St}_{\text{Aut}(\mathcal{T})}(\mathcal{L}_0) & \longleftarrow & \text{St}_{\text{Aut}(\mathcal{T})}(\mathcal{L}_1) & \longleftarrow & \text{St}_{\text{Aut}(\mathcal{T})}(\mathcal{L}_2) & \longleftarrow & \dots \\ \times & & \times & & \times & & \\ & & & & & & \\ \text{Aut}_{[0]}(\mathcal{T}) & \longleftarrow & \text{Aut}_{[1]}(\mathcal{T}) & \longleftarrow & \text{Aut}_{[2]}(\mathcal{T}) & \longleftarrow & \dots \end{array}$$

The homomorphisms in the bottom row are the natural restrictions, and  $\text{Aut}(\mathcal{T})$  is the inverse limit of the inverse system represented by this row. Thus  $\text{Aut}(\mathcal{T})$  is a profinite group with topology that coincides with the Tychonoff product topology.

In this topological setting, we recall the *Hausdorff dimension* of a subgroup of  $\text{Aut}(\mathcal{T})$ :

DEFINITION 1.11 ([BS97]). *Let  $G < \text{Aut}(\mathcal{T})$  be a closed subgroup. Its Hausdorff dimension is*

$$\limsup_{n \rightarrow \infty} \frac{\log |G / \text{St}_G(\mathcal{L}_n)|}{\log |\text{Aut}(\mathcal{T}) / \text{St}_{\text{Aut}(\mathcal{T})}(\mathcal{L}_n)|},$$

a real number in  $[0, 1]$ .

Note that, according to our agreements, the iterated permutational wreath product

$$\prod_{i=1}^n (\wr) \text{Sym}(m_i) = ((\dots (\text{Sym}(m_n) \wr \text{Sym}(m_{n-1})) \wr \dots) \wr \text{Sym}(m_1)),$$

naturally acts on  $Y_1 \times Y_2 \times \dots \times Y_n$  which is exactly the set of words of length  $n$ . The action is by permutations  $f$  that respect prefixes in the sense that

$$|u \wedge v| = |u^f \wedge v^f|,$$

for all words  $u$  and  $v$  of length  $n$ . This allows us to define the action on the set of all words of length at most  $n$ , which is exactly why we may think of  $\prod_{i=1}^n (\wr) \text{Sym}(m_i)$  as being the automorphism group  $\text{Aut}(\mathcal{T}_{[n]})$  of the finite tree  $\mathcal{T}_{[n]}$ .

Since  $\prod_{i=1}^n (\wr) \text{Sym}(m_i)$  acts on the words of length  $n$ , the inverse limit  $\varprojlim_n \prod_{i=1}^n (\wr) \text{Sym}(m_i)$  acts on the set of infinite words by isometries, which is one of the interpretations of  $\text{Aut}(\mathcal{T})$  we already mentioned.

We agree on a simplified notation concerning the word  $u = y_{1,j_1} y_{2,j_2} \dots y_{n,j_n}$  over  $\bar{Y}$ , the section  $f_u$  of the automorphism  $f$  and the homomorphism  $\varphi_u$ . We will write sometimes just  $u = j_1 j_2 \dots j_n$  since the sequence of indices  $j_1 j_2 \dots j_n$  uniquely determines and is uniquely determined by the word  $u$ . Also, we will write  $f_{j_1 j_2 \dots j_n}$  and  $\varphi_{j_1 j_2 \dots j_n}$  for the appropriate section  $f_u$  and section homomorphism  $\varphi_u$ . Actually, we could agree that  $Y_i = \{1, 2, \dots, m_i\}$ , for  $i \in \mathbb{N}_+$ , in which case the original and the simplified notation are the same.

### 1.3. Geometric definition of a branch group

For the length of this section we make an important assumption that  $G$  is a group of automorphisms of  $\mathcal{T}$  that acts transitively on each level of the tree. In this case we say that  $G$  acts *spherically transitively*.

It follows easily from (9), that all vertex stabilizers  $\text{St}_G(u)$  corresponding to vertices  $u$  on the same level are conjugate in  $G$ . Indeed, if  $h$  fixes  $u$  then  $h^g$  fixes  $u^g$  and  $(h_u)^{g_u} = (h^g)_{u^g}$ . This also shows that, in case of a spherically transitive action, the upper companion groups  $U_u$ ,  $u$  a vertex in the level  $\mathcal{L}_n$ , are conjugate in  $\text{Aut}(\mathcal{T}_n)$ , we denote by  $U_n^G$ , or just by  $U_n$ , their isomorphism type, and we call it the *upper companion group* of  $G$  at level  $n$  or the  $n$ -th upper companion:

$$\begin{array}{ccccc} L_u = \text{Rst}_G(u) & \xleftarrow{\triangleleft} & \text{St}_G(u) & \xrightarrow{\varphi_u} & U_u \\ \downarrow (\cdot)^g & & \downarrow (\cdot)^g & & \downarrow (\cdot)^{g_u} \\ L_{u^g} = \text{Rst}_G(u^g) & \xleftarrow{\triangleleft} & \text{St}_G(u^g) & \xrightarrow{\varphi_{u^g}} & U_{u^g} \end{array} \quad (h_u)^{g_u} = (h^g)_{u^g}$$

Moreover, we note that if the section  $g_u$  is trivial then the upper companion groups  $U_u$  and  $U_{u^g}$  are not only conjugate, but they are equal.

Similarly, the rigid vertex stabilizers of vertices on the same level are also conjugate in  $G$ , we denote by  $L_n^G$ , or just by  $L_n$ , their isomorphism type, and we call it the *lower companion group* of

$G$  at level  $n$  or the  $n$ -th lower companion (see the above diagram). We note that the rigid vertex stabilizer  $\text{Rst}_G(u)$  is a normal subgroup in the corresponding vertex stabilizer  $\text{St}_G(u)$ . Moreover, the lower companion group  $L_u$  naturally embeds via the section homomorphisms  $\varphi_u$  in the upper companion group  $U_u$  as a normal subgroup. In case of the full automorphism group, we already remarked that this embedding is an isomorphism. In general, this is not true, and we will study more closely the “next best case” when the embedded subgroup has a finite index.

**PROPOSITION 1.12.** *Let  $G$  be a group of automorphisms of  $\mathcal{T}$  acting spherically transitively. If  $\text{Rst}_G(\mathcal{L}_n)$  has finite index in  $G$  for all  $n$ , then  $G$  is branch group with branch structure  $(L_n^G, \text{Rst}_G(\mathcal{L}_n))_{i=1}^\infty$ .*

**DEFINITION 1.13.** *Let  $G$  be a group of automorphisms of  $\mathcal{T}$  acting spherically transitively. We say that  $G$  is a*

- (1) branch group acting on a tree if all rigid stabilizers of  $G$  have finite index in  $G$ .
- (2) weakly branch group acting on a tree if all rigid stabilizers of  $G$  are non-trivial (which implies that they are infinite).
- (3) rough group acting on a tree if all rigid stabilizers of  $G$  are trivial.

The branch structure from the previous proposition is not unique, as usual, and we see that for  $G$  to be branch group it is enough if we require that each rigid vertex stabilizer group  $\text{Rst}_G(u)$ ,  $u$  a vertex in  $\mathcal{T}$ , has a subgroup  $L(u)$  such that  $H_n = \prod \{L(u) \mid u \in \mathcal{L}_n\}$  is normal of finite index in  $G$ , for all  $n$ .

Particularly important type of branch groups is introduced by the following definitions.

**DEFINITION 1.14.** *A fractal branch group  $G$  acting on the regular tree  $\mathcal{T}^{(m)}$  is a regular branch group if there exists a finite index subgroup  $K$  of  $G$  such that  $K^m$  is contained in  $(K)\psi$  as a subgroup of finite index. In such a case, we say that  $G$  is branching over  $K$ . We also say that  $K$  geometrically contains  $K^m$ . In case  $K$  contains  $K^m$  but the index is infinite we say that  $G$  is weakly regular branch over  $K$ .*

**DEFINITION 1.15.** *Let  $G$  be a regular branch group generated as a monoid by a finite set  $S$ , and consider the induced word metric on  $G$ . We say  $G$  is contracting if there exist positive constants  $\lambda < 1$  and  $C$  such that for every word  $w \in S^*$  representing an element of  $\text{St}_G(\mathcal{L}_1)$ , writing  $(w)\psi = (w_1, \dots, w_m)$ , we have*

$$|w_i| < \lambda|w| \text{ for all } i \in Y, \text{ as soon as } |w| > C. \quad (10)$$

The constant  $\lambda$  is called a contracting constant.

In a loose sense, the abstract branch groups are groups that remind us of the full automorphism groups of the spherically homogeneous rooted trees. Any branch group has a natural action on a rooted tree. Indeed, let  $G$  be a branch group with branch structure  $(L_i, H_i)_{i \in \mathbb{N}}$ . The set of subgroups  $\{L_i^{(j)} \mid i \in \mathbb{N}, j = 1, \dots, k_i\}$  ordered by inclusion forms a spherically homogeneous tree with branching sequence  $\overline{m} = m_1, m_2, \dots$ , where  $m_i = k_i/k_{i-1}$ . The group  $G$  acts on this set by conjugation and, because of the refinement conditions, the resulting permutation is a tree automorphism (see Figure 1.1)

The action of the branch group  $G$  on the tree determined by the branch structure is not faithful in general. Indeed, it is known that a branch group that satisfies the conditions in Proposition 1.12 is centerless (see [Gri00]). On the other hand, a direct product of a branch group  $G$  in the sense of our algebraic definition with a finite group  $H$  is still a branch group. If  $H$  has non-trivial center, then  $G \times H$  is a branch group with non-trivial center.

It would be interesting to understand the nature of the kernel of the action in the passing from an abstract branch group to a group that acts on a tree. In particular, is it correct that this kernel is always in the center (Question 2)? We note that the kernel is trivial in case of a just-infinite branch group (see Chapter 5).

### 1.4. Portraits and branch portraits

A tree automorphism can be described by its *portrait*, already defined before, and repeated here in the following form:

DEFINITION 1.16. *Let  $f$  be an automorphism of  $\mathcal{T}$ . The portrait of  $g$  is a decoration of the tree  $\mathcal{T}$ , where the decoration of the vertex  $u$  belongs to  $\text{Sym}(Y_{|u|+1})$ , and is defined inductively as follows: first, there is  $\pi_\emptyset \in \text{Sym}(Y_1)$  such that  $g = h\pi_\emptyset$  and  $h$  stabilizes the first level. This  $\pi_\emptyset$  is the label of the root vertex. Then, for all  $y \in Y_1$ , label the tree below  $y$  with the portrait of the section  $g_y$ .*

The following notion of a branch portrait based on the branch structure of the group in question is useful in some considerations:

DEFINITION 1.17. *Let  $G$  be a branch group, with branch structure  $(L_i, H_i)_{i \in \mathbb{N}}$ . The branch portrait of  $g \in G$  is a decoration of the tree  $\mathcal{T}^{(Y)}$ , where the decoration of the root vertex belongs to  $G/H_1$  and the decoration of the vertex  $y_1 \dots y_i$  belongs to  $L^{(y_1 \dots y_i)} / \prod_{y \in Y_{i+1}} L^{(y_1 \dots y_i y)}$ . Fix once and for all transversals for the above coset spaces. The branch portrait of  $g$  is defined inductively as follows: the decoration of the root vertex is  $H_1 g$ , and the choice of the transversal gives us an element  $(g_{y_1})_{y_1 \in Y_1}$  of  $H_1$ . Decorate then  $y_1 \in Y_1$  by  $\prod_{y_2 \in Y_2} L^{(y_1 y_2)} g_{y_1}$ ; again the choice of transversals gives us elements  $g_{y_1 y_2} \in L^{(y_1 y_2)}$ ; etc.*

There are uncountably many possible branch portraits that use the chosen transversals, even when  $G$  is a countable branch group. We therefore introduce the following notion:

DEFINITION 1.18. *Let  $G$  be a branch group. Its tree completion  $\overline{G}$  is the inverse limit*

$$\varprojlim G / \text{St}_G(n).$$

*This is also the closure in  $\text{Aut } \mathcal{T}$  of  $G$  in the topology given by its action on the tree  $\mathcal{T}$ .*

Note that since  $\overline{G}$  is closed in  $\text{Aut}(\mathcal{T})$  it is a profinite group, and thus is compact, and totally disconnected. If  $G$  has the congruence subgroup property [Gri00], then  $\overline{G}$  is also the profinite completion of  $G$ .

LEMMA 1.19. *Let  $G$  be a branch group and  $\overline{G}$  its tree completion. Then Definition 1.17 yields a bijection between the set of branch portraits over  $G$  and  $\overline{G}$ .*

Branch portraits are very useful to express, for instance, the lower central series. They appear also, in more or less hidden manner, in most results on growth and torsion.

### 1.5. Groups of finite automata and recursively defined automorphisms

We introduce two more ways to think about tree automorphisms in the case of a regular tree. It is not impossible to extend the definitions to more general cases, but we choose not to do so. Thus for the length of this section we set  $Y = \{1, \dots, m\}$  and we work with the regular tree  $\mathcal{T} = \mathcal{T}^{(Y)} = \mathcal{T}^{(m)}$ .

**1.5.1. Recursively defined automorphisms.** Let  $X = \{x^{(1)}, \dots, x^{(n)}\}$  be a set of symbols and  $F$  a finite set of finitary automorphisms of  $\mathcal{T}$ . The following equations

$$x^{(i)} = (w_{i,1}, \dots, w_{i,m})f^{(i)}, \quad i \in \{1, \dots, n\}, \quad (11)$$

where  $w_{i,j}$  are words over  $X \cup F$  and  $f^{(i)}$  are elements in  $F$ , define an automorphism of  $\mathcal{T}$ , still written  $x^{(i)}$ , for each symbol  $x^{(i)}$  in  $X$ . The way in which the equations (11) define automorphisms recursively is as follows: we interpret the  $m$ -tuple  $(w_{i,1}, \dots, w_{i,m})$  as an automorphism fixing the first level and  $w_{i,j}$  are just the sections at  $j$ ,  $j \in \{1, \dots, m\}$ ; the equations (11) clearly define the vertex permutation at the root for all  $x^{(i)}$ ; if the vertex permutations of all the  $x^{(i)}$  are defined for the first  $k$  levels then the vertex permutations of all the  $w_{i,j}$  are defined for the first  $k$  levels, which in turn defines all the vertex permutations of all the  $x^{(i)}$  for the first  $k+1$  levels.

Every automorphism of  $\mathcal{T}$  that can be defined as a member of some set  $X$  of recursively defined automorphisms as above is called a *recursively definable* automorphism of  $\mathcal{T}$ . The set of recursively definable automorphisms of  $\mathcal{T}$  forms a subgroup  $\text{Aut}_r(\mathcal{T})$  of  $\text{Aut}(\mathcal{T})$ . The group  $\text{Aut}_r(\mathcal{T})$  is a regular branch group which properly contains  $\text{Aut}_f(\mathcal{T})$ .

When one defines automorphisms recursively it is customary to choose all finitary automorphisms  $f^{(i)}$  to be rooted automorphisms (see Definition 1.23). The advantage in that case is that  $w_{i,j}$  is exactly the section of  $x^{(i)}$  at  $j$ . As an example of a recursively defined automorphisms consider

$$b = (a, b)$$

acting on the binary tree  $\mathcal{T}^{(2)}$ , where  $a = ((1, 2))$  is the non-trivial rooted automorphism of  $\mathcal{T}^{(2)}$ . Clearly, the diagram in Figure 1.6 represents  $b$  through its vertex permutations.

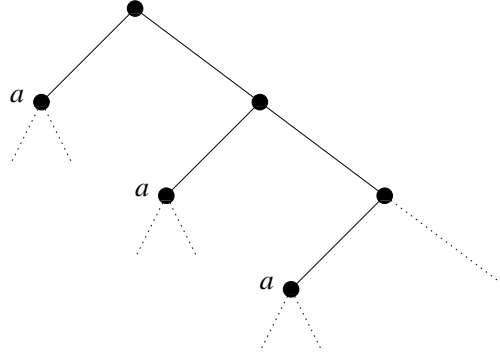


FIGURE 1.6. The recursively defined automorphism  $b$

It is easy to see that all tree automorphisms are recursively definable if we extend our definition to allow infinite sets  $X$ . Indeed

$$g_u = (g_{u1}, \dots, g_{um})(u)g, \quad u \in Y^*$$

defines recursively  $g$  and all of its sections.

**1.5.2. Groups of finite automata.** Since we want to define automata that behave like tree automorphisms we need automata that transform words rather than recognize them, i.e., we will be working with transducers. The fact that we want our automata to preserve lengths and permute words while preserving prefixes strongly suggests the choices made in the following definition.

DEFINITION 1.20. A synchronous invertible finite transducer is a quadruple  $T = (Q, Y, \tau, \lambda)$  where

- (1)  $Q$  is a finite set (set of states of  $T$ ),
- (2)  $Y$  is a finite set (the alphabet of  $T$ ),
- (3)  $\tau$  is a map  $\tau : Q \times Y \rightarrow Q$  (the transition function of  $T$ ), and
- (4)  $\lambda$  is a map  $\lambda : Q \times Y \rightarrow Y$  (the output function of  $T$ ) such that the induced map  $\lambda_q : Y \rightarrow Y$  obtained by fixing a state  $q$  is a permutation of  $Y$ , for all states  $q \in Q$ .

If  $T$  is a synchronous invertible finite transducer and  $q$  a state in  $Q$  we sometimes give  $q$  a distinguished status, call it the initial state and define the initial synchronous invertible finite transducer  $T_q$  as the transducer  $T$  with initial state  $q$ .

We say just (initial) transducer in the sequel rather than (initial) synchronous invertible finite transducer.



FIGURE 1.7. An example of a transducer

It is customary to represent transducers with directed labelled graphs where  $Q$  is the set of vertices and there exists an edge from  $q_0$  to  $q_1$  if and only if  $\tau(q_0, y) = q_1$ , for some  $y \in Y$ , in which case the edge is labelled by  $y|\lambda(q_0, y)$ . The diagram in Figure 1.7 gives an example.

Informally speaking, given an initial transducer  $T_q$  and an input word  $w$  over  $Y$  we start at the vertex  $q$  and we travel through the graph by reading  $w$  one letter at a time and following the values of the transition function. Thus if we find ourselves at the state  $q'$  and we read the letter  $y$  we move to the state  $\tau(q', y)$  by following the edge labelled by  $y|\lambda(q', y)$ . In the same time, we write down an output word, one letter at a time, simply by writing down the letters after the vertical bar in the labels of the edges we used in our journey.

More formally, given an initial transducer  $T_q$  we define recursively the maps  $\tau_q : Y^* \rightarrow Q$  and  $\lambda_q : Y^* \rightarrow Y^*$  as follows:

$$\begin{aligned} \tau_q(\emptyset) &= q \\ \tau_q(wy) &= \tau(\tau_q(w), y) \quad \text{for } w \in Y^* \\ \lambda_q(\emptyset) &= \emptyset \\ \lambda_q(wy) &= \lambda_q(w)\lambda(\tau_q(w), y) \quad \text{for } w \in Y^* \end{aligned}$$

It is not difficult to see that  $\lambda_q$ , the *output function* of the initial transducer  $T_q$ , represents an automorphism of  $\mathcal{T}$ . The set of all tree automorphisms  $\text{Aut}_{ft}(\mathcal{T})$  that can be realized as output functions of some initial transducer forms a subgroup of  $\text{Aut}(\mathcal{T})$  and this subgroup is a regular branch group sitting properly between the group of finitary and the group of recursively definable automorphisms of  $\mathcal{T}$ .

If we allow infinitely many states, then every automorphism  $g$  of  $\mathcal{T}$  can be realized by an initial transducer. Indeed, we may define the set of states  $Q$  to be the set of sections of  $g$ , i.e.  $Q = \{g_u | u \in Y^*\}$  and

$$\tau(g_u, y) = g_{uy} \quad \text{and} \quad \lambda(g_u, y) = y^{(u)g}.$$

We could index the states by the vertices in  $\mathcal{T}$ , but by indexing them by the sections of  $g$  we see that an automorphism  $g$  can be defined by a finite initial transducer if it has only finitely many distinct sections. The converse is also true.

**PROPOSITION 1.21.** *An automorphism of  $\mathcal{T}$  is the output function of some initial transducer if and only if it has finitely many distinct sections.*

Every transducer  $T$  defines a group  $G_T$  of tree automorphisms generated by the initial transducers of  $T$  (one for each state). Groups that are defined by transducers are known as *groups of automata*.

Note that the notion of a group of automata is different from the notion of automatic group in the sense of Jim Cannon. For more information on groups of automata we refer the reader to [GNS00].

### 1.6. Examples of branch groups

We have already seen a couple of examples of branch groups acting on a regular tree  $\mathcal{T}$ . Namely, the groups  $\text{Aut}_f(\mathcal{T})$ ,  $\text{Aut}_{ft}(\mathcal{T})$ ,  $\text{Aut}_r(\mathcal{T})$  and the full automorphism group  $\text{Aut}(\mathcal{T})$ .

PROPOSITION 1.22. *Let  $\mathcal{T}$  be regular tree and  $G$  be any of the groups  $\text{Aut}_f(\mathcal{T})$ ,  $\text{Aut}_{ft}(\mathcal{T})$ ,  $\text{Aut}_r(\mathcal{T})$  or  $\text{Aut}(\mathcal{T})$ . Then  $G$  is a regular branch group with  $G = G \wr_{Y_1} \text{Sym}(Y_1)$ .*

None of the groups in the previous proposition is finitely generated, but the first three are countable. Another example of a regular branch group is, for a permutation group  $A$  of  $Y$ , the group  $\text{Aut}_A(\mathcal{T}^{(Y)})$  that consists of those automorphisms of the regular tree  $\mathcal{T}^{(Y)}$  whose vertex permutations come from  $A$ . A special case of the last example was mentioned before as the infinitely iterated wreath product of copies of the cyclic group  $\mathbb{Z}/p\mathbb{Z}$  and the full automorphism group is another special case.

In the sequel we give some examples of finitely generated branch groups. We make use of rooted and directed automorphisms.

DEFINITION 1.23. *An automorphism of  $\mathcal{T}$  is rooted if all of its vertex permutations that correspond to non-empty words are trivial.*

Clearly, the rooted automorphisms are precisely the finitary automorphisms from  $\text{Aut}_{[1]}$ . A rooted automorphism  $f$  just permutes rigidly the  $m_1$  trees  $\mathcal{T}_1, \dots, \mathcal{T}_{m_1}$  as prescribed by the root permutation  $(\emptyset)f$ . It is convenient not to make too much difference between the root vertex permutation  $(\emptyset)f$  and the rooted automorphism  $f$  defined by it. Therefore, if  $a$  is a permutation of  $Y_1$  we also say that  $a$  is a rooted automorphism of  $\mathcal{T}$ . More generally, if  $a$  is the vertex permutation of  $f$  at  $u$  and all the vertex permutations below  $u$  are trivial, then we do not distinguish  $a$  from the section  $f_u$  defined by it, i.e., we write  $(f)\varphi_u = f_u = a = (u)f$ .

DEFINITION 1.24. *Let  $\ell = y_1y_2y_3\dots$  be an infinite ray in  $\mathcal{T}$ . We say that the automorphism  $f$  of  $\mathcal{T}$  is directed along  $\ell$  and we call  $\ell$  the spine of  $f$  if all vertex permutations along the ray  $\ell$  and all vertex permutations corresponding to vertices whose distance to the ray  $\ell$  is at least 2 are trivial.*

In the sequel, we define many directed automorphisms that use the rightmost infinite ray in  $\mathcal{T}$  as a spine, i.e., the spine is  $m_1m_2m_3\dots$ . Therefore, the only vertices that can have a nontrivial permutation are the vertices of the form  $m_1m_2\dots m_nj$  where  $j \neq m_{n+1}$ . Note that directed automorphisms fix the first level, i.e., their root vertex permutation is trivial.

**1.6.1. The first Grigorchuk group  $\mathfrak{G}$ .** A description of the first Grigorchuk group, denoted by  $\mathfrak{G}$ , appeared for the first time in 1980 in [Gri80]. Since then, the group  $\mathfrak{G}$  has been used as an example or counter-example in many non-trivial situations.

The group  $\mathfrak{G}$  acts on the rooted binary tree  $\mathcal{T}^{(2)}$  and it is generated by the four automorphisms  $a$ ,  $b$ ,  $c$  and  $d$  defined below. The automorphism  $a$  is the only possible rooted automorphism  $a = ((1, 2))$  that permutes rigidly the two subtrees below the root. Parts of the portraits along the spine of the generators  $b$ ,  $c$ , and  $d$  are depicted in Figure 1.8, Figure 1.9 and Figure 1.10. We implicitly assume that the patterns that are visible in the diagrams repeat indefinitely along the spine, i.e., along the rightmost ray.

Another way to define the directed generators of  $\mathfrak{G}$  is by the following recursive definition:

$$b = (a, c) \quad c = (a, d) \quad d = (1, b).$$

It is clear from this recursive definition that  $\mathfrak{G}$  can also be defined as a group of automata.

The group  $\mathfrak{G}$  is a 2-group, has a solvable word problem and intermediate growth (see [Gri84]). The best known estimates of the growth of the first Grigorchuk group are given by the first author in [Bar98] and [Bar01] (see also [Leo98b, Leo00]). The subgroup structure of  $\mathfrak{G}$  is a subject of many articles (see [Roz96, BG02] and Chapter 7) and it turns out that  $\mathfrak{G}$  has finite width. An infinite set of defining relations is given by Igor Lysionok in [Lys85], and the second author shows that this system is minimal in [Gri99]. The conjugacy problem is solved by Yuriĭ Leonov in [Leo98a]

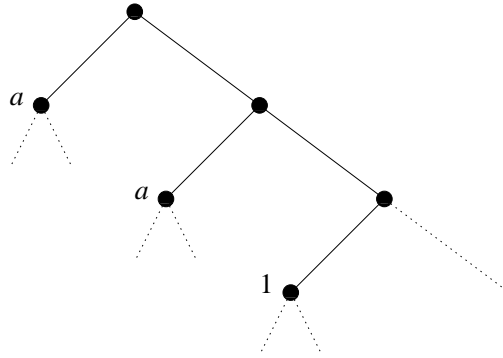


FIGURE 1.8. The directed automorphism  $b$

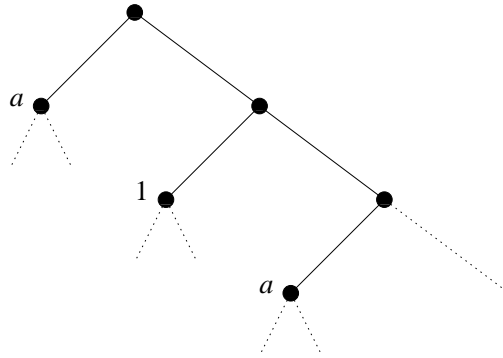


FIGURE 1.9. The directed automorphism  $c$

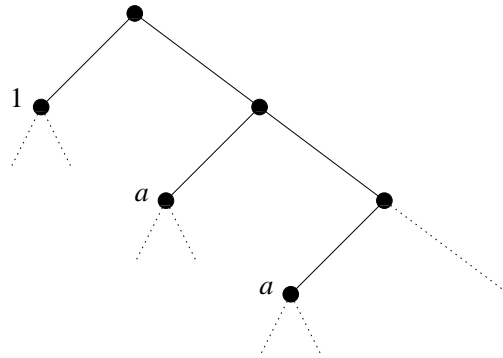


FIGURE 1.10. The directed automorphism  $d$

and Alexander Rozhkov in [Roz98]. A detailed exposition of many of the known properties of  $\mathfrak{G}$  is included in the book [Har00] by Pierre de la Harpe. Another exposition (in Italian) appears in [CMS01].

Most properties of  $\mathfrak{G}$ , in one way or another, follow from the following

PROPOSITION 1.25. *The group  $\mathfrak{G}$  is a regular branch group over the subgroup  $K = \langle [a, b] \rangle^{\mathfrak{G}}$ .*

PROOF. The first step is to prove that  $K$  has finite index in  $\mathfrak{G}$ . We check that  $a^2, b^2, c^2, d^2, bcd, (ad)^4$  are relators in  $\mathfrak{G}$ . It follows that  $B = \langle b \rangle^{\mathfrak{G}}$  has index 8, since  $\mathfrak{G}/B$  is  $\langle aB, dB \rangle$ , a dihedral group of order 8. Consequently  $B = \langle b, b^a, b^{a^2}, b^{a^3} \rangle$ . Now  $B/K$  is  $\langle bK \rangle$ , of order 2, so  $K$  has index 2 in  $B$  and thus 16 in  $\mathfrak{G}$ .

Then we consider  $L = \langle [b, d^a] \rangle^{\mathfrak{G}}$  in  $K$ . A simple computation gives  $([b, d^a])\psi = ([a, b], 1)$ , so  $(L)\psi = K \times 1$ , and we get  $(K)\psi \geq (L \times L^a)\psi = K \times K$ .  $\square$

The index of  $K$  in  $\mathfrak{G}$  is 16 and the index of  $K \times K$  in  $(K)\psi$  is 4.

**1.6.2. The second Grigorchuk group.** The second Grigorchuk group was introduced in the same paper as the first one ([Gri80]). It acts on the 4-regular tree  $\mathcal{T}^{(4)}$  and it is generated by the cyclic rooted automorphism  $a = ((1, 2, 3, 4))$  and the directed automorphism  $b$  whose portrait is given in Figure 1.11.

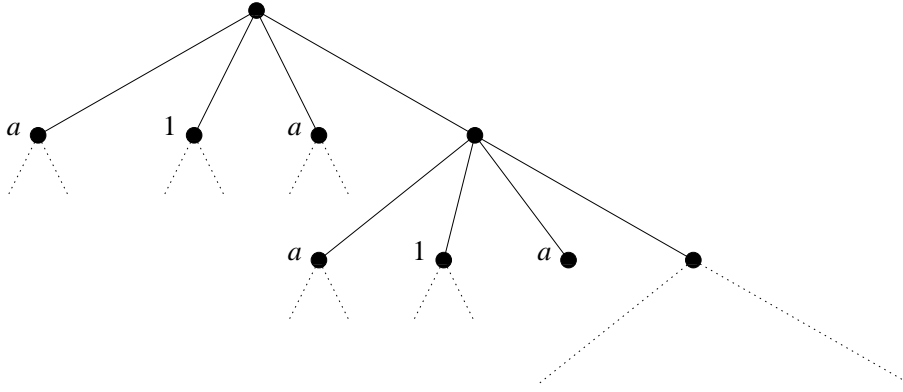


FIGURE 1.11. The directed automorphism  $b$  in the second Grigorchuk group

A recursive definition of  $b$  is

$$b = (a, 1, a, b).$$

The second Grigorchuk group is not investigated nearly as thoroughly as the first one. It is finitely generated, infinite, residually finite and centerless. In addition, it is not finitely presented, and is a torsion group with a solvable word problem. More on this group can be found in [Vov00].

**1.6.3. Gupta-Sidki  $p$ -groups.** The first Gupta-Sidki  $p$ -groups were introduced in [GS83a]. For odd prime  $p$ , we define the rooted automorphism  $a = ((1, 2, \dots, p))$  and the directed automorphism  $b$  of  $\mathcal{T}^{(p)}$  whose portrait is given in Figure 1.12.

The group  $G = \langle a, b \rangle$  is a 2-generated  $p$ -group with solvable word problem and no finite presentation (consider [Sid87b]). In case  $p = 3$  the automorphism group, centralizers and derived groups were calculated by Said Sidki in [Sid87a].

More examples of 2-generated  $p$ -groups along the same lines were constructed by Narain Gupta and Said Sidki in [GS83b]. In this paper the directed automorphism  $b$  is defined recursively by

$$b = (a, a^{-1}, a, a^{-1}, \dots, a, a^{-1}, b).$$

It is shown in [GS83b] that these groups are just-infinite  $p$ -groups. Also, every finite  $p$ -group is contained in the corresponding Gupta-Sidki infinite  $p$ -group.

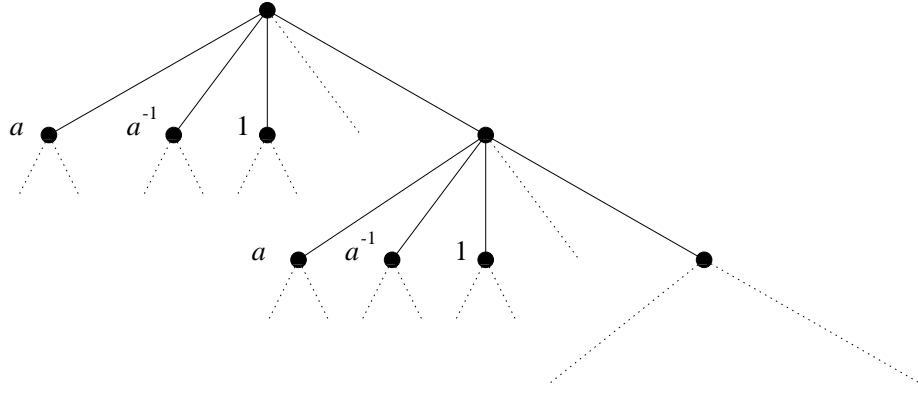


FIGURE 1.12. The directed automorphism  $b$  in the Gupta-Sidki groups

**1.6.4. Three groups acting on the ternary tree.** We define three groups acting on the ternary tree  $\mathcal{T}^{(3)}$ . Each of them is 2-generated, with generators  $a$  and  $b$ , where  $a$  is the rooted automorphism  $a = ((1, 2, 3))$  and  $b$  is one of the following three directed automorphisms

$$b = (a, 1, b) \quad \text{or} \quad b = (a, a, b) \quad \text{or} \quad b = (a, a^2, b).$$

The corresponding group  $G = \langle a, b \rangle$  is denoted by  $\Gamma$ ,  $\bar{\Gamma}$  and  $\bar{\bar{\Gamma}}$ , respectively. The group  $\Gamma$  is called the Fabrykowski-Gupta group and is the first example of a group of intermediate growth that is not constructed by the second author. The construction appears in [FG85], with an incorrect proof, and in [FG91]. The group  $\bar{\Gamma}$  is called the Bartholdi-Grigorchuk group and is studied in [BG02]. The article shows that both  $\Gamma$  and  $\bar{\Gamma}$  are virtually torsion-free with a torsion-free subgroup of index 3. The group  $\bar{\bar{\Gamma}}$  is known as the Gupta-Sidki group, it is the first one of the three to appear in print in [GS83a]. All three groups have intermediate growth. The first author has calculated the central lower series for  $\Gamma$  and  $\bar{\bar{\Gamma}}$  (see [Bar00a, Bar00b]). We note here that  $\bar{\Gamma}$  is not branch group but only weakly branch group, and the other two are regular branch groups over their commutator subgroups.

**1.6.5. Generalization of the Fabrykowski-Gupta example.** The following examples of branch groups acting on the regular tree  $\mathcal{T}^{(m)}$ , for  $m \geq 3$ , are studied in [Gri00]. The group  $G = \langle a, b \rangle$  is generated by the rooted automorphism  $a = ((1, 2, \dots, m))$  and the directed automorphism  $b$  whose portrait is given in Figure 1.13.

The group  $G$  is a regular branch group over its commutator. Moreover, the rigid vertex stabilizers are isomorphic to the commutator subgroup. Clearly, for  $m = 3$  we obtain the Fabrykowski-Gupta group  $\Gamma$ . For  $m \geq 5$ ,  $G$  is just-infinite, and for a prime  $m \geq 7$  the group  $G$  has the congruence subgroup property. The last two results can probably be extended to other branching indices.

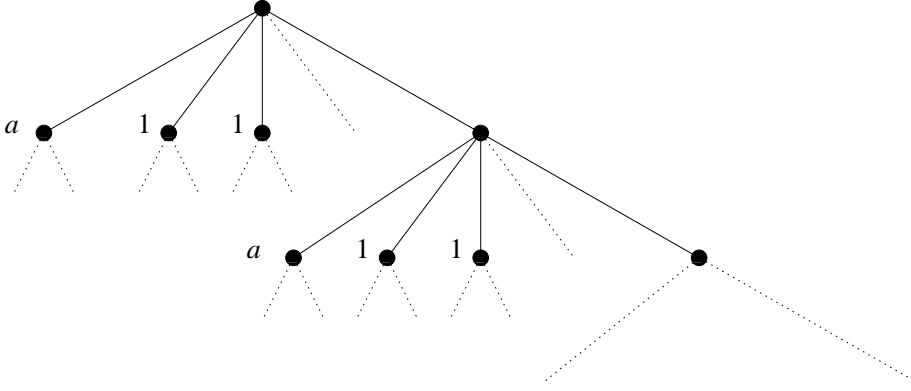
**1.6.6. Examples of Peter Neumann.** The following example is constructed in [Neu86]. Let  $A = \text{Alt}(6)$  be the alternating group acting on the alphabet  $Y = \{1, \dots, 6\}$ . For each pair  $(a, y)$  with  $y \in Y$  and  $a \in \text{St}_A(y)$ , define an automorphism  $b_{(a,y)}$  of the regular tree  $\mathcal{T}^{(6)}$  recursively by

$$b_{(a,y)} = (1, \dots, 1, b_{(a,y)}, 1, \dots, 1)a,$$

where the only nontrivial section appears at the vertex  $y$ . Let  $G$  be the group generated by all these tree automorphisms, i.e.

$$G = \langle \{b_{(a,y)} \mid y \in Y, a \in \text{St}_A(y)\} \rangle.$$

Since  $G$  is generated by 6 perfect subgroups (one copy of  $\text{Alt}(5)$  for each  $y \in Y$ ) we see that  $G$  is perfect. It is also easy to see that if  $a$  and  $a'$  fix both  $x$  and  $y$ , with  $x \neq y$ , then  $[b_{(a,x)}, b_{(a',y)}] = [a, a']$ .

FIGURE 1.13. The directed automorphism  $b$ 

Since

$$\left\langle \bigcup_{x \neq y \in Y} [\text{St}_A(\{x, y\}), \text{St}_A(\{x, y\})] \right\rangle = A,$$

we see that  $G$  contains  $A$  as a rooted subgroup. It follows that  $G \cong G \wr_Y A$ .

**THEOREM 1.26** (Peter Neumann [Neu86]). *Let  $A$  be a non-abelian finite simple group acting faithfully and transitively on the set  $Y$ . If  $G$  is a perfect, residually finite group such that  $G \cong G \wr_Y A$  then*

- (1) *All non-trivial normal subgroups subgroups of  $G$  have finite index, i.e.,  $G$  is just-infinite (see Chapter 5).*
- (2) *Every subnormal subgroup of  $G$  is isomorphic to a finite direct power of  $G$ , but  $G$  does not satisfy the ascending chain condition on subnormal subgroups.*
- (3)  *$G$  is minimal (see Chapter 5 again).*

A group that satisfies the conditions of the previous theorem is a regular branch group over itself, acting on the regular tree  $\mathcal{T}^{(Y)}$ . Furthermore, the only normal subgroups of  $G$  are the (rigid) level stabilizers  $\text{St}_G(\mathcal{L}_n) = \text{Rst}_G(\mathcal{L}_n) \cong G^{m^n}$ .

**1.6.7. The examples of Dan Segal.** For more details on the following examples check [Seg01]. Precursors can be found in [Seg00].

For  $i \in \mathbb{N}$ , let  $A_i$  be a finite, perfect, transitive permutation subgroup of  $\text{Sym}(Y_{i+1})$ , where  $Y_{i+1}$  is a set of  $m_{i+1}$  elements. We assume that all stabilizers  $\text{St}_{A_i}(y)$  are distinct, for fixed  $i$  and  $y \in Y_{i+1}$ . Let  $A_i = \langle a_i^{(1)}, \dots, a_i^{(k)} \rangle$ . For  $j \in \{1, \dots, k\}$ , the diagram in Figure 1.14 represents the directed automorphism  $b_0^{(j)}$  of  $\mathcal{T}^{(Y)}$ , which is recursively defined through

$$b_i^{(j)} = (a_{i+1}^j, 1, \dots, 1, b_{i+1}^{(j)}), \quad i \in \mathbb{N}, \quad j \in \{1, \dots, k\}.$$

We define a group  $G_i = \langle A_i \cup B_i \rangle$  where  $B_i = \langle b_i^{(1)}, \dots, b_i^{(k)} \rangle$ , for  $i \in \mathbb{N}$ . Let  $x$  be an element in  $A_0$  such that  $x$  fixes 1 but not  $m_1$ . Then

$$[b_0^{(i)}, (b_0^{(j)})^x] = ([a_1^{(i)}, a_1^{(j)}], 1, \dots, 1)$$

which, by the perfectness of  $A_1$ , shows that the rigid stabilizers of the vertices on the first level contain the rooted subgroup  $A_1$ . Since  $((a_1^{(j)})^{-1}, 1, \dots, 1)b_0^j = (1, \dots, b_1^{(j)})$  we see that the rigid vertex stabilizers of the vertices on the first level are exactly the upper companion groups. Similar claims hold for the other levels and we see that  $G = G_0$  is a branch group with

$$\text{St}_G(\mathcal{L}_n) = \text{Rst}_G(\mathcal{L}_n) \cong \prod^{m_1 m_2 \dots m_n} G_n.$$

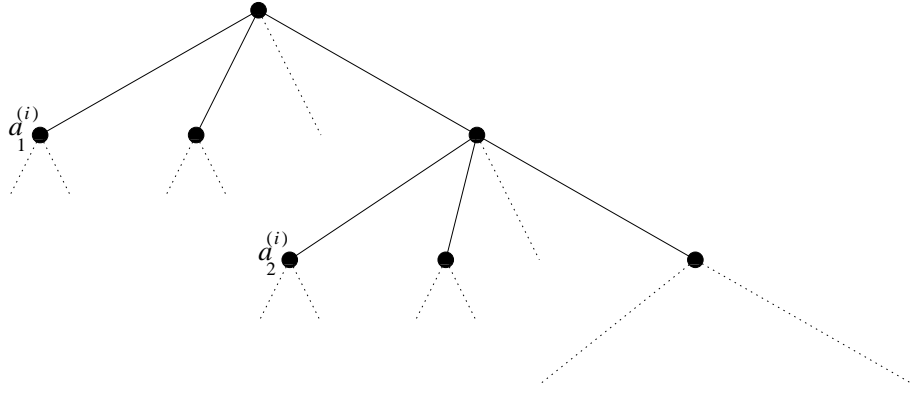


FIGURE 1.14. The directed automorphism in the examples of Segal

In a similar fashion, in case  $k = 2$  we can construct a branch group on only three generators as follows. For each  $i \in \mathbb{N}$  choose a permutation  $\mu_i \in A_i$  of  $Y_{i+1}$  that fixes  $m_{i+1}$  but  $\mu_i^2$  does not fix the symbol  $r_{i+1} \in Y_{i+1}$ . Then define the directed automorphism  $b_0$  recursively through

$$b_i = (1, \dots, 1, a_{i+1}^{(1)}, 1, \dots, 1, a_{i+1}^{(2)}, 1, \dots, 1, b_{i+1}), \quad i \in \mathbb{N},$$

where  $a_{i+1}^{(1)}$  is on position  $r_{i+1}$  and  $a_{i+1}^{(2)}$  is on position  $r_{i+1}^{\mu_i}$ , for  $i \in \mathbb{N}$ . Define  $G_i = \langle A_i \cup \{b_i\} \rangle$ , for  $i \in \mathbb{N}$ . Then we also get a branch group  $G = G_0$  with

$$\text{St}_G(\mathcal{L}_n) = \text{Rst}_G(\mathcal{L}_n) \cong \prod^{m_1 m_2 \dots m_n} G_n.$$

Different choices of groups  $A_i$  together with appropriate actions give various groups with various interesting properties. We list two interesting results from [Seg01], one below and another in Theorem 8.21 that use the above examples:

**THEOREM 1.27.** *For every collection  $\mathcal{S}$  of finite non-abelian simple groups, there exists a 63-generated just-infinite group  $G$  whose upper composition factors (composition factors of the finite quotients) are precisely the members of  $\mathcal{S}$ . In addition, there exists a 3-generated just-infinite group  $\bar{G}$  whose non-cyclic upper composition factors are precisely the members of  $\mathcal{S}$ .*

## Spinal Groups

In this chapter we introduce the class of spinal groups of tree automorphisms. This class is rich in examples of finitely generated branch groups with various exceptional properties, constructed by the second author in [Gri80, Gri84, Gri85a], Narain Gupta and Said Sidki in [GS83a, GS83b], Alexander Rozhkov in [Roz86], Jacek Fabrykowski and Narain Gupta in [FG85, FG91], and more recently the first and the third author in [BŠ01, Bar00a, Bar00b, Šun00] and Dan Segal in [Seg00]. We will discuss some of these examples in the sequel, after we give a definition that covers all of them. Many of these examples were already presented in Section 1.6.

All examples of finitely generated branch groups that we mentioned by now are spinal groups, except for the examples of Peter Neumann from [Neu86] (see Section 1.6). It is not known at present if the groups of Peter Neumann are conjugate to spinal groups (Question 4).

### 2.1. Construction, basic tools and properties

**2.1.1. Definition of spinal groups.** Let  $\omega$  be a triple consisting of a group of rooted automorphisms  $A_\omega$ , a group  $B$  and a doubly indexed family  $\bar{\omega}$  of homomorphisms:

$$\omega_{ij} : B \rightarrow \text{Sym}(Y_{i+1}), \quad i \in \mathbb{N}, j \in \{1, \dots, m_i - 1\}.$$

Such a triple is called a *defining triple*. Each  $b \in B$  defines a directed automorphism  $b_\omega$  whose portrait is depicted in the diagram in Figure 2.1.

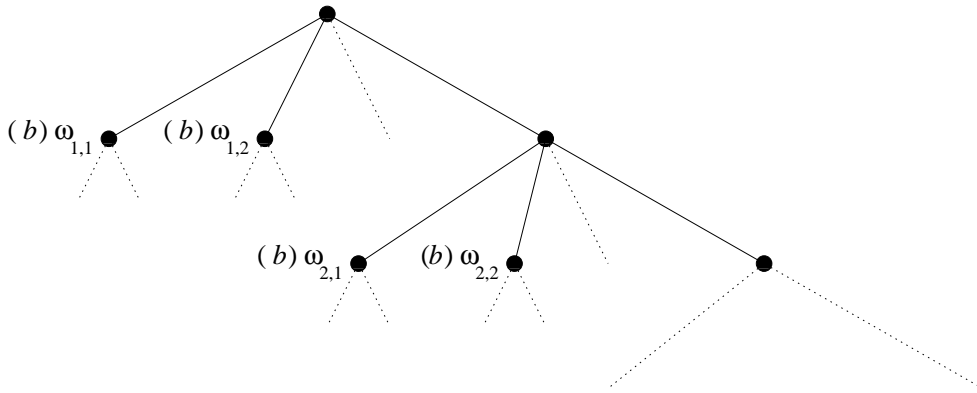


FIGURE 2.1. The directed automorphism  $b_\omega$

Therefore,  $B_\omega = \{b_\omega | b \in B\}$  is a set of directed automorphisms. We can think of  $B$  as acting on the tree  $\mathcal{T}$  by automorphisms. We define now the group  $G_\omega$ , where  $\omega$  is a defining triple, as the group of tree automorphisms generated by  $A_\omega$  and  $B_\omega$ , namely  $G_\omega = \langle A_\omega \cup B_\omega \rangle$ . We call  $A_\omega$  the *rooted part*, or the *root group*, and  $B$  the *directed part* of  $G_\omega$ . The family  $\bar{\omega}$  is sometimes referred to as the *defining family of homomorphisms*.

Let us define the *shifted triple*  $\sigma^r \omega$ , for  $r \in \mathbb{N}_+$ . The triple  $\sigma^r \omega$  consists of the group  $A_{\sigma^r \omega}$  of rooted automorphisms of  $\mathcal{T}_r$  defined by

$$A_{\sigma^r \omega} = \left\langle \bigcup_{j=1}^{m_r-1} (B)\omega_{rj} \right\rangle,$$

the same group  $B$  as in  $\omega$ , and the shifted family  $\sigma^r \bar{\omega}$  of homomorphisms

$$\omega_{i+r,j} : B \rightarrow \text{Sym}(Y_{i+r+1}), \quad i \in \mathbb{N}, \quad j \in \{1, \dots, m_{i+r} - 1\}.$$

With the natural agreement that  $\sigma^0 \omega = \omega$ , we see that  $\sigma^r \omega$  defines a group  $G_{\sigma^r \omega}$  of tree automorphisms of  $\mathcal{T}_r$  for each  $r \in \mathbb{N}$ . Note that the diagram in Figure 2.2 describes the action of  $b_{\sigma \omega}$  on the shifted tree  $\mathcal{T}^{(\sigma \bar{Y})} = \mathcal{T}_1$ , and that  $b_{\sigma \omega}$  is just the section of  $b_\omega$  at  $m_1$ .

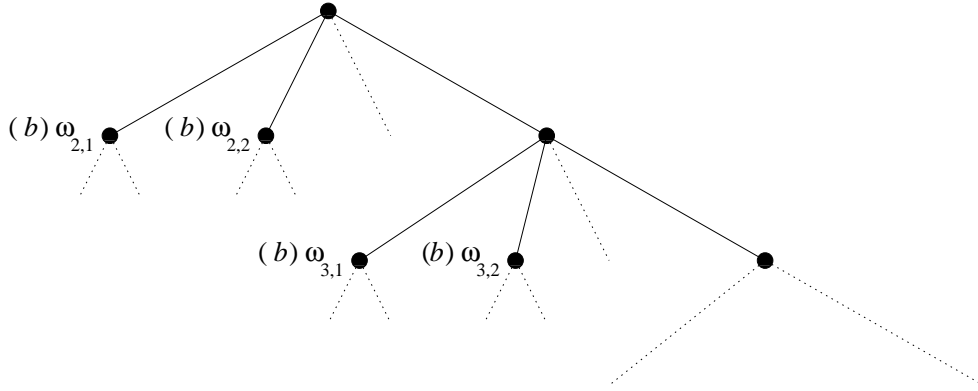


FIGURE 2.2. The directed automorphism  $b_{\sigma \omega}$

We are ready now for the

DEFINITION 2.1. Let  $\omega = (A_\omega, B, \bar{\omega})$  be a defining triple. The group

$$G_\omega = \langle A_\omega \cup B_\omega \rangle$$

is called the spinal group defined by  $\omega$  if the following two conditions are satisfied:

- (1) spherical transitivity condition:  $A_{\sigma^r \omega}$  acts transitively on the corresponding alphabet  $Y_{r+1}$ , for all  $r \in \mathbb{N}$ .
- (2) strong kernel intersection condition:

$$\bigcap_{i \geq r} \bigcap_{j=1}^{m_i-1} \text{Ker}(\omega_{ij}) = 1, \quad \text{for all } r.$$

The spherical transitivity condition guarantees that  $G_\omega$  acts spherically transitively on  $\mathcal{T}$ , as well as that the same is true for the actions of the shifted groups  $G_{\sigma^r \omega}$  on the corresponding shifted trees. Similarly, the strong intersection condition guarantees that the action of  $B$  on  $\mathcal{T}$  is faithful, and that the same is true for the actions of the shifted groups  $G_{\sigma^r \omega}$  on the corresponding shifted trees.

The class of defining triples  $\omega$  that satisfy the above two conditions will be denoted by  $\Omega$ . The above considerations indicate that  $\Omega$  is closed under the shift, i.e., if  $\omega$  is in  $\Omega$  then so is any shift  $\sigma^r \omega$ . This fact is crucial in many arguments involving spinal groups, but we will rarely mention it explicitly.

In the following subsections we introduce the tools and constructions we use in the investigation of the spinal groups along with some basic properties that follow quickly from the given considerations.

**2.1.2. Simple reductions.** The abstract group  $B$  is canonically isomorphic to the group of tree automorphisms  $B_{\sigma^r\omega}$ , for any  $r$ , so that we will not make too much difference between them and will frequently omit the index in the notation. Letters like  $b, b_1, b', \dots$  are exclusively reserved for the nontrivial elements in  $B$  and are called  $B$ -letters. Letters like  $a, a_1, a', \dots$  are exclusively reserved for the nontrivial elements in  $A_{\sigma^r\omega}$ ,  $r \in \mathbb{N}$ , and are called  $A$ -letters. Note that the groups  $A_{\sigma^r\omega}$ ,  $r \in \mathbb{N}$  are not necessarily isomorphic but we omit the index sometimes anyway.

The set  $S_\omega = (A_\omega \cup B_\omega) \setminus \{1\}$  is the *canonical generating set* of  $G_\omega$ . The generators in  $A_\omega \setminus \{1\}$  are called  $A$ -generators and the generators in  $B_\omega \setminus \{1\}$  are called  $B$ -generators. Note that  $S_\omega$  does not contain the identity and generates  $G_\omega$  as a monoid, since it is closed under inversion.

We will not be very careful to distinguish the group elements in  $G_\omega$  from the words in the canonical generators that represent them. At first, it is a sacrifice to the clarity of presentation, but our opinion is that in the long run we only gain by avoiding useless distinctions.

Define the *length of an element  $g$  of  $G_\omega$*  to be the shortest length of a word over  $S$  that represents  $g$ , and denote this length by  $|g|$ . There may be more than one word of shortest length representing the same element.

Clearly,  $G_\omega$  is a homomorphic image of the free product  $A_\omega * B_\omega$ . Therefore, every  $g$  in  $G_\omega$  can be written in the form

$$[a_0]b_1a_1b_2a_2 \dots a_{k-1}b_k[a_k] \quad (12)$$

where the appearances of  $a_0$  and  $a_k$  are optional. Relations of the following 4 types:

$$a_1a_2 \rightarrow 1, \quad a_3a_4 \rightarrow a_5, \quad b_1b_2 \rightarrow 1, \quad b_3b_4 \rightarrow b_5,$$

that follow from the corresponding relations in  $A$  and  $B$  are called *simple relations*. A *simple reduction* is any single application of a simple relation from left to right (indicated above by the arrows). Any word of the form (12) is called a *reduced word* and any word can be rewritten in unique reduced form using simple reductions. Among all the words that represent an element  $g$ , the ones of shortest length are necessarily reduced, but those that are reduced do not necessarily have the shortest length.

Note that the system of reductions described above is complete, i.e., it always terminates with a word in reduced form and the order in which we apply the reductions does not change the final reduced word obtained by the reduction.

In some considerations one needs to perform cyclic reductions. A reduced word  $F$  over  $S$  of the form  $F = s_1us_2$  for some  $s_1, s_2 \in S$ ,  $u \in S^*$ , is *cyclically reduced* if the word  $us_2s_1$  obtained from  $F$  by a cyclic shift is also reduced. If  $F = s_1us_2$  is reduced but not cyclically reduced, the word obtained from  $us_2s_1$  after one application of a simple reduction is said to be obtained from  $F$  by one *cyclic reduction*. On the group level the simple cyclic reduction described above corresponds to conjugation by  $s_1$ .

**2.1.3. Level stabilizers.** In order to simplify the notation we denote the level stabilizer  $\text{St}_{G_\omega}(\mathcal{L}_n)$  by  $\text{St}_\omega(\mathcal{L}_n)$ . In the sequel we often simplify notation by replacing  $G_\omega$  as a superscript or subscript just by  $\omega$ , and we do this without warning. Since each element in  $B$  fixes the first level, a word  $u$  over  $S$  represents an element in  $\text{St}_\omega(\mathcal{L}_1)$  if and only if the word in  $A$ -letters obtained after deleting all  $B$ -letters in  $u$  represents the identity element.

Further,  $\text{St}_\omega(\mathcal{L}_1)$  is the normal closure of  $B_\omega$  in  $G_\omega$ ,  $G_\omega = \text{St}_\omega(\mathcal{L}_1) \rtimes A_\omega$ , and  $\text{St}_\omega(\mathcal{L}_1)$  is generated by the elements  $b_\omega^g = g^{-1}b_\omega g$ , for  $b_\omega$  in  $B_\omega$  and  $g$  in  $A_\omega$ .

Clearly,  $(b_\omega)\psi = ((b)\omega_{1,1}, (b)\omega_{1,2}, \dots, (b)\omega_{1,m_1-1}, b_{\sigma\omega})$ . For any  $a$  in  $A_\omega$ ,  $(b_\omega^a)\psi$  has the same components as  $(b_\omega)\psi$  does, but in different positions depending on  $a$ . More precisely:

LEMMA 2.2. *For any  $h$  in  $\text{St}_\omega(\mathcal{L}_1)$ ,  $g$  in  $A_\omega$ ,  $b$  in  $B$  and  $i \in \{1, \dots, m_1\}$ , we have*

- (1)  $(h^g)\varphi_i = (h)\varphi_{ig^{-1}}$ .
- (2) *The coordinates of  $(b^g)\psi$  are:  $(b)\omega_{1,j}$  at the coordinate  $j^g$ , for  $j \in \{1, \dots, m_1 - 1\}$ , and  $b$  at  $m_1^g$ .*

For example, if  $a$  is the cyclic permutation  $a = ((m_1, \dots, 2, 1))$ , the images of  $b_\omega^{a^j}$  under various  $\varphi_i^\omega$  are given in Table 1.

	$\varphi_1$	$\varphi_2$	$\varphi_3$	$\cdots$	$\varphi_{m_1-1}$	$\varphi_{m_1}$
$b_\omega$	$(b)\omega_{1,1}$	$(b)\omega_{1,2}$	$(b)\omega_{1,3}$	$\cdots$	$(b)\omega_{1,m_1-1}$	$b_{\sigma\omega}$
$b_\omega^a$	$(b)\omega_{1,2}$	$(b)\omega_{1,3}$	$(b)\omega_{1,4}$	$\cdots$	$b_{\sigma\omega}$	$(b)\omega_{1,1}$
$b_\omega^{a^2}$	$(b)\omega_{1,3}$	$(b)\omega_{1,4}$	$(b)\omega_{1,5}$	$\cdots$	$(b)\omega_{1,1}$	$(b)\omega_{1,2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$b_\omega^{a^{m_1-2}}$	$(b)\omega_{1,m_1-1}$	$b_{\sigma\omega}$	$(b)\omega_{1,1}$	$\cdots$	$(b)\omega_{1,m_1-3}$	$(b)\omega_{1,m_1-2}$
$b_\omega^{a^{m_1-1}}$	$b_{\sigma\omega}$	$(b)\omega_{1,1}$	$(b)\omega_{1,2}$	$\cdots$	$(b)\omega_{1,m_1-2}$	$(b)\omega_{1,m_1-1}$

TABLE 1. The maps  $\varphi_i$  associated with the permutation  $a = ((m_1, \dots, 2, 1))$

Since the root group  $A_\omega$  acts transitively on  $Y_1$  we get all the elements from  $A_{\sigma\omega}$  and all the elements from  $B_{\sigma\omega}$  in the image of  $\text{St}_\omega(\mathcal{L}_1)$  under  $\varphi_i$ , for any  $i$ . Therefore, the shifted group  $G_{\sigma\omega}$  is precisely the image of the first level stabilizer  $\text{St}_\omega(\mathcal{L}_1)$  under any of the section homomorphisms  $\varphi_i$ ,  $i \in \{1, \dots, m_1\}$ .

Let  $g$  be an element in  $G_\omega$ . There are unique elements  $h$  in  $\text{St}_\omega(\mathcal{L}_n)$  and  $a$  in  $A_\omega$  such that  $g = ha$ . Clearly,  $a$  is the vertex permutation of  $g$  at the root, i.e.,  $a = (\emptyset)g$ , and  $(h)\psi = (g_1, \dots, g_{m_1})$  where  $g_i$  is the section automorphisms of  $g$  at the vertex  $y_i$ ,  $i \in \{1, \dots, m_1\}$ . Therefore, the section  $G_{y_i}$  is contained in the image of  $\text{St}_\omega(\mathcal{L}_1)$  under  $\varphi_i$ , and we already established that this image is the shifted group  $G_{\sigma\omega}$ . Therefore, for all  $i = 1, \dots, m_1$ ,

$$G_{y_i} = U_{y_i} = G_{\sigma\omega},$$

i.e., the sections on the first level, upper companion group and the shifted  $G_{\sigma\omega}$  group coincide for spinal groups.

Further, the  $n$ -th upper companion group of  $G_\omega$  is the shifted group  $G_{\sigma^n\omega}$  and the homomorphism

$$\psi_n : \text{St}_\omega(\mathcal{L}_n) \rightarrow \prod_{i=1}^{m_1 m_2 \dots m_n} G_{\sigma^n\omega}$$

given by

$$(g)\psi_n = ((g)\varphi_{1\dots 1}, \dots, (g)\varphi_{m_1\dots m_n}) = (g_{1\dots 1}, \dots, g_{m_1\dots m_n})$$

is an embedding.

We end this subsection with an easy proposition:

PROPOSITION 2.3. *Every spinal group  $G_\omega$  is infinite.*

PROOF. The proper subgroup  $\text{St}_\omega(\mathcal{L}_1)$  of  $G_\omega$  maps under  $\varphi_1$  onto  $G_{\sigma\omega}$ , which is also spinal.  $\square$

The above proof is very simple, so let us use the opportunity here to point out an important feature that is shared by many of the proofs involving spinal groups. The proof does not work with a fixed sequence  $\omega$ , but rather involves arguments and facts about all the shifts of  $\omega$  (note the importance of the fact that  $\Omega$  is closed for shifts). In other words, the group  $G_\omega$  is always considered together with all of its companions, and the only way we extract some information about some spinal group  $G_\omega$  is through such a synergic cooperation between the group and its companions. Unavoidably, we also make observations about the companion groups.

**2.1.4. Portraits.** The following construction corresponds to the constructions exhibited in [Gri84, Gri85a, Bar98], but it is more general and allows different modifications and applications.

A *profile* is a sequence  $\overline{P} = (P_t)_{t \in \mathbb{N}}$  of sets of automorphisms, called *profile sets*, where  $P_t \subseteq \text{Aut}(\mathcal{T}_t)$ . We define now a *portrait* of an element  $f$  of  $G_\omega$  with respect to the profile  $\mathcal{P}$ . The portrait is defined inductively as follows: if  $f$  belongs to the profile set  $P_0$  then the portrait of  $f$  is the tree that consists of one vertex decorated by  $f$ ; otherwise the portrait of  $f$  is the tree that is decorated by  $a = (\emptyset)f$  at the root and has the portraits of the sections  $f_1, \dots, f_{m_1}$ , with respect to the shifted sequence  $\sigma\overline{P}$  of profile sets, hanging on the  $m_1$  labelled vertices below the root. Therefore, the portrait of  $f$  is a subtree (finite or infinite) of the tree on which  $f$  acts, its interior vertices are decorated by the corresponding vertex permutations of  $f$ , and its leaves are decorated by elements in the chosen profile sets and are equal to the corresponding sections.

For example, if all the profile sets are empty, we obtain the portrait representation of  $f$  through its vertex permutations that we already defined (see Figure 1.3). We sometimes refer to this portrait as the *full portrait* of  $f$ . If  $P_t$  is empty for  $t = 0, \dots, r-1$  and equal to  $G_{\sigma^r \omega}$  for  $t = r$ , then the portrait of  $f$  is the subtree of  $\mathcal{T}$  that consists of the first  $r$  levels, the vertices at the levels 0 through  $r-1$  are decorated by their vertex permutations and the vertices at level  $r$  are decorated by their corresponding sections. Such a portrait is called the *depth- $r$  decomposition* of  $f$ . The depth-1, depth-2 and depth-3 decompositions of the element  $g = abacadacabadac$  in  $\mathfrak{G}$  are given in the Figure 2.3, Figure 2.4 and Figure 2.5. The decompositions can be easily calculated from Table 2 that describes

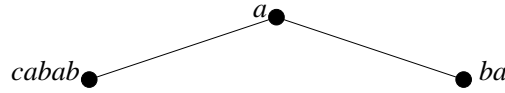


FIGURE 2.3. Depth-1 decomposition of  $g$

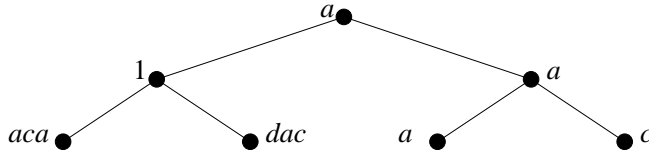


FIGURE 2.4. Depth-2 decomposition of  $g$

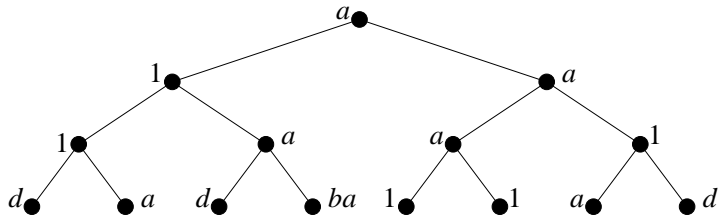


FIGURE 2.5. Depth-3 decomposition of  $g$

$\varphi_1$  and  $\varphi_2$  (recall the definition of  $\mathfrak{G}$  from Subsection 1.6.1. The calculations follow, in which we

	$\varphi_1$	$\varphi_2$
$b$	$a$	$c$
$c$	$a$	$d$
$d$	$1$	$b$
$b^a$	$c$	$a$
$c^a$	$d$	$a$
$d^a$	$b$	$1$

TABLE 2. The table for  $\varphi_1$  and  $\varphi_2$  in  $\mathfrak{G}$ 

identified the elements and their images under the decomposition map  $\psi$ :

$$\begin{aligned}
g &= abacadacabadac = b^a cd^a cb^a dc^a a = (cabac1d, ad1daba)a = (cabab, ba)a \\
cabab &= cb^a b = (aca, dac) \\
ba &= (a, c)a \\
aca &= c^a = (d, a) \\
dac &= dc^a a = (d, ba)a \\
a &= (1, 1)a.
\end{aligned}$$

Let us note that, in general, the leaves of a portrait do not have to be all at the same level, and it is possible that some paths from the root end with a leaf and some are infinite. The various types of portraits carry important information about the elements they represent, which is not surprising, since different elements have different portraits.

Consider the construction of a portrait more closely. Let  $f$  be represented by a reduced word

$$F = [a_0]b_1 a_1 b_2 a_2 \dots a_{k-1} b_k [a_k]$$

over  $S$ . We rewrite  $f = F$  in the form

$$\begin{aligned}
f &= b_1^{[a_0^{-1}]} b_2^{([a_0]a_1)^{-1}} \dots b_k^{([a_0]a_1 \dots a_{k-1})^{-1}} [a_0]a_1 \dots a_{k-1} [a_k] \\
&= b_1^{g_1} b_2^{g_2} \dots b_k^{g_k} g,
\end{aligned} \tag{13}$$

where  $g_i = ([a_0]a_1 \dots a_{i-1})^{-1}$  and  $[a_0]a_1 \dots a_{k-1} [a_k] = g$  are in  $A_\omega$ . Next, using the homomorphisms  $\omega_{1,j}$ ,  $j \in \{1, \dots, m_1 - 1\}$ , and a table similar to Table 1 (but for all possible  $a$ ) we compute the (not necessarily reduced) words  $\overline{F_1}, \dots, \overline{F_{m_1}}$  representing the first level sections  $f_1, \dots, f_{m_1}$ , respectively. Then we reduce these  $m_1$  words using simple reductions and obtain the reduced words  $F_1, \dots, F_{m_1}$ . Note that, on the word level, the order in which we perform the reductions is unimportant since the system of simple reductions is complete.

Let us consider any of the words  $\overline{F_1}, \dots, \overline{F_{m_1}}$ , say  $\overline{F_j}$ , and study its possible content. Each factor  $b^g$  from (13) contributes to the word  $\overline{F_j}$  either

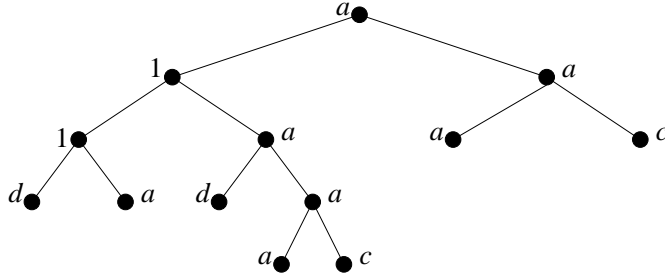
- one appearance of the  $B$ -letter  $b$ , in case  $j = m_1^g$  or
- one appearance of the  $A$ -letter  $(b)\omega_{1,j^{g^{-1}}}$ , in case  $j \neq m_1^g$  and  $b \notin \text{Ker}(\omega_{1,j^{g^{-1}}})$  or
- the empty word, in case  $j \neq m_1^g$  and  $b \in \text{Ker}(\omega_{1,j^{g^{-1}}})$ .

Therefore, the length of any of the reduced words  $F_1, \dots, F_{m_1}$  does not exceed  $k$ , i.e does not exceed  $(n+1)/2$  where  $n$  is the length of  $F$ . As a consequence we obtain the following:

LEMMA 2.4. For any  $f \in G_\omega$ ,

$$|f_i| \leq (|f| + 1)/2.$$

The *canonical profile* is the profile  $\overline{P} = (S_{\sigma^i \omega} \cup \{1\})_{i \in \mathbb{N}}$ , whose profile set, on each level, consists of the canonical generators together with the identity. For example the canonical portrait of  $g = abacadacabadac$  is given in Figure 2.6.

FIGURE 2.6. The canonical portrait of  $g$ 

COROLLARY 2.5. *The depth of the canonical portrait of a word  $w$  over  $S_\omega$  is no larger than  $\lceil \log_2(|w|) \rceil + 1$ .*

## 2.2. G groups

The G groups are natural generalizations of the first Grigorchuk group  $\mathfrak{G}$  from [Gri80] and of the groups  $G_\omega$  from [Gri84, Gri85a]. The idea of these examples is based on the strong covering property (see Definition 2.6). The section presents uncountable family of spinal groups of G type. All of them are finitely generated, just-infinite, residually finite, centerless, amenable, not elementary amenable, recursively but not finitely presented torsion groups of intermediate growth (a definition of intermediate growth will come later in Chapter 10). For proofs see [Gri84, Gri85a], as well as the later chapters in this text.

These examples were generalized in [BŠ01, Šun00]. The generalized examples share many of the properties of Grigorchuk groups mentioned above.

In all Grigorchuk examples the homomorphisms  $\omega_{i,j}$  are trivial for all  $j \neq 1$ , and we denote  $\omega_i = \omega_{i,1}$ , for all  $i \geq 1$ . Therefore, the only homomorphisms in the defining family of homomorphisms  $\bar{\omega}$  that we need to specify are the homomorphisms in the sequence  $\omega_1\omega_2\omega_3\dots$ , which we call the *defining sequence of the triple  $\omega$* , and we avoid complications in our notation by simply writing  $\bar{\omega} = \omega_1\omega_2\omega_3\dots$ .

**2.2.1. Grigorchuk  $p$ -groups.** The Grigorchuk 2-groups, which are a natural generalization of  $\mathfrak{G}$ , were introduced in [Gri84]. They act on the rooted binary tree  $\mathcal{T}^{(2)}$ . The rooted group  $A = \{1, a\}$  and the group  $B = \{1, b, c, d\} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  are still the same as for  $\mathfrak{G}$ . There are three non-trivial homomorphisms from  $B$  to  $\text{Sym}(2) = \{1, a\}$ , and we denote them as follows:

$$\begin{aligned} 0 &= \begin{pmatrix} 1 & b & c & d \\ 1 & a & a & 1 \end{pmatrix}, \\ 1 &= \begin{pmatrix} 1 & b & c & d \\ 1 & a & 1 & a \end{pmatrix}, \\ 2 &= \begin{pmatrix} 1 & b & c & d \\ 1 & 1 & a & a \end{pmatrix}. \end{aligned}$$

Note that the vectors representing the elements of  $P$  also correspond to the lines in the projective 2-dimensional space over the finite Galois field  $\mathbb{F}_p$ . A Grigorchuk 2-group is defined by any infinite sequence of homomorphisms  $\bar{\omega} = \omega_1\omega_2\omega_3\dots$ , where the homomorphisms  $\omega_1, \omega_2, \omega_3, \dots$  come from the set of homomorphisms  $H = \{0, 1, 2\}$  and each homomorphism from  $H$  occurs infinitely many times in  $\bar{\omega}$ . In this setting, the first Grigorchuk group  $\mathfrak{G}$  is defined by the periodic sequence of homomorphisms  $\bar{\omega} = 012012\dots$ .

Grigorchuk  $p$ -groups, introduced in 1985 in [Gri85a], act on the rooted  $p$ -ary tree  $\mathcal{T}^{(p)}$ , for  $p$  a prime. The rooted group  $A = \langle a \rangle$  is the cyclic group of order  $p$  generated by the cyclic permutation

$a = ((1, 2, \dots, p))$  and the group  $B$  is the group  $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ . Denote by  $\begin{bmatrix} u \\ v \end{bmatrix}$  the homomorphism from  $B$  to  $\text{Sym}(p)$  sending  $(x, y)$  to  $a^{ux+vy}$  and let  $P$  be the set of homomorphisms

$$P = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \dots, \begin{bmatrix} p-1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

A Grigorchuk  $p$ -group is defined by any infinite sequence of homomorphisms  $\bar{\omega} = \omega_1\omega_2\omega_3\dots$ , where the homomorphisms  $\omega_1, \omega_2, \omega_3, \dots$  come from the set  $P$  and each homomorphism from  $P$  occurs infinitely many times in  $\bar{\omega}$ .

Only those Grigorchuk  $p$ -groups that are defined by a recursive sequence  $\bar{\omega}$  have solvable word problem (see [Gri84]). The conjugacy problem is solvable under the same condition (see Chapter 3).

**2.2.2. G groups.** The following examples generalize the class of Grigorchuk groups from the previous subsection. A subclass of the class of groups we are about to define is the subject of [BŠ01] and the general case is considered in [Sun00].

We are now going to specify the triples  $\omega$  that define the groups in the class of G groups. As before, the homomorphisms  $\omega_{i,j}$  are trivial for all  $j \neq 1$ , and we denote  $\omega_i = \omega_{i,1}$ , for all  $i \geq 1$ . For  $b \in B$ , the corresponding directed automorphism  $b_\omega$  is given by the diagram in Figure 2.7

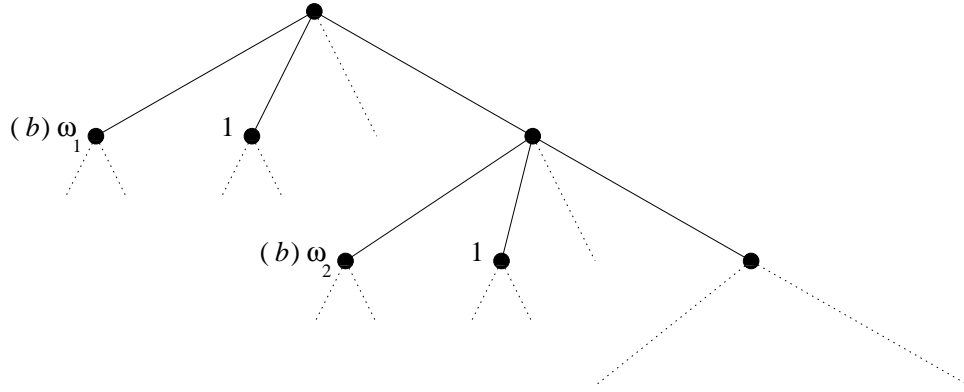


FIGURE 2.7. The directed automorphism  $b_\omega$  in G groups

For all positive  $r$ , we have

$$A_{\sigma^r \omega} = (B)\omega_r$$

and we denote

$$K_i = \text{Ker}(\omega_i).$$

Therefore, for positive  $r$ , each of the rooted groups  $A_{\sigma^r \omega}$  is a homomorphic image of  $B$ .

**DEFINITION 2.6.** Let  $G_\omega$  be a spinal group defined by the triple  $\omega = (A_\omega, B, \bar{\omega})$  with  $\omega_{i,j} = 1$  whenever  $j \neq 1$ . The group  $G_\omega$  is a G group if the following strong covering condition is satisfied:

$$\bigcup_{i=r}^{\infty} K_i = B, \quad \text{for all } r.$$

The strong kernel intersection condition is equivalent to a statement that  $(b)\omega_i$  is nontrivial for infinitely many indices, while the strong covering condition is equivalent to a statement that  $(b)\omega_i$  is trivial for infinitely many indices. The class of triples  $\omega$  that satisfy the above conditions and thus define G groups will be denoted by  $\widehat{\Omega}$ . Clearly,  $\widehat{\Omega}$  is closed for shifting under  $\sigma$ , and the use of this fact is essential but hardly ever emphasized.

Let us show an easy way to build examples of  $\mathbf{G}$  groups. Start with a group  $B$  that has a covering by a family of proper normal subgroups  $\{N_\alpha \mid \alpha \in I\}$  of finite index and with trivial intersection, i.e.,

$$\bigcup_{\alpha \in I} N_\alpha = B \text{ and } \bigcap_{\alpha \in I} N_\alpha = 1.$$

Choose a sequence of normal subgroups  $N_1 N_2 N_2 \dots$ , where  $N_i$  come from the above family of normal subgroups, such that the strong intersection and covering conditions hold, i.e.,

$$\bigcup_{i=r}^{\infty} N_i = B, \text{ and } \bigcap_{i=r}^{\infty} N_i = 1, \quad \text{for all } r.$$

For each  $i$ , let the factor group  $B/N_i$  act transitively and faithfully as a permutation group on some alphabet  $Y_{i+1}$ , and let  $\omega_i$  be the natural homomorphism from  $B$  to the permutation group  $B/N_i$  followed by the embedding of  $B/N_i$  in the symmetric group  $\text{Sym}(Y_{i+1})$ . Choose a group  $A_\omega$  and an alphabet  $Y_1$  on which  $A_\omega$  acts transitively and faithfully. The triple  $\omega$  that consist of  $A_\omega$ ,  $B$  and the sequence of homomorphisms  $\bar{\omega} = \omega_1 \omega_2 \omega_3 \dots$  defines a spinal group  $G_\omega$  which is a  $\mathbf{G}$  group acting on the tree  $\mathcal{T}(\bar{Y})$ . Clearly, the strong intersection and covering conditions are satisfied since  $K_i = N_i$ , and the spherical transitivity condition is satisfied since  $A_{\sigma^i \omega} = B/N_i$ .

It is of special interest to consider the case when  $B$  is finite. The family of all groups  $B$  that have a covering by a finite family of proper normal subgroups can be characterized, according to a theorem of Marc Brodie Robert Chamberlain and Luise-Charlotte Kappe from [BCK88], as the family of those finite groups that have  $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$  as a factor group, for some prime  $p$  (actually the theorem holds in general and characterizes both finite and infinite groups  $B$  that have coverings with a finite family of proper normal subgroups). In addition, to make sure that there exist a covering with trivial intersection, we must add the condition that  $B$  is not subdirectly irreducible, i.e., the intersection of all non-trivial normal subgroups of  $B$  must be trivial.

However, for some reason (aesthetics or something deeper) we might want to restrict our attention only to  $\mathbf{G}$  groups that act on regular trees. For example, such a case appears in [BŠ01] where the added restriction is that all the factors  $B/N_i = A_{\sigma^i \omega}$  are isomorphic to a fixed group  $A$ , and they all act in the same way on the same alphabet  $\{1, 2, \dots, m\}$ . The  $\mathbf{G}$  groups defined in this way act on the regular tree  $\mathcal{T}^{(m)}$ . It is fairly easy to construct examples of  $\mathbf{G}$  groups with this added restrictions in case of an abelian group  $B$ . Any group that is a direct product of proper powers of cyclic groups can be used as  $B$  in the above construction, i.e., any group of the form

$$B = (\mathbb{Z}/n_1\mathbb{Z})^{k_1} \times (\mathbb{Z}/n_2\mathbb{Z})^{k_2} \times \dots \times (\mathbb{Z}/n_s\mathbb{Z})^{k_s}$$

with  $k_1, \dots, k_s \geq 2$  has a family of normal subgroups of the required type such that all the factors are isomorphic to

$$A = (\mathbb{Z}/n_1\mathbb{Z})^{k_1-1} \times (\mathbb{Z}/n_2\mathbb{Z})^{k_2-1} \times \dots \times (\mathbb{Z}/n_s\mathbb{Z})^{k_s-1}.$$

In particular, we see that any finite abelian group can be used in the role of the rooted group  $A$ .

Characterizing the family of finite groups  $B$  that have a covering with a family of proper normal subgroups with trivial intersection and such that all factors are isomorphic is an interesting problem. The smallest known non-abelian example so far was communicated to the authors by Derek Holt through the Group Pub Forum (see <http://www.bath.ac.uk/~masgcs/gpf.html>)

Let  $B = \langle b_1, b_2, b_3, b_4, b_5, b_6, x_{12}, x_{34} \rangle$  where  $b_1, b_2, b_3, b_4, b_5, b_6$  all have order 3 and commute with each other,  $x_{12}$  and  $x_{34}$  have order 2, commute and

$$b_i^{x_{jk}} = \begin{cases} b_i & \text{if } i \in \{j, k\}, \\ b_i^{-1} & \text{otherwise.} \end{cases}$$

In other words,  $B$  is the semidirect product  $(\mathbb{Z}/3\mathbb{Z})^6 \rtimes (\mathbb{Z}/2\mathbb{Z})^2$  where  $(\mathbb{Z}/2\mathbb{Z})^2 = \langle x_{12}, x_{34} \rangle$ ,  $x_{12}$  fixes the first two and acts by inversion on the last four coordinates of  $(\mathbb{Z}/3\mathbb{Z})^6$ ,  $x_{34}$  fixes the middle two and acts by inversion on the other four coordinates and, consequently,  $x_{56} = x_{12}x_{34}$  fixes the last two and inverts the first four coordinates.

The following 12 subgroups are normal in  $B$ , their intersection is trivial, their union is  $B$ , and each factor is isomorphic to the symmetric group  $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \text{Sym}(3)$ :

$$\begin{aligned} &\langle b_1, b_3, b_4, b_5, b_6, x_{12} \rangle, & \langle b_1, b_2, b_3, b_5, b_6, x_{34} \rangle, & \langle b_1, b_2, b_3, b_4, b_5, x_{56} \rangle, \\ &\langle b_2, b_3, b_4, b_5, b_6, x_{12} \rangle, & \langle b_1, b_2, b_4, b_5, b_6, x_{34} \rangle, & \langle b_1, b_2, b_3, b_4, b_6, x_{56} \rangle, \\ &\langle b_1 b_2, b_3, b_4, b_5, b_6, x_{12} \rangle, & \langle b_1, b_2, b_3 b_4, b_5, b_6, x_{34} \rangle, & \langle b_1, b_2, b_3, b_4, b_5 b_6, x_{56} \rangle, \\ &\langle b_1 b_2^2, b_3, b_4, b_5, b_6, x_{12} \rangle, & \langle b_1, b_2, b_3 b_4^2, b_5, b_6, x_{34} \rangle, & \langle b_1, b_2, b_3, b_4, b_5 b_6^2, x_{56} \rangle. \end{aligned}$$

The smallest example of a non-abelian group  $B$  that has a covering by normal subgroups with trivial intersection is the dihedral group  $D_6$  on 12 elements. Indeed,  $D_6$  is covered by its 3 normal subgroups, call them  $N_1, N_2$  and  $N_3$ , of index 2. The intersection of these 3 groups is not trivial, but if we include the center  $Z(D_6)$  in the covering, we do get trivial intersection. The corresponding factors are  $\mathbb{Z}/2\mathbb{Z} = D_6/N_1 = D_6/N_2 = D_6/N_3$  and the dihedral group on six elements  $D_3 = D_6/Z(D_6)$ . The first three factors can act as the symmetric group on  $\{1, 2\}$ , and the last one can act either regularly on  $\{1, 2, 3, 4, 5, 6\}$  by the right regular representation or as the symmetric group of  $\{1, 2, 3\}$ .

We can define even more involved examples of  $\mathbf{G}$  groups. For example, we can let  $B$  itself be the first Grigorchuk group  $\mathfrak{G}$ . It has 7 subgroups of index 2, and 3 of them cover the group. Since we need trivial intersection we can use the level stabilizers to accomplish this. Therefore, we can easily define a  $\mathbf{G}$  in which the directed part itself is a  $\mathbf{G}$  group, for example  $\mathfrak{G}$ .

It is clear that in the examples when  $B$  is infinite the branching indices of the trees on which the group acts is not bounded. On the other hand, in case of a finite  $B$ , the branching indices are bounded by the order of  $B$  and they have to be divisors of the order of  $B$ , except for the first branching index  $m_1$ , which can be arbitrarily large.

### 2.3. GGS groups

The GGS groups (Grigorchuk-Gupta-Sidki groups, the terminology comes from [Bau93]) are natural generalizations of the second Grigorchuk group from [Gri80] and the Gupta-Sidki examples from [GS83a]. They act on a regular tree  $\mathcal{T}^{(m)}$ , where in most of the examples we present  $m$  is prime or a prime power, and they have a special *stabilization* property, namely if an element  $g \in G$  is not in the level stabilizer  $\text{St}_G(\mathcal{L}_1)$  then the first level sections of the power  $g^m$  are either in the stabilizer or are closer to be in the stabilizer than the original element  $g$ .

In all examples that we give here, except the general case considered by Bartholdi in [Bar00b], the rooted part  $A = \langle a \rangle$  is the cyclic group of order  $m$  generated by the permutation  $a = ((1, 2, \dots, m))$ . The group  $B = \langle b \rangle$  is also a cyclic group of order  $m$ . All homomorphisms  $\omega_{i,j}$  map the elements from  $B$  to powers of  $a$  and  $\omega_{i,j} = \omega_{i',j}$ , for all indices. Therefore, in order to define a spinal group we only need to specify a vector  $E = (\varepsilon_1, \dots, \varepsilon_{m-1})$ , where  $\varepsilon_j$  are integers, and let  $(b)\omega_{i,j} = a^{\varepsilon_j}$ , for all indices. The group defined by the vector  $E$  is denoted by  $G_E$ .

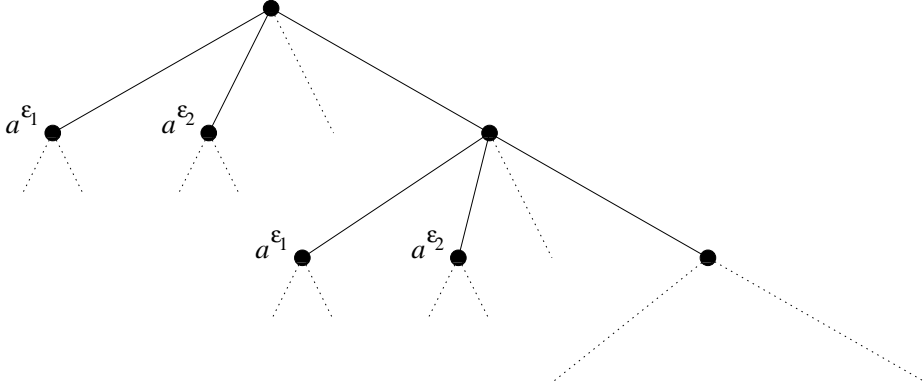
Therefore,  $G_E = \langle a, b \rangle$  where  $a$  is the rooted automorphism defined by the cyclic permutation  $a = ((1, 2, 3, \dots, m))$ , and the directed automorphism  $b = b_E$  is defined by the diagram in Figure 2.8.

In order for  $E$  to define a spinal group it is necessary and sufficient that  $\text{gcd}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m-1}, m) = 1$ . Note that in this situation  $\omega = \sigma\omega$ ,  $G_\omega = G_{\sigma\omega}$ , etc., and we have fractal groups.

**2.3.1. GGS groups with small branching index.** In the case  $m = 2$ , the only possible non-trivial vector  $E = (1)$  defines the infinite dihedral group. This is clear since this spinal group is infinite and generated by two elements of order 2.

There are only three essentially different vectors in case  $m = 3$ , and they are  $(1, 0)$ ,  $(1, 1)$  and  $(1, 2)$ . The corresponding groups were already introduced as the Fabrykowski-Gupta group  $\Gamma$ , Bartholdi-Grigorchuk group  $\overline{\Gamma}$  and Gupta-Sidki group  $\overline{\overline{\Gamma}}$ .

We have already mentioned one example of GGS group in case  $m = 4$  and it is the second Grigorchuk group. The defining vector for the second Grigorchuk group is  $E = (1, 0, 1)$ .

FIGURE 2.8. The directed automorphism  $b_E$ 

**2.3.2. Gupta-Sidki examples.** The Gupta-Sidki groups from [GS83a] act on the regular tree  $\mathcal{T}^{(p)}$  where  $p$  is an odd prime and are defined by the vector  $E = (1, -1, 0, 0, \dots, 0)$ . The  $p$ -groups introduced in [GS83b] are defined by the vector  $E = (1, -1, \dots, 1, -1)$ .

**2.3.3. More examples of GGS groups.** The defining vector  $E = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m-1})$ , where  $m = p^n$  is a prime power, defines an infinite 2-generated  $p$ -group if and only if

$$\sum_{s \in O_k(m)} \varepsilon_s \equiv 0 \pmod{p^{k+1}},$$

for  $k = 0, \dots, n-1$ , where

$$O_k(m) = \{ p^k, 2p^k, \dots, (p^{n-k} - 1)p^k \}.$$

The sufficiency in the above claim (in a more general setting) is proved by Narain Gupta and Said Sidki in [GS84] and the necessity by Vovkivsky in [Vov00]. The latter article also shows that in case the defining vector  $E$  does satisfy the condition above, the obtained  $p$ -group is just-infinite, not finitely presented branch group.

**2.3.4. General version of GGS groups.** One chapter of the Ph.D. dissertation of Laurent Bartholdi [Bar00b] is devoted to a class of groups that comes as a natural generalization of all of the previous examples of GGS groups. The groups act on the tree  $\mathcal{T}^{(m)}$ , where  $m$  is arbitrary. A defining vector  $E = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m-1})$  is a vector of permutations of the alphabet  $Y = \{1, 2, \dots, m\}$  of the tree such that the group of permutations  $A = \langle \varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m-1} \rangle$  acts transitively on  $Y$ , and the spinal group  $G_E$  is generated by the rooted automorphisms from  $A$  together with the directed automorphism  $b_E$  (simply written  $b$ ) defined by the diagram in Figure 2.9

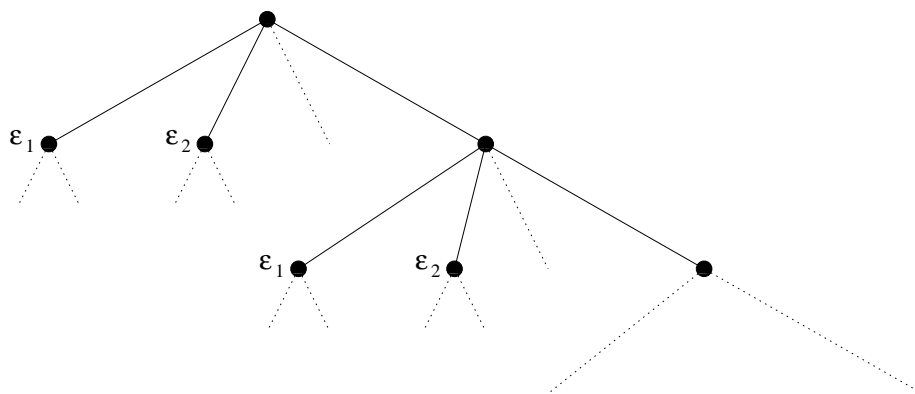
To see how these examples fit in our general scheme, note that the group  $B = \langle b \rangle$  is a cyclic group of order equal to the least common multiple of the orders of  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m-1}$ , and the homomorphisms in the triple  $\omega$  are defined by  $(b)\omega_{i,j} = \varepsilon_j$  for all indices.

The first author shows in his dissertation ([Bar00b]) that unless  $m = 2$  (in which case the group is the infinite dihedral group) the GGS groups are finitely generated, residually finite, centerless and not finitely presented groups. He also presents a sufficient and necessary condition for a GGS group to be torsion group (see Chapter 6).

**THEOREM 2.7.** *Let  $G$  be a GGS group. Then  $G$  is weakly regular branch on the non-trivial subgroup*

$$K = (\cup_{|A|}(G'))',$$

where  $\cup_{|A|}(G')$  denotes the subgroup of  $G'$  generated by the  $|A|$ -th powers.

FIGURE 2.9. The directed automorphism  $b_E$  in GGS groups

SKETCH OF A PROOF.  $G'$  contains all elements of the form  $[b, b^a]$ , which can be written through  $\psi$  as

$$(a_1, \dots, [a_i, b], a_{i+1}, \dots, [b, a_m]).$$

Write  $n = |A|$ . then  $\mathcal{U}_n(G')$  contains all elements of the form  $(1, \dots, [a_i, b]^n, 1, \dots, [b, a_m]^n)$ , and its derived subgroup contains all elements of the form  $(1, \dots, 1, [[b, a]^n, [b, a']^n])$ . We therefore have  $1 \times \dots \times 1 \times K \leq \psi(K)$ .

To show that  $K$  is not trivial, we first argue that  $G'$  is non-trivial, since it has finite index in an infinite group. Then, since the torsion has unbounded order in  $G$ ,  $\mathcal{U}_n(G')$  is non-trivial; now this last group has a subgroup mapping onto  $G$  (namely  $\text{St}_G(\mathcal{L}_N)$  for large  $N$ ), and therefore cannot be abelian.  $\square$



Part 2

## Algorithmic Aspects

## Word and Conjugacy Problem

Branch groups have good algorithmic properties. A universal algorithm solving the word problem for  $\mathbf{G}$  and  $\mathbf{GGS}$  groups and many other branch groups was mentioned in [Gri80] and described in [Gri84] (see also [Gri98, Gri99]). This algorithm is very fast and needs a minimal amount of space and we will describe it below.

However, there are branch groups with unsolvable word problem. For instance, the following claim is proved in [Gri84]

**THEOREM 3.1.** *The group  $G_\omega$  has solvable word problem if and only if  $\bar{\omega}$  is a recursive sequence.*

Note that the proof in [Gri84] is given for the considered case of 2-groups (see Section 2.2.1), but it can be easily extended to the other cases.

This result inspired the second author to use Kolmogorov complexity to study word problems in branch groups and other classes of groups in [Gri85b].

The generalized word problem (is there an algorithm which for any element  $g \in G$  and a finitely generated subgroup  $H \leq G$  given by a generating set decides if  $g$  belongs to  $H$ ) was considered only recently in [GW01] for the first Grigorchuk group  $\mathfrak{G}$  and it was shown that  $\mathfrak{G}$  has a solvable generalized word problem.

The solution of the conjugacy problem for the basic examples of branch groups came much later. First, Wilson and Zaleskii solved the conjugacy problem in  $\mathbf{GGS}$   $p$ -groups, for  $p$  an odd prime, by using the notion of Maltcev's conjugacy separability and pro- $p$  methods (see [WZ97]. Slightly later, simultaneously and independently, Leonov in [Leo98a] and Rozhkov in [Roz98] solved the conjugacy problem for  $p = 2$ . The paper of Rozhkov deals only with the first Grigorchuk group  $\mathfrak{G}$ , while Leonov considers all 2-groups  $G_\omega$  from [Gri84]. Also, the results of Leonov are stronger, since upper bounds on the length of conjugating elements are given in terms of depth.

The idea of Leonov and Rozhkov was developed in [GW00] in different directions. Still, there are some  $\mathbf{GGS}$  groups with unsettled conjugacy problem. For example, the results from [GW00] do not apply to the group  $\bar{\Gamma}$  (see 1.6.4), which is not branch but only weakly branch.

One of the improvements reached in [GW00] is that the conjugacy problem is solvable not only for the considered groups, but for their subgroups of finite index (note that there are groups with solvable conjugacy problem that contain subgroups of index 2 with unsolvable conjugacy problem, see [CM77])

Theorem C from [GW00] is the strongest result on the conjugacy problem. The corresponding algorithm in Corollary C in the same article uses the principle of Dirichlet and is quite different from the Leonov-Rozhkov algorithm.

There is also a result on the isomorphism problem for the spinal 2-groups  $G_\omega$  defined in [Gri84]. Namely, for every sequence  $\bar{\omega}$  there are only countably many sequences  $\bar{\omega}'$  with  $G_\omega \cong G_{\omega'}$ . Thus, there are uncountably many finitely generated branch groups.

### 3.1. The word problem

We describe an algorithm that solves the word problem in the case of spinal groups modulo an oracle that has effective knowledge of the defining triple  $\omega$  in a sense that we make clear below. In fact, the way in which we answer the question “does the word  $F$  over  $S$  represent the identity in

$G_\omega$ ?" is by constructing the canonical portrait of  $F$  and we need tables similar to Table 1, for all possible  $a$ , in order to do that.

Let us assume that we have an oracle  $\mathcal{O}_\omega(n)$  that has *effective knowledge* of the first  $n$  levels of the defining triple  $\omega$ , meaning that

- (1) the multiplication table of  $B$  is known,
- (2) the permutation groups  $A_\omega, \dots, A_{\sigma^n \omega}$  are known, in the sense that we can actually perform the permutations,
- (3) the homomorphisms in the first  $n$  levels in  $\bar{\omega}$  are known, i.e. for all  $b \in B$ ,  $i \in \{1, \dots, n\}$ ,  $j \in \{1, \dots, m_i - 1\}$ , it is known exactly what permutation in  $A_{\sigma^i \omega}$  is equal to  $(b)_{\omega_{i,j}}$ .

Note that if we have an oracle  $\mathcal{O}_\omega(n)$ , then we also have oracles  $\mathcal{O}_{\sigma^t \omega}(n - t)$ , for  $t \in \{0, \dots, n\}$ .

**PROPOSITION 3.2.** *Let  $G_\omega$  be a spinal group and let the oracle  $\mathcal{O}_\omega(n)$  have effective knowledge of the first  $n$  levels of  $\omega$ . The word problem in  $G_\omega$  is solvable, by using the oracle, for all words of length at most  $2^{n+1} + 1$ .*

**PROOF.** If  $F$  is not reduced we can reduce it since we have the multiplication tables for  $B$  and  $A_\omega$  (even in case  $n = 0$ ). So assume that  $F$  is reduced and let

$$F = [a_0]b_1a_1 \dots a_{k-1}b_k[a_k].$$

Note that if  $[a_0]a_1 \dots a_{k-1}[a_k] \neq 1$  in  $A_\omega$  then  $F$  does not stabilize the first level of  $\mathcal{T}$  and does not represent the identity.

We prove the claim by induction on  $n$ , for all spinal groups simultaneously.

No reduced non-empty word  $F$  of length at most 3 represents the identity. This is true because either  $F$  does not fix the first level or it represents a conjugate of  $b$ , for some  $b \in B$ . This completes the base case  $n = 0$ .

Let  $F$  be reduced,  $4 \leq |F| \leq 2^{n+1} + 1$  and  $[a_0]a_1 \dots a_{k-1}[a_k] = 1$  in  $A_\omega$ . Rewrite  $F$  as

$$F = b_1^{g_1} \dots b_k^{g_k},$$

where  $g_i = ([a_0]a_1 \dots a_{i-1})^{-1}$  and then use the oracle's knowledge of the first level of  $\omega$  to construct tables similar to Table 1 and calculate the possibly unreduced words  $\bar{F}_i$ ,  $i \in \{1, \dots, m_1\}$ , that represent the first level sections of  $F$ . Each of these words has length no greater than  $\frac{|F|+1}{2} \leq 2^n + 1$  and we may apply the inductive hypothesis to solve the word problem for each of them by using the oracle  $\mathcal{O}_{\sigma \omega}(n - 1)$ . The word  $F$  represents the identity in  $G_\omega$  if and only if each of the words  $F_1, \dots, F_{m_1}$  represents the identity in  $G_{\sigma \omega}$  which is clear since  $(F)\psi = (F_1, \dots, F_{m_1})$  and  $\psi$  is an embedding.  $\square$

**COROLLARY 3.3.** *If an oracle  $\mathcal{O}_\omega$  has effective knowledge of all the levels of  $G_\omega$ , the word problem in  $G_\omega$  is solvable.*

For many spinal groups the converse is also true. Namely, if the word problem is solvable, one can use the strategy from [Gri84] and construct test-words that would help to recover the kernels of the homomorphisms used in the definition of  $\omega$ , which is then enough to recover the actual homomorphisms. For example, if we know that  $G_\omega$  is one of the Grigorchuk 2-groups from [Gri84] and we have an oracle that solves the word problem, we can check the test-words  $(ab)^4$ ,  $(ac)^4$  and  $(ad)^4$  and find out which one represents the identity, which then tells us that  $\omega_1$  was 2, 1 or 0, respectively. Depending on this result, we construct longer test-words that give us information about  $\omega_2$ , and then longer for  $\omega_3$ , etc.

### 3.2. The conjugacy problem in $\mathfrak{G}$

Two algorithms which solve the conjugacy problem for regular branch groups satisfying some natural conditions involving length functions are described in [GW00]. The first one accumulates the ideas from [Leo98a, Roz98] and is presented here in the simplest form for the first Grigorchuk

group  $\mathfrak{G}$ . The second one, based on the Dirichlet principle, is quite different and has more potential for applications.

We describe now the first algorithm for the case of  $\mathfrak{G}$ . Recall that  $\mathfrak{G}$  is fractal regular branch group with  $K \times K \preceq K \leq G$ , where  $K$  is the normal closure of  $[a, b]$ . For  $g, h \in \mathfrak{G}$  we define

$$Q(g, h) = \{ Kf \mid g^f = h \}.$$

Clearly,  $Q(g, h) = \emptyset$  if and only if  $g$  and  $h$  are not conjugate in  $\mathfrak{G}$ .

**THEOREM 3.4.** *The conjugacy problem is solvable for  $\mathfrak{G}$ .*

**PROOF.**

**LEMMA 3.5.** *Let  $f, g, h \in \mathfrak{G}$  and let*

$$g^f = h. \tag{14}$$

*Let  $g = (g_1, g_2), h = (h_1, h_2) \in \text{St}_{\mathfrak{G}}(\mathcal{L}_1)$ .*

(a) *If  $f \in \text{St}_{\mathfrak{G}}(\mathcal{L}_1)$  and  $f = (f_1, f_2)$  then (14) is equivalent to*

$$\begin{cases} g_1^{f_1} = h_1 \\ g_2^{f_2} = h_2 \end{cases}$$

b) *If  $f \notin \text{St}_{\mathfrak{G}}(\mathcal{L}_1)$  and  $f = (f_1, f_2)a$  then (14) is equivalent to*

$$\begin{cases} g_1^{f_1} = h_2 \\ g_2^{f_2} = h_1 \end{cases}$$

(2) *Let  $g = (g_1, g_2)a, h = (h_1, h_2)a \notin \text{St}_{\mathfrak{G}}(\mathcal{L}_1)$ .*

a) *If  $f \in \text{St}_{\mathfrak{G}}(\mathcal{L}_1)$  and  $f = (f_1, f_2)$  then (14) is equivalent to*

$$\begin{cases} (g_1 g_2)^{f_1} = h_1 h_2 \\ f_2 = g_2 f_1 h_2^{-1} \end{cases}$$

b) *If  $f \notin \text{St}_{\mathfrak{G}}(\mathcal{L}_1)$  and  $f = (f_1, f_2)a$  then (14) is equivalent to*

$$\begin{cases} (g_1 g_2)^{f_1} = h_2 h_1 \\ f_2 = g_1^{-1} f_1 h_2 \end{cases}$$

**LEMMA 3.6.** *Let  $x = (x_1, y_1)$  and  $y = (y_1, y_2)$  be elements of  $\mathfrak{G}$ . Let*

$$Q(x, y) = \{ Kz_i \mid i \in I \}$$

$$Q(x_1, y_1) = \{ Kz_j \mid j \in J \}$$

$$Q(x_2, y_2) = \{ Kz_{j'} \mid j' \in J' \}$$

*For every  $i \in I$  there exists  $j \in J$  and  $j' \in J'$  such that the element  $z_{jj'} = (z_j, z_{j'})$  is in  $\mathfrak{G}$  and  $Kz_i = Kz_{jj'}$ .*

The meaning of Lemma 3.6 is that if we have  $Q(x_1, y_1)$  and  $Q(x_2, y_2)$  already calculated, then we can calculate  $Q(x, y)$ . We just need to check all pairs  $(z_j, z_{j'})$  that are in  $\mathfrak{G}$  and chose one in each coset of  $K$ .

We describe an algorithm that solves the conjugacy problem in  $\mathfrak{G}$  by calculating  $Q(g, h)$ , given  $g$  and  $h$ .

Split  $Q(g, h)$  as

$$Q(g, h) = Q_{1a}(g, h) \cup Q_{1b}(g, h)$$

or as

$$Q(g, h) = Q_{2a}(g, h) \cup Q_{2b}(g, h)$$

according to the cases described in Lemma 3.5.

If we knew how to calculate  $Q(g_1, h_1)$  and  $Q(g_2, h_2)$  for the case 1a, we could compute  $Q_{1a}(g, h)$ . Thus to prove the theorem we need only to show that there is a reduction in length in each of the cases. This is obvious for the cases 1a and 1b as

$$|g_i| + |h_j| < |g| + |h|,$$

for  $i, j \in \{1, 2\}$ , except when  $|g| = |h| = 1$ , which case can be handled directly. For the cases 2a and 2b we only have

$$|g_1g_2| + |h_1h_2| \leq |g| + |h|,$$

and equality is possible only if the letter  $d$  does not appear in the word representing  $g$  nor in the one representing  $f$ . In case of an equality we repeat the argument with the pair  $(g_1g_2, h_1h_2)$  or  $(g_1g_2, h_2h_1)$  depending on the case.

After at most three steps there must be some reduction in the length and we may apply induction on the length.  $\square$

## Presentations and endomorphic presentations of branch groups

Branch groups are probably never finitely presented and this is established for the known examples. There are many approaches one can use to prove that a given branch group cannot have a finite presentation and we will say more about them below, along with some historical remarks.

Every branch group that has a solvable word problem, as all basic examples do, has a recursive presentation.

An important discovery was made by Igor Lysionok in 1985, who showed in [Lys85] that the first Grigorchuk group can be given by the presentation (15). No due attention was given to this fact for a long time, until the second author used this presentation in [Gri98] to construct an example of finitely presented amenable but not elementary amenable group, thus answering a question of Mahlon Day from [Day57].

The idea of Igor Lysionok was developed in a different direction in [Gri99], where the presentations of the form (15), which are finite presentations modulo the iteration of a single endomorphism, were called  $L$ -presentations. It was shown in [Gri99] that some GGS groups (for example the Gupta-Sidki  $p$ -groups, for  $p > 3$ ) have such presentations.

The study of  $L$ -presentations was continued, and some aspects were completely resolved in [Bar]. First of all, the notion of  $L$ -presentations was extended to the notion of endomorphic presentation in a way that allows several endomorphisms to be included in the presentation. On the basis of this extended definition, a general result was obtained, claiming that all regular branch groups satisfying certain natural requirements have finite endomorphic presentations (see Theorem 4.7 below). It would be interesting to answer the question what branch groups have  $L$ -presentations in the sense of [Gri99] and, in particular, resolve the status of the Gupta-Sidki group  $\overline{\Gamma}$  (Question 7).

We finish the chapter by providing several examples of groups with finite endomorphic presentations, mostly taken from [Bar].

### 4.1. Non-finite presentability

In his early work, the second author proved in various ways that  $\mathfrak{G}$  is not finitely presented. We review briefly these ideas, since they generalize differently to various other examples.

**THEOREM 4.1.** *The first Grigorchuk group  $\mathfrak{G}$  is not finitely presented.*

**FIRST PROOF, FROM [Gri80].** More details can be found in [Gri99]. Assume for contradiction that  $\mathfrak{G}$  is finitely presented, say as  $\langle S | R \rangle$ . The Reidemeister-Schreier method then gives a presentation of  $\text{St}_{\mathfrak{G}}(1)$  with relators  $R \cup R^a$ ; writing for each relator  $(r)\psi = (r_1, r_2)$  we obtain a presentation of  $\mathfrak{G}$  with relators  $\{r_1, r_2\}_{r \in R}$ . Now since  $|r_i| \leq |r|/2$  for all  $r \in R$  of length at least 2 (using cyclic reductions), we obtain after enough applications of the above process a presentation of  $\mathfrak{G}$  with relations of length 1, i.e. a free group.

This contradicts almost every property of  $\mathfrak{G}$ : that it is torsion, of subexponential growth, just-infinite, or contains elements  $(x, 1)$  and  $(1, y)$  that commute.  $\square$

**SECOND PROOF, FROM [Gri84].** The set of groups with given generator set  $\{a_1, \dots, a_m\}$  is a topological space, with the “weak topology”: a sequence  $(G_i)$  of groups converges to  $G$  if for all

radii  $R$  the sequence of balls  $B_{G_i}(R)$  in the Cayley graphs of the corresponding groups stabilize to the ball  $B_G(R)$ .

Assume for contradiction that  $\mathfrak{G}$  is finitely presented, say as  $\langle S \mid R \rangle$ . Then for any sequence  $G_i \rightarrow \mathfrak{G}$  the groups  $G_i$  are quotients of  $\mathfrak{G}$  for  $i$  large enough. However, if one considers all Grigorchuk groups  $\mathfrak{G}_\omega$  from [Gri84], defined through infinite sequences  $\bar{\omega} \in \{0, 1, 2\}^{\mathbb{N}}$ , one notes that the map  $\omega \rightarrow \mathfrak{G}_\omega$  is continuous, with the Tychonoff topology on  $\{0, 1, 2\}^{\mathbb{N}}$ . There are therefore infinite groups converging to  $\mathfrak{G}$ , which contradicts  $\mathfrak{G}$ 's just-infiniteness.  $\square$

A third proof involves a complete determination of the presentation, and of its Schur multiplier. The results are:

THEOREM 4.2 (Lysionok, [Lys85]). *The Grigorchuk group  $\mathfrak{G}$  admits the following presentation:*

$$\mathfrak{G} = \langle a, c, d \mid \phi^i(a^2), \phi^i(ad)^4, \phi^i(adacac)^4 \ (i \geq 0) \rangle, \quad (15)$$

where  $\phi : \{a, c, d\}^* \rightarrow \{a, c, d\}^*$  is defined by  $\phi(a) = aca, \phi(c) = cd, \phi(d) = c$ .

THEOREM 4.3 (Grigorchuk, [Gri99]). *The Schur multiplier  $H_2(\mathfrak{G}, \mathbb{Z})$  of the first Grigorchuk group is  $(\mathbb{Z}/2)^\infty$ , with basis  $\{\phi^i[d, d^a], \phi^i[d^{ac}, d^{aca}]\}_{i \in \mathbb{N}}$ .*

Given a presentation  $G = F/R$ , where  $F$  is a free group, the Schur multiplier may be computed as  $H_2(G, \mathbb{Z}) = (R \cap [F, F])/[F, R]$  (see [Kar87] or [Bro94] for details). This implies instantly that  $\mathfrak{G}$  may not be finitely presented, and moreover that no relation can be omitted.

Another approach that deserves attention is demonstrated in [Gup84]. Namely, certain recursively presented groups are constructed there and the strategy is to build an increasing sequence of normal subgroups  $(R_n)_{n \in \mathbb{N}}$  of the free group  $F$  whose union is the kernel  $R$  of the presentation of the constructed group  $G$  as  $F/R$ .

The most general result, at least for spinal groups, is given in the third author's dissertation [Šun00], and it follows the "topological" approach from [Gri84], but without the explicit use of the topology.

THEOREM 4.4. *Let  $\mathcal{C}$  be a class of groups that is closed under homomorphic images and subgroups (of finite index) and  $\omega = (A_\omega, B, \bar{\omega})$  be a sequence that defines a spinal group in  $\mathcal{C}$ . Further, assume that, for every  $r$ , there exists a triple  $\eta^{(r)}$  of the form  $\eta^{(r)} = (A_{\sigma^r \omega}, B, \bar{\eta})$ , where  $\bar{\eta}^{(r)}$  is a doubly indexed family of homomorphisms*

$$\eta_{i,j} : B \rightarrow \text{Sym}(Y_{j+1}), \quad i \in \{r+1, r+2, \dots\}, \quad j \in \{1, \dots, m_i - 1\}$$

defining a group of tree automorphisms (not necessarily spinal)  $G_{\eta^{(r)}}$  that acts on the shifted tree  $\mathcal{T}^{(\sigma^r Y)}$  and is not in  $\mathcal{C}$ . Then, the spinal group  $G_\omega$  is not finitely presented.

PROOF. Assume, on the contrary, that  $G_\omega$  is finitely presented.

Further, assume that the length of the longest relator in the finite presentation of  $G_\omega$  is no greater than  $2^{n+1} + 1$ . If  $\omega'$  is any triple (not necessarily defining a spinal group) that agrees with  $\omega$  on the first  $n$  levels then any word of length no greater than  $2^{n+1} + 1$  representing the identity in  $G_\omega$  represents the identity in  $G_{\omega'}$ . Thus all relators from the finite presentation of  $G_\omega$  represent the identity in  $G_{\omega'}$ , so that  $G_{\omega'}$  is a homomorphic image of  $G_\omega$ , and, therefore, a member of  $\mathcal{C}$ .

Define  $\omega'$  so that it agrees with  $\omega$  on the first  $n$  levels and it uses the definition of  $\eta^{(n)}$  to define the rest of the levels (just concatenate the definition of  $\eta^{(n)}$  to the definition of the first  $n$  levels of  $\omega$ ). Since  $G_{\omega'}$  is a member of  $\mathcal{C}$ , so is each of its upper companion groups, including  $G_{\eta^{(n)}}$ , a contradiction.  $\square$

Since the class of torsion groups is closed for subgroups and images and since it is fairly easy to construct triples  $\eta^{(r)}$  that define groups containing elements of infinite order, we obtain the following:

COROLLARY 4.5. *No torsion spinal group is finitely presented.*

## 4.2. Endomorphic presentations of branch groups

The recursive structure of branch groups appears explicitly in their presentations by generators and relators, and such presentations have been described since the mid-80's for the first example, the Grigorchuk group.

In this section, we will mainly consider finitely generated, regular branch groups, the reason being that the regularity of presentations becomes much more apparent in these cases. The main result is best formulated in terms of “endomorphic presentations” [Gri99, Bar]:

DEFINITION 4.6. *An endomorphic presentation is an expression of the form*

$$L = \langle S \mid Q \mid \Phi \mid R \rangle,$$

where  $S$  is an alphabet (i.e., a set of symbols),  $Q, R \subset F_S$  are sets of reduced words (where  $F_S$  is the free group on  $S$ ), and  $\Phi$  is a set of group homomorphisms  $\phi : F_S \rightarrow F_S$ .

$L$  is finite if  $Q, R, S, \Phi$  are finite. It is ascending if  $Q$  is empty.

$L$  gives rise to a group  $G_L$  defined as

$$G_L = F_S / \left\langle Q \cup \bigcup_{\phi \in \Phi^*} (R)\phi \right\rangle^\#,$$

where  $\langle \cdot \rangle^\#$  denotes normal closure and  $\Phi^*$  is the monoid generated by  $\Phi$ , i.e., the closure of  $\{1\} \cup \Phi$  under composition.

As is customary, we identify the endomorphic presentation  $L$  and the group  $G_L$  it defines.

An endomorphic presentation that has exactly one homomorphism in  $\Phi$  is called  $L$ -presentation.

The geometric interpretation of endomorphic presentations in the context of branch groups is the following: one has a finite generating set ( $S$ ), a finite collection of relations, some of which ( $R$ ) are related to the branching and therefore can be “moved from one tree level down to the next” by endomorphisms ( $\Phi$ ).

The main result of this chapter is:

THEOREM 4.7 ([Bar]). *Let  $G$  be a finitely generated, contracting, regular branch group. Then  $G$  has a finite endomorphic presentation. However,  $G$  is not finitely presented.*

The motivations in studying group presentations of branch groups are the following:

- They exhibit a regularity that closely parallels the branching structure;
- They allow explicit embeddings of branch groups in finitely presented groups (see Theorem 4.9);
- They may give an explicit basis for the Schur multiplier of branch groups (see Theorem 4.3).

This section sums up the proof of Theorem 4.7. Details may be found in [Bar].

Let  $G$  be regular branch on its subgroup  $K$ , and fix generating sets  $S$  for  $G$  and  $T$  for  $K$ . Without loss of generality, assume  $K \leq \text{St}_G(\mathcal{L}_1)$ , since one may always replace  $K$  by  $K \cap \text{St}_G(\mathcal{L}_1)$ .

First, there exists a finitely presented group  $\Gamma = \langle S \mid Q \rangle$  with subgroups  $\Delta$  and  $\Upsilon = \langle T \rangle$  corresponding to  $\text{St}_G(\mathcal{L}_1)$  and  $K$ , such that the map  $\psi : \text{St}_G(\mathcal{L}_1) \rightarrow G^m$  lifts to a map  $\Delta \rightarrow \Gamma^m$ .

The data are summed up in the following diagram:

$$\begin{array}{ccccccc}
 & & \Gamma & \longrightarrow & G & & \\
 & & \downarrow & & \downarrow & & \\
 & & \Delta & \longrightarrow & \text{St}_G(\mathcal{L}_1) & \xrightarrow{\psi} & G^m \\
 & \longleftarrow \tilde{\psi} & & & & & \\
 \Gamma^m & & & & & & \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \Upsilon^m & & \Upsilon & \longrightarrow & K & \longleftarrow & K^m
 \end{array}$$

Since  $\text{Im } \tilde{\psi}$  contains  $\Upsilon^m$ , it has finite index in  $\Gamma^m$ . Since  $\Gamma^m$  is finitely presented,  $\text{Im } \tilde{\psi}$  too is finitely presented. Similarly,  $\Delta$  is finitely presented, and we may express  $\text{Ker } \tilde{\psi}$  as the normal closure  $\langle R_1 \rangle^\#$  in  $\Delta$  of those relators in  $\text{Im } \tilde{\psi}$  that are not relators in  $\Delta$ . Clearly  $R_1$  may be chosen to be finite.

We now use the assumption that  $G$  is contracting, with constant  $C$ . Let  $R_2$  be the set of words over  $S$  of length at most  $C$  that represent the identity in  $G$ . Set  $R = R_1 \cup R_2$ , which clearly is finite.

We consider  $T$  as a set distinct from  $S$ , and not as a subset of  $S^*$ . We extend each  $\varphi_y$  to a monoid homomorphism  $\tilde{\varphi}_y : (S \cup T)^* \rightarrow (S \cup T)^*$  by defining it arbitrarily on  $S$ .

Assume  $\Gamma = \langle S \mid Q \rangle$ , and let  $w_t \in S^*$  be a representation of  $t \in T$  as a word in  $S$ . We claim that the following is an endomorphic presentation of  $G$ :

$$G = \langle S \cup T \mid Q \cup \{t^{-1}w_t\}_{t \in T} \mid \{\tilde{\varphi}_y\}_{y \in Y} \mid R_1 \cup R_2 \rangle. \quad (16)$$

For this purpose, consider the following subgroups  $\Xi_n$  of  $\Gamma$ : first  $\Xi_0 = \{1\}$ , and by induction

$$\Xi_{n+1} = \{ \gamma \in \Delta \mid (\gamma)\tilde{\psi} \in \Xi_n^m \}.$$

We computed  $\Xi_1 = \langle R \rangle^\#$ . Since  $G$  acts transitively on the  $n$ -th level of the tree, a set of normal generators for  $\Xi_n$  is given by  $\bigcup_{y \in Y^n} (R)\varphi_{y_1} \cdots \varphi_{y_n}$ . We also note that  $(\Xi_{n+1})\tilde{\psi} = \Xi_n^m$ .

We will have proven the claim if we show  $G = \Gamma / \bigcup_{n \geq 0} \Xi_n$ . Let  $w \in \Gamma$  represent the identity in  $G$ . After  $\psi$  is applied  $|w|$  times, we obtain  $m^{|w|}$  words that are all of length at most  $C$ , that is, they belong to  $\Xi_1$ . Then since  $(\Xi_{n+1})\tilde{\psi} = \Xi_n^m$ , we get  $w \in \Xi_{|w|+1}$ , and (16) is a presentation of  $G$ .

As a bonus, the presentation (16) expresses  $K$  as the subgroup of  $G$  generated by  $T$ .

### 4.3. Examples

We describe here a few examples of branch groups' presentations. As a first example, let us consider the group  $\text{Aut}_f(\mathcal{T}^{(2)})$  of finitary automorphisms of the binary tree.

**THEOREM 4.8.** *A presentation of  $\text{Aut}_f(\mathcal{T}^{(2)})$  is given by*

$$T = \langle x_0, x_1, \dots \mid x_i^2, [x_j, x_k^{x_i}] \quad \forall j, k > i \rangle,$$

and these relators are independent.

**PROOF.** The generator  $x_i$  is interpreted as the element whose portrait has a single non-trivial label, at level  $i$ . The relations are easily checked, and they yield a presentation because they are sufficient to put words in the  $x_i$  in wreath product normal form.

Finally, if  $G_n$  is the quotient of  $\text{Aut}_f(\mathcal{T})$  acting on the  $n$ -th level,  $H_2(G_n, \mathbb{Z}) = (\mathbb{Z}/2)^{\binom{n+1}{3}}$  by [Bla72] or [Kar87].  $\square$

We now stick to the  $L$ -presentation notation, and give presentations for the following examples:

**The “first Grigorchuk group”:** The group  $\mathfrak{G}$  admits the ascending  $L$ -presentation

$$\mathfrak{G} = \langle a, c, d \mid \phi \mid a^2, [d, d^a], [d^{ac}, (d^{ac})^a] \rangle,$$

where  $\phi : \{a, c, d\}^* \rightarrow \{a, c, d\}^*$  is defined by

$$\phi(a) = aca, \quad \phi(c) = cd, \quad \phi(d) = c.$$

These relators are independent, and  $H_2(\mathfrak{G}, \mathbb{Z}) = (\mathbb{Z}/2)^\infty$ .

**The “Grigorchuk supergroup” [BG02]:** The group  $\tilde{\mathfrak{G}} = \langle a, \tilde{b}, \tilde{c}, \tilde{d} \rangle$  acting on the binary tree, where  $a$  is the rooted automorphism  $a = ((1, 2))$  and the other three generators are the directed automorphisms defined recursively by

$$\tilde{b} = (a, \tilde{c}) \quad \tilde{c} = (1, \tilde{d}) \quad \tilde{d} = (1, \tilde{b}),$$

admits the ascending  $L$ -presentation

$$\tilde{\mathfrak{G}} = \langle \langle a, \tilde{b}, \tilde{c}, \tilde{d} \mid \tilde{\phi} \mid a^2, [\tilde{b}, \tilde{c}], [\tilde{c}, \tilde{c}^a], [\tilde{c}, \tilde{d}^a], [\tilde{d}, \tilde{d}^a], [\tilde{c}^{a\tilde{b}}, (\tilde{c}^{a\tilde{b}})^a], [\tilde{c}^{a\tilde{b}}, (\tilde{d}^{a\tilde{b}})^a], [\tilde{d}^{a\tilde{b}}, (\tilde{d}^{a\tilde{b}})^a] \rangle \rangle,$$

where  $\tilde{\phi} : \{a, \tilde{b}, \tilde{c}, \tilde{d}\}^* \rightarrow \{a, \tilde{b}, \tilde{c}, \tilde{d}\}^*$  is defined by

$$a \mapsto a\tilde{b}a, \quad \tilde{b} \mapsto \tilde{d}, \quad \tilde{c} \mapsto \tilde{b}, \quad \tilde{d} \mapsto \tilde{c}.$$

These relators are independent, and  $H_2(\tilde{\mathfrak{G}}, \mathbb{Z}) = (\mathbb{Z}/2)^\infty$ .

**The “Fabrykowski-Gupta group” [FG91, BG02]:** The group  $\Gamma$  admits the ascending endomorphic presentation

$$\langle a, r \mid \phi, \chi_1, \chi_2 \mid a^3, [r^{1+a^{-1}-1+a+1}, a], [a^{-1}, r^{1+a+a^{-1}}][r^{a+1+a^{-1}}, a] \rangle,$$

where  $\sigma, \chi_1, \chi_2 : \{a, r\}^* \rightarrow \{a, r\}^*$  are given by

$$\begin{aligned} \phi(a) &= r^{a^{-1}}, & \phi(r) &= r, \\ \chi_1(a) &= a, & \chi_1(r) &= r^{-1}, \\ \chi_2(a) &= a^{-1}, & \chi_2(r) &= r. \end{aligned}$$

These relators are independent, and  $H_2(\Gamma, \mathbb{Z}) = (\mathbb{Z}/3)^\infty$ .

**The “Gupta-Sidki group” [Sid87b]:** The Gupta-Sidki group  $\overline{\overline{\Gamma}}$  admits the endomorphic presentation

$$\langle a, t, u, v \mid a^3, t^3, u^{-1}t^a, v^{-1}t^{a^{-1}} \mid \phi, \chi \mid (tuv)^3, [v, t][vt, u^{-1}tv^{-1}u], [t, u]^3[u, v]^3[t, v]^3 \rangle,$$

where  $\phi, \chi : \{t, u, v\}^* \rightarrow \{t, u, v\}^*$  are given by

$$\phi : \begin{cases} t \mapsto t, \\ u \mapsto [u^{-1}t^{-1}, t^{-1}v^{-1}]t = u^{-1}tv^{-1}tuv^{-1}, \\ v \mapsto t[tv, ut] = t^{-1}vutv^{-1}tu^{-1}, \end{cases} \quad \chi : \begin{cases} t \mapsto t^{-1}, \\ u \mapsto u^{-1}, \\ v \mapsto v^{-1}. \end{cases}$$

These relators are independent, and  $H_2(\overline{\overline{\Gamma}}, \mathbb{Z}) = (\mathbb{Z}/3)^\infty$ . Note that  $\chi$  is induced by the automorphism of  $\overline{\overline{\Gamma}}$  defined by  $a \mapsto a$ ,  $t \mapsto t^{-1}$ ; however,  $\phi$  does not extend to an endomorphism of  $\overline{\overline{\Gamma}}$ .

It is precisely for that reason that no ascending endomorphic presentation of  $\overline{\overline{\Gamma}}$  is known.

**The “Brunner-Sidki-Vieira group” [BSV99]:** Consider the group  $G = \langle \mu, \tau \rangle$  acting on the binary tree, where  $\mu$  and  $\tau$  are defined recursively by

$$\mu = (1, \mu^{-1})a, \quad \tau = (1, \tau)a.$$

Note that  $G$  is not branch, but it is weakly branch. Writing  $\lambda = \tau\mu^{-1}$ ,  $G$  admits the ascending  $L$ -presentation

$$G = \langle \lambda, \tau \mid \phi \mid [\lambda, \lambda^\tau], [\lambda, \lambda^{\tau^3}] \rangle,$$

where  $\phi$  is defined by  $\tau \mapsto \tau^2$  and  $\lambda \mapsto \tau^2 \lambda^{-1} \tau^2$ .

The above  $L$ -presentation for  $\mathfrak{G}$  allowed the second author to answer a question of Mahlon Day [Day57] for the class of finitely presented groups (the question is formulated in [CFP96] in this special setting):

**THEOREM 4.9** (Grigorchuk, [Gri98]). *There exists a finitely presented amenable group that is not elementarily amenable.*

Recall that a group  $G$  is amenable if it admits a left-invariant finitely-additive measure. Examples include the finite groups, the abelian groups and all groups obtained from previous examples by short exact sequences and direct limits. The smallest class containing the finite and abelian groups and closed for the mentioned basic constructions is known as the class of *elementarily amenable* groups.

**PROOF.** Consider the presentation of the first Grigorchuk group given above, and form the HNN extension  $H$  amalgamating  $\mathfrak{G}$  with  $\phi(\mathfrak{G})$ . It is an ascending HNN extension, so  $H$  is amenable; and  $H$  admits the (ordinary) finite presentation

$$H = \langle a, c, d, t \mid a^2, [d, d^a], [d^{ac}, d^{aca}], a^t a c a, c^t c d, d^t c \rangle.$$

□

Another presentation of the group  $H$  from the previous proof, due to the first author, is given in [CGH99]

$$H = \langle a, t \mid a^2, T a T a t a t a T a t a T a t a T, (T a t a)^8, (T^2 a t a T a t^2 a T a t a)^4 \rangle,$$

where  $T$  denotes the inverse of  $t$ .



## Part 3

# Algebraic Aspects

## Just-Infinite Branch Groups

DEFINITION 5.1. A group  $G$  is just-infinite if it is infinite but all of its proper quotients are finite, i.e., if all of its nontrivial normal subgroups have finite index.

The following simple criterion from [Gri00] characterizes the just-infinite branch groups acting on a tree.

THEOREM 5.2. Let  $G$  be a branch group acting on a tree and let  $(L_i, H_i)_{i \in \mathbb{N}}$  be a corresponding branch structure. The following three conditions are equivalent

- (1)  $G$  is just-infinite.
- (2) the abelianization  $H_i^{\text{ab}}$  of  $H_i$  is finite, for each  $i \in \mathbb{N}$ .
- (3) the commutator subgroup  $H'_i$  of  $H_i$  has finite index in  $G$ , for each  $i \in \mathbb{N}$ .

The statement is a corollary of the fact that  $H'_i$ , being characteristic in the normal subgroup  $H_n$ , is a normal subgroup of  $G$ , for each  $i \in \mathbb{N}$ , and the following useful lemma that says that weakly branch groups satisfy the following property:

LEMMA 5.3. Let  $G$  be a weakly branch group acting on a tree and let  $(L_i, H_i)_{i \in \mathbb{N}}$  be a corresponding branch structure. Then every non-trivial normal subgroup  $N$  of  $G$  contains the commutator subgroup  $H'_n$ , for some  $n$  depending on  $N$ .

PROOF. Let  $g$  be an element in  $G \setminus \text{St}_G(\mathcal{L}_1)$  and let  $N = \langle g \rangle^G$  be its normal closure in  $G$ . Then  $g = ha$  for some  $h \in \text{St}_{\text{Aut}(\mathcal{T})}(\mathcal{L}_1)$  with decomposition  $h = (h_1, \dots, h_{m_1})$  and  $a$  a nontrivial rooted automorphism of  $\mathcal{T}$ . Without loss of generality we may assume that  $1^a = m_1$ .

For arbitrary elements  $\xi, \nu \in L_1$ , we define  $f, t \in H_1$  by  $f = (\xi, 1, \dots, 1)$  and  $t = (\nu, 1, \dots, 1)$  and calculate

$$\begin{aligned} [g, f] &= (\xi, 1, \dots, 1, (\xi^{-1})^{h_1}), \\ [[g, f], t] &= ([\xi, \nu], 1, \dots, 1). \end{aligned}$$

Since  $[[g, f], t]$  is always in  $N = \langle g \rangle^G$ , we obtain  $L'_1 \times L'_1 \times \dots \times L'_1 \preceq N$  and, by the spherical transitivity of  $G$ , it follows that

$$L'_1 \times L'_1 \times \dots \times L'_1 = H'_1 \preceq N.$$

Thus any normal subgroup of  $G$  that contains  $g$  also contains  $H'_1$ .

Similarly, if  $g$  is an element in  $\text{St}_G(\mathcal{L}_n) \setminus \text{St}_G(\mathcal{L}_{n+1})$  and  $N$  is the normal closure  $N = \langle g \rangle^G$ , then  $N$  contains  $H'_{n+1}$ .  $\square$

In particular, the above results imply that all finitely generated torsion weakly branch groups are just-infinite branch groups.

The study of just-infinite groups is motivated by their minimality. More precisely, we have the following

THEOREM 5.4. [Gri00] Every finitely generated infinite group has a just-infinite quotient.

Therefore, if  $\mathcal{C}$  is a class of groups closed for taking quotients and it contains a finitely generated infinite group, then it contains a finitely generated just-infinite group.

Note that there are non-finitely generated groups that do not have just-infinite quotients, for example, the additive group of rational numbers  $\mathbb{Q}$ .

It is known (see [Wil71]) that a just-infinite group with non-trivial Baer radical is a finite extension of a free abelian group of finite rank (recall that the *Baer radical* of the group  $G$  is the subgroup of  $G$  generated by the cyclic subnormal subgroups of  $G$ ). Moreover, the only just-infinite group with non-trivial center is the infinite cyclic group  $\mathbb{Z}$ . Therefore, an abelian group has just-infinite quotient if and only if it can be mapped onto  $\mathbb{Z}$ . In particular, no abelian torsion group has just-infinite quotients. The last fact is in a sharp contrast with the fact that there are large classes of centerless, torsion, just-infinite, branch groups, for instance G groups with finite directed part  $B$  (see Chapter 6) and many GGS groups.

**DEFINITION 5.5.** *A group  $G$  is hereditarily just-infinite if it is residually finite and all of its non-trivial normal subgroups are just-infinite.*

We mention that our definition of hereditarily just-infinite group differs from the one given in [Wil00] in that we require residual finiteness. Note that all non-trivial normal subgroups of a group  $G$  are just infinite if and only if all subgroups of finite index in  $G$  are just infinite. This is true since every subgroup of finite index in  $G$  contains a normal subgroup of  $G$  of finite index.

Examples of hereditarily just-infinite groups are the infinite cyclic group  $\mathbb{Z}$ , the infinite dihedral group  $D_\infty$  and the projective special linear groups  $\mathrm{PSL}(n, \mathbb{Z})$ , for  $n \geq 3$ . However, the whole class is far from well understood and described.

The following result from [Gri00], which modifies the result of John Wilson from [Wil71] (see also [Wil00]), strongly motivates the study of the branch groups.

**THEOREM 5.6** (Trichotomy of just-infinite groups). *Let  $G$  be a finitely generated just-infinite group. Then exactly one of the following holds:*

- (1)  $G$  is a branch group.
- (2)  $G$  has a normal subgroup  $H$  of finite index of the form

$$H = L^{(1)} \times \cdots \times L^{(k)} = L^k,$$

where the factors are copies of a group  $L$ , the conjugations by the elements in  $G$  transitively permute the factors of  $H$  and  $L$  has exactly one of the following two properties:

- (a)  $L$  is hereditary just-infinite (in case  $G$  is residually finite).
- (b)  $L$  is simple (in case  $G$  is not residually finite).

The proof of this theorem presented in [Gri00] uses only the statement from [Wil71] that every subnormal subgroup in a just-infinite group with trivial Baer radical has a near complement (but this is probably one of the most important facts from Wilson's theory). The proof actually works for any (not necessarily finitely generated) just-infinite group with trivial Baer radical, for instance just-infinite groups which are not virtually cyclic.

The results of Wilson in [Wil71] (see also [Wil00]) combined with the above trichotomy result show that the following characterization of just-infinite branch groups is possible. Define an equivalence relation on the set of subnormal subgroups of a group  $G$  by,  $H \sim K$  if the intersection  $H \cap K$  has a finite index booth in  $H$  and in  $K$ . The set of equivalence classes of subnormal subgroups, ordered by the order induced by inclusion, forms a Boolean lattice, which, following John Wilson, is called the *structure lattice* of  $G$ .

**THEOREM 5.7.** *Let  $G$  be a just-infinite group. Then  $G$  is branch group if and only if it has infinite structure lattice.*

*Moreover, in such a case, the structure lattice is isomorphic to the lattice of closed and open subsets of the Cantor set.*

On the intuitive level, the just-infinite groups should be considered as “small” groups in contrast to, say, the free or non-elementary hyperbolic groups, which are “large” groups.

There is a rigorous approach to the concept of largeness in groups. Namely, following Pride ([Pri80]), we say that a group  $G$  is *larger* than a group  $H$ , and we denote  $G \succeq H$ , if  $H$  has a subgroup of finite index that is a homomorphic image of a subgroup of  $G$  of finite index. The groups  $G$  and  $H$  are *equally large* (or *Pride equivalent*) if  $G \succeq H$  and  $H \succeq G$ . The set of equivalence classes of equally large groups is partially ordered by  $\succeq$  and the class of finite groups is the obvious smallest element.

We denote the class of groups equally large to  $G$  by  $[G]$ . A group  $G$  is called *minimal* if the only class below  $[G]$  is the class of finite groups [1]. The *height* of a group  $G$  is the height of the class  $[G]$  in the ordering, i.e. the length of a maximal chain between [1] and  $[G]$ . Therefore, the minimal groups are the groups of height 1. Such groups are called *atomic* in [Neu86]

**THEOREM 5.8 ([GW]).** *The first Grigorchuk group  $\mathfrak{G}$  and the Gupta-Sidki  $p$ -groups are minimal.*

A number of questions about just-infinite groups was asked in [Pri80, EP84]. Positive answer to Problem 5 from [Pri80] (Problem 4' in [EP84]) that asks if there exist finitely generated just-infinite groups that do not satisfy the ascending chain condition on subnormal subgroups was provided in [Gri84]. Later, Peter Neumann constructed in [Neu86] more examples of finitely generated just-infinite regular branch groups answering the same question (and also some other questions) raised by Martin Edjvet and Steve Pride in [EP84]. In particular, Peter Neumann provided negative answer to the question if every finitely generated minimal group is finite-by- $D_2$ -by-finite (here  $D_2$  denotes the class of groups in which every nontrivial subnormal subgroup has finite index). Negative answer to this last question also follows from Theorem 5.8 above.

The question of possible heights of finitely generated just-infinite groups is an interesting one. All hereditarily just-infinite and all infinite simple groups are minimal. It is plausible that the Grigorchuk 2-groups from [Gri84] that are defined by non-periodic sequences have infinite height (see Question 13).

## Torsion Branch Groups

As mentioned in the introduction, the most elegant examples of finitely generated infinite torsion groups were constructed within the class of branch groups ([Gri80, GS83a]). Therefore, branch groups play important role in problems of Burnside type. At the moment, there is a number of constructions of torsion branch groups. Besides the early works ([Ale72, Suš79, Gri80, Gri83, Gri84, GS83a, GS83b, GS84] there are more modern and general constructions ([BŠ01, Gri00, Šun00]). Nevertheless, all these constructions follow the same idea of stabilization and length reduction.

Recall that finitely generated torsion branch groups acting on a tree are just-infinite (Theorem 5.2).

We provide a proof here that all  $\mathbf{G}$  groups with torsion directed group  $B$  are themselves torsion groups, and the argument follows the mentioned general scheme. This is an improvement over the result in [BŠ01] that deals only with regular trees and all the root actions (the actions of  $A_{\sigma^t\omega}$ , for all  $t$ ) are isomorphic and regular. Thus whenever  $B$  is a finitely generated torsion group the corresponding  $\mathbf{G}$  group is a Burnside group and there are uncountably many non-isomorphic examples. We know that torsion spinal groups cannot have finite presentation (see Corollary 4.5. The results in Theorem 6.9 below show that these groups cannot have finite exponent. Therefore, in the finitely generated case, one of our goals is to give upper bounds on the order of an element depending on its length. This leads to the notion of torsion growth.

Let  $G$  be finitely generated infinite torsion group and let  $S$  be a finite generating set that generates  $G$  as a monoid. For any non-negative real number  $n$ , the maximal order of an element of length at most  $n$  is finite, and we denote it by  $\pi_G^S(n)$ . The function  $\pi_G^S$ , defined on the non-negative real numbers, is called the *torsion growth* function of  $G$  with respect to  $S$ .

The group order of an element  $g$  is denoted by  $\pi(g)$ . In case  $F$  is a word  $\pi(F)$  denotes the order of the element represented by the word  $F$ .

We describe a step in a procedure that successfully implements the ideas and constructions introduced in [Gri84]. The final result of the procedure is a tree that helps us to show that all  $\mathbf{G}$  groups whose directed part  $B$  is torsion are themselves torsion groups and also to determine some upper bounds on the torsion growth of the constructed groups. The construction presented here is slightly more complicated than the construction from [Gri84] because of the fact that we need to take into account the possibility of non-cyclic (and even non-regular) actions of the groups  $A_{\sigma^t\omega}$  on their corresponding alphabets.

Let  $G'_\omega$ , be a  $\mathbf{G}$  group defined by the triple  $\omega$  (recall Definition 2.1 and Definition 2.6) and let  $F$  be a reduced word of even length of the form

$$F = b_1 a_1 \dots b_k a_k, \tag{17}$$

where  $a_i$  represent non-trivial rooted automorphisms in  $A_\omega$  and  $b_j$  represent non-trivial directed automorphisms in  $B_\omega$ . Rewrite  $F$  in the form  $F = b_1 b_2^{g_2} \dots b_k^{g_k} a_1 \dots a_k$ , where  $g_i = (a_1 \dots a_{i-1})^{-1}$ ,  $i = 2, \dots, k$ . Set  $g = a_1 \dots a_k \in A_\omega$  and let its order be  $s$ . Note that  $g = 1$  corresponds to  $F \in \text{St}_\omega(\mathcal{L}_1)$ . Put

$$H = b_1 b_2^{g_2} \dots b_k^{g_k},$$

consider the word  $F^s = (Hg)^s \in \text{St}_\omega(\mathcal{L}_1)$  and rewrite it in the form  $F^s = H H^{g^{-1}} H^{g^{-2}} \dots H^{g^{-(s-1)}}$ . Next, by using tables similar to Table 1, but for all possible  $a$ , we calculate the possibly unreduced

words  $\overline{F_1}, \dots, \overline{F_{m_1}}$  representing the first level sections  $(F^s)\varphi_1, \dots, (F^s)\varphi_{m_1}$ , respectively. We have

$$\overline{F_i} = H_i H_{i^g} \dots H_{i^{g^{s-1}}},$$

for  $i = 1, \dots, m_1$ , where  $H_j$  represents the corresponding section of  $H$ . Note that any two words  $\overline{F_i}$  and  $\overline{F_j}$  that correspond to two indices in the same orbit of  $g$  represent conjugate elements of  $G_\omega$ . This is clear since

$$\overline{F_{i^g}} = H_{i^g} H_{i^{g^2}} \dots H_{i^{g^{s-1}}} H_i = \overline{F_i}^{H_i}.$$

For  $i = 1, \dots, m_1$ , let the length of the cycle of  $i$  in  $g$  be  $t_i$ . Then

$$\overline{F_i} = (H_i H_{i^g} \dots H_{i^{g^{t_i-1}}})^{s/t_i}.$$

and let  $F_i$  be a cyclically reduced word obtained after applying simple reductions (including the cyclic ones) to the word  $\tilde{F}_i = H_i H_{i^g} \dots H_{i^{g^{t_i-1}}}$ . Clearly

$$(F^s)\psi = (\tilde{F}_1^{s/t_1}, \dots, \tilde{F}_{m_1}^{s/t_{m_1}})$$

and  $F$  has finite order if and only if  $F_1, \dots, F_{m_1}$  all have finite order. In the case of a finite order, the order  $\pi(F)$  of  $F$  is a divisor of  $s \cdot \text{lcm}(\pi(F_1), \dots, \pi(F_{m_1}))$ , since the order  $\pi(F_i^{s/t_i})$  divides the order  $\pi(F_i)$ .

Let us make a couple of simple observations on the structure of the possibly unreduced words  $\tilde{F}_1, \dots, \tilde{F}_{m_1}$  used to obtain the reduced words  $F_1, \dots, F_{m_1}$ . We have

$$\begin{aligned} \tilde{F}_i &= H_i H_{i^g} \dots H_{i^{g^{t_i-1}}} = \\ &= (b_1)\varphi_i (b_2^{g_2})\varphi_i \dots (b_k^{g_k})\varphi_i (b_1)\varphi_{i^g} (b_2^{g_2})\varphi_{i^g} \dots (b_k^{g_k})\varphi_{i^g} \dots \\ &\quad (b_1)\varphi_{i^{g^{t_i-1}}} (b_2^{g_2})\varphi_{i^{g^{t_i-1}}} \dots (b_k^{g_k})\varphi_{i^{g^{t_i-1}}}. \end{aligned} \quad (18)$$

It is important to note that the indices  $i, i^g, \dots, i^{g^{t_i-1}}$  are all distinct, since the length of the cycle of  $i$  in  $g$  is  $t_i$ . This means that at most one of  $(b_1)\varphi_i, (b_1)\varphi_{i^g}, \dots, (b_1)\varphi_{i^{g^{t_i-1}}}$  can be equal to  $b_1$ , at most one can be equal to  $(b_1)\omega_1$ , and all others are empty. Thus we conclude that the word  $\tilde{F}_i$  consists of some of the letters  $b_1, \dots, b_k$ , possibly not in that order, and no more than  $k$   $A$ -letters. In particular, it is possible to get  $k$   $A$ -letters only if none of the letters  $b_1, \dots, b_k$  is in the kernel  $K_1$ .

We have already observed that any two, possibly unreduced, words  $\tilde{F}_i$  and  $\tilde{F}_j$  such that the indices  $i$  and  $j$  are in the same cycle of  $g$  are conjugate. Choose a representative for each cycle of  $g$ , and let  $\tilde{F}_{j_1}, \dots, \tilde{F}_{j_c}$  be such representatives. The important feature of our construction is that each letter  $b_i$  from the representation of  $F$  as in (17), as well as each  $(b_i)\omega_1$ , will appear in exactly one of the representatives  $\tilde{F}_{j_1}, \dots, \tilde{F}_{j_c}$  before reduction.

The tree that has  $F$  at its root and the cyclically reduced cycle representatives  $F_{j_1}, \dots, F_{j_c}$  at the vertices below the root is called the *pruned period decomposition* of  $F$ .

**THEOREM 6.1.** *A  $G$  group is a torsion group if and only if its directed part  $B$  is a torsion group.*

**PROOF.** We prove that the order of any element  $g$  in  $G_\omega$  is finite in case  $B$  is a torsion group. The proof is by induction on the length  $n$  of  $g$ , for all  $G$  groups simultaneously.

The statement is clear for  $n = 0$  and  $n = 1$ . Assume that it is true for all words of length less than  $n$ , where  $n \geq 2$ , and consider an element  $g$  of length  $n$ .

If  $n$  is odd the element  $g$  is conjugate to an element of smaller length and we are done by the inductive hypothesis. Assume then that  $n$  is even. Clearly,  $g$  is conjugate to an element that can be represented by a word of the form

$$F = b_1 a_1 \dots b_k a_k.$$

If all the cycle representatives  $F_i$  from the pruned period decomposition of  $F$  have length shorter than  $n$  we are done by the inductive hypothesis.

Assume that some of the cycle representatives  $F_i$  have length  $n$ . This is possible only when  $F$  does not have any  $B$ -letters from  $K_1$ . Also, the words  $\tilde{F}_i$  corresponding to the words  $F_i$  of length  $n$

must be reduced, so that the words  $F_i$  that have length  $n$  must have the same  $B$ -letters as  $F$  does. For each of these finitely many words we repeat the discussion above. Either all of the constructed words  $F_{i_j}$  are strictly shorter than  $n$ , and we get the result by induction; or some have length  $n$ , but the  $B$ -letters appearing in them do not come from  $K_1 \cup K_2$ .

This procedure cannot go on forever since  $K_1 \cup K_2 \cup \dots \cup K_r = B$  holds for some  $r \in \mathbb{N}$ . Therefore at some stage we get a shortening in all the words and we conclude that the order of  $F$  is finite.  $\square$

In the sequel we just list some estimates on the period growth in case the directed part  $B$  is a finite group. The proofs can be found in [BŠ01].

A finite subsequence  $\omega_{i+1}\omega_{i+2}\dots\omega_{i+r}$  of the defining sequence  $\bar{\omega} = \omega_1\omega_2\dots$  is *complete* if each element of  $B$  is sent to the identity by at least one homomorphism from the sequence  $\omega_{i+1}\omega_{i+2}\dots\omega_{i+r}$ , i.e., if  $\bigcup_{j=1}^r K_{i+j} = B$ . We note that the complete sequence  $\omega_{i+1}\omega_{i+2}\dots\omega_{i+r}$  must have length at least  $m+1$ , where  $m$  is the minimal branching index in the branching sequence, since each kernel  $K_{i+j}$  has index  $|A_{\sigma^{i+j}\omega}| \geq m_{i+j+1} \geq m$  in  $B$ , for all  $j = 1, \dots, r$ . In particular, the length of a complete sequence is never shorter than 3. By the definition of a G group, all sequences that define a G group can be factored into finite complete subsequences.

A defining sequence  $\bar{\omega}$  is *r-homogeneous*, for  $r \geq 3$ , if all of its finite subsequences of length  $r$  are complete. A defining sequence  $\bar{\omega}$  is *r-factorable*, for  $r \geq 3$ , if it can be factored in complete subsequences of length at most  $r$ .

**THEOREM 6.2 (Period  $\eta$ -Estimate).** *If  $\bar{\omega}$  is an r-homogeneous sequence and B has exponent q, then there exist a positive constant C such that the torsion growth function of the group  $G_\omega$  satisfies*

$$\pi_\omega(n) \leq Cn^{\log_{1/\eta_r}(q)}$$

where  $\eta_r$  is the positive root of the polynomial  $x^r + x^{r-1} + x^{r-2} - 2$ .

**THEOREM 6.3 (Period 3/4-Estimate).** *If  $\bar{\omega}$  is an r-factorable sequence and B has exponent q, then there exists a positive constant C such that the torsion growth function of the group  $G_\omega$  satisfies*

$$\pi_\omega(n) \leq Cn^{r \log_{4/3}(q)}.$$

**THEOREM 6.4 (Period 2/3-Estimate).** *If  $\bar{\omega}$  is an r-factorable sequence such that each factor contains three letters whose kernels cover B and B has exponent q, then there exists a positive constant C such that the torsion growth function of the group  $G_\omega$  satisfies*

$$\pi_\omega(n) \leq Cn^{r \log_{3/2}(q)}.$$

Let us assume now that all the branching indices are prime numbers and the groups  $A_{\sigma^t\omega}$  are cyclic of prime order, for all  $t$ . There is no loss in generality if we assume that  $A_{\sigma^t\omega}$  is generated by the cyclic permutation  $a = (1, 2, \dots, m_{t+1})$ . Note that our assumptions force  $B$  to be abelian group, since  $B$  is always a subdirect product of several copies of the root groups  $A_{\sigma^t\omega}$ .

**THEOREM 6.5.** *Let the branching sequence consists only of primes, B have exponent q and  $\bar{\omega}$  be an r-homogeneous word. There exists a positive constant C such that the torsion growth function of  $G_\omega$  satisfies*

$$\pi_\omega(n) \leq Cn^{(r-1) \log_2(q)}.$$

Finally we give a tighter upper bound on the period growth of the Grigorchuk 2-groups (as defined in [Gri84]).

**THEOREM 6.6.** *Let  $G_\omega$  be a Grigorchuk 2-group. If  $\bar{\omega}$  is an r-homogeneous word, then there exists a positive constant C such that the torsion growth function of the group  $G_\omega$  satisfies*

$$\pi_\omega(n) \leq Cn^{r/2}.$$

In addition to the above estimates [BŠ01] provides lower bounds on the torsion growth function in some cases and, in particular, shows that some Grigorchuk 2-groups have torsion growth functions  $\pi(n)$  that are at least linear in  $n$  (for example the group defined by the sequence  $\bar{\omega} = 01020102\dots$ ). Previous results of Igor Lysionok and Yuriĭ Leonov ([Lys98, Leo97a] already established a lower bound of  $C_1 n^{\frac{1}{2}}$  for the torsion growth function of the first Grigorchuk group, while Theorem 6.6 establishes an upper bound of  $C_2 n^{\frac{3}{2}}$ .

The result following the definition below finds its predecessor in the work of Narain Gupta and Said Sidki [GS83b], where they prove that the Gupta-Sidki  $p$ -groups contain arbitrary long iterated wreath products of cyclic groups of order  $p$  and therefore contain a copy of each finite  $p$ -group. The general argument below follows the approach from [GHZ00] and applies well to more broad settings.

**DEFINITION 6.7.** *Let  $\mathcal{P}$  be a non-empty set of primes. A group  $G$  of automorphisms of  $\mathcal{T}$  has omnipresent  $\mathcal{P}$ -torsion if, for every vertex  $u \in \mathcal{T}$ , the lower companion group  $L_u^G$  (the rigid stabilizer at  $u$ ) has non-trivial elements of  $\mathcal{P}$ -order.*

In case  $\mathcal{P}$  consists of all primes we just say that  $G$  has *omnipresent torsion* and in case  $\mathcal{P} = \{p\}$  we say that  $G$  has *omnipresent  $p$ -torsion*. We note that every spherically transitive group with omnipresent  $\mathcal{P}$ -torsion must be a weakly branch group. We also note that if  $G$  has omnipresent  $\mathcal{P}$ -torsion then so does each of its lower companion groups  $L_u$  considered as a group of automorphisms of  $\mathcal{T}_u$ .

**LEMMA 6.8.** *Let  $\mathcal{P}$  be a non-empty set of primes and  $G$  a group of tree automorphisms with omnipresent  $\mathcal{P}$ -torsion. Then  $G$  contains arbitrary long iterated wreath products of cyclic groups of the form*

$$((\mathbb{Z}/p_1\mathbb{Z} \wr \dots) \wr \mathbb{Z}/p_{n-1}\mathbb{Z}) \wr \mathbb{Z}/p_n\mathbb{Z}$$

where  $n \in \mathbb{N}$  and each  $p_i$  is a  $\mathcal{P}$ -prime, for  $i \in \{1, \dots, n\}$ .

**SKETCH OF A PROOF.** We prove the claim by induction on  $n$ , simultaneously for all groups of tree automorphisms with omnipresent  $\mathcal{P}$ -torsion. The claim is obvious for  $n = 1$ . Assume that  $n \geq 2$  and that the claim holds for all positive natural numbers  $< n$ .

Choose an arbitrary non-trivial element  $g$  of  $G$  of finite  $\mathcal{P}$ -prime order  $p_n$ . Let  $g$  fix the level  $\mathcal{L}_k$  but not  $\mathcal{L}_{k+1}$  and let  $u$  be a vertex on level  $k$  with non-trivial vertex permutation  $(u)g = a$ . All non-trivial cycles of  $a$  have length  $p$  and, without loss of generality, we may assume that one such cycle is  $(1, 2, \dots, p)$ . Without loss of generality we may also assume that the sections  $g_{u1}, \dots, g_{up}$  are trivial (we may accomplish this by conjugation if necessary). By the inductive hypothesis, the lower companion group  $L_{uy}$  contains an iterated wreath product of length  $n - 1$

$$Q = ((\mathbb{Z}/p_1\mathbb{Z} \wr \dots) \wr \mathbb{Z}/p_{n-2}\mathbb{Z}) \wr \mathbb{Z}/p_{n-1}\mathbb{Z}$$

of the required form. But then

$$\langle Q, g \rangle \cong Q \wr \langle g \rangle = ((\mathbb{Z}/p_1\mathbb{Z} \wr \dots) \wr \mathbb{Z}/p_{n-1}\mathbb{Z}) \wr \mathbb{Z}/p_n\mathbb{Z}.$$

□

The above lemma has many corollaries, some of which are summed up in the following theorem:

**THEOREM 6.9.** *Let  $G$  be a group of tree automorphisms. If, for each vertex  $u$ , the lower companion group  $L_u$  of  $G$  has an element of finite order, then  $G$  has elements of unbounded finite order. Further,*

- (1) *Every weakly branch torsion group has infinite exponent.*
- (2) *Every weakly branch  $p$ -group contains a copy of every finite  $p$ -group.*
- (3) *Every weakly regular branch group is weakly branched over a torsion free group or contains a copy of every finite  $p$ -group, for some prime  $p$ .*

- (4) *A regular branch group is either virtually torsion free or it contains a copy of every finite  $p$ -group, for some prime  $p$ .*

Finally, we note that the class of GGS groups is also rich with examples of torsion groups, starting with the second Grigorchuk group from [Gri80], Gupta-Sidki  $p$ -groups from [GS83a, GS83b] and certain Gupta-Sidki extensions from [GS84]. For a wide class of torsion branch GGS groups see Subsection 2.3.3.

## Subgroup Structure

We study in this chapter some subgroup series (derived series, powers series) and general facts about branch groups. We then describe important small-index subgroups in the examples  $\mathfrak{G}, \Gamma, \overline{\Gamma}, \overline{\overline{\Gamma}}$  (recall the definitions from Section 1.6. The lower central series is treated in the next chapter. Most of the results come from [BG02].

### 7.1. The derived series

Let  $G$  be a group. The *derived series*  $(G^{(n)})_{n \in \mathbb{N}}$  of  $G$  is defined by  $G^{(0)} = G$  and  $G^{(n)} = [G^{(n-1)}, G^{(n-1)}]$ . A group is *solvable* if  $G^{(n)} = \{1\}$  for some  $n \in \mathbb{N}$ . It is *residually solvable* if  $\bigcap_{n \in \mathbb{N}} G^{(n)} = \{1\}$ . Note that if  $(\gamma_n(G))_{n \in \mathbb{N}}$  is the lower central series of  $G$ , then a general result states that  $G^{(n)} \leq \gamma_{2^n}(G)$  holds for all  $n$ .

**7.1.1. The derived series of  $\mathfrak{G}$ .** Since  $\mathfrak{G}$  is regular branch over  $K = \langle x \rangle^G$ , where  $x = [a, b]$ , we consider the finite-index subgroups  $\text{Rist}_{\mathfrak{G}}(\mathcal{L}_n) = K \times \cdots \times K$  with  $2^n$  factors, for  $n \geq 2$ .

**THEOREM 7.1.**  $\mathfrak{G}^{(n)} = \text{Rist}_{\mathfrak{G}}(\mathcal{L}_{2n-3})$  for all  $n \geq 3$ , and  $K^{(n)} = \text{Rist}_{\mathfrak{G}}(\mathcal{L}_{2n})$  for all  $n \geq 1$ .

**PROOF.** First, one may check by elementary means that  $\mathfrak{G}^{(3)} = \text{Rist}_{\mathfrak{G}}(\mathcal{L}_3) = K^{\times 8}$ . Then  $K'$  is the normal closure in  $\mathfrak{G}$  of  $[x^d, x]$ , and  $[x^d, x]\psi = ([ca]^b, ca), 1$ , and  $[ca]^b, ca]\psi = (x, 1)$ . Therefore  $K' = K^{\times 4}$ .  $\square$

**7.1.2. The derived series of  $\overline{\overline{\Gamma}}$ .** The result is even slightly simpler for  $\overline{\overline{\Gamma}}$ , which is regular branch over  $\overline{\overline{\Gamma}}$ :

**THEOREM 7.2.**  $\overline{\overline{\Gamma}}^{(n)} = (\overline{\overline{\Gamma}}')^{\times 3^{n-2}}$  for all  $n \geq 2$ .

**PROOF.** The core of the argument is to show that  $\overline{\overline{\Gamma}}^{(3)} = \overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}'$ . This follows from  $\overline{\overline{\Gamma}}^{(3)} = \gamma_8(\overline{\overline{\Gamma}})$  and  $\overline{\overline{\Gamma}}' = \gamma_5(\overline{\overline{\Gamma}})$ , but the computations are tricky — see Subsection 8.2.3 for details.  $\square$

### 7.2. The powers series

Let  $G$  be a group and  $d$  an integer. The *powers series*  $(\mathcal{U}_d^n(G))_{n \in \mathbb{N}}$  is defined by  $\mathcal{U}_d^0(G) = G$  and

$$\mathcal{U}_d^n(G) = \langle x^d \text{ for all } x \in \mathcal{U}_d^{n-1}(G) \rangle.$$

**THEOREM 7.3.** *The 2-powers series of  $\mathfrak{G}$  is as follows  $\mathcal{U}_2(\mathfrak{G}) = \mathfrak{G}'$  and*

$$\mathcal{U}_2^n(\mathfrak{G}) = \langle \Delta((\mathcal{U}_2 K)^{\times 2^{n-2}}), \Delta(K^{\times 2^{n-1}}) \rangle,$$

where  $\Delta(G^j) = \{(g, \dots, g) \mid g \in G\}$  is the diagonal subgroup.

### 7.3. Parabolic subgroups

In the context of groups acting on a hyperbolic space, a parabolic subgroup is the stabilizer of a point on the boundary. We give here a few general facts concerning parabolic subgroups of branch groups, and recall some results on growth of groups and sets on which they act.

More information and uses of parabolic subgroups appear in the context of representations (Subsection 9.0.1), Schreier graphs (Section 10.3) and spectrum (Chapter 11).

**DEFINITION 7.4.** *A ray  $e$  in  $\mathcal{T}$  is an infinite geodesic starting at the root of  $\mathcal{T}$ , or equivalently an element of  $\partial\mathcal{T} = Y^{\mathbb{N}}$ .*

*Let  $G \leq \text{Aut}(\mathcal{T})$  be any subgroup acting spherically transitively and  $e$  be a ray. The associated parabolic subgroup is  $P_e = \text{St}_G(e)$ .*

The following important facts are easy to prove:

- For any  $e \in \partial\mathcal{T}$ , we have  $\bigcap_{f \in \partial\mathcal{T}} P_f = \bigcap_{g \in G} P_e^g = 1$ .
- Let  $e = e_1 e_2 \dots$  be an infinite ray and define the subgroups  $P_n = \text{St}_G(e_1 \dots e_n)$ . Then  $P_n$  has index  $m_1 m_2 \dots m_n$  in  $G$  (since  $G$  acts transitively) and

$$P_e = \bigcap_{n \in \mathbb{N}} P_n.$$

- $P$  has infinite index in  $G$ , and has the same image as  $P_n$  in the quotient  $G_n = G/\text{St}_G(\mathcal{L}_n)$ .

**DEFINITION 7.5.** *Two infinite sequences  $\sigma, \tau : \mathbb{N} \rightarrow Y$  are confinal if there is an  $N \in \mathbb{N}$  such that  $\sigma_n = \tau_n$  for all  $n \geq N$ .*

*Confinality is an equivalence relation, and equivalence classes are called confinality classes.*

The following result is due to Volodymyr Nekrashevych and Vitaly Sushchansky.

**PROPOSITION 7.6.** *Let  $G$  be a group acting on a regular rooted tree  $\mathcal{T}^{(m)}$ , and assume that for any generator  $g \in G$  and infinite sequence  $\tau$ , the sequences  $\tau$  and  $\tau^g$  differ only in finitely many places. Then the confinality classes are unions of orbits of the action of  $G$  on  $\partial\mathcal{T}$ . If moreover for all  $u \in \mathcal{T}$  and  $v \in \mathcal{T} \setminus u$  there is some  $a \in \text{St}_G(u) \cap \text{St}_G(v)$  transitive on the  $m$  subtrees below  $v$ , then the orbits of the action are the confinality classes.*

**DEFINITION 7.7.** *The subgroup  $H$  of  $G$  is weakly maximal if  $H$  is of infinite index in  $G$ , but all subgroups of  $G$  strictly containing  $H$  are of finite index in  $G$ .*

Note that every infinite finitely generated group admits maximal subgroups, by Zorn's lemma.

However, some branch groups may not contain any infinite-index maximal subgroups; this is the case for  $\mathfrak{G}$ , as was shown by Ekaterina Pervova (see [Per00])

**PROPOSITION 7.8.** *Let  $P$  be a parabolic subgroup of a branch group  $G$  with branch structure  $(L_i, H_i)_{i \in \mathbb{N}}$ . Then  $P$  is weakly maximal.*

**PROOF.** Let  $P = \text{Stab}_G(e)$  where  $e = e_1 e_2 \dots$ . Recall that  $G$  contains a product of  $k_n$  copies of  $L_n$  at level  $n$ , and clearly  $P$  contains a product of  $k_n - 1$  copies of  $L_n$  at level  $n$ , namely all but the one indexed by the vertex  $e_1 \dots e_n$ .

Take  $g \in G \setminus P$ . There is then an  $n \in \mathbb{N}$  such that  $(e_1 \dots e_n)^g \neq e_1 \dots e_n$ , so  $\langle P, P^g \rangle$  contains the product  $H_n = L_n^1 \times \dots \times L_n^{k_n}$  of  $k_n$  copies of  $L_n$  at level  $n$ , hence is of finite index in  $G$ .  $\square$

### 7.4. The structure of $\mathfrak{G}$

Recall that  $\mathfrak{G}$ , introduced in Subsection 1.6.1, is the group acting on the binary tree, generated by the rooted automorphism  $a$  and the directed automorphisms  $b, c, d$  satisfying  $\psi(b) = (a, c)$ ,  $\psi(c) = (a, d)$  and  $\psi(d) = (1, b)$ .

$\mathfrak{G}$  has 7 subgroups of index 2:

$$\begin{aligned} &\langle b, ac \rangle, && \langle c, ad \rangle, \quad \langle d, ab \rangle, \\ &\langle b, a, a^c \rangle, && \langle c, a, a^d \rangle, \langle d, a, a^b \rangle, \\ &\text{St}_G(1) = \langle b, c, b^a, c^a \rangle. \end{aligned}$$

As can be computed from its presentation [Lys85] and a computer algebra system [S+93],  $\mathfrak{G}$  has the following subgroup count:

Index	Subgroups	Normal	In $\text{St}_{\mathfrak{G}}(\mathcal{L}_1)$	Normal
1	1	1	0	0
2	7	7	1	1
4	19	7	9	4
8	61	7	41	7
16	237	5	169	5
32	843	3	609	3

See [Bar00c] and [CST01] for more information.

**7.4.1. Normal closures of generators.** They are as follows:

$$\begin{aligned} A = \langle a \rangle^{\mathfrak{G}} &= \langle a, a^b, a^c, a^d \rangle, && \mathfrak{G}/A \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \\ B = \langle b \rangle^{\mathfrak{G}} &= \langle b, b^a, b^{ad}, b^{ada} \rangle, && \mathfrak{G}/B \cong D_8, \\ C = \langle c \rangle^{\mathfrak{G}} &= \langle c, c^a, c^{ad}, c^{ada} \rangle, && \mathfrak{G}/C \cong D_8, \\ D = \langle d \rangle^{\mathfrak{G}} &= \langle d, d^a, d^{ac}, d^{aca} \rangle, && \mathfrak{G}/D \cong D_{16}. \end{aligned}$$

**7.4.2. Some other subgroups.** To complete the picture, we introduce the following subgroups of  $\mathfrak{G}$ :

$$\begin{aligned} K &= \langle (ab)^2 \rangle^{\mathfrak{G}}, && L = \langle (ac)^2 \rangle^{\mathfrak{G}}, && M = \langle (ad)^2 \rangle^{\mathfrak{G}}, \\ \overline{B} &= \langle B, L \rangle, && \overline{C} = \langle C, K \rangle, && \overline{D} = \langle D, K \rangle, \\ T &= K^2 = \langle (ab)^4 \rangle^{\mathfrak{G}}, \\ T_{(m)} &= \underbrace{T \times \cdots \times T}_{2^m}, && K_{(m)} = \underbrace{K \times \cdots \times K}_{2^m}, && N_{(m)} = T_{(m-1)}K_{(m)}. \end{aligned}$$

**THEOREM 7.9.** • In the Lower Central Series,  $\gamma_{2^m+1}(\mathfrak{G}) = N_{(m)}$  for all  $m \geq 1$ .

- In the Derived Series,  $K^{(n)} = \text{Rist}_{\mathfrak{G}}(2n)$  for all  $n \geq 2$  and  $\mathfrak{G}^{(n)} = \text{Rist}_{\mathfrak{G}}(2n-3)$  for all  $n \geq 3$ .
- The rigid stabilizers satisfy

$$\text{Rist}_{\mathfrak{G}}(n) = \begin{cases} D & \text{if } n = 1, \\ K_{(n)} & \text{if } n \geq 2. \end{cases}$$

- The level stabilizers satisfy

$$\text{St}_{\mathfrak{G}}(\mathcal{L}_n) = \begin{cases} \langle b, c, d \rangle^{\mathfrak{G}} & \text{if } n = 1, \\ \langle D, T \rangle & \text{if } n = 2, \\ \langle N_{(2)}, (ab)^4(adabac)^2 \rangle & \text{if } n = 3, \\ \underbrace{\text{St}_{\mathfrak{G}}(\mathcal{L}_3) \times \cdots \times \text{St}_{\mathfrak{G}}(\mathcal{L}_3)}_{2^{n-3}} & \text{if } n \geq 4. \end{cases}$$

Consequently, the index of  $\text{St}_{\mathfrak{G}}(\mathcal{L}_n)$  is

$$|\mathfrak{G}/\text{St}_{\mathfrak{G}}(\mathcal{L}_n)| = 2^{5 \cdot 2^{n-3} + 2}. \quad (19)$$

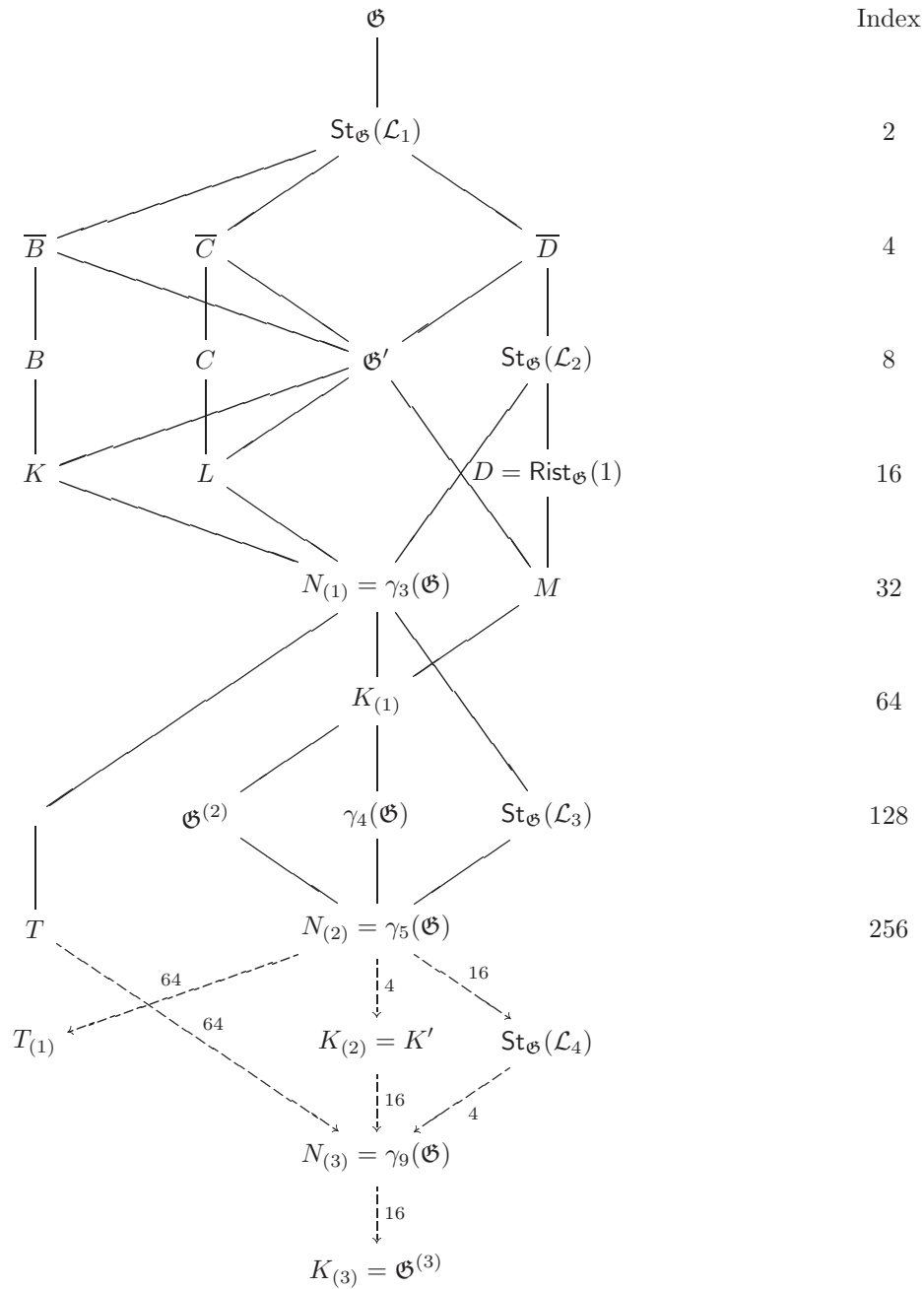


TABLE 1. The top of the lattice of normal subgroups of  $\mathfrak{G}$  below  $\text{St}_{\mathfrak{G}}(\mathcal{L}_1)$ . The index of the inclusions are indicated next to the edges.

- There is for all  $\sigma \in Y^n$  a surjection  $\cdot|_{\sigma} : \text{St}_{\mathfrak{G}}(\mathcal{L}_n) \rightarrow \mathfrak{G}$  given by projection on the factor indexed by  $\sigma$ .

The top of the lattice of normal subgroups of  $\mathfrak{G}$  below  $\text{St}_{\mathfrak{G}}(\mathcal{L}_1)$  is given in Table 1.

COROLLARY 7.10. The closure of  $\mathfrak{G}$  in  $\text{Aut}(T)$  has Hausdorff dimension  $5/8$ .

**7.4.3. The Subgroup  $P$ .** Let  $e$  be the ray  $2^\infty$  and let  $P$  be the corresponding parabolic subgroup. We describe completely its structure as follows:

**THEOREM 7.11.**  $P/P'$  is an infinite elementary 2-group generated by the images of  $c$ ,  $d = (1, b)$  and of all elements of the form  $(1, \dots, 1, (ac)^4)$  in  $\text{Rist}_{\mathfrak{G}}(n)$  for  $n \in \mathbb{N}$ . The following decomposition holds:

$$P = \left( B \times \left( \left( K \times \left( (K \times \dots) \times \langle (ac)^4 \rangle \right) \right) \times \langle b, (ac)^4 \rangle \right) \right) \times \langle c, (ac)^4 \rangle,$$

where each factor (of nesting  $n$ ) in the decomposition acts on the subtree just below some  $e_n$  but not containing  $e_{n+1}$ .

Note that we use the same notation for a subgroup  $B$  or  $K$  acting on a subtree, keeping in mind the identification of a subtree with the original tree. Note also that  $\psi$  is omitted when it would make the notations too heavy.

**PROOF.** Define the following subgroups of  $\mathfrak{G}_n$ :

$$\begin{aligned} H_n &= \langle b, c \rangle^{\mathfrak{G}_n}; & B_n &= \langle b \rangle^{\mathfrak{G}_n}; & K_{(n)} &= \langle (ab)^2 \rangle^{\mathfrak{G}_n}; \\ Q_n &= B_n \cap P_n; & R_n &= K_{(n)} \cap P_n. \end{aligned}$$

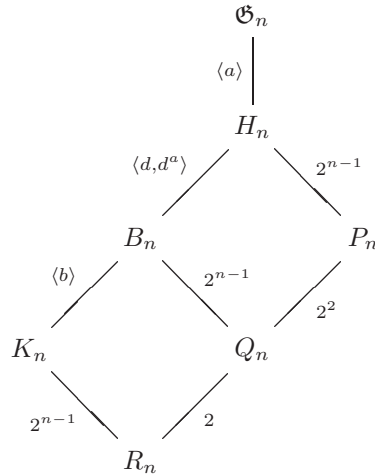
Then the theorem follows from the following proposition. □

**PROPOSITION 7.12.** These subgroups have the following structure:

$$\begin{aligned} P_n &= (B_{n-1} \times Q_{n-1}) \times \langle c, (ac)^4 \rangle; \\ Q_n &= (K_{n-1} \times R_{n-1}) \times \langle b, (ac)^4 \rangle; \\ R_n &= (K_{n-1} \times R_{n-1}) \times \langle (ac)^4 \rangle. \end{aligned}$$

**PROOF.** A priori,  $P_n$ , as a subgroup of  $H_n$ , maps in  $(B_{n-1} \times B_{n-1}) \times \langle (a, d), (d, a) \rangle$ . Restricting to those pairs that fix  $e_n$  gives the result. Similarly,  $Q_n$ , as a subgroup of  $B_n$ , maps in  $(K_{n-1} \times K_{n-1}) \times \langle (a, c), (c, a) \rangle$ , and  $R_n$ , as a subgroup of  $K_n$ , maps in  $(K_{n-1} \times K_{n-1}) \times \langle (ac, ca), (ca, ac) \rangle$ . □

**COROLLARY 7.13.** The group  $\mathfrak{G}_n$  and its subgroups  $H_n, B_n, K_n, P_n, R_n, Q_n$  are arranged in a lattice



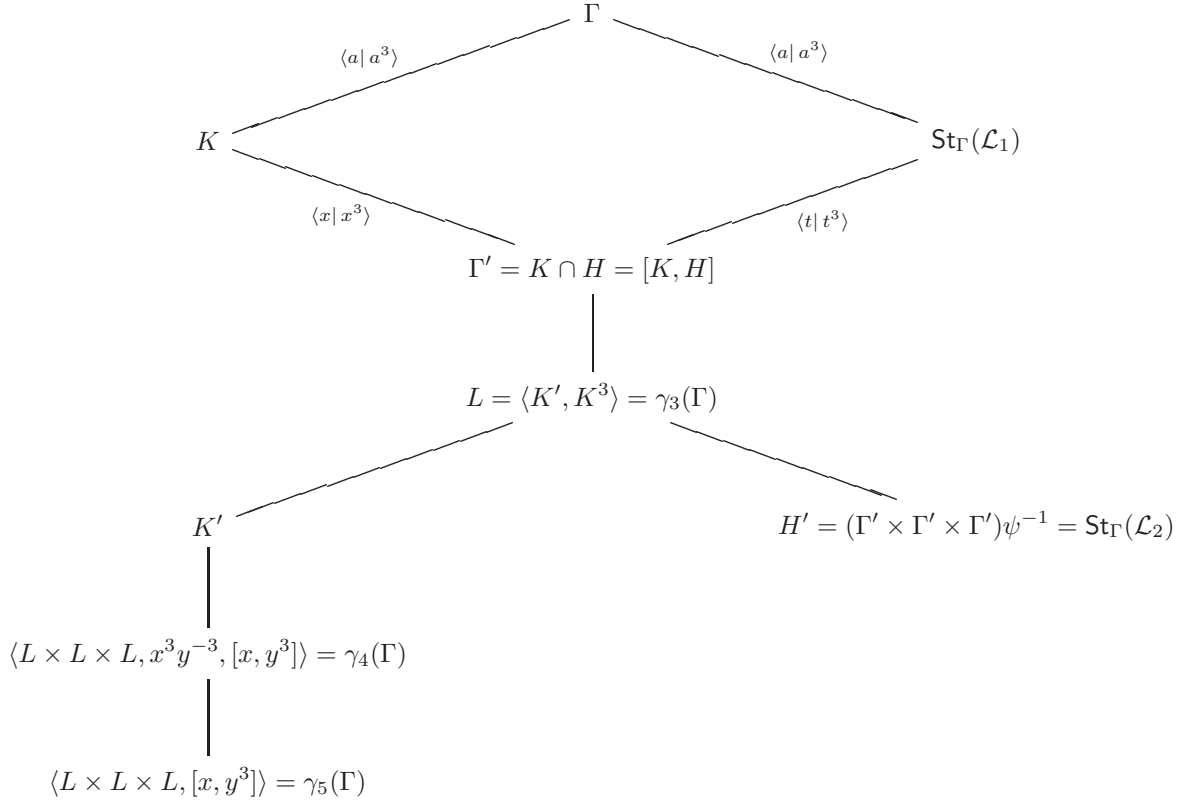
where the quotients or the indices are represented next to the edges.

### 7.5. The structure of $\Gamma$

Recall that  $\Gamma$  is the group acting on the ternary tree, generated by the rooted automorphism  $a = ((1, 2, 3))$  and the directed automorphism  $t$  satisfying  $(t)\psi = (a, 1, t)$ .

Define the elements  $x = at$ ,  $y = ta$  of  $\Gamma$ . Let  $K$  be the subgroup of  $\Gamma$  generated by  $x$  and  $y$ , and let  $L$  be the subgroup of  $K$  generated by  $K'$  and cubes in  $K$ . Write  $H = \text{St}_\Gamma(\mathcal{L}_1)$ .

PROPOSITION 7.14. *We have the following diagram of normal subgroups:*



where the quotients are represented next to the edges; all edges represent normal inclusions of index 3. Furthermore  $L = K \cap (K \times K \times K)\psi^{-1}$ .

PROOF. First we prove  $K$  is normal in  $\Gamma$ , of index 3, by writing  $y^t = x^{-1}y^{-1}$ ,  $y^{a^{-1}} = y^{-1}x^{-1}$ ,  $y^{t^{-1}} = y^a = x$ ; similar relations hold for conjugates of  $x$ . A transversal of  $K$  in  $\Gamma$  is  $\langle a \rangle$ . All subgroups in the diagram are then normal.

Since  $[a, t] = y^{-1}x = t^a t^{-1}$ , we clearly have  $\Gamma' < K \cap H$ . Now as  $\Gamma' \neq K$  and  $\Gamma' \neq H$  and  $\Gamma'$  has index  $3^2$ , we must have  $\Gamma' = K \cap H$ . Finally  $[a, t] = [x, t]^{t^{-1}}$ , so  $\Gamma' = [K, H]$ .

Next  $x^3 = [a, t][t, a^{-1}][a^{-1}, t^{-1}]$  and similarly for  $y$ , so  $K^3 < \Gamma'$  and  $L < \Gamma'$ . Also,  $[x, y]\psi = (y^{-1}, y^{-1}, x^{-1})$  and  $(x^3)\psi = (y, x, y)$  both belong to  $K \times K \times K$ , while  $[a, t]$  does not; so  $L$  is a proper subgroup of  $\Gamma'$ , of index 3 (since  $K/L$  is the elementary abelian group  $(\mathbb{Z}/3\mathbb{Z})^2$  on  $x$  and  $y$ ).

Consider now  $H'$ . It is in  $\text{St}_\Gamma(\mathcal{L}_2)$  since  $H = \text{St}_\Gamma(\mathcal{L}_1)$ . Also,  $[t, t^a] = y^3[y^{-1}, x]$  and similarly for other conjugates of  $t$ , so  $H' < L$ , and  $[t, t^a]\psi = ([a, t], 1, 1)$ , so  $(H')\psi = \Gamma' \times \Gamma' \times \Gamma'$ . Finally  $H'$  is of index 3 in  $L$  (since  $H/H' = (\mathbb{Z}/3\mathbb{Z})^3$  on  $t, t^a, t^{a^{-1}}$ ), and since  $\text{St}_\Gamma(\mathcal{L}_2)$  is of index  $3^4$  in  $\Gamma$  (with quotient  $\mathbb{Z}/3\mathbb{Z} \wr \mathbb{Z}/3\mathbb{Z}$ ) we have all the claimed equalities.  $\square$

PROPOSITION 7.15.  *$\Gamma$  is a just-infinite fractal group, is regular branch over  $\Gamma'$ , and has the congruence property.*

PROOF.  $\Gamma$  is fractal by Lemma 1.7 and the nature of the map  $\psi$ . By direct computation,  $[\Gamma : \Gamma'] = [\Gamma' : (\Gamma' \times \Gamma' \times \Gamma')\psi^{-1}] = [(\Gamma' \times \Gamma' \times \Gamma')\psi^{-1} : \Gamma''] = 3^2$ , so  $\Gamma$  is branched on  $\Gamma'$ . Then  $\Gamma'' = \gamma_5(\Gamma)$ , as is shown in [Bar00c], so  $\Gamma''$  has finite index and  $\Gamma$  is just-infinite by Theorem 5.2.  $\Gamma' \geq \text{St}_\Gamma(\mathcal{L}_2)$ , so  $\Gamma$  has the congruence property.  $\square$

PROPOSITION 7.16. *Writing  $\langle S \rangle$  for the 3-abelian quotient of  $\langle S \rangle$ , we have exact sequences*

$$\begin{aligned} 1 \rightarrow \Gamma' \times \Gamma' \times \Gamma' \rightarrow (H)\psi \rightarrow \langle t, t^a, t^{a^2} \rangle_{3-ab}, \\ 1 \rightarrow \Gamma' \times \Gamma' \times \Gamma' \rightarrow (\Gamma')\psi \rightarrow \langle [a, t], [a^2, t] \rangle_{3-ab}. \end{aligned}$$

THEOREM 7.17. *The subgroup  $K$  of  $\Gamma$  is torsion-free; thus  $\Gamma$  is virtually torsion-free.*

PROPOSITION 7.18. *The finite quotients  $\Gamma_n = \Gamma / \text{St}_\Gamma(\mathcal{L}_n)$  of  $\Gamma$  have order  $3^{3^{n-1}+1}$  for  $n \geq 2$ , and 3 for  $n = 1$ .*

PROOF. Follows immediately from  $[\Gamma : \Gamma'] = 3^2$  and  $[\Gamma' : (\Gamma' \times \Gamma' \times \Gamma')\psi^{-1}] = 3^2$ .  $\square$

COROLLARY 7.19. *The closure of  $\Gamma$  in  $\text{Aut}(\mathcal{T})$  is isomorphic to the profinite completion  $\widehat{\Gamma}$  and is a pro-3-group. It has Hausdorff dimension  $1/3$ .*

## 7.6. The structure of $\overline{\Gamma}$

Recall that  $\overline{\Gamma}$  is the group acting on the ternary tree, generated by the rooted automorphism  $a = ((1, 2, 3))$  and the directed automorphism  $t$  defined by  $(t)\psi = (a, a, t)$ .

Define the elements  $x = ta^{-1}$ ,  $y = a^{-1}t$  of  $\overline{\Gamma}$ , and let  $K$  be the subgroup of  $\overline{\Gamma}$  generated by  $x$  and  $y$ . Then  $K$  is normal in  $\overline{\Gamma}$ , because  $x^t = y^{-1}x^{-1}$ ,  $x^a = x^{-1}y^{-1}$ ,  $x^{t^{-1}} = x^{a^{-1}} = y$ , and similar relations hold for conjugates of  $y$ . Moreover  $K$  is of index 3 in  $\overline{\Gamma}$ , with transversal  $\langle a \rangle$ . Write  $H = \text{St}_{\overline{\Gamma}}(\mathcal{L}_1)$ .

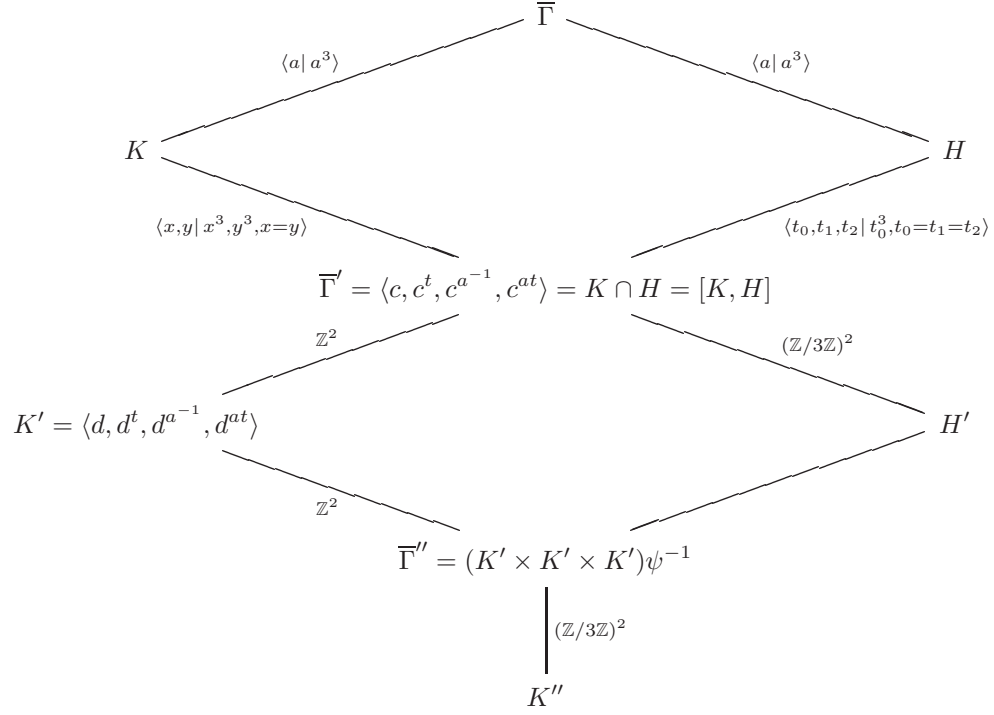
LEMMA 7.1.  *$H$  and  $K$  are normal subgroups of index 3 in  $\overline{\Gamma}$ , and  $\overline{\Gamma}' = \text{St}_K(\mathcal{L}_1) = H \cap K$  is of index 9; furthermore  $(H \cap K)\psi \triangleleft K \times K \times K$ . For any element  $g = (u, v, w) \in (H \cap K)\psi$  one has  $wvu \in H \cap K$ .*

PROOF. First note that  $\text{St}_K(\mathcal{L}_1) = \langle x^3, y^3, xy^{-1}, y^{-1}x \rangle$ , for every word in  $x$  and  $y$  whose number of  $a$ 's is divisible by 3 can be written in these generators. Then compute

$$\begin{aligned} (x^3)\psi &= (y, x^{-1}y^{-1}, x), & (y^3)\psi &= (x^{-1}y^{-1}, x, y), \\ (xy^{-1})\psi &= (1, x^{-1}, x), & (y^{-1}x)\psi &= (y, 1, y^{-1}). \end{aligned}$$

The last assertion is also checked on this computation.  $\square$

PROPOSITION 7.20. *Writing  $c = [a, t] = x^{-1}y^{-1}x^{-1}$  and  $d = [x, y]$ , we have the following diagram of normal subgroups:*



where the quotients are represented next to the edges; additionally,

$$\begin{aligned}
 K/K' &= \langle x, y | [x, y] \rangle \cong \mathbb{Z}^2, \\
 \bar{\Gamma}'/\bar{\Gamma}'' &= \langle c, c^t, c^{a^{-1}}, c^{at} | [c, c^t], \dots \rangle \cong \mathbb{Z}^4, \\
 K'/K'' &= \langle d, d^t, d^{a^{-1}}, d^{at} | [d, d^t], \dots, (d/d^{at})^3, (d^{a^{-1}}/d^t)^3 \rangle \cong \mathbb{Z}^2 \times (\mathbb{Z}/3\mathbb{Z})^2.
 \end{aligned}$$

Writing each subgroup in the generators of the groups above it, we have

$$\begin{aligned}
 K &= \langle x = at^{-1}, y = a^{-1}t \rangle, \\
 H &= \langle t, t_1 = t^a, t_2 = t^{a^{-1}} \rangle, \\
 \bar{\Gamma}' &= \langle b_1 = xy^{-1}, b_2 = y^{-1}x, b_3 = x^3, b_4 = y^3 \rangle = \langle c_1 = tt_1^{-1}, c_2 = tt_1t, c_3 = tt_2^{-1}, c_4 = tt_2t \rangle.
 \end{aligned}$$

COROLLARY 7.21. *The congruence property does not hold for  $\bar{\Gamma}$ ; nor is it regular branch.*

PROPOSITION 7.22.  *$\bar{\Gamma}$  is a fractal group, is weakly branch, and just-nonsolvable.*

PROOF.  $\bar{\Gamma}$  is fractal by Lemma 1.7 and the nature of the map  $\psi$ . The subgroup  $K$  described above has an infinite-index derived subgroup  $K'$  (with infinite cyclic quotient), from which we conclude that  $\bar{\Gamma}$  is not just-infinite; indeed  $K'$  is normal in  $\bar{\Gamma}$  and  $\bar{\Gamma}/K' \cong \mathbb{Z}^2 \times \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$  is infinite.  $\square$

PROPOSITION 7.23. *The subgroup  $K$  of  $\bar{\Gamma}$  is torsion-free; thus  $\bar{\Gamma}$  is virtually torsion-free.*

PROOF. For  $1 \neq g \in K$ , let  $|g|_t$ , the  $t$ -length of  $g$ , denote the minimal number of  $t^{\pm 1}$ 's required to write  $g$  as a word over the alphabet  $\{a^{\pm 1}, t^{\pm 1}\}$ . We will show by induction on  $|g|_t$  that  $g$  is of infinite order.

First, if  $|g|_t = 1$ , i.e.,  $g \in \{x^{\pm 1}, y^{\pm 1}\}$ , we conclude from  $(x^3)\psi = (*, *, x)$  and  $(y^3)\psi = (*, *, y)$  that  $g$  is of infinite order.

Suppose now that  $|g|_t > 1$ , and  $g \in \text{St}_{\overline{\Gamma}}(\mathcal{L}_n) \setminus \text{St}_{\overline{\Gamma}}(\mathcal{L}_{n+1})$ . Then there is some sequence  $\sigma$  of length  $n$  that is fixed by  $g$  and such that  $g|_{\sigma} \notin H$ . By Lemma 7.1,  $g|_{\sigma} \in K$ , so it suffices to show that all  $g \in K \setminus H$  are of infinite order.

Such a  $g$  can be written as  $(u, v, w)\psi^{-1}z$  for some  $(u, v, w) \in (K \cap H)\psi$  and  $z \in \{x^{\pm 1}, y^{\pm 1}\}$ ; by symmetry let us suppose  $z = x$ . Then  $g^3 = (uavawt, vawtua, wtuava)\psi^{-1} = (g_0, g_1, g_2)\psi^{-1}$ , say. For any  $i$ , we have  $|g_i|_t \leq |g|_t$ , because all the components of  $(x)\psi$  and  $(y)\psi$  have  $t$ -length  $\leq 1$ . We distinguish three cases:

- (1)  $g_i = 1$  for some  $i$ . Then consider the image  $\overline{g}_i$  of  $g_i$  in  $\overline{\Gamma}/\overline{\Gamma}'$ . By Lemma 7.1,  $wvu \in G'$ , so  $\overline{g}_i = 1 = \overline{a^2t}$ . But this is a contradiction, because  $\overline{\Gamma}/\overline{\Gamma}'$  is elementary abelian of order 9, generated by the independent images  $\overline{a}$  and  $\overline{t}$ .
- (2)  $0 < |g_i|_t < |g|_t$  for some  $i$ . Then by induction  $g_i$  is of infinite order, so  $g^3$  too, and  $g$  too.
- (3)  $|g_i|_t = |g|_t$  for all  $i$ . We repeat the argument with  $g_i$  substituted for  $g$ . As there are finitely many elements  $h$  with  $|h|_t = |g|_t$ , we will eventually reach either an element of shorter length or an element already considered. In the latter case we obtain a relation of the form  $(g^{3^n})\psi^n = (\dots, g, \dots)$  from which  $g$  is seen to be of infinite order.

□

PROPOSITION 7.24. *The finite quotients  $\overline{\Gamma}_n = \overline{\Gamma}/\text{St}_{\overline{\Gamma}}(\mathcal{L}_n)$  of  $\overline{\Gamma}$  have order  $3^{\frac{1}{4}(3^n+2n+3)}$  for  $n \geq 2$ , and  $3^{\frac{1}{2}(3^n-1)}$  for  $n \leq 2$ .*

PROOF. Define the following family of two-generated finite abelian groups:

$$A_n = \begin{cases} \langle x, y | x^{3^{n/2}}, y^{3^{n/2}}, [x, y] \rangle & \text{if } n \equiv 0[2], \\ \langle x, y | x^{3^{(n+1)/2}}, y^{3^{(n+1)/2}}, (xy^{-1})^{3^{(n-1)/2}}, [x, y] \rangle & \text{if } n \equiv 1[2]. \end{cases}$$

First suppose  $n \geq 2$ ; Consider the diagram of groups described above, and quotient all the groups by  $\text{St}_{\overline{\Gamma}}(\mathcal{L}_n)$ . Then the quotient  $K/K'$  is isomorphic to  $A_n$ , generated by  $x$  and  $y$ , and the quotient  $K'/\overline{\Gamma}''$  is isomorphic to  $A_{n-1}$ , generated by  $[x, y]$  and  $[x, y]^t$ . As  $|A_n| = 3^n$ , the index of  $K'_n$  in  $\overline{\Gamma}_n$  is  $3^{n+1}$  and the index of  $\overline{\Gamma}''_n$  is  $3^{2n}$ . Then as  $\overline{\Gamma}''_n \cong K_{n-1}^3$  and  $|\overline{\Gamma}''_2| = 1$  we deduce by induction that  $|\overline{\Gamma}''_n| = 3^{\frac{1}{4}(3^n-6n+3)}$  and  $|K'_n| = 3^{\frac{1}{4}(3^n-2n-1)}$ , from which  $|\overline{\Gamma}_n| = 3^{2n} + |\overline{\Gamma}''_n| = 3^{\frac{1}{4}(3^n+2n+3)}$  follows.

For  $n \leq 2$  we have  $\overline{\Gamma}_n = \text{Aut}(T)_n = \mathbb{Z}/3 \wr \dots \wr \mathbb{Z}/3$ . □

COROLLARY 7.25. *The closure of  $\overline{\Gamma}$  in  $\text{Aut}(T)$  has Hausdorff dimension  $1/2$ .*

PROPOSITION 7.26. *We have exact sequences*

$$\begin{aligned} 1 \rightarrow K' \times K' \times K' \rightarrow (H)\psi \rightarrow \mathbb{Z}^4 \rtimes \mathbb{Z}/3\mathbb{Z} \rightarrow 1, \\ 1 \rightarrow K' \times K' \times K' \rightarrow (K')\psi \rightarrow \mathbb{Z}^2 \rightarrow 1. \end{aligned}$$

### 7.7. The structure of $\overline{\overline{\Gamma}}$

Recall that  $\overline{\overline{\Gamma}}$  is the group acting on the ternary tree, generated by the rooted automorphism  $a = ((1, 2, 3))$  and the directed automorphism  $t$  satisfying  $(t)\psi = (a, a^{-1}, t)$ . Write  $H = \text{St}_{\overline{\overline{\Gamma}}}(\mathcal{L}_1)$ .

PROPOSITION 7.27. *We have the following diagram of normal subgroups:*

$$\begin{array}{c}
 \overline{\Gamma} \\
 \left| \langle a \mid a^3 \rangle \right. \\
 H = \text{St}_{\overline{\Gamma}}(\mathcal{L}_1) \\
 \left| \langle t \mid t^3 \rangle \right. \\
 \overline{\Gamma}' = [G, H] \\
 \left| [a, t] \right. \\
 \gamma_3(\overline{\Gamma}) = \overline{\Gamma}^3 = \text{St}_{\overline{\Gamma}}(\mathcal{L}_2) \\
 \left| (at)^3 \right. \\
 H' = (\overline{\Gamma}' \times \overline{\Gamma}' \times \overline{\Gamma}')\psi^{-1}
 \end{array}$$

where the quotients are represented next to the arrows; all edges represent normal inclusions of index 3.

PROPOSITION 7.28.  $\overline{\Gamma}$  is a just-infinite fractal group, and is a regular branch group over  $\overline{\Gamma}'$ .

PROPOSITION 7.29.  $\overline{\Gamma}' \geq \text{St}_{\overline{\Gamma}}(\mathcal{L}_2)$ , so  $\overline{\Gamma}$  has the congruence property.

## Central Series, Finiteness of Width and Associated Lie Algebras

In this chapter we study the lower central, lower  $p$ -central, and dimension series of basic examples of branch groups, and describe the associated Lie algebras. This chapter is very much connected to the previous one.

We exhibit two branch groups of finite width:  $\mathfrak{G}$  and  $\Gamma$ , and describe the “Lie graph” of their associated Lie algebras. We show that the Gupta-Sidki 3-group has unbounded width, and its Lie algebra has growth of degree  $n^{\log 3 / \log(1+\sqrt{2})-1}$ ; we also describe its Lie graph.

For all regular branch groups, the corresponding Lie algebras have polynomial growth (usually of non-integral degree), see Theorem 8.9. The technique used below, described in [Bar00c], is an extension of the methods of [BG00a].

We start by recalling the famous construction, due to Wilhelm Magnus [Mag40].

### 8.1. $N$ -series

**DEFINITION 8.1.** *Let  $G$  be a group. An  $N$ -series is series  $\{G_n\}$  of normal subgroups with  $G_1 = G$ ,  $G_{n+1} \leq G_n$  and  $[G_m, G_n] \leq G_{m+n}$  for all  $m, n \geq 1$ . The associated Lie ring is*

$$\mathcal{L}(G) = \bigoplus_{n=1}^{\infty} \mathcal{L}_n,$$

with  $\mathcal{L}_n = G_n/G_{n+1}$  and the bracket operation  $\mathcal{L}_n \otimes \mathcal{L}_m \rightarrow \mathcal{L}_{m+n}$  induced by commutation in  $G$ .

For  $p$  a prime, an  $N_p$ -series is an  $N$ -series  $\{G_n\}$  such that  $\mathcal{U}_p(G_n) \leq G_{pn}$ , and the associated Lie ring is a restricted Lie algebra over  $\mathbb{F}_p$  [Jac41],

$$\mathcal{L}_{\mathbb{F}_p}(G) = \bigoplus_{n=1}^{\infty} \mathcal{L}_n,$$

with the Frobenius operation  $\mathcal{L}_n \rightarrow \mathcal{L}_{pn}$  induced by raising to the power  $p$  in  $G$ . (Recall the definition of  $\mathcal{U}_d(G)$  from Section 7.2.)

The standard example of  $N$ -series is the *lower central series*,  $\{\gamma_n(G)\}_{n=1}^{\infty}$ , given by  $\gamma_1(G) = G$  and

$\gamma_n(G) = [G, \gamma_{n-1}(G)]$ , or the *lower  $p$ -central series* or *Frattini series* given by  $P_1(G) = G$  and  $P_n(G) = [G, P_{n-1}(G)]\mathcal{U}_p(P_{n-1}(G))$ . It differs from the lower central series in that its successive quotients are all elementary  $p$ -groups.

The standard example of  $N_p$ -series is the *dimension series*, also known as the Zassenhaus [Zas40], Jennings [Jen41], Lazard [Laz53] or Brauer series, given by  $G_1 = G$  and  $G_n = [G, G_{n-1}]\mathcal{U}_p(G_{\lceil n/p \rceil})$ , where  $\lceil n/p \rceil$  is the least integer greater than or equal to  $n/p$ . It can alternately be described, by a result of Lazard [Laz53], as

$$G_n = \prod_{i: p^j \geq n} \mathcal{U}_{p^j}(\gamma_i(G)),$$

or as

$$G_n = \{g \in G \mid g - 1 \in \Delta^n\},$$

where  $\Delta$  is the augmentation (or fundamental) ideal of the group algebra  $\mathbb{F}_p G$ . Note that this last definition extends to characteristic 0, giving a graded Lie algebra  $\mathcal{L}_{\mathbb{Q}}(G)$  over  $\mathbb{Q}$ . In that case, the subgroup  $G_n$  is the isolator of  $\gamma_n(G)$ :

$$G_n = \sqrt{\gamma_n(G)} = \{g \in G \mid \langle g \rangle \cap \gamma_n(G) \neq \{1\}\}.$$

We mention finally for completeness another  $N_p$ -series, the *Lie dimension series*  $L_n(G)$  defined by

$$L_n(G) = \{g \in G \mid g - 1 \in \Delta^{(n)}\},$$

where  $\Delta^{(n)}$  is the  $n$ -th Lie power of  $\Delta < \mathbb{k}G$ , given by  $\Delta^{(1)} = \Delta$  and  $\Delta^{(n)} = [\Delta^{(n-1)}, \Delta] = \{xy - yx \mid x \in \Delta^{(n-1)}, y \in \Delta\}$ . It is then known [PS75] that

$$L_n(G) = \prod_{(i-1) \cdot p^j \geq n} \mathfrak{U}_{p^j}(\gamma_i(G))$$

if  $\mathbb{k}$  is of characteristic  $p$ , and

$$L_n(G) = \sqrt{\gamma_n(G)} \cap [G, G]$$

if  $\mathbb{k}$  is of characteristic 0.

**DEFINITION 8.2.** *An  $N$ -series  $\{G_n\}$  has finite width if there is a uniform constant  $W$  such that  $l_n := \text{rank } G_n/G_{n+1} \leq W$  holds for all  $n$ , where  $\text{rank } A$  is the minimal number of generators of the abelian group  $A$ . A group has finite width if its lower central series has finite width — this definition comes from [KLP97].*

The following result is well-known, and shows that sometimes the Lie ring  $\mathcal{L}(G)$  is actually a Lie algebra over  $\mathbb{F}_p$ .

**LEMMA 8.3.** *Let  $G$  be a group generated by a set  $S$ . Let  $\mathcal{L}(G)$  be the Lie ring associated to the lower central series.*

- (1) *If  $S$  is finite, then  $\mathcal{L}_n$  is a finite-rank  $\mathbb{Z}$ -module for all  $n$ .*
- (2) *If there is a prime  $p$  such that all generators  $s \in S$  have order  $p$ , then the Lie algebra associated to the lower  $p$ -central series coincides with  $\mathcal{L}$ . As a consequence,  $\mathcal{L}_n$  is a vector space over  $\mathbb{F}_p$  for all  $n$ .*

We return to the lower  $p$ -central series of  $G$ . Consider the graded algebra

$$\overline{\mathbb{F}_p G} = \bigoplus_{n \in \mathbb{N}} \Delta^n / \Delta^{n+1}.$$

A fundamental result connecting  $\mathcal{L}_{\mathbb{F}_p}(G)$  and  $\overline{\mathbb{F}_p G}$  is the

**PROPOSITION 8.4** (Quillen [Qui68]).  *$\overline{\mathbb{F}_p G}$  is the enveloping  $p$ -algebra of  $\mathcal{L}_{\mathbb{F}_p}(G)$ .*

The Poincaré-Birkhoff-Witt Theorem then gives a basis of  $\overline{\mathbb{F}_p G}$  consisting of monomials over a basis of  $\mathcal{L}_{\mathbb{F}_p}(G)$ , with exponents at most  $p-1$ . As a consequence, we have the

**PROPOSITION 8.5** (Jennings [Jen41]). *Let  $G$  be a group, and let  $\sum_{n \geq 1} l_n \hbar^n$  be the Hilbert-Poincaré series of  $\mathcal{L}_{\mathbb{F}_p}(G)$ . Then*

$$\text{growth}(\overline{\mathbb{F}_p G}) = \prod_{n=1}^{\infty} \left( \frac{1 - \hbar^{pn}}{1 - \hbar^n} \right)^{l_n}.$$

As a consequence, we have the following proposition, firstly observed by Berezniĭ (for a proof see [Pet99] and [BG00a]):

**PROPOSITION 8.6.** *Let  $G$  be a group and expand the power series  $\text{growth}(\mathcal{L}_{\mathbb{F}_p}(G)) = \sum_{n \geq 1} l_n \hbar^n$  and  $\text{growth}(\overline{\mathbb{F}_p G}) = \sum_{n \geq 0} f_n \hbar^n$ . Then*

(1)  $\{f_n\}$  grows exponentially if and only if  $\{l_n\}$  does, and we have

$$\limsup_{n \rightarrow \infty} \frac{\ln l_n}{n} = \limsup_{n \rightarrow \infty} \frac{\ln f_n}{n}.$$

(2) If  $l_n \sim n^d$ , then  $f_n \sim e^{n^{(d+1)/(d+2)}}$ .

Finally, we recall a connection between the growth of  $G$  and that of  $\overline{\mathbb{F}_p G}$ :

PROPOSITION 8.7 ([BG00a], Lemma 2.5). *Let  $G$  be a group generated by a finite set  $S$ . Then*

$$\frac{\text{growth}(G)}{1 - \hbar} \geq \frac{\text{growth}(\overline{\mathbb{F}_p G})}{1 - \hbar},$$

the inequality being valid coefficient-wise.

The following result exhibits a “gap in the spectrum” of growth, for residually- $p$  groups:

COROLLARY 8.8 ([Gri89, BG00a]). *Let  $G$  be a residually- $p$  group for some prime  $p$ . Then the growth of  $G$  is either polynomial, in case  $G$  is virtually nilpotent, or is at least  $e^{\sqrt{n}}$ .*

## 8.2. Lie algebras of branch groups

Our main purpose, in this section, is to illustrate the following result by examples:

THEOREM 8.9. *Let  $G$  be a finitely generated regular branch group and  $\mathcal{L}G$  the Lie ring associated to its lower central series. Then  $\text{growth}(\mathcal{L}G)$  has polynomial growth (not necessarily of integer degree).*

Its proof relies on branch portraits, introduced in Section 1.4.

SKETCH OF PROOF. Let  $G$  be regular branch over  $K$ . The Lie algebra of  $G$  is isomorphic to that of  $\overline{G}$ , so we consider the latter. For each  $n \in \mathbb{N}$ , consider the set of branch portraits associated to  $\gamma_n(G)$ . Since  $K$  has finite index, it suffices to consider only the  $\gamma_n(G) \leq K$ . Let  $n(\ell)$  be minimal such that the branch portraits of  $\gamma_{n(\ell)}(G)$  are trivial in their first  $\ell$  levels. It suffices to show that this function is exponential. Consider the portraits of  $\gamma_n(G)$ , for  $n(\ell) \leq n \leq n(\ell+1)$ . For  $n$  close to  $n(\ell)$ , there will be all portraits that are trivial except in a subtree at level  $\ell$ . Then for larger  $n$  there will be, using commutation with a generator that is nontrivial at the root vertex, portraits trivial except in two subtrees, where they have labels  $P$  and  $P^{-1}$  respectively. As  $n$  becomes larger and larger, the only remaining portraits will be those whose labels in all subtrees at level  $\ell$  are identical. This passage from one level to the next is exponential.  $\square$

We obtain an explicit description of the lower central series in several cases, and show:

- For the first Grigorchuk group  $\mathfrak{G}$ , the Grigorchuk supergroup  $\tilde{\mathfrak{G}}$  and the Fabrykowski-Gupta group  $\Gamma$ , the Lie algebras  $L$  and  $\mathcal{L}_{\mathbb{F}_p}$  have finite width.
- For the Gupta-Sidki group  $\overline{\Gamma}$ , the Lie algebras  $L$  and  $\mathcal{L}_{\mathbb{F}_p}$  have polynomial growth of degree  $d = \log 3 / \log(1 + \sqrt{2}) - 1$ .

The first result obtained in that direction is due to Alexander Rozhkov. He proved in [Roz96] that for the first Grigorchuk group  $\mathfrak{G}$  one has

$$\text{rank } \gamma_n(\mathfrak{G}) / \gamma_{n+1}(\mathfrak{G}) = \begin{cases} 3 & \text{if } n = 1, \\ 2 & \text{if } n = 2^m + 1 + r, \text{ with } 0 \leq r < 2^{m-1}, \\ 1 & \text{if } n = 2^m + 1 + r, \text{ with } 2^{m-1} \leq r < 2^m. \end{cases}$$

However, the Lie algebra structure contained in an  $N$ -series  $\{G_n\}$  is much richer than the series  $\{\text{rank } G_n / G_{n+1}\}$ , and we will give a fuller description of the  $\gamma_n(\mathfrak{G})$  below.

All our examples will satisfy the following conditions:

- (1)  $G$  is finitely generated by a set  $S$ ;

(2) there is a prime  $p$  such that all  $s \in S$  have order  $p$ .

It then follows from Lemma 8.3 that  $\gamma_n(G)/\gamma_{n+1}(G)$  is a finite-dimensional vector space over  $\mathbb{F}_p$ , and therefore that  $\mathcal{L}(G)$  is a Lie algebra over  $\mathbb{F}_p$  that is finite at each dimension. Clearly the same property holds for the restricted algebra  $\mathcal{L}_{\mathbb{F}_p}(G)$ .

We describe such Lie algebras as oriented labelled graphs, in the following notation:

DEFINITION 8.10. Let  $\mathcal{L} = \bigoplus_{n \geq 1} \mathcal{L}_n$  be a graded Lie algebra over  $\mathbb{F}_p$ , and choose a basis  $B_n$  and a scalar product  $\langle | \rangle$  of  $\mathcal{L}_n$  for all  $n \geq 1$ .

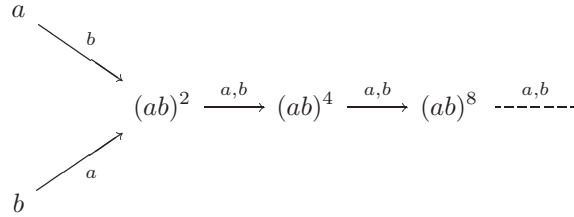
The Lie graph associated to these choices is an abstract graph. Its vertex set is  $\bigcup_{n \geq 1} B_n$ , and each vertex  $x \in B_n$  has a degree,  $n = \deg x$ . Its edges are labelled as  $\alpha x$ , with  $x \in B_1$  and  $\alpha \in \mathbb{F}_p$ , and may only connect a vertex of degree  $n$  to a vertex of degree  $n + 1$ . For all  $x \in B_1$ ,  $y \in B_n$  and  $z \in B_{n+1}$ , there is an edge labelled  $\langle [x, y] | z \rangle x$  from  $y$  to  $z$ .

If  $\mathcal{L}$  is a restricted algebra of  $\mathbb{F}_p$ , there are additional edges from vertices of degree  $n$  to vertices of degree  $pn$ . For all  $x \in B_n$  and  $y \in B_{pn}$ , there is an edge labelled  $\langle x^p | y \rangle \cdot p$  from  $x$  to  $y$ .

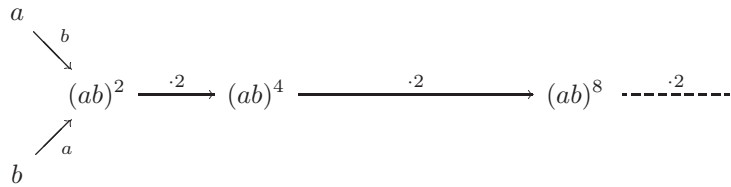
Edges labelled  $0x$  are naturally omitted, and edges labelled  $1x$  are simply written  $x$ .

There is some analogy between this definition and that of a Cayley graph — this topic will be developed in Section 10.3. The generators (in the Cayley sense) are simply chosen to be the  $\text{ad}(x)$  with  $x$  running through  $B_1$ , a basis of  $G/[G, G]$ .

As an example of Lie graph, let  $G$  be the infinite dihedral group  $D_\infty = \langle a, b | a^2, b^2 \rangle$ . Then  $\gamma_n(G) = \langle (ab)^{2^{n-1}} \rangle$  for all  $n \geq 2$ , and its Lie ring is again a Lie algebra over  $\mathbb{F}_2$ , with Lie graph



Note that the lower 2-central series of  $G$  is different: we have  $G_{2^n} = G_{2^{n+1}} = \dots = G_{2^{n+1}-1} = \gamma_{n+1}(G)$ , so the Lie graph of  $\mathcal{L}_{\mathbb{F}_2}(G)$  is



We shall also need the following notation: let  $G$  be a regular branch group over  $K$ , embedded in  $G \wr (\mathbb{Z}/m\mathbb{Z})$ . For all  $i \in \mathbb{N}$  and all  $g \in G$  define the maps

$$i(g) = \text{ad}((1, \dots, m))^i(g, 1, \dots, 1) = \left( g^{\binom{i}{0}}, g^{-\binom{i}{1}}, \dots, g^{(-1)^{m-1} \binom{i}{m-1}} \right);$$

concretely, for  $m = 2$  one has

$$0(g) = (g, 1), \quad 1(g) = (g, g)$$

and for  $m = 3$  one has

$$0(g) = (g, 1, 1), \quad 1(g) = (g, g^{-1}, 1), \quad 2(g) = (g, g^{-2}, g) \equiv (g, g, g) \pmod{\mathcal{U}_3(G)}.$$

When  $m$  is prime, one clearly has  $i(g) = 0$  for all  $i \geq m$ , and if  $g \in K$  then  $i(g) \in K$  for all  $i \in \{0, 1, \dots, d-1\}$ .

**8.2.1. The group  $\mathfrak{G}$ .** We give an explicit description of the Lie algebra of  $\mathfrak{G}$ , and compute its Hilbert-Poincaré series. These results were obtained in [BG00a].

Set  $x = (ab)^2$ . Then  $\mathfrak{G}$  is branch over  $K = \langle x \rangle^{\mathfrak{G}}$ , and  $K/(K \times K)$  is cyclic of order 4, generated by  $x$ .

Extend the generating set of  $\mathfrak{G}$  to a formal set  $S = \{a, b, c, d, \{ \begin{smallmatrix} b \\ c \end{smallmatrix} \}, \{ \begin{smallmatrix} c \\ d \end{smallmatrix} \}, \{ \begin{smallmatrix} d \\ b \end{smallmatrix} \} \}$ , whose meaning shall be made clear later. Define the transformation  $\sigma$  on words in  $S^*$  by

$$\sigma(a) = a \begin{smallmatrix} b \\ c \end{smallmatrix}, \quad \sigma(b) = d, \quad \sigma(c) = b, \quad \sigma(d) = c,$$

extended to subsets by  $\sigma \{ \begin{smallmatrix} x \\ y \end{smallmatrix} \} = \{ \begin{smallmatrix} \sigma x \\ \sigma y \end{smallmatrix} \}$ . Note that for any fixed  $g \in G$ , all elements  $h \in \mathbf{St}_{\mathfrak{G}}(1)$  such that  $\psi(h) = (g, *)$  are obtained by picking a letter from each set in  $\sigma(g)$ . This motivates the definition of  $S$ .

**THEOREM 8.11.** *Consider the following Lie graph: its vertices are the symbols  $X(x)$  and  $X(x^2)$ , for words  $X \in \{0, 1\}^*$ . Their degrees are given by*

$$\begin{aligned} \deg X_1 \dots X_n(x) &= 1 + \sum_{i=1}^n X_i 2^{i-1} + 2^n, \\ \deg X_1 \dots X_n(x^2) &= 1 + \sum_{i=1}^n X_i 2^{i-1} + 2^{n+1}. \end{aligned}$$

There are four additional vertices:  $a, b, d$  of degree 1, and  $[a, d]$  of degree 2.

Define the arrows as follows: an arrow labelled  $\{ \begin{smallmatrix} x \\ y \end{smallmatrix} \}$  stands for two arrows, labelled  $x$  and  $y$ , and the arrows labelled  $c$  are there to expose the symmetry of the graph (indeed  $c = bd$  is not in our chosen basis of  $G/[G, G]$ ):

$$\begin{array}{ll} a \xrightarrow{b,c} x & a \xrightarrow{c,d} [a, d] \\ b \xrightarrow{a} x & d \xrightarrow{a} [a, d] \\ x \xrightarrow{a,b,c} x^2 & x \xrightarrow{c,d} 0(x) \\ [a, d] \xrightarrow{b,c} 0(x) & 0* \xrightarrow{a} 1* \\ 1^n(x) \xrightarrow{\sigma^n \{ \begin{smallmatrix} c \\ d \end{smallmatrix} \}} 0^{n+1}(x) & 1^n(x) \xrightarrow{\sigma^n \{ \begin{smallmatrix} b \\ d \end{smallmatrix} \}} 0^n(x^2) \\ 1^n 0* \xrightarrow{\sigma^n \{ \begin{smallmatrix} c \\ d \end{smallmatrix} \}} 0^n 1* & \text{if } n \geq 1. \end{array}$$

Then the resulting graph is the Lie graph of  $\mathcal{L}(\mathfrak{G})$ . A slight modification gives the Lie graph of  $\mathcal{L}_{\mathbb{F}_2}(\mathfrak{G})$ : the degree of  $X_1 \dots X_n(x^2)$  is  $2 \deg X_1 \dots X_n(x)$ ; and the square maps are given by

$$\begin{aligned} X(x) &\xrightarrow{\cdot 2} X(x^2), \\ 1^n(x^2) &\xrightarrow{\cdot 2} 1^{n+1}(x^2). \end{aligned}$$

The subgraph spanned by  $a, t$ , the  $X_1 \dots X_i(x)$  for  $i \leq n-2$  and the  $X_1 \dots X_i(x^2)$  for  $i \leq n-4$  is the Lie graph associated to the finite quotient  $\mathfrak{G}/\mathbf{St}_{\mathfrak{G}}(n)$ .

**PROOF.** The proof proceeds by induction on length of words, or, what amounts to the same, on depth in the lower central series.

First, the assertion is checked “manually” up to degree 3. The details of the computations are the same as in [BG00a].

We claim that for all words  $X, Y$  with  $\deg Y(x) > \deg X(x)$  we have  $Y(x) \in \langle X(x) \rangle^{\mathfrak{G}}$ , and similarly  $Y(x^2) \in \langle X(x^2) \rangle^{\mathfrak{G}}$ . The claim is verified by induction on  $\deg X$ .

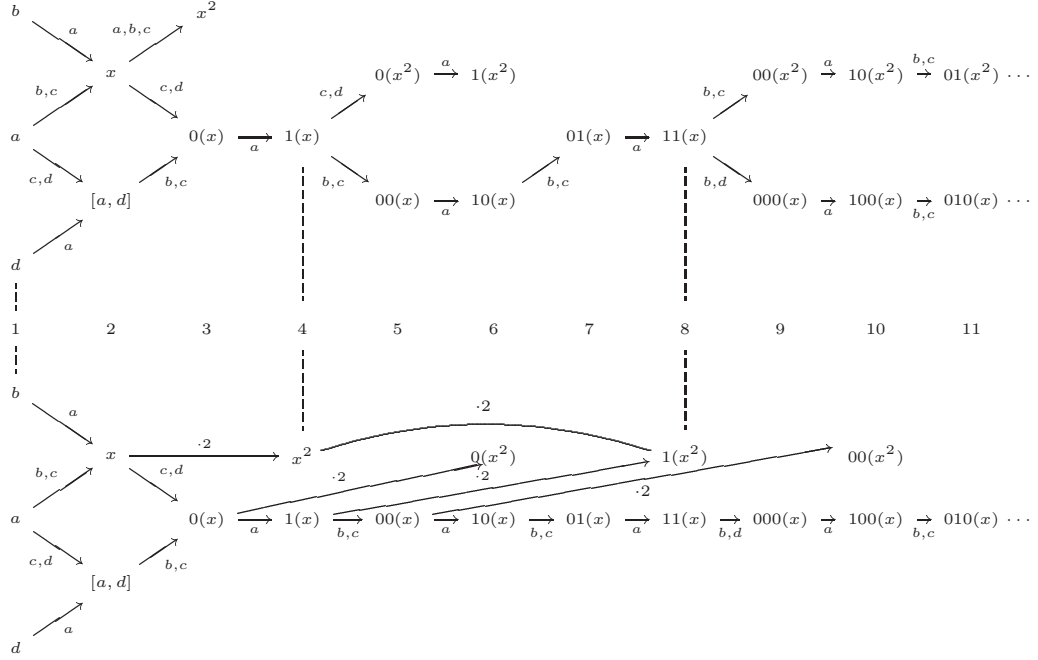


FIGURE 8.1. The beginning of the Lie graphs of  $\mathcal{L}(\mathfrak{G})$  (top) and  $\mathcal{L}_{\mathbb{F}_2}(\mathfrak{G})$  (below).

We then claim that for any non-empty word  $X$ , either  $\text{ad}(a)X(*) = 0$  (if  $X$  starts by “1”) or  $\text{ad}(v)X(*) = 0$  for  $v \in \{b, c, d\}$  (if  $X$  starts by “0”). Again this holds by induction.

We then prove that the arrows are as described above. For instance, for the last one,

$$\text{ad}(\sigma^n \begin{Bmatrix} c \\ d \end{Bmatrix}) 1^n 0 * = \begin{cases} (\text{ad}(\sigma^n \begin{Bmatrix} d \\ b \end{Bmatrix}) 1^{n-1} 0 *, \text{ad}(\begin{Bmatrix} a \\ 1 \end{Bmatrix}) 1^{n-1} 0 *) \\ = 0 \text{ad}(\sigma^{n-1} \begin{Bmatrix} c \\ d \end{Bmatrix}) 1^{n-1} 0 * = 0^n 1 * & \text{if } n \geq 2, \\ (\text{ad}(\begin{Bmatrix} b \\ c \end{Bmatrix}) 0 *, \text{ad}(a) 0 *) = 01 * & \text{if } n = 1. \end{cases}$$

Finally we check that the degrees of all basis elements are as claimed. Fix a word  $X(*)$ , and consider the largest  $n$  such that  $X(*) \in \gamma_n(\mathfrak{G})$ . Thus there is a sequence of  $n - 1$  arrows leading from the left of the Lie graph of  $\mathcal{L}(\mathfrak{G})$  to  $X(*)$ , and no longer sequence, so  $\text{deg } X(*) = n$ .

The modification giving the Lie graph of  $\mathcal{L}_{\mathbb{F}_2}(\mathfrak{G})$  is justified by the fact that in  $\mathcal{L}(\mathfrak{G})$  we always have  $\text{deg } X(x^2) \leq 2 \text{deg } X(x)$ , so the element  $X(x^2)$  appears always last as the image of  $X(x)$  through the square map. The degrees are modified accordingly. Now  $X(x^2) = X1(x^2)$ , and  $2 \text{deg } X1(x) \geq 4 \text{deg } X(x)$ , with equality only when  $X = 1^n$ . This gives an additional square map from  $1^n(x^2)$  to  $1^{n+1}(x^2)$ , and requires no adjustment of the degrees.  $\square$

COROLLARY 8.12. *Define the polynomials*

$$\begin{aligned} Q_2 &= -1 - \hbar, \\ Q_3 &= \hbar + \hbar^2 + \hbar^3, \\ Q_n(\hbar) &= (1 + \hbar)Q_{n-1}(\hbar^2) + \hbar + \hbar^2 \text{ for } n \geq 4. \end{aligned}$$

Then  $Q_n$  is a polynomial of degree  $2^{n-1} - 1$ , and the first  $2^{n-3} - 1$  coefficients of  $Q_n$  and  $Q_{n+1}$  coincide. The term-wise limit  $Q_\infty = \lim_{n \rightarrow \infty} Q_n$  therefore exists.

The Hilbert-Poincaré series of  $\mathcal{L}(\mathfrak{G}/\text{St}_{\mathfrak{G}}(n))$  is  $3\hbar + \hbar^2 + \hbar Q_n$ , and the Hilbert-Poincaré series of  $\mathcal{L}(\mathfrak{G})$  is  $3\hbar + \hbar^2 + \hbar Q_\infty$ .

The Hilbert-Poincaré series of  $\mathcal{L}_{\mathbb{F}_2}(\mathfrak{G})$  is  $3 + \frac{2\hbar + \hbar^2}{1 - \hbar^2}$ .

As a consequence,  $\mathfrak{G}/\text{St}_{\mathfrak{G}}(n)$  is nilpotent of class  $2^{n-1}$ , and  $\mathfrak{G}$  has finite width.

**8.2.2. The group  $\Gamma$ .** We give here an explicit description of the Lie algebra of  $\Gamma$ , and compute its Hilbert-Poincaré series.

**THEOREM 8.13 ([Bar00c]).** In  $\Gamma$  write  $c = [a, t]$  and  $u = [a, c] \equiv 2(at)$ . For words  $X = X_1 \dots X_n$  with  $X_i \in \{0, 1, 2\}$  define symbols  $\overline{X_1 \dots X_n}(c)$  (representing elements of  $\Gamma$ ) by

$$\begin{aligned} \overline{i0}(c) &= i0(c)/i(u), \\ \overline{i2^{m+1}1^n}(c) &= i(\overline{2^{m+1}1^n}(c) \cdot 01^m 0^n(u)^{(-1)^n}), \\ \text{and } \overline{iX}(c) &= i\overline{X}(c) \text{ for all other } X. \end{aligned}$$

Consider the following Lie graph: its vertices are the symbols  $\overline{X}(c)$  and  $X(u)$ . Their degrees are given by

$$\begin{aligned} \deg \overline{X_1 \dots X_n}(c) &= 1 + \sum_{i=1}^n X_i 3^{i-1} + \frac{1}{2}(3^n + 1), \\ \deg X_1 \dots X_n(u) &= 1 + \sum_{i=1}^n X_i 3^{i-1} + (3^n + 1). \end{aligned}$$

There are two additional vertices, labelled  $a$  and  $t$ , of degree 1.

Define the arrows as follows, for all  $n \geq 1$ :

$$\begin{array}{ll} a \xrightarrow{-t} c & t \text{ -----}^a \text{-----} \rightarrow c \\ c \xrightarrow{-t} 0(c) & c \xrightarrow{a} u \\ u \xrightarrow{-t} 1(c) & \overline{2^n}(c) \xrightarrow{-t} \overline{0^{n+1}}(c) \\ 0* \text{ -----}^a \text{-----} \rightarrow 1* & 1* \text{ -----}^a \text{-----} \rightarrow 2* \\ 2^n 0* \xrightarrow{t} 0^n 1* & 2^n 1* \xrightarrow{t} 0^n 2* \end{array}$$

$$\overline{X_1 \dots X_n}(c) \xrightarrow{-(-1)^{\sum X_i} t} (X_1 - 1) \dots (X_n - 1)(u)$$

Then the resulting graph is the Lie graph of  $\mathcal{L}(\Gamma)$ .

The subgraph spanned by  $a, t$ , the  $\overline{X_1 \dots X_i}(c)$  for  $i \leq n - 2$  and the  $X_1 \dots X_i(u)$  for  $i \leq n - 3$  is the Lie graph associated to the finite quotient  $\Gamma/\text{St}_{\Gamma}(n)$ .

**COROLLARY 8.14.** Define the integers  $\alpha_n = \frac{1}{2}(5 \cdot 3^{n-2} + 1)$ , and the polynomials

$$\begin{aligned} Q_2 &= 1, \\ Q_3 &= 1 + 2\hbar + \hbar^2 + \hbar^3 + \hbar^4 + \hbar^5 + \hbar^6, \\ Q_n(\hbar) &= (1 + \hbar + \hbar^2)Q_{n-1}(\hbar^3) + \hbar + \hbar^{\alpha_n - 2} \text{ for } n \geq 4. \end{aligned}$$

Then  $Q_n$  is a polynomial of degree  $\alpha_n - 2$ , and the first  $3^{n-2} + 1$  coefficients of  $Q_n$  and  $Q_{n+1}$  coincide. The term-wise limit  $Q_{\infty} = \lim_{n \rightarrow \infty} Q_n$  therefore exists.



Define the arrows as follows, for all  $n \geq 1$ :

$$\begin{array}{ll}
a \xrightarrow{-t} c & t \dashrightarrow^a c \\
c \xrightarrow{-t} 0(c) & c \xrightarrow{a} u \\
u \xrightarrow{-t} 1(c) & \overline{2^n}(c) \xrightarrow{-t} \overline{0^{n+1}}(c) \\
0* \dashrightarrow^a 1* & 1* \dashrightarrow^a 2* \\
2^n 0* \xrightarrow{t} 0^n 1* & 2^n 1* \xrightarrow{t} 0^n 2* \\
\overline{X_1 \dots X_n}(c) \xrightarrow{-(-1)^{\sum X_i} t} (X_1 - 1) \dots (X_n - 1)(u) & \\
c \xrightarrow{\cdot 3} \overline{00}(c) & \overline{2^n}(u) \xrightarrow{\cdot 3} \overline{2^{n+1}}(u) \\
*0(c) \xrightarrow{\cdot 3} *2(u) \text{ if } 3 \deg *0(c) = \deg *2(u) & 
\end{array}$$

Then the resulting graph is the Lie graph of  $\mathcal{L}_{\mathbb{F}_3}(\Gamma)$ .

The subgraph spanned by  $a, t$ , the  $\overline{X_1 \dots X_i}(c)$  for  $i \leq n-2$  and the  $X_1 \dots X_i(u)$  for  $i \leq n-3$  is the Lie graph of the Lie algebra  $\mathcal{L}_{\mathbb{F}_3}(\Gamma/\text{St}_\Gamma(n))$ .

As a consequence, the dimension series of  $\Gamma/\text{St}_\Gamma(n)$  has length  $3^{n-1}$  (the degree of  $\overline{2^n}(u)$ ), and  $\Gamma$  has finite width.

**8.2.3. The group  $\overline{\Gamma}$ .** We give here an explicit description of the Lie algebra of  $\overline{\Gamma}$ , and compute its Hilbert-Poincaré series.

Introduce the following sequence of integers:

$$\alpha_1 = 1, \quad \alpha_2 = 2, \quad \alpha_n = 2\alpha_{n-1} + \alpha_{n-2} \text{ for } n \geq 3,$$

and  $\beta_n = \sum_{i=1}^n \alpha_i$ . One has

$$\begin{aligned}
\alpha_n &= \frac{1}{2\sqrt{2}} \left( (1 + \sqrt{2})^n - (1 - \sqrt{2})^n \right), \\
\beta_n &= \frac{1}{4} \left( (1 + \sqrt{2})^{n+1} + (1 - \sqrt{2})^{n+1} - 2 \right).
\end{aligned}$$

The first few values are

$n$	1	2	3	4	5	6	7	8
$\alpha_n$	1	2	5	12	29	70	169	398
$\beta_n$	1	3	8	20	49	119	288	686

**THEOREM 8.17 ([Bar00c]).** In  $\overline{\Gamma}$  write  $c = [a, t]$  and  $u = [a, c] = 2(t)$ . Consider the following Lie graph: its vertices are the symbols  $X_1 \dots X_n(x)$  with  $X_i \in \{0, 1, 2\}$  and  $x \in \{c, u\}$ . Their degrees are given by

$$\begin{aligned}
\deg X_1 \dots X_n(c) &= 1 + \sum_{i=1}^n X_i \alpha_i + \alpha_{n+1}, \\
\deg X_1 \dots X_n(u) &= 1 + \sum_{i=1}^n X_i \alpha_i + 2\alpha_{n+1}.
\end{aligned}$$

There are two additional vertices, labelled  $a$  and  $t$ , of degree 1.

Define the arrows as follows:

$$\begin{array}{ccc}
a \xrightarrow{-t} c & & c \xrightarrow{t} 0(c) \\
t \dashrightarrow^a c & & c \dashrightarrow^a u \\
& & u \xrightarrow{t} 1(c) \\
0* \dashrightarrow^a 1* & & 1* \dashrightarrow^a 2* \\
& & 2* \xrightarrow{t} 0\# \text{ whenever } * \xrightarrow{t} \# \\
2(c) \xrightarrow{t} 1(u) & & 1(c) \xrightarrow{-t} 0(u) \\
10* \xrightarrow{-t} 01* & & 11* \xrightarrow{-t} 02* \\
20* \xrightarrow{t} 11* & & 21* \xrightarrow{t} 12*
\end{array}$$

(Note that these last 3 lines can be replaced by the rules  $2* \xrightarrow{t} 1\#$  and  $1* \xrightarrow{-t} 0\#$  for all arrows  $* \xrightarrow{a} \#$ .)

Then the resulting graph is the Lie graph of  $\mathcal{L}(\overline{\Gamma})$ . It is also the Lie graph of  $\mathcal{L}_{\mathbb{F}_3}(\overline{\Gamma})$ , with the only non-trivial cube maps given by

$$2^n(c) \xrightarrow{-3} 2^n 00(c), \quad 2^n(c) \xrightarrow{-3} 2^n 1(u).$$

The subgraph spanned by  $a, t$ , the  $X_1 \dots X_i(c)$  for  $i \leq n-2$  and the  $X_1 \dots X_i(u)$  for  $i \leq n-3$  is the Lie graph associated to the finite quotient  $\overline{\Gamma}/\text{St}_{\overline{\Gamma}}(n)$ .

COROLLARY 8.18. Define the following polynomials:

$$\begin{aligned}
Q_1 &= 0, \\
Q_2 &= \hbar + \hbar^2, \\
Q_3 &= \hbar + \hbar^2 + 2\hbar^3 + \hbar^4 + \hbar^5, \\
Q_n &= (1 + \hbar^{\alpha_n - \alpha_{n-1}})Q_{n-1} + \hbar^{\alpha_{n-1}}(\hbar^{-\alpha_{n-2}} + 1 + \hbar^{\alpha_{n-2}})Q_{n-2} \text{ for } n \geq 3.
\end{aligned}$$

Then  $Q_n$  is a polynomial of degree  $\alpha_n$ , and the polynomials  $Q_n$  and  $Q_{n+1}$  coincide on their first  $2\alpha_{n-1}$  terms. The coefficient-wise limit  $Q_\infty = \lim_{n \rightarrow \infty} Q_n$  therefore exists.

The largest coefficient in  $Q_{2n+1}$  is  $2^n$ , at position  $\frac{1}{2}(\alpha_{2n+1} + 1)$ , so the coefficients of  $Q_\infty$  are unbounded. The integers  $k$  such that  $\hbar^k$  has coefficient 1 in  $Q_\infty$  are precisely the  $\beta_n + 1$ .

The Hilbert-Poincaré series of  $\mathcal{L}(\overline{\Gamma}/\text{St}_{\overline{\Gamma}}(n))$  is  $\hbar + Q_n$ , and the Hilbert-Poincaré series of  $\mathcal{L}(\overline{\Gamma})$  is  $\hbar + Q_\infty$ . The same holds for the Lie algebra  $\mathcal{L}_{\mathbb{F}_3}(\overline{\Gamma}/\text{St}_{\overline{\Gamma}}(n))$  and  $\mathcal{L}_{\mathbb{F}_3}(\overline{\Gamma})$ .

As a consequence,  $\overline{\Gamma}/\text{St}_{\overline{\Gamma}}(n)$  is nilpotent of class  $\alpha_n$ , and  $\overline{\Gamma}$  does not have finite width.

We note as an immediate consequence that

$$[\overline{\Gamma} : \gamma_{\beta_{n+1}}(\overline{\Gamma})] = 3^{\frac{1}{2}(3^n + 1)},$$

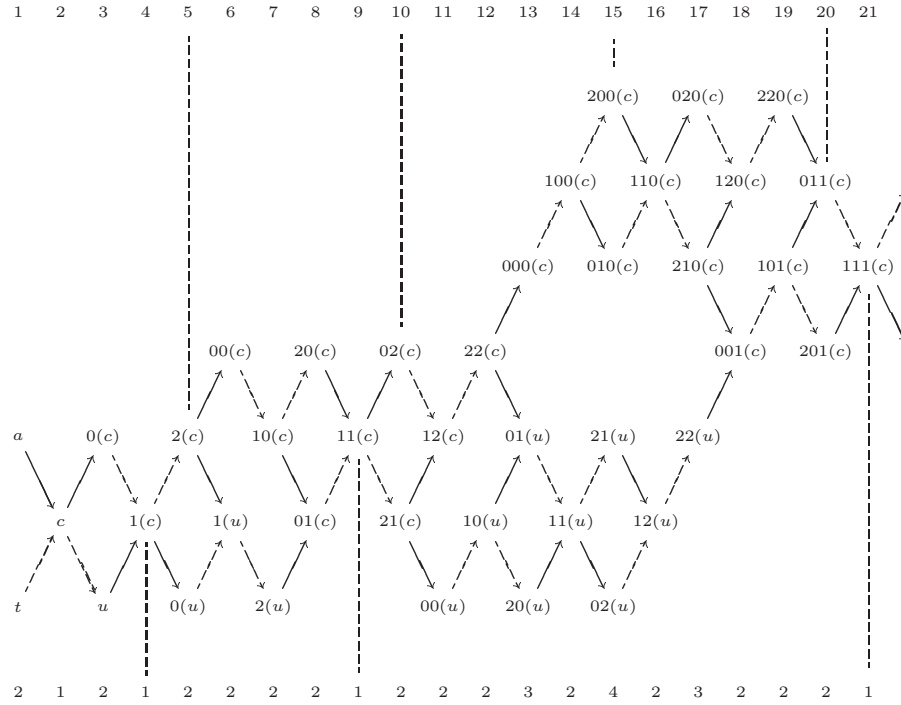


FIGURE 8.3. The beginning of the Lie graph of  $\mathcal{L}(\overline{\Gamma})$ . The generator  $\text{ad}(t)$  is shown by plain/blue arrows, and the generator  $\text{ad}(a)$  is shown by dotted/red arrows.

so that the asymptotic growth of  $l_n = \dim(\gamma_n(\overline{\Gamma}))/\gamma_{n+1}(\overline{\Gamma})$  is polynomial of degree  $d = \log 3 / \log(1 + \sqrt{2}) - 1$ , meaning that  $d$  is minimal such that

$$\limsup_{n \rightarrow \infty} \frac{\sum_{i=1}^n l_i}{\sum_{i=1}^n i^d} < \infty.$$

We then have by Proposition 8.6 the

COROLLARY 8.19. *The growth of  $\overline{\Gamma}$  is at least  $e^{n \frac{\log 3}{\log(1+\sqrt{2})+\log 3}} \cong e^{n \cdot 0.554}$ .*

We may also improve the general result  $\overline{\Gamma}^{(k)} \leq \gamma_{2^k}(\overline{\Gamma})$  to the following

THEOREM 8.20. *For all  $k \in \mathbb{N}$  we have*

$$\overline{\Gamma}^{(k)} \leq \gamma_{\alpha_{k+1}}(\overline{\Gamma}).$$

### 8.3. Subgroup growth

For a finitely generated group  $G$ , its *subgroup growth function* is  $a_n(G) = |\{H \leq G \mid [G : H] = n\}|$ , and its *normal subgroup growth* is the function  $b_n(G) = |\{N \triangleleft G \mid [G : N] = n\}|$ . Building on their earlier papers [Seg86a, Seg86b, GSS88, MS90, LM91] Alex Lubotzky, Avinoam Mann and Dan Segal obtained in [LMS93] a characterization of finitely generated groups with polynomial subgroup growth. Namely, the finitely generated groups of polynomial subgroup growth are precisely the virtually solvable groups of finite rank. For a well written survey refer to [Lub95b].

Since  $a_n$  and  $b_n$  only count finite-index subgroups, it is especially interesting to estimate the subgroup growth of just-infinite groups, and branch groups appear naturally in this context. A recent result lends support to that view:

**THEOREM 8.21** ([**Seg01**]). *Let  $f : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$  be non-decreasing and such that  $\log(f(n))/\log(n)$  is unbounded as  $n \rightarrow \infty$ . Then there exists a 4-generator branch (spinal) group  $G$  whose subgroup growth is not polynomial, but satisfies*

$$a_n(G) \lesssim n^{f(n)}.$$

In order to prove the above theorem Dan Segal uses the construction described in Subsection 1.6.7 with  $A_i = \mathrm{PSL}(2, p_i)$ ,  $i \in \mathbb{N}$ , as rooted subgroups and the action is the natural doubly transitive action of  $\mathrm{PSL}(2, p_i)$  on a set of  $p_i + 1$  elements. The subgroup growth function can be made slow by a choice of a sequence of primes  $(p_i)_{i \in \mathbb{N}}$  that grows quickly enough. In addition, Dan Segal shows that there exists a continuous range of possible “slow” subgroup growths.

## Representation Theory of Branch Groups

This chapter deals with the representation theory of branch groups and of their finite quotients over vector spaces of finite dimension and over Hilbert spaces. For infinite discrete groups the theory of infinite-dimensional unitary representations is a quite a difficult subject. Fortunately, for branch groups the situation is easier to handle, and we produce several results, all taken from [BG00b], confirming this.

The study of unitary representations of profinite branch groups is of great importance and is motivated from several directions. For just-infinite branch groups it is equivalent to the study of irreducible representations of finite quotients  $G/\text{St}_G(\mathcal{L}_n)$ . The first step in this direction is to consider the quasi-regular representations  $\rho_{G/P_n}$ , where  $P_n$  is the stabilizer of a point of level  $n$ , and this will be done below, again following [BG00b]. Another direction is the study of representations of all 3 types of groups (discrete branch  $p$ -groups, branch pro- $p$ -groups, and their finite quotients) in vector spaces over the finite field  $\mathbb{F}_p$ . This last study was initiated by Donald Passman and Will Temple in [PT96].

The spectral properties of the quasi-regular representations  $\rho_{G/P}$  will be described in Chapter 11. We introduce the following notion:

**DEFINITION 9.1.** *Let  $G$  be a group, and  $\mathbb{k}$  an algebraically closed field. We define  $F_G(n) \in \mathbb{N} \cup \{\infty\}$  as the number of irreducible representations of degree at most  $n$  of  $G$  over  $\mathbb{k}$ . Similarly,  $f_G(n)$  denotes the number of such representations of degree exactly  $n$ .*

Therefore,  $f_G(n)$  is the growth function of the representation ring of  $G$  over  $\mathbb{k}$ , whose degree- $n$  component is generated by  $\mathbb{k}G$ -modules of dimension  $n$ , and whose addition and multiplication are  $\oplus$  and  $\otimes$ .

First, we remark that if  $G$  is finitely generated,  $\mathbb{k}$  is algebraically closed, and  $G$  does not have any  $\text{char}(\mathbb{k})$ -torsion, then  $F_G(n)$  is finite for all  $n$ . This follows from a theorem of Weil (see [Far87]). We assume these conditions are satisfied by  $G$ .

If  $H$  is a finite-index subgroup of  $G$ , we have  $F_G \sim F_H$ , as shown in [PT96].

The first lower bound on  $F(n)$  appears in a paper by Donald Passman and Will Temple [PT96], where it was stated for the Gupta-Sidki  $p$ -groups. We improve slightly the result:

**THEOREM 9.2** ([PT96]). *Let  $G$  be a finitely-generated  $m$ -regular branch group over  $K$ , and consider its representations over any field  $\mathbb{k}$ . Then*

$$F_G \succsim n^{(m-1) \log_{[\psi(K):K^m]} [K:K'] - 1}.$$

The proof given in [PT96] extends easily to all regular branch groups. We note that this result is obtained by considering all possible inductions of degree-1 representations from  $K$  up to  $G$ ; it may well be that the function  $F_G$  grows significantly faster than claimed, and the whole representation theory of  $(K)\psi/K^m$  should be taken into account.

As a consequence, we obtain:

- $F_{\mathfrak{G}} \succsim n^2$ , since  $[K : K'] = 64$  and  $[(K)\psi : K^2] = 4$ ;
- $F_{\Gamma} \succsim n^3$ , since  $[\Gamma' : \Gamma''] = 81$  and  $[(\Gamma')\psi : (\Gamma')^2] = 9$ ;
- $F_{\mathbb{F}} \succsim n^3$ , for the same reason;
- for the Gupta-Sidki  $p$ -groups  $G_p$  of Subsection 1.6.3, the general result  $F_{G_p} \succsim n^{p-2}$ .

Note that since these groups are just-infinite, non-faithful representations in vector spaces must factor through a finite quotient; and since these groups are of intermediate growth, they cannot be linear (by Tits' alternative), so in fact all finite-dimensional representations factor through a finite quotient of  $G$ , which may even be taken to be of the form  $G_n = G/\text{St}_G(\mathcal{L}_n)$  if  $G$  has the congruence subgroup property.

For concrete cases, like  $\mathfrak{G}$  and  $\Gamma$ , we may exhibit some unitary irreducible representations as follows:

**9.0.1. Quasi-regular representations.** The representations we consider here are associated to parabolic subgroups, i.e., stabilizers of an infinite ray in the tree (see Section 7.3).

For  $G$  a group acting on a tree and  $P$  a parabolic subgroup, we let  $\rho_{G/P}$  denote the quasi-regular representation of  $G$ , acting by right-multiplication on the space  $\ell^2(G/P)$ . This representation is infinite-dimensional, and a criterion for irreducibility, due to George Mackey, follows:

DEFINITION 9.3. *The commensurator of a subgroup  $H$  of  $G$  is*

$$\text{comm}_G(H) = \{g \in G \mid H \cap H^g \text{ is of finite index in } H \text{ and } H^g\}.$$

*Equivalently, letting  $H$  act on the left on the right cosets  $\{gH\}$ ,*

$$\text{comm}_G(H) = \{g \in G \mid H \cdot (gH) \text{ and } H \cdot (g^{-1}H) \text{ are finite orbits}\}.$$

THEOREM 9.4 (Mackey [Mac76, BH97]). *Let  $G$  be an infinite group and let  $P$  be any subgroup of  $G$ . Then the quasi-regular representation  $\rho_{G/P}$  is irreducible if and only if  $\text{comm}_G(P) = P$ .*

The following results appear in [BG02]:

THEOREM 9.5. *If  $G$  is weakly branch, then  $\text{comm}_G(P) = P$ , and therefore  $\rho_{G/P}$  is irreducible.*

Note that the quasi-regular representations we consider are good approximants of the regular representation, in the sense that  $\rho_G$  is a subrepresentation of  $\bigotimes_{P \text{ parabolic}} \rho_{G/P}$ . We have a continuum of parabolic subgroups  $P_e = \text{St}_G(e)$ , where  $e$  runs through the boundary of a tree, so we also have a continuum of quasi-regular representations. If  $G$  is countable, there are uncountably many non-equivalent representations, because among the uncountably many  $P_e$  only countably many are conjugate. As a consequence,

THEOREM 9.6. *There are uncountably many non-equivalent representations of the form  $\rho_{G/P}$ , where  $P$  is a parabolic subgroup.*

We now consider the finite-dimensional representations  $\rho_{G/P_n}$ , where  $P_n$  is the stabilizer of the vertex at level  $n$  in the ray defining  $P$ . These are permutational representations on the sets  $G/P_n$  of cardinality  $m_1 \dots m_n$ . The  $\rho_{G/P_n}$  are factors of the representation  $\rho_{G/P}$ . Noting that  $P = \bigcap_{n \geq 0} P_n$ , it follows that

$$\rho_{G/P_n} \Rightarrow \rho_{G/P},$$

in the sense that for any non-trivial  $g \in G$  there is an  $n \in \mathbb{N}$  with  $\rho_{G/P_n}(g) \neq 1$ .

We describe now the decomposition of the finite quasi-regular representations  $\rho_{G/P_n}$ . It turns out that it is closely related to the orbit structure of  $P_n$  on  $G/P_n$ . We state the result for the examples  $\mathfrak{G}, \tilde{\mathfrak{G}}, \Gamma, \bar{\Gamma}, \bar{\bar{\Gamma}}$ :

THEOREM 9.7 ([BG02]).  *$\rho_{\mathfrak{G}/P_n}$  and  $\rho_{\tilde{\mathfrak{G}}/P_n}$  decompose as a direct sum of  $n+1$  irreducible components, one of degree  $2^i$  for each  $i \in \{1, \dots, n-1\}$  and two of degree 1.*

*$\rho_{\Gamma/P_n}, \rho_{\bar{\Gamma}/P_n}$  and  $\rho_{\bar{\bar{\Gamma}}/P_n}$  decompose as a direct sum of  $2n+1$  irreducible components, two of degree  $2^i$  for each  $i \in \{1, \dots, n-1\}$  and three of degree 1.*



## Part 4

# Geometric and Analytic Aspects

## Growth

The notion of growth in finitely generated groups was introduced by Efremovich in [Efr53] and Shvarts in [Šva55] in their study of Riemannian manifolds. The works of John Milnor in the late sixties ([Mil68b, Mil68a]) contributed to the current reinforced interest in the topic. Before we make brief historical remarks on the research made in connection to word growth in finitely generated groups, let us introduce the necessary definitions. We concentrate solely on finitely generated infinite groups.

Let  $S = \{s_1, \dots, s_k\}$  be a non-empty set of symbols. A *weight function* on  $S$  is any function  $\tau : S \rightarrow \mathbb{R}_{>0}$ . Therefore, each symbol in  $S$  is assigned a positive weight. The *weight* of any word over  $S$  is then defined by the extension of  $\tau$  to a homomorphism, still written  $\tau : S^* \rightarrow \mathbb{R}_{\geq 0}$ , defined on the free monoid  $S^*$  of words over  $S$ . Therefore, for any word over  $S$  we have

$$\tau(s_{i_1} s_{i_2} \dots s_{i_\ell}) = \sum_{j=1}^{\ell} \tau(s_{i_j}).$$

Note that the empty word is the only word of weight 0. For any non-negative real number  $n$  there are only finitely many words in  $S^*$  of weight at most  $n$ .

Let  $G$  be an infinite group and  $\rho : S^* \twoheadrightarrow G$  a surjective monoid homomorphism. Therefore,  $G$  is finitely generated and  $\rho(S) = \{\rho(s_1), \dots, \rho(s_k)\}$  generates  $G$  as a monoid. The *weight* of an element  $g$  in  $G$  with respect to the triple  $(S, \tau, \rho)$  is, by definition, the smallest weight of a word  $u$  in  $S^*$  that represents  $g$ , i.e., the smallest weight of a word in  $\rho^{-1}(g)$ . The weight of  $g$  with respect to  $(S, \tau, \rho)$  is denoted by  $\partial_G^{(S, \tau, \rho)}(g)$ .

For  $n$  a non-negative real number, the elements in  $G$  that have weight at most  $n$  with respect to  $(S, \tau, \rho)$  constitute the *ball* of radius  $n$  in  $G$  with respect to  $(S, \tau, \rho)$ , denoted by  $B_G^{(S, \tau, \rho)}(n)$ .

Let  $G$  act transitively on a set  $X$  on the right, and let  $x$  be an element of  $X$ . We define  $B_{x,G}^{(S, \tau, \rho)} = \{x^g \mid g \in B_G^{(S, \tau, \rho)}(n)\}$  to be the *ball* in  $X$  of radius  $n$  with center at  $x$ .

The number of elements in  $B_{x,G}^{(S, \tau, \rho)}(n)$  is finite and we denote it by  $\gamma_{x,G}^{(S, \tau, \rho)}(n)$ . The function  $\gamma_{x,G}^{(S, \tau, \rho)}$ , defined on the non-negative real numbers, is called the *growth function* of  $X$  at  $x$  as a  $G$ -set with respect to  $(S, \tau, \rho)$ .

The equivalence class of  $\gamma_{x,G}^{(S, \tau, \rho)}$  under  $\sim$  (recall the notation from the introduction) is called the *degree of growth* of  $G$  and it does not depend on the (finite) set  $S$ , the weight function  $\tau$  defined on  $S$ , the homomorphism  $\rho$ , nor the choice of  $x$ .

**PROPOSITION 10.1** (Invariance of the growth function). *If  $\gamma_{x,G}^{(S, \tau, \rho)}$  and  $\gamma_{x',G}^{(S', \tau', \rho')}$  are two growth functions of the group  $G$ , then they are equivalent with respect to  $\sim$ .*

When we define a weight function on a group  $G$  we usually pick a finite generating subset of  $G$  closed for inversion and not containing the identity, assign a weight function to those generating elements and extend the weight function to the whole group  $G$  in a natural way, thus blurring the distinction between a word over the generating set and the element in  $G$  represented by that word, and completely avoiding the discussion of  $\rho$ .

A standard way to assign a weight function is to assign the weight 1 to each generator. In that case we use the standard notation and terminology, i.e., we denote the weight of a word  $u$  by  $|u|$

and call it the *length* of  $u$ . In this setting, the length of the group element  $g$  is the distance from  $g$  to the identity in the Cayley graph of the group with respect to the generating set  $S$ .

If we let  $G$  act on itself by right multiplication we see that the growth function of  $G$  just counts the number of elements in the corresponding ball in  $G$ , i.e.  $\gamma_{1,G}^{(S,\tau,\rho)}(n) = |B_G^{(S,\tau,\rho)}(n)|$ . Since the degree of growth is an invariant of the group we are more interested in it than in the actual growth function for a given generating set.

In the next few examples the weight 1 is assigned to the generating elements. If  $G = \mathbb{Z}$  and  $S = \{1, -1\}$  then  $\gamma_G(n) = 2n + 1 \sim n$ . More generally, if  $G$  is the free abelian group of rank  $k$ , we have  $\gamma_G \sim n^k$ . If  $G$  is the free group of rank  $k \geq 2$  with the standard generating set together with the inverses, then  $\gamma_G(n) = \frac{k(2k-1)^n - 1}{k-1} \sim e^n$ . It is clear that the growth functions of groups on  $k$  generators are bounded above by the growth function of the free group on  $k$  generators. Therefore, the exponential degree of growth is the largest possible degree of growth.

For any finitely generated infinite group  $G$ , the following trichotomy exists:  $G$  is of

- *polynomial growth* if  $\gamma_G(n) \lesssim n^d$ , for some  $d \in \mathbb{N}$ ;
- *intermediate growth* if  $n^d \not\lesssim \gamma_G(n) \not\lesssim e^n$ , for all  $d \in \mathbb{N}$ ;
- *exponential growth* if  $e^n \sim \gamma_G(n)$ .

We say  $G$  is of *subexponential growth* if  $\gamma_G(n) \not\lesssim e^n$  and of *superpolynomial growth* if  $n^d \not\lesssim \gamma_G(n)$  for all  $d \in \mathbb{N}$ .

By the results of John Milnor, Joseph Wolf and Brian Hartley (see [Mil68a] and [Wol68]), solvable groups have exponential growth unless they are virtually nilpotent in which case the growth is polynomial. There is a formula giving the degree of polynomial growth in terms of the lower central series of  $G$ , due to Yves Guivarc'h and Hyman Bass [Gui70, Bas72]. Namely, if  $d_j$  represents the torsion-free rank of the  $j$ -th factor in the lower central series of the finitely generated nilpotent group  $G$ , then the degree of growth of  $G$  is polynomial of degree  $\sum j d_j$ .

By the Tits' alternative [Tit72], a finitely generated linear group is either virtually solvable or it contains the free group of rank two. Therefore, the growth of a linear group must be exponential or polynomial. In the same spirit, Ching Chou showed in [Cho80] that the growth of elementary amenable groups must be either polynomial or exponential (recall that the class of elementary amenable groups is the smallest class containing all finite and all abelian groups that is closed under subgroups, homomorphic images, extensions and directed unions). Any non-elementary hyperbolic group must have exponential growth ([GH90]).

A fundamental result of Mikhael Gromov [Gro81] states that every group of polynomial growth is virtually nilpotent. The results of John Milnor, Joseph Wolf, Yves Guivarc'h, Hyman Bass and Mikhael Gromov together imply that a group has a polynomial growth if and only if it is virtually nilpotent, and in this case the growth function is equivalent to  $n^d$  where  $d$  is the integer  $\sum j d_j$ , as stated above.

We recall the definition of amenable group:

**DEFINITION 10.2.** *Let  $G$  be a group acting on a set  $X$ . This action is amenable in the sense of von Neumann [vN29] if there exists a finitely additive measure  $\mu$  on  $X$ , invariant under the action of  $G$ , with  $\mu(X) = 1$ .*

*A group  $G$  is amenable if its action on itself by right-multiplication is amenable.*

The following criterion, due to Følner, can sometimes be used to show that a certain action is amenable.

**THEOREM 10.3.** *Let  $G$  act transitively on a set  $X$  and let  $S$  be a generating set for  $G$ . The action is amenable if and only if, for every positive real number  $\lambda$  there exists a finite set  $F$  such that  $|F\Delta Fs| < \lambda|F|$ , for all  $s \in S$ , where  $\Delta$  denotes the symmetric difference.*

Using the Følner's criterion one can show that groups of subexponential growth are always amenable.

In [Mil68c] John Milnor asked the following question: "Is the function  $\gamma(n)$  necessarily equivalent either to a power of  $n$  or to the exponential function  $e^n$ ?" In other words, Milnor asked if the growth is always polynomial or exponential.

The first examples of groups of intermediate growth were constructed by the second author in [Gri80], and they are known as the first and the second Grigorchuk group. Both examples are 2-groups of intermediate growth and they are both amenable but not elementary. These examples show that the answer to Milnor's question is "no". In the same time, these examples show that there exist amenable but not elementary amenable groups, thus answering a question of Mahlon Day from [Day57].

Other examples of groups of intermediate growth are  $\Gamma$ , see Section 7.5 and the first torsion-free example from [Gri85a], which is based on the first Grigorchuk group  $\mathfrak{G}$ .

It was shown in [Gri84, Gri85a] that the Grigorchuk  $p$ -groups are of intermediate growth and that there are uncountably large chains and antichains of growth functions associated with these examples, which are all branch groups.

An important unanswered question in the theory of groups of intermediate growth is the existence of a group whose degree of growth is  $e^{\sqrt{n}}$ . We remarked before that residually- $p$  groups that have degree of growth strictly below  $e^{\sqrt{n}}$  must be virtually nilpotent, and therefore of polynomial growth (see 8.8). The same conclusion holds for residually nilpotent groups (see [LM91]).

The historical remarks on the research on growth above are biased towards the existence and development of examples of groups of intermediate growth. There is plenty of great research on growth in group theory that is not concerned with this aspect. An excellent review of significant results and a good bibliography can be found in [Har00] and [GH97].

Before we move on to more specific examples, we make an easy observation.

**PROPOSITION 10.4.** *No weakly branch group has polynomial growth, i.e. no weakly branch group is virtually nilpotent.*

### 10.1. Growth of $\mathbf{G}$ groups with finite directed part

We will concentrate on the case of  $\mathbf{G}$  groups with finite directed part  $B$ . Note that all branching indices, except maybe the first one, are bounded above by  $|B|$  since each homomorphic image  $A_{\sigma^r\omega} = (B)_{\omega_r}$  acts transitively on a set of  $m_{r+1}$  elements. Therefore,  $|B| \geq |A_{\sigma^r\omega}| \geq m_{r+1}$ , for all  $r$ . Let us denote, once and for all, the largest branching index by  $M$  and the smallest one by  $m$ .

All the estimates of word growth that we give in this and the following sections are done with respect to the canonical generating set  $S_\omega = (A_\omega \cup B_\omega) - 1$ . As a shorthand, we use  $\gamma_\omega(n)$  instead of  $\gamma_{G_\omega}(n)$ .

In order to express some of the results we use the notions of complete subsequence and  $r$ -homogeneous and  $r$ -factorable sequence (see the remarks before Theorem 6.2). We recall that all sequences in  $\widehat{\Omega}$ , i.e. sequences that define  $\mathbf{G}$  groups (see Definition 2.6), can be factored into finite complete subsequences.

**THEOREM 10.5.** *All  $\mathbf{G}$  groups with finite directed part have subexponential growth.*

The following lemma is a direct generalization of [Gri85a, Lemma 1]. The proof is similar, but adapted to our more general setting.

**LEMMA 10.6 (3/4-Shortening).** *Let  $\bar{\omega}$  be a sequence that starts with a complete sequence of length  $r$ . Then the following inequality holds for every reduced word  $F$  representing an element in  $\text{St}_\omega(\mathcal{L}_r)$ :*

$$|L_r(F)| \leq \frac{3}{4}|F| + M^r,$$

where  $|L_r(F)|$  represents the total length of the words on the level  $r$  of the  $r$ -level decomposition of  $F$ .

PROOF. Define  $\xi_i$  to be the number of  $B$ -letters from  $K_i \setminus (K_{i-1} \cup \dots \cup K_1)$  appearing in the words at the level  $i - 1$ , and  $\nu_i$  to be the number of simple reductions performed to get the words  $F_{j_1 \dots j_i}$  on the level  $i$  from their unreduced versions  $\overline{F}_{j_1 \dots j_i}$ .

A reduced word  $F$  of length  $n$  has at most  $(n + 1)/2$   $B$ -letters. Every  $B$ -letter in  $F$  that is in  $K_1$  contributes one  $B$ -letter and no  $A$ -letters to the unreduced words  $\overline{F}_1, \dots, \overline{F}_{m_1}$ . The  $B$ -letters in  $F$  that are not in  $K_1$ , and there are at most  $(n + 1)/2 - \xi_1$  such letters, contribute one  $B$ -letter and one  $A$ -letter. Finally, the  $\nu_1$  simple reductions reduce the number of letters on level 1 by at least  $\nu_1$ . Therefore,

$$|L_1(F)| \leq 2((n + 1)/2 - \xi_1) + \xi_1 - \nu_1 = n + 1 - \xi_1 - \nu_1.$$

In the same manner, each of the  $\xi_2$   $B$ -letters on level 1 that is in  $K_2 \setminus K_1$  contributes one  $B$ -letter to the unreduced words on level 2. The other  $B$ -letters, and there are at most  $(|L_1(F)| + M)/2 - \xi_2$  of them, contribute at most 2 letters, so, after simple reductions, we have

$$|L_2(F)| \leq n + 1 + M - \xi_1 - \xi_2 - \nu_1 - \nu_2.$$

Proceeding in the same manner, we obtain the estimate

$$|L_r(F)| \leq n + 1 + M + \dots + M^{r-1} - \xi_1 - \xi_2 - \dots - \xi_r - \nu_1 - \nu_2 - \dots - \nu_r. \quad (20)$$

If  $\nu_1 + \nu_2 + \dots + \nu_r \geq n/4$ , the claim of the lemma immediately follows.

Let us therefore consider the case when

$$\nu_1 + \nu_2 + \dots + \nu_r < n/4. \quad (21)$$

For  $i = 0, \dots, r - 1$ , define  $|L_i(F)|^+$  to be the number of  $B$ -letters from  $B \setminus (K_1 \cup \dots \cup K_i)$  appearing in the words at the level  $i$ . Clearly,  $|L_0(F)|^+$  is the number of  $B$ -letters in  $F$  and

$$|L_0(F)|^+ \geq \frac{n-1}{2}.$$

Going from the level 0 to the level 1, each  $B$ -letter contributes one  $B$  letter of the same type. Therefore, the words  $\overline{F}_1, \dots, \overline{F}_{m_1}$  from the first level before the reduction takes place have exactly  $|L_0(F)|^+ - \xi_1$  letters that come from  $B - K_1$ . Since we lose at most  $2\nu_1$  letters due to the simple reductions, we obtain

$$|L_1(F)|^+ \geq \frac{n-1}{2} - \xi_1 - 2\nu_1.$$

Next, we go from level 1 to level 2. There are  $|L_1(F)|^+$   $B$ -letters on level 1 that come from  $B \setminus K_1$ , so there are exactly  $|L_1(F)|^+ - \xi_2$   $B$ -letters from  $B \setminus (K_1 \cup K_2)$  in the words  $\overline{F}_{11}, \dots, \overline{F}_{m_1 m_2}$ , and then we lose at most  $2\nu_2$   $B$ -letters due to the simple reductions. We get

$$|L_2(F)|^+ \geq \frac{n-1}{2} - \xi_1 - \xi_2 - 2\nu_1 - 2\nu_2,$$

and, by proceeding in a similar manner,

$$|L_{r-1}(F)|^+ \geq \frac{n-1}{2} - \xi_1 - \dots - \xi_{r-1} - 2\nu_1 - \dots - 2\nu_{r-1}. \quad (22)$$

Since  $\omega_1 \dots \omega_r$  is complete, we have  $K_r \setminus (K_1 \cup \dots \cup K_{r-1}) = B \setminus (K_1 \cup \dots \cup K_{r-1})$  and  $\xi_r = |L_{r-1}(F)|^+$  so that the inequalities (20), (21) and (22) yield

$$|L_r(F)| \leq \frac{n}{2} + \frac{1}{2} + 1 + M + \dots + M^{r-1} + \nu_1 + \dots + \nu_{r-1} - \nu_r,$$

which implies the claim.  $\square$

Now we can finish the proof of Theorem 10.5 using the approach used by the second author in [Gri85a] (see also [Har00, Theorem VIII.61]). We use the following easy lemma, which follows from the fact every subgroup of index  $L$  has a transversal whose representatives have length at most  $L - 1$  (use a Schreier transversal).

LEMMA 10.7. *Let  $G$  be a group and  $H$  be a subgroup of finite index  $L$  in  $G$ . Let  $\gamma(n)$  denote the growth function of  $G$  with respect to some finite generating set  $S$  and the standard length function on  $S$ , and let  $\beta(n)$  denote the number of words of length at most  $n$  that are in  $H$ , i.e.*

$$\beta(n) = |\{g \mid g \in H, |g| \leq n\}| = |B_G^S(n) \cap H|.$$

Then

$$\gamma(n) \leq L\beta(n + L - 1).$$

Let

$$e_\omega = \lim_{n \rightarrow \infty} \sqrt[n]{\gamma_\omega(n)}$$

denote the *exponential growth rate* of  $G_\omega$ . It is known that this rate is 1 if and only if the group in question has subexponential growth. Therefore, all we need to show is that this rate is 1.

PROOF OF THEOREM 10.5. For any  $\epsilon > 0$  there exists  $n_0$  such that

$$\gamma_\omega(n) \leq (e_\omega + \epsilon)^n,$$

for all  $n \geq n_0$ . If we denote  $C_\omega = \gamma_\omega(n_0)$ , we obtain

$$\gamma_\omega(n) \leq C_\omega(e_\omega + \epsilon)^n,$$

for all  $n$ . Note that  $C_\omega$  depends on  $\epsilon$ .

Let  $\omega$  start with a complete sequence of length  $r$ . Denote

$$\beta_\omega(n) = |\{g \mid g \in \text{St}_\omega(\mathcal{L}_r), |g| \leq n\}|.$$

By Lemma 10.6 and the fact that  $\psi_r$  is an embedding we have

$$\beta_\omega(n) \leq \sum \gamma_{\sigma^r \omega}(n_1) \gamma_{\sigma^r \omega}(n_2) \cdots \gamma_{\sigma^r \omega}(n_s),$$

where  $s = m_1 \cdot m_2 \cdots m_r$ , and the summation is over all tuples  $(n_1, \dots, n_s)$  of non-negative integers with  $n_1 + n_2 + \cdots + n_s \leq \frac{3}{4}n + M^r$ . Let  $L$  be the index of  $\text{St}_\omega(\mathcal{L}_r)$  in  $G_\omega$ . By the previous lemma and the above discussions we have

$$\gamma_\omega(n) \leq L\beta_\omega(n + L - 1) \leq LC_{\sigma^r \omega}^s \sum (e_{\sigma^r \omega} + \epsilon)^{n_1 + \cdots + n_s}.$$

where the summation is over all tuples  $(n_1, \dots, n_s)$  of non-negative integers with  $n_1 + \cdots + n_s \leq \frac{3}{4}(n + L - 1) + M^r$ . The number of such tuples is polynomial  $P(n)$  in  $n$  (depending also on the constants  $L$ ,  $M$  and  $r$ ). Therefore,

$$\gamma_\omega(n) \leq LC_{\sigma^r \omega}^s P(n) (e_{\sigma^r \omega} + \epsilon)^{\frac{3}{4}(n+L-1)+M^r}.$$

Taking the  $n$ -th root on both sides and the limit as  $n$  tends to infinity gives

$$e_\omega \leq (e_{\sigma^r \omega} + \epsilon)^{\frac{3}{4}},$$

and since this inequality holds for all positive  $\epsilon$ , we obtain

$$e_\omega \leq (e_{\sigma^r \omega})^{\frac{3}{4}}.$$

Since the exponential growth rate  $e_{\sigma^t \omega}$  is bounded above by  $|A_{\sigma^t \omega}| + |B_{\sigma^t \omega}| - 1 \leq 2|B| - 1$ , for all  $t > 0$ , it follows that  $e_\omega = 1$  for all  $\omega \in \widehat{\Omega}$ .  $\square$

A general lower bound, tending to  $e^n$  when  $m \rightarrow \infty$ , exists on the word growth, and holds for all  $G$  groups:

THEOREM 10.8. *All  $G$  groups with finite directed part have superpolynomial growth. Moreover, the growth of  $G_\omega$  satisfies*

$$e^{n^\alpha} \lesssim \gamma_\omega(n),$$

where  $\alpha = \frac{\log(m)}{\log(m) - \log \frac{1}{2}}$ .

A proof can be found in [BŠ01] for the more special case that is considered there. Note that  $\alpha > \frac{1}{2}$ , as long as  $m \neq 2$ . Putting the last several results together gives

**THEOREM 10.9.** *All G groups with finite directed part have intermediate growth.*

For the special case of  $\mathfrak{G}$  a slightly better lower bounds exist, due to Yuriĭ Leonov [Leo98b] who obtained  $e^{n^{0.5041}} \preceq \gamma(n)$ , and to the first author [Bar01] who obtained  $e^{n^{0.5157}} \preceq \gamma(n)$ .

## 10.2. Growth of G groups defined by homogeneous sequences

Implicitly, all defining triples  $\omega$  have  $r$ -homogeneous defining sequence  $\bar{\omega}$  for some fixed  $r$ . The following estimate holds:

**THEOREM 10.10** ( $\eta$ -Estimate). *If  $\bar{\omega}$  is an  $r$ -homogeneous sequence, then the growth function of the group  $G_\omega$  satisfies*

$$\gamma_\omega(n) \preceq e^{n^\alpha}$$

where  $\alpha = \frac{\log(M)}{\log(M) - \log(\eta_r)} < 1$  and  $\eta_r$  is the positive root of the polynomial  $x^r + x^{r-1} + x^{r-2} - 2$ .

We mentioned already that the weight assignment is irrelevant as far as the degree of growth is concerned. But, appropriately chosen weight assignments can make calculations easier, and this is precisely how the above estimates are obtained.

Let  $G$  be a group that is generated as a monoid by the set of generators  $S$  that does not contain the identity. A weight function  $\tau$  on  $S$  is called triangular if  $\tau(s_1) + \tau(s_2) \geq \tau(s_3)$  whenever  $s_1, s_2, s_3 \in S$  and  $s_1 s_2 = s_3$ .

In the case of spinal groups, in order to define a triangular weight we must have

$$\tau(a_1) + \tau(a_2) \geq \tau(a_1 a_2) \quad \text{and} \quad \tau(b_1) + \tau(b_2) \geq \tau(b_1 b_2),$$

for all  $a_1, a_2 \in A$  and  $b_1, b_2 \in B$  such that  $a_1 a_2 \neq 1$  and  $b_1 b_2 \neq 1$ .

Every  $g$  in  $G_\omega$  admits a minimal form with respect to a triangular weight  $\tau$

$$[a_0] b_1 a_1 b_2 a_2 \dots a_{k-1} b_k [a_k]$$

where all  $a_i$  are in  $A - 1$ , all  $b_i$  are in  $B - 1$ , and the appearances of  $a_0$  and  $a_k$  are optional.

The following weight assignment generalizes the approach taken in [Bar98] by the first author in order to estimate the growth of  $\mathfrak{G}$  (see also [BŠ01]).

The linear system of equations in the variables  $\tau_0, \dots, \tau_r$ :

$$\begin{cases} \eta_r(\tau_0 + \tau_i) = \tau_0 + \tau_{i-1} & \text{for } i = r, \dots, 2, \\ \eta_r(\tau_0 + \tau_1) = \tau_r. \end{cases} \quad (23)$$

has a solution, up to a constant multiple, given by

$$\begin{cases} \tau_i = \eta_r^i + \eta_r^{r-i} - 1 & \text{for } i = r, \dots, 1, \\ \tau_0 = 1 - \eta_r^r. \end{cases} \quad (24)$$

We also require  $\tau_1 + \tau_2 = \tau_r$  and we get that  $\eta_r$  must be a root of the polynomial  $x^r + x^{r-1} + x^{r-2} - 2$ . We choose  $\eta_r$  to be the root of this polynomial that is between 0 and 1 obtain that the solution (24) of the system (23) satisfies the additional properties

$$0 < \tau_1 < \dots < \tau_r < 1, \quad 0 < \tau_0 < 1, \quad (25)$$

$$\tau_i + \tau_j \geq \tau_k \text{ for all } 1 \leq i, j, k \leq r \text{ with } i \neq j. \quad (26)$$

The index  $r$  in  $\eta_r$  will be omitted without warning.

Now, given  $\omega \in \Omega^{(r)}$ , we define the weight of the generating elements in  $S_\omega$  as follows:  $\tau(a) = \tau_0$ , for  $a$  in  $A - 1$  and  $\tau(b_\omega) = \tau_i$ , where  $i$  is the smallest index with  $(b)_\omega i = 1$ , i.e., the smallest index with  $b \in K_i = \text{Ker}(\omega_i)$ .

Clearly,  $\tau$  is a triangular weight function. The only point worth mentioning is that if  $b$  and  $c$  are two  $B$ -letters of the same weight and  $bc = d \neq 1$  then  $d$  has no greater weight than  $b$  or  $c$  (this holds because  $b_\omega, c_\omega \in K_i$  implies  $d_\omega \in K_i$ ).

For obvious reasons, the weight  $\partial_{G_\omega}^{(S_\omega, \tau, \rho)}(g)$ , for  $g \in G_\omega$ , is denoted by  $\partial^\tau(g)$  and, more often, just by  $\partial(g)$ .

We define the *portrait of an element of  $G_\omega$  of size  $n$*  with respect to the weight  $\tau$  as the portrait with respect to the sequence of profile sets  $B_{G_\omega}^\tau(n), B_{G_{\sigma^\omega}}^\tau(n), \dots$ , which is the sequence of balls of radius  $n$  in the corresponding companion groups. Therefore, in the process of building a portrait only those vertices that would be decorated by an element that has weight larger than  $n$  are decomposed further at least one more level and become interior vertices decorated by vertex permutations.

Just as an easy example, we give the portraits of size 3 and size 2 with respect to the standard word length of the element  $g = abacadacabadac$  in  $\mathfrak{G}$  in Figure 10.1 and Figure 10.2. Other

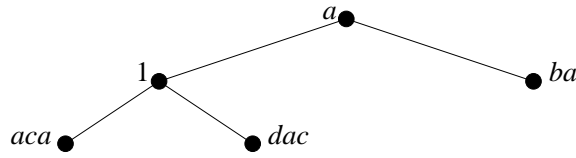


FIGURE 10.1. Portrait of  $g$  of size 3 (same as the portrait of size 4)

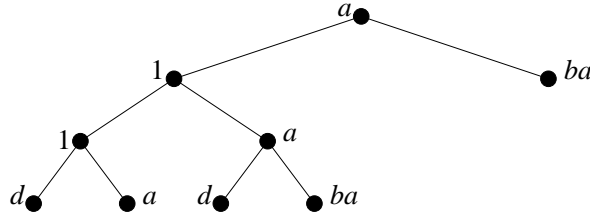


FIGURE 10.2. Portrait of  $g$  of size 2

portraits of the same element are given in Subsection 2.1.4.

The following lemma says that the total sum of the weights of the sections of an element  $f$  is significantly shorter (by a factor less than 1) than the weight of  $f$ . This observation leads to upper bounds on the word growth in  $\mathbf{G}$  groups.

LEMMA 10.11 ( $\eta$ -Shortening). *Let  $f \in G_\omega$  and  $\bar{\omega}$  be an  $r$ -homogeneous sequence. Then*

$$\sum_{i=1}^{m_1} \partial^\tau(f_i) \leq \eta_r(\partial^\tau(f) + \tau_0).$$

PROOF. Let a minimal form of  $f$  be

$$f = [a_0]b_1a_1 \dots b_{k-1}a_{k-1}b_k[a_k].$$

Further let  $f = hg$  where  $g \in A_\omega$  and  $h \in \text{St}_\omega(\mathcal{L}_1)$ . Then  $h$  can be written in the form  $h = [a_0]b_1a_1 \dots a_{k-1}b_k[a_k]g^{-1}$  and rewritten in the form

$$h = b_1^{g_1} \dots b_k^{g_k}, \tag{27}$$

where  $g_i = ([a_0]a_1 \dots a_{i-1})^{-1} \in A_\omega$ . Clearly,  $\partial(f) \geq (k-1)\tau_0 + \sum_{j=1}^k \tau(b_j)$ , which yields

$$\sum_{i=1}^k \eta(\tau_0 + \tau(b_j)) \leq \eta(\partial(f) + \tau_0). \tag{28}$$

Now, observe that if the  $B$ -generator  $b$  is of weight  $\tau_i$  with  $i > 1$  then  $(b^g)\psi$  has as components one  $B$ -generator of weight  $\tau_{i-1}$  and one  $A$ -generator (of weight  $\tau_0$  of course) with the rest of the components trivial. Therefore, such a  $b^g$  (from (27)) contributes at most  $\tau_0 + \tau_{i-1} = \eta(\tau_0 + \tau(b))$  to the sum  $\sum \partial(f_i)$ . On the other hand, if  $b$  is a  $B$ -generator of weight  $\tau_1$  then  $(b^g)\psi$  has as components one  $B$ -generator of weight at most  $\tau_r$ , and the rest of the components are trivial. Such a  $b^g$  contributes at most  $\tau_r = \eta(\tau_0 + \tau(b))$  to the sum  $\sum \partial(f_i)$ . Therefore

$$\sum_{i=1}^{m_1} \partial(f_i) \leq \sum_{j=1}^k \eta(\tau_0 + \tau(b_j)) \quad (29)$$

and the claim of the lemma follows by combining (28) and (29).  $\square$

For a chosen  $C$  we can now construct the portraits of size  $C$  of the elements of  $G_\omega$ . In case  $C$  is large enough, the previous lemma guarantees that these portraits are finite.

LEMMA 10.12. *There exists a positive constant  $C$  such that*

$$L(n) \lesssim n^\alpha,$$

with  $\alpha = \frac{\log(M)}{\log(M) - \log(\eta_r)}$ , where  $L(n)$  is the maximal possible number of leaves in the portrait of size  $C$  of an element of weight at most  $n$ .

We give a proof that does not work exactly, but gives the right idea. One can find a complete proof in [BŠ01].

SKETCHY PROOF. We present how the proof would work if

$$\sum_{i=1}^{m_1} \partial(f_i) \leq \eta \partial(f) \quad (30)$$

holds rather than the inequality in Lemma 10.11.

We choose  $C$  big enough so that the portraits of size  $C$  are finite. Define a function  $L'(n)$  on  $\mathbb{R}_{\geq 0}$  by

$$L'(n) = \begin{cases} 1 & \text{if } n \leq C, \\ n^\alpha & \text{if } n > C. \end{cases}$$

We prove, by induction on  $n$ , that  $L(n) \leq L'(n)$ . If the weight  $n$  of  $g$  is  $\leq C$ , the portrait has 1 leaf and  $L'(n) = 1$ . Otherwise, the portrait of  $g$  is made up of those of  $g_1, \dots, g_{m_1}$ . Let the weights of these  $m_1$  elements be  $n_1, \dots, n_{m_1}$ . By induction, the number of leaves in the portrait of  $g_i$  is at most  $L'(n_i)$ ,  $i = 1, \dots, m_1$ , and the number of leaves in the portrait of  $g$  is, therefore, at most  $\sum_{i=1}^{m_1} L'(n_i)$ .

There are several cases, but let us just consider the case when all of the numbers  $n_1, \dots, n_{m_1}$  are greater than  $C$ . Using Jensen's inequality, the inequality 30, the facts that  $\eta^\alpha = M^{\alpha-1}$ ,  $0 < \alpha < 1$ , and direct calculation, we see that

$$\begin{aligned} \sum_{i=1}^{m_1} L'(n_i) &= \sum_{i=1}^{m_1} n_i^\alpha \leq m_1 \left( \frac{1}{m_1} \sum_{i=1}^{m_1} n_i \right)^\alpha \leq \\ &\leq \frac{1}{(m_1)^{\alpha-1}} (\eta n)^\alpha = \left( \frac{M}{m_1} \right)^{\alpha-1} n^\alpha \leq n^\alpha = L'(n). \end{aligned}$$

$\square$

PROOF OF THEOREM 10.10. The number of labelled, rooted trees with at most  $L(n)$  leaves, whose branching indices do not exceed  $M$ , is  $\lesssim e^{n^\alpha}$ . A tree with  $L(n)$  leaves has  $\sim L(n)$  interior vertices, so there are  $\lesssim e^{n^\alpha}$  ways to decorate the interior vertices. The decoration of the leaves can also be chosen in  $\lesssim e^{n^\alpha}$  ways. Therefore  $\gamma_\omega(n) \lesssim e^{n^\alpha}$ .  $\square$

**10.2.1. Growth in the case of factorable sequences.** An upper bound on the degree of word growth in case of  $r$ -factorable sequences can thus be obtained from Theorem 10.10, since every  $r$ -factorable sequence is  $(2r - 1)$ -homogeneous, but we can do slightly better if we combine Lemma 10.6 with the idea of portrait of an element. We omit the proof because of its similarity to the other proofs in this chapter.

**THEOREM 10.13 (3/4-Estimate).** *If  $\bar{\omega}$  is an  $r$ -factorable sequence, then the growth function of the group  $G_\omega$  satisfies*

$$\gamma_\omega(n) \lesssim e^{n^\alpha}$$

where  $\alpha = \frac{\log(M^r)}{\log(M^r) - \log(3/4)} = \frac{\log(M)}{\log(M) - \log(\sqrt[r]{3/4})} < 1$ .

The 3/4-Estimate was obtained only for the class of Grigorchuk  $p$ -groups defined by  $r$ -homogeneous (not  $r$ -factorable as above) sequences by Roman Muchnik and Igor Pak in [MP01] by different means. The same article contains some sharper considerations in case  $p = 2$ .

We can provide a small improvement in a special case that includes all Grigorchuk 2-groups. Namely, we are going to assume that  $\bar{\omega}$  is an  $r$ -factorable sequence such that each factor contains three homomorphisms whose kernels cover  $B$ .

**LEMMA 10.14 (2/3-Shortening).** *Let  $\omega \in \widehat{\Omega}$  defines a group acting on the rooted binary tree. In addition, let the defining sequence  $\bar{\omega}$  be such that there exist 3 terms  $\omega_k, \omega_\ell$  and  $\omega_s, 1 \leq k < \ell < m \leq r$ , with the property that  $K_k \cup K_\ell \cup K_s = B$ . Then the following inequality holds for every reduced word  $F$  representing an element in  $\text{St}_\omega(\mathcal{L}_r)$ :*

$$|L_r(F)| < \frac{2}{3}|F| + 3 \cdot M^r.$$

We note that the above shortening lemma cannot be improved, unless one starts paying attention to reductions beyond the simple ones. Indeed, in the first Grigorchuk group  $\mathfrak{G}$  the word  $F = (abadac)^{4k}$  has length  $24k$ , while  $|L_3(F)| = 16k = \frac{2}{3}|F|$ . On the other hand  $F = (abadac)^{16} = 1$  in  $\mathfrak{G}$ , so by taking into account other relations the multiplicative constant of Lemma 10.14 could possibly be sharpened.

As a corollary to the shortening lemma above, we obtain:

**THEOREM 10.15 (2/3-Estimate).** *If  $\bar{\omega}$  is an  $r$ -factorable sequence such that each factor contains three letters whose kernels cover  $B$ , then the growth function of the group  $G_\omega$  satisfies*

$$\gamma_\omega(n) \lesssim e^{n^\alpha}$$

where  $\alpha = \frac{\log(M^r)}{\log(M^r) - \log(2/3)} = \frac{\log(M)}{\log(M) - \log(\sqrt[r]{2/3})} < 1$ .

### 10.3. Parabolic space and Schreier graphs

We describe here the aspects of the parabolic subgroups (introduced in Section 7.3) related to growth. The  $G$ -space we study is defined as follows:

**DEFINITION 10.16.** *Let  $G$  be a branch group, and let  $P = \text{St}_G(e)$  be a parabolic subgroup of  $G$ . The associated parabolic space is the  $G$ -set  $G/P$ .*

We consider in this section only finitely-generated, contracting groups. We assume a branch group  $G$ , with fixed generating set  $S$ , has been chosen.

The proof of the following result appears in [BG00b].

**PROPOSITION 10.17.** *Let  $G$ , a group of automorphisms of the regular tree of branching index  $m$ , satisfy the conditions of Proposition 7.6, and suppose it is contracting (see Definition 1.15). Let  $P$  be a parabolic subgroup. Then  $G/P$ , as a  $G$ -set, is of polynomial growth of degree at most  $\log_{1/\lambda}(m)$ , where  $\lambda$  is the contracting constant. If moreover  $G$  is spherically transitive, then  $G/P$ 's asymptotical growth is polynomial of degree  $\log_{1/\lambda'}(m)$ , with  $\lambda'$  the infimum of the  $\lambda$  as above.*

The growth of  $G/P$  is also connected to the growth of the associated Lie algebra (see Chapter 8). The following appears in [Bar00c]:

**THEOREM 10.18.** *Let  $G$  be a branch group, with parabolic space  $G/P$ . Then there exists a constant  $C$  such that*

$$\frac{C \text{growth}(G/P)}{1 - \bar{h}} \geq \frac{\text{growth} \mathcal{L}(G)}{1 - \bar{h}},$$

where  $\text{growth}(X)$  denotes the growth formal power series of  $X$ .

Anticipating, we note that in the above theorem equality holds for  $\mathfrak{S}$ , but does not hold for  $\Gamma$  nor  $\overline{\Gamma}$  (see Corollaries 8.8 and 8.19).

The most convenient way to describe the parabolic space  $G/P$  by giving it a graph structure:

**DEFINITION 10.19.** *Let  $G$  be a group generated by a set  $S$  and  $H$  a subgroup of  $G$ . The Schreier graph  $\mathcal{S}(G, H, S)$  of  $G/H$  is the directed graph on the vertex set  $G/H$ , with for every  $s \in S$  and every  $gH \in G/H$  an edge from  $gH$  to  $sgH$ . The base point of  $\mathcal{S}(G, H, S)$  is the coset  $H$ .*

Note that  $\mathcal{S}(G, 1, S)$  is the Cayley graph of  $G$  relative to  $S$ . It may happen that  $\mathcal{S}(G, P, S)$  have loops and multiple edges even if  $S$  is disjoint from  $H$ . Schreier graphs are  $|S|$ -regular graphs, and any degree-regular graph  $\mathcal{G}$  containing a 1-factor (i.e., a regular subgraph of degree 1; there is always one if  $\mathcal{G}$  has even degree) is a Schreier graph [Lub95a, Theorem 5.4].

For all  $n \in \mathbb{N}$  consider the finite quotient  $G_n = G / \text{St}_G(n)$ , acting on the  $n$ -th level of the tree. We first consider the finite graphs  $\mathcal{G}_n = \mathcal{S}(G, P_n \text{St}_G(n), S) = \mathcal{S}(G_n, P / (P \cap \text{St}_G(n)), S)$ . Due to the fractal (or recursive) nature of branch groups, there are simple local rules producing  $\mathcal{G}_{n+1}$  from  $\mathcal{G}_n$ , the limit of this process being the Schreier graph of  $G/P$ . Before stating a general result, we start by describing these rules for two of our examples:  $\mathfrak{S}$  and  $\Gamma$ .

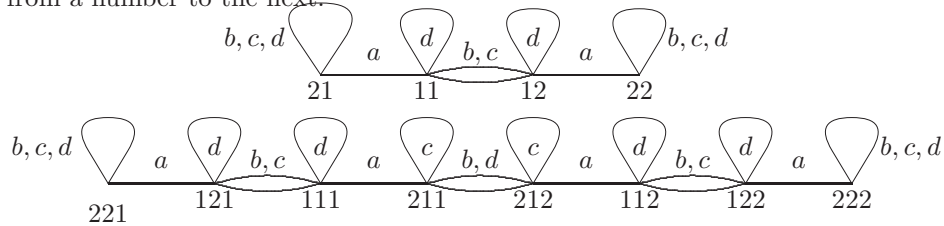
**10.3.1.  $\mathcal{S}(\mathfrak{S}, P, S)$ .** Assume the notation of Section 7.4. The graphs  $\mathcal{G}_n = \mathcal{S}(\mathfrak{S}, P_n, S)$  will have edges labelled by  $S = \{a, b, c, d\}$  (and not oriented, because all  $s \in S$  are involutions) and vertices labelled by  $Y^n$ , where  $Y = \{1, 2\}$ .

First, it is clear that  $\mathcal{G}_0$  is a graph on one vertex, labelled by the empty sequence  $\emptyset$ , and four loops at this vertex, labelled by  $a, b, c, d$ . Next,  $\mathcal{G}_1$  has two vertices, labelled by 1 and 2; an edge labelled  $a$  between them; and three loops at 1 and 2 labelled by  $b, c, d$ .

Now given  $\mathcal{G}_n$ , for some  $n \geq 1$ , perform on it the following transformation: replace the edge-labels  $b$  by  $d$ ,  $d$  by  $c$ ,  $c$  by  $b$ ; replace the vertex-labels  $\sigma$  by  $2\sigma$ ; and replace all edges labelled by  $a$  connecting  $\sigma$  and  $\tau$  by: edges from  $2\sigma$  to  $1\sigma$  and from  $2\tau$  to  $1\tau$ , labelled  $a$ ; two edges from  $1\sigma$  to  $1\tau$  labelled  $b$  and  $c$ ; and loops at  $1\sigma$  and  $1\tau$  labelled  $d$ . We claim the resulting graph is  $\mathcal{G}_{n+1}$ .

To prove the claim, it suffices to check that the letters on the edge-labels act as described on the vertex-labels. If  $b(\sigma) = \tau$ , then  $d(2\sigma) = 2\tau$ , and similarly for  $c$  and  $d$ ; this explains the cyclic permutation of the labels  $b, c, d$ . The other substitutions are verified similarly.

As an illustration, here are  $\mathcal{G}_2$  and  $\mathcal{G}_3$  for  $\mathfrak{S}$ . Note that the sequences in  $Y^*$  that appear correspond to ‘‘Gray enumeration’’, i.e., enumeration of integers in base 2 where only one bit is changed from a number to the next:



**10.3.2.  $\mathcal{S}(\Gamma, P, S)$ .** Assume the notation of Section 7.5. First,  $\mathcal{G}_0$  has one vertex, labelled by the empty sequence  $\emptyset$ , and four loops, labelled  $a, a^{-1}, t, t^{-1}$ . Next,  $\mathcal{G}_1$  has three vertices, labelled

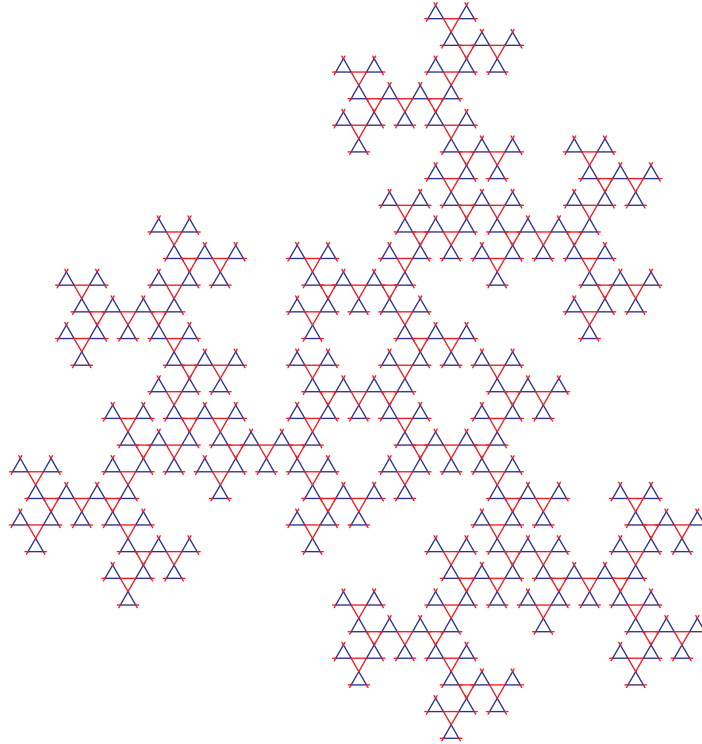


FIGURE 10.3. The Schreier Graph of  $\Gamma_6$ . The edges represent the generators  $s$  and  $a$ .

1, 2, 3, cyclically connected by a triangle labelled  $a, a^{-1}$ , and with two loops at each vertex, labelled  $t, t^{-1}$ . In the pictures only geometrical edges, in pairs  $\{a, a^{-1}\}$  and  $\{t, t^{-1}\}$ , are represented.

Now given  $\mathcal{G}_n$ , for some  $n \geq 1$ , perform on it the following transformation: replace the vertex-labels  $\sigma$  by  $2\sigma$ ; replace all triangles labelled by  $a, a^{-1}$  connecting  $\rho, \sigma, \tau$  by: three triangles labelled by  $a, a^{-1}$  connecting respectively  $1\rho, 2\rho, 3\rho$  and  $1\sigma, 2\sigma, 3\sigma$  and  $1\tau, 2\tau, 3\tau$ ; a triangle labelled by  $t, t^{-1}$  connecting  $1\rho, 1\sigma, 1\tau$ ; and loops labelled by  $t, t^{-1}$  at  $2\rho, 2\sigma, 2\tau$ . We claim the resulting graph is  $\mathcal{G}_{n+1}$ .

As above, it suffices to check that the letters on the edge-labels act as described on the vertex-labels. If  $a(\rho) = \sigma$  and  $t(\rho) = \tau$ , then  $t(1\rho) = 1\sigma$ ,  $t(2\rho) = 2\rho$  and  $t(3\rho) = 3\tau$ . The verification for  $a$  edges is even simpler.

**10.3.3. Substitutional graphs.** The two Schreier graphs presented in the previous subsection are special cases of *substitutional graphs*, which we define below.

Substitutional graphs were introduced in the late 70's to describe growth of multicellular organisms. They bear a strong similarity to  $L$ -systems [RS80], as was noted by Mikhael Gromov [Gro84]. Another notion of graph substitution has been studied by [Pre98], where he had the same convergence preoccupations as us.

Let us make a convention for this section: all graphs  $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$  shall be *labelled*, i.e., endowed with a map  $E(\mathcal{G}) \rightarrow C$  for a fixed set  $C$  of colors, and *pointed*, i.e., shall have a distinguished vertex  $* \in V(\mathcal{G})$ . A *graph embedding*  $\mathcal{G}' \hookrightarrow \mathcal{G}$  is just an injective map  $E(\mathcal{G}') \rightarrow E(\mathcal{G})$  preserving the adjacency operations.

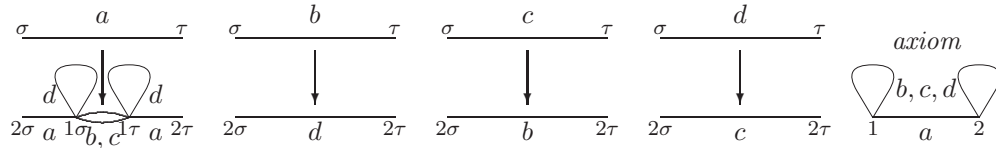
DEFINITION 10.20. A substitutional rule is a tuple  $(U, R_1, \dots, R_n)$ , where  $U$  is a finite  $m$ -regular edge-labelled graph, called the axiom, and each  $R_i$ ,  $i = 1, \dots, n$ , is a rule of the form  $X_i \rightarrow Y_i$ , where  $X_i$  and  $Y_i$  are finite edge-labelled graphs. The graphs  $X_i$  are required to have no common edge. Furthermore, for each  $i = 1, \dots, n$ , there is an inclusion, written  $\iota_i$ , of the vertices of  $X_i$  in the vertices of  $Y_i$ ; the degree of  $\iota_i(x)$  is the same as the degree of  $x$  for all  $x \in V(X_i)$ , and all vertices of  $Y_i$  not in the image of  $\iota_i$  have degree  $m$ .

Given a substitutional rule, one sets  $\mathcal{G}_0 = U$  and constructs iteratively  $\mathcal{G}_{n+1}$  from  $\mathcal{G}_n$  by listing all embeddings of all  $X_i$  in  $\mathcal{G}_n$  (they are disjoint), and replacing them by the corresponding  $Y_i$ . If the base point  $*$  of  $\mathcal{G}_n$  is in a graph  $X_i$ , the base point of  $\mathcal{G}_{n+1}$  will be  $\iota_i(*)$ .

Note that this expansion operation preserves the degree, so  $\mathcal{G}_n$  is a  $m$ -regular finite graph for all  $n$ . We are interested in fixed points of this iterative process.

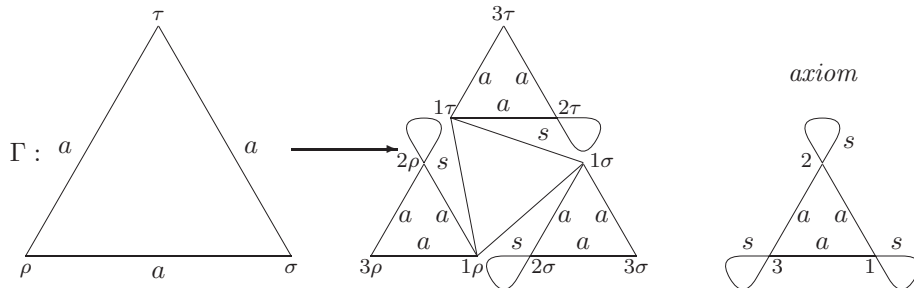
For any  $R \in \mathbb{N}$ , consider the balls  $B_{*,n}(R)$  of radius  $R$  at the base point  $*$  in  $\mathcal{G}_n$ . Since there is only a finite number of rules, there is an infinite sequence  $n_0 < n_1 < \dots$  such that the balls  $B_{*,n_i}(R) \subseteq \mathcal{G}_{n_i}$  are all isomorphic. We consider  $\mathcal{G}$ , a limit graph in the sense of [GZ97] (the limit exists), and call it a *substitutional graph*.

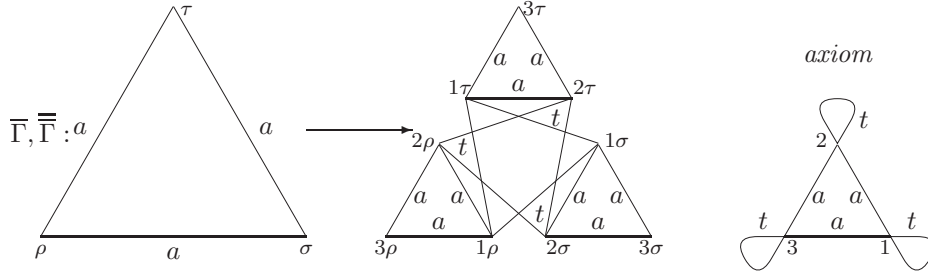
THEOREM 10.21 ([BG00b]). The following four substitutional rules describe the Schreier graph of  $\mathfrak{S}$ :



where the vertex inclusions are given by  $\sigma \mapsto 2\sigma$  and  $\tau \mapsto 2\tau$ . The base point is the vertex  $22\dots$

THEOREM 10.22 ([BG00b]). The substitutional rules producing the Schreier graphs of  $\Gamma$ ,  $\overline{\Gamma}$  and  $\overline{\overline{\Gamma}}$  are given below. Note that the Schreier graphs of  $\overline{\Gamma}$  and  $\overline{\overline{\Gamma}}$  are isomorphic:





where the vertex inclusions are given by  $\rho \mapsto 3\rho$ ,  $\sigma \mapsto 3\sigma$  and  $\tau \mapsto 3\tau$ . The base point is the vertex  $333\dots$ .

By Proposition 10.17, these two limit graphs have asymptotically polynomial growth of degree no higher than  $\log_2(3)$ .

Note that there are maps  $\pi_n : V(\mathcal{G}_{n+1}) \rightarrow V(\mathcal{G}_n)$  that locally (i.e., in each copy of some right-hand rule  $Y_i$ ) are the inverse of the embedding  $\iota_i$ . In case these  $\pi_n$  are graph morphisms one can consider the projective system  $\{\mathcal{G}_n, \pi_n\}$  and its inverse limit  $\widehat{\mathcal{G}} = \varprojlim \mathcal{G}_n$ , which is a profinite graph [RZ00]. We devote our attention to the discrete graph  $\mathcal{G} = \varinjlim \mathcal{G}_n$ .

The growth series of  $\mathcal{G}$  can often be described as an infinite product. We give such an expression for the graph in Figure 10.3.2, making use of the fact that  $\mathcal{G}$  “looks like a tree” (even though it is amenable).

Consider the finite graphs  $\mathcal{G}_n$ ; recall that  $\mathcal{G}_n$  has  $3^n$  vertices. Let  $D_n$  be the diameter of  $\mathcal{G}_n$  (maximal distance between two vertices), and let  $\gamma_n = \sum_{i \in \mathbb{N}} \gamma_n(i) X^i$  be the growth series of  $\mathcal{G}_n$  (here  $\gamma_n(i)$  denotes the number of vertices in  $\mathcal{G}_n$  at distance  $i$  from the base point  $*$ ).

The construction rule for  $\mathcal{G}$  implies that  $\mathcal{G}_{n+1}$  can be constructed as follows: take three copies of  $\mathcal{G}_n$ , and in each of them mark a vertex  $V$  at distance  $D_n$  from  $*$ . At each  $V$  delete the loop labelled  $s$ , and connect the three copies by a triangle labelled  $s$  at the three  $V$ 's. It then follows that  $D_{n+1} = 2D_n + 1$ , and  $\gamma_{n+1} = (1 + 2X^{D_n+1})\gamma_n$ . Using the initial values  $\gamma_0 = 1$  and  $D_0 = 0$ , we obtain by induction

$$D_n = 2^n - 1, \quad \gamma_n = \prod_{i=0}^{n-1} (1 + 2X^{2^i}).$$

We have also shown that the ball of radius  $2^n$  around  $*$  contains  $3^n$  points, so the growth of  $\mathcal{G}$  is at least  $n^{\log_2(3)}$ . But Proposition 10.17 shows that it is also an upper bound, and we conclude:

PROPOSITION 10.23.  $\Gamma$  is an amenable 4-regular graph whose growth function is transcendental, and admits the product decomposition

$$\gamma(X) = \prod_{i \in \mathbb{N}} (1 + 2X^{2^i}).$$

It is planar, and has polynomial growth of degree  $\log_2(3)$ .

Any graph is a metric space when one identifies each edge with a disjoint copy of an interval  $[0, L]$  for some  $L > 0$ . We turn  $\mathcal{G}_n$  in a diameter-1 metric space by giving to each edge in  $\mathcal{G}_n$  the length  $L = \text{diam}(\mathcal{G}_n)^{-1}$ . The family  $\{\mathcal{G}_n\}$  then converges, in the following sense:

Let  $A, B$  be closed subsets of the metric space  $(X, d)$ . For any  $\epsilon$ , let  $A_\epsilon = \{x \in X \mid d(x, A) \leq \epsilon\}$ , and define the Hausdorff distance

$$d_X(A, B) = \inf\{\epsilon \mid A \subseteq B_\epsilon, B \subseteq A_\epsilon\}.$$

This defines a metric on closed subsets of  $X$ . For general metric spaces  $(A, d)$  and  $(B, d)$ , define their Gromov-Hausdorff distance

$$d^{GH}(A, B) = \inf_{X, i, j} d_X(i(A), j(B)),$$

where  $i$  and  $j$  are isometric embeddings of  $A$  and  $B$  in a metric space  $X$ .

We may now rephrase the considerations above as follows: the sequence  $\{\mathcal{G}_n\}$  is convergent in the Gromov-Hausdorff metric. The limit set  $\mathcal{G}_\infty$  is a compact metric space.

The limit spaces are then: for  $\mathfrak{G}$  and  $\tilde{\mathfrak{G}}$ , the limit  $\mathcal{G}_\infty$  is the interval  $[0, 1]$  (in accordance with its linear growth, see Proposition 10.17). The limit spaces for  $\Gamma$ ,  $\bar{\Gamma}$  and  $\overline{\bar{\Gamma}}$  are fractal sets of dimension  $\log_2(3)$ .

## Spectral Properties of Unitary Representations

We describe here some explicit computations of the spectrum of the Schreier graphs defined in Section 10.3. The natural viewpoint is that of spectra of representations, which we define now:

**DEFINITION 11.1.** *Let  $G$  be a group generated by a finite symmetric set  $S$ . The spectrum  $\text{spec}(\tau)$  of a representation  $\tau : G \rightarrow \mathcal{U}(\mathcal{H})$  with respect to the given set of generators is the spectrum of  $\Delta_\tau = \sum_{s \in S} \tau(s)$  seen as an bounded operator on  $\mathcal{H}$ .*

(The condition that  $S$  be symmetric ensures that  $\text{spec}(\tau)$  is a subset of  $\mathbb{R}$ .)

Let  $H$  be a subgroup of  $G$ . Then the spectrum of the ( $\ell^2$ -adjacency operator of the) Schreier graph  $\mathcal{S}(G, H, S)$  is the spectrum of the quasi-regular representation  $\rho_{G/H}$  of  $G$  in  $\ell^2(G/H)$ . This establishes the connection with the previous chapter.

### 11.1. Unitary representations

Let  $G$  act on the rooted tree  $\tau$ . Then  $G$  also acts on the boundary  $\partial\mathcal{T}$  of the tree. Since  $G$  preserves the uniform measure on this boundary, we have a unitary representation  $\pi$  of  $G$  in  $L^2(\partial\mathcal{T}, \nu)$ , where  $\nu$  is the Bernoulli measure; equivalently, we have a representation in  $L^2([0, 1], \text{Lebesgue})$ . Let  $\mathcal{H}_n$  be the subspace of  $L^2(\partial\mathcal{T}, \nu)$  spanned by the characteristic functions  $\chi_\sigma$  of the rays  $e$  starting by  $\sigma$ , for all  $\sigma \in Y^n$ . It is of dimension  $k_n$ , and can equivalently be seen as spanned by the characteristic functions in  $L^2([0, 1], \text{Lebesgue})$  of intervals of the form  $[(i-1)/k_n, i/k_n]$ ,  $1 \leq i \leq k_n$ . These  $\mathcal{H}_n$  are invariant subspaces, and afford representations  $\pi_n = \pi|_{\mathcal{H}_n}$ . As clearly  $\pi_{n-1}$  is a subrepresentation of  $\pi_n$ , we set  $\pi_n^\perp = \pi_n \ominus \pi_{n-1}$ , so that  $\pi = \bigoplus_{n=0}^\infty \pi_n^\perp$ .

Let  $P$  be a parabolic subgroup (see Section 7.3), and set  $P_n = P\text{St}_G(n)$ . Denote by  $\rho_{G/P}$  the quasi-regular representation of  $G$  in  $\ell^2(G/P)$  and by  $\rho_{G/P_n}$  the finite-dimensional representations of  $G$  in  $\ell^2(G/P_n)$ , of degree  $k_n$ . Since  $G$  is level-transitive, the representations  $\pi_n$  and  $\rho_{G/P_n}$  are unitary equivalent.

The  $G$ -spaces  $G/P$  are of polynomial growth when the conditions of Proposition 10.17 are fulfilled, and therefore, according to the criterion of Følner given in Theorem 10.3, they are amenable.

The following result belongs to the common lore:

**PROPOSITION 11.2.** *Let  $H$  be a subgroup of  $G$ . Then the quasi-regular representation  $\rho_{G/H}$  is weakly contained in  $\rho_G$  if and only if  $H$  is amenable.*

(“Weakly contained” is a topological extension of “contained”; it implies for instance inclusion of spectra.)

**THEOREM 11.3.** *Let  $G$  be a group acting on a regular rooted tree, and let  $\pi$ ,  $\pi_n$  and  $\pi_n^\perp$  be as above.*

- (1) *If  $G$  is weakly branch, then  $\rho_{G/P}$  is an irreducible representation of infinite dimension.*
- (2)  *$\pi$  is a reducible representation of infinite dimension whose irreducible components are precisely those of the  $\pi_n^\perp$  (and thus are all finite-dimensional). Moreover*

$$\text{spec}(\pi) = \overline{\bigcup_{n \geq 0} \text{spec}(\pi_n)} = \overline{\bigcup_{n \geq 0} \text{spec}(\pi_n^\perp)}.$$

- (3) The spectrum of  $\rho_{G/P}$  is contained in  $\overline{\cup_{n \geq 0} \text{spec}(\rho_{G/P_n})} = \overline{\cup_{n \geq 0} \text{spec}(\pi_n)}$ , and thus is contained in the spectrum of  $\pi$ . If moreover either  $P$  or  $G/P$  are amenable, these spectra coincide, and if  $P$  is amenable, they are contained in the spectrum of  $\rho_G$ :

$$\text{spec}(\rho_{G/P}) = \text{spec}(\pi) = \overline{\bigcup_{n \geq 0} \text{spec}(\pi_n)} \subseteq \text{spec}(\rho_G).$$

- (4)  $\Delta_\pi$  has a pure-point spectrum, and its spectral radius  $r(\Delta_\pi) = s \in \mathbb{R}$  is an eigenvalue, while the spectral radius  $r(\Delta_{\rho_{G/P}})$  is not an eigenvalue of  $\Delta_{\rho_{G/P}}$ . Thus  $\Delta_{\rho_{G/P}}$  and  $\Delta_\pi$  are different operators having the same spectrum.

**11.1.1. Can one hear a representation?** We end by turning to a question of Mark Kać [Kac66]: “Can one hear the shape of a drum?” This question was answered in the negative in [GWW92], and we here answer by the negative to a related question: “Can one hear a representation?” Indeed  $\rho_{G/P}$  and  $\pi$  have same spectrum (i.e., cannot be distinguished by hearing), but are not equivalent. Furthermore, if  $G$  is a branch group, there are uncountably many nonequivalent representations within  $\{\rho_{G/\text{St}_G(e)} \mid e \in \partial \mathcal{T}\}$ , as is shown in [BG02].

The same question may be asked for graphs: “are there two non-isomorphic graphs with same spectrum?” There are finite examples, obtained through the notion of *Sunada pair* [Lub95a]. Cédric Béguin, Alain Valette and Andrzej Żuk produced the following example in [BVZ97]: let  $\Gamma$  be the integer Heisenberg group (free 2-step nilpotent on 2 generators  $x, y$ ). Then  $\Delta = x + x^{-1} + y + y^{-1}$  has spectrum  $[-2, 2]$ , which is also the spectrum of  $\mathbb{Z}^2$  for an independent generating set. As a consequence, their Cayley graphs have same spectrum, but are not quasi-isometric (they do not have the same growth).

Using the result of Nigel Higson and Gennadi Kasparov [HK97] (giving a partial positive answer to the Baum-Connes conjecture), we may infer the following

PROPOSITION 11.4. *Let  $\Gamma$  be a torsion-free amenable group with finite generating set  $S = S^{-1}$  such that there is a map  $\phi : \Gamma \rightarrow \mathbb{Z}/2\mathbb{Z}$  with  $\phi(S) = \{1\}$ . Then*

$$\text{spec}\left(\sum_{s \in S} \rho(s)\right) = [-|S|, |S|].$$

In particular, there are countably many non-quasi-isometric graphs with the same spectrum, including the graphs of  $\mathbb{Z}^d$ , of free nilpotent groups and of suitable torsion-free groups of intermediate growth (for the first examples, see [Gri85a]).

In contrast to Proposition 11.4 stating that the spectrum of the regular representation is an interval, the spectra of the representation  $\pi$  may be totally disconnected. The first two authors prove in [BG00b] the following

THEOREM 11.5. *For  $\lambda \in \mathbb{R}$ , define*

$$J(\lambda) = \pm \sqrt{\overline{\overline{\overline{\overline{\lambda \pm \sqrt{\lambda \pm \sqrt{\lambda \pm \sqrt{\dots}}}}}}}}}$$

(note the closure operator written in bar notation on the top). Then we have the following spectra:

$\mathbf{G}$	$\text{spec}(\pi)$
$\mathfrak{G}$	$[-2, 0] \cup [2, 4]$
$\check{\mathfrak{G}}$	$[0, 4]$
$\Gamma$	$\{4, 1\} \cup 1 + J(6)$
$\overline{\Gamma}, \overline{\overline{\Gamma}}$	$\{4, -2, 1\} \cup 1 \pm \sqrt{\frac{9}{2} \pm 2J(\frac{45}{16})}$

In particular, the spectrum of the graph in Figure 10.3.2 is the totally disconnected set  $\{4, 1\} \cup 1 + J(6)$ , which is the union of a Cantor set of null Lebesgue measure and a countable collection of isolated points.

### 11.2. Operator recursions

We now describe the computations of the spectra for the examples in Theorem 11.5, which share the property of acting on an  $m$ -regular tree  $\mathcal{T}$ . Let  $\mathcal{H}$  be an infinite-dimensional Hilbert space, and suppose  $\Phi : \mathcal{H} \rightarrow \mathcal{H} \oplus \cdots \oplus \mathcal{H}$  is an isomorphism, where the domain of  $\Phi$  is a sum of  $m \geq 2$  copies of  $\mathcal{H}$ . Let  $S$  be a finite subset of  $\mathcal{U}(\mathcal{H})$ , and suppose that for all  $s \in S$ , if we write  $\Phi^{-1}s\Phi$  as an operator matrix  $(s_{i,j})_{i,j \in \{1, \dots, d\}}$  where the  $s_{i,j}$  are operators in  $\mathcal{H}$ , then  $s_{i,j} \in S \cup \{0, 1\}$ .

This is precisely the setting in which we will compute the spectra of our five example groups: for  $\mathfrak{G}$ , we have  $m = 2$  and  $S = \{a, b, c, d\}$  with

$$\begin{aligned} a &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & b &= \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}, \\ c &= \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, & d &= \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}. \end{aligned}$$

For  $\tilde{\mathfrak{G}}$ , we also have  $m = 2$ , and  $S = \{a, \tilde{b}, \tilde{c}, \tilde{d}\}$  given by

$$b = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, \quad d = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}.$$

For  $\Gamma = \langle a, s \rangle$ ,  $\bar{\Gamma} = \langle a, t \rangle$  and  $\bar{\bar{\Gamma}} = \langle a, r \rangle$ , we have  $m = 3$  and

$$\begin{aligned} a &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, & s &= \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{pmatrix}, \\ t &= \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & t \end{pmatrix}, & r &= \begin{pmatrix} a & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & r \end{pmatrix}. \end{aligned}$$

Each of these operators is unitary. The families  $S = \{a, b, c, d\}, \dots$  generate subgroups  $G(S)$  of  $\mathcal{U}(\mathcal{H})$ . The choice of the isomorphism  $\Phi$  defines a unitary representation of  $\langle S \rangle$ .

We note, however, that the expression of each operator as a matrix of operators does not uniquely determine the operator, in the sense that different isomorphisms  $\Phi$  can yield non-isomorphic operators satisfying the same recursions. Even if  $\Phi$  is fixed, it may happen that different operators satisfy the same recursions. We considered two types of isomorphisms:  $\mathcal{H} = \ell^2(G/P)$ , where  $\Phi$  is derived from the decomposition map  $\psi$ ; and  $\mathcal{H} = L^2(\partial\mathcal{T})$ , where  $\Phi : \mathcal{H} \rightarrow \mathcal{H}^Y$  is defined by  $\Phi(f)(\sigma) = (f(0\sigma), \dots, f((m-1)\sigma))$ , for  $f \in L^2(\partial\mathcal{T})$  and  $\sigma \in \partial\mathcal{T}$ . There are actually uncountably many non-equivalent isomorphisms, giving uncountably many non-equivalent representations of the same group, as indicated in Subsection 11.1.1.

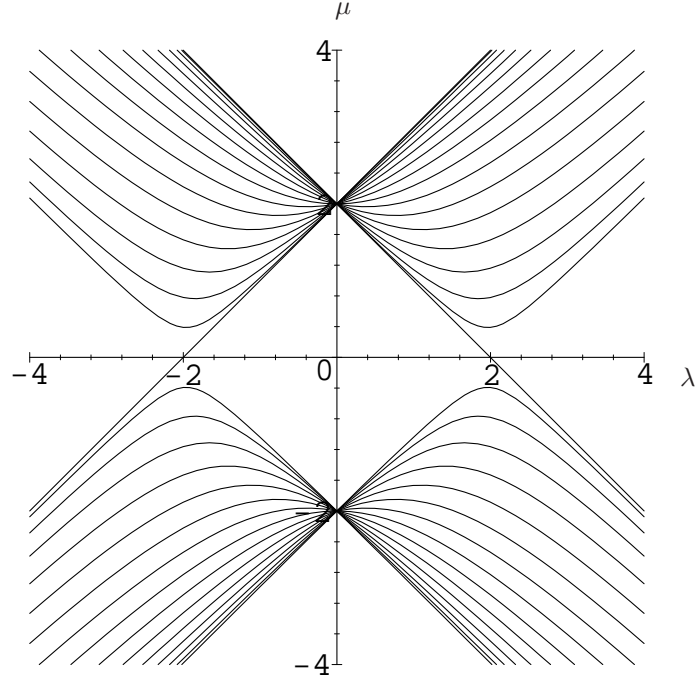
**11.2.1. The spectrum of  $\mathfrak{G}$ .** For brevity we shall only describe the spectrum of  $\pi$  for the first Grigorchuk group. Details and computations for the other examples appear in [BG00b].

Denote by  $a_n, b_n, c_n, d_n$  the permutation matrices of the representation  $\pi_n = \rho_{G/P_n}$ . We have

$$\begin{aligned} a_0 &= b_0 = c_0 = d_0 = (1), \\ a_n &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & b_n &= \begin{pmatrix} a_{n-1} & 0 \\ 0 & c_{n-1} \end{pmatrix}, \\ c_n &= \begin{pmatrix} a_{n-1} & 0 \\ 0 & d_{n-1} \end{pmatrix}, & d_n &= \begin{pmatrix} 1 & 0 \\ 0 & b_{n-1} \end{pmatrix}. \end{aligned}$$

The Hecke-Laplace operator of  $\pi_n$  is

$$\Delta_n = a_n + b_n + c_n + d_n = \begin{pmatrix} 2a_{n-1} + 1 & 1 \\ 1 & \Delta_{n-1} - a_{n-1} \end{pmatrix},$$

FIGURE 11.1. The spectrum of  $Q_n(\lambda, \mu)$  for  $\mathfrak{G}$ 

and we wish to compute its spectrum. We start by proving a slightly stronger result: define

$$Q_n(\lambda, \mu) = \Delta_n - (\lambda + 1)a_n - (\mu + 1)$$

and

$$\begin{aligned} \Phi_0 &= 2 - \mu - \lambda, \\ \Phi_1 &= 2 - \mu + \lambda, \\ \Phi_2 &= \mu^2 - 4 - \lambda^2, \\ \Phi_n &= \Phi_{n-1}^2 - 2(2\lambda)^{2^{n-2}} \quad (n \geq 3). \end{aligned}$$

Then the following steps compute the spectrum of  $\pi_n$ : first, for  $n \geq 2$ , we have

$$|Q_n(\lambda, \mu)| = (4 - \mu^2)^{2^{n-2}} \left| Q_{n-1} \left( \frac{2\lambda^2}{4 - \mu^2}, \mu + \frac{\mu\lambda^2}{4 - \mu^2} \right) \right| \quad (n \geq 2).$$

Therefore, for all  $n$  we have

$$|Q_n| = \Phi_0 \Phi_1 \cdots \Phi_n.$$

Then, for all  $n$  we have

$$\begin{aligned} \{(\lambda, \mu) : Q_n(\lambda, \mu) \text{ non invertible}\} &= \{(\lambda, \mu) : \Phi_0(\lambda, \mu) = 0\} \cup \{(\lambda, \mu) : \Phi_1(\lambda, \mu) = 0\} \\ &\cup \{(\lambda, \mu) : 4 - \mu^2 + \lambda^2 + 4\lambda \cos\left(\frac{2\pi j}{2^n}\right) = 0 \text{ for some } j = 1, \dots, 2^{n-1} - 1\}. \end{aligned}$$

In the  $(\lambda, \mu)$  system, the spectrum of  $Q_n$  is thus a collection of 2 lines and  $2^{n-1} - 1$  hyperbolæ.

The spectrum of  $\Delta_n$  is precisely the set of  $\theta$  such that  $|Q_n(-1, \theta-1)| = 0$ . From the computations above we obtain

PROPOSITION 11.6.

$$\text{spec}(\Delta_n) = \{1 \pm \sqrt{5 - 4 \cos \phi} : \phi \in 2\pi\mathbb{Z}/2^n\} \setminus \{0, -2\}.$$

Therefore the spectrum of  $\pi$ , for the group  $\mathfrak{G}$ , is

$$\text{spec}(\Delta) = [-2, 0] \cup [2, 4].$$

The first eigenvalues of  $\Delta_n$  are 4; 2;  $1 \pm \sqrt{5}$ ;  $1 \pm \sqrt{5 \pm 2\sqrt{2}}$ ;  $1 \pm \sqrt{5 \pm 2\sqrt{2 \pm \sqrt{2}}}$ ; etc.

The numbers of the form  $\pm\sqrt{\lambda \pm \sqrt{\lambda \pm \sqrt{\dots}}}$  appear as preimages of the quadratic map  $z^2 - \lambda$ , and after closure produce a Julia set for this map (see [Bar88]). In the given example this Julia set is just an interval. For the groups  $\Gamma, \bar{\Gamma}, \overline{\bar{\Gamma}}$ , however, the spectra are simple transformations of Julia sets which are totally disconnected, as similar computations show.

## Open Problems

This chapter collects questions for which no answer is yet known. Here we use the geometric definition of branch group given in Definition 1.13, except in Question 2.

The theory of branch group is a recent development and the questions arise and die almost every day, so there are no longstanding problems in the area and there is no easy way to say which questions are difficult and which are not. The list below just serves as a list of problems that one should naturally ask at this given moment. We kindly invite the interested readers to solve as many of the problems below.

**Added in the proof.** The authors had a chance to proofread the article 18 months after the initial submission. Some of the proposed problems have been solved in the meantime and we include appropriate comments to that effect.

QUESTION 1. *Is there a finitely generated fractal regular branch group  $G$ , branched over  $K$ , acting on the binary tree, such that the index of the geometric embedding of  $K \times K$  into  $K$  is two?*

QUESTION 2. *Every branch group from Definition 1.1 acts canonically on the tree determined by its branch structure as a group of tree automorphisms. Is the kernel of this action necessarily central?*

QUESTION 3. *Does every finitely generated branch  $p$ -group, where  $p$  is a prime, satisfy the congruence subgroup property?*

QUESTION 4. *Is every finitely generated branch group isomorphic to a spinal group?*

QUESTION 5. *Is the conjugacy problem solvable in all branch groups with solvable word problem?*

QUESTION 6. *When do the defining triples  $\omega$  and  $\omega'$  define non-isomorphic examples of branch groups in [Gri84, Gri85a] and more generally in  $G$  and GGS groups?*

QUESTION 7. *Which branch groups have finite  $L$ -presentations? Finite ascending  $L$ -presentations? In particular, what is the status of the Gupta-Sidki 3-group?*

All that is known at present is that the Gupta-Sidki 3-group has a finite endomorphic presentation.

QUESTION 8. *Do there exist finitely presented branch groups?*

QUESTION 9. *Is it correct that there are no finitely generated hereditary just-infinite torsion groups?*

QUESTION 10. *Is every finitely generated just-infinite group of intermediate growth necessarily a branch group?*

QUESTION 11. *Is every finitely generated hereditarily just-infinite group necessarily linear?*

QUESTION 12. *Recall that  $G$  has bounded generation if there exist elements  $g_1, \dots, g_k$  in  $G$  such that every element in  $G$  can be written as  $g_1^{n_1} \dots g_k^{n_k}$  for some  $n_1, \dots, n_k \in \mathbb{Z}$ . Can a just-infinite branch group have bounded generation? Can infinite simple group have bounded generation?*

QUESTION 13. *What is the height (in the sense of [Pri80], see Chapter 5) of a Grigorchuk 2-group  $G_\omega$  when the defining sequence  $\bar{\omega}$  is not periodic? Same question for arbitrary  $\mathbf{G}$  groups. In particular, can the height be infinite?*

QUESTION 14. *Is every maximal subgroup in a finitely generated branch group necessarily of finite index?*

QUESTION 15. *Is there is a finitely generated branch group containing the free group  $F_2$  on two generators?*

Positive answer is provided by Said Sidki and John Wilson in [SW02]

QUESTION 16. *Are there finitely generated branch groups with exponential growth that do not contain the free group  $F_2$ ?*

QUESTION 17. *Is there a finitely generated branch group whose degree of growth is  $e^{\sqrt{n}}$ ? Is there such a group in the whole class of finitely generated groups?*

QUESTION 18. *What is the exact degree of growth of any of the basic examples of regular branch groups (for example  $\mathfrak{G}$ ,  $\Gamma$ ,  $\bar{\Gamma}$ , ...)?*

QUESTION 19. *What is the growth of the Brunner-Sidki-Vieira group (see [BSV99] and Section 4.3)?*

It is known that the Brunner-Sidki-Vieira group does not contain any non-abelian free groups, but it is not known whether it contains a non-abelian free monoid. Note that this group is not a branch group, but it is a weakly branch group.

QUESTION 20. *Are there finitely generated non-amenable branch groups not containing the free group  $F_2$ ?*

QUESTION 21. *Is it correct that in each finitely generated fractal branch group  $G$  every finitely generated subgroup is either finite or Pride equivalent with  $G$ ?*

A stronger property holds for  $\mathfrak{G}$ , namely, John Wilson and the second author have proved in [GW01] that every finitely generated subgroup of  $\mathfrak{G}$  is either finite or commensurable with  $\mathfrak{G}$ . Claas Röver has announced that the answer is also positive for the Gupta-Sidki group  $\bar{\bar{\Gamma}}$ . However, the answer is negative in general.

QUESTION 22. *Do there exist branch groups with the property (T)?*

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