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ERGODIC PROPERTIES OF BOUNDARY ACTIONS AND NIELSEN–SCHREIER THEORY

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Abstract. We study the basic ergodic properties (ergodicity and conservativity) of the action of a subgroup H of a free group F on the boundary ∂F with respect to the uniform measure. Our approach is geometrical and combinatorial, and it is based on choosing a system of Nielsen–Schreier generators in H associated with a geodesic spanning tree in the Schreier graph $X = H\backslash F$. We give several (mod 0) equivalent descriptions of the Hopf decomposition of the boundary into the conservative and the dissipative parts. Further we relate conservativity and dissipativity of the action with the growth of the Schreier graph X and of the subgroup H ($\equiv$ cogrowth of X), respectively. On the other hand, our approach sheds a new light on entirely algebraic properties of subgroups of a free group. We also construct numerous examples illustrating the connections between various relevant notions.

Introduction

In 1921 Jacob Nielsen [Nie21] proved that any finitely generated subgroup of a free group is itself a free group. His proof was based on a rewriting procedure which allows one to reduce an arbitrary finite system of elements of a free group to a system of free generators. Since then Nielsen's method has become one of the main tools in the combinatorial group theory [MKS76, LS01]. It is used in the study of the group of automorphisms of a free group, for solving equations in free groups and in numerous other applications. Its scope is by no means restricted to free groups and extends to the combinatorial group theory at large, K-theory and topology.

In 1927 Nielsen's result was extended by Otto Schreier [Sch27] to arbitrary subgroups in another seminal work (where, in particular, what is currently known as Schreier graphs was introduced). Under the name of the Nielsen–Schreier theorem it is now one of the bases of the theory of infinite groups. Schreier's method is at first glance quite different from Nielsen's and uses families of coset representatives (transversals). That Nielsen and Schreier actually arrived at essentially the same generating systems became clear much later and was proved in [HR48, KS58].

In this work we show that the Nielsen–Schreier theory is useful in the ergodic theory, and our main result is its application to the study of the ergodic properties of the boundary action of subgroups of a finitely generated free group. On the other hand, our point of view "from infinity" (based on using dynamical invariants of the boundary action) sheds new light on certain purely algebraic properties of subgroups of free groups.

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The boundary theory occupies an important place in various mathematical fields: geometric group theory, rigidity theory, theory of Kleinian groups, potential analysis, Markov chains, to name just a few. The free group is one of the central objects in the study of boundaries of groups. Its simple combinatorial structure makes of it a convenient test-case which contributes to the understanding of general concepts, both in the group-theoretic (as the free group is the universal object in the category of discrete groups) and geometric (as its Cayley graph, the homogeneous tree, is a discrete analogue of the constant curvature hyperbolic space) frameworks.

There exist many different boundaries of a group (either coming from the corresponding compactification or not): the space of ends, the Martin boundary, the visual boundary, the Busemann boundary, the Floyd boundary, the Poisson(–Furstenberg) boundary, etc. In the case of the free group F freely generated by a finite set $\mathcal{A}$, all these notions coincide, and the *boundary* ∂F can be realized as the space $\breve{\mathcal{A}}_r^\infty$ of infinite freely reduced words in the alphabet $\breve{\mathcal{A}} = \mathcal{A} \cup \mathcal{A}^{-1}$. The action of the group on itself extends by continuity to a continuous action on ∂F.

The choice of the generating set $\mathcal{A}$ determines a natural *uniform* probability measure $\mathfrak{m}$ on ∂F which is quasi-invariant under the action of F. This measure can also be interpreted in a number of other ways. Namely, as the measure of maximal entropy of the unilateral Markov shift in the space of infinite irreducible words, as a conformal density (Patterson measure), or as the hitting ($\equiv$ harmonic) measure of the simple random walk on the group. In the latter interpretation the measure space $(\partial F, \mathfrak{m})$ is actually isomorphic to the Poisson boundary of the random walk, and it is this interpretation that plays an important role in our work.

The main goal of the present paper is to study the basic ergodic properties, i.e., ergodicity and conservativity, of the action of a subgroup $H \le F$ on the boundary ∂F with respect to the measure $\mathfrak{m}$. Our principal results are:

- An explicit combinatorial description of the Hopf decomposition of the boundary action (Theorem 1.21 and Theorem 2.12) and the identification of its conservative part with the horospheric limit set (Theorem 3.21);
- A sufficient condition of complete dissipativity of the boundary action (Theorem 4.2) and a necessary and sufficient condition of its conservativity (Theorem 4.12) in terms of the growth of the group H ($\equiv$ the cogrowth of the associated Schreier graph X) and of X, respectively;
- Numerous new examples illustrating and clarifying the interrelations between various related notions (Section **3.G** and Section **4.D**).

On the other hand, we expect our approach to be useful for purely algebraic problems as well. For instance, our analysis of the ergodic properties of the boundary action allows us to give a conceptual proof of an old theorem of Karrass–Solitar on finitely generated subgroups of a free group (Remark 3.33).

Recall that an action of a countable group is called *ergodic* with respect to a quasi-invariant measure if it has no non-trivial invariant sets. Any action (on a Lebesgue space) admits a unique *ergodic decomposition* into its *ergodic components*. An action is called *conservative* if it admits no non-trivial *wandering set* (i.e, such that its translations are pairwise disjoint). There is always a maximal wandering set, and the union of its translations is called the *dissipative part* of the action. Any action admits the so-called *Hopf decomposition* into the conservative and dissipative parts. These parts can also be described as the unions of all the purely non-atomic, and, respectively, of all the atomic ergodic components. It is important to keep in mind that the Hopf decomposition (as well as other measure theoretic notions) is defined (mod 0), i.e., up to measure 0 subsets.

It is pretty straightforward to see that ergodicity of the boundary action is equivalent to the *Liouville property* of the simple random walk on the Schreier graph X (i.e., to the absence of non-constant bounded harmonic functions on X), see Section **3.E**. On the other hand, as was shown by Kaimanovich [Kai95], the boundary action of a non-trivial normal subgroup is always conservative. In particular, if $G = F/H$ is any non-Liouville (for example, non-amenable) group, then the action of the normal subgroup H on $(\partial F, \mathfrak{m})$ is conservative without being ergodic. The only other previously known example of the Hopf decomposition of a boundary action was the one of completely dissipative $\mathbb{Z}$-actions [Kai95].

The starting point of our approach is the *Schreier graph* structure on the quotient homogeneous space $X = H\backslash F$. To quote [Sti93, Section 2.2.6], Schreier's method "begs to be interpreted in terms of spanning trees" in the Schreier graph. Indeed, there is a one-to-one correspondence between Schreier generating systems for the subgroup H and spanning trees in X rooted at the origin $o = H$, which we remind in Section **1.A** and Section **1.B**. This correspondence consists in assigning the associated cycle in X to any edge removed when passing to the spanning tree (Theorem 1.8).

By interpreting points of the boundary ∂F as infinite paths without backtracking issued from the origin $o = H$ in the Schreier graph X, we define two subsets of ∂F: the *Schreier limit set* Ω and the *Schreier fundamental domain* Δ (Definition 1.15). The set Ω corresponds to the paths which pass infinitely many times through $\mathsf{Edges}(X) \setminus \mathsf{Edges}(T)$ and is homeomorphic to the set ∂H of infinite irreducible words in the alphabet $\breve{\mathcal{S}} = S \sqcup S^{-1}$, whereas the set Δ corresponds to the rays issued from the origin in the tree T, and is homeomorphic to the boundary ∂T of T. These sets give rise to a decomposition $\partial F = \left(\bigsqcup_{h\in H} h\Delta\right) \sqcup \Omega$ (Theorem 1.21).

However, in order to study this decomposition further we have to impose an additional condition on the Schreier generating system by requiring it to be *minimal*, which means that the corresponding spanning tree T is geodesic (the class of minimal Schreier systems coincides with the class of *Nielsen* generating systems, see Section **1.E** and the references therein). Under this assumption we prove that the above decomposition, indeed, coincides (mod 0) with the Hopf decomposition of the boundary action (Theorem 2.12).

The topological counterpart of the Hopf decomposition of the boundary action is the decomposition of the boundary ∂F into a union of the closed H-invariant *limit set* $\Lambda = \Lambda_H$ (the closure of H in the compactification $\widehat{F} = F \cup \partial F$) and its complement Λ^c. According to a general result (valid for all Gromov hyperbolic spaces), the restriction of the H-action to Λ is *minimal* (any orbit is dense), whereas its restriction to Λ^c is *properly discontinuous* (no orbit has accumulation points). The decomposition $\partial F = \Lambda \sqcup \Lambda^c$ corresponds to the decomposition of the Schreier graph X into a union of its *core* X_* and the collection of *hanging branches* (Theorem 3.8; see Section **3.A** for the definitions). In particular, $\Lambda = \partial F$ if and only if X has no hanging branches.

The Schreier limit set Ω is contained in the full limit set Λ, which corresponds to the fact that proper discontinuity of the boundary action on Λ^c implies its complete dissipativity with respect to any quasi-invariant measure (in particular, the uniform measure $\mathfrak{m}$). Geometrically, any hanging branch in X gives rise to a non-trivial wandering set in Λ^c. However, the action on Λ may also have a non-trivial dissipative part, or even be completely dissipative. For instance, it may so happen that the Schreier graph X has no hanging branches at all (i.e., $\Lambda = \partial F$), but nonetheless the boundary action is completely dissipative (Example 4.19).

We introduce the *small* (resp., *big*) *horospheric limit set* $\Lambda^{horS} = \Lambda_H^{horS}$ (resp., $\Lambda^{horB} = \Lambda_H^{horB}$) of the subgroup H as the set of all the points $\omega \in \Lambda$ such that any (resp., a certain) horoball centered at ω contains infinitely many points from H, and show that the Schreier limit set Ω is sandwiched between Λ^{horS} and Λ^{horB}, but coincides with them (mod 0) with respect to the measure $\mathfrak{m}$ (Theorem 3.20 and Theorem 3.21). We also establish certain other inclusions and show by appropriate examples that all of them are strict (Section **3.G**).

If the subgroup H is finitely generated (i.e, if the core X_* is finite), then the *Hopf alternative* between conservativity and complete dissipativity holds: either the Schreier graph X is finite and the boundary action of H is ergodic (therefore, conservative), or X is infinite and the boundary action is completely dissipative (Theorem 3.30). However, for infinitely generated subgroups the relationship between the ergodic properties of the boundary action and the geometry of the Schreier graph X is much more complicated (as illustrated by numerous examples in Section **4.D**).

We prove that if the exponential growth rate of H ($\equiv$ the *cogrowth* of X) satisfies the inequality $v_H < \sqrt{2m-1}$, where m is the number of generators of F (i.e., if $v_H < \sqrt{v_F}$), then the boundary action of H is completely dissipative (Theorem 4.2). On the other hand, we show (Theorem 4.12) that the boundary action of H is conservative if and only if $\lim_n |S_X^n|/|S_F^n| = 0$, where S_X^n (resp., S_F^n) is the radius n sphere in X (resp., F) centered at the origin o (resp., at the identity e). In particular, if the exponential growth rate v_X of the Schreier graph X satisfies the inequality $v_X < 2m-1$, then the boundary action is conservative (Corollary 4.14).

Markov chains (not only the aforementioned simple random walks, but also the other chains described in Section **5**) play an important role in understanding the ergodic properties of the boundary action. Another measure-theoretical tool which we use in this paper is the relationship of the boundary action with two other natural actions of the subgroup H (see Section **3.F** for references and more details).

The first one is the action on the square $\partial^2 F$ of the boundary ∂F endowed with the square of the uniform measure $\mathfrak{m}$. The ergodic properties of this action are the same as for the (discrete) *geodesic flow* on the Schreier graph X and are described by the *classical Hopf alternative* (Theorem 3.35): the action of H on $(\partial^2 F, \mathfrak{m}^2)$ is either ergodic (therefore, conservative) or completely dissipative. Moreover, ergodicity of this action is equivalent to divergence of the *Poincaré series* $\sum_{h\in H}(2m-1)^{-|h|}$. Note that the ergodic behaviour of the action on ∂F is much more complicated than of that on $\partial^2 F$: for instance, the Hopf alternative for the former, generally speaking, holds only in the finitely generated case. It is interesting that one of our descriptions of the Hopf decomposition of the action on ∂F deals with a series similar to the Poincaré series arising for the action on $\partial^2 F$. However, once again, it is more complicated as it involves the *Busemann functions* rather than plain distances (see Theorem 2.12 and Theorem 3.21).

The second auxiliary action is the action of the group H on the space of horospheres in F, i.e., the $\mathbb{Z}$-extension of the action on ∂F determined by the *Busemann cocycle*. Geometrically, this action corresponds to what could be called (by analogy with Fuchsian groups) *"horocycle flow"* on the Schreier graph. We use the fact that (unlike for the action on $\partial^2 F$) the ergodic properties of this action are precisely the same as for the original action on ∂F (Theorem 3.38).

It is a commonplace that the homogeneous tree is a "rough sketch" of the hyperbolic plane. Both these spaces are Gromov hyperbolic (even CAT(-1)), and their isometry groups are "large enough" (so that the rotations around any reference point inside act transitively on the hyperbolic boundary). The subgroups of the free group F, which are the object of our consideration, are just the torsion free discrete groups of isometries of the Cayley tree of F. Thus, the question about analogous results for discrete isometry groups in the hyperbolic setup — be it for the usual hyperbolic plane (Fuchsian groups), higher dimensional simply connected spaces of constant negative curvature (Kleinian groups), arbitrary non-compact rank 1 symmetric spaces, general CAT(-1) or Gromov hyperbolic spaces, even for spaces which are hyperbolic in a weaker form — cannot fail to be asked.

The work on the present article prompted the second author to show that the identification of the conservative part of the boundary action with the big horospheric limit set Λ^{horB} is actually valid in the full generality of a discrete group of isometries of an arbitrary Gromov hyperbolic space endowed with a quasi-conformal boundary measure [Kai08] (see the references therein for a list of earlier particular cases of this result). The proof uses the fact that, by definition, the logarithms of the Radon–Nikodym derivatives of such a measure are (almost) proportional to the Busemann cocycle, in combination with the description of the Hopf decomposition of an arbitrary action in terms of the orbitwise sums of the Radon–Nikodym derivatives (Theorem 2.2). However, this is the only situation in our paper when the cases of the free group and of the hyperbolic plane are specializations of a common general result. Even here we obtain, in terms of Nielsen–Schreier generators, a much more detailed information about the Hopf decomposition than in the general case (Theorem 3.21).

Two other occasions when our results have analogues for Fuchsian or Kleinian groups are Theorem 4.2 and Theorem 4.12 from Section **4** which give qualitative criteria of complete dissipativity and conservativity of the boundary action, respectively. Here common general results are unknown, and our methods are completely different from those used in the hyperbolic situation by Patterson [Pat77] and Matsuzaki [Mat05] in the first case (see Remark 4.6) and by Sullivan [Sul81] in the second case (see Remark 4.16). Although the "hyperbolic" techniques most likely might be carried over to our situation as well, our approach is much more appropriate in the discrete case as it uses combinatorial tools not readily available in the continuous case. For instance, we obtain Theorem 4.12 as a corollary of Theorem 4.10 which gives an explicit formula for the measure of a certain canonical wandering set; a hyperbolic analogue of Theorem 4.10 is unknown.

Let us finally mention a couple of open questions arising in connection with the present work. The most obvious one is to what extent our results can be carried over to other boundary measures. The first candidate would be the conformal (Patterson) measures which are singular with respect to the uniform measure $\mathfrak{m}$ in the case when the growth v_H of H is strictly smaller than the growth of the ambient group F. By a general result from [Kai08], in this case the conservative part can still be identified with the big horospheric limit set (see above), but we do not know to what extent the combinatorial machinery developed in the present paper can be adapted to this situation. Of course, one can also try to generalize our technique to the nearest relatives of free groups, i.e., to word hyperbolic groups, or even to general discrete groups of isometries of Gromov hyperbolic spaces.

The other question is more concrete and concerns existence of conservative boundary actions (with respect to the uniform measure) with $v_H = \sqrt{2m-1}$. The analogous question is also open for Fuchsian groups, see Remark 4.4 and Remark 4.6.

1. Nielsen–Schreier theory and the boundary action

Let F denote the free group freely generated by a finite set $\mathcal{A}$ with $|\mathcal{A}| = m \geq 2$, and let $\breve{\mathcal{A}} = \mathcal{A} \sqcup \mathcal{A}^{-1}$. The Cayley graph $\Gamma(F, \breve{\mathcal{A}})$ is a homogeneous tree of degree $|\breve{\mathcal{A}}| = 2|\mathcal{A}|$.

We shall use the notation $\breve{\mathcal{A}}^*$ (resp., $\breve{\mathcal{A}}^\infty$) for the sets of all finite (resp., right infinite) words in the alphabet $\breve{\mathcal{A}}$. The length of a word w is denoted by $|w|$. For the subsets of $\breve{\mathcal{A}}^*$ and $\breve{\mathcal{A}}^\infty$ consisting of freely reduced words we shall add the subscript r, so that there is a canonical map

$$\sigma : F \to \breve{\mathcal{A}}_r^* \tag{1.1}$$

identifying F and $\breve{\mathcal{A}}_r^*$.

Any subgroup $H \leq F$ of a free group F is also free. It was proved by Nielsen [Nie21] (for finitely generated subgroup) and Schreier [Sch27] by giving two different constructions of free generating sets in H, see [MKS76] and the references therein. As it turned out, Nielsen's generating systems are just a particular case of Schreier's systems (see Theorem 1.24 below). We shall begin by recasting the original symbolic construction of Schreier (described in [MKS76, Section 2.3]) in terms of spanning trees in the Schreier graph $X \cong H \backslash F$ (cf. [Sti93, Section 2.2.6] and [KM02, Section 6]). Further we shall construct a decomposition of the boundary ∂F naturally associated with such a spanning tree (Theorem 1.21), which is the main goal of this Section.

1.A. Spanning trees and Schreier transversals. Given a subgroup $H \leq F$, denote by $\Gamma(X, \breve{\mathcal{A}})$ the *Schreier graph* of the homogeneous space $X = H \backslash F$ with respect to $\breve{\mathcal{A}}$, i.e., two cosets Hg_1, Hg_2 are connected with an edge if and only if $g_1^{-1} g_2 \in \breve{\mathcal{A}}$, in which case the oriented edge $[Hg_1, Hg_2]$ is labelled with $g_1^{-1} g_2$. Notice that, unlike the Cayley graph, the Schreier graph may have multiple edges with the same endpoints (but different labels). The extreme example is $H = F$, when X consists of just one vertex with $|\mathcal{A}|$ attached loops. In the sequel we shall always assume that X is endowed with the Schreier graph structure.

The Schreier graph X has a distinguished vertex $o = H$, it is connected, $|\breve{\mathcal{A}}|$-regular (loops attached to points $x \in X$ with $xg = g$ for certain $g \in \breve{\mathcal{A}}$ are also counted!), and the set of edge labels around each vertex is precisely $\breve{\mathcal{A}}$. Moreover, the labels assigned to two different orientations of any edge are mutually inverse. Conversely, any graph with the above properties is the Schreier graph associated with a subgroup of F.

Remark 1.2. By a theorem of Gross any regular graph of even degree can be realized as the Schreier graph associated to a subgroup of a free group (i.e., its edges can be labelled in the aforementioned way). It is explained in [Lub95] for finite graphs; an inductive argument can be used to carry the proof over to infinite graphs (also see [dlH00]).

It will be convenient to use the following geometric interpretation of the set of irreducible words in the alphabet $\breve{\mathcal{A}}$:

Proposition 1.3. *The set $\breve{\mathcal{A}}_r^* \cong F$ is in one-to-one correspondence $g \leftrightarrow \pi(g)$ with the set $\mathsf{Paths}_o(X)$ of finite paths without backtracking in the Schreier graph X, starting from the origin $o = H$. This correspondence amounts to consecutive reading of the edge labels along the path, starting from the origin.*

Consider a *spanning tree* T in X rooted at the point o, so that the origin o can be connected with any vertex $x \in X$ by a unique path $[o, x] = [o, x]_T$ which only uses the edges from T (such a tree can easily be constructed for any connected graph, e.g., see [Sti93, Section 2.1.5]). Then the set of words associated to these paths as x runs through the whole set X (see Proposition 1.3), is a collection $\mathcal{T}$ of coset representatives (a *transversal*) for the group H. The transversal $\mathcal{T}$ has the property that any initial segment of an element of $\mathcal{T}$ is itself an element of $\mathcal{T}$. Such transversals are said to satisfy the *Schreier property*. Conversely, any Schreier transversal obviously determines a spanning tree in X.

1.B. Schreier generating systems. Any Schreier transversal $\mathcal{T}$ (equivalently, the associated spanning tree T) gives rise to a system of free generators for H parameterized by the edges of X which are not in T. Indeed, any such edge $\mathcal{E} = [x, y] \in \mathsf{Edges}(X) \setminus \mathsf{Edges}(T)$ determines a non-trivial cycle $\varsigma_{\mathcal{E}} = [o, x]_T[x, y][y, o]_T$ in X obtained by joining the endpoints x, y with o in T by unique paths $[o, x]_T$ and $[y, o]_T$, respectively, see Figure 1. The corresponding generator $s = s_{\mathcal{E}} = \pi^{-1}(\varsigma_{\mathcal{E}})$ is presented by the word $\sigma(s)$ which consists of the edge labels read along the path $\varsigma_{\mathcal{E}}$ (see Proposition 1.3). We shall denote by $\sigma_-(s), \sigma_0(s), \sigma_+(s) \in \check{\mathcal{A}}$ the words which correspond to the parts $[o, x]_T, [x, y], [y, o]_T$ of the path $\varsigma_{\mathcal{E}}$, respectively, so that

$$\sigma(s) = \sigma_-(s)\sigma_0(s)\sigma_+(s) \; . \tag{1.4}$$

In particular,

$$|\sigma_-(s)| = |x|_T \; , \qquad |\sigma_+(s)| = |y|_T \; , \tag{1.5}$$

where $|\cdot|_T = d_T(o, \cdot)$ is the graph distance from the origin o in the tree T. We denote by $\check{\mathcal{S}}$ the set of all the generators $s = s_{\mathcal{E}}$ of H obtained in this way, and by $\mathcal{S} \subset \check{\mathcal{S}}$ the set of generators which correspond to those edges $\mathcal{E}$ which are labelled with elements of $\mathcal{A}$, i.e., for which $\sigma_0(s) \in \mathcal{A}$. Two different orientations of the same edge give a pair of mutually inverse generators, so that $\check{\mathcal{S}} = \mathcal{S} \sqcup \mathcal{S}^{-1}$.

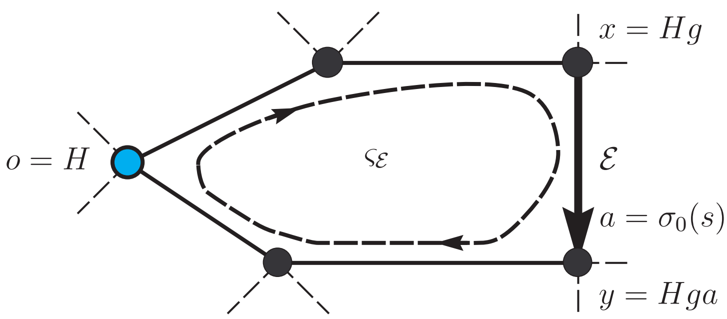

FIGURE 1.

Obviously, any cycle ς in H issued from o can be presented as a composition of the cycles $\varsigma_{\mathcal{E}}$ (which correspond to the sequence of edges from $\mathsf{Edges}(X) \setminus \mathsf{Edges}(T)$ through which ς passes), so that $\mathcal{S}$ is a generating system for H. The fact that it generates H freely follows from a general theorem of Schreier [MKS76, Theorem 2.9]. In our case, however, there is a more explicit argument.

Lemma 1.6. *For any two elements $s, s' \in \check{\mathcal{S}}$ with $s' \neq s^{-1}$ denote by*

$$\alpha(s, s') = \overline{\sigma_+(s)\sigma_-(s')} \in \check{\mathcal{A}}_r^*$$

the result of the free reduction of the concatenation of the components $\sigma_+(s)$ *and* $\sigma_-(s')$ *from the decomposition* (1.4)*. Then for any* $h = s_1 s_2 \dots s_n \in \check{\mathcal{S}}_r^* \cong H$ *one has*

$$\sigma(h) = \sigma_-(s_1)\sigma_0(s_1)\alpha(s_1, s_2)\sigma_0(s_2)\dots\alpha(s_{n-1}, s_n)\sigma_0(s_n)\sigma_+(s_n) \ . \tag{1.7}$$

Proof. Look at the decompositions $\sigma(s_i) = \sigma_-(s_i)\sigma_0(s_i)\sigma_+(s_i)$ (1.4) for all $i = 1, 2, \dots, n$. The word $\sigma_-(s_1)\sigma_0(s_1)$ ends with the letter $\sigma_0(s_1)$ which corresponds to passing through the edge $\mathcal{E}_{s_1} \in \mathsf{Edges}(X) \setminus \mathsf{Edges}(T)$ associated with the generator s_1. Since no other generator passes through this edge, the letter $\sigma_0(s_1)$ does not cancel. In the same way the middle letters $\sigma_0(s_i)$ do not cancel for all the other s_i.

More geometrically, let $[x, y] = \mathcal{E}_s$ and $[x', y'] = \mathcal{E}_{s'}$, then $\alpha(s, s')$ is the word obtained be reading the labels along the geodesic segment $[y, x']$ joining the points y and x' in the spanning tree T. Thus, letting $[x_i, y_i] = \mathcal{E}_{s_i}$ for $1 \le i \le n$, we have that the path (1.7) consists of the consecutive segments

$$[o, x_1], [x_1, y_1], [y_1, x_2], [x_2, y_2], \dots, [y_{n-1}, x_n], [x_n, y_n], [y_n, o] \ ,$$

see Figure 2 where $n = 5$. □

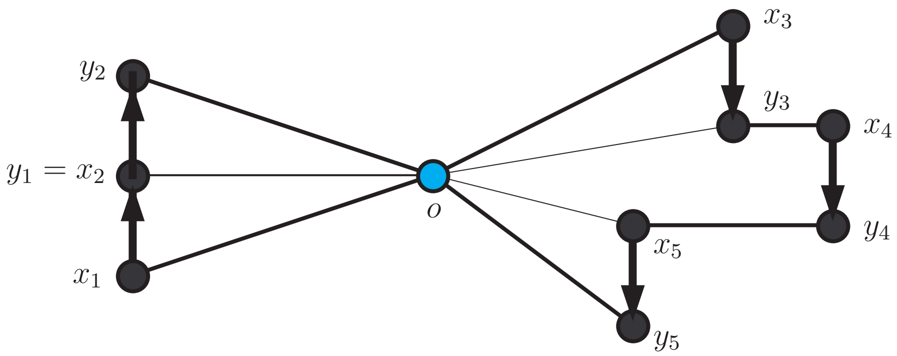

Figure 2.

We can summarize this discussion in the following way:

Theorem 1.8. *Any spanning tree* T *in the Schreier graph* X *determines a one-to-one correspondence* $\mathcal{E} \mapsto s_{\mathcal{E}}$*,* $s \mapsto \mathcal{E}_s$ *between the set of oriented edges of* X*, which are not in* T*, and the set* $\check{\mathcal{S}} = \mathcal{S} \sqcup \mathcal{S}^{-1}$ *of the associated free generators of* H *and their inverses.*

Definition 1.9. The free generating system $\mathcal{S}$ of the subgroup H is called the *Schreier system* associated with the spanning tree T (equivalently, with the corresponding Schreier transversal $\mathcal{T}$).

1.C. The boundary map. There is a natural *compactification* $\widehat{F} = F \cup \partial F$ of the group F. It does not depend on the choice of the generating set $\mathcal{A}$ and admits a number of interpretations, for instance, as the *end* or as the *hyperbolic* compactifications of the Cayley graph $\Gamma(F, \check{\mathcal{A}})$. The action of the group F on itself extends to a continuous action of F on the boundary ∂F.

In symbolic terms, the map $\sigma : F \to \check{\mathcal{A}}_r^*$ (1.1) can be extended to the *boundary* ∂F. This extension (also denoted by σ)

$$\sigma : \partial F \to \check{\mathcal{A}}_r^\infty$$

identifies ∂F with the set $\breve{\mathcal{A}}_r^\infty$ of infinite freely reduced words endowed with the product topology of pointwise convergence. A sequence $g_n \in F$ converges to a boundary point $\omega \in \partial F$ if and only if the finite words $\sigma(g_n)$ converge to the infinite word $\sigma(\omega)$. The action $(g,\omega) \mapsto g\omega$ consists then in concatenation of the associated words with a subsequent free reduction.

Given a point $\omega \in \partial F$ we shall denote by $[\omega]_n$ its n-th *truncation*, i.e., the element of F corresponding to the initial length n segment of the word $\sigma(\omega)$. Geometrically, the sequence $\{[\omega]_n\}_{n=0}^\infty$ is the *geodesic ray* $[e,\omega)$ in the Cayley graph $\Gamma(F,\breve{\mathcal{A}})$ joining the group identity e with the boundary point ω, see Figure 3.

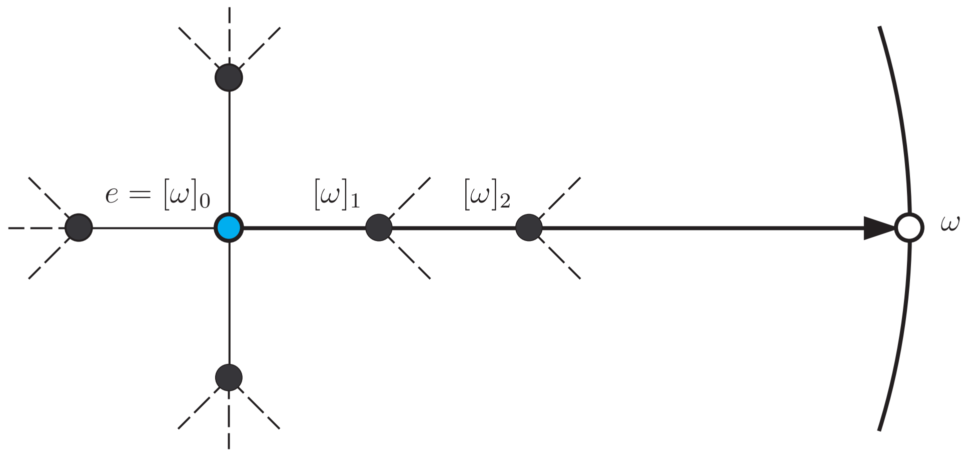

FIGURE 3.

In the same way as for the ambient group F, we shall denote by $\partial H \cong \breve{\mathcal{S}}_r^\infty$ the space of infinite freely reduced words in the alphabet $\breve{\mathcal{S}}$ endowed with the product topology of pointwise convergence.

Remark 1.10. The space ∂H is compact if and only if the alphabet $\breve{\mathcal{S}}$ is finite, i.e., the group H is finitely generated.

Theorem 1.11. *Let $\mathcal{S}$ be the free generating system of a subgroup $H \le F$ determined by a spanning tree T in the associated Schreier graph X (see Theorem 1.8). Then the restriction $\sigma : H \cong \breve{\mathcal{S}}_r^* \to \breve{\mathcal{A}}_r^*$ of the map σ (1.1) extends by continuity to a map $\sigma^\infty : \partial H \cong \breve{\mathcal{S}}_r^\infty \to \partial F \cong \breve{\mathcal{A}}_r^\infty$ as*

$$\sigma^\infty(\xi) = \lim_{n\to\infty} \sigma([\xi]_n)\ , \tag{1.12}$$

where $\xi = s_1 s_2 \cdots \in \breve{\mathcal{S}}_r^\infty \cong \partial H$ is an infinite freely reduced word in the alphabet $\breve{\mathcal{S}}$ and $[\xi]_n = s_1 s_2 \dots s_n \in \breve{\mathcal{S}}_r^* \cong H$ are its truncations. The extended map σ^∞ is an H-equivariant homeomorphism of ∂H onto its image

$$\Omega = \sigma^\infty(\partial H) \subset \partial F\ . \tag{1.13}$$

Proof. As follows from Lemma 1.6, the limit (1.12) exists, and

$$\sigma^\infty(\xi) = \sigma_-(s_1)\sigma_0(s_1)\alpha(s_1,s_2)\sigma_0(s_2)\alpha(s_2,s_3)\sigma_0(s_3)\dots\ . \tag{1.14}$$

The H-equivariance of the limit map σ^∞ is obvious. It order to check its invertibility note that the initial segments $\sigma_-(s)\sigma_0(s)$ of all the words $\sigma(s)$, $s \in \breve{\mathcal{S}}$ are *isolated* in the sense that they do not occur as initial segments of any other word $\sigma(s')$, $s' \in \breve{\mathcal{S}}$ (this is one of the defining properties of a *Nielsen system*, see below Section **1.E**; in our case it directly follows from the construction of $\breve{\mathcal{S}}$). Therefore, the word $\sigma^\infty(\xi)$ uniquely determines the letter $s_1 \in \mathcal{S}$ such that $\sigma^\infty(\xi)$ begins with the segment $\sigma_-(s_1)\sigma_0(s_1)$, i.e., the initial letter of ξ. By using the H-equivariance and applying the same consideration to $\xi' = s_1^{-1}\xi$ we recover then the second letter of ξ and so on. Finally, continuity of σ^∞ follows from formula (1.14), whereas continuity of the inverse map follows from its description in the previous sentence. □

1.D. A boundary decomposition. Along with the set Ω (1.13) we also define a subset $\Delta \subset \partial F \cong \breve{\mathcal{A}}_r^\infty$ as the set of all the infinite words which do not begin with any of the segments $\sigma_-(s)\sigma_0(s)$, $s \in \breve{\mathcal{S}}$.

Definition 1.15. The sets $\Omega, \Delta \subset \partial F$ are called the *Schreier limit set* and the *Schreier fundamental domain*, respectively. They are determined by the choice of a spanning tree T in the Schreier graph X.

Proposition 1.16. *The Schreier limit set Ω is G_δ in ∂F, and the Schreier fundamental domain Δ is closed in ∂F.*

Proof. Given a point $\xi \in \partial H$, denote by $\mathcal{C}_n(\xi) \subset \partial F$ the cylinder set consisting of all the infinite words beginning with the initial segment $\sigma_-(s_1)\sigma_0(s_1)\alpha(s_1, s_2)\ldots\alpha(s_{n-1}, s_n)\sigma_0(s_n)$ of $\sigma^\infty(\xi)$ in the expansion (1.14). Then

$$\Omega = \bigcap_n \bigcup_{\xi \in \partial H} \mathcal{C}_n(\xi) , \qquad \Delta = \overline{\bigcup_{\xi \in \partial H} \mathcal{C}_1(\xi)} .$$

The cylinders $\mathcal{C}_n(\xi)$ are all open in ∂F, whence the claim. □

Remark 1.17. The Schreier limit set Ω is closed in ∂F ($\equiv$ compact in the relative topology) if and only if ∂H is compact, i.e., if and only if H is finitely generated (cf. Remark 1.10).

The identification π of the group $F \cong \breve{\mathcal{A}}_r^*$ with the set $\mathsf{Paths}_o(X)$ (Proposition 1.3) obviously extends to an identification (also denoted by π) of the boundary $\partial F \cong \breve{\mathcal{A}}_r^\infty$ with the set $\mathsf{Paths}_o^\infty(X)$ of infinite paths without backtracking in X issued from o. In terms of this identification the sets Ω and Δ admit the following descriptions:

Proposition 1.18. *The Schreier limit set Ω corresponds to the set of infinite paths without backtracking in X which pass infinitely often through the edges not in the spanning tree T. The Schreier fundamental domain Δ corresponds to the set of paths which always stay in T, i.e., which never pass through any of the edges from $\mathsf{Edges}(X) \setminus \mathsf{Edges}(T)$.*

Proof. In view of the correspondence from Theorem 1.8, formula (1.14) shows that if $\omega = \sigma^\infty(\xi) \in \Omega$, then the associated path $\pi(\omega)$ passes through the edges $\mathcal{E}_{s_1}, \mathcal{E}_{s_2}, \ldots$ at the moments which correspond to the letters $\sigma_0(s_1), \sigma_0(s_2), \ldots$. Conversely, let us record consecutively the edges $\mathcal{E}_{s_1}, \mathcal{E}_{s_2}, \ldots$ through which the path corresponding to $\omega \in \partial F$ passes. Then $\omega = \sigma^\infty(\xi)$ for $\xi = s_1 s_2 \ldots$.

In the same way one verifies the description of the set Δ. A word $\omega \in \partial F$ begins with the segment $\sigma_-(s)\sigma_0(s)$ for a certain $s \in \breve{\mathcal{S}}$ if and only if the edge $\mathcal{E}_s$ is the first edge not in T through which the associated path passes. □

Following the above argument one also obtains a description of the translates $h\Delta$ of the Schreier fundamental domain (cf. Lemma 1.6 and Figure 2):

Proposition 1.19. *For any $h = s_1 s_2 \dots s_n \in \breve{\mathcal{S}}_r^* \cong H$ the set $h\Delta$ corresponds to the set of paths in X which, starting from o, pass through the edges $\mathcal{E}_{s_1}, \mathcal{E}_{s_2}, \dots, \mathcal{E}_{s_n}$ and follow edges in T at all the other times.*

Since the origin o can be joined with any point $x \in X$ by a unique path in the spanning tree T, the correspondence described in Proposition 1.3 determines a natural embedding of T into the Cayley graph $\Gamma(F, \breve{\mathcal{A}})$ such that o is mapped to the identity $e \in F$. Then the boundary ($\equiv$ the space of ends) ∂T becomes a subset of ∂F, and Proposition 1.18 implies

Proposition 1.20. *Under the above identification the Schreier fundamental domain $\Delta \subset \partial F$ is homeomorphic to the boundary ∂T of the spanning tree T.*

Proposition 1.18 and Proposition 1.19 yield

Theorem 1.21. *Given a spanning tree in the Schreier graph $X \cong H\backslash F$, the associated Schreier limit set Ω and the translates of the Schreier fundamental domain Δ provide a disjoint decomposition of the boundary*

$$\partial F = \left(\bigsqcup_{h \in H} h\Delta \right) \sqcup \Omega \,. \tag{1.22}$$

1.E. Geodesic spanning trees and minimal Schreier systems. A spanning tree T in a graph X is called *geodesic* (with respect to a root vertex o), if $d_T(o, x) = d_X(o, x)$ for every vertex x of X. A geodesic spanning tree exists in any connected graph. For a Schreier graph X, one possible way to construct a geodesic spanning tree is to use the fact that its edges are labelled with letters from $\breve{\mathcal{A}}$. Then, taking for any vertex $x \in X$ the lexicographically minimal among all the geodesic segments joining the origin o with x, the union of all such minimal segments is a geodesic spanning tree in X.

In terms of the discussion from Section **1.A**, a spanning tree in the Schreier graph X is geodesic if and only if the corresponding Schreier transversal is *minimal*, i.e., the length of each representative is minimal in its coset. The Schreier system of free generators $\mathcal{S}$ associated with a minimal Schreier transversal (equivalently, with a geodesic spanning tree in X) is called a *minimal Schreier system*.

An important consequence of minimality is the inequality

$$\big| |\sigma_-(s)| - |\sigma_+(s)| \big| \le 1 \qquad \forall\, s \in \breve{\mathcal{S}} \,, \tag{1.23}$$

which follows at once from the geometric interpretation given in Section **1.B** (more precisely, from formula (1.5)).

We refer the reader to [MKS76, Section 3.2] for a definition and construction of *Nielsen systems* of free generators in a subgroup of a free group.

Theorem 1.24 ([MKS76, Theorem 3.4])**.** *Any minimal Schreier system of generators in a subgroup H of a free group F is a Nielsen system. Conversely, any Nielsen system of generators is (up to a possible inversion of some elements) a minimal Schreier system.*

Remark 1.25. The interpretation of minimal Schreier systems in geometric terms of geodesic spanning trees allows one to make the proof given in [MKS76] (and reproducing the argument from [KS58]) significantly simpler. For instance, the "minimal Schreier $\Longrightarrow$ Nielsen" part (another proof of which was first given in [HR48]) becomes in these terms completely obvious.

From now on, when considering a generating set $\mathcal{S}$ in a subgroup $H \le F$, we shall always assume that $\mathcal{S}$ is a minimal Schreier system associated with a geodesic spanning tree T in the Schreier graph X.

2. Hopf decomposition of the boundary action

The aim of this Section is to show that the decomposition of the boundary ∂F into a disjoint union of the Schreier limit set and the translates of the Schreier fundamental domains obtained in Theorem 1.21 in fact provides the Hopf decomposition of the boundary action of the subgroup H with respect to the uniform measure $\mathfrak{m}$ (Theorem 2.12).

2.A. Conservativity and dissipativity. Let G be a countable group acting by *measure class preserving transformations* on a measure space $(\mathcal{X}, m)$, i.e., the measure m is *quasi-invariant* under this action (for any group element $g \in G$ the corresponding translated measure defined as $gm(A) = m(g^{-1}A)$ is equivalent to m).

Unless otherwise specified, all the identities, properties etc. related to measure spaces will be understood mod 0 *(i.e., up to null sets).*

A measurable set $A \subset \mathcal{X}$ is called *recurrent* if for a.e. point $x \in A$ the trajectory Gx eventually returns to A, i.e., $gx \in A$ for a certain element $g \in G$ other than the group identity e. Equivalently, A is recurrent iff $A \subset \bigcup_{g \in G \setminus \{e\}} gA$. The opposite notion is that of a *wandering set*, i.e., a measurable set $A \subset \mathcal{X}$ with pairwise disjoint translates gA, $g \in G$.

The action of G on $(\mathcal{X}, m)$ is called *conservative* if any measurable subset of positive measure is recurrent, and it is called *dissipative* if there exists a wandering set of positive measure. If the whole action space is the union of translates of a certain wandering set, then the action is called *completely dissipative*.

Remark 2.1. If a set $A \subset \mathcal{X}$ is recurrent with respect to a subgroup $G' \subset G$, then it is obviously recurrent with respect to the whole group G. Therefore, conservativity of the action of G' implies conservativity of the action of G.

The action space always admits a unique *Hopf decomposition* $\mathcal{X} = \mathcal{C} \sqcup \mathcal{D}$ into a union of two disjoint G-invariant measurable sets $\mathcal{C}$ and $\mathcal{D}$ (called the *conservative* and the *dissipative parts* of the action, respectively) such that the restriction of the action to $\mathcal{C}$ is conservative and the restriction of the action to $\mathcal{D}$ is completely dissipative, see [Aar97] and the references therein. Hopf was also the first to notice that for certain classes of dynamical systems the above decomposition is trivial: any system from these classes is either conservative or completely dissipative. In this situation one talks about the *Hopf alternative* (see Section **3.F** for more details).

There is an important class of measure spaces called *Lebesgue spaces* (e.g., see [Roh52, CFS82]). Measure-theoretically these are the measure spaces such that their non-atomic part is isomorphic to an interval with the Lebesgue measure on it. There is also an intrinsic definition of Lebesgue spaces based on their separability properties. However, for our purposes it is enough to know that any *Polish topological space* (i.e., separable, metrizable, complete) endowed with a Borel measure is a Lebesgue measure space, so that *all the measure spaces considered in this paper are Lebesgue.*

For Lebesgue spaces the Hopf decomposition can also be described in terms of the ergodic components of the action, see [Kai08]. In the case when the action is (essentially) *free*, i.e., the stabilizers of almost all points are trivial, this description is especially simple. Namely, $\mathcal{C}$ is the union of all the ergodic components for which the corresponding conditional measure is purely non-atomic, whereas $\mathcal{D}$ is the union of all the purely atomic ergodic components (i.e., of the ergodic components which consist of a single G-orbit; we shall call such orbits *dissipative*).

Below we shall use the following explicit description of the conservative part of an action in terms of its Radon–Nikodym derivatives.

Theorem 2.2 ([Kai08]). *Let $(\mathcal{X}, m)$ be a Lebesgue space endowed with a free measure class preserving action of a countable group G. Denote by μ_x, $x \in \mathcal{X}$, the measure on the orbit Gx defined as*

$$\mu_x(gx) = \frac{dg^{-1}m}{dm}(x) = \frac{dm(gx)}{dm(x)}$$

(the measures μ_x corresponding to different points x from the same G-orbit are obviously proportional). Then for a.e. point $x \in X$ the following conditions are equivalent:

(i) *The orbit Gx is dissipative;*
(ii) *The measure μ_x is finite.*

2.B. The uniform measure and the Busemann function. Given a group element $g \in F$ we shall denote by $C_g = C_{\sigma(g)} \subset \partial F$ the associated *cylinder set* of dimension $|g|$, which is the set of all the infinite words which begin with the word $\sigma(g)$:

$$C_g = \{\omega \in \partial F : [\omega]_{|g|} = g\} .$$

Geometrically, C_g is the "shadow" of g, i.e., the set of the endpoints of all the geodesic rays issued from the group identity e and passing through the point g.

Denote by $\mathfrak{m}$ the probability measure on $\partial F \cong \breve{\mathcal{A}}_r^\infty$ which is *uniform with respect to the generating set $\mathcal{A}$*. In other words, all the cylinder sets of the same dimension have equal measure

$$\mathfrak{m}C_g = \frac{1}{2m(2m-1)^{|g|-1}} \qquad \forall\, g \in F \setminus \{e\} , \tag{2.3}$$

where $m = |\mathcal{A}|$ is the number of generators of the group F.

The Radon–Nikodym derivatives of $\mathfrak{m}$ have a natural geometric interpretation. Let us first remind the corresponding notions. Given a point $\omega \in \partial F$, the associated *Busemann cocycle* is defined as

$$\beta_\omega(g_1, g_2) = \lim_{g \to \omega} \big[d(g_2, g) - d(g_1, g)\big] = d(g_2, g_1 \wedge_\omega g_2) - d(g_1, g_1 \wedge_\omega g_2) , \qquad g_1, g_2 \in F ,$$

where $d(g, g') = |g^{-1}g'|$ is the distance in the Cayley graph $\Gamma(F, \breve{\mathcal{A}})$, and $g_1 \wedge_\omega g_2$ denotes the starting point of the common part of the geodesic rays $[g_1, \omega)$ and $[g_2, \omega)$, see Figure 4. Thus, $\beta_\omega(g_1, g_2)$ is a regularization of the formal expression "$d(g_2, \omega) - d(g_1, \omega)$". It can also be written as

$$\beta_\omega(g_1, g_2) = b_\omega(g_2) - b_\omega(g_1) ,$$

where

$$b_\omega(g) = \beta_\omega(e, g) = \lim_{n \to \infty} \big[d(g, [\omega]_n) - n\big]$$

is the *Busemann function* associated with the point $\omega \in \partial F$ (or, geometrically, with the corresponding geodesic ray $[e, \omega)$).

For two words $w_1, w_2 \in \breve{\mathcal{A}}^* \cup \breve{\mathcal{A}}^\infty$ denote by $w_1 \wedge w_2$ their *confluent* (i.e., the longest common initial segment), and by

$$(w_1|w_2) = |w_1 \wedge w_2| \tag{2.4}$$

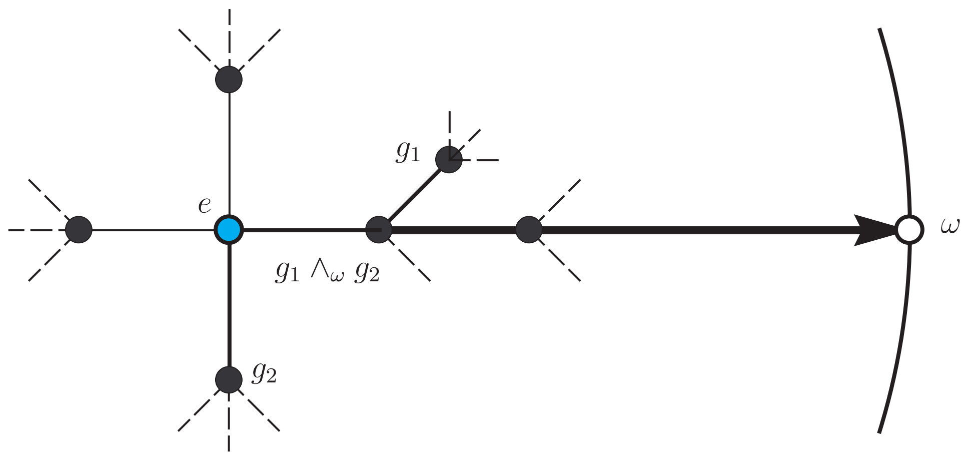

FIGURE 4.

their *Gromov product* [Gro87], see Figure 5. Then the Busemann function is connected with the Gromov product by the formula

$$b_\omega(g) = |g| - 2(g|\omega) \ . \tag{2.5}$$

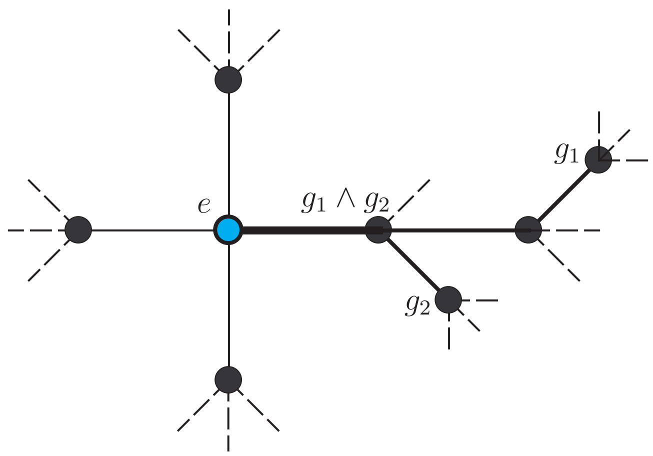

FIGURE 5.

The following property of the measure $\mathfrak{m}$ is well-known, and can easily be established by comparing measures of cylinder sets.

Proposition 2.6. *The measure $\mathfrak{m}$ is quasi-invariant with respect to the action of F on ∂F, and its Radon–Nikodym cocycle is*

$$\frac{dg\mathfrak{m}}{d\mathfrak{m}}(\omega) = (2m-1)^{-b_\omega(g)} \qquad \forall\, g \in F \ \forall\, \omega \in \partial F \ . \tag{2.7}$$

Proof. Since $gC_{g'} = C_{gg'}$ for any $g' \in F$ with $|g'| > |g|$, we have that under this condition

$$g\mathfrak{m}\,(C_{g'}) = \mathfrak{m}\left(g^{-1}C_{g'}\right) = \mathfrak{m}C_{g^{-1}g'} = \frac{1}{2m(2m-1)^{|g^{-1}g'|-1}} \ ,$$

whence
$$\frac{g\mathfrak{m}C_{g'}}{\mathfrak{m}C_{g'}} = (2m-1)^{|g'|-|g^{-1}g'|} = (2m-1)^{-b_\omega(g)} \qquad \forall\, \omega \in C_{g'} \ .$$
□

Remark 2.8. The objects appearing in the left-hand and the right-hand sides of formula (2.7) are of different nature. The Radon–Nikodym derivatives in the left-hand side are *a priory* defined almost everywhere only, whereas the Busemann function in the right-hand side is a *bona fide* continuous function on the boundary. It is a commonplace that such an equality is interpreted as saying that there is a version of the Radon–Nikodym derivative in the left-hand side given by the individually defined function in the right-hand side.

2.C. Inequalities for the Busemann function. The following auxiliary properties of the Busemann function are used in the proof of Theorem 2.12 below and later on.

Proposition 2.9. *If $\omega = \sigma^\infty(\xi) \in \Omega$ for $\xi = s_1 s_2 \cdots \in \check{\mathcal{S}}_r^\infty \cong \partial H$, and $h = [\xi]_n = s_1 s_2 \dots s_n$, then $b_\omega(h) \le 0$.*

Proof. Denote by $[x_i, y_i] = \mathcal{E}_{s_i} \in \mathsf{Edges}(X) \setminus \mathsf{Edges}(T)$ the oriented edges corresponding to the generators $s_i \in \check{\mathcal{S}}$ (see Theorem 1.8). The path $\pi(h)$ in X (see Proposition 1.3) starts from the origin o, passes consecutively through the points $x_1, y_1, \dots, x_n, y_n$, and returns to o. The segments $[o, x_1], [y_1, x_2], \dots, [y_{n-1}, x_n], [y_n, o]$ in this path are obtained by joining their endpoints in the spanning tree T, and the segments $[x_1, y_1], \dots, [x_n, y_n]$ are just the corresponding edges $\mathcal{E}_{s_i}$, see Lemma 1.6. Denote by D the length of the path $\pi(h)$ from the beginning until the point y_n, and by L the length of the remaining segment $[y_n, o]$, so that the total length of this path is $|h| = D + L$. Since T is a *geodesic* spanning tree (it is here that we use this condition), L is the distance between y_n and o in the graph X, so that by the triangle inequality $L \le D$.

The infinite path $\pi(\omega)$ also starts from the point o. It passes consecutively through the points $x_1, y_1, \dots, x_n, y_n, \dots$. The segments $[o, x_1], [y_1, x_2], \dots, [y_{n-1}, x_n], \dots$ are obtained by joining their endpoints in the spanning tree T, and the segments $[x_1, y_1], \dots, [x_n, y_n], \dots$ are the edges $\mathcal{E}_{s_i}$. Therefore, $(h|\omega) \ge D$, and by (2.5)
$$b_\omega(h) = |h| - 2(h|\omega) \le (D + L) - 2D = L - D \le 0 \ .$$
□

Proposition 2.10. *If $\omega \in \Delta$, then $b_\omega(h) \ge 0$ for any $h \in H$.*

Proof. This is the same argument as in the proof of Proposition 2.9. In view of formula (2.5) we have to show that $|h| \ge 2(h|\omega)$. Let us split the cycle $\pi(h)$ in X associated with h into two parts. The first one is the geodesic segment from o to the beginning x_1 of the oriented edge $[x_1, y_1]$ corresponding to the first letter $s_1 \in \check{\mathcal{S}}$ of h, and the second one is the rest of $\pi(h)$. By the triangle inequality the length of the second part is at least $d_X(o, x_1)$, so that the total length $|h|$ of the path $\pi(h)$ is at least $2d_X(o, x_1)$. On the other hand, $(h|\omega) \le d_X(o, x_1)$, because ω does not pass through $[x_1, y_1]$. □

Proposition 2.11. *If a point $\omega \in \partial F$ corresponds to a geodesic ray in X, then $b_\omega(h) \ge 0$ for any $h \in H$.*

Proof. This is again the same argument consisting in comparing the length of a geodesic subsegment in the cycle $\pi(h)$ with its total length. Let $g = h \wedge \omega$, and $x = og$. It means that the cycle $\pi(h)$ first follows the path ρ determined by ω, until it reaches the point

x, after which it somehow returns to the origin o. Since ρ is a geodesic ray, the length of the first part of the cycle (from the origin o to the point x along the ray ρ) does not exceed the length of the remaining part, whence $(h|\omega) = d_X(o, x) \le |h|/2$ which implies the claim. □

2.D. The Hopf decomposition of the boundary action. Now we are ready to prove the main result of this Section:

Theorem 2.12. *The Schreier limit set $\Omega \subset \partial F$ determined by a geodesic spanning tree in the Schreier graph $X \cong H\backslash F$ ($\equiv$ by a minimal Schreier generating system) coincides* (mod 0) *with the conservative part of the action of the subgroup $H \le F$ on the boundary ∂F with respect to the uniform measure $\mathfrak{m}$.*

Proof. Since the Schreier fundamental domain Δ is measurable (Proposition 1.16), Theorem 1.21 implies that the dissipative part $\mathcal{D}$ of the action is at least the union $\bigcup_h h\Delta$, so that the conservative part $\mathcal{C}$ of the action is contained in Ω. For showing that $\mathcal{C} = \Omega$ we shall introduce the H-invariant set

$$\Sigma = \left\{ \omega \in \partial F : \sum_{h \in H} (2m-1)^{-b_\omega(h)} = \infty \right\} . \tag{2.13}$$

Any non-trivial element $g \in F$ has two fixed points on ∂F (the attracting and the repelling ones). Therefore, the boundary action is essentially free with respect to any purely non-atomic quasi-invariant measure. Thus, by Proposition 2.6 and the criterion from Theorem 2.2, the set Σ coincides (mod 0) with the conservative part $\mathcal{C}$, and it remains to show that $\Omega \subset \Sigma$, which follows at once from Proposition 2.9 above. □

Remark 2.14. A priori, the set Ω depends on the choice of Nielsen–Schreier generating system $\mathcal{S}$ in H ($\equiv$ of a geodesic spanning tree in the Schreier graph $\equiv$ of a minimal Schreier transversal). However, Theorem 2.12 shows that the sets Ω for different choices of $\mathcal{S}$ only differ by a subset of $\mathfrak{m}$-measure 0.

Remark 2.15. It is likely that Theorem 2.12 also holds for many other boundary measures. It would be interesting to investigate this question further. What (if any) are the examples of purely non-atomic quasi-invariant measures on ∂F, for which the conservative part is strictly smaller than the Schreier limit set Ω?

3. Limit sets and the core

In this Section we compare the Schreier limit set Ω with several other limit sets. In particular, we show that Ω (mod 0) coincides with the horospheric limit sets (Theorem 3.21). In this discussion we use connections with random walks and with several extensions of the boundary action.

3.A. Hanging branches and the core. A *branch* of a regular tree is a subtree which has one vertex (the *root*) of degree 1 and all the other vertices (which form its *interior*) are of full degree. Such a branch is uniquely determined by its *stem*, which is the oriented edge going from the root to the interior of the branch, see Figure 6.

Definition 3.1. A subgraph of the Schreier graph X isomorphic to a branch in the Cayley graph of F (with its labelling) is called a *hanging branch*. The subgraph $X_* \subset X$ obtained by removing from X all the hanging branches (i.e., all their edges and all the interior vertices) is called the *core* of X.

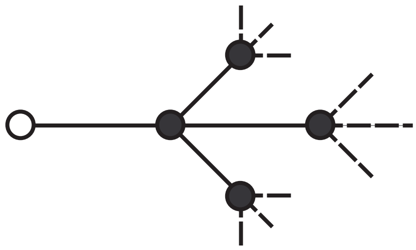

FIGURE 6.

With the exception of the trivial case when the Schreier graph X is a tree, i.e., $H = \{e\}$ (which we shall always exclude below), the core X_* is non-empty. Any hanging branch is contained in a unique maximal hanging branch, and maximal hanging branches are precisely those whose root belongs to the core. In other words, the graph X is obtained from the core X_* by filling the deficient valencies of the vertices of X_* with maximal hanging branches (so that all the degrees of the resulting graph have the full valency $|\breve{\mathcal{A}}|$). Thus, since the Schreier graph X is connected, its core X_* is also connected.

Remark 3.2. The definition of the core of a graph as what is left after removing all the subtrees is due to Gersten [Ger83, Sta83]. See [Sta91, KM02] for an exposition of the ensuing approach of Stallings to the study of subgroups of free groups based on the notion of a *folding* of graphs. Note that, following [Sta83, Section 7.2] we are talking about the *absolute* core of a graph which is independent of the choice of a reference vertex. Some authors (e.g., [BO09]) use a different definition, according to which the (relative) core is the union of all reduced loops in the Schreier graph X starting from a chosen reference point $o \in X$. The absolute and the relative cores coincide if and only if the reference vertex o lies in the absolute core.

The following property is, of course, known to specialists (moreover, it is basically the *raison d'être* of the definition of the core).

Proposition 3.3. *A subgroup $H \leq F$ is finitely generated if and only if the core X_* of the associated Schreier graph X is finite.*

Proof. There is a natural one-to-one correspondence between spanning trees in the Schreier graph X and in the core X_*. Indeed, the restriction of any spanning tree in X to X_* is a spanning tree in X_*. Conversely, any spanning tree in X_* uniquely extends to a spanning tree in X by attaching to it all the maximal hanging branches. Now, if the core is finite, then any spanning tree in it ($\equiv$ the associated spanning tree in X) is obtained by removing finitely many edges, so that the number of generators of H is finite (see Theorem 1.8). Conversely, if H is finitely generated, then the core is contained in the finite union of the cycles in X corresponding to the generators of H. $\square$

The definition of the core directly implies

Lemma 3.4. *All the paths $\varphi \in \mathsf{Paths}_o(X) \cup \mathsf{Paths}_o^\infty(X)$ (i.e., both finite and infinite paths without backtracking issued from o) can be uniquely split into the following three consecutive parts (some of which may be missing), see Figure 7:*

(i) *the geodesic segment joining the origin o with the root $o' \in X_*$ of the maximal hanging branch which contains o (if $o \notin X_*$ and φ passes through X_*);*
(ii) *the part of φ (possibly infinite) which is contained in X_*;*

(iii) *the part of φ (possibly infinite) entirely contained inside a certain hanging branch.*

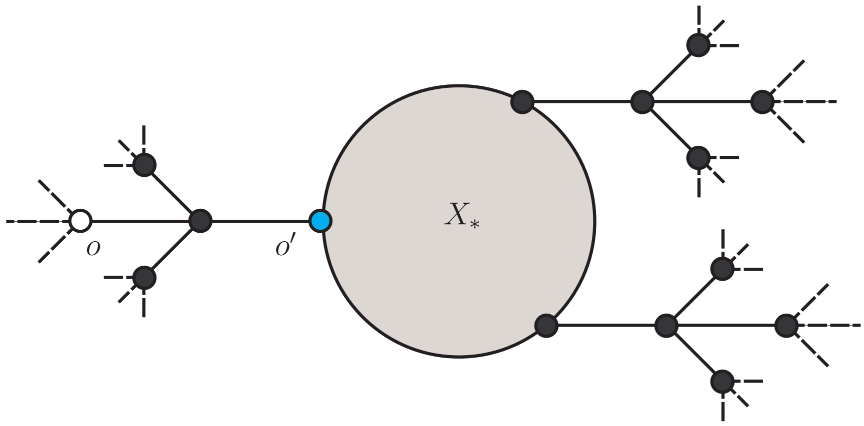

Figure 7.

Below we shall also use the following

Lemma 3.5. *Let $\mathcal{S}$ be the system of generators of a subgroup $H \le F$ determined by a spanning tree T in the Schreier graph X. Then for any $s \in \mathcal{S}$ the Schreier graph X' of the group $H' = \langle \mathcal{S} \setminus \{s\} \rangle$ is obtained by deleting from X the edge $\mathcal{E}_s$ and attaching a hanging branch to each of its endpoints.*

Proof. Being a Schreier graph, the edges of X are labelled with letters from $\breve{\mathcal{A}}$. This labelling extends to a labelling of X' which makes of it the Schreier graph of a certain subgroup of F. We can choose a spanning tree T' in X' by taking the union of T and of the hanging branches added during the construction of X'. The tree T' determines then a set of generators $\mathcal{S}'$, which, since the labellings of X and X' agree, coincides with $\mathcal{S} \setminus \{s\}$. □

3.B. The full limit set.

Definition 3.6. The *limit set* $\Lambda = \Lambda_H \subset \partial F$ of a subgroup $H \le F$ is the set of all the limit points of H with respect to the compactification $\widehat{F} = F \cup \partial F$ described in Section **1.C** (below we shall sometimes call this limit set *full* in order to distinguish it from other limit sets).

The limit set Λ is closed and H-invariant. The following description is actually valid for an arbitrary discrete group of isometries of a Gromov hyperbolic space, see [Gro87, Bou95]:

Theorem 3.7. *The action of H on Λ is* minimal *(there are no proper H-invariant closed subsets), whereas the action of H on the complement $\partial F \setminus \Lambda$ is* properly discontinuous *(no orbit has accumulation points).*

In our concrete situation the limit set Λ and a certain natural fundamental domain in $\partial F \setminus \Lambda$ admit the following very explicit description (similar to Proposition 1.18) in terms of the correspondence $\omega \mapsto \pi(\omega)$ between ∂F and the set $\mathsf{Paths}_o^\infty(X)$ of infinite paths without backtracking in X starting from the origin o (see Proposition 1.3 and the comment before Proposition 1.18).

Theorem 3.8.

(i) *The full limit set* $\Lambda \subset \partial F$ *corresponds to the set of paths from* $\mathsf{Paths}_o^\infty(X)$ *which eventually stay inside the core* X_* *(i.e., for these paths the component (ii) from Lemma 3.4 is infinite).*

(ii) *The full limit set* Λ *is the closure* $\overline{\Omega}$ *of the Schreier limit set.*

(iii) *The complement* $\partial F \setminus \Lambda$ *is a disjoint union of* H*-translates of the fundamental domain* $\Theta = \Delta \cap (\partial F \setminus \Lambda)$*. The set* Θ *is open and corresponds to the set of paths from* Paths_o^∞ *which do not pass through any of the edges from* $\mathsf{Edges}(X) \setminus \mathsf{Edges}(T)$ *and eventually stay inside a hanging branch (i.e., for which the component (iii) from Lemma 3.4 is infinite).*

(iv) $\Lambda = \Omega$ *(equivalently,* $\Delta = \Theta$*) if and only if* H *is finitely generated.*

Proof. (i) Elements of H correspond to cycles in $\mathsf{Paths}_o(X)$. By Lemma 3.4, if $o \in X_*$ then any cycle from $\mathsf{Paths}_o(X)$ is entirely contained in the core, and if $o \notin X_*$ then for any such cycle the components (i) and (iii) described in Lemma 3.4 are the geodesic segments $[o, o']$ and $[o', o]$, respectively. Thus, the pointwise limit of any sequence of such cycles (as their lengths go to infinity) is a path from $\mathsf{Paths}_o^\infty(X)$ with infinite component (ii). Conversely, if the n-th point $\varphi(n)$ of a path $\varphi \in \mathsf{Paths}_o^\infty(X)$ belongs to X_*, then the corresponding truncation $[\varphi]_n$ can be extended to a cycle without backtracking (as otherwise $\varphi(n)$ must be inside a hanging branch), so that φ is a pointwise limit of cycles from $\mathsf{Paths}_o(X)$.

(ii) As it follows from Theorem 1.11, $\Omega \subset \Lambda$, it is non-empty and H-invariant. Thus, in view of the minimality of Λ (Theorem 3.7), $\overline{\Omega} = \Lambda$.

(iii) Since $\Theta \subset \Delta$, the fact that its H-translates are pairwise disjoint and that their union is the complement of Λ follows at once from Theorem 1.21. The description of Θ in terms of the associated subset of $\mathsf{Paths}_o^\infty(X)$ is a combination of the description of the complement of Λ, which is (i) above, and of the description of the Schreier fundamental domain Δ (Proposition 1.18). Finally, Θ is open because hanging branches contain no edges from $\mathsf{Edges}(X) \setminus \mathsf{Edges}(T)$, so that if a path $\varphi \in \mathsf{Paths}_o^\infty$ corresponds to a point $\omega \in \Theta$, then the whole open cylinder $C_{[\omega]_n}$ is also contained in Θ for a sufficiently large n.

(iv) We shall prove this claim in terms of the descriptions of the sets Δ and Θ from Proposition 1.18 and from (iii) above, respectively. Therefore, in view of Proposition 3.3 we have to show that the core X_* is finite if and only if all the paths from $\mathsf{Paths}_o^\infty(X)$ confined to the spanning tree T eventually hit a certain hanging branch. Indeed, if X_* is finite, then the restriction of the spanning tree T to X_* is also finite, so that any path as above must eventually leave X_* and enter a hanging branch. Conversely, if X_* is infinite, then the restriction of the spanning tree T to X_* is also infinite, so that there is an infinite path without backtracking obtained by first joining o with X_* (if $o \notin X_*$) and then staying inside the restriction of T to X_*. □

Remark 3.9. Although the Schreier fundamental domain Δ is closed (Proposition 1.16) and contains the fundamental domain Θ, it is *not* necessarily the closure of Θ. For instance, it may happen that Θ is empty (i.e., $\Lambda = \partial F$), although Δ is not (and even has positive measure, see Remark 3.12).

Corollary 3.10. $\Lambda = \partial F$ *if and only if the Schreier graph* X *has no hanging branches (i.e.,* $X_* = X$*).*

Corollary 3.11. *If* X *has a hanging branch, then the boundary action of* H *has a non-trivial dissipative part.*

Proof. If X has a hanging branch, then by Corollary 3.10 $\Lambda \neq \partial F$, i.e., the fundamental domain Θ is non-empty. Since Θ is open, and the measure $\mathfrak{m}$ has full support, $\mathfrak{m}(\Theta) > 0$, so that Θ is a non-trivial wandering set. □

Remark 3.12. The converse of Corollary 3.11 is not true, see Example 4.19.

3.C. Radial limit set. Specializing the type of convergence in the definition of the full limit set (Definition 3.6) one obtains subsets of Λ with various geometric properties.

Definition 3.13. The *radial limit set* $\Lambda^{rad} \subset \partial F$ is the set of all the accumulation points of the sequences of elements of H which stay within bounded distance from a certain geodesic ray in F.

This definition in combination with Theorem 3.8(i) implies

Proposition 3.14. *The radial limit set $\Lambda^{rad} \subset \partial F$ corresponds to the set of paths $\varphi \in \mathsf{Paths}_o^\infty(X)$ which eventually stay inside the core X_* and do not go to infinity (i.e., $\liminf_n d_X(o, \varphi(n)) < \infty$).*

Proposition 3.15. *The radial limit set Λ^{rad} coincides with the full limit set Λ if and only if the group H is finitely generated.*

Proof. By Proposition 3.3, H is finitely generated if and only if the core X_* is finite, which implies the claim in view of Proposition 3.14. □

Remark 3.16. According to a result of Beardon and Maskit [BM74], a Fuchsian group is finitely generated if and only if its limit set is the union of the radial limit set and the set of parabolic fixed points. In our situation there are no parabolic points, so that Proposition 3.15 is a complete analogue of this result. In other words, a subgroup of a finitely generated free group is *geometrically finite* (see [Bow93]) if and only if it is finitely generated. Note that the core can also be defined as the quotient of the *geodesic convex hull* of the full limit set Λ by the action of H, so that in our situation geometrical finiteness of H coincides with its *convex cocompactness* (i.e., finiteness of the core).

3.D. Horospheric limit sets.

Definition 3.17. The *horosphere* passing through a point $g \in F$ and centered at a point $\omega \in \partial F$ is the corresponding level set of the Busemann cocycle β_ω:

$$\mathrm{Hor}_\omega(g) = \{g' \in F : \beta_\omega(g, g') = 0\} .$$

In the same way, the *horoballs* in F are defined as

$$\mathrm{HBall}_\omega(g) = \{g' \in F : \beta_\omega(g, g') \leq 0\} .$$

Restricting converging sequences to horoballs in F provides us with *horospheric limit points*. Unfortunately, the situation here is more complicated than with the radial limit points, and we have to define two different horospheric limit sets.

Definition 3.18. The *small* (resp., *big*) *horospheric limit set* $\Lambda^{horS} = \Lambda^{horS}_H$ (resp., $\Lambda^{horB} = \Lambda^{horB}_H$) of a subgroup $H \leq F$ is the set of all the points $\omega \in \partial F$ such that any (resp., a certain) horoball centered at ω contains infinitely many points from H.

In terms of the Busemann function a point $\omega \in \partial F$ belongs to Λ^{horS} (resp., to Λ^{horB}) if for any (resp., a certain) $N \in \mathbb{Z}$ there are infinitely many points $h \in H$ with $b_\omega(h) \leq N$.

Remark 3.19. Usually our small horospheric limit set is called just the *horospheric limit set*, and in the context of Fuchsian and Kleinian groups its definition, along with the definition of the radial limit set, goes back to Hedlund [Hed36]. Following [Mat02] we call it *small* in order to better distinguish it from the *big* one, which, although apparently first explicitly introduced by Tukia [Tuk97], essentially appears already in Pommerenke's paper [Pom76]. See [Sta95, DS00] for a detailed discussion of various kinds of limit points for Fuchsian groups.

The horospheric limit sets $\Lambda^{horS}, \Lambda^{horB}$ are obviously H-invariant and contained in the full limit set Λ (since the only boundary accumulation point of any horoball is just its center). The following theorem describes the relationship between the full limit set Λ, the radial limit set Λ^{rad}, the both horospheric limit sets, the Schreier limit set Ω and the set Σ (2.13).

Theorem 3.20. *One has the inclusions*

$$\Lambda^{rad} \subset \Lambda^{horS} \subset \Omega \subset \Lambda^{horB} \subset \Sigma \subset \Lambda \ .$$

Proof.

$\Lambda^{rad} \subset \Lambda^{horS}$. Obvious.

$\Lambda^{horS} \subset \Omega$. It follows from Theorem 1.21 and Proposition 2.10.

$\Omega \subset \Lambda^{horB}$. This inclusion was actually already established in the course of the proof of Theorem 2.12.

$\Lambda^{horB} \subset \Sigma$. Obvious.

$\Sigma \subset \Lambda$. Clearly, we may assume that $\Lambda \neq \partial F$. If $\omega \notin \Lambda$, then $\max_{h\in H}(h|\omega) < \infty$ (for, if $(h_n|\omega) \to \infty$, then $h_n \to \omega$). Thus, by formula (2.5) for any $\omega \notin \Lambda$ convergence of the series (2.13) from the definition of the set Σ is equivalent to convergence of the *Poincaré series* $\sum_{h\in H}(2m-1)^{-|h|}$. Now, if $\Lambda \neq \partial F$, then by Corollary 3.10 and Corollary 3.11 the boundary action has a non-trivial dissipative part, and the Poincaré series is convergent by Corollary 3.36 below. □

We shall show in Section **3.G** later on that all the inclusions in Theorem 3.20 are, generally speaking, strict. Nonetheless,

Theorem 3.21. *The sets* $\Lambda^{horS}, \Omega, \Lambda^{horB}, \Sigma$ *all coincide* $\mathfrak{m}$-mod 0.

Proof. As it follows from Theorem 2.2, Proposition 2.6 and Theorem 2.12, the sets Σ and Ω coincide $\mathfrak{m}$-mod 0. We shall show that $\mathfrak{m}(\Lambda^{horB} \setminus \Lambda^{horS}) = 0$ which would imply the claim. Indeed, on the H-invariant set $A = \Lambda^{horB} \setminus \Lambda^{horS}$, which is contained (mod 0) in the conservative part of the action, the projection $\widetilde{\partial F} \cong \partial F \times \mathbb{Z} \to \partial F$ admits a measurable H-equivariant section (which consists in assigning to a boundary point ω the "smallest" horosphere centered at ω and containing an infinite number of points from H). It implies that the ergodic components of the skew action of H on $A \times \mathbb{Z} \subset \widetilde{\partial F}$ are given by taking this section and its shifts over the ergodic components of the action of H on A, the latter being impossible by Theorem 3.38 on a set of positive measure. □

Remark 3.22. Coincidence (mod 0) of the conservative part of the boundary action with the big horospheric limit set Λ^{horB} is actually true in much greater generality of an arbitrary Gromov hyperbolic space endowed with a quasi-conformal boundary measure [Kai08]. The proof uses the fact that, by definition, the logarithms of the Radon–Nikodym derivatives of this measure are (almost) proportional to the Busemann cocycle, in combination with the criterion from Theorem 2.2.

3.E. Boundary action and random walks. An important aspect of the boundary behaviour is related to the asymptotic properties of *random walks* on the group F and on the Schreier graph X (see [KV83, Kai00b] and the references therein for the general background), and, in particular, to the fact that the uniform measure on the boundary $\mathfrak{m}$ can be interpreted as the harmonic measure of the simple random walk on F.

Let μ be the probability measure on the group F equidistributed on the generating set $\breve{\mathcal{A}}$. Then the random walk (F, μ) is precisely the *simple random walk* on the Cayley graph $\Gamma(F, \breve{\mathcal{A}})$, i.e., for any $g \in F$ the transition probability $\pi_g = g\mu$ is equidistributed on the set of neighbors of g in the graph $\Gamma(F, \breve{\mathcal{A}})$. Moreover, at each point $g \in F$ the increment $\zeta \in \breve{\mathcal{A}}$ is precisely the label of the edge along which the random walk moves to a new position.

The following result (which we reformulate in modern terms) is due to Dynkin and Maljutov [DM61]:

Theorem 3.23. *Sample paths of the simple random walk (F, μ) converge a.e. to the boundary ∂F, the hitting distribution is the uniform measure $\mathfrak{m}$, and the space $(\partial F, \mathfrak{m})$ is isomorphic to the Poisson boundary of this random walk.*

There is an obvious one-to-one correspondence between the harmonic functions of the simple random walk on the Schreier graph X (the definition of which — the same as for the simple random walk (F, μ) — takes into account eventual loops and multiple edges in X) and H-invariant μ-harmonic functions on F, which implies (see [Kai95] for a more detailed argument):

Theorem 3.24. *The space of ergodic components of the action of H on $(\partial F, \mathfrak{m})$ is canonically isomorphic to the Poisson boundary of the simple random walk on the Schreier graph X. In particular, the action is ergodic if and only if this random walk is Liouville ($\equiv$ has no non-constant bounded harmonic functions).*

Another direct corollary of the general theory of Poisson boundaries of covering Markov operators from [Kai95] is

Theorem 3.25. *The action of any non-trivial normal subgroup $H \lhd F$ on $(\partial F, \mathfrak{m})$ is conservative.*

Remark 3.26. A similar result in the hyperbolic setup is due to Matsuzaki [Mat93],[Mat02]: the boundary action of any normal subgroup of a Kleinian group of divergent type is conservative with respect to any Patterson measure class.

The situation when $\mathfrak{m}\Lambda = 0$ can be completely characterized in terms of the simple random walk on the Schreier graph.

Proposition 3.27. $\mathfrak{m}\Lambda = 0$ *if and only if a.e. sample path of the simple random walk on X eventually stays inside a certain hanging branch.*

Proof. Using the labelling of edges, any sample path (x_n) of the simple random walk on X can be uniquely lifted to a sample path (g_n) of the simple random walk on F. The latter a.e. converges to a boundary point $\omega \in \partial F$, and the distribution of ω is precisely the measure $\mathfrak{m}$ (Theorem 3.23). A sample path (x_n) eventually stays inside a hanging branch if and only if the path $\pi(\omega)$ does so, whence the claim in view of Theorem 3.8(i). □

Corollary 3.28. *If the simple random walk on the Schreier graph X is such that a.e. sample path eventually stays inside a certain hanging branch, then the boundary action is completely dissipative.*

Remark 3.29. The converse of Corollary 3.28 is not true, see Example 4.19.

Corollary 3.11 implies that the boundary action of any finitely generated group H of infinite index has a non-trivial dissipative part. Indeed, since H is finitely generated, the core X_* of the Schreier graph X is finite (Proposition 3.3). As H is of infinite index, the graph X is infinite and so necessarily has a hanging branch.

In fact, the following dichotomy completely describes the conservativity properties of the boundary action for finitely generated groups:

Theorem 3.30. *If H is finitely generated, then the Hopf alternative holds: either*

(i) *H is of finite index and its boundary action is ergodic (therefore, conservative),*

or

(ii) *H is of infinite index and its boundary action is completely dissipative.*

Proof. If a subgroup H has a finite index, then the associated Schreier graph X is finite, and therefore the simple random walk on it is Liouville, so that the boundary action of H is ergodic by Theorem 3.24.

If H is finitely generated of infinite index, then X in an infinite graph consisting of a finite core X_* and some hanging branches glued to it (see Proposition 3.3). Since the simple random walk in any hanging branch is transient, the simple random walk on X is also transient, which implies that a.e. sample path eventually stays inside a certain hanging branch, whence the claim by Corollary 3.28. □

Remark 3.31. In view of Proposition 3.27 the dichotomy from Theorem 3.30 can be reformulated in the following way: for a finitely generated group H either the Schreier graph X is finite, or else $\mathfrak{m}\Lambda = 0$.

Remark 3.32. For infinitely generated subgroups this dichotomy does not hold, for instance, see Example 4.27.

Remark 3.33. An unexpected application of Theorem 3.30 is a one line conceptual proof of an old theorem of Karrass and Solitar [KS57]: if $H \le F$ is finitely generated and contains a non-trivial normal subgroup, then H is of finite index in F (if H itself is a normal subgroup, this was first proved by Schreier in his famous 1927 paper [Sch27]). Indeed, if H contains a normal subgroup, then its boundary action is conservative by Theorem 3.25 and Remark 2.1. Since H is finitely generated, by Theorem 3.30 it must be of finite index in F. Note that a recent far-reaching generalization of the Karrass–Solitar theorem [PT07, Corollary 5.13] in particular extends it to all subgroups $H \le F$ with the property that the set $gH \cap Hg$ is infinite for all $g \in F$. It would be interesting to compare this property with the conservativity of the boundary action.

Remark 3.34. Any infinitely generated subgroup H of infinite index with conservative boundary action readily provides the following example: the action of any finitely generated subgroup of H is completely dissipative by Theorem 3.30 in spite of conservativity of the action of the whole group H.

3.F. Extensions of the boundary action. There are two extensions of the boundary action which have natural geometric interpretations. The first one is the action of H on the space $\partial^2 F := (\partial F \times \partial F) \setminus \mathrm{diag}$, or, in other words, on the space of bi-infinite geodesics in the Cayley graph $\Gamma(F, \breve{\mathcal{A}})$. We shall endow it with the square $\mathfrak{m}^2$ of the measure $\mathfrak{m}$.

Ergodicity of this action is equivalent to ergodicity of the (discrete) geodesic flow on X. The study of the ergodic properties of the action of a discrete group of hyperbolic isometries of $\mathbb{H}^n$ on $\partial^2\mathbb{H}^n$ has a long history beginning with the pioneering works of Hedlund and E. Hopf in the 30's for Fuchsian groups. Its current state is given by the so-called *Hopf–Tsuji–Sullivan theorem*, e.g., see [Sul81, Nic89, Kai94] and the references therein. Analogous results for the action of a subgroup H of a free group on $\partial^2 F$ with respect to the measure $\mathfrak{m}^2$ were obtained by Coornaert and Papadopoulos [CP96]. The general case of the harmonic measure of a covering Markov operator on a Gromov hyperbolic space was treated by Kaimanovich [Kai94]. In our situation the results can be summarized in the following analogue of the Hopf–Tsuji–Sullivan theorem:

Theorem 3.35. *The action of H on $(\partial^2 F, \mathfrak{m}^2)$ is either ergodic (therefore, conservative) or completely dissipative (the* Hopf alternative*). If the action is ergodic, then*

(i) $\mathfrak{m}\Lambda^{rad} = 1$;
(ii) *The simple random walk on the Schreier graph X is recurrent;*
(iii) *The* Poincaré series $\sum_{h\in H}(2m-1)^{-|h|}$ *diverges.*

Alternatively, if the action is completely dissipative, then

(i′) $\mathfrak{m}\Lambda^{rad} = 0$;
(ii′) *The simple random walk on the Schreier graph X is transient;*
(iii′) *The Poincaré series converges.*

Corollary 3.36. *If the action of H on $(\partial F, \mathfrak{m})$ is dissipative, then the Poincaré series converges.*

Proof. Since the action of H on $(\partial^2 F, \mathfrak{m}^2)$ projets to the action on $(\partial F, \mathfrak{m})$, dissipativity of the latter implies dissipativity of the former. □

Remark 3.37. The action of F on $\partial^3 F$ (and therefore on all the higher products) is properly discontinuous in view of the existence of an equivariant *barycenter map* $\partial^3 F \to F$. Hence, it is dissipative for any purely non-atomic measure.

Yet another extension of the boundary action is obtained by taking the skew action of H on $\widetilde{\partial F} := \partial F \times \mathbb{Z}$ determined by the Busemann cocycle. Geometrically this action is just the action of H on the space of horospheres in F (see Section **3.D**). The space $\widetilde{\partial F}$ is endowed with the natural measure $\widetilde{\mathfrak{m}}$ which is the product of $\mathfrak{m}$ and the counting measure on $\mathbb{Z}$. The ergodic properties of this action essentially coincide with the ergodic properties of the boundary action, namely:

Theorem 3.38. *Let $\partial F = \mathcal{C} \cup \mathcal{D}$ be the decomposition of the boundary ∂F into the conservative and the dissipative parts of the H-action. Then the conservative and the dissipative parts of the action of H on the space $\widetilde{\partial F}$ are $\mathcal{C} \times \mathbb{Z}$ and $\mathcal{D} \times \mathbb{Z}$, respectively. The conservative ergodic components of the H-action on $\widetilde{\partial F}$ are the preimages of the ergodic components of the H-action on $\mathcal{C}$ under the projection $\widetilde{\partial F} \to \partial F$. In particular, the action of H on $\mathcal{C} \times \mathbb{Z}$ is ergodic if and only if the action of H on $\mathcal{C}$ is ergodic.*

The proof of this theorem almost *verbatim* coincides with the proof of the analogous result for the boundary actions of Fuchsian groups [Kai00a, Theorem 4.2] based in turn on Sullivan's proof in [Sul82] (see also [Sul81]) of the fact that the boundary action of a Kleinian group is of type III_1 on its conservative part with respect to the Lebesgue measure. The only difference is that in our situation the range of the Busemann cocycle is $\mathbb{Z}$ (rather than $\mathbb{R}$ as in the case of manifolds), so that, in particular, the type of the boundary action of H on the conservative part is III_λ with $\lambda = \log(2m-1)$.

3.G. Examples. We shall now give examples showing that all the sets from Theorem 3.20 are, in general, pairwise distinct.

Example 3.39. $\Lambda^{rad} \neq \Lambda^{horS}$. By Theorem 2.12, Theorem 3.21, Theorem 3.24 and Theorem 3.35, if the simple random walk on X is transient, but still has the Liouville property, then $\mathfrak{m}\Lambda^{rad} = 0$, whereas $\mathfrak{m}\Lambda^{horS} = 1$. Such examples are readily available already in the case when the subgroup H is normal and X is the quotient group, see [KV83].

Example 3.40. $\Lambda^{horS} \neq \Omega$. Let us take two geodesic rays $\rho_1, \rho_2 \cong \mathbb{Z}_+$ joined at the origin o, so that $\rho_1(0) = \rho_2(0) = o$. We construct the Schreier graph X by first adding the edges $[\rho_1(n+1), \rho_2(n)]$ for all $n > 0$ and then filling all the deficient valencies with hanging branches. The geodesic spanning tree T is obtained from X by removing all the edges $[\rho_1(n), \rho_1(n+1)]$ for $n > 0$, see Figure 8. Let $\omega_1 \in \partial F$ be the boundary point which corresponds to the ray ρ_1 considered as a path without backtracking in X. Then $\omega_1 \in \Omega$ because ρ_1 passes through an infinite number of edges from $\mathsf{Edges}(X) \setminus \mathsf{Edges}(T)$. On the other hand, since ω_1 corresponds to a geodesic in X, $\omega \notin \Lambda^{horS}$ by Proposition 2.11.

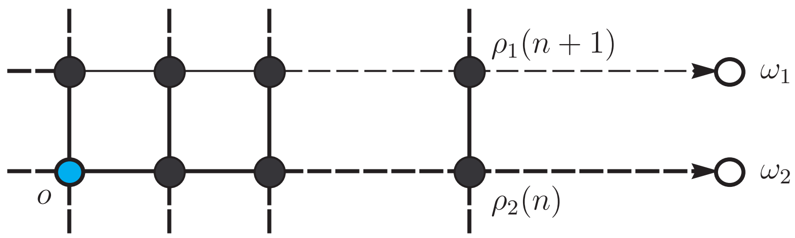

FIGURE 8.

Example 3.41. $\Omega \neq \Lambda^{horB}$. Let X be the same graph as in the previous example. Take the boundary point $\omega_2 \in \partial F$ which corresponds this time to the ray ρ_2, see Figure 8. Then $\omega_2 \notin \Omega$. On the other hand, for any $n > 0$ the generator s_n corresponding to the edge $[\rho_1(n), \rho_1(n+1)]$ has the property that $b_{\omega_2}(s_n) = 2$, whence $\omega_2 \in \Lambda^{horB}$.

Example 3.42. $\Lambda^{horB} \neq \Sigma$. Let us take a geodesic ray $\rho \cong \mathbb{Z}_+$ starting at the origin $o = \rho(0)$ and an integer sequence d_n (to be specified later). We construct the Schreier graph X by first attaching to each vertex $\rho(n)$, $n > 0$, a loop of length $2d_n + 1$ and then filling all the deficient valencies with hanging branches. The spanning geodesic tree T is obtained by removing from each such loop the middle edge $\mathcal{E}_n$, so that in T there are two segments of length d_n attached to each point $\rho(n)$, see Figure 9. Denote by s_n the corresponding generators. Let $\omega \in \partial F$ be the boundary point corresponding to the ray ρ. Then for any $h \in H \cong \check{\mathcal{S}}_r^*$ one has the inequality $b_\omega(h) \geq \sum_n t_n(2d_n+1)$, where t_n is the number of occurrences of $s_n^{\pm 1}$ in h. Indeed, by (2.5) $b_\omega(h) = |h| - 2(h|\omega)$. We can write $|h| = l + \sum_n t_n(2d_n+1)$, where l is the sum of the lengths of the pieces of the associated path in X which correspond to moving along the ray ρ. The latter sum contains the term $(h|\omega)$ which corresponds to the confluent of h and ρ, and, since h is a cycle, one has $l \geq 2(h|\omega)$, which implies the desired inequality. Now, if $d_n \uparrow \infty$, then $\omega \notin \Lambda^{horB}$. On the other hand, $b_\omega(s_n) = 2d_n + 1$, and one can still choose d_n is such a way that $\sum_n (2m-1)^{-2d_n} = \infty$, so that $\omega \in \Sigma$.

Example 3.43. $\Sigma \neq \Lambda$. Since Σ coincides (mod 0) with the conservative part of the action (see Theorem 3.21), whenever the boundary action is completely dissipative we

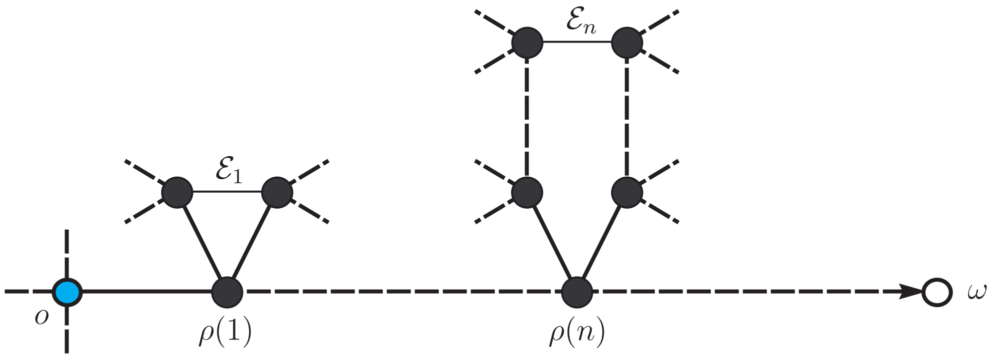

FIGURE 9.

have $\mathfrak{m}\Sigma = 0$. On the other hand, in this situation it is still possible that $\Lambda = \partial F$, i.e., that the Schreier graph has no hanging branches (Corollary 3.10), see Example 4.19.

4. GEOMETRY OF THE SCHREIER GRAPH

In this Section we shall study the relationship between the ergodic properties of the action of the subgroup H on the boundary $(\partial F, \mathfrak{m})$ and the geometry of the Schreier graph $X = H\backslash F$ (in particular, its quantitative characteristics).

4.A. Growth and cogrowth. Denote by S_F^n and B_F^n (resp., S_X^n and B_X^n) the sphere and the ball of radius n in the Cayley graph $\Gamma(F, \breve{\mathcal{A}})$ (resp., in the Schreier graph $X = \Gamma(H\backslash F, \breve{\mathcal{A}})$) centered at the group identity e (resp., at the point $o = H$). The *exponential volume growth rate* of X is

$$v_X = \limsup |B_X^n|^{1/n} \le v_F = 2m - 1 .$$

The *cogrowth rate* of X ($\equiv$ the growth rate of H in F) is

$$v_H = \limsup_{n\to\infty} |H \cap B_F^n|^{1/n} \le 2m - 1 ,$$

and it is the inverse of the radius of convergence of the *cogrowth series*

$$\mathcal{G}_H(z) = \sum_n |H \cap S_F^n| z^n .$$

Denote by ρ_F (resp., ρ_X) the spectral radius of the simple random walk on F (resp., on $X = \Gamma(H\backslash F, \breve{\mathcal{A}})$), so that $\sqrt{2m-1}/m = \rho_F \le \rho_X \le 1$ [Kes59]. By [Gri78, Gri80]

$$\rho_X = \begin{cases} \frac{\sqrt{2m-1}}{m} & , \quad 1 \le v_H \le \sqrt{2m-1} , \\ \frac{\sqrt{2m-1}}{2m}\left(\frac{\sqrt{2m-1}}{v_H} + \frac{v_H}{\sqrt{2m-1}}\right) & , \quad \sqrt{2m-1} \le v_H \le 2m-1 , \end{cases} \tag{4.1}$$

which implies that $\rho_X = \rho_F$ if and only if $v_H \le \sqrt{2m-1}$, and $\rho_X = 1$ (i.e., the graph X is *amenable*) if and only if $v_H = 2m - 1$. It was also shown in [Gri78, Gri80] that the simple random walk on X is transient if and only if $\mathcal{G}_H(1/(2m-1)) < \infty$.

4.B. Dissipativity.

Theorem 4.2. *If $v_H < \sqrt{2m-1}$ then the boundary action of H is completely dissipative.*

Proof. For $h = s_1 s_2 \dots s_n \in \breve{\mathcal{S}}_r^* \cong H$ let $C_h \subset \partial H$ be the corresponding cylinder set, so that

$$\Omega = \bigcup_{h \in H} \sigma^\infty(C_h) .$$

By (1.14)

$$\sigma^\infty(C_h) \subset C_g ,$$

where $C_g \subset \partial F$ is the cylinder set based at the word

$$g = \sigma_-(s_1)\sigma_0(s_1)\alpha(s_1, s_2)\sigma_0(s_2) \dots \alpha(s_{n-1}, s_n)\sigma_0(s_n) \in \breve{\mathcal{A}}_r^* \cong F .$$

Since $|g| \geq |\sigma(h)|/2$ (see the first half of the proof of Proposition 2.9),

$$\mathfrak{m}\sigma^\infty(C_h) \leq \mathfrak{m}C_g = \frac{1}{2m(2m-1)^{|g|-1}} \leq \frac{1}{(2m-1)^{|\sigma(h)|/2}} ,$$

whence

$$\sum_{h \in H} \mathfrak{m}\sigma^\infty(C_h) < \infty ,$$

which implies the claim by Borel–Cantelli lemma, because any point from ∂H belongs to infinitely many cylinders C_h. □

Remark 4.3. The converse of Theorem 4.2 is not true, see Example 4.25.

Remark 4.4. The upper bound $\sqrt{2m-1}$ in Theorem 4.2 is optimal. Indeed, if $N \lhd F$ is a non-trivial normal subgroup, then its boundary action is always conservative, see Theorem 3.25. On the other hand, N being non-amenable, $\rho_{F/N} > \rho_F$ by Kesten's theorem [Kes59], whence by formula (4.1) $v_N > \sqrt{2m-1}$. There are numerous examples of normal subgroups N for which $\rho_{F/N}$ is arbitrarily close to ρ_F, and therefore by (4.1) v_N is arbitrarily close to $\sqrt{2m-1}$. For instance, if N_l is the kernel of the natural homomorphism $F = \mathbb{Z} * \mathbb{Z} * \dots * \mathbb{Z} \to \mathbb{Z}_l * \mathbb{Z}_l * \dots * \mathbb{Z}_l$, then $\rho_{F/N_l} \to \rho_F$ as $l \to \infty$ by explicit formulas from [Woe00, Section 9].

Remark 4.5. We do not now whether there exist subgroups $H \leq F$ whose boundary action is not completely dissipative (or, even better, is conservative), but $v_H = \sqrt{2m-1}$ (equivalently, $\rho_X = \rho_F$). It is impossible when the Schreier graph X is radially symmetric. [For, then the radial part of the simple random walk on X would also have spectral radius ρ_F and would therefore have a positive ρ_F-invariant function, e.g., see [MW89] and the references therein. On the other hand, an explicit calculation shows that if such a function exists, then the number of cycles in X must be finite, and therefore the action must be completely dissipative by Theorem 3.30.]

Remark 4.6. Results similar to Theorem 4.2 are known for Fuchsian groups (where completely different methods were used). The fact that if the critical exponent $\delta = \delta_H$ ($\equiv$ the logarithmic rate of growth with respect to the Riemannian metric) of a Fuchsian group H satisfies inequality $\delta < \frac{1}{2}$, then the boundary action is completely dissipative with respect to the Lebesgue measure class goes back to Patterson [Pat77]. Another proof is given in a recent paper of Matsuzaki [Mat05], where examples of Fuchsian groups with conservative boundary action for which δ is arbitrarily close to $\frac{1}{2}$ are constructed. However, it is unknown whether such an example exists with the critical exponent precisely $\frac{1}{2}$ (cf. Remark 4.4).

4.C. Conservativity. Denote by $\gamma(x)$ the number of edges from the set $\mathsf{Edges}(X) \setminus \mathsf{Edges}(T)$ incident with a vertex $x \in X$, so that the degree of x in the spanning tree T is

$$\deg_T(x) = \deg_X(x) - \gamma(x) = 2m - \gamma(x) \ . \tag{4.7}$$

Proposition 4.8.

$$\mathfrak{m}\Delta \ = 1 - \frac{1}{2m} \sum_{x \in X} \frac{\gamma(x)}{(2m-1)^{|x|}} \ . \tag{4.9}$$

Proof. By Proposition 1.19 and Theorem 1.21

$$\partial F \setminus \Delta = \bigsqcup_{s \in \breve{\mathcal{S}}} C_{\sigma_-(s)\sigma_0(s)} \ ,$$

whence by formula (2.3)

$$\mathfrak{m}\Delta = 1 - \frac{1}{2m} \sum_{s \in \breve{\mathcal{S}}} \frac{1}{(2m-1)^{|\sigma_-(s)|}} \ ,$$

which implies the claim in view of the one-to-one correspondence between the set $\breve{\mathcal{S}}$ and the oriented edges not in T established in Theorem 1.8. □

Theorem 4.10. *The sequence*

$$a_n = \frac{|S_X^n|}{|S_F^n|} = \frac{|S_X^n|}{2m(2m-1)^{n-1}}$$

monotonously decreases and converges to $\mathfrak{m}\Delta$.

Proof. By Proposition 4.8

$$\mathfrak{m}\Delta = 1 - \frac{1}{2m} \sum_{n=0}^{\infty} \frac{\gamma_n}{(2m-1)^n} \ , \tag{4.11}$$

where

$$\gamma_n = \sum_{x \in S_X^n} \gamma(x) \ .$$

In view of (4.7) the numbers $|S_X^n|$ and γ_n are connected by the formulas

$$|S_X^1| = \deg_T o = \deg_X o - \gamma_0 = 2m - \gamma_0$$

and

$$|S_X^{n+1}| = \sum_{x \in S_X^n} (\deg_T x - 1) = (2m-1)|S_X^n| - \gamma_n \ , \qquad n > 0 \ ,$$

which imply monotonicity of a_n. Substituting

$$\gamma_0 = 2m - |S_X^1| \ ,$$

and

$$\gamma_n = (2m-1)|S_X^n| - |S_X^{n+1}| \ , \qquad n > 0$$

into formula (4.11) yields the claim. □

As a corollary we immediately obtain:

Theorem 4.12. *The boundary action of H is conservative if and only if*

$$\lim_n \frac{|S_X^n|}{|S_F^n|} = \lim_n \frac{|S_X^n|}{2m(2m-1)^{n-1}} = 0 \ .$$

Remark 4.13. In different terms this result is also independently proved in the recent paper [BO09, Theorem 9].

Corollary 4.14. *If $v_X < 2m-1$, then the boundary action of H is conservative.*

Remark 4.15. The converse of Corollary 4.14 is not true, see Example 4.26.

Remark 4.16. Theorem 4.12 can also be reformulated as saying that the boundary action is conservative if and only if $|B_X^n|/|B_F^n| \to 0$. In the context of Fuchsian groups this result (obtained by entirely different means) with B_F^n (resp., $|B_X^n|$) replaced with the area of the n-ball in the hyperbolic plane (resp., in the quotient surface) was proved by Sullivan [Sul81, Theorem IV], and essentially goes back to Hopf [Hop39].

Theorem 4.2 and Theorem 4.12 imply

Corollary 4.17. *If $|S_X^n|/|S_F^n| \to 0$ (in particular, if $v_X < 2m-1$), then $v_H \geq \sqrt{2m-1}$.*

Remark 4.18. Since $v_{F/N} < v_F$ for any non-trivial normal subgroup $N \lhd F$ (e.g., see [GdlH01]), as another corollary one obtains a "quantitative" proof of Theorem 3.25.

4.D. Examples.

Example 4.19 (counterexample to the converse of Corollary 3.11). The group H is infinitely generated, the graph X has no hanging branches, but v_H is arbitrarily close to 1, so that the action of H on $(\partial F, \mathfrak{m})$ is completely dissipative (see Theorem 4.2).

We construct the graph X inductively by starting from the homogeneous tree $X_0 = T_{2m}$ of degree $2m$ with origin o. We shall think of the spheres $S_{X_k}^R$ centered at o in the graphs X_k as *levels*, and follow the perverse tradition according to which trees are allowed to grow downwards, so that the level 0 (consisting just of the origin) is the highest (actually, in Figure 10 below we shall draw it from left to right). In this construction we shall need an increasing integer sequence $1 = d_1 < d_2 < \dots$ to be specified later.

The inductive step of the construction consists in choosing two points $x_{k+1} \neq y_{k+1} \in S_{X_k}^{d_{k+1}}$ for a certain integer d_{k+1} to be specified later in such a way that their "predecessors" in $S_{X_k}^{d_k}$ are distinct. The graph X_{k+1} is then obtained from X_k in the following way:

- remove from X_k one of the branches growing from x_{k+1} downwards;
- do the same with the point y_{k+1};
- add the edge $\mathcal{E}_{k+1}$ joining x_{k+1} and y_{k+1}.

Thus, all the graphs X_k are also $2m$-regular and have the same origin o. Finally, the graph X is the limit of the sequence X_k, see Figure 10.

The graph X has a natural geodesic spanning tree T which is obtained by removing all the horizontal edges $\mathcal{E}_k$ added during the construction of X. Denote by s_k the corresponding generator of H, and let $H_k = \langle s_1, s_2, \dots, s_k \rangle$.

Obviously, the sequences of points x_k, y_k can be chosen in such a way that X has no hanging branches as the latter condition is equivalent to the property that the intersection of any shadow in the spanning tree T (with respect to the origin o) with the set $\{x_2, y_2, x_3, y_3, \dots\}$ is non-empty.

We shall now explain (once again inductively) how to choose the sequence d_k. More precisely, we shall show that once the numbers $d_1, d_2, \dots, d_k$ and the group H_k have already been chosen, and the group H_k has the property that

$$|H_k \cap B_F^n| \leq C\alpha_k^n \qquad \forall n \geq 0 \tag{4.20}$$

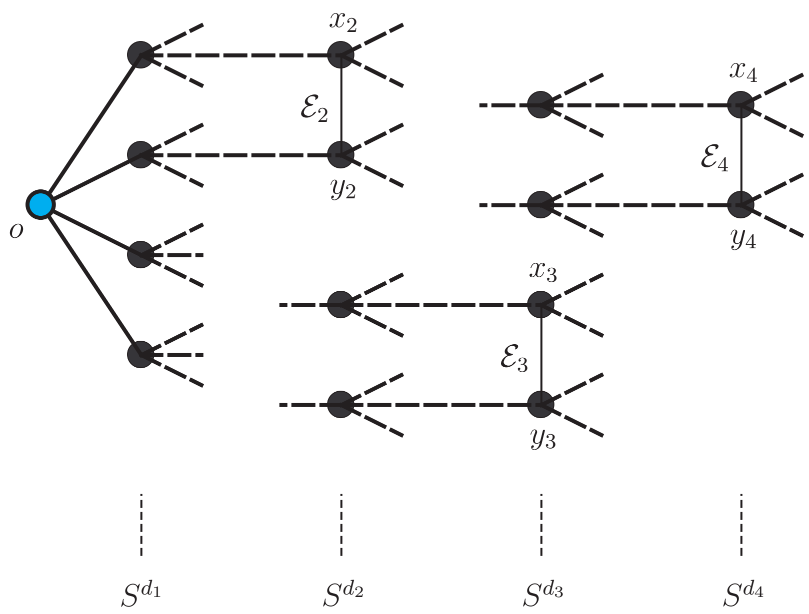

FIGURE 10.

for certain constants $C, \alpha_k > 1$, then for any $\alpha_{k+1} > \alpha_k$ the distance d_{k+1} can be chosen in such a way that for the group H_{k+1} also

$$|H_{k+1} \cap B_F^n| \le C\alpha_{k+1}^n \qquad \forall n \ge 0 \tag{4.21}$$

(with the same constant C!). Then, starting from the group $H_1 \cong \mathbb{Z}$ (which has subexponential growth), and taking an arbitrary sequence

$$1 < \alpha_1 < \alpha_2 < \cdots < \alpha_k \cdots \nearrow \alpha$$

we would be able to conclude that $v_H \le \alpha$.

For notational simplicity we put $s = s_{k+1}$. Any element of the group $H_{k+1} = \langle H_k, s\rangle$ can be presented as

$$g = h_0 s^{\varepsilon_1} h_1 s^{\varepsilon_2} \dots h_{t-1} s^{\varepsilon_t} h_t \tag{4.22}$$

for certain $t \ge 0$, $h_i \in H$ and $\varepsilon_i = \pm 1$ such that $\varepsilon_i = \varepsilon_{i+1}$ whenever $h_i = e$. As it follows from the definition of s, the length of cancellations on each side between any two consecutive terms in the above expansion does not exceed d_k. Since $|s| = 2d_{k+1} + 1$, we obtain the inequality

$$|g| \ge \sum_{i=0}^{t} |h_i| + t|s| - 4td_k = \sum_{i=0}^{t} |h_i| + tD\ , \tag{4.23}$$

where

$$D = |s| - 4d_k = 2d_{k+1} + 1 - 4d_k\ . \tag{4.24}$$

We shall now estimate $|H_{k+1} \cap B_F^n|$, i.e., the number of elements g of the form (4.22) with $|g| \le n$, by using the inequality (4.23). We have to control the following numbers:

(a) The number t of occurrences of $s^{\pm 1}$ in the expansion (4.22);
(b) The number N_b of choices of the signs ε_i for a given value of t;
(c) The number N_c of possible sets of lengths $|h_i| = l_i$ of the words h_i for given t;

(d) The number N_d of the choices of the words $h_i \in H_k$ with the prescribed lengths l_i. Let us find the corresponding estimates one by one.

(a) By (4.23)

$$t \le \tau = n/D\ .$$

(b) Trivially,

$$N_b \le 2^\tau\ .$$

(c) By (4.23), N_c does not exceed the number of ordered partitions of $n - tD$ into not more than $t+1$ integer summands, so that

$$N_c \le \binom{n - tD + t}{t}\ .$$

(d) Finally, by (4.20) and (4.23)

$$N_d \le C^\tau \alpha_k^n\ .$$

Thus,

$$|H_{k+1} \cap B_F^n| \le \tau 2^\tau \max_{t \le \tau} \binom{n - tD + t}{t} C^\tau \alpha_k^n\ ,$$

and in order to conclude it is sufficient just to estimate the max in the above product. Since

$$\begin{aligned}\frac{1}{n}\log\binom{n - tD + t}{t} &= \frac{1}{n}\log\binom{n - tD'}{t}\\ &\le \frac{1}{n}\log\frac{(n - tD')^t}{(t/e)^t} = \frac{t}{n}\left(1 + \log\left(\frac{n}{t} - D'\right)\right) = \frac{1 + \log(z - D')}{z}\ ,\end{aligned}$$

where $D' = D - 1$ and $z = n/t \ge D$, we can choose D (equivalently, d_{k+1}, in view of formula (4.24)) in such a way that

$$\sup_{z \ge D} \frac{1 + \log(z - D')}{z}$$

is arbitrarily small, whence the claim.

Example 4.25 (counterexample to the converse of Theorem 4.2). The cogrowth $v_H = 2m - 1$ is maximal, i.e., the Schreier graph X is amenable (see Section **4.A**), but the boundary action is completely dissipative.

Take the homogeneous tree T_{2m} with the root o, remove one of the branches rooted at o, and replace it with a geodesic ray $\rho \cong \mathbb{Z}_+$. Then attach to all the vertices of ρ other than the origin $m-1$ length 1 loops, so that the resulting graph X is $2m$-regular: it is a union of $2m-1$ hanging branches and the ray ρ (with attached loops) joined at the point o, see Figure 11. The graph X is obviously amenable because of the presence of the ρ branch. On the other hand, the simple random walk on it eventually stays inside one of the hanging branches (since the simple random walk on $\mathbb{Z}_+$ is recurrent), whence the claim in view of Corollary 3.28.

Example 4.26 (counterexample to the converse of Corollary 4.14). The growth $v_X = 2m-1$ is maximal, but the boundary action is conservative.

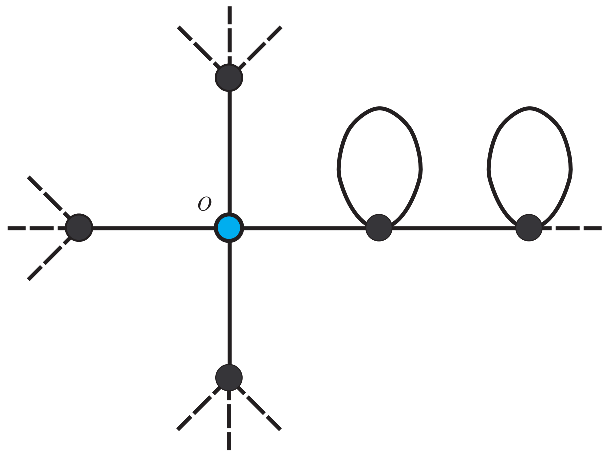

FIGURE 11.

Take the homogeneous tree T_{2m} with the root o and take an increasing sequence d_n with density 0, i.e., such that $d_n/n \to \infty$. Then add one new vertex in the middle of each edge of T_{2m} joining the spheres of radii d_n and $d_n + 1$ and attach to every such vertex $m-1$ length 1 loops. Then the resulting graph X is radially symmetric, $2m$-regular, and has the growth $v_X = 2m - 1$, although it satisfies the condition of Theorem 4.12. The radial part of this graph is presented on Figure 12.

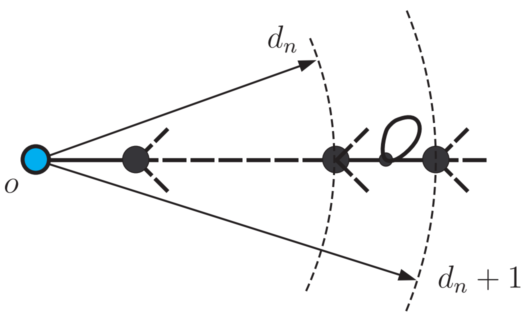

FIGURE 12.

Example 4.27 (counterexample to an extension of Theorem 3.30). An infinitely generated subgroup H such that its boundary action has both conservative and dissipative parts of positive measure.

Take a group $\widetilde{H} \le F$ such that the simple random walk on the associated Schreier graph $\widetilde{X}$ is transient and the boundary action of $\widetilde{H}$ is conservative (for instance, this is the case when $\widetilde{H}$ is a normal subgroup with transient quotient, see Theorem 3.25). Then necessarily $\widetilde{H}$ is infinitely generated by Theorem 3.30. Let $\widetilde{\mathcal{S}}$ be the system of generators of $\widetilde{H}$ determined by a geodesic spanning tree $\widetilde{T}$. Then for any $s \in \widetilde{\mathcal{S}}$ the group $H = \langle \mathcal{S} \rangle$, $\mathcal{S} = \widetilde{\mathcal{S}} \setminus \{s\}$ has the desired property.

Indeed, by Lemma 3.5 the Schreier graph $X = H \backslash F$ has hanging branches, which implies that the dissipative part of the boundary action of H is non-trivial (see Corollary 3.11).

Let us now prove non-triviality of the conservative part. Let $\mathcal{E} = \mathcal{E}_s \in \mathsf{Edges}(\widetilde{X}) \setminus \mathsf{Edges}(\widetilde{T})$ be the edge corresponding to the generator s, and let $A \subset \partial F$ be the set of all points ω such that the associated path $\widetilde{\pi}(\omega)$ in $\widetilde{X}$ never passes through the edge $\mathcal{E}$ (in either direction). Since the boundary action of $\widetilde{H}$ is conservative, by Theorem 2.12 for a.e. point $\omega \in A$ the path $\widetilde{\pi}(\omega)$ passes nonetheless through infinitely many edges not in $\widetilde{T}$.

Transience of the simple random walk implies that $\mathfrak{m}A > 0$. Namely, by using labelling along edges every trajectory (x_n) of the random walk on X lifts to a trajectory (g_n) of the simple random walk on F. Denote by $g_\infty \in \partial F$ its limit point. Then the set of edges through which the path $\widetilde{\pi}(g_\infty)$ passes is contained in the analogous set for the original sample path (x_n). Thus, if (x_n) never passes through $\mathcal{E}$, then the path $\widetilde{\pi}(g_\infty)$ also has this property. By transience the probability of the former event is positive, and since the image of the measure in the space of sample paths under the above transformation is $\mathfrak{m}$, the claim follows.

Now, for any $\omega \in A$ the associated path $\pi(\omega)$ in the Schreier graph X passes through the same edges as the path $\widetilde{\pi}(\omega)$ in $\widetilde{X}$ (under the natural identification described in Lemma 3.5), i.e., it passes through the edges which are not in the spanning tree T of X infinitely many times. Since $\mathfrak{m}A > 0$, it means that the conservative part of the boundary action of H is also non-trivial.

Remark 4.28. One can also show that in the above construction the ergodic components of the boundary action of $\widetilde{H}$ are in one-to-one correspondence with the ergodic components of the conservative part of the boundary action of H. In particular, if the boundary action of $\widetilde{H}$ is ergodic, then the conservative part of the boundary action of H is also ergodic. It follows from the fact that the Poisson boundary of the simple random walk on $\widetilde{X}$ does not change after removing the edge $\mathcal{E}$.

Remark 4.29. In the context of covering Markov operators a similar example with both conservative and dissipative components in the Poisson boundary was constructed in [Kai95].

Remark 4.30. In the above example the measure $\mathfrak{m}$ of the full limit set Λ is intermediate between 0 and 1. Indeed, $\mathfrak{m}(\Lambda) < 1$ because the Schreier graph has a hanging branch. On the other hand, $\mathfrak{m}(\Lambda) > 0$ because the boundary action has a non-trivial conservative part.

5. Associated Markov chains

As we have already seen, the simple random walk on the Schreier graph plays the crucial role in understanding the ergodic properties of the boundary action. In this Section we shall look at two other Markov chains closely connected with the considered problems.

5.A. Random walk on edges. The uniform boundary measure $\mathfrak{m}$ can be interpreted as the measure in the path space of the following Markov chain $(\mathcal{E}_n)$ on the set of oriented edges of the Schreier graph X. Its initial distribution is uniform on the set of $2m$ edges issued from the origin o, and the transition probability from an arbitrary edge $\mathcal{E}$ is equidistributed on the set of $2m-1$ edges $\mathcal{E}'$ issued from the endpoint of $\mathcal{E}$ without backtracking (in other words, the labels on $\mathcal{E}$ and $\mathcal{E}'$ do not cancel). Then Proposition 1.18 and Theorem 2.12 imply

Theorem 5.1. *The boundary action of H is conservative (resp., dissipative) if a.e. sample path of the chain $\mathcal{E}_n$ visits the set $\mathsf{Edges}(X) \setminus \mathsf{Edges}(T)$ infinitely often (resp., finitely many times).*

Remark 5.2. Deciding whether a given set is visited or not with positive probability by sample paths of a certain Markov chain is a classical problem in probability (and potential theory) which goes back to Kakutani. Explicit estimates for the probabilities of visiting the set infinitely many times or not visiting it at all are given in terms of various kinds of *capacity*, see [BPP95].

5.B. Markov chain on cycles. We shall now assume that the conservative part of the action of H on $(\partial F, \mathfrak{m})$ is non-trivial, i.e., $\mathfrak{m}\Omega > 0$. We give its symbolic interpretation and show that its ergodicity is equivalent to the Liouville property for a certain naturally associated Markov chain.

Let us endow ∂H with the probability measure $\mathfrak{m}_*$ which is the preimage of the normalized restriction $\mathfrak{m}|_\Omega/\mathfrak{m}\Omega$ with respect to the map σ^∞ (see Theorem 1.11). Obviously, the dynamical systems $(H, \Omega, \mathfrak{m}|_\Omega)$ and $(H, \partial H, \mathfrak{m}_*)$ are isomorphic (up to the constant multiplier used to normalize the measure $\mathfrak{m}|_\Omega$). In particular, the action of H on the space $(\partial H, \mathfrak{m}_*)$ is conservative.

Recall that the space ∂H, being the set of infinite irreducible words in the alphabet $\breve{\mathcal{S}}$, is the state space of a topological Markov chain. The alphabet $\breve{\mathcal{S}}$ of this chain is in one-to-one correspondence with a set of cycles in the Schreier graph X (see Section **1.B**).

Theorem 5.3. *The measure $\mathfrak{m}_*$ on ∂H is Markov in the alphabet $\breve{\mathcal{S}}$. It corresponds to the initial distribution*

$$\theta(s) = \mathfrak{m}_* C_s \ , \qquad s \in \breve{\mathcal{S}} \ ,$$

and the transition matrix

$$M(s, s') = (2m-1)^{|s'|-|ss'|} \cdot \frac{\theta(s')}{\theta(s)} \ . \tag{5.4}$$

Proof. The argument basically consists in noticing that, due to the special properties of the generating set $\mathcal{S}$, the Radon–Nikodym derivative

$$\frac{d\mathfrak{m}_*(s_2 s_3 \dots)}{d\mathfrak{m}_*(s_1 s_2 s_3 \dots)} = \frac{ds_1\mathfrak{m}_*}{d\mathfrak{m}_*}(\xi) \ , \quad \xi = s_1 s_2 s_3 \dots \in \partial H \ , \tag{5.5}$$

of the measure $\mathfrak{m}_*$ is *Markov* in the sense that it depends on the letters s_1, s_2 only. [In the language of symbolic dynamics this derivative, or, more rigorously, its logarithm, is called the *potential* of the measure $\mathfrak{m}$.] Indeed, by Proposition 2.6 and Proposition 2.9

$$\frac{d\mathfrak{m}_*(s_2 s_3 \dots)}{d\mathfrak{m}_*(s_1 s_2 s_3 \dots)} = (2m-1)^{|s_1 s_2| - |s_2|} \ ,$$

whence for an arbitrary cylinder set $C_{s_1 s_2 \dots s_n}$

$$\frac{\mathfrak{m}_* C_{s_2 \dots s_n}}{\mathfrak{m}_* C_{s_1 s_2 \dots s_n}} = (2m-1)^{|s_1 s_2| - |s_2|} \ .$$

Comparing the formula

$$\begin{aligned} \mathfrak{m}_* C_{s_1 s_2 \dots s_n} &= \frac{\mathfrak{m}_* C_{s_1 s_2 \dots s_n}}{\mathfrak{m}_* C_{s_2 \dots s_n}} \cdot \frac{\mathfrak{m}_* C_{s_2 \dots s_n}}{\mathfrak{m}_* C_{s_3 \dots s_n}} \cdot \dots \cdot \frac{\mathfrak{m}_* C_{s_{n-1} s_n}}{\mathfrak{m}_* C_{s_n}} \cdot \mathfrak{m}_* C_{s_n} \\ &= (2m-1)^{(|s_2| - |s_1 s_2|) + (|s_3| - |s_2 s_3|) + \dots + (|s_n| - |s_{n-1} s_n|)} \cdot \theta(s_n) \end{aligned}$$

with the analogous expansion for $\mathfrak{m}_* C_{s_1 s_2 \dots s_n s_{n+1}}$ we get the claim. □

5.C. Applications to the boundary action.

Theorem 5.6. *The following measure spaces are canonically isomorphic:*

(i) *The Poisson boundary of the Markov chain on $\breve{\mathcal{S}}$ described in Theorem 5.3;*
(ii) *The space of ergodic components of the (one-sided) time shift in the measure space $(\partial H, \mathfrak{m}_*)$;*
(iii) *The space of ergodic components of the action of the group H on the measure space $(\partial H, \mathfrak{m}_*)$.*

Proof. The isomorphism of the spaces (i) and (ii) is a general fact from the theory of Markov chains, e.g., see [Kai92], whereas the isomorphism of the spaces (ii) and (iii) follows from the coincidence of the orbit equivalence relations of the time shift and of the H-action on ∂H. □

Remark 5.7. Of course, Theorem 5.6 remains valid for an arbitrary Markov measure on ∂H with the full support.

Corollary 5.8. *The Schreier graph X is Liouville if and only if the boundary action of H on $(\partial F, \mathfrak{m})$ is conservative and the Markov chain described in Theorem 5.3 is Liouville.*

Remark 5.9. There are examples when the Markov chain described in Theorem 5.3 is Liouville although the Schreier graph X is not. They correspond to the situation when the boundary action of H on $(\partial F, \mathfrak{m})$ has both conservative and dissipative parts, and the conservative part is ergodic, see Example 4.27 and Remark 4.28.

Remark 5.10. Yet another example of a Markov measure on ∂H is provided by the harmonic (hitting) measure of a random walk (H, μ) with $\operatorname{supp} \mu = \breve{\mathcal{S}}$. It was known already to Dynkin and Malyutov [DM61] (also see [LM71]), that if H is finitely generated, then the sample paths converge a.e. to the boundary ∂H, the hitting measure λ is Markov, and the space $(\partial H, \lambda)$ is the Poisson boundary of the random walk (H, μ). A recent addition to these facts is the observation that the hitting measure λ is in fact *multiplicative Markov*, and not just plain Markov [Mai05, MM07], i.e., the transition probability from any $s \in \breve{\mathcal{S}}$ is the normalized restriction of the initial distribution onto the set $\breve{\mathcal{S}} \setminus \{s^{-1}\}$ of all letters admissible from s. These results readily generalize to the situation when H is infinitely generated by using the approximation of $H = \langle s_1, s_2, \dots \rangle$ by finitely generated subgroups $H_n = \langle s_1, s_2, \dots, s_n \rangle$. In particular, the hitting measure λ on ∂H is just the projective limit of the hitting measures λ_n on ∂H_n; since each of the measures λ_n is multiplicative Markov, the limit measure λ is also multiplicative Markov. This fact can be used to show that, generally speaking, the measure $\mathfrak{m}_*$ is not equivalent to the hitting measure of a random walk on H. Indeed, the Markov measure with the transition probabilities (5.4) is multiplicative only if for any $s' \neq s'' \in \breve{\mathcal{S}}$ the difference $|ss'| - |ss''|$ is the same for all $s \in \breve{\mathcal{S}}$ with $s', s'' \neq s^{-1}$, which is a quite restrictive condition.

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