Equivariant principal bundles over spheres and cohomogeneity one manifolds

Hambleton, Ian; Hausmann, Jean-Claude

How to cite

HAMBLETON, Ian, HAUSMANN, Jean-Claude. Equivariant principal bundles over spheres and cohomogeneity one manifolds. In: Proceedings of the London Mathematical Society, 2003, vol. 86, n° 1, p. 250–272. doi: 10.1112/S0024611502013722

This publication URL: https://archive-ouverte.unige.ch/unige:12282

Publication DOI: 10.1112/S0024611502013722

EQUIVARIANT PRINCIPAL BUNDLES OVER SPHERES AND COHOMOGENEITY ONE MANIFOLDS

IAN HAMBLETON AND JEAN-CLAUDE HAUSMANN

ABSTRACT. We classify $SO(n)$-equivariant principal bundles over S^n in terms of their isotropy representations over the north and south poles. This is an example of a general result classifying equivariant (Π, G)-bundles over cohomogeneity one manifolds.

1. INTRODUCTION

Let Π and G be Lie groups. A principal (Π, G)-bundle is a locally trivial, principal G-bundle $p\colon E \to X$ such that E and X are left Π-spaces. The projection map p is Π-equivariant and $\gamma(e \cdot g) = (\gamma e) \cdot g$, where $\gamma \in \Pi$ and $g \in G$ acts on $e \in E$ by the principal action. Equivariant principal bundles, and their natural generalizations, were studied by T. E. Stewart [24], T. tom Dieck [5], [6, I (8.7)], R. Lashof [15], [16] together with P. May [17] and G. Segal [18].

These authors study equivariant principal bundles by homotopy theoretic methods. There exists a classifying space $B(\Pi, G)$ for principal (Π, G)-bundles [5], so the classification of equivariant bundles in particular cases can be approached by studying the Π-equivariant homotopy type of $B(\Pi, G)$. If the structural group G of the bundle is *abelian*, then the main result of [18] states that equivariant bundles over a Π-space X are classified by the ordinary homotopy classes of maps $[X \times_\Pi E\Pi, BG]$. In practice, this program leads to an obstruction theory rather than a classification. See, however, the results of Lashof in the special cases where Π acts transitively [14] or semi-freely [16] on the base space X.

Another approach to equivariant principal bundles uses the "local" invariants arising from isotropy representations at singular points of (X, Π), together with equivariant gauge theory [1], [8], [9], [10]. By an isotropy representation at a Π-fixed point $x_0 \in X$ we mean the homomorphism $\alpha_{x_0}\colon \Pi \to G$ defined by the formula

$$\gamma \cdot e_0 = e_0 \cdot \alpha(\gamma)$$

where $e_0 \in p^{-1}(x_0)$. The homomorphism α is independent of the choice of e_0 up to conjugation in G. The relationship between the local invariants and the homotopy classification (in the form of a Localization Theorem ?) deserves further study.

Date: Jan. 20, 2001.

The authors thank the Swiss National Fund for Scientific Research, the Université de Génève, and the Max Planck Institut für Mathematik in Bonn for hospitality and support.

In this paper, we use the second approach for $\Pi = SO(n)$ acting in the standard way on $X = S^n$. In this concrete situation, we obtain a complete classification by relatively elementary geometric methods. It turns out that the local isotropy representations at the north and south poles of S^n explicitly determine the classification of $(SO(n), G)$ principal bundles over S^n (for short (n, G)-bundles). One surprising consequence is that the set $\mathcal{E}(n, G)$ of (n, G)-bundles is finite for $n \geq 3$. In contrast, the set of (non-equivariant) principal G-bundles over S^n is often infinite. A detailed statement of these results is given in the next section and their proofs, essentially self-contained, are explained in Sections 3 to 6. Several examples are given in Section 7. In Section 8 we show how these results fit into the more general setting of equivariant (Π, G)-bundles over certain Π-manifolds studied by K. Jänich [13] and E. Straume [25]. In particular, we obtain a classification of (Π, G)-bundles over cohomogeneity 1 manifolds.

The authors would like to thank P. de la Harpe for helpful discussions.

2. Statement of results

Let S^n be the n-dimensional sphere of radius 1 in $\mathbf{R}^{n+1}$. We consider the action on S^n of the group $SO(n)$, by orthogonal transformations fixing the poles $(0, \dots, 0, \pm 1)$.

Let G be a Lie group. We denote by $\mathcal{R}(n, G)$ the set of smooth homomorphisms from $SO(n)$ to G modulo the conjugations by elements of G. Unless specified, all maps between manifolds are smooth of class C^∞.

By a G-principal bundle η over S^n, we mean, as usual, a smooth map $p \colon E \to S^n$ from a manifold $E = E(\eta)$ and a free right action $E \times G \to E$ so that $p(z \cdot g) = p(z)$ with the standard local triviality condition. The isomorphism classes of G-bundles over S^n are in bijection with $\pi_{i-1}(G)/\pi_0(G)$, the quotient of the homotopy group $\pi_{n-1}(G)$ (based on the neutral element e of G) by the action of $\pi_0(G)$ induced by the conjugation of G on itself. The bijection associates to a bundle η the class $C(\eta) := [\partial(\mathrm{id}_{S^n})] \in \pi_{n-1}(G)/\pi_0(G)$, where $\partial \colon \pi_n(S^n) \to \pi_{n-1}(G)$ is the boundary operator in the homotopy exact sequence of η [23, Th. 18.5].

A *$SO(n)$-equivariant principal G-bundle* ξ over S^n (or an *(n, G)-bundle* for short) is a G-principal bundle $\xi^\flat$ over S^n together with a left action $SO(n) \times E(\xi) \to E(\xi)$ commuting with the free right action of G and such that the projection to S^n is $SO(n)$-equivariant (we write $E(\xi)$ for $E(\xi^\flat)$). Two (n, G)-bundles ξ_1, ξ_2 are *isomorphic* if there exists a diffeomorphism $h : E(\xi_1) \to E(\xi_2)$ which is both $SO(n)$ and G-equivariant and which commutes with the projections to S^n. We will compute the set $\mathcal{E}(n, G)$ of isomorphism classes of (n, G)-bundles.

Let ξ be a (n, G)-bundle. Choose points $a, b \in E(\xi)$ such that $p(a) = (0, \dots, -1)$ and $p(b) = (0, \dots, 1)$. Let α, β be the maps from $SO(n)$ to G determined by the formulae $A \cdot a = a \cdot \alpha(A)$ and $A \cdot b = b \cdot \beta(A)$. We shall prove in Lemma 3.2 that α and β are smooth homomorphisms and that their class in $\mathcal{R}(n, G)$

depend only on $[\xi] \in \mathcal{E}(n,G)$. We call α and β the *isotropy representations* (associated to a and b). This defines a map $J\colon \mathcal{E}(n,G) \to \mathcal{R}(n,G) \times \mathcal{R}(n,G)$ by $J(\xi) := ([\alpha],[\beta])$. When $n = 2$ and G is connected, $J(\xi)$ is a complete invariant which, in particular, determines the (non-equivariant) isomorphism class of $\xi^\flat$. More precisely, let $\psi\colon \mathcal{R}(2,G) \times \mathcal{R}(2,G) \to \pi_1(G)$ be the map determined by $\psi(\alpha,\beta)(z) := [\alpha(z)\beta(z)^{-1}]$ (one uses that $SO(2) \approx S^1$ and that ψ is well defined if G is connected).

Theorem A. *Suppose that G is connected Lie group. Then,*

(i) *the map $J\colon \mathcal{E}(2,G) \to \mathcal{R}(2,G) \times \mathcal{R}(2,G)$ is a bijection.*
(ii) *if $J(\xi) = ([\alpha],[\beta])$, then $\psi(\alpha,\beta) = C(\xi^\flat)$.*

We shall now generalize Theorem A for $n \geq 2$ or G any Lie group. In general, J is then neither injective nor surjective and $C(\xi^\flat)$ is not determined by $J(\xi)$. Consider $SO(n-1)$ as the subgroup of $SO(n)$ fixing the last coordinate. The restriction $[\mu] \mapsto [\mu\,|_{SO(n-1)}]$ gives a map $\mathrm{Res} : \mathcal{R}(n,G) \to \mathcal{R}(n-1,G)$. Denote by $\mathcal{R}(n,G) \times_{(n-1)} \mathcal{R}(n,G)$ the set of $([\alpha],[\beta]) \in \mathcal{R}(n,G) \times \mathcal{R}(n,G)$ such that $\mathrm{Res}\,[\alpha] = \mathrm{Res}\,[\beta]$. If $\varphi\colon H \to G$ is a group homomorphism, we denote by $Z_\varphi \subset G$ the centralizer of $\varphi(H)$.

Theorem B. *Let G be any Lie group G. Then*

(i) *the image of J is $\mathcal{R}(n,G) \times_{(n-1)} \mathcal{R}(n,G)$.*
(ii) *let $\alpha,\beta\colon SO(n-1) \to G$ be two smooth homomorphisms such that $\alpha\,|_{SO(n-1)} = \beta\,|_{SO(n-1)} =: \gamma$. Then $J^{-1}([\alpha],[\beta])$ is in bijection with with the set of double cosets $\pi_0(Z_\alpha)\backslash\pi_0(Z_\gamma)/\pi_0(Z_\beta)$.*

Remark 2.1. The compatibility statement in Part (i) of Theorem B was also observed by K. Grove and W. Ziller [7, Prop. 1.6]. In § 8, Theorem B is extended to a more general setting, to include equivariant principal bundles over "special" Π-manifolds in the sense of Jänich [13]. In particular this provides a classification of the equivariant bundles considered by Grove and Ziller.

Since $SO(1)$ is trivial, Theorem B reduces to Part (i) of Theorem A when $n = 2$. To determine $C(\xi^\flat)$ as in Part (ii) of Theorem A, we must choose particular representatives of $[\alpha]$ and $[\beta]$ (in general, $J(\xi)$ does not determine $\xi^\flat$: see examples 7.2 and 7.5). An *isotropic lifting* for ξ is a smooth curve $\tilde{c}\colon [-1,1] \to E(\xi)$ lifting the meridian arc $c(t) = (0,\ldots,\cos(\pi t/2),\sin(\pi t/2))$ and such that $B \cdot \tilde{c}(t) = \tilde{c}(t)\alpha(B)$ for all $B \in SO(n-1)$. Isotropic lifting always exist (see Lemma 3.5). Choosing $a := \tilde{c}(-1)$ and $b := \tilde{c}(-1)$ leads to isotropy representations $\alpha,\beta\colon SO(n) \to G$ such that $\alpha\,|_{SO(n-1)} = \beta\,|_{SO(n-1)}$. The map $\psi(\alpha,\beta)\colon SO(n) \to G$ constructed as in Theorem A then satisfies $\psi(\alpha,\beta)(AB) = \psi(\alpha,\beta)(A)$ when $B \in SO(n-1)$. It thus induces a map

$$\bar{\psi}(\alpha,\beta)\colon S^{n-1} \cong SO(n)/SO(n-1) \to G.$$

Note that $\bar{\psi}$ is well defined since α and β are actual homomorphisms and not conjugacy classes.

Proposition C. *Let ξ be a (n, G)-bundle. Let $\alpha, \beta\colon SO(n) \to G$ be the isotropy representation associated to the end points of an isotropic lifting. Then, $[\bar{\psi}(\alpha, \beta)] = C(\xi^\flat)$ in $\pi_{n-1}(G)/\pi_0(G)$.*

We shall prove two consequences of Theorem B and Proposition C which emphasize the constrast between the cases $n = 2$ and $n \geq 3$.

Proposition D. *Let η be a principal G-bundle over S^2 with G a non-trivial Lie group. Then there exist infinitely many $\xi \in \mathcal{E}(2, G)$ such that $\xi^\flat \cong \eta$.*

Proposition E. *For G a compact Lie group, the set $\mathcal{E}(n, G)$ is finite when $n \geq 3$.*

These results are proved in § 5, while the former sections are devoted to preliminary material. In Section 6, we determine which (n, G)-bundles come from an $SO(n + 1)$-equivariant bundles. Examples are given in § 7.

3. Preliminary constructions

3.1. *J is well defined.* This follows from the following lemma.

Lemma 3.2. *Let ξ be a (n, G)-bundle. Let $a, b \in E(\xi)$ such that $p(a) = (0, \ldots, -1)$ and $p(b) = (0, \ldots, 1)$. Let β, α be the maps from $SO(n)$ to G determined by the formulae $A \cdot a = a \cdot \alpha(A)$ and $A \cdot b = b \cdot \beta(A)$. Then α and β are smooth homomorphisms and their class in $\mathcal{R}(n, G)$ depends only on $[\xi] \in \mathcal{E}(n, G)$.*

Proof. Let $A, B \in SO(n)$. One has

$$\begin{aligned} a \cdot \alpha(BA) &= (BA) \cdot a = B \cdot (A \cdot a) = B \cdot (a \cdot \alpha(A)) = \\ &= (B \cdot a) \cdot \alpha(A) = a \cdot (\alpha(B)\alpha(A)). \end{aligned}$$

Therefore, α and similarly, β, are homomorphism. They are smooth because the action of $SO(n)$ is smooth. If a' is another choice instead of a, there exists $g \in G$ such that $a' = a \cdot g$ and one has

$$a \cdot (g\alpha'(A)) = a' \cdot \alpha'(A) = A \cdot a' = A \cdot a \cdot g = a \cdot (\alpha(A)g), \tag{3.3}$$

whence $\alpha'(A) = g^{-1}\alpha(A)g$. This proves that the class of (α, β) in $\mathcal{R}(n, G)$ does not depend on the choice of a and b. Now, if $h\colon E(\xi) \xrightarrow{\approx} E(\xi')$ is a $(SO(n), G)$-equivariant diffeomorphism over the identity of S^n, then, by choosing $a' := h(a)$ and $b' := h(b)$, one has $(\alpha', \beta') = (\alpha, \beta)$. The proof of Lemma 3.2 is then complete. □

3.4. *Isotropic liftings.* Let $I := [-1, 1]$ and $c\colon I \to S^n$ be the parametrisation of the meridian arc $c(t) = (0, \ldots, \cos(\pi t/2), \sin(\pi t/2))$. Let $\tilde{c}\colon I \to E = E(\xi)$ be a (smooth) lifting of c. As $c(t)$ is fixed by $SO(n-1)$, one has $B \cdot \tilde{c}(t) = \tilde{c}(t) \cdot \alpha_t(B)$, for $B \in SO(n-1)$. As in the proof of Lemma 3.2, one checks that this gives a smooth path α_t ($t \in I$) of homomorphisms from $SO(n-1)$ to G, which depends on the lifting $\tilde{c}$. Call $\tilde{c}$ *isotropic* if α_t is constant: $\alpha_t(B) = \alpha(B)$ for all $B \in SO(n-1)$.

Lemma 3.5. *Any (n, G)-bundle admits an isotropic lifting.*

We shall make use of connections on (n, G)-bundles which are $SO(n)$-invariant. These can be obtained by averaging any connection (see [1, p. 522]), since the space of connections is an affine space. If a curve $u(t)$ in $E(\xi)$ is horizontal for a $SO(n)$-invariant connection, then $u(t) \cdot g$ and $A \cdot u(t)$ are horizontal. Lemma 3.5 then follows from the following

Lemma 3.6. *Let ξ be a (n, G)-bundle endowed with a $SO(n)$-invariant connection. Then, any lifting $\tilde{c}$ of c which is horizontal is isotropic.*

Proof. If $\tilde{c}$ is an horizontal lifting , so are $B \cdot \tilde{c}$ and $\tilde{c} \cdot \alpha(B)$ for $B \in SO(n-1)$. As $B \cdot \tilde{c}(-1) = \tilde{c}(-1) \cdot \alpha(B)$, one has $B \cdot \tilde{c}(t) = \tilde{c}(t) \cdot \alpha(B)$ for all $t \in I$. □

3.7. *The (n, G)-bundles $\xi_{\alpha,\beta}$.* If X is a topological space, the unreduced suspension ΣX is

$$\Sigma X := I \times X \big/ \{(-1, x) \sim (-1, x') \text{ and } (1, x) \sim (1, x'),\ \forall\ x, x' \in X\}.$$

We donote by $C_{-}X$ the image of $[-1, 1) \times X$ in ΣX and by $C_{+}X$ those of $(-1, 1] \times X$.

Let (α, β) be a pair of smooth homomorphisms from $SO(n)$ to G. Define the space $\widehat{E}_{\alpha,\beta}$ by

$$\begin{aligned} \widehat{E}_{\alpha,\beta} \ :=\ & I \times SO(n) \times G \big/ \{(-1, A', g) \sim (-1, A, \alpha(A^{-1}A')\, g) \text{ and} \\ & \qquad (1, A', g) \sim (1, A, \beta(A^{-1}A')\, g),\ \forall A \in SO(n)\}. \end{aligned}$$

The space $\widehat{E}_{\alpha,\beta}$ admits an obvious free right action of G and a map $p\colon \widehat{E}_{\alpha,\beta} \to \Sigma SO(n)$. This makes a principal G-bundle over $\Sigma SO(n)$; indeed, trivializations on $C_{\pm}SO(n)$ are given by

$$\begin{array}{lllll} \hat{\varphi}_{-}: & [t, A, g] & \mapsto & ([t, A], \alpha(A)\, g) & \text{if} \quad -1 \le t < 1 \\ \hat{\varphi}_{+}: & [t, A, g] & \mapsto & ([t, A], \beta(A)\, g) & \text{if} \quad -1 < t \le 1. \end{array} \tag{3.8}$$

The change of trivializations is

$$\hat{\varphi}_{-} \circ \hat{\varphi}_{+}^{-1}([t, A], g) = ([t, A], \alpha(A)\beta(A)^{-1} g). \tag{3.9}$$

Now, suppose that $\alpha\,_{|SO(n-1)} = \beta\,_{|SO(n-1)}$. Form the space $E_{\alpha,\beta}$ as the quotient

$$E_{\alpha,\beta} := \widehat{E}_{\alpha,\beta} \big/ \{[t, AB, g] \sim [t, A, \alpha(B) g],\ \forall\, B \in SO(n-1)\}.$$

Let $\varepsilon\colon SO(n) \to S^{n-1}$ be the map which associates to a matrix its last column; it is also the projection $\varepsilon\colon SO(n) \to SO(n)/SO(n-1) \cong S^{n-1}$. There is an map $p\colon E_{\alpha,\beta} \to \Sigma S^{n-1}$ given by $p([t, A, g]) = [t, \varepsilon(A)]$ and a free G-action given by $[t, A, g] \cdot g_1 := [t, A, g g_1]$ As above, we check that this defines a G-principal bundle over ΣS^{n-1}; the trivializations $\hat{\varphi}_{\pm}$ descend to trivializations $\check{\varphi}_{\pm}$ over $C_{\pm}S^{n-1}$.

The map $[t, A] \mapsto A \cdot c(t)$ descends to a homeomorphism $f\colon \Sigma S^{n-1} \xrightarrow{\approx} S^n$. By replacing p by $f \circ p$, we obtain a (topological) principal G-bundle

$$\xi_{\alpha,\beta}\colon E_{\alpha,\beta} \xrightarrow{p} S^n.$$

Let $S^n_\pm$ be the punctured spheres $S^n_\pm := f(C_\pm S^{n-1})$. The trivializations given by the compositions

$$\varphi_\pm\colon p^{-1}(S^n_\pm) \xrightarrow{\check{\varphi}_\pm} C_\pm S^{n-1} \times G \xrightarrow{f \times \mathrm{id}} S^n_\pm \times G \tag{3.10}$$

are homeomorphisms from $p^{-1}(S^n_\pm)$ onto manifolds. The change of trivialization is a diffeomorphism, being obtained by conjugating that of (3.9) by f. Therefore, $\varphi_\pm$ produce a smooth manifold structure on $E_{\alpha,\beta}$. The map p and the G action are smooth. One checks that the map

$$SO(n) \times \widehat{E}_{\alpha,\beta} \to \widehat{E}_{\alpha,\beta} \quad \text{given by} \quad C \cdot [t, A, g] := [t, CA, g]$$

descends to a smooth $SO(n)$-action on $E_{\alpha,\beta}$ which makes $\xi_{\alpha,\beta}$ a (n, G)-bundle.

3.11. *Proof of Part (i) of Theorem B.* Let ξ be a (n, G)-bundle. By Lemma 3.5 there exists an isotropic lifting $\tilde{c}\colon I \to E(\xi)$ of c. Choosing $a := \tilde{c}(-1)$ and $b := \tilde{c}(1)$ produces a representative (α, β) of $J(\xi)$ with $\alpha\,|_{SO(n-1)} = \beta\,|_{SO(n-1)}$. Therefore, the image of J is contained in $\mathcal{R}(n, G) \times_{(n-1)} \mathcal{R}(n, G)$.

Conversely, a class $P \in \mathcal{R}(n, G) \times_{(n-1)} \mathcal{R}(n, G)$ is represented by a pair (α, β) with $\alpha\,|_{SO(n-1)} = \beta\,|_{SO(n-1)}$. Let $\mathbf{1}$ be the identity matrix in $SO(n)$ and e be the unit element of G. Computing $J(\xi_{\alpha,\beta})$ with the points $a := [-1, \mathbf{1}, e]$ and $b := [1, \mathbf{1}, e]$ in $E_{\alpha,\beta}$ shows that $J(\xi_{\alpha,\beta}) = P$. □

4. The map $\tilde{J}_\gamma$

Let $\gamma\colon SO(n-1) \to G$ be a smooth homomorphism. Define a set $\mathcal{R}_\gamma(n, G)$ as follows: an element of $\mathcal{R}_\gamma(n, G)$ is represented by a pair (α, β) of smooth homomorphisms from $SO(n)$ to G such that $\alpha\,|_{SO(n-1)} = \beta\,|_{SO(n-1)} = \gamma$. Two pairs (α_1, β_1) and (α_2, β_2) represent the same element of $\mathcal{R}_\gamma(n, G)$ if there is a smooth path $\{g_t \mid t \in [-1, 1]\}$ in the centralizer Z_γ of $\gamma(SO(n-1))$ such that $\alpha_2(A) = g_{-1}\alpha_1(A)g_{-1}^{-1}$ and $\beta_2(A) = g_1\beta_1(A)g_1^{-1}$. There is an obvious map $j\colon \mathcal{R}_\gamma(n, G) \to \mathcal{R}(n, G) \times_{(n-1)} \mathcal{R}(n, G)$.

Part (i) of Theorem B, already proven in (3.11), permits us to define a map $\bar{J}\colon \mathcal{E}(n, G) \to \mathcal{R}(n-1, G)$ by $\bar{J}(\xi) := \mathrm{Res}\,[\alpha] = \mathrm{Res}\,[\beta]$. We shall now compute the preimage $\bar{J}^{-1}([\gamma])$.

Proposition 4.1. *Let $\gamma\colon SO(n-1) \to G$ be a smooth homomorphism. Then there exists a bijection $\tilde{J}_\gamma\colon \bar{J}^{-1}([\gamma]) \xrightarrow{\approx} \mathcal{R}_\gamma(n, G)$ such that $j \circ \tilde{J}_\gamma = J$.*

The proof divides into several steps.

4.2. *Definition of* $\tilde{J}_\gamma$. Let ξ be a (n, G)-bundle with $\bar{J}(\xi) = [\gamma]$. Choose, using Lemma 3.5, an isotropic lifting $\tilde{c}_0 \colon I \to E(\xi)$ of c. As $\bar{J}(\xi) = [\gamma]$, the constant path $\alpha_t^0 \colon SO(n-1) \to G$ associated to $\tilde{c}_0$ is conjugated to γ: there exists $g \in G$ such that $\alpha_t^0(B) = g\gamma(B)g^{-1}$. Let $\tilde{c} := \tilde{c}_0 \cdot g$. Like in Equation (3.3), one checks that $\tilde{c}$ is γ-*isotropic*, i.e. $\alpha_t = \gamma$. Choosing $a := \tilde{c}(-1)$ and $b := \tilde{c}(1)$ then produces a pair (α, β) of smooth homomorphisms from $SO(n)$ to G which represents a class $\tilde{J}_\gamma(\xi)$ in $\mathcal{R}_\gamma(n, G)$.

To see that $\tilde{J}_\gamma$ is thus well defined, let $\tilde{c}'$ be another γ-isotropic lifting of c. The smooth path $t \mapsto g_t \in G$ defined by $\tilde{c}'(t) = \tilde{c}(t) \cdot g_t$ then satisfies, for all $B \in SO(n-1)$:

$$\gamma(B) = \alpha_t'(B) = g_t^{-1}\alpha_t(B)g_t = g_t^{-1}\gamma(B)g_t.$$

Therefore, $g_t \in Z_\gamma$. One has $\alpha'(A) = g_{-1}^{-1}\alpha_t(A)g_{-1}$ and $\beta'(A) = g_1^{-1}\beta_t(A)g_1$, for all $A \in SO(n)$, which proves that $\tilde{J}_\gamma(\xi)$ does not depend on the choice of a γ-isotropic lifting.

Now, if $h \colon E(\xi) \xrightarrow{\approx} E(\xi')$ is a $(SO(n), G)$-equivariant diffeomorphism over the identity of S^n and $\tilde{c} \colon I \to E(\xi)$ is a γ-isotropic lifting for ξ, then $c' := h \circ \tilde{c}$ is a γ-isotropic lifting for ξ' giving $(\alpha', \beta') = (\alpha, \beta)$. This proves that $\tilde{J}_\gamma$ is well defined.

4.3. *Surjectivity of* $\tilde{J}_\gamma$. Let (α, β) represent a class P in $\mathcal{R}_\gamma(n, G)$. One checks that $\tilde{J}_\gamma(\xi_{\alpha,\beta}) = P$, using that the path $t \mapsto [t, \mathbf{1}, e]$ is a γ-isotropic lifting for $\xi_{\alpha,\beta}$.

4.4. *Injectivity of* $\tilde{J}_\gamma$. Let $\gamma \colon SO(n-1) \to G$ be a smooth homomorphism. and let ξ be a (n, G)-bundle with $\bar{J}(\xi) = [\gamma]$. There exists $a \in E(\xi)$ with $p(a) = (0, \dots, -1)$ and $B \cdot a = a \cdot \gamma(B)$ for all $B \in SO(n-1)$. Choose a $SO(n)$-invariant connection on ξ and let $\tilde{c}$ an horizontal lifting of c with $\tilde{c}(-1) = a$. By Lemma 3.6, $\tilde{c}$ is γ-isotropic. If $\beta \colon SO(n) \to G$ is defined by $A \cdot \tilde{c}(1) = \tilde{c}(1) \cdot \beta(A)$, then (α, β) represents $\tilde{J}_\gamma(\xi)$.

Consider the map $\widehat{\lambda} \colon \widehat{E}_{\alpha,\beta} \to E(\xi)$ given by

$$\widehat{\lambda}([t, A, g]) := A \cdot \tilde{c}(t) \cdot g.$$

The map $\widehat{\lambda}$ descends to a continuous map $\lambda \colon E_{\alpha,\beta} \to E(\xi)$ which is both $SO(n)$ and G-equivariant and which covers the identity of S^n. Therefore ξ and $\xi_{\alpha,\beta}$ are isomorphic as topological (n, G)-bundles. What remains to prove is that λ is a diffeomorphism, which is only non-trivial around the fibers $E_\pm$ above the north and south poles.

The connection on ξ provides a smooth trivialization of ξ restricted to the punctured sphere S^n_- (see 3.7) in the following way. Consider the map $s_- \colon p^{-1}(S^n_-) \to E_-$ assigning to z the end point in E_- of the horizontal path through z above the meridian arc through $p(z)$. Define the G-equivariant map $\sigma_- \colon p^{-1}(S^n_-) \to G$

by $s_-(z) = a \cdot \sigma_-(z)$. The required trivialization $\tau_-\colon p^{-1}(S^n_-) \to S^n_- \times G$ is $\tau_-(z) := (p(z), \sigma_-(z))$.

Take the trivialization φ_- for $\xi_{\alpha,\beta}$ defined in Equations (3.10) of (3.7). As $\tilde{c}$ is horizontal, one has

$$\tau_- \circ \lambda \circ \varphi_-^{-1}(x, g) = (x, g).$$

This, and the same for E_+, prove that λ is a diffeomorphism. We have thus established that if $\tilde{J}_\gamma(\xi)$ is represented by (α, β) then the (n, G)-bundle ξ is isomorphic to $\xi_{\alpha,\beta}$, which proves the injectivity of $\tilde{J}_\gamma$.

The proof of Proposition 4.1 is now complete. □

5. Proof of the main results

This section contains the proofs of the results stated in § 2.

5.1. *Proof of Theorem B.* Part (i) has already been proven in (3.11). We shall now prove Part (ii). Let $\alpha, \beta\colon SO(n-1) \to G$ be two smooth homomorphisms such that $\alpha\,|_{SO(n-1)} = \beta\,|_{SO(n-1)} = \gamma$. The pair (α, β) defines a class $[\alpha, \beta] \in \mathcal{R}_\gamma(n, G)$. The group $Z_\gamma \times Z_\gamma$ acts on $\mathcal{R}_\gamma(n, G)$ by

$$(g, h) \cdot [\alpha, \beta] := [g\alpha g^{-1}, h\beta h^{-1}].$$

The set $J^{-1}([\alpha], [\beta]) \subset \bar{J}^{-1}([\gamma])$ is, via the bijection $\tilde{J}_\gamma\colon \bar{J}^{-1}([\gamma]) \xrightarrow{\approx} \mathcal{R}_\gamma(n, G)$ of Proposition 4.1, in bijection with an orbit of the above action. Let "∼" be the equivalence relation on $Z_\gamma \times Z_\gamma$ defined by $(g_1, h_1) \sim (g_2, h_2)$ iff $(g_1, h_1)[\alpha, \beta] = (g_2, h_2)[\alpha, \beta]$. Let

$$\phi\colon Z_\gamma \times Z_\gamma \to \pi_0(Z_\alpha)\backslash\pi_0(Z_\gamma)/\pi_0(Z_\beta)$$

be the map defined by $\phi(g, h) := [g^{-1}h]$. Part (ii) of Theorem B then follows from the following lemma.

Lemma 5.2. *$(g_1, h_1) \sim (g_2, h_2)$ iff $\phi(g_1, h_1) = \phi(g_2, h_2)$.*

Proof. Suppose that $(g_1, h_1) \sim (g_2, h_2)$. This means that there exist $s_-, s_+ \in Z_\gamma$, with $[s_-] = [s_+]$ in $\pi_0(Z_\gamma)$, such that the following equality

$$(g_1\alpha g_1^{-1}, h_1\beta h_1^{-1}) = (s_-g_2\alpha g_2^{-1}s_-^{-1}, s_+h_2\beta h_2^{-1}s_+^{-1})$$

holds in $Z_\gamma \times Z_\gamma$. This implies that

$$g_1 = s_-\, g_2\, A \quad\text{and}\quad h_1 = s_+\, h_2\, B$$

with $A \in Z_\alpha$ and $B \in Z_\beta$ (the centralizers of the images of α and β). Therefore $g_1^{-1}h_1 = A^{-1}g_2^{-1}s_-^{-1}s_+\, h_2\, B$, which implies $\phi(g_1, h_1) = \phi(g_2, h_2)$.

To prove the converse, observe that

- $(g, h) \sim (C\, g, C\, h)$ for $C \in Z_\gamma$.
- $(g, h) \sim (gA, hB)$ for $A \in Z_\alpha$ and $B \in Z_\beta$.
- $(g, h) \sim (g, uh)$ for u in the identity component of Z_γ.

Suppose that $\phi(g_1,h_1)=\phi(g_2,h_2)$. This means that there are $A\in Z_\alpha$, $B\in Z_\beta$ and u in the identity component of Z_γ auch that $g_1^{-1}h_1=A^{-1}g_2^{-1}uh_2B$ (u can be put in the middle since the identity component of Z_γ is a normal subgroup of Z_γ). One then has

$$(g_1,h_1)\sim(e,g_1^{-1}h_1)=(e,A^{-1}g_2^{-1}uh_2B)\sim(g_2A,u\,h_2B)\sim(g_2,h_2)\ .$$

□

Proof of Proposition C. Let ξ, α and β as in the statement of Proposition C. Let $\gamma:=\alpha_{|SO(n-1)}=\beta_{|SO(n-1)}$. Then, $\xi\in\bar J^{-1}([\gamma])$ and, by 4.2, one has $\tilde J(\xi)=[\alpha,\beta]$ in $\mathcal{R}_\gamma(n,G)$. By 4.4, $\xi=\xi_{\alpha,\beta}$. Therefore, $C(\xi^\flat)=C(\xi^\flat_{\alpha,\beta})=[\bar\psi(\alpha,\beta)]$, the last equality coming from Equation (3.9) of 3.7 and that $C(\xi^\flat)$ can be represented by the transition function [23, Th. 18.4]. □

Proof of Theorem A. Since $SO(1)$ is trivial, $Z_\gamma=G$ which is suppose to be connected. Therefore, Part (i) is a particular case of Part (i) of Theorem B. Let $c\colon I\to S^n$ parametrizing the meridian arc, as in (3.4). Let $\alpha,\beta\colon SO(2)\to G$ be two homomorphisms representing $J(\xi)$. One can find a and b so that α and β are the isotropy representations associated to a and b. As G is connected, the submanifold $P_0:=p^{-1}(c(I))$ of $E(\xi)$ is connected and there is a smooth lifting $\tilde c$ of c such that $\tilde c(-1)=a$ and $\tilde c(1)=b$. As $SO(1)$ is trivial, $\tilde c$ is isotropic. Part (ii) of Theorem A then follows from Proposition C. □

Proof of Proposition D. Recall that any element of $\pi_1(G,e)$ can be represented by a homomorphism (a geodesic in a maximal compact subgroup K of G, with a K-bi-invariant Riemannian metric, being a 1-parameter subgroup [11, Ch. IV, § 6]). Therefore, if η is a G-bundle over S^2, there exists a homomorphism $\alpha\colon S^1\to G$ such that $C(\eta)=[\alpha]$. For $q\in\mathbf{N}$, let $\alpha_q\colon S^1\to G$ given by $\alpha_q(z):=\alpha(z)^q$. If α is not trivial, the classes $[\alpha_q]$ are all distinct in $\mathcal{R}(2,G)$. Indeed, the set $\mathcal{R}(2,G)$ is in bijection with lattice points in a Weyl chamber of the Lie algebra of a maximal torus of G and the point representing α_q is q times those representing α.

Suppose first that η is not trivial. Hence, α is not trivial and $[\alpha_{q+1},\alpha_q]$ are all different classes in $\mathcal{R}_\gamma(2,G)$ with $[\bar\psi(\alpha_{q+1},\alpha_q])=C(\eta)$. The result then follows from Propositions C and 4.1.

When η is trivial, one takes any non-trivial homomorphism $\alpha\colon SO(2)\to G$. The classes $[\alpha_q,\alpha_q]$ in $\mathcal{R}_\gamma(2,G)$ represent infinitely many distinct $SO(2)$-equivariant G-bundles ξ_q with trivial $\xi_q^\flat$. □

Proof of Proposition E. If $n\geq 3$, the group $SO(n)$ is semi-simple and the set $\mathcal{R}(n,G)$ is finite. The latter follows from the following known results:

- a homomorphism is determined by its tangent map at the identity (as a homomorphism of Lie algebras).
- the Lie algebra of G contains only finitely many semi-simple Lie subalgebras, up to inner automorphism [21, Prop. 12.1].

- there are only finitely many homomorphisms between two semisimple Lie algebras.

Also, if G is compact, the group Z_γ is compact and then $\pi_0(Z_\gamma)$ is finite. Proposition E then follows from Theorem B. □

Remark 5.3. To remove the hypothesis "G compact" from Proposition E, it is enough to consider the case G connected. Indeed, $\mathcal{R}_\gamma(n,G)$ is a quotient of $\mathcal{R}_\gamma(n,G_e)$, where G_e is the connected component of e. One would then need the following kind of result: if H is a compact Lie subgroup of a connected Lie group G, then $\pi_0(Z(H))$ is finite. We do not know whether this true.

6. $SO(n+1)$-equivariant bundles

In this section, we describe the (n,G)-bundles which are $SO(n+1)$-equivariant G-bundles, for the natural action of $SO(n+1)$ on S^n. Let $\delta\colon SO(n)\xrightarrow{\approx} SO(n)$ be the conjugation by the diagonal $(n\times n)$-matrix $\mathrm{Diag}\,(1,\dots,1,-1)$ (or, equivalently, $\mathrm{Diag}\,(-1,\dots,-1,1)$). If $\alpha\colon SO(n)\to G$ is a smooth homomorphism, observe that $\mathrm{Res}\,[\alpha]=\mathrm{Res}\,[\alpha\circ\delta]$ in $\mathcal{R}(n-1,G)$.

Theorem 6.1. *Let ξ be a (n,G)-bundle. If ξ comes from an $SO(n+1)$-equivariant G-bundle then $J(\xi)$ is of the form $([\alpha],[\alpha\circ\delta])$.*

For any $[\alpha]\in\mathcal{R}(n-1,G)$ there is a unique $\xi\in\mathcal{R}(n,G)$ which comes from a $SO(n+1)$-equivariant G-bundle and such that $J(\xi)=([\alpha],[\alpha\circ\delta])$.

Proof. For $\theta\in[0,\pi]$, let $R_\theta\in SO(n+1)$ be the rotation of angle θ in the plane of the last 2 coordinates. Let $R:=R_\pi$, the diagonal matrix with entries $(1,\dots,1,-1,-1)$.

Let ξ be an $SO(n+1)$-equivariant bundle. Choose $a,b\in E(\xi)$, with $p(a)=(0,\dots,-1)$ and let $b:=R\cdot a$. For $A\in SO(n)$, one has $R^{-1}AR=\delta(A)$ and

$$
\begin{aligned}
b\,\beta(A) &= A\cdot b = A\cdot(R\cdot a) = R\cdot(R^{-1}AR)\cdot a\\
&= R\cdot a\,\alpha(\delta(A)) = b\,\alpha(\delta(A)),
\end{aligned}
\tag{6.2}
$$

whence $\beta=\alpha\circ\delta$, which proves Part (i).

Let $\alpha\colon SO(n)\to G$ be a smooth homomorphism and set $\gamma:=\alpha_{|SO(n-1)}$. Suppose that ξ is an $SO(n+1)$-equivariant G-bundle with $a\in E(\xi)$ such that $p(a)=(0,\dots,-1)$ and $A\cdot a=a\,\alpha(A)$ for $A\in SO(n)$. Then $\xi\in\bar J^{-1}([\gamma])$. let $c\colon I\to E(\xi)$ be the curve $c(t):=R_{\theta(t)}\cdot a$, where $\theta(t):=\frac{\pi}{2}(t-1)$. Using that $R_\theta^{-1}BR_\theta=B$ for $B\in SO(n-1)$, on checks, as in equation (6.2), that c is a γ-isotropic lifting. By (4.2) and equation (6.2), one has $\tilde J_\gamma(\xi)=([\alpha],[\alpha\circ\delta])$ in $\mathcal{R}_\gamma(n,G)$. By Proposition 4.1, this proves that ξ is determined by α, which proves the uniqueness statment of Part (ii).

It remains to construct, for a smooth homomorphism $\alpha\colon SO(n)\to G$, an $SO(n+1)$-equivariant G-bundle with $J(\xi)=([\alpha],[\alpha\circ\delta])$. Consider the map

$p\colon SO(n{+}1) \to S^n$ sending a matrix to its last column. This makes an $SO(n{+}1)$-equivariant $SO(n)$-bundle (the principal $SO(n)$-bundle associated to the tangent bundle of S^n). Let ξ be the G-bundle obtained by the Borel construction, using the homomorphism $\alpha\circ\delta\colon SO(n) \to G$

$$E(\xi) := SO(n+1) \times G/\{(BA, g) = (B, \alpha(\delta(A))g)\}. \tag{6.3}$$

This is an $SO(n+1)$-equivariant G-bundle. Choosing $a := (R, e)$ and $b := (I, e)$, one sees that

$$A \cdot a = (AR, e) = (R\delta(A), e) = (R, \alpha(A)e) = a\,\alpha(A)$$

and

$$A \cdot b = (A, e) = (I, \alpha(\delta(A))e) = b\,\alpha(\delta(A)).$$

Therefore, ξ is an $SO(n+1)$-equivariant G-bundle with $J(\xi) = ([\alpha], [\alpha\circ\delta])$. □

Remark 6.4. Theorem 6.1 and its proof show that any $SO(n+1)$-equivariant G bundle is derived from the tangent bundle to S^n by the Borel construction (formula (6.3)). This can be compared with [10, § 6].

7. Examples and applications

Notation: if X is a set, we denote by ΔX the diagonal in $X \times X$.

7.1. *$n = 2$ and $G = U(m)$.* A homomorphism $\alpha\colon S^1 \to U(m)$ has, up to conjugacy, a unique diagonal form $\alpha(z) = \mathrm{Diag}\,(z^{p_1}, \dots, z^{p_m})$, with $p_1 \geq \dots \geq p_m$. The same holds for β. By Theorem A, $\mathcal{E}(2, U(m))$ is then in bijection with the set of pairs (p, q) of m-tuples of non-decreasing integers. In $\pi_1(U(m)) = \mathbf{Z}$, one has $[\alpha] = \sum_{i=1}^m p_i$ and $[\beta] = \sum_{i=1}^m q_i$, so, by Proposition C:

$$C(\xi^\flat) = \sum_{i=1}^m (p_i - q_i).$$

If one wishes instead to characterize $\xi^\flat$ by its first Chern number $c(\xi^\flat) \in H^2(S^2) = \mathbf{Z}$, then $c(\xi^\flat) = -C(\xi^\flat)$ [19, p. 445].

For instance, if τ is the unit tangent bundle over S^2 with the natural action, then $\alpha(z) = z$, $\beta(z) = z^{-1}$, so $C(\tau) = -2$ and $c(\tau) = 2 = \chi(S^2)$.

Note that, if ξ comes from a $SO(3)$-bundle, then, by Theorem 6.1, one has $q_i = -p_i$ (like for τ above). In particular $c(\xi^\flat)$ must be even (see also [10, (6.3)]).

7.2. *$n = 2$ and $G = O(2)$.* For $q \in \mathbf{Z}$, let $\alpha_q\colon SO(2) \to O(2)$ be the homomorphism $A \to A^q$. The set $\mathcal{R}(2, O(2))$ is in bijection with $\mathbf{N}$ given by $\alpha_q \mapsto |q|$. The same recipe produces a bijection $\pi_1(O(2), e)/\pi_0(O(2)) \cong \mathbf{N}$. Let $\xi_{p,q} := \xi_{\alpha_p,\alpha_q}$ be the $(2, O(2))$-bundles constructed in (3.7). By Proposition 4.1 and (4.3), each $\mathcal{E}(2, O(2))$ is represented by some $\xi_{p,q}$, with the only relation $\xi_{p,q} = \xi_{-p,-q}$. One has $J(\xi_{p,q}) = (|p|, |q|)$ and $C(\xi_{p,q}) = |p - q|$. Therefore, $J^{-1}(r, s)$ contains 1 element if $rs = 0$ and 2 otherwise.

7.3. *$n = 2$ and $G = SO(m)$ with $m \geq 3$.* A maximal torus of $SO(m)$ is formed by matrices containing 2-blocks concentrated around the diagonal, so isomorphic to $SO(2)^k$ where $k = [m/2]$. As in 7.1, by Theorem A, $\mathcal{E}(2, SO(m))$ is then in bijection with the set of pairs (p, q) of k-tuples of non-decreasing integers. The bundle $\xi^\flat$ is determined by its second Stiefel-Whitney number $w(\xi^\flat) \in \mathbf{Z}_2$ which is then given by

$$w(\xi^\flat) = \sum_{i=1}^{[\frac{m}{2}]} (p_i - q_i) \qquad (\mathrm{mod}\, 2).$$

Again, ξ comes from a $SO(3)$-equivariant bundle iff $q_i = -p_i$ and then $\xi^\flat$ is trivial.

7.4. *$n = 2k + 1 \geq 3$ and G is a compact classical group other than $SO(2m)$.* The important thing is that $SO(n-1)$ contains a maximal torus of $SO(n)$. Therefore, by [3, Ch. 6, Corollary 2.8], for any embedding $\psi\colon G \hookrightarrow U(m)$, the representations $\psi \circ \alpha$ and $\psi \circ \beta$ are conjugate in in $U(M)$. For G a compact classical group other than $SO(2m)$, this implies that α and β are conjugate in G [20, Pro. 8, p.56]. Therefore, the image of J is the diagonal $\Delta\mathcal{R}(n, G)$. We do not know whether this true for $G = SO(2m)$ (see [20, Rm. p.57] for a possible source of counter-examples).

7.5. *$n = 2k + 1 \geq 3$ and $G = SO(n)$.* The set $\mathcal{R}(n, SO(n))$ has just two elements, represented by the trivial homomorphism and the identity id of $SO(n)$. If $\iota\colon SO(n-1) \subset SO(n)$ denotes the inclusion, then $Z_{[\iota]}$ contains 2 elements, represented by the identity matrix and the diagonal matrix D with entries $(-1, \ldots, -1, 1)$. Let δ be the inner automorphism of $SO(n)$ given by the conjugation by D. By Theorem B, the set $\mathcal{E}(n, SO(n))$ for $n = 2k + 1 \geq 3$ then contains 3 elements:

(i) the trivial bundle $S^n \times SO(n)$ with the action $A \cdot (z, B) = (A \cdot z, B)$. The isotropy representations are both trivial.
(ii) the trivial bundle $S^n \times SO(n)$ with the action $A \cdot (z, B) = (A \cdot z, AB)$. It is characterized by $\bar{J}(\xi) = [\iota]$ and $J_\iota(\xi) = ([\mathrm{id}], [\mathrm{id}])$. This does not come from an $SO(n+1)$-equivariant bundle.
(iii) the principal $SO(n)$-bundle $\mathcal{T}S^n$ associated with the tangent bundle of S^n. It is characterized by $\bar{J}(\mathcal{T}S^n) = [\iota]$ and $J_\iota(\mathcal{T}S^n) = ([\mathrm{id}], [\delta])$. This comes from an $SO(n+1)$-equivariant bundle.

Observe that $([\mathrm{id}], [\delta]) = ([\delta], [\mathrm{id}])$ in $\mathcal{R}_\iota(n, SO(n))$. By Proposition C, this implies that $C(\mathcal{T}S^n) = -C(\mathcal{T}S^n)$. This proves again the classical fact that $C(\mathcal{T}S^{2k+1}) \in \pi_{2k}(SO(2k+1))$ is of order 2 [4, Cor. IV.1.11].

7.6. *$n = 2k \geq 6$ and $G = SO(n)$.* The set $\mathcal{R}(n, SO(n))$ contains 3 elements, represented by the trivial homomorphism, the identity id of $SO(n)$ and the conjugation δ by the diagonal matrix with entries $(-1, \ldots, -1, 1)$. The non-trivial homomorphisms restrict to the inclusion $\iota\colon SO(n-1) \subset SO(n)$. The group Z_ι is trivial. Therefore, J is injective and the set $\mathcal{E}(n, SO(n))$ for $n = 2k$ then contains 5 elements:

(i) the trivial bundles $S^n \times SO(n)$ with the actions $A \cdot (z, B) = (A \cdot z, B)$, $A \cdot (z, B) = (A \cdot z, AB)$ and $A \cdot (z, B) = (A \cdot z, \delta(A)B)$. Their images by J is the diagonal $\Delta\mathcal{R}(n, SO(n))$.

(ii) the principal $SO(n)$-bundle $\mathcal{T}S^n$ associated with the tangent bundle of S^n. One has $J(\mathcal{T}S^n) = ([\mathrm{id}], [\delta])$.

(iii) the (n, G)-bundle $-\mathcal{T}S^n$ with $J(-\mathcal{T}S^n) = ([\delta], [\mathrm{id}])$. Its underlying $SO(n)$-principal bundle is stably trival with Euler number -2.

The trivial bundle with action $A \cdot (z, B) = (A \cdot z, B)$, as well as the bundles in (ii) and (iii) are the ones coming from $SO(n+1)$-equivariant bundles.

7.7. *$n = 4$ and $G = SO(3)$.* The groups $SO(4)$ and $SO(3)$ are built up out of the unit quaternions S^3 by $SO(4) \cong (S^3 \times S^3)/\{(1, 1), (-1, -1)\}$ and $SO(3) \cong S^3/\{\pm 1\}$. Recall that these isomorphisms are constructed as follows: the orthogonal transformation $A_{p,q} \in SO(4)$ associated to $(p, q) \in S^3 \times S^3$ is $A_{p,q}(x) := px\bar{q}$, where $x \in \mathbf{H}$ is a quaternion and $\mathbf{H}$ is made isomorpohic to $\mathbf{R}^4$ by choosing $(i, j, k, 1)$ as a basis. The correspondence $p \to A_{p,p}$ then induces the inclusion $\iota\colon SO(3) \subset SO(4)$. As for the automorphism δ of $SO(4)$ of § 6, the conjugation by $D := \mathrm{Diag}\,(-1, \dots, -1, 1)$, as $Dx = \bar{x}$, one checks easily that $\delta(A_{p,q}) = A_{q,p}$.

The non-equivariant isomorphism class of a $SO(3)$-principal bundle η is characterized by $C(\eta) \in \pi_3(SO(3)) = \pi_3(S^3) = \mathbf{Z}$. It is also determined by its first Pontrjagin number $p(\eta) \in 4\,\mathbf{Z}$, with the relation $p(\eta) = 4\,C(\eta)$.

The set $\mathcal{R}(4, SO(3))$ contains 3 elements represented by the trivial homomorphism and those induced by the projections $S^3 \times S^3 \to S^3$ given by $\sigma_1(p, q) := p$, $\sigma_2(p, q) := q$. The last two restrict over $SO(3)$ to the identity id of $SO(3)$. The group Z_ι being trivial, the map J is injective. This shows that the set $\mathcal{E}(4, SO(3))$ contains 5 elements:

(i) the trivial bundles $S^4 \times SO(3)$ with the actions $A \cdot (z, B) = (A \cdot z, B)$ and $A \cdot (z, B) = (A \cdot z, \sigma_i(A)B)$ for $i = 1, 2$. Their images by J is the diagonal $\Delta\mathcal{R}(4, SO(3))$.

(ii) the principal $SO(3)$-bundle $\mathcal{H}\colon \mathbf{R}P^7 \to S^4$ coming from the quaternionic Hopf bundle $S^7 \to S^4$; the $SO(4)$-action comes from the $SU(2) \times SU(2)$-action on S^7 given by $(p, q) \cdot (z_1, z_2) = (pz_1, qz_2)$. One has $J(\mathcal{H}) = ([\sigma_1], [\sigma_2])$ and $p(\mathcal{H}^\flat) = 4$.

(iii) the (n, G)-bundle $-\mathcal{H}$ with $J(-\mathcal{H}) = ([\sigma_2], [\sigma_1])$ and $p(\mathcal{H}^\flat) = -4$.

The trivial bundle with action $A \cdot (z, B) = (A \cdot z, B)$, as well as the bundles in (ii) and (iii) are the ones coming from $SO(5)$-equivariant bundles.

7.8. *$n = 4$ and $G = SO(4)$.* Taking the notations of 7.7, the set $\mathcal{R}(4, SO(4))$ contains 5 elements represented by:

- the trivial homomorphism.
- those induced by $\sigma_1(p,q) := (p,p)$ and $\sigma_2(p,q) := (q,q)$.
- the identity id of $SO(4)$.
- the homomorphism $\delta(p,q) := (q,p)$.

The non-trivial homomorphisms all restrict to to ι over $SO(3)$. The group Z_ι is trivial and then J is injective. The non-equivariant isomorphism class of a $SO(4)$-principal bundle η is characterized by $C(\eta) \in \pi_3(SO(4)) = \pi_3(S^3 \times S^3) = \mathbf{Z} \oplus \mathbf{Z}$. More usually, one takes the pair of integers $(p(\eta), e(\eta))$ formed by the first Pontrjagin number ($\in 2\mathbf{Z}$) and the Euler number of η. The linear map which sends $C(\eta)$ to $(p(\eta), e(\eta))$ has matrix $\left(\begin{smallmatrix} 2 & 2 \\ 1 & -1 \end{smallmatrix}\right)$ (as can be checked on the examples below).

One sees that the set $\mathcal{E}(4, SO(3))$ then contains 17 elements:

(i) the trivial bundles $S^4 \times SO(3)$ with the 5 actions using the above representations. Their images by J are just the diagonal elements $\Delta\mathcal{R}(4, SO(4))$.

(ii) the principal $SO(4)$-bundle $\widehat{\mathcal{H}}$ whose total space is $E(\widehat{\mathcal{H}}) := \mathbf{R}P^7 \times_{SO(3)} SO(4)$. One has $J(\widehat{\mathcal{H}}) = ([\sigma_1], [\sigma_2])$ and $C(\widehat{\mathcal{H}}^\flat) = (1,1)$ therefore its characteristic classes are $(p(\mathcal{H}^\flat), e(\mathcal{H}^\flat)) = (4,0)$. Also its "inverse" $-\widehat{\mathcal{H}}$, with $J(-\widehat{\mathcal{H}}) = ([\sigma_2], [\sigma_1])$ and $(p(\mathcal{H}^\flat), e(\mathcal{H}^\flat)) = (-4,0)$.

(iii) the principal $SO(4)$-bundle $\mathcal{T}S^4$ associated with the tangent bundle of S^4. One has $J(\mathcal{T}S^4) = ([\mathrm{id}], [\delta])$. Therefore, $C(\mathcal{T}S^4) = (1,-1)$ and $(p(\mathcal{T}^\flat), e(\mathcal{T}^\flat)) = (0,2)$. Again, one can consider its inverse.

(iv) the (n,G)-bundles ξ_i $(i = 1,2)$ with $J(\xi_i) = ([\mathrm{id}], [\sigma_i])$ and their inverses $-\xi_i$. They satisfy $C(\xi_1) = (0,-1)$ and $C(\xi_2) = (1,0)$, or, equivalently:

$$(p(\xi_1^\flat), e(\xi_1^\flat)) = (-2,1) \quad \text{and} \quad (p(\xi_2^\flat), e(\xi_2^\flat)) = (2,1).$$

(v) the (n,G)-bundle $\xi_{i,\delta}$ $(i = 1,2)$ with $J(\xi_i) = ([\delta], [\sigma_i])$ and their inverses. They satisfy $C(\xi_{1,\delta}) = (-1,0)$ and $C(\xi_{1,\delta}) = (0,1)$, or, equivalently:

$$(p(\xi_{1,\delta}^\flat), e(\xi_{1,\delta}^\flat)) = (-2,-1) \quad \text{and} \quad (p(\xi_{2,\delta}^\flat), e(\xi_{2,\delta}^\flat)) = (2,-1).$$

Only the trivial bundle with action $A \cdot (z,B) = (A \cdot z, B)$ and the bundles in (ii) and (iii) come from $SO(5)$-equivariant bundles.

7.9. $G = U(m)$. In order to have non-trivial $(n, U(m))$-bundles, one must have $\dim U(m) > \dim SO(n)$. We check that we are then in the stable range, where, by Bott periodicity,

$$\pi_{n-1}(U(m)) \approx \pi_{n-1}(U(m+k)) \approx \begin{cases} 0 & \text{if } n \text{ is odd} \\ \mathbf{Z} & \text{if } n \text{ is even.} \end{cases}$$

Problem: *which integers occur as $C(\xi^\flat)$ for a $(n, U(m))$-bundle ξ ?*

8. A more general setting

The orthogonal action of $SO(n)$ on S^n is an example of the *special* Π-manifolds defined by Jänich [13, 1.2]. Other examples include the "cohomogeneity one" actions studied by E. Straume [25] (see [7] for a recent application and other references). In this section we give the classification of equivariant (Π, G)-bundles over special Π-manifolds. We will assume in this section that Π and G are both *compact* Lie groups.

Let X be a smooth, connected, closed n-dimensional manifold with a smooth Π-action. Choose a Π-invariant Riemannian metric on X, and then each tangent space T_xX contains a Π_x invariant subspace V_x perpendicular to the orbit $\Pi \cdot x$. Then X is called a *special* Π-manifold if for each $x \in X$ the representation of Π_x on the normal space V_x is the direct sum of a trivial representation and a transitive representation.

It follows that the orbit space $M = X/\Pi$ admits a natural structure as a topological manifold with boundary [13, 1.3], with dimension equal to $n - \dim(\Pi/H)$ where H is the principal isotropy type. Under the "functional" smooth structure [2, VI.6], the orbit space M is a smooth manifold with boundary. The pair $(X, \pi\colon X \to M)$ is called a special Π-manifold over M.

Special Π manifolds over M were classified by Jänich [13, 3.2], and independently by W.-C. Hsiang and W.-Y. Hsiang [12] (see also [2, V.5, VI.6]).

Let $\partial M_A = \{B_\alpha\}_{\alpha\in A}$ denote the set of boundary components of M. An *admissible* isotropy group system (H, U_A) over M consists of a closed subgroup H of Π and a set $U_A = \{U_\alpha\}_{\alpha\in A}$ of closed subgroups in Π containing H, with the property that for each $\alpha \in A$ there exists a transitive representation in which H appears as the isotropy group of a non-zero vector. Let $\Gamma = N(H)/H$ and $\Omega_\alpha = N(U_\alpha) \cap N(H)/H$ for each $\alpha \in A$. The idea of the classification is to re-construct X from the unique principal Γ-bundle P over M such that $P_{|M_0} = \{x \in X \,|\, \Pi_x = H\}$, and a reduction of the structural group of $P|B_\alpha$ to Ω_α over each of the boundary components B_α.

Let $B_\alpha \times [0,1]$ be a collar neighbourhood of some component B_α in M, and let $Y_\alpha = \pi^{-1}(B_\alpha \times [0,1])$ denote its pre-image in X. The key fact is the following identification of Y_α as a Π-space.

Theorem 8.1. (Tube Theorem [2, V.4.2]) *Let $Y = Y_\alpha$, $B = B_\alpha$ and $\Omega = \Omega_\alpha$. There exists a right Ω-principal bundle $Q = Q_\alpha$ over B, and a Π-equivariant diffeomorphism*

$$M_\psi \times_\Omega Q \xrightarrow{\approx} Y$$

commuting with the projection to $[0,1]$, where M_ψ denotes the mapping cylinder of the canonical projection $\psi\colon \Pi/H \to \Pi/U$.

Let $X_0 = X - \bigcup_\alpha \pi^{-1}(B_\alpha \times [0,1/2))$ and $M_0 = X_0/\Pi$. The Tube Theorem shows that X is Π-diffeomorphic to the union

$$X = X_0 \cup \bigcup_\alpha Y_\alpha = \Pi/H \times_\Gamma P \cup \bigcup_\alpha M_{\psi_\alpha} \times_{\Omega_\alpha} Q_\alpha$$

with the identification on the overlaps $B_\alpha \times [1/2, 1]$ induced by a reduction of structural groups $P|B_\alpha \cong \Gamma \times_{\Omega_\alpha} Q_\alpha$.

Two special Π-manifolds (X_1, π_1) and (X_2, π_2) over M are called *isomorphic* when there exists a Π-equivariant diffeomorphism $f\colon X_1 \to X_2$ so that the induced diffeomorphism $\bar{f}\colon M \to M$ is the identity. By the the smooth isotopy lifting theorem of G. Schwarz [22, Corollary 2.4] this is equivalent to Jänich's original definition where f was assumed to be only isotopic to the identity, by an isotopy fixing ∂M pointwise. Let $\mathcal{S}[H, U_A]$ denote the set of isomorphism classes of special Π-manifolds over M, with isotropy group system fine equivalent to (H, U_A) (cf. [13, §2]).

Theorem 8.2. ([13, 3.2]) *Let (H, U_A) be an admissible isotropy group system over M, where M is a smooth, compact connected manifold with boundary. Then*

$$\mathcal{S}[H, U_A] \cong [M, \partial M_A; B\Gamma, B\Omega_A] \ .$$

In order to classify equivariant (Π, G)-bundles (E, p) over a special Π-manifold X, called *special* (Π, G)*-bundles* for short, we will generalize the results of Lashof [16] to describe the bundles over Π-spaces X_0 and Y_α with one orbit, and then follow Jänich's method to glue the pieces together.

First some general definitions: if $\rho\colon H \to G$ is a (smooth) homomorphism, $[\rho]$ denotes the set of homomorphisms $\rho'\colon H \to G$ such that $\rho'(h) = g\rho(h)g^{-1}$ for some $g \in G$ and all $h \in H$. We will say that *the fibre over x belongs to* $[\rho]$ if for each $z \in p^{-1}(x)$ there exists $\rho' \in [\rho]$ such that $hz = z \cdot \rho'(h)$ for all $h \in H$. Then let

$$X^{[\rho]} = \{x \in X^H \mid \text{ the fibre over } x \text{ belongs to } [\rho]\}$$

Let $X_0^{[\rho]} = X_0 \cap X^{[\rho]}$ and notice that $X_0^{[\rho]} \subset P = \{x \in X \mid \Pi_x = H\}$. By [16, Lemma 1.2], the space

$$E^\rho = \{z \in E \mid hz = z \cdot \rho(h), \ \forall h \in H\}$$

is an Z_ρ-bundle over $X^{[\rho]}$, where Z_ρ is the centralizer of ρ in G. The group $\widehat{G} = \Pi \times G$ has a left action on the total space E given by the formula $(\gamma, g) \cdot z = \gamma z \cdot g^{-1}$ for any $(\gamma, g) \in \widehat{G}$ and any $z \in E$. Let us set

$$H\langle\rho\rangle := \{(h, \rho(h)) \mid h \in H\} \subset \Pi \times G$$

and

$$\Gamma\langle\rho\rangle := N(H\langle\rho\rangle)/H\langle\rho\rangle \ .$$

Then E^ρ is just the fixed set of $H\langle\rho\rangle$ in E under this action.

Two special (Π, G)-bundles (E_1, p_1) and (E_2, p_2) over M are called *equivalent* when there exists a Π-equivariant G-bundle isomorphism $\phi\colon E_1 \to E_2$, covering a Π-equivariant diffeomorphism $f\colon X_1 \to X_2$, so that the induced diffeomorphism $\bar{f}\colon M \to M$ is the identity.

We now give the classification of the part of (E, p) lying over M_0. After the remarks above, we see that it follows directly from the slice theorem [2, II.5.8].

Theorem 8.3. ([16, 1.9]) *Let $\rho\colon H \to G$ be a smooth homomorphism. The equivalence classes of (Π, G)-equivariant bundles (E_0, p_0) over M_0 with all fibres belonging to $[\rho]$ are in bijection with the homotopy classes of maps $[M_0, B\Gamma\langle\rho\rangle]$.*

We next define an *admissible* (Π, G)-isotropy group system over M to be a set $(H, U_A; \rho, \rho_A)$, where H and U_A are as above, $\rho\colon H \to G$ is a homomorphism, and $\rho_A = \{\rho_\alpha\}_{\alpha\in A}$ is a set of homomorphisms $\rho_\alpha\colon U_\alpha \to G$ such that $\rho_\alpha | H = \rho$. We define

$$\Omega_\alpha\langle\rho_\alpha\rangle = N(U_\alpha\langle\rho_\alpha\rangle) \cap N(H\langle\rho\rangle)/H\langle\rho\rangle$$

and

$$\Omega_A\langle\rho_A\rangle = \{\Omega_\alpha\langle\rho_\alpha\rangle\}_{\alpha\in A} \ .$$

A special (Π, G)-bundles (E, p) *realizes* an admissible (Π, G)-isotropy group system $(H, U_A; \rho, \rho_A)$ over M if

(i) there exists points $\{y_\alpha \in Y_\alpha\}$ such that $\Pi_{y_\alpha} = U_\alpha$,
(ii) for each y_α, the normal space V_{y_α} to the orbit $\Pi \cdot y_\alpha$ has a point $z_\alpha \in V_{y_\alpha}$ with $\Pi_{z_\alpha} = H$,
(iii) the images $c_\alpha(t)$ of rays in V_{y_α} joining z_α to y_α, have isotropic liftings $\tilde{c}_\alpha(t)$ to E such that $\tilde{c}_\alpha(t) \subset E^\rho$ for $0 \leq t < 1$ and $\tilde{c}_\alpha(1) \subset E^{\rho_\alpha}$.

It is not difficult to check (following [13, §2]) that every special (Π, G)-bundle (E, p) over M realizes some admissible (Π, G)-isotropy group system $(H, U_A; \rho, \rho_A)$. This isotropy group system is unique up to a natural notion of equivalence, extending the "fine-orbit structure" of Jänich.

We say that two isotropy group systems $(H, U_A; \rho, \rho_A)$ and $(H', U'_A; \rho', \rho'_A)$ are *fine-equivalent* if the following conditions hold:

(i) there exists an element $\gamma \in \Pi \times G$ such that $H'\langle\rho'\rangle = \gamma H\langle\rho\rangle\gamma^{-1}$, and
(ii) there exist $n_\alpha \in NH\langle\rho\rangle$ such that $U'_\alpha\langle\rho'_\alpha\rangle = (\gamma n_\alpha)U_\alpha\langle\rho_\alpha\rangle(\gamma n_\alpha)^{-1}$.

Let $\mathcal{S}[H, U_A; \rho, \rho_A]$ denote the set of equivalence classes of special (Π, G)-bundles over M realizing the given (Π, G)-isotropy group system, up to fine equivalence.

Theorem 8.4. *Let $(H, U_A; \rho, \rho_A)$ be an admissible (Π, G)-isotropy group system over M, where M is a smooth, compact connected manifold with boundary. Then*

$$\mathcal{S}[H, U_A; \rho, \rho_A] \cong \left[M, \partial M_A; B\Gamma\langle\rho\rangle, B\Omega_A\langle\rho_A\rangle\right] \ .$$

Proof. Suppose that we are given an admissible (Π, G)-isotropy group system over M. Let (E, p) be a special (Π, G)-bundle over M realizing the given (Π, G)-isotropy group system. By restricting the bundle to M_0, we get a map $\omega_0\colon M_0 \to$

$B\Gamma\langle\rho\rangle$ classifying the principal $\Gamma\langle\rho\rangle$-bundle $P\langle\rho\rangle$ (which completely determines (E_0, p_0)) by Theorem 8.3. We can apply the Tube Theorem 8.1 to the $\widehat{G} = \Pi \times G$ action on $p^{-1}(Y_\alpha) \subset E$, since $z \in E^\rho$ means that $\widehat{G}_z = H\langle\rho\rangle$ and similarly for $z \in E^{\rho_\alpha}$. This identifies the restriction of our bundle to the part over $B_\alpha \times [0,1]$ as $M_{\varphi_\alpha} \times_{\Omega_\alpha\langle\rho_\alpha\rangle} Q\langle\rho_\alpha\rangle$, where

$$\varphi_\alpha \colon \widehat{G}/H\langle\rho\rangle \to \widehat{G}/U_\alpha\langle\rho_\alpha\rangle$$

is the $\widehat{G}$-equivariant projection, and $Q\langle\rho_\alpha\rangle$ is a principal right $\Omega_\alpha\langle\rho_\alpha\rangle$-bundle over B_α. The classifying map ω_0 for $P\langle\rho\rangle$ therefore extends to a map

$$\omega \colon (M, \partial M_A) \to (B\Gamma\langle\rho\rangle, B\Omega_A\langle\rho_A\rangle)$$

where the notation means that each boundary component B_α is mapped into the α-component of $B\Omega_A\langle\rho_A\rangle$. The restriction of ω to B_α classifies $Q\langle\rho_\alpha\rangle$. This shows that (E, p) is determined up to equivalence by ω.

Conversely, if we are given a map ω as above we can reconstruct a special (Π, G)-equivariant bundle over M realizing the isotropy group system, up to fine equivalence. It can be checked that this bundle is unique up to equivalence. □

Remark 8.5. Note that a special $\Pi \times G$-manifold over M is a special (Π, G)-bundle over M precisely when the subgroup $1 \times G$ acts freely on the total space. The isotropy group system for the bundle is just the collection of isotropy groups for the $\Pi \times G$-action. This observation shows that Theorem 8.4 follows from Jänich's results.

Corollary 8.6. *Let Π and G be compact Lie groups. The set of special (Π, G)-equivariant bundles over M is finite, provided that $\dim M \leq 1$ and the isotropy groups are semi-simple.*

Proof. This is proved using [21], as for the finiteness of $\mathcal{E}(n, G)$. □

To conclude this section, we discuss the connection between these results and Theorem B. Let X be a special Π-manifold over M, and let $\mathcal{E}(X, \mathcal{F})$ denote the set of bundle isomorphism classes of principal (Π, G)-bundles over X with isotropy group system fine equivalent to $\mathcal{F} := (H, U_A; \rho, \rho_A)$. Here a bundle isomorphism is a Π-equivariant G-bundle isomorphism $\phi \colon E_1 \to E_2$ covering the *identity* on X.

Lemma 8.7. *There is a commutative diagram:*

$$\begin{array}{ccccc}
\mathcal{E}(X, \mathcal{F}) & \longrightarrow & \mathcal{S}[H, U_A; \rho, \rho_A] & \xrightarrow{\lambda} & \mathcal{S}[H, U_A] \\
\downarrow{\scriptstyle v} & & \downarrow{\scriptstyle \approx} & & \downarrow{\scriptstyle \approx} \\
\big[M, \partial M_A; BZ_\rho, BZ_{\rho_A}\big] & \xrightarrow{u} & \big[M, \partial M_A; B\Gamma\langle\rho\rangle, B\Omega_A\langle\rho_A\rangle\big] & \longrightarrow & \big[M, \partial M_A; B\Gamma, B\Omega_A\big]
\end{array}$$

where the horizontal composites have images represented by the element $[X]$.

Proof. The exact sequences $1 \to Z_\rho \to \Gamma\langle\rho\rangle \to \Gamma$ and $1 \to Z_{\rho_\alpha} \to \Omega_\alpha\langle\rho\rangle \to \Omega_\alpha$ of groups induce a map $u\colon \left[M, \partial M_A; BZ_\rho, BZ_{\rho_A}\right] \to \left[M, \partial M_A; B\Gamma\langle\rho\rangle, B\Omega_A\langle\rho_A\rangle\right]$. By Theorem 8.4 and Theorem 8.2, we also have a *surjective* map

$$v\colon \mathcal{E}(X,\mathcal{F}) \to \left[M, \partial M_A; BZ_\rho, BZ_{\rho_A}\right]$$

induced by u and our construction of equivariant bundles. □

Theorem 8.8. *Let X be a special Π-manifold over M, and $\mathcal{F} = (H, U_A; \rho, \rho_A)$ an admissible isotropy group system. Then*

$$v\colon \mathcal{E}(X,\mathcal{F}) \cong \left[M, \partial M_A; BZ_\rho, BZ_{\rho_A}\right] .$$

Proof. The map v is given in the diagram above, and we have already observed that it is surjective. Suppose that ξ_1, $\xi_2 \in \mathcal{E}(X,\mathcal{F})$ with $v(\xi_1) = v(\xi_2)$. Then we have a continuous map

$$(M \times I, \partial(M \times I)) \to (BZ_\rho, BZ_{\rho_A})$$

realizing the homotopy between the classifying maps for ξ_1 and ξ_2. By the surjectivity of v for (Π, G)-bundles over $M \times I$, we get a bundle (E, p) over $X \times I$ which restricts to ξ_1 and ξ_2 at the ends $X \times \partial I$. Since $E \cong E_0 \times I$, we get $\xi_1 \cong \xi_2$. □

Let $\mathrm{Aut}(X)$ be the group of Π-equivariant isotopy classes of Π-equivariant diffeomorphisms of X over the identity of M. The group $\mathrm{Aut}(X)$ preserving acts on $\mathcal{E}(X,\mathcal{F})$ by pulling back: $f \cdot \xi := (f^{-1})^*\xi$. This action is well defined by the equivariant Covering Homotopy Theorem of Palais [2, II.7.3]. If

$$\lambda\colon \mathcal{S}[H, U_A; \rho, \rho_A] \to \mathcal{S}[H, U_A]$$

denote the natural forgetful map, then there is an induced map

$$\psi\colon \mathcal{E}(X,\mathcal{F}) \to \lambda^{-1}(X)$$

given by applying our stronger equivalence relation on bundles (which allows ϕ to cover a self-diffeomorphism of X).

Proposition 8.9. *The map ψ induces a bijection between $\lambda^{-1}(X)$ and the quotient of $\mathcal{E}(X,\mathcal{F})$ by the action of* $\mathrm{Aut}(X)$.

Proof. The map from one set of bundles to the other is defined by regarding a (Π, G)-bundle over X as an element of $\lambda^{-1}(X)$, and this is well-defined since the equivalent relation in $\mathcal{S}[H, U_A; \rho, \rho_A]$ is stronger. Moreover, two bundles with isotropy group system $\mathcal{F}$ over X are equivalent if and only if they are in the same orbit of the action of $\mathrm{Aut}(X)$, hence our correspondence is injective. On the other hand, if (E', p') is a bundle with base space X' in $\lambda^{-1}(X)$, then there exists a Π-equivariant diffeomorphism $h\colon X \to X'$ covering the identity on M. Then $E := h^*(E')$ is an equivalent element in $\lambda^{-1}(X)$, and is a bundle over X, so our correspondence is surjective. □

These results and Theorem 8.4 can sometimes be used for explicit classification of equivariant (Π, G)-bundles over special Π-manifolds. Notice that Bredon in [2, V.7] together with [2, Theorem V.6.4] has determined $\mathrm{Aut}(X)$ in many cases of interest.

We shall now specialize to special Π-manifolds over $I := [-1, 1]$, and extend Theorem B to this setting. Examples include cohomogeneity 1 actions on spheres, classified in [25, Thm. C, Table II]. The two components $\{\pm 1\}$ of ∂I are denoted $\{\pm\}$, and the notation $\mathcal{F} = (H, U_\pm; \rho, \rho_\pm)$ will be used for the admissible (Π, G)-isotropy group systems, as well as Γ, $\Omega_\pm$, etc. The classification of special Π-manifolds over I takes the following form.

Theorem 8.10. *Let $(H, U_\pm)$ be an admissible isotropy group system over I. Then $\mathcal{S}[H, U_\pm]$ is in bijection with the double cosets $\pi_0(\Omega_-)\backslash\pi_0(\Gamma)/\pi_0(\Omega_+)$.*

Proof. This follows directly from Theorem 8.2, since

$$[I, \partial I; B\Gamma, B\Omega_\pm] \cong \pi_1(B\Omega_-)\backslash\pi_1(B\Gamma)/\pi_1(B\Omega_+) \cong \pi_0(\Omega_-)\backslash\pi_0(\Gamma)/\pi_0(\Omega_+). \quad \square$$

Remark 8.11. The bijection of Theorem 8.10 can be seen in a more constructive way. Given a special Π-manifold X over I, we can choose an *H-meridian* $c\colon I \to X$, i.e. a smooth section of $X \to I$ so that $\Pi_{c(t)} = H$ for t in the interior of I; this can be obtained from a smooth section of the (trivial) principal Γ-bundle $P \to I$. Let $U_\pm := \Pi_{c(\pm 1)}$. We say that $(H, U_\pm)$ is an *H-meridian isotropy group system* for X. Choosing another smooth section of P gives isotropy groups conjugate to $U_\pm$ by elements in the same connected component of $N(H)$.

As in (3.7), the special Π-manifold X can be reconstructed as a quotient of $I \times \Pi$:

$$X = (I \times \Pi/H) \big/ \{(\pm 1, g) \sim (\pm 1, gu_\pm), \forall\, u_\pm \in U_\pm\}$$

Therefore, X is determined by the subgroups $U_\pm$ (compare [7, Prop. 1.6]). Moreover, any set $(H, U'_\pm)$, where the $U'_\pm$ are conjugate to $U_\pm$, occurs as an H-meridian isotropy group system for some special Π-manifold X' over I (proved as in (4.3)). In this way, $\mathcal{S}[H, U_\pm]$ is a quotient of $\pi_0(N(H)) \times \pi_0(N(H))$. The diagonal group acts trivially, and by carefully examining the relevant equivalence relation one sees that $\mathcal{S}[H, U_\pm]$ is in bijection with $\pi_0(\Omega_-)\backslash\pi_0(\Gamma)/\pi_0(\Omega_+)$.

Let X be a special Π-manifold over I. We will now compute the set $\mathcal{E}(X, G)$ of isomorphism classes of Π-equivariant principal G-bundles over X. Fix an H-meridian isotropy group system $(H, U_\pm)$ for X induced by a smooth H-meridian $c\colon I \to X$. If $\xi = (E, p)$ is a (Π, G)-bundle over X, then we can choose an isotropic lift $\tilde{c}\colon I \to E$, and obtain an admissible isotropy group system $\mathcal{F} = (H, U_\pm; \rho, \rho_\pm)$ for the bundle.

Since the composition of an isotropic lift with a bundle isomorphism is again isotropic, the conjugacy classes $[\rho_\pm] \in \mathcal{R}(U_\pm, G)$ depend only on $[\xi] \in \mathcal{E}(X, G)$. This defines a map

$$J\colon \mathcal{E}(X, G) \to \mathcal{R}(U_-, G) \times \mathcal{R}(U_+, G)\ .$$

We write $\mathcal{R}(U_-,G)\times_H\mathcal{R}(U_+,G)$ for the set of pairs $([\rho_-],[\rho_+])\in\mathcal{R}(U_-,G)\times\mathcal{R}(U_+,G)$ such that $\operatorname{Res}[\rho_-]=\operatorname{Res}[\rho_+]$ in $\mathcal{R}(H,G)$.

Theorem B generalizes to special manifolds over I as follows:

Theorem 8.12. *Let X be a special Π-manifold over I realizing the isotropy group system $(H,U_\pm)$. The set $\mathcal{E}(X,G)$ of isomorphism classes of (Π,G)-bundles over X is determined by the following properties:*

(i) *the image of J is $\mathcal{R}(U_-,G)\times_H\mathcal{R}(U_+,G)$.*

(ii) *Let $\rho_\pm\colon U_\pm\to G$ be two smooth homomorphisms such that $\rho_-|_H=\rho_+|_H=:\rho$. Then there is a bijection between $J^{-1}([\rho_-],[\rho_+])$ and the set of double cosets $\pi_0(Z_{\rho_-})\backslash\pi_0(Z_\rho)/\pi_0(Z_{\rho_+})$.*

Proof. Since $\rho_\pm$ comes from an admissible system, the image of J is contained in $\mathcal{R}(U_-,G)\times_H\mathcal{R}(U_+,G)$. On the other hand, let $\rho_\pm\colon U_\pm\to G$ be two smooth homomorphisms such that $\rho_-|_H=\rho_+|_H=:\rho$. The special $(\Pi\times G)$-manifold constructed as in Remark 8.11, with isotropy group system $(H\langle\rho\rangle,U_\pm\langle\rho_\pm\rangle)$ (associated to an $H\langle\rho\rangle$-meridian) is a (Π,G)-bundle ξ over X (the special Π-manifold with H-meridian isotropy group system $(H,U_\pm)$). One has $J(\xi)=([\rho_-],[\rho_+])$, which proves Part (i).

If $(E,p)\in J^{-1}([\rho_-],[\rho_+])$ then its isotropy group system is fine equivalent to $\mathcal{F}=(H,U_\pm;\rho,\rho_\pm)$. In the notation introduced earlier, we have

$$\mathcal{E}(X,\mathcal{F})=J^{-1}([\rho_-],[\rho_+])$$

and the result now follows from Theorem 8.8. □

We also have a more explicit version of Proposition 8.9. Note that $J(f\cdot\xi)=J(\xi)$, for $f\in\operatorname{Aut}(X)$, so we must investigate the action of $\operatorname{Aut}(X)$ on a preimage $\mathcal{E}(X,\mathcal{F})\cong J^{-1}([\rho_-],[\rho_+])$. The group $\operatorname{Aut}(X)$ has a homotopy description: choose a base point $(\bullet)\in\Omega_-\backslash\Gamma/\Omega_+$ which corresponds to the class of X in $\pi_0(\Omega_-)\backslash\pi_0(\Gamma)/\pi_0(\Omega_+)$.

Proposition 8.13. (Bredon [2, V.7.3, VI.6.4]) *There is a group anti-isomorphism*

$$\operatorname{Aut}(X)\cong[I,\partial I;\Gamma,\Omega_\pm]_\bullet$$

Proof. The pointed maps $(I,\partial I)\to(\Gamma,\Omega_\pm)$ send ∂I into the component of $\Omega_-\backslash\Gamma/\Omega_+$ containing the base point $(\bullet)$. Let $f\in\operatorname{Aut}(X)$. Using our H-meridian c, one defines a smooth path $d\colon I\to\Pi/H$ by the formula

$$f(c(t))=d(t)\cdot c(t).$$

The map f being Π-equivariant, one has, for all $h\in H$

$$hd(t)\cdot c(t)=h\cdot f(c(t))=f(hc(t))=f(c(t))=d(t)c(t)$$

This implies, for all $t\in I$, that $d(t)\in N(H)/H=\Gamma$. For $t=\pm1$, one gets in addition that $d(\pm1)\in\Omega_\pm$. We check that this defines an anti-homomorphism from $\mathrm{Aut}(X)$ to the group $[I, \partial_- I, \partial_+ I; \Gamma, \Omega_-, \Omega_+]$. Now, if $d(t)$ represents a class in the latter, the formula

$$f_d(\alpha c(t)) := \alpha d(t) \cdot c(t) \quad , \quad \alpha \in \Pi, t \in I$$

defines an element f_d of $\mathrm{Aut}(X)$ and constitutes an inverse to the above anti-homomorphism. □

Suppose that the homomorphisms $\Gamma\langle\rho\rangle \to \Gamma$ and $\Omega_\pm\langle\rho\rangle \to \Omega_\pm$ are surjective. Then we have a fibre bundle

$$Z_{\rho_-}\backslash Z_\rho / Z_{\rho_+} \to \Omega_-\langle\rho\rangle\backslash\Gamma\langle\rho\rangle/\Omega_+\langle\rho\rangle \to \Omega_-\backslash\Gamma/\Omega_+. \tag{8.14}$$

Therefore, $\pi_1(\Omega_-\backslash\Gamma/\Omega_+)$ acts on $\pi_0(Z_{\rho_-}\backslash Z_\rho/Z_{\rho_+}) = \pi_0(Z_{\rho_-})\backslash\pi_0(Z_\rho)/\pi_0(Z_{\rho_+})$.

Theorem 8.15. *Suppose that the homomorphisms $\Gamma\langle\rho\rangle \to \Gamma$ and $\Omega_\pm\langle\rho\rangle \to \Omega_\pm$ are surjective. Let $\rho_\pm \colon U_\pm \to G$ be two smooth homomorphisms such that $\rho_- {}_{|H} = \rho_+ {}_{|H} =: \rho$. Then the quotient of $J^{-1}([\rho_-], [\rho_+])$ by the action of $\mathrm{Aut}(X)$ is in bijection with with the quotient of $\pi_0(Z_{\rho_-})\backslash\pi_0(Z_\rho)/\pi_0(Z_{\rho_+})$ by the action of $\pi_1(\Omega_-\backslash\Gamma/\Omega_+)$.*

Proof. By Theorem 8.8, there is a surjective map $\psi\colon J^{-1}([\rho_-], [\rho_+]) \to \lambda^{-1}([X])$. Since $\Gamma\langle\rho\rangle \to \Gamma$ and $\Omega_\pm\langle\rho\rangle \to \Omega_\pm$ are surjective, we have the fibration (8.14). From the homotopy exact sequence of this fibration, we see that $\lambda^{-1}([X])$ is in bijection with the quotient of $\pi_0\big(Z_{\rho_-}\backslash Z_\rho/Z_{\rho_+}\big)$ by the action of $\pi_1(\Omega_-\backslash\Gamma/\Omega_+)$. But this is just the action of $\mathrm{Aut}(X)$ by Proposition 8.13. □

Example 8.16. Consider the standard action of $\Pi = SO(n)$ on S^n, with $H = SO(n-1)$. As $U_\pm = \Pi$, the homomorphisms $\Gamma\langle\rho\rangle \to \Gamma$ and $\Omega_\pm\langle\rho\rangle \to \Omega_\pm$ are surjective.

When $n = 2$, H is the trivial group. If G is connected, then $Z_\rho = G$ is connected and the map J is a bijection, as seen in Theorem A. Also, the group of Π-automorphism of S^2 is equal to Π, so $\mathrm{Aut}(S^2)$ is trivial.

If $n \geq 3$, the group $\Gamma = \Omega_\pm$ has 2 elements, the non-trivial one represented by the diagonal matrix $D := \mathrm{Diag}\,(1, \dots 1, -1, -1)$. The space $\Omega_-\backslash\Gamma/\Omega_+$ being reduced to a point, it follows from Theorem 8.15 that the group $\mathrm{Aut}(S^n)$ acts trivially on $\mathcal{E}(S^n, G)$. This has the following consequence:

Proposition 8.17. *Let $\rho_\pm \colon SO(n) \to G$ be two representations into a compact Lie group G such that $\rho_{\pm|SO(n-1)} = \rho$. Then, the element $\rho_-(D)\rho_+(D)^{-1}$ belongs to Z_ρ and represent the trivial element in $\pi_0(Z_{\rho_-})\backslash\pi_0(Z_\rho)/\pi_0(Z_{\rho_+})$.*

Proof. As $\rho_{\pm|H} = \rho$, one has, for all $h \in H$,

$$\rho_+(D)^{-1}\rho(h)\rho_+(D) = \rho_-(D)^{-1}\rho(h)\rho_-(D).$$

Therefore, $\rho_-(D)\rho_+(D)^{-1} \in Z_\rho$.

Recall that our H-meridian for S^n is $c(t) = (0, \dots, \cos(\pi t/2), \sin(\pi t/2))$. By Proposition 4.1 and its proof there exists a (Π, G)-bundle ξ over S^n, with a ρ-isotropic lifting $\tilde{c}\colon I \to E(\xi)$ of c such that the isotropy representations associated to $\tilde{c}(\pm 1)$ are $\rho_\pm$. The curve $\tilde{c}$ is the horizontal lifting of c for a Π-invariant connection. There is no problem to extend the definitions of $c(t)$ and $\tilde{c}(t)$ for $t \in \mathbf{R}$. This defines $\mu \in G$ by

$$\tilde{c}(3) = \tilde{c}(-1) \cdot \mu \quad \text{and} \quad \tilde{c}(5) = \tilde{c}(1) \cdot \mu.$$

For $t \in [-1, 1]$, one has $D \cdot \tilde{c}(t) = \tilde{c}(2-t)\rho_+(D)$ (since this is true for $t = 1$ and both side are horizontal). For $t = -1$, this gives

$$\tilde{c}(-1)\mu\rho_+(D) = \tilde{c}(3)\rho_+(D) = D \cdot \tilde{c}(-1) = \tilde{c}(-1)\rho_-(D)$$

Therefore, $\mu = \rho_-(D)\rho_+(D)^{-1}$.

Now, let $\xi_1 := \delta^*\xi$; one has

$$E(\xi_1) = \{(x, u) \in S^n \times E(\xi) \mid \delta(x) = \pi(u)\}$$

and the Π-action on $E(\xi_1)$ comes from the diagonal action. A ρ-isotropic lifting $\tilde{c}_1 \colon I \to E(\xi_1)$ of c is then given by

$$\tilde{c}_1(t) := (c(t), \tilde{c}(t-2)).$$

By Proposition 4.1, one has $\xi = \xi_1$ in $\mathcal{E}(S^n, G)$ iff $\tilde{J}_\rho(\xi) = \tilde{J}_\rho(\xi_1)$ in $\mathcal{R}_\rho(n, G)$. By 4.2, using the curves $\tilde{c}$ and $\tilde{c}_1$, one has in $\mathcal{R}_\rho(n, G)$

$$\tilde{J}_\rho(\xi) = [\rho_-, \rho_+] \quad \text{and} \quad \tilde{J}_\rho(\xi_1) = [\mu^{-1}\rho_-\mu, \rho_+].$$

This proves that μ represents the unit element in $\pi_0(Z_{\rho_-})\backslash\pi_0(Z_\rho)/\pi_0(Z_{\rho_+})$. $\square$

Example 8.18. Consider the action of $SO(3)$ on the traceless (3×3)-symmetric matrixes by conjugation. Restricting this action to the unit sphere (for the invariant scalar product $\operatorname{tr}(AB)$) makes S^4 a special $SO(3)$-manifold over I (see [7, §3]) for more details on this classical cohomogeneity one $SO(3)$-action on S^4). Here $H = S(O(1)\times O(1)\times O(1))$, $U_- = S(O(1)\times O(2))$ and $U_+ = S(O(2)\times O(1))$. Observe that any homomorphism $\rho_\pm \colon U_\pm \to G$ is trivial unless its restriction to H is injective.

Take $G = SO(3)$ and use the standard inclusions of $U_\pm$ into $SO(3)$. Any non-trivial homomorphism ρ_- is conjugate to $\tilde{\rho}_- \colon (\varepsilon, r_\theta) \mapsto (\varepsilon, r_\theta^p)$ and any non-trivial ρ_+ is conjugate to $\tilde{\rho}_+ \colon (r_\theta, \varepsilon) \mapsto (r_\theta^q, \varepsilon)$, where $p, q \in \mathbf{N}_{\mathrm{odd}}$, the set of positive odd integers. Both $\tilde{\rho}_\pm$ restrict on H to the identification ρ of H with the diagonal matrices of $SO(3)$. One has $Z_\rho = H$, $Z_{\tilde{\rho}_-} = \{\operatorname{diag}(1,1,1), \operatorname{diag}(1,-1,-1)\}$ and $Z_{\tilde{\rho}_+} = \{\operatorname{diag}(1,1,1), \operatorname{diag}(-1,-1,1)\}$. Therefore, the set of double cosets $\pi_0(Z_{\tilde{\rho}_-})\backslash\pi_0(Z_\rho)/\pi_0(Z_{\tilde{\rho}_+})$ is trivial and $\mathcal{E}([S^4 \to I], SO(3))$ is in bijection, via the map J of theorem 8.12, with $\{0, 0\} \cup \mathbf{N}_{\mathrm{odd}} \times \mathbf{N}_{\mathrm{odd}}$.

In [7, § 3], $(SO(3), SO(3))$-bundles $P_{p,q}$ over S^4 are constructed for $p, q \in \mathbf{Z}$ with $p \equiv 3 (\operatorname{mod} 4)$ (those come from $(S^3 \times S^3)$-bundles). These bundles satisfy $J(P_{p,q}) = (|p|, |q|)$, so, up to isomorphism, the sign of p and q does not matter.

Example 8.19. This is the complex analogue of Example 8.18. One considers the action of $SU(3)$ on the traceless (3×3)-Hermitian matrixes by conjugation and restrict it to the unit sphere. One thus gets a special $SU(3)$-manifold over I diffeomorphic to S^7. The isotropy groups are $H = S(U(1) \times U(1) \times U(1))$, $U_- = S(U(1) \times U(2))$ and $U_+ = S(U(2) \times U(1))$. As in Example 1, any homomorphism $\rho_\pm \colon U_\pm \to G$ is trivial unless its restriction to H is injective.

For $G = SU(3)$, one checks that any non trivial homomorphism of $U_\pm$ to G is conjugate to the standard inclusion. As $Z(H) = H$ in $SU(3)$, the map J is injective and $\mathcal{E}(S^7, SU(3))$ consists of two elements.

References

1. D. Brandt and J.-C. Hausmann, *Théorie de jauge et symétries des fibrés*, Ann. Inst. Fourier (Grenoble) **43** (1993), no. 2, 509–537.
2. Glen E. Bredon, *Introduction to compact transformation groups*, Academic Press, New York, 1972, Pure and Applied Mathematics, Vol. 46.
3. Theodor Bröcker and Tammo tom Dieck, *Representations of compact Lie groups*, Springer-Verlag, New York, 1995.
4. William Browder, *Surgery on simply-connected manifolds*, Springer-Verlag, New York, 1972, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 65.
5. Tammo tom Dieck, *Faserbündel mit Gruppenoperation*, Arch. Math. (Basel) **20** (1969), 136–143.
6. ______, *Transformation groups*, Walter de Gruyter & Co., Berlin, 1987.
7. Karsten Grove and Wolfgang Ziller, *Curvature and symmetry of Milnor spheres*, Ann. of Math. (2) **152** (2000), no. 1, 331–367.
8. Ian Hambleton and Ronnie Lee, *Perturbation of equivariant moduli spaces*, Math. Ann. **293** (1992), no. 1, 17–37.
9. Ian Hambleton, Ronnie Lee, and Ib Madsen, *Rigidity of certain finite group actions on the complex projective plane*, Comment. Math. Helv. **64** (1989), no. 4, 618–638.
10. Jean-Claude Hausmann, *Théorie de jauge et groupoïdes*, Université de Genève preprint (2000),"www.unige.ch/math/folks/hausmann".
11. Sigurdur Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1978.
12. Wu-chung Hsiang and Wu-yi Hsiang, *Differentiable actions of compact connected classical groups. I*, Amer. J. Math. **89** (1967), 705–786.
13. Klaus Jänich, *Differenzierbare Mannigfaltigkeiten mit Rand als Orbiträume differenzierbarer G-Mannigfaltigkeiten ohne Rand*, Topology **5** (1966), 301–320.
14. R. K. Lashof, *Lifting semifree actions*, Proc. Amer. Math. Soc. **80** (1980), no. 1, 167–171.
15. ______, *Equivariant bundles*, Illinois J. Math. **26** (1982), no. 2, 257–271.
16. ______, *Equivariant bundles over a single orbit type*, Illinois J. Math. **28** (1984), no. 1, 34–42.
17. R. K. Lashof and J. P. May, *Generalized equivariant bundles*, Bull. Soc. Math. Belg. Sér. A **38** (1986), 265–271 (1987).
18. R. K. Lashof, J. P. May, and G. B. Segal, *Equivariant bundles with abelian structural group*, Proceedings of the Northwestern Homotopy Theory Conference (Evanston, Ill., 1982) (Providence, R.I.), Amer. Math. Soc., 1983, pp. 167–176.
19. John Milnor, *Some consequences of a theorem of Bott*, Ann. of Math. (2) **68** (1958), 444–449.

20. Arkadi L. Onishchik, *Topology of transitive transformation groups*, Johann Ambrosius Barth Verlag GmbH, Leipzig, 1994.
21. R. W. Richardson, Jr., *A rigidity theorem for subalgebras of Lie and associative algebras*, Illinois J. Math. **11** (1967), 92–110.
22. Gerald W. Schwarz, *Lifting smooth homotopies of orbit spaces*, Inst. Hautes Études Sci. Publ. Math. (1980), no. 51, 37–135.
23. Norman Steenrod, *The Topology of Fibre Bundles*, Princeton University Press, Princeton, N. J., 1951, Princeton Mathematical Series, vol. 14.
24. T. E. Stewart, *Lifting group actions in fibre bundles*, Ann. of Math. (2) **74** (1961), 192–198.
25. Eldar Straume, *Compact connected Lie transformation groups on spheres with low cohomogeneity. I*, Mem. Amer. Math. Soc. **119** (1996), no. 569, vi+93.

Department of Mathematics & Statistics
McMaster University
Hamilton, ON L8S 4K1, Canada
E-mail address: ian@math.mcmaster.ca

Section de Mathématiques
Université de Genève, B.P. 240
CH-1211 Genève 24, Switzerland
E-mail address: hausmann@math.unige.ch