On the resummation of the Lee-Yang edge singularity coupled to gravity

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Codesido Sanchez, Santiago

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Santiago Codesido Sánchez

ABSTRACT: We study the Borel-Padé resummation of the asymptotic series for the string equation of the Lee-Yang edge singularity. Numerical methods are provided to compute a high accuracy exact solution. We find the resummation matches the numerical integration without need for further non-perturbative corrections.

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1   Introduction

As it is well known, many of the usual perturbative series used in physical problems are only asymptotic in nature. A standard tool to make sense of these formal expansions is Borel resummation [1]. However, it is often the case that one has to add additional "non-perturbative" sectors to the original perturbative series, in order to reconstruct the exact answer by resummation. This leads to the theory of transseries and resurgent asymptotics. For example, in the theory of ODEs, the asymptotic expansion near an irregular singular point can be extended to a transseries which might then be used to reconstruct the exact solution (see [2] for a comprehensive review of these and other examples).

The genus expansion of string theory is no exception to this divergent behaviour [3], but in this case it is not clear what is the exact, non-perturbative answer behind the perturbative series. In some examples, one can use non-trivial dualities, such as the AdS/CFT correspondence, to provide an exact answer from the dual side. This opens up the possibility of comparing the Borel resummation of the divergent string series with an exact definition.

This comparison was done for the free energy on the three sphere of ABJM theory in [4], which can be computed exactly by localization techniques. The genus expansion of string theory is simply the $1/N$ expansion of this free energy. It turns out that this expansion is Borel summable, i.e. there are no obstructions to the Borel resummation of the series. However, and in contrast to what happens in many simple models in Quantum Mechanics, this resummation does not agree with the exact result. Of course, there is no obvious *a priori* reason why there should be such an agreement. The difference between the two results was understood in terms of complex instantons, which show up as complex poles in the Borel integration plane. A closely related example was recently studied in the context of topological string theory on toric Calabi–Yau threefolds. In these models, one can provide a non-perturbative definition of the topological string free energy by using spectral theory

[5] (see [6] for a review). Once again, Borel resummation of the asymptotic series differs from this exact result. However, the perturbative solution can be upgraded to a transseries including instanton corrections [7]. By a systematic Borel resummation of the full transseries [8], the exact answer can be indeed reproduced.

These non-trivial facts led us to review earlier models of strings with an exact, non-perturbative definition. We consider in this paper minimal models coupled to 2D gravity [9–11], which can be defined through the double scaling limit of a theory of randomly triangulated surfaces [12]. One can even write an ODE for the partition function of these non-critical string models, depending on a parameter essentially equivalent to the string coupling constant. This ODE is called in this context the string equation of the model. The asymptotic expansion of the solution to this ODE corresponds the genus expansion of the corresponding string.

The existence of the string equation is not enough to define the theory non-perturbatively (see [13] for a review), since one has to provide in addition appropriate boundary conditions. In the case of the Lee-Yang edge singularity coupled to gravity, such a boundary condition can be found, and one can use it to calculate the free energy of the non-critical string exactly, by numerical integration of the string equation [14]. The corresponding asymptotic series is known to be Borel summable, and in [13] it is asked whether its Borel resummation reproduces the true solution to the string equation. In view of the recent results in [4, 8], this is not obviously the case. We will show in this paper, by a detailed numerical comparison of the resummation and the numerical integration of the string equation, that the solutions seem to be equal, at least with a very high numerical accuracy. This is the case *despite* the presence of complex Borel poles like the ones identified in [4] as a source for the corrections to the Borel resummation.

This paper is organized as follows. We will first briefly review the analytic structure of the solutions and their large order behaviour, and then we will compare the Borel resummation with the exact numerical integration.

2 The string equation of the Lee-Yang edge singularity

2.1 Perturbative solution

The string equation of the Lee-Yang edge singularity itself is given, as in [14], by

$$P[f, x] = f^3 + \frac{1}{10} f^{(4)} + f f'' + \frac{1}{2} (f')^2 - x = 0. \tag{2.1}$$

The physical conditions we must impose to the solution, following [14], are absence of poles (which would mean zeros in the partition function) and most importantly, an asymptotic behaviour that gives the right large N limit for the string picture. It is

$$f(x) \underset{|x| \to \infty}{=} \text{sign} (x) |x|^{1/3} + o\left(|x|^{1/3}\right). \tag{2.2}$$

In [14] a numerical plot of a numerical approximation to the pole-free solution is provided. For negative values of x, the solution oscillates. This is an Airy-like symptom of the Stokes phenomenon (see [2]). Essentially, complex exponentially small corrections take over, and their interference produces the oscillation. This will set our focus on the $x > 0$ region. There, the series expansion at infinity is resummable. Since no oscillations are present there is no *a priori* need to have exponentially small corrections to the Borel sum. The question is: does the resummation, with no further corrections, agree with the numerical answer?

We will begin by computing this resummation. The limit in which we implement the asymptotic conditions is $x \to \infty$. For convenience, we redefine $f(x) = x^{1/3} g(t)$ and $x = t^{-3/7}$. In this variable,

(2.1) is solved by the (formal) series

$$g(t) = \sum_{k=0}^{\infty} \mu_k t^k. \tag{2.3}$$

The coefficients can be recovered algebraically order by order, and we find

$$\mu_0 = 1, \ \ \mu_1 = \frac{1}{18}, \ \ \mu_2 = -\frac{7}{108}, \ \ \mu_3 = \frac{4199}{17496}, \ \ \mu_4 = -\frac{409297}{262440}, \ \ \mu_5 = \frac{101108329}{9447840},$$
$$\mu_6 = \frac{25947984239}{191318760}, \ \ \mu_7 = -\frac{3760665121759}{204073344}, \ \ \mu_8 = \frac{1158425083469857567}{826497043200}, \ldots \tag{2.4}$$

It is easy to generate them with computer aid at much higher orders, as will be required later for numerics.

2.2 Borel transform

That (2.3) is only an asymptotic series is hinted at already by the first few coefficients. The divergence can also be seen in figure 1. Our objective now is to make sense of this formal, divergent, series by using Borel resummation.

Figure 1: Growth of coefficients

(a) $|\mu_k|$ (b) $\text{sign}\,(\mu_k)$
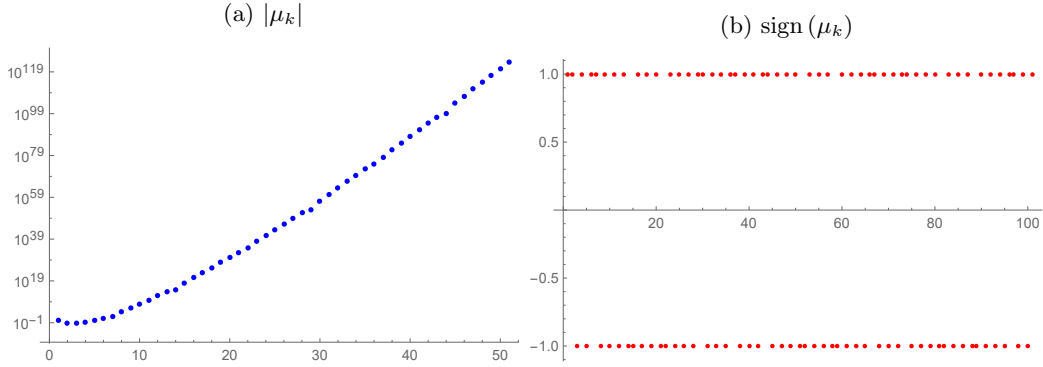
The idea of the Borel transform is to fix a $(\beta k)!$ divergence in a formal power series by defining the (now converging) series

$$(Bg)\,(t) = \sum_{k=0}^{\infty} \frac{\mu_k}{(\beta k)!} t^k := \sum_{k=0}^{\infty} \tilde{\mu}_k t^k. \tag{2.5}$$

and then reintroducing the divergence "inside" of the sum by commuting it with the integral form of the gamma function. This defines the Borel resummation of $f(x)$,

$$F(x) = x^{\frac{1}{3}+\frac{7}{3\beta}} \int_0^{\infty} e^{-x^{\frac{7}{3\beta}} s} \,(Bg)\,\left(s^{\beta}\right)\,\mathrm{d}s. \tag{2.6}$$

The motivation for this definition is that, should everything converge, the Borel resummation gives the actual value of the original series.

The particular rate of divergence that we have, as is the usual case with string theory, is $\beta = 2$. This asymptotic behaviour can also be found by looking at the first exponentially small correction as done in [13].

In figure 2 we have the coefficients of (Bg), which now have a behaviour similar to $|\tilde{\mu}_k| \sim A^k$, with a finite radius of convergence. Still, to have a well defined integral in (2.6), we need the integration path in (2.6) to be free of obstructions.

Suppose we had $\tilde{\mu}_k = A^{-k}$ with $A > 0$. This can be directly resummed, and (Bg) would have a pole as $(t - A)^{-1}$, right in the middle of the integration path. In our case, like we saw in figure 1, the signs are alternating. In the simplified version $\tilde{\mu}_k = A^{-k}$, it would happen when $A < 0$ – the pole would lie on the *negative* real axis, leaving the path of integration free.

Figure 2: Borel coefficients $\frac{|\mu_k|}{(2k)!}$
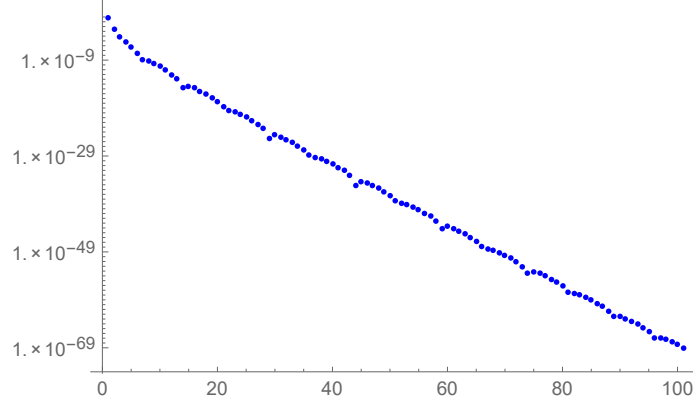
We will need the derivatives $F^{(m)}$ of (2.6) for our numerical analysis. Since for $x \neq 0$ the integrand of $F(x)$ and its derivatives are continuous, there are no problems in commuting ∂_x with the s-integral, and one gets expressions like

$$F^{(1)}(x) = x^{\frac{1}{3} + \frac{7}{3\beta}} \int_0^\infty \frac{\beta - 7sx^{\frac{7}{3\beta}} + 7}{3\beta x} e^{-x^{\frac{7}{3\beta}} s} (Bg) \left(s^\beta\right) \, \mathrm{d}s. \tag{2.7}$$

2.3 Padé approximant and Borel plane

In the previous section we considered the simple case of a Borel transform behaving exactly like $\tilde{\mu}_k = A^{-k}$. In general, one cannot find such a closed expression for (Bg) – let alone when we do not even have a closed form for the μ_k, and are generating them recursively. However, a good (and integrable) substitute to an exact analytical continuation is given by the Padé approximant, as detailed in [1]. In short, it is the rational function whose Taylor expansion agrees with the original series at a given order. We denote by $P^{(n/n)}$ the diagonal approximant matching the series (Bg) at order $O(t^{2n+1})$. That is, the one with equal degree in numerator and denominator. For Borel-Padé resummation, it is the one[1] that usually gives the best numerical convergence with n.

As we see in figure 3, where we represent the poles and zeros of $P^{(n/n)}(t)$ in the Borel $\mathbb{C}$-plane, there is a nice structure that remains stable as we improve the approximation. In particular, a branch cut develops along the negative axis, leaving no obstruction for the integral along $x > 0$.

Let us remark that the closest poles to the origin are located precisely at the values of the action A^2 given by the first exponential correction. This is well known, and [15] provides a general way to compute the action for an arbitrary (p, q) model. A detailed calculation for our particular case is given in [13]. For illustrative purposes consider simply the ansatz

$$g(t) = g_{(0)}(t) + g_{(1)}(t)e^{-\frac{A}{\sqrt{t}}} + \ldots \tag{2.8}$$

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[1] To work with derivatives such as (2.7) it is more convenient to use shifted Padé approximants such as $P^{(n-1/n+1)}$, to compensate the s factors introduced by the x-derivative, so that we still get good convergence in the numerical integration for the relatively smaller values of x. Of course, the picture we get of the Borel plane for them is essentially the same as in figure (3).

Figure 3: Borel-Padé t-plane

(a) Poles (left) and zeros (right) of $P^{(20/20)}$

(b) Poles (left) and zeros (right) of $P^{(70/70)}$
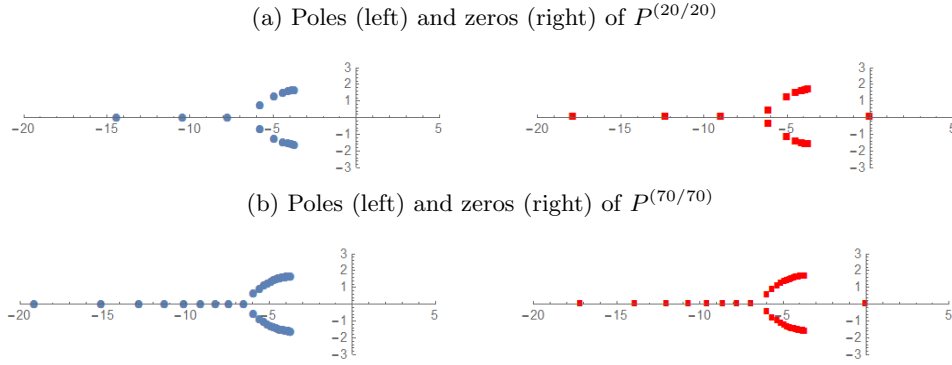
where $g_{(0)}$ is the perturbative solution of (2.3), and $g_{(1)}$ the power series for the exponentially suppressed part – the *one-instanton transseries*. Inserting it into (2.1), the first-order condition is

$$\frac{2401A^4}{12960} + \frac{49A^2}{36} + 3 = 0. \tag{2.9}$$

The solutions[2] to this are $A^2 = -\frac{36}{49}\left(5 \pm i\sqrt{5}\right) \simeq -3.67 \pm 1.64\,i$, plotted as a star in figure 4, matching the two closest poles to the origin in the Borel plane.

The natural variable for the transseries is, however, $\sqrt{t}$. In that plane, easily obtained by unfolding the one in figure (3), there are two branch cuts along the imaginary axis, and we actually have four actions, two of which have positive real part. As discussed in [8], the transseries associated to such actions is related to the difference between Borel resummation and their exact answer. Remarkably, we will see in the next section that despite the presence of positive real part poles, we will not need any transseries correction.

Figure 4: Borel-Padé t-plane and action A^2
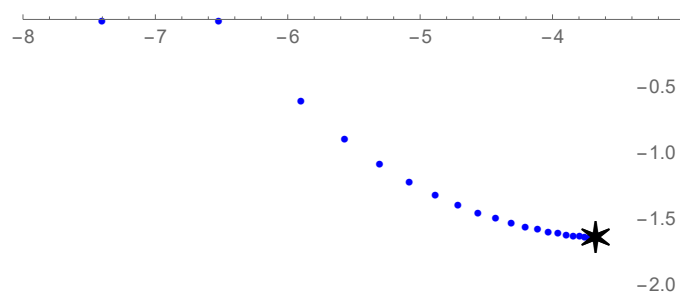
3 Numerical analysis
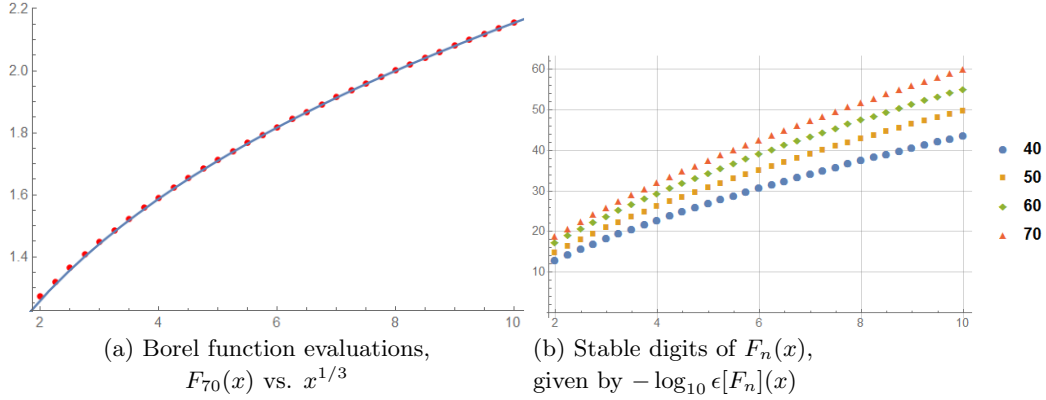
3.1 Convergence of the resummation

We integrate (2.6) numerically. We will truncate $F(x)$ at order n by using the $P^{(n/n)}$ approximant in (2.6). This gives an approximation $F_n(x)$ to $F(x)$. Assuming that the error made by truncating decreases with n, we can estimate the relative uncertainty due to truncation by

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[2] A itself is of course chosen by having positive real part, so the instanton correction is small.

$$\epsilon[F_n](x) := \frac{\Delta[F_n](x)}{|F_n(x)|} := \left| \frac{F_n(x) - F_{n-1}(x)}{F_n(x)} \right|. \tag{3.1}$$

In figure 5 we plot the calculated values of (2.6), and the asymptotic function $x^{1/3}$ – the difference becomes only obvious at $x \sim 1$. Also, for different truncations, the number of reliable digits is given. These are computed by looking at how many digits remain stable from one truncation to the previous, which is in essence the logarithm of the relative error. We can easily see the convergence as we increase n, with stable digits ranging between 20 to 60, depending on the value of x. One finds similar plots for the first four derivatives $F_n^{(m)}(x)$. This validates, in retrospect, the assumption that (3.1) is a good estimate for the truncation error.

Figure 5: Numerical Borel resummation, as a function of x

(a) Borel function evaluations,
$F_{70}(x)$ vs. $x^{1/3}$

(b) Stable digits of $F_n(x)$,
given by $-\log_{10} \epsilon[F_n](x)$

3.2 First approach: error propagation

We will try to answer whether the resummation solves (2.1) without requiring direct knowledge of the exact (numerically integrated) solution. We recast the equation into a normalized form $Q[f, x] = P[f, x]/x = 0$, so that it measures in a homogeneous way the failure of F_n (and its derivatives) to be a solution of (2.1),

$$E_n(x) = |Q[F_n, x]|. \tag{3.2}$$

The error E_n should be zero in the $n \to \infty$ limit. That is, if $F(x)$ –the Borel sum– is really a solution of (2.1). The question is then whether the deviations thereof at finite n are only due to F_n itself being only an approximation of the full F, or rather to F being close to but not a solution of (2.1). To check this, we can propagate the uncertainty of $F_n(x)$ in (3.1). For the sake of simplicity we use the approximation of the small-error propagation formula,

$$\Delta[Q[F_n, x]] = \sqrt{\sum_{i=0}^{4} \left(\frac{\partial Q}{\partial f^{(i)}}[F_n, x] \right)^2 \left(\Delta\left[F_n^{(i)}\right] \right)^2}. \tag{3.3}$$

Figure 6 gives the expected correct digits from the uncertainty propagation. The plot for the error E_n is fundamentally the same, and we empirically find (with n up to 70) that

$$0 < \log_{10} \frac{\Delta[Q[F_n, x]]}{E_n(x)} < 1, \tag{3.4}$$

that is to say, the failure of $F_n(x)$ to be a solution is just somewhat smaller than what one would expect from simply propagating the truncation errors. This not surprising since (3.3) can slightly
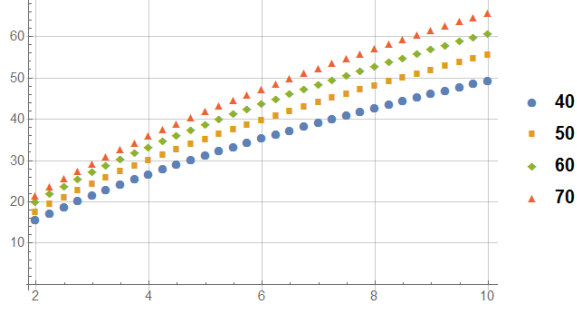
Figure 6: $-\log_{10}\Delta\left[Q[F_n,x]\right]$

overestimate the error due to correlations. Therefore we cannot say $F(x)$ does not solve (2.1) up to the numerical accuracy. In other words, it is a *sufficiently* good solution

Finally, figure 6 also shows this error decreases steadily with the Padé order, across all values of x. This is important to rule out the possibility that simply the trivial, leading part is responsible for the matching: the Borel sum is in fact *necessary* to tend to the correct answer. We will test this in more detail in the next section.

3.3 Second approach: numerical integration

Although it is useful being able to face the problem with a minimal amount of information, a direct comparison is of course a stronger test. In order to find the exact numerical solution, we first turn our attention to how would it be possible to implement the boundary conditions of the problem. The series being asymptotical in nature, our best bet is to take some *optimal truncation* as a starting guess in a region where it allows for reasonable precision – at least, of the order of the Borel resummations we were considering.

Figure 7: $\mu_k\,x^{\frac{1-7k}{3}}$, as a function of k
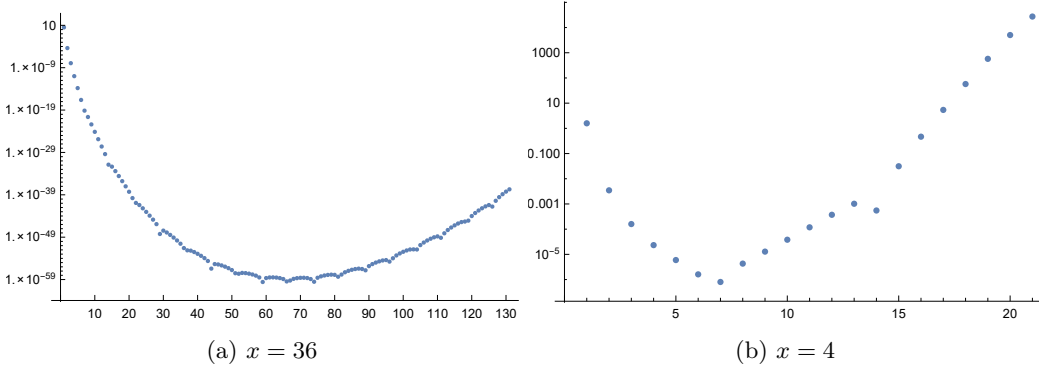

(a) $x=36$ (b) $x=4$

Figure 7 shows how much optimal truncation breaks down at small values of x. Including terms up to Padé order $n=65$ (or Taylor order 130) for $x=36$, we get to precisions of ~ 60 digits. To give an example,

$$f(36)|_{n=65} = 3.3019701041589074417361002236242101151246356402503182361949 0617\ldots$$
$$f(36)|_{n=64} = 3.3019701041589074417361002236242101151246356402503182361949 0258\ldots$$

(3.5)

The derivatives, computed as in (2.7), have the same amount of stable digits. For the sake of comparison, if we try at $x=4$, the "optimal" truncation now happens at order 6, with only 6 reliable digits.

The standard procedure to backtrack along the x axis would now be to use a finite step numerical algorithm. However, the Lee-Yang equation is particularly troublesome, displaying stiffness problems that cannot be consistently avoided even by Runge-Kutta integrators. This is made worse by the kind of precision we require to meaningfully compare with Borel-Padé.

We will resort to a method similar to what is described in [16], in spirit a finite step method of arbitrarily high order. Given a set of "initial conditions" $\{f^{(n)}(x_i), n = 0 \ldots 3\}$, one builds the power series expansion of $f(x)$ around x_i,

$$f(x) = \sum_{k=0}^{\infty} \nu_k(x_i) \left(x - x_i\right)^k, \tag{3.6}$$

and find the rest of the $\nu_k(x_i)$ recursively with (2.1), by requiring

$$P\left[\sum_{k=0}^{N} \nu_k(x_i) \left(x - x_i\right)^k, x\right] = 0 + O\left(x^{N+1}\right). \tag{3.7}$$

At finite x_i the solution is regular and has a finite radius of convergence of (3.7). We can use that to compute the first three coefficients at the next step

$$\nu_k(x_{i+1}) = \frac{1}{k!} \sum_{l=k}^{N} \nu_l(x_i) \frac{l!}{(l-k)!} \left(x_{i+1} - x_i\right)^{l-k} + O(\Delta x^{N+1}), \quad k = 0 \ldots 3, \tag{3.8}$$

calculate the rest $k = 4 \ldots N$ with condition (3.7), and repeat.

This procedure allows one to take step sizes only limited by the radius of convergence in (3.8). Notice that precision can be arbitrarily increased by the order N of the approximation, and we can take relatively big "jumps" that avoid the decreasing step size problem inherent to stiff equations. It should be remarked, however, that numerical rounding errors accumulate fast when doing the recursion from (3.7) and the propagation (3.8) – very high numerical working precisions are needed to avoid them.

In the appendix we provide the values for both the Borel resummation at order $n_{\text{Pade}} = 60$, and for the numeric integration at order $N = 70$, calculated from the asymptotic guess (3.5) with order $n_{\text{guess}} = 65$. For the Borel sum, the stable digits are determined by contrast with the order $n_{\text{Pade}} = 59$ approximant. For the numeric integration, they are given by comparison with a truncation at $N = 69$ *and* an initial guess at $n_{\text{guess}} = 64$.

Figure 8: Digit comparison (Borel vs. integration), as a function of x
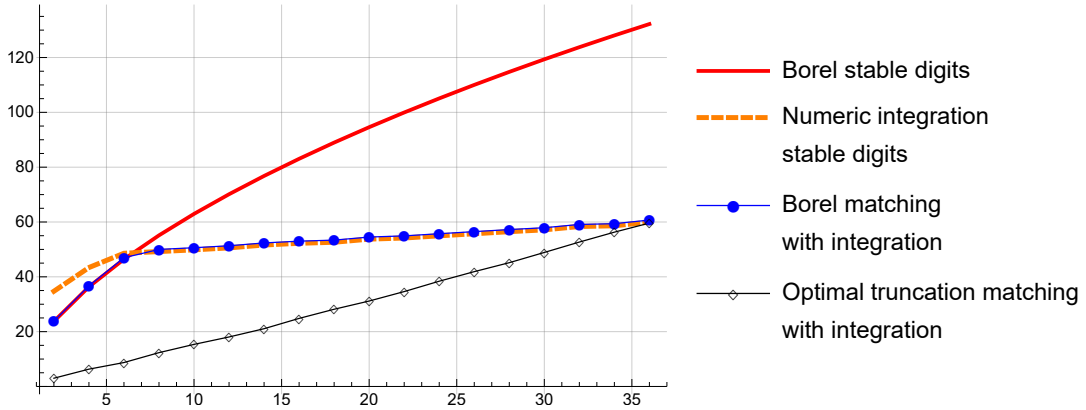
In figure 8 we have plotted the stable digits of each of the solutions, for values $x = 2, 4, \ldots, 36$. The numeric integration is mostly limited by the error of the initial guess. At very small values

of x (~ 5) it starts to slowly break down due to an insufficient truncation order N, but by then the Borel sum is performing worse. Of course, N was chosen taking this into account. One could in principle go to higher orders in the asymptotic guess by using higher starting points in x. This implies a tradeoff with the numerical integration, and we would need even higher N (and thus working precision) to be able to reach accurately the low x region. In any case, around $x = 36$ the convergence of Borel-Padé already outperforms greatly the precision of the numerical integration. After all, the initial guess (and the initial error of the integration) is given precisely by optimal truncation, and comparing it with the resummation does not say anything useful. The interesting section of the plot lies at the smaller values of x, where optimal truncation becomes a really bad approximation to the numerical result.

In general, the coincident digits between Borel and integration are just slightly better than expected by the corresponding error of the worse of the two. Therefore, up to numerical accuracies, the Borel resummation coincides with the solution of (2.1).

The question again is if the agreement is trivial. With that in mind we also plot the matching between the optimal truncation and the integration (which is mostly determined by the precision of the truncation itself). This serves as a baseline for comparison with the number of matching digits between numeric integration and Borel-Padé. The difference tells how much of the agreement is purely due to resummation – going up to ~ 40 digits at small values of x.

Finally, one could think that there are exponentially small corrections hidden behind the precisions used. As detailed in [2], the size of such a exponentially small term is precisely that of the error committed by optimal truncation. These grow as we go to $x \to 0$, and unless the problem has some fine-tuning of the order of 10^{40}, we can rule out the need for non-perturbative corrections to the Borel resummation.

4   Conclusions and outlook

The relatively straightforward nature of the problem (as opposed to, say, topological string theories) is at the core of its simple Borel plane structure and well behaved resummation. The agreement between numerical integration and Borel-Padé is excellent, and no additional non-perturbative corrections are needed.

The missing necessary corrections in [4] were understood to be coming from complex poles in the positive real part of the Borel plane. Even though they are present in our case, their corresponding transseries is not needed to complete the Borel resummation. However, that this kind of equations can be studied numerically with high precision opens the door to asking the same question about different problems. To begin with, one can think of the other low-dimensional string equations that are studied in [13], along with the Lee-Yang edge singularity. After finding a case with positive real part instantons, one could ideally use the transseries framework in [8] to include them and reproduce exact results.

Of course, one could also study the $x < 0$ region here, where the oscillations should be obtained by transseries corrections. This lies outside of the purpose of this note, which is studying a case where a *well defined* Borel sum exists, who can be directly compared to the real numerical answer.

Acknowledgements

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Appendix: Numerical tables

All values shown up to their respective precisions.

Optimally truncated $f(x)$

x	$f(x)$
2	1.3
4	1.59073
6	1.8186377
8	2.00086036150
10	2.15498728734783
12	2.2898129373328068
14	2.4104250162655861467
16	2.52005872342591954554422
18	2.620912627400761925786175567
20	2.714556356752330205888517990062
22	2.802154016385710992710466847641073
24	2.884595523655590860455675901408236762 2
26	2.96257820339351782973953091483865993803 09
28	3.0366597988814206650903255567321680526928 070
30	3.107294208630669981547595488516102336196050 22702
32	3.17485633796112185866258467303845709176605743 2352291
34	3.239659844759811219192249415003574529035280231261 226044
36	3.30197010415890744173610022362421011512463564025031 82361949

Numerically integrated $f(x)$, Taylor order $N = 60$

2	1.2717432154878315125890487374340645
4	1.5907325048941121260771523803118030424757552
6	1.8186376998969563251325685438715476478566616293557
8	2.0008603614944979899725674308079860331094740607790
10	2.1549872873478251122542299372497687073270122470114
12	2.2898129373328067703868554483156339361490438251 8568
14	2.4104250162655861467396369304375149232086918069 28593
16	2.5200587234259195455442186288253624909850407144 08864
18	2.6209126274007619257861756675188334307442061180 631239
20	2.7145563567523302058885179906201132270197493419 4522064
22	2.8021540163857109927104668476410730453972326875 7593681
24	2.8845955236555908604556759014082367622249654156 156161162
26	2.9625782033935178297395309148386599380309061579 618881205
28	3.0366597988814206650903255567321680526928069817 5073696155
30	3.1072942086306699815475954885161023361960502270 2175596313
32	3.1748563379611218586625846730384570917660574323 52290860755
34	3.2396598447598112191922494150035745290352802312 6122604639537
36	3.3019701041589074417361002236242101151246356402 5031823619491

Borel resummed $f(x)$, Padé order $n = 50$

2	1.2717432154878315125890
4	1.5907325048941121260771523803118030 4
6	1.8186376998969563251325685438715476478566616294
8	2.0008603614944979899725674308079860331094746077 8981706
10	2.1549872873478251122542299372497687073270122470 113775153943730
12	2.2898129373328067703868554483156339361490438251 85680728653612311757313
14	2.4104250162655861467396369304375149232086918069 28593265489460125817396092736
16	2.5200587234259195455442186288253624909850407144 0884664323299083450891453189631457939
18	2.6209126274007619257861756675188334307442061180 631239317333404672554232346555039383392 19
20	2.7145563567523302058885179906201132270197493419 4522064079741664013737881143655155752021 1509958
22	2.8021540163857109927104668476410730453972326875 75936812994395789301287425257446651035528 83022107125
24	2.8845955236555908604556759014082367622249654156 15616116116171479511953545884032680063881633 5812264607082 3437
26	2.9625782033935178297395309148386599380309061579 61888120558524213290837350276265980241275144 735302072141389449
28	3.0366597988814206650903255567321680526928069817 50736961547883677064257186256386099685586253 58473003228282943624621
30	3.1072942086306699815475954885161023361960502270 21755963133317624387010184876075226167662195168 94069324048707479561 5680
32	3.1748563379611218586625846730384570917660574323 52290860755134057592990968119208374844715690098 367396462004218765610 75101448
34	3.2396598447598112191922494150035745290352802312 61226043953694013529333761213421861391431387238 7417336272841298488458703 14366538
36	3.3019701041589074417361002236242101151246356402 50318236194906812166617174034284057950216012334 14316279943655966291049433 95672639467