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Interactions of cosmological gravitational waves and magnetic fields

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The energy momentum tensor of a magnetic field always contains a spin-2 component in its anisotropic stress and therefore generates gravitational waves. It has been argued in the literature (Caprini & Durrer [1]) that this gravitational wave production can be very strong and that back-reaction cannot be neglected. On the other hand, a gravitational wave background does affect the evolution of magnetic fields. It has also been argued (Tsagas et al. [2], [15]) that this can lead to a very strong amplification of a primordial magnetic field. In this paper we revisit these claims and study back reaction to second order.

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I. INTRODUCTION

Wherever we can measure them in the Universe, magnetic fields of 0.5 to several micro Gauss are present. They have been found in ordinary galaxies [4] like ours, but also in galaxies at relatively high redshift [5] and in galaxy clusters [6]. It is still unknown where these cosmological magnetic fields come from. Are they primordial, *i.e.* generated in the early universe [7], or did they form later on by some non-linear aspect of structure formation, like the Harrison mechanism which works once vorticity or, more generically, turbulence has developed [8]?

In addition, once initial fields are generated, it is still unclear whether they are strongly amplified by a dynamo mechanism or only moderately by contraction. Since the cosmic plasma is an excellent conductor, the magneto-hydrodynamic (MHD) approximation can be employed which implies that the magnetic field lines are frozen in during structure formation. Therefore, as long as non-linear magnetic field generation can be neglected, the magnetic field scales inversely proportional to the area, so that $B/\rho^{2/3}$ is roughly constant during structure formation. Here ρ is the energy (or matter) density of the cosmic plasma. For galaxies, with a density of about $\rho_{\rm gal} \sim 10^5\bar{\rho}$ this means that simple contraction will enhance magnetic fields by approximately 10^3, $\bar{\rho}$ is the mean density. Hence, if no dynamo is active during galaxy formation, initial fields of $B_{\rm in} \sim 10^{-9}$Gauss are needed. On the other hand, non-linear dynamo action can enhance the magnetic field exponentially by a factor up to 10^{15}, so that initial fields as tiny as $B_{\rm in} \sim 10^{-21}$Gauss might suffice [9]. However, since this enhancement is exponentially sensitive to the age of the Universe, it remains unclear how it can generate the magnetic fields in galaxies at redshifts of $z \sim 1$ or more, where the age of the Universe was at most half its present value reducing the amplification factor to less than 10^8.

Another problem of cosmic magnetic fields is that primordial generation of fields usually leads to a very blue magnetic field energy spectrum,

$$\frac{{\rm d}\rho_B}{{\rm d}\log k} \propto k^{M+3} \, , \tag{1}$$

where $M = 2$ for "causally" produced magnetic fields [10] and $M \sim 0$ for typical inflationary production mechanisms [11]. Such blue magnetic field spectra are strongly constrained by their gravity wave production [1] and cannot lead to the large scale fields observed today. The only solution to the problem might lie in an "inverse cascade" by which energy is transferred from small to larger scales. Since within the linearized approximation each Fourier mode evolves independently, such a cascade is inherently non-linear. Within standard MHD is has been shown that only helical magnetic fields can lead to inverse cascade [9].

In this work, we want to address a weakly non-linear effect which has not been considered in [9], namely the interaction of gravitational waves and magnetic fields. We shall study how this interaction can modify the magnetic field spectrum. We also re-interpret a finding by Tsagas [15], where the interaction between gravitational waves and magnetic fields has been interpreted as "resonant amplification". Similar conclusions are drawn in Refs. [12], [13]. However, in this last article it is noted that the amplification can take place only on super-horizon scales. And even though Ref.[13] does mention that there is no amplification in the long-wavelength limit, they do not really quantify this statement.

We show that the build up of magnetic fields due to their interaction with gravitational waves is at most logarithmic and thus comparable to the generation of gravitational waves by magnetic fields.

Furthermore, as it has been pointed out also in Ref. [13], this super-horizon "amplification" is independent of the fact whether the plasma is highly conducting or not. This seems physically reasonable as currents generated by electromagnetic fields can act only causally,

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i.e. on sub-horizon scales. An animated discussion on this subject follwed the above publications and can be found in Refs. [15], [14]. Here the role of a finite conductivity in an expanding Universe is addressed but controversal final conclusions have been reached.

The main advantage of our treatment is that we express the relevant results entirely in terms of physical, measurable quantities, which renders the interpretation straight forward. We actually find for the density parameters of second order perturbations that, once the scales considered are inside the horizon,

$$\Omega_B^{(2)} \simeq \Omega_B^{(1)}\Omega_{\rm GW}^{(1)} \simeq \Omega_B^{(1)}\left(\frac{H_{\rm inf}}{M_{\rm P}}\right)^2 , \qquad (2)$$

$$\Omega_{\rm GW}^{(2)} \simeq \left[\Omega_B^{(1)}\right]^2 + \left[\Omega_{\rm GW}^{(1)}\right]^2 , \qquad (3)$$

as one probably would expect naively. Even though most parts of this result can already be found in the above cited papers, they are interpreted there in a different way, and especially in Eq. (2) it is not always noted that the factor $\Omega_{\rm GW}^{(1)}$ always has to remain small.

The paper is organized as follows. In the next section we set up the fully non-linear equations for the evolution of magnetic fields in the relativistic MHD approximation. We use the 3+1 formalism and closely follow the derivation given in Ref. [16]. Since we are mainly interested in gravitational waves, we specialize to the vorticity free case. In Section III we consider linear perturbations. We solve the linear perturbation equations for gravitational waves and magnetic fields with given initial conditions. We also derive the evolution of the corresponding energy densities. This part is not new and mainly included for completeness and to fix the notation for the subsequent Section IV, where we solve the second order equations. On this level the gravitational waves interact with the magnetic field. We calculate the second order magnetic field generated by this interaction and show that for reasonable values for the first order perturbations, its energy density remains always much smaller than the energy density of the first order contributions. In this sense, one cannot speak of resonant amplification. In Section V we summarize our results and draw some conclusions.

Throughout this work we use the metric signature $(-,+,+,+)$. Conformal time is denoted by t and we neglect the background curvature of the Universe, $K=0$. Spacetime indices are denoted by lower case Greek letters, μ,ν, while lower case Latin letters, i,j are used for spatial indices. Most of our calculations are performed in the radiation dominated era and we shall often use the expression

$$a(t) = H_{\rm in} a_{\rm in}^2 t$$

for the scale factor.

II. THE BASIC EQUATIONS

We work in the MHD approximation, where we assume high conductivity. The electric field is then small compared to the magnetic field in the baryon rest-frame which we take to be the frame of our "fundamental observer". In addition, we assume the velocity u^μ of this fundamental observer to be vorticity-free and we neglect acceleration. According to Frobenius' theorem u is hyper-surface orthogonal and we can choose spatial coordinates in the three-space orthogonal to u. Furthermore, in the early Universe which is of interest to us, the dominant radiation and baryons are tightly coupled so that the energy flux is also given by u and we can set the heat flux $q=0$. In our vorticity free frame, the magnetic part of the Weyl tensor, H_{ij} is related to the shear simply by

$$H_{ij} = {\rm curl}\sigma_{ij} ,$$

where curl is the 3-dim curl on the hyper-surface normal to u. Here σ is the shear of u given by

$$\begin{aligned}\sigma_{\mu\nu} &\equiv \frac{1}{2}\left(u_{\mu;\nu} + u_{\mu;\nu}\right) - \frac{1}{3}\Theta\tilde{p}_{\mu\nu} ,\\ \Theta &\equiv u^\mu_{;\mu} \text{ and}\\ \tilde{p}_{\mu\nu} &\equiv g_{\mu\nu} + u_\mu u_\nu .\end{aligned}$$

Note that the normalization of u implies $0 = u^\nu u_{\nu;\mu} \propto u_{0;\mu}$. The gravito-magnetic interaction can then be described by the following equations, see [16]

$$\begin{aligned}\nabla_u E_{ij} &= -\Theta E_{ij} - \frac{1}{2}\kappa\left(\rho + p + \frac{1}{6\pi}B^2\right)\sigma_{ij}\\ &\quad -D^2\sigma_{ij} - \kappa\frac{1}{2}\nabla_u\Pi_{ij} - \frac{1}{6}\Theta\kappa\Pi_{ij}\\ &\quad +3\sigma_{\langle i}{}^n E_{j\rangle n} - \frac{1}{2}\kappa\sigma_{\langle i}{}^n\Pi_{j\rangle n} , \qquad (4)\\ \nabla_u\sigma_{ij} &= -E_{ij} + \frac{1}{2}\kappa\Pi_{ij} - \sigma_{\langle i}{}^n\sigma_{j\rangle n} - \frac{2}{3}\Theta\sigma_{ij} , \qquad (5)\\ \nabla_u B_i &= -\frac{2}{3}\tilde{p}_{ij}\Theta B^j + \sigma_{ij}B^j , \qquad (6)\\ \nabla_u\Theta &= -\frac{1}{3}\Theta^2 - \frac{1}{2}\kappa\left(\rho + 3p + \frac{1}{4\pi}B^2\right) - 2\sigma^2 . \qquad (7)\end{aligned}$$

Here E_{ij} is the electric part of the Weyl tensor, ρ and p are the energy density and pressure of the cosmic fluid which is assumed to follow the motion of the baryons (like, *e.g.* radiation before decoupling), $\kappa = 8\pi G$ is the gravitational coupling constant and B_i is the magnetic field. We have neglected the electric field in the above equations, since we assumed it to be much smaller that the magnetic field, *i.e.* $B^2 \gg E^2$. The covariant derivative in direction u is denoted by ∇_u and the brackets indicate symmetrization and trace subtraction,

$$X_{\langle ij\rangle} = \frac{1}{2}\left(X_{ij} + X_{ji}\right) - \frac{1}{3}\tilde{p}_{ij}X_m{}^m .$$

D^2 is the Laplace operator on the hyper-surface orthogonal to u. The scalars B^2 and σ^2 are simply $\sigma^2 \equiv \sigma_{ij}\sigma^{ij}/2$ and $B^2 \equiv B_iB^i$.

In addition to this we have the Einstein equation, the spatial part of which yields

$$\mathcal{R}_{ij} = E_{ij} + \frac{2}{3}\left(\kappa\rho + \frac{1}{8\pi}\kappa B^2 - \frac{1}{3}\Theta^2 + \sigma^2\right)\tilde{p}_{ij} + \frac{1}{2}\kappa\Pi_{ij} - \frac{1}{3}\Theta\sigma_{ij} + \sigma_{n\langle i}\sigma^n{}_{j\rangle} \ , \qquad (8)$$

and its trace, the generalized Friedmann equation,

$$\frac{1}{3}\Theta^2 + \frac{1}{2}\mathcal{R} = \kappa\rho + \frac{1}{8\pi}\kappa B^2 + \sigma^2 \ . \qquad (9)$$

Here $\mathcal{R}_{ij}$ is the Ricci tensor on the spatial hyper-surface and $\mathcal{R}$ is its trace.

From this system we can derive second order equations for σ_{ij} and B_i which are

$$\begin{aligned}&\nabla_u\nabla_u\sigma_{ij} - D^2\sigma_{ij} + \frac{5}{3}\Theta\nabla_u\sigma_{ij} \\ &+\left(\frac{4}{9}\Theta^2 - \frac{3}{2}\kappa p - \frac{5}{6}\kappa\rho - \frac{1}{6\pi}\kappa B^2 - \frac{4}{3}\sigma^2\right)\sigma_{ij} = \\ &\quad \kappa\nabla_u\Pi_{ij} + \frac{2}{3}\Theta\kappa\Pi_{ij} + \frac{2}{3}\kappa B^2\sigma_{ij} + \Theta\sigma_{\langle i}{}^n\sigma_{j\rangle n} \\ &\quad +2\sigma_{\langle i}{}^n\nabla_u\sigma_{j\rangle n} - \nabla_u\sigma_{\langle i}{}^n\sigma_{j\rangle n} - \kappa\sigma_{\langle i}{}^n\Pi_{j\rangle n} \\ &\quad +\frac{1}{3}\sigma^2\sigma_{ij} + 3\sigma_{\langle i}{}^n\left[\frac{1}{2}\left(\sigma_{j\rangle}{}^m\sigma_{nm} + \sigma_n{}^m\sigma_{j\rangle m}\right)\right. \\ &\quad \left. -\frac{2}{3}\delta_{j\rangle n}\sigma^2\right] \ , \end{aligned} \qquad (10)$$

and

$$\begin{aligned}&\nabla_u\nabla_u B_i - D^2 B_i + \frac{5}{3}\Theta\nabla_u B_i \\ &+\left(\frac{1}{3}\kappa\rho - \kappa p + \frac{2}{9}\Theta^2 + \frac{1}{12\pi}\kappa B^2 + \frac{2}{3}\sigma^2\right)B_i = \\ &\quad \sigma_{ij}\nabla_u B^j + 2\Theta\sigma_{ij}B^j + +2(\nabla_u\sigma_{ij})B^j \\ &\quad -\frac{3}{2}\kappa\Pi_{ij}B^j + \sigma^n_{\langle i}\sigma_{j\rangle n}B^j + \mathrm{curl}J_i \ . \end{aligned} \qquad (11)$$

Eq. (11) can be obtained from Eq. (40) of [19] when setting $A_i = 0$, $\omega_{ij} = 0$ and $q_i = 0$. J_i stands for the 3-dimensional current. Eq. (11) is obtained without neglecting the electric field. The term $\mathrm{curl}E_i$, which is present in the original Maxwell equation which reduces to Eq. (6) if $E_i = 0$ [16] results in the Laplacian term D^2B_i and terms proportional to the wavenumber k times the electric field [see Eq. (40) of [19]]. We have neglected these latter contributions in the above equation, since they are only relevant inside the horizon ($kt \gg 1$), where we can neglect the source term of the equations, as we shall argue in the following.

In a regime of low conductivity we can neglect also the current in Eq. (11) and the magnetic field obeys to the above wave equation, while in a very hight conductivity case we should directly set the electric field $E_i = 0$ from the beginning and solve Eq. (6), obtaining a power-low behaviour with respect to time for B_i. In both cases we find that the behaviour in time of the induced second order magnetic field $B_i^{(2)}$ is the same on super-horizon scales (up to uncertain logarithmic corrections). We interpret this as the insensitivity of super-horizon perturbations to plasma properties like conductivity.

Inside the horizon, we neglect the source term. This is motivated by the Green function of the damped wave equation obtained when linearizing (11), which rapidly oscillates on sub-horizon scales. For Eq. (6) it is not the Green function but the source term $\sigma_{ij}^{(1)}B^j_{(1)}$ which oscillates when $kt \gg 1$, since gravity waves start oscillating at horizon crossing. Therefore again, the sub-horizon amplification is unimportant. The same conclusion is actually drawn in Ref. [13], where the fluid velocities are not neglected.

In the following we shall consider these equations in first and second perturbative orders with respect to a spatially flat Friedmann background,

$$\mathrm{d}s^2 = a^2(-\mathrm{d}t^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j) \, .$$

We neglect a possible spatial curvature of the background and work with conformal time t. The time dependence of the scale factor a is determined by the Friedmann equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa}{3}\rho a^2 \quad \text{and}$$

$$\dot{\rho} = -3(1+w)\rho\left(\frac{\dot{a}}{a}\right) \ , \quad w = p/\rho \ .$$

III. FIRST ORDER PERTURBATIONS

A. Magnetic fields

A background Friedmann Universe can of course not contain a magnetic field since the latter always generates anisotropic stresses $\Pi_{ij} \neq 0$ which break isotropy. When considering a magnetic field as a first order perturbation, Eq. (6) leads in first order to

$$\dot{B}_i^{(1)} = -\frac{\dot{a}}{a}B_i^{(1)} \ . \qquad (12)$$

For this we use that to lowest order $u = a^{-1}\partial_t$ and $(\nabla_u B)_i = a^{-1}(\partial_t - \dot{a}/a)B_i$. Furthermore $\tilde{p}_{ij} = g_{ij} = a^2\delta_{ij}$ and $\Theta = 3\dot{a}/a^2$. This is solved by

$$\begin{aligned} B_i^{(1)}(\mathbf{x},t) &= B_{i\,\mathrm{in}}^{(1)}(\mathbf{x})\frac{a_{\mathrm{in}}}{a(t)} \ , \\ B^{i(1)}(\mathbf{x},t) &= B_{\mathrm{in}}^{i(1)}(\mathbf{x})\frac{a_{\mathrm{in}}^3}{a^3(t)} \ . \end{aligned} \qquad (13)$$

The average energy density of the first order magnetic field is then given by

$$\langle\rho_B^{(1)}\rangle = \frac{1}{8\pi}\langle B_{(\mathrm{in})}^{(1)2}(\mathbf{x})\rangle\frac{a_{\mathrm{in}}^4}{a^4(t)} \ . \qquad (14)$$

Here, we assume that the first order magnetic field has been generated by some random process. Hence $B^{(1)}_{i\,\rm in}$ is a random variable and $\langle\cdots\rangle$ denotes the expectation value. We assume also that this random process is statistically homogeneous so that $\langle\rho_B^{(1)}\rangle$ is independent of position.

B. Gravitational waves

For the gravity wave equation we consider a Fourier component

$$\begin{aligned}\sigma^{(1)k}_{ij}(\mathbf{x},t) &= \sigma^{(1)}(\mathbf{k},t)Q_{ij}(\hat{\mathbf{k}})\exp(i\mathbf{k}\cdot\mathbf{x}) \ ,\\ D^2\sigma^{(1)k}_{ij}(\mathbf{x},t) &= -\frac{k^2}{a^2(t)}\sigma^{(1)k}_{ij}(\mathbf{x},t) \ .\end{aligned}$$

Here $Q_{ij}(\hat{\mathbf{k}})$ is a transverse traceless polarization tensor. We assume that the gravity waves are statistically isotropic and parity invariant so that both polarizations have the same averaged square amplitudes. For the amplitude $\sigma^{(1)}(\mathbf{k},t)$ we obtain to first order the usual tensor perturbation propagation equation (neglecting anisotropic stresses of the cosmic fluid)

$$\ddot{\sigma}^{(1)} + \left[k^2 - \frac{3}{2}(1+w)\mathcal{H}^2\right]\sigma^{(1)} = 0 \ , \qquad (15)$$

where $\mathcal{H} = \dot{a}/a$ denotes the co-moving Hubble parameter, $\mathcal{H} = aH$, where H is the physical Hubble parameter. We now rewrite this equation in terms of the dimensionless variable $\Sigma_{(1)}(\mathbf{k},t) \equiv \sigma_{(1)}(\mathbf{k},t)/(a^2_{\rm in}\Theta) = \sigma_{(1)}(\mathbf{k},t)/(3Ha^2_{\rm in})$. We have normalized by the factor $1/a^2_{\rm in}$ in order for the quantity Σ to be independent of the normalization of the scale factor. This is not true for σ which is $\sigma \propto a^2_{\rm in}$. In this way, Σ can be directly related to observable quantities which are of course independent of the normalization of the scale factor. Equivalently, we will make use of the variable $\mathcal{B}$ that is defined as $\mathcal{B} \equiv \sqrt{\kappa}B/(3Ha_{\rm in})$ in order to be independent of the normalization of the scale factor, as well as Σ. In terms of Σ the above equation becomes

$$\begin{aligned}&\ddot{\Sigma}_{(1)} - 3(1+w)\mathcal{H}\dot{\Sigma}_{(1)}\\ &\quad + \left[k^2 + \left(\frac{3}{2} + 6w + \frac{9}{2}w^2\right)\mathcal{H}^2\right]\Sigma_{(1)} = 0 \ . \quad (16)\end{aligned}$$

In the matter or radiation era, the solutions to this linear homogeneous differential equation are well known in terms of Bessel functions. We are mainly interested in the radiation epoch, where $w = 1/3$. During radiation domination the Universe expands like $a(t) \propto t$ such that $\mathcal{H} = Ha = 1/t$. We can therefore express the scale factor as

$$a(t) = H_{\rm in}a^2_{\rm in}t \ . \qquad (17)$$

In the radiation dominated Universe Eq. (16) reduces to

$$\ddot{\Sigma}_{(1)} - \frac{4}{t}\dot{\Sigma}_{(1)} + \left(k^2 + \frac{4}{t^2}\right)\Sigma_{(1)} = 0 \ , \qquad (18)$$

with solution

$$\Sigma_{(1)} \propto (kt)^3\left[j_1(kt) + y_1(kt)\right] \ , \qquad (19)$$

where j_n and y_n denote the spherical Bessel functions of index n [18].

We distinguish the super- and sub-horizon behaviors. In the long wavelengths limit, $z \equiv kt \ll 1$, we have

$$\lim_{z\to 0} z^3 j_1(z) \simeq \frac{z^4}{3} \ , \qquad \lim_{z\to 0} z^3 y_1(z) \propto -z \ .$$

Taking into account only the faster growing mode, we obtain

$$\Sigma_{(1)}(t) \simeq \Sigma^{\rm in}_{(1)}\left(\frac{kt}{kt_{\rm in}}\right)^4 \ , \qquad kt \ll 1 \ , \qquad (20)$$

or equivalently

$$\Sigma_{(1)}(t) \simeq \Sigma^{\rm in}_{(1)}\left(\frac{a}{a_{\rm in}}\right)^4 \ , \qquad kt \ll 1 \ . \qquad (21)$$

The quantity directly related to gravity waves is however given by $\sigma_{(1)} = 3Ha^2_{\rm in}\Sigma_{(1)}$, for which we obtain on super-horizon scales

$$\sigma_{(1)}(t) \simeq \sigma^{\rm in}_{(1)}\left(\frac{a}{a_{\rm in}}\right)^2 \ , \qquad kt \ll 1 \ . \qquad (22)$$

A direct consequence of this is that the "gravity wave energy density" is constant in time outside the horizon, as we show in the next sub-section. Of course the notion of "gravity wave energy density" and "gravity wave" is not strictly well defined for wavelengths larger than the size of the Hubble horizon. We shall just use the expression which is valid inside the horizon and call this the "gravity wave energy density" by analogy. It has a physical interpretation as a true energy density only once it enters the horizon. However, whenever this quantity becomes of the order of the background energy density, we know that perturbations become large and we can no longer trust linear perturbation theory.

Let us also consider the short wavelengths limit where $kt \gg 1$. In this limit we can approximate

$$\Sigma_{(1)}(t) \simeq (kt)^2\frac{\cos(kt)}{\cos(1)}\Sigma_{(1)}(kt=1) \ , \qquad kt \gg 1 \ , \quad (23)$$

where the initial constant $\Sigma_{(1)}(kt = 1)$ stands for the value of $\Sigma_{(1)}$ when it enters the horizon and can be obtained from Eq. (20),

$$\Sigma_{(1)}(kt=1) \simeq \Sigma^{\rm in}_{(1)}\left(\frac{1}{kt_{\rm in}}\right)^4 \ .$$

The behavior of gravity waves on sub-horizon scales, $kt \gg 1$, is then given by

$$\sigma_{(1)}(t) \simeq 3a^2_{\rm in}H(kt)^2\frac{\cos(kt)}{\cos(1)}\Sigma_{(1)}(kt=1) \ . \qquad (24)$$

We shall see that in this case the gravity waves energy density decreases like $1/a^4$, as it has to be for true gravity waves which are massless modes.

C. Energy Densities

As a first physically important quantity, let us discuss the energy densities of these first order perturbations and the corresponding density parameters.

The magnetic energy density is

$$\rho_B^{(1)} \equiv \frac{B_{(1)}^2}{8\pi} = \frac{B_i^{(1)} B_{(1)}^i}{8\pi} , \tag{25}$$

with Eq. (14), this becomes

$$\rho_B^{(1)}(t) = \frac{1}{8\pi} B_{(1)\,\mathrm{in}}^2 \left(\frac{a_{\mathrm{in}}^4}{a^4}\right) . \tag{26}$$

In the radiation dominated universe under consideration, the density parameter of the first order magnetic field is therefore given by

$$\Omega_B^{(1)}(t) \equiv \frac{\rho_B^{(1)}}{\rho_c} = \frac{8\pi G \rho_B^{(1)}}{3H^2} = \frac{G}{3} \frac{B_{(1)\,\mathrm{in}}^2}{H_{\mathrm{in}}^2} = \Omega_{B\,\mathrm{in}}^{(1)} . \tag{27}$$

The density parameter $\Omega_B^{(1)}$ is constant in time. Both, the background radiation and the magnetic field which is frozen in, scale in the same way with the expansion of the Universe. As long as the magnetic field density parameter $\Omega_B^{(1)}$ is much smaller than 1, the magnetic field can be considered a small perturbation.

This is the result for a constant magnetic field. We also want to consider a stochastic magnetic field. In this case $\mathbf{B}(x)$ is a random variable and its spectrum is given by [1]

$$a^2(t)\mathbf{B}(\mathbf{x},t) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3k \mathbf{B}(\mathbf{k}) e^{i\mathbf{x}\cdot\mathbf{k}} , \tag{28}$$

$$\langle B_i(\mathbf{k}) B_j^*(\mathbf{q})\rangle = (2\pi)^3 \delta(\mathbf{k}-\mathbf{q}) \mathcal{P}_{ij}(\hat{\mathbf{k}}) \mathcal{P}_{B\,\mathrm{in}}^{(1)}(k) . \tag{29}$$

Here the basic time evolution of the magnetic field $\propto a^{-2}$ has been removed so that, to first order $\mathbf{B}(\mathbf{k})$ is independent of time. $\mathcal{P}_{ij}(\hat{\mathbf{k}}) = \delta_{ij} - k^{-2}k_ik_j$ is the projection tensor onto the plane normal to $\mathbf{k}$. The tensorial form of the spectrum is dictated by statistical isotropy which also requires that $\mathcal{P}_{B\,\mathrm{in}}^{(1)}$ depends only on the absolute value $k = |\mathbf{k}|$, and by the fact that $\mathbf{B}$ is divergence free. The Dirac delta is a consequence of statistical homogeneity[1]. In this case we obtain

$$\begin{aligned}\langle \rho_B^{(1)} \rangle &= \frac{1}{(2\pi)^6 8\pi} \int \mathrm{d}^3k \int \mathrm{d}^3q \langle \mathbf{B}(\mathbf{k})\mathbf{B}(\mathbf{q})\rangle e^{i\mathbf{x}\cdot(\mathbf{k}-\mathbf{q})} \\ &= \frac{1}{(2\pi)^3} \int \frac{\mathrm{d}k}{k} k^3 \mathcal{P}_{B\,\mathrm{in}}^{(1)}(k) \\ &= \int \frac{\mathrm{d}k}{k} \frac{\mathrm{d}\rho_B^{(1)}(k)}{\mathrm{d}\log k} .\end{aligned}$$

[1] One could also add a term which is odd under parity but we disregard this possibility in this work [20].

For the magnetic field density parameter at scale k this yields

$$\frac{\mathrm{d}\Omega_B^{(1)}(k,t)}{\mathrm{d}\log k} = \frac{8\pi G}{3(2\pi)^3} \frac{k^3 \mathcal{P}_{B\,\mathrm{in}}^{(1)}(k)}{H_{\mathrm{in}}^2} = \frac{\mathrm{d}\Omega_{B\,\mathrm{in}}^{(1)}(k)}{\mathrm{d}\log k} . \tag{30}$$

Let us now consider gravity waves. The gravity wave energy density in real space is given by

$$\rho_{GW}^{(1)} \equiv \frac{\langle \dot{h}_{ij}\dot{h}^{ij}\rangle}{8\pi G} \frac{1}{a^2} , \tag{31}$$

where the factor $1/a^2$ comes from the fact that the dot denotes the derivative with respect to conformal time and the difference of a factor 4 in the normalization as compared *e.g.* to [17] comes from our definition of the perturbation variable $[g_{ij} = a^2(\delta_{ij} + 2h_{ij})]$. In Eq. (31) h_{ij} is considered as tensor field with respect to the spatial metric δ_{ij} so that there are no scale factors involved in raising or lowering indices, $h_{ij} = h_i{}^j = h^{ij}$. For simplicity we shall keep this convention is this section for all spatial tensors.

To lowest order the shear is given by $\sigma_{ij}^{(1)} = a\dot{h}_{ij}$. Furthermore, the fact that $\sigma_{ij}^{(1)}$ is transverse and traceless together with statistical isotropy determines entirely the tensor structure of the power spectrum.

$$\begin{aligned}&\langle \sigma_{ij}^{(1)\,\mathrm{in}}(\mathbf{k}) \sigma_{lm}^{(1)\,\mathrm{in}}(\mathbf{q})\rangle = \\ &\qquad (2\pi)^3 \delta(\mathbf{k}-\mathbf{q}) \mathcal{M}_{ijlm}(\hat{\mathbf{k}}) \mathcal{P}_{\sigma\,\mathrm{in}}^{(1)}(k) ,\end{aligned}$$

where [1]

$$\begin{aligned}\mathcal{M}_{ijlm}(\hat{\mathbf{k}}) \equiv\ & \delta_{il}\delta_{jm} + \delta_{im}\delta_{jl} - \delta_{ij}\delta_{lm} + k^{-2}(\delta_{ij}k_lk_m + \\ & \delta_{lm}k_ik_j - \delta_{il}k_jk_m - \delta_{im}k_lk_j - \delta_{jl}k_ik_m \\ & -\delta_{jm}k_lk_i) + k^{-4}k_ik_jk_lk_m .\end{aligned} \tag{32}$$

We have $\mathcal{M}^{ij}{}_{ij} = 4$, which takes into account the two polarization degrees of freedom. Therefore, considering that also for the shear we do not multiply by the scale factor while raising or lowering indices, $\sigma_{ij} = \sigma^{ij}$, we can write the gravity waves energy density in terms of σ_{ij} as

$$\rho_{GW}^{(1)} = \frac{\langle \sigma_{ij}\sigma^{ij}\rangle}{8\pi G} \frac{1}{a^4} . \tag{33}$$

For the contribution to the energy density per logarithmic frequency interval we then obtain

$$\begin{aligned}\frac{\mathrm{d}\rho_{GW}^{(1)}(k,t)}{\mathrm{d}\log k} &= \frac{2}{(2\pi)^3 G} \left[k^3 \mathcal{P}_\sigma^{(1)}(k,t)\right] \frac{1}{a^4} \\ &= \frac{18}{(2\pi)^3 G} \left[k^3 \mathcal{P}_\Sigma^{(1)}(k,t)\right] H^2 \left(\frac{a_{\mathrm{in}}}{a}\right)^4 ,\end{aligned} \tag{34}$$

where we have used the relation $\sigma_{ij} = 3Ha_{\mathrm{in}}^2\Sigma_{ij}$ or equivalently $\mathcal{P}_\sigma^{(1)}(k,t) = 9H^2a_{\mathrm{in}}^4\mathcal{P}_\Sigma^{(1)}(k,t)$. Finally, we can write the gravity wave density parameter as

$$\frac{\mathrm{d}\Omega^{(1)}_{\rm GW}(k,t)}{\mathrm{d}\log k} \equiv \frac{1}{\rho_c}\frac{\mathrm{d}\rho^{(1)}_{GW}}{\mathrm{d}\log k} = \frac{48\pi}{(2\pi)^3}\left[k^3\mathcal{P}^{(1)}_{\Sigma}(k,t)\right]\left(\frac{a_{\rm in}}{a}\right)^4 . \quad (35)$$

We have now to distinguish between super- and sub-horizon modes. Using our super-horizon result for $\Sigma_{(1)} = \sigma_{(1)}/(3Ha^2_{\rm in})$ where $kt \ll 1$

$$\Sigma^{(1)}_{ij}(\mathbf{k},t) \simeq \Sigma^{(1)}_{ij\,{\rm in}}(\mathbf{k})\left(\frac{t}{t_{\rm in}}\right)^4 ,$$

we find

$$\frac{\mathrm{d}\rho^{(1)}_{GW}(k,t)}{\mathrm{d}\log k} = \frac{18}{(2\pi)^3 G}\left[k^3\mathcal{P}^{(1)}_{\Sigma\,{\rm in}}(k)\right]\left(\frac{a}{a_{\rm in}}\right)^8 H^2\left(\frac{a_{\rm in}}{a}\right)^4 = \frac{18}{(2\pi)^3 G}\left[k^3\mathcal{P}^{(1)}_{\Sigma\,{\rm in}}(k)\right]H^2_{\rm in} = \frac{\mathrm{d}\rho^{(1)}_{GW\,{\rm in}}}{\mathrm{d}\log k} . \quad (36)$$

For the last equal sign we made use of Eq. (17). Hence on super-horizon scales the "gravity wave energy density" is time independent. Then, of course the gravity wave density parameter grows like a^4,

$$\frac{\mathrm{d}\Omega^{(1)}_{\rm GW}(k,t)}{\mathrm{d}\log k} = \frac{\mathrm{d}\Omega^{(1)\,{\rm in}}_{GW}}{\mathrm{d}\log k}\left(\frac{a}{a_{\rm in}}\right)^4 , \quad kt \ll 1 , \quad (37)$$

where

$$\frac{\mathrm{d}\Omega^{(1)\,{\rm in}}_{GW}(k)}{\mathrm{d}\log k} = \frac{48\pi}{(2\pi)^3}\left[k^3\mathcal{P}^{(1)}_{\Sigma\,{\rm in}}(k)\right] . \quad (38)$$

Inside the horizon, $kt \gg 1$, we have to insert the expression of $\Sigma_{(1)}$ given by Eq. (23) in Eq. (34), which yields

$$\frac{\mathrm{d}\rho^{(1)}_{GW}(k,t)}{\mathrm{d}\log k} \simeq \frac{9}{(2\pi)^3 G}\left[k^3\mathcal{P}^{(1)}_{\Sigma\,{\rm in}}(k)\right]\frac{H^2_{\rm in}}{(kt_{\rm in})^4}\left(\frac{a_{\rm in}}{a}\right)^4 \propto \frac{1}{a^4} . \quad (39)$$

For the density parameter we obtain in a radiation dominated background

$$\frac{\mathrm{d}\Omega^{(1)}_{GW}(k)}{\mathrm{d}\log k} \simeq \frac{24\pi}{(2\pi)^3}\left(\frac{1}{kt_{\rm in}}\right)^4\left[k^3\mathcal{P}^{(1)}_{\Sigma\,{\rm in}}(k)\right] \simeq \frac{1}{2}\left(\frac{1}{kt_{\rm in}}\right)^4\frac{\mathrm{d}\Omega^{(1)\,{\rm in}}_{GW}}{\mathrm{d}\log k} , \quad kt \gg 1 . \quad (40)$$

Inside the horizon, the gravity wave density parameter is constant in time as is natural in a radiation dominated Universe. Note that this agrees, up to the factor 1/2 which comes from averaging $\cos^2(kt)$, with Eq.(37) at horizon entry, where $(a/a_{\rm in})^4 = (kt_{\rm in})^{-4}$. Large scale gravity waves from inflation, are "amplified" for a long time before entering the horizon, *i.e.* they have $kt_{\rm in} \ll 1$. Only if $\left[\mathrm{d}\Omega^{(1)}_{GW}(k)/\mathrm{d}\log k\right]$ is small for all values of k, perturbation theory is justified. Therefore it is not sufficient if $\left[\mathrm{d}\Omega^{(1)\,{\rm in}}_{GW}/\mathrm{d}\log k\right]$ is small, but we actually need that $(kt_{\rm in})^{-4}\left[\mathrm{d}\Omega^{(1)\,{\rm in}}_{GW}/\mathrm{d}\log k\right]$ be small. This is better understood if we write the energy density in terms of the metric perturbation. In a radiation dominated Universe the "growing" (not decaying) mode solution for the metric perturbation is

$$h_{ij}(k,t) = e_{ij}(\mathbf{k})h_{\rm in}j_0(kt) ,$$

where $e_{ij}(\mathbf{k})$ is transverse traceless and j_0 is the spherical Bessel function of order 0. Using $j'_0 = -j_1$ and Eq. (31) yields

$$\rho^{(1)}_{GW} = k^3\frac{k^2\langle|h_{\rm in}|^2\rangle j_1^2(kt)}{8\pi G a^2}$$

With $\rho_c = 3H^2/(8\pi G) = 3/(8\pi G a^2 t^2)$ and $\langle|h_{\rm in}|^2\rangle \equiv \mathcal{P}_h$, we find

$$\frac{\mathrm{d}\Omega_{GW}}{\mathrm{d}\log k} = 3[(kt)^2 j_1^2(kt)]k^3 P_h \simeq 3(kt)^4 k^3\mathcal{P}_h , \quad \text{if } kt \ll 1 . \quad (41)$$

Hence if the metric perturbations are small for all values of k, *i.e.* $k^3\mathcal{P}_h \ll 1$ this implies

$$\frac{\mathrm{d}\Omega_{GW}}{\mathrm{d}\log k} \ll (kt)^4 .$$

Therefore the requirement $(kt_{\rm in})^{-4}\left[\mathrm{d}\Omega^{(1)\,{\rm in}}_{GW}/\mathrm{d}\log k\right] \ll 1$ is equivalent to the requirement that the metric perturbations be small on super horizon scales [note that $j_0(z) \simeq 1$ for $z \ll 1$].

Before we go to the second order, let us stress this point once more, because it is the origin of the confusion in the literature. Inflation generates gravitational waves with an amplitude

$$k^3\mathcal{P}_h \simeq \left(\frac{H_{\rm inf}}{M_{\rm P}}\right)^2 \leq 10^{-10} ,$$

where $M_{\rm P}$ is the Planck mass and $H_{\rm inf}$ denotes the scale factor during inflation. The maximum value of 10^{-10} is the maximum tensor fluctuation from inflation allowed by the cosmic microwave background anisotropies.

However, the density parameter on super-horizon scale is given by, see Eq. (41)

$$\frac{\mathrm{d}\Omega^{(1)}_{GW}}{\mathrm{d}\log k} \simeq (kt)^4\left(\frac{H_{\rm inf}}{M_{\rm P}}\right)^2 , \qquad kt \ll 1 .$$

This equation is correct for any power law background, $a \propto t^q$, also for matter and even for inflation. 0nly at horizon crossing, can the density parameter become of the order 10^{-10}. Inside the horizon it stays constant if the background is radiation. Hence Eq. (40) can be written as

$$\frac{\mathrm{d}\Omega^{(1)}_{GW}(k)}{\mathrm{d}\log k} \simeq \left(\frac{H_{\rm inf}}{M_{\rm P}}\right)^2 , \qquad kt \gg 1 . \quad (42)$$

IV. SECOND ORDER PERTURBATIONS

In this section we include all terms of second order in the perturbations, and we shall insert our first order results for them; *i.e.* in terms of the form $\sigma_{ij}B^j$ we insert $\sigma_{ij}^{(1)}B^j_{(1)}$ or for Π_{ij} we insert the first order magnetic fields, $\Pi^{(1)}_{ij} = B^{(1)}_i B^{(1)}_j - (1/3)\tilde{p}^{(0)}_{ij} B^{(1)2}$ in Eqs. (10) and (11). We obtain the following differential equations for the evolution of the second order perturbations $B^{(2)}_i(\mathbf{x},t)$ and $\sigma^{(2)}_{ij}(\mathbf{x},t)$:

$$\begin{aligned}
&\nabla_u\nabla_u B^{(2)}_i - D^2 B^{(2)}_i + \frac{5}{3}\Theta\nabla_u B^{(2)}_i + \frac{1}{3}\Theta^2(1-w)B^{(2)}_i = \\
&\quad \sigma^{(1)}_{ij}\nabla_u B^j_{(1)} + 2\Theta\sigma^{(1)}_{ij}B^j_{(1)} + 2\nabla_u\sigma^{(1)}_{ij}B^j_{(1)} \\
&\quad +(D^2)^{(1)}B^{(1)}_i + \mathrm{curl}J_i \,, \qquad (43)\\
&\nabla_u\nabla_u \sigma^{(2)}_{ij} - D^2\sigma^{(2)}_{ij} + \frac{5}{3}\Theta\nabla_u\sigma^{(2)}_{ij} + \frac{1}{6}\Theta^2(1-3w)\sigma^{(2)}_{ij} = \\
&\quad \kappa\nabla_u\Pi^{(1)}_{ij} + \frac{2}{3}\Theta\kappa\Pi^{(1)}_{ij} + \Theta\sigma_{\langle i(1)}{}^{n}\sigma^{(1)}_{j\rangle n} + (D^2)^{(1)}\sigma^{(1)}_{ij} \\
&\quad +2\sigma^{n(1)}_{\langle i}\nabla_u\sigma^{(1)}_{j\rangle n} - \nabla_u\sigma_{\langle i(1)}{}^{n}\sigma^{(1)}_{j\rangle n} \,. \qquad (44)
\end{aligned}$$

Taking into account that $\nabla_u B_{i(1)} = -(2/3)\Theta B_{i(1)}$ together with $\nabla_u\Pi^{(1)}_{ij} = -(4/3)\Theta\Pi^{(1)}_{ij}$, Eqs. (43), (44) can be simplified to

$$\begin{aligned}
&\nabla_u\nabla_u B^{(2)}_i - D^2 B^{(2)}_i + \frac{5}{3}\Theta\nabla_u B^{(2)}_i + \frac{1}{3}\Theta^2(1-w)B^{(2)}_i = \\
&\quad \left[\frac{4}{3}\Theta\sigma_{ij(1)} + 2\nabla_u\sigma^{(1)}_{ij}\right]B^j_{(1)} + (D^2)^{(1)}B^{(1)}_i \,, \qquad (45)\\
&\nabla_u\nabla_u \sigma^{(2)}_{ij} - D^2\sigma^{(2)}_{ij} + \frac{5}{3}\Theta\nabla_u\sigma^{(2)}_{ij} + \frac{1}{6}\Theta^2(1-3w)\sigma^{(2)}_{ij} = \\
&\quad -\frac{2}{3}\Theta\kappa\Pi^{(1)}_{ij} + \Theta\sigma_{\langle i(1)}{}^{n}\sigma^{(1)}_{j\rangle n} + (D^2)^{(1)}\sigma^{(1)}_{ij} \\
&\quad +2\sigma^{n(1)}_{\langle i}\nabla_u\sigma^{(1)}_{j\rangle n} - \nabla_u\sigma_{\langle i(1)}{}^{n}\sigma^{(1)}_{j\rangle n} \,. \qquad (46)
\end{aligned}$$

We have also neglected the term $\mathrm{curl}J_i$ in Eq. (43). Since it is proportional to k in the Fourier space, its contribution is important only on sub-horizon scales, where we anyway neglect the source part. Outside the horizon, $kt \ll 1$, it is negligible.

A. The second order magnetic field from gravity waves and a constant magnetic field

For simplicity, and to gain intuition, we first consider a constant first order magnetic field,

$$\begin{aligned}
B^{(1)}_i(\mathbf{x},t) &= B^{(1)}_{i\,\mathrm{in}}\frac{a_{\mathrm{in}}}{a} \,,\\
B^{(1)}_i(\mathbf{k},t) &= B^{(1)}_{i\,\mathrm{in}}\frac{a_{\mathrm{in}}}{a}\delta^3(\mathbf{k}) \,.
\end{aligned}$$

In this case, the convolution of $B_{(1)}$ and $\sigma_{(1)}$ into which the products in ordinary space transform under Fourier transformation, become normal products and the second order magnetic field $B^{(2)}_i$ has the same wavelength as the first order gravity wave which generates it.

Remembering that $\sigma_{ij} \propto a^{-4}\sqrt{\mathcal{P}^{(1)}_\sigma} \equiv a^{-4}\sigma^{(1)}$ one obtains

$$\ddot{B}^{(2)}_i + 2\mathcal{H}\dot{B}^{(2)}_i + B^{(2)}_i\left[k^2 + \frac{1}{2}\mathcal{H}^2(1-3w)\right] = 2\dot{\sigma}^{(1)}_{ij}B^{(1)}_{j\,\mathrm{in}}\frac{a_{\mathrm{in}}}{a^2} \,. \qquad (47)$$

In principle, one has to consider the corrections to the orthogonal spatially projected covariant derivative $(D^2)^{(1)}B^{(1)}_i$ due to the tensor perturbations h_{ij} in the metric tensor $g_{\mu\nu}$. Computing these corrections, they turn out to be equal to zero, since the magnetic field is transverse. This remains valid even if $\mathbf{B}^{(1)}$ is not constant.

Considering the expansion-normalized dimensionless variable $\mathcal{B}^{(2)}_i \equiv \sqrt{\kappa}B^{(2)}_i/(\Theta a_{\mathrm{in}})$, we obtain

$$\begin{aligned}
&\ddot{\mathcal{B}}^{(2)}_i - \mathcal{H}(1+3w)\dot{\mathcal{B}}^{(2)}_i \\
&\quad +\mathcal{B}^{(2)}_i\left[k^2 + \mathcal{H}^2\left(\frac{1}{2} + 3w + \frac{9}{2}w^2\right)\right] = f_i \,,\\
&f_j \equiv 2\sqrt{\kappa}\left[\dot{\Sigma}^{(1)}_{ij} - \frac{3}{2}\mathcal{H}(1+w)\Sigma^{(1)}_{ij}\right]B^{(1)}_{j\,\mathrm{in}}\left(\frac{a_{\mathrm{in}}}{a}\right)^2 \,.
\end{aligned} \qquad (48)$$

We investigate the behavior of the second order perturbation in the radiation dominated phase.

Moreover, since the source $f_i(\mathbf{k},t)$ and therefore also $\mathcal{B}^{(2)}_i(\mathbf{k},t)$ are random variables, we want to determine their spectra. The first order gravity wave spectrum is

$$\begin{aligned}
\langle\Sigma^{(1)\,\mathrm{in}}_{ij}(\mathbf{k})\Sigma^{*(1)\,\mathrm{in}}_{ln}(\mathbf{q})\rangle &= (2\pi)^3\mathcal{M}_{ijln}(\hat{\mathbf{k}})\delta^3(\mathbf{k}-\mathbf{q})\mathcal{P}^{(1)}_{\Sigma\,\mathrm{in}}(k) \,,\\
\langle\Sigma^{(1)\,\mathrm{in}}_{ij}(\mathbf{k})\Sigma^{*ij}_{(1)\,\mathrm{in}}(\mathbf{q})\rangle &= 4(2\pi)^3\delta^3(\mathbf{k}-\mathbf{q})\mathcal{P}^{(1)}_{\Sigma\,\mathrm{in}}(k) \,,
\end{aligned}$$

where $\mathcal{M}_{ijlm}$ is the gravity waves polarization tensor defined in Eq. (32). It can also be expressed in terms of the projection tensor $\mathcal{P}_{ij}(\hat{\mathbf{k}})$, $\mathcal{M}_{ijlm} \equiv \mathcal{P}_{il}\mathcal{P}_{jm} + \mathcal{P}_{im}\mathcal{P}_{jl} - \mathcal{P}_{ij}\mathcal{P}_{lm}$. Actually $(1/2)\mathcal{M}_{ij}{}^{lm}$ is the projection tensor onto the two transverse traceless modes of a rank 2 symmetric tensor. The power spectrum of the second order magnetic field $\mathcal{B}_{(2)}$ is of the form

$$\langle\mathcal{B}^{(2)}_i(\mathbf{k},t)\mathcal{B}^{*(2)}_j(\mathbf{p},t)\rangle = (2\pi)^3\mathcal{P}_{ij}(\hat{\mathbf{k}})\delta^3(\mathbf{k}-\mathbf{p})\mathcal{P}^{(2)}_{\mathcal{B}}(k,t) \,. \qquad (49)$$

We obtain the solution for $\mathcal{B}^{(2)}_i(\mathbf{k},t)$ with the help of Green function method,

$$\mathcal{B}^{(2)}_i(\mathbf{k},t) = \int_{t_{\mathrm{in}}}^{t}\mathrm{d}t'\mathcal{G}(t,t',\mathbf{k})f_i(\mathbf{k},t') \,. \qquad (50)$$

Here $\mathcal{G}$ is the Green function of the second order linear differential operator acting on $\mathcal{B}^{(2)}_i$ which depends on the cosmological background. It can be determined in terms of the homogeneous solutions which in the radiation dominated era are simply spherical Bessel functions and powers. More precisely, in terms of $z = kt$, Eq. (48) in the radiation dominated case, $w = 1/3$, becomes

$$\mathcal{B}_i^{(2)\prime\prime} - \frac{2}{z}\mathcal{B}_i^{(2)\prime} + \left(1 + \frac{2}{z^2}\right)\mathcal{B}_i^{(2)} = k^{-2} f_i(z, \mathbf{k}) \ , \quad (51)$$

where the prime denotes a derivative w.r.t. z. Two homogenous solutions to this equation are $P_1(z) = z^2 j_0(z)$ and $P_2(z) = z^2 y_0(z)$. Defining the Wronskian, $W(z) = P_1'(z)P_2(z) - P_1(z)P_2'(z) = z^2$, a possible Green function is

$$\mathcal{G}(z, z', \mathbf{k}) = \frac{P_1(z')P_2(z) - P_1(z)P_2(z')}{W(z')} \ . \quad (52)$$

The solution obtained by integrating with this Green function satisfies the initial condition $\mathcal{B}_i^{(2)}(z_{\rm in}, \mathbf{k}) = \mathcal{B}_i^{(2)\prime}(z_{\rm in}, \mathbf{k}) = 0$. Any other solution can be obtained by adding a homogeneous solution to this one. We discuss the physically correct choice of initial conditions in more detail in the Appendix A. For the magnetic field, the initial conditions chosen with this Green function seem adequate to us. We can now write the magnetic field spectrum as

$$\langle \mathcal{B}_i^{(2)}(\mathbf{k}, t)\mathcal{B}_j^{*(2)}(\mathbf{p}, t)\rangle = \int_{z_{\rm in}}^{z} {\rm d}z' \int_{z_{\rm in}}^{z} {\rm d}z'' k^{-2} p^{-2} \times \\ \mathcal{G}(z, z', \mathbf{k})\mathcal{G}^*(z, z'', \mathbf{p})\langle f_i(\mathbf{k}, z') f_j^*(\mathbf{p}, z'')\rangle \ . \quad (53)$$

We solve Eq. (51), distinguishing the sub- and super-horizon regimes. In the long wavelength limit, $kt = z \ll 1$, we have to insert the solution obtained for gravity waves $\Sigma_{ij}^{(1)}$ on super-horizon scales and given in Eq. (20). Therefore, the source term $f_i(\mathbf{k}, t)$ reads

$$f_i(\mathbf{k}, t') = 4\sqrt{\kappa}\mathcal{P}_i^s(\hat{\mathbf{k}})\left[\Sigma_{sn}^{(1)\,{\rm in}}(\mathbf{k}) B_n^{(1)\,{\rm in}}\right](H_{\rm in} a_{\rm in})^2 t' \ , \quad (54)$$

and equivalently for $f_j^*(\mathbf{q}, t'')$. The power spectrum of f_i can then be written as

$$\begin{aligned} \langle f_i(\mathbf{k}, z') f_j^*(\mathbf{p}, z'')\rangle &= \\ & 16\kappa\mathcal{P}_i^s(\hat{\mathbf{k}})\mathcal{P}_j^l(\hat{\mathbf{q}})\langle \Sigma_{sn}^{(1)\,{\rm in}}(\mathbf{k})\Sigma_{lr}^{*(1)\,{\rm in}}(\mathbf{k})\rangle \\ & B_n^{(1)\,{\rm in}} B_r^{*(1)\,{\rm in}}(H_{\rm in} a_{\rm in})^4 z' z'' k^{-2} \\ &\equiv (2\pi)^3\delta^3(\mathbf{k} - \mathbf{p})\mathcal{P}_{ij}(\hat{\mathbf{k}}) h(z', z'', k) \ . \end{aligned} \quad (55)$$

For the function $h(z', z'', k)$ we obtain the following expression

$$\begin{aligned} & h(z', z'', k) \simeq F(k) g(z') g(z'') \ , \\ & F(k) = \kappa B_{(1)\,{\rm in}}^2 \mathcal{P}_{\Sigma\,{\rm in}}^{(1)}(k) k^{-2} \ , \\ & g(z') = 4H_{\rm in}^2 a_{\rm in} z' \ . \end{aligned}$$

The solution for the power spectrum of the second order perturbation of the magnetic field can then be written as

$$\langle \mathcal{B}_i^{(2)}(\mathbf{k}, t)\mathcal{B}_j^{*(2)}(\mathbf{p}, t)\rangle = (2\pi)^3\mathcal{P}_{ij}(\hat{\mathbf{k}})\delta^3(\mathbf{k}-\mathbf{p}) \times \\ \left[\int_{z_{\rm in}}^{z} {\rm d}z'\mathcal{G}(z, z', \mathbf{k})\sqrt{F(k)} g(z')\right]^2 \ . \quad (56)$$

The square $[\cdots]^2$ is simple the power spectrum $\mathcal{P}_{\mathcal{B}}^{(2)}(k, t)$ which we want to determine. Of course, the integrals in the square bracket are solutions to our magnetic field Eq. (51) with source $\sqrt{F(k)} g(z)$. Hence $\sqrt{\mathcal{P}_{\mathcal{B}}^{(2)}}$ satisfies the equation

$$\begin{aligned} & P'' - \frac{2}{z}P' + \left(1 + \frac{2}{z^2}\right)P = \frac{\alpha}{k^3} z \ , \quad (57) \\ & |P| \equiv \sqrt{\mathcal{P}_{\mathcal{B}}^{(2)}(k, t)} \ , \qquad z \equiv kt \ , \\ & \alpha \equiv 4H_{\rm in}^2 a_{\rm in}\sqrt{\kappa B_{(1)\,{\rm in}}^2 \mathcal{P}_{\Sigma\,{\rm in}}^{(1)}(k)} \ . \end{aligned}$$

Solving the above equation with the Wronskian method in the regime $z = kt \ll 1$, one finds

$$P(z) \simeq \frac{\alpha}{2k^3} z^3 \ , \qquad z = kt \ll 1 \ .$$

This yields

$$\begin{aligned} k^3\mathcal{P}_{\mathcal{B}}^{(2)}(k, t) \ &\simeq \ 4\kappa\frac{B_{(1)}^{{\rm in}\,2}}{H_{\rm in}^2}\left[k^3\mathcal{P}_{\Sigma\,{\rm in}}^{(1)}(k)\right]\left(\frac{a}{a_{\rm in}}\right)^6 \ , \\ & kt \ll 1 \ . \end{aligned} \quad (58)$$

This is the second order magnetic field power spectrum induced by the presence of a first order field and a gravitational wave. It is the growth $\propto t^6$ of this induced field which has been interpreted in Refs. [2, 12, 15] as strong amplification. But before drawing such conclusions, we want to compare the energy density parameter of $B^{(2)}$ with the one of $\sigma^{(1)}$ and $B^{(1)}$ inside the horizon, where these quantities have a simple physical interpretation.

Inside the horizon, $kt \gg 1$, we can no longer use the above simple approximation for the source term. The solution of Eq. (57) with a generic source term,

$$\left[\sqrt{\mathcal{P}_{\mathcal{B}}^{(2)}(k, z)}\right]'' - \frac{2}{z}\left[\sqrt{\mathcal{P}_{\mathcal{B}}^{(2)}(k, z)}\right]' \\ + \left[\sqrt{\mathcal{P}_{\mathcal{B}}^{(2)}(k, z)}\right] = \mathcal{S}(k, z) \ , \quad (59)$$

can be written as

$$\sqrt{\mathcal{P}_{\mathcal{B}}^{(2)}(k, z)} = \int_{z_{\rm in}}^{z} dz'\mathcal{S}(k, z')\mathcal{G}(z, z', \mathbf{k}) \, . \quad (60)$$

But, once the gravity waves enter the horizon, the source and the Green function start oscillating and the contribution to the above integral becomes negligible. We therefore neglect the source inside the horizon and simply match the solution at horizon crossing with the homogeneous solutions of Eq. (57) given above, that are $P_1(z) = z^2 j_0(z)$ and $P_2(z) = z^2 y_0(z)$ ($z = kt$). Considering the limit $z \gg 1$, this yields

$$k^3\mathcal{P}^{(2)}_{\mathcal{B}}(k,t) \simeq 2\kappa \frac{B^{\rm in\,2}_{(1)}}{H^2_{\rm in}} \left[k^3\mathcal{P}^{(1)}_{\Sigma\,{\rm in}}(k)\right] \times \left(\frac{a}{a_{\rm in}}\right)^2 \frac{1}{(kt_{\rm in})^4}\ , \quad kt \gg 1\ . \qquad (61)$$

1. The energy density

To analyze this amplification which happens mainly on super-horizon scales, let us compare energy densities after horizon entry. The energy density of our second order magnetic field is

$$\frac{{\rm d}\rho^{(2)}_B(k,t)}{{\rm d}\log k} \equiv \frac{1}{(2\pi)^3}\left[k^3\mathcal{P}^{(2)}_B(k,t)\right]\frac{1}{a^2} = \frac{1}{(2\pi)^3}\left[k^3\mathcal{P}^{(2)}_{\mathcal{B}}(k,t)\right]\frac{9H^2}{\kappa}\left(\frac{a_{\rm in}}{a}\right)^2 . \qquad (62)$$

The factor $1/a^2$ comes from the fact that we have to raise one index of $\langle B^{(2)}_i B^{(2)}_i\rangle$ in order to compute the energy density, while $a^2_{\rm in}$ is due to the definition of $\mathcal{B}^{(2)}_i \propto B^{(2)}_i/a_{\rm in}$ that we gave above. The density parameter for $B^{(2)}$ then reads

$$\frac{{\rm d}\Omega^{(2)}_B(k,t)}{{\rm d}\log k} = \frac{3}{(2\pi)^3}\left(\frac{a_{\rm in}}{a}\right)^2\left[k^3\mathcal{P}^{(2)}_{\mathcal{B}}(k,t)\right]\ . \qquad (63)$$

With $H = H_{\rm in}a^2_{\rm in}/a^2$ we find that even though $\mathcal{P}^{(2)}_{\mathcal{B}}(k,t)$ is growing like t^6 on super-horizon scales, the density parameter grows like $\Omega^{(1)}_{\rm GW}$. After horizon entry, this growth stops and $\Omega^{(2)}_B$ remains constant. Inserting the solutions (58) and (61) for $k^3\mathcal{P}^{(2)}_{\mathcal{B}}(k,t)$ gives

$$\frac{{\rm d}\Omega^{(2)}_B(k,t)}{{\rm d}\log k} = 6\frac{d\Omega^{(1)}_{\rm GW}(k,t)}{d\log k}\Omega^{(1)}_B \qquad (64)$$

on super- and sub-horizon scales.

Hence, even though the second order magnetic field $\mathcal{B}_{(2)}$ is growing considerably, this reflects only the growth of the unphysical density parameter $\Omega^{(1)}_{\rm GW}$ on super-horizon scales. Once this is factored in, the magnetic field density parameter is not. The values for both, $\left[{\rm d}\Omega^{(1)\,{\rm in}}_{\rm GW}(k)/{\rm d}\log k\right](kt_{\rm in})^{-4} = \left[{\rm d}\Omega^{(1)}_{\rm GW}(k)/{\rm d}\log k\right]$ and $\Omega^{(1)}_B$ are at most of the order of 10^{-5} and smaller. For the gravity waves, we have seen that $\left[{\rm d}\Omega^{(1)\,{\rm in}}_{\rm GW}(k)/{\rm d}\log k\right](kt_{\rm in})^{-4}$ is just the square amplitude of the metric perturbations on super horizon scales, which has to be $k^3P_h \lesssim 10^{-10}$ in order not to overproduce CMB anisotropies on large scales (integrated Sachs–Wolfe effect). Similar arguments yield $\Omega^{(1)}_B < 10^{-5}$ on large scales (see, *e.g.* [21, 22]). Therefore, even though we agree with the calculation in Ref. [15], we do not agree with the interpretation. If the gravitational wave energy density is as small as required by the measurements of CMB anisotropies, $\Omega^{(2)}_B$ always remains smaller than $\Omega^{(1)}_B$. Furthermore, up to logarithmic corrections, $B^{(2)}$ inherits the spectrum of the first order gravity waves.

In the next section we show that this conclusion persists also if we allow for a stochastic magnetic field. Just the computation becomes more involved.

B. The second order magnetic field from gravity waves and a stochastic magnetic field

In the case in which the first order magnetic field is not spatially constant, all the products $\Sigma^{(1)}_{ij}(\mathbf{x},t)B^j_{(1)}(\mathbf{x},t)$ become convolutions in Fourier space

$$\int {\rm d}^3x e^{i\mathbf{k}\cdot\mathbf{x}}\Sigma^{(1)}_{ij}(\mathbf{x},t)B^j_{(1)}(\mathbf{x},t) = \frac{1}{(2\pi)^3}\mathcal{P}_i{}^n(\hat{\mathbf{k}})\int {\rm d}^3q\Sigma^{(1)}_{nj}(\mathbf{q},t)B^j_{(1)}(\mathbf{k}-\mathbf{q},t)\ ,$$

where the projector $\mathcal{P}_i{}^n \equiv \delta^n_i - \hat{k}_i\hat{k}^n$ projects onto the transverse modes. The result of this convolution is a magnetic field and therefore transverse. Hence this projector is not strictly necessary. But as we shall see, it simplifies the calculations.

Our equations are written in terms of the dimensionless expansion-normalized variables $\mathcal{B}^{(2)}_i(\mathbf{x},t)$ and $\Sigma^{(2)}_{ij}(\mathbf{x},t)$, and we want to express their power spectra in terms of the power spectra of the first order random variables $B^{(1)}_i(\mathbf{x},t)$ and $\Sigma^{(1)}_{ij}(\mathbf{x},t)$ for which we assume simple power laws,

$$B^{(1)}_i(\mathbf{k},t) = B^{\rm in}_{i(1)}(\mathbf{k})\frac{a_{\rm in}}{a}\ ,$$
$$B^i_{(1)}(\mathbf{k},t) = B^{{\rm in}\,i}_{(1)}(\mathbf{k})\frac{a^3_{\rm in}}{a^3}\ ,$$
$$a^2_{\rm in}\langle B^{(1)\,{\rm in}}_i(\mathbf{k})B^{*(1)\,{\rm in}}_j(\mathbf{q})\rangle = (2\pi)^3\mathcal{P}_{ij}(\hat{\mathbf{k}})\delta^3(\mathbf{k}-\mathbf{q})\mathcal{P}^{(1)}_{B\,{\rm in}}(k)\ ,$$
$$\langle B^{(1)\,{\rm in}}_i(\mathbf{k})B^{*i}_{(1)\,{\rm in}}(\mathbf{q})\rangle = 2(2\pi)^3\delta^3(\mathbf{k}-\mathbf{q})\mathcal{P}^{(1)}_{B\,{\rm in}}(k)\ ,$$
$$\mathcal{P}^{(1)}_{B\,{\rm in}}(k) = \begin{cases}[B^2_{(1)\,{\rm in}}\lambda^3](\lambda k)^M & \text{for } k < k_d,\\ 0 & \text{for } k > k_d,\end{cases} \qquad (65)$$

where k_d is the damping scale which we assume to be always much smaller than the Hubble scale. The scale λ is arbitrary, e.g., the scale at which we want to calculate the magnetic field. With this normalization $B^{\rm in}_{(1)}$ is simply the amplitude of the magnetic field at scale λ at time $t_{\rm in}$. At any other moment, the magnetic field at scale λ is given by $B^{\rm in}_{(1)}a^2_{\rm in}/a^2(t)$.

Equivalently we have for the gravity wave power spectrum

$$\Sigma_{ij}^{(1)}(\mathbf{k},t) = \Sigma_{ij}^{(1)\,\mathrm{in}}(\mathbf{k})T(k,t)\ ,$$
$$\langle \Sigma_{ij}^{(1)\,\mathrm{in}}(\mathbf{k})\Sigma_{ln}^{*(1)\,\mathrm{in}}(\mathbf{q})\rangle = (2\pi)^3\mathcal{M}_{ijln}(\hat{\mathbf{k}})\delta^3(\mathbf{k}-\mathbf{q})\mathcal{P}_{\Sigma\,\mathrm{in}}^{(1)}(k)\ ,$$
$$\langle \Sigma_{ij}^{(1)\,\mathrm{in}}(\mathbf{k})\Sigma_{(1)\,\mathrm{in}}^{*ij}(\mathbf{q})\rangle = 4(2\pi)^3\delta^3(\mathbf{k}-\mathbf{q})\mathcal{P}_{\Sigma\,\mathrm{in}}^{(1)}(k)\ ,$$
$$\mathcal{P}_{\Sigma\,\mathrm{in}}^{(1)}(k) = [\Sigma_{(1)\,\mathrm{in}}^2\lambda^3](\lambda k)^A\ . \qquad (66)$$

Here the transfer function $T(k,t)$ keeps track of the deterministic time-dependence of the gravity waves. In the previous section we have derived the well known behavior of the gravity wave transfer function which oscillates on sub-horizon scales, $kt \gg 1$, and behaves like a power law on super-horizon scales. For the radiation dominated case,

$$T(k,t) \ \simeq \ \left(\frac{a}{a_{\mathrm{in}}}\right)^4 \ , \qquad kt \ll 1\ . \qquad (67)$$

Starting from Eq. (45), we can write the following evolution equation for the second order perturbation

$$\ddot{B}_i^{(2)}(\mathbf{x},t) + 2\mathcal{H}\dot{B}_i^{(2)}(\mathbf{x},t) - a^2D^2B_i^{(2)}(\mathbf{x},t) + \frac{1}{2}\mathcal{H}^2(1-3w)B_i^{(2)}(\mathbf{x},t) = 2a\dot{\sigma}_{ij}^{(1)}(\mathbf{x},t)B_{(1)}^j(\mathbf{x},t)\ . \qquad (68)$$

Replacing $B_i^{(2)} = 3Ha_{\mathrm{in}}\mathcal{B}_i^{(2)}/\sqrt{\kappa}$ and $\sigma_{ij}^{(1)} = 3Ha_{\mathrm{in}}^2\Sigma_{ij}^{(1)}$, we obtain

$$\ddot{\mathcal{B}}_i^{(2)}(\mathbf{x},t) - (1+3w)\mathcal{H}\dot{\mathcal{B}}_i^{(2)}(\mathbf{x},t) - a^2D^2\mathcal{B}_i^{(2)}(\mathbf{x},t) + \left(\frac{1}{2}+3w+\frac{9}{2}w^2\right)\mathcal{H}^2\mathcal{B}_i^{(2)}(\mathbf{x},t) = 2\sqrt{\kappa}a_{\mathrm{in}}aB_{(1)}^j(\mathbf{x},t)\left[\dot{\Sigma}_{ij}^{(1)}(\mathbf{x},t) - \frac{3}{2}\mathcal{H}(1+w)\Sigma_{ij}^{(1)}(\mathbf{x},t)\right]\ . \qquad (69)$$

This is the same differential equation as for the constant magnetic field. In Fourier space this equation becomes

$$\ddot{\mathcal{B}}_i^{(2)}(\mathbf{k},t) - (1+3w)\mathcal{H}\dot{\mathcal{B}}_i^{(2)}(\mathbf{k},t) + \mathcal{B}_i^{(2)}(\mathbf{k},t)\times\left[k^2 + \left(\frac{1}{2}+3w+\frac{9}{2}w^2\right)\mathcal{H}^2\right] = f_i(\mathbf{k},t)\ , \qquad (70)$$

where the source $f_i(\mathbf{k},t)$ is now given by a convolution

$$f_i(\mathbf{k},t) \equiv \frac{2}{(2\pi)^3}\sqrt{\kappa}a_{\mathrm{in}}a{\mathcal{P}_i}^r(\hat{\mathbf{k}})\times\left[\int \mathrm{d}^3q\dot{\Sigma}_{rj}^{(1)}(\mathbf{q},t)B_{(1)}^j(\mathbf{k}-\mathbf{q},t) - \frac{3}{2}(1+w)\mathcal{H}\times\int \mathrm{d}^3q\Sigma_{rj}^{(1)}(\mathbf{q},t)B_{(1)}^j(\mathbf{k}-\mathbf{q},t)\right]\ . \qquad (71)$$

In terms of the variable $z = kt$ we obtain again Eq. (51). As in the previous section we solve it with the Green function method. Therefore, the power spectrum of $\mathcal{B}_i^{(2)}$ is given by

$$\langle \mathcal{B}_i^{(2)}(\mathbf{k},t)\mathcal{B}_j^{*(2)}(\mathbf{p},t)\rangle = (2\pi)^3\delta^3(\mathbf{k}-\mathbf{p})\times(\delta_{ij}-\hat{k}_i\hat{k}_j)\mathcal{P}_{\mathcal{B}}^{(2)}(k,t)\ ,$$

with

$$\mathcal{P}_{\mathcal{B}}^{(2)}(k,t) \ = \ \int_{z_{\mathrm{in}}}^z \mathrm{d}z'\int_{z_{\mathrm{in}}}^z \mathrm{d}z''\mathcal{G}(z,z',\mathbf{k})\times\mathcal{G}^*(z,z'',\mathbf{p})\langle f_i(\mathbf{k},z')f_j^*(\mathbf{p},z'')\rangle\ ,$$

where $z = kt$. In the radiation dominated epoch ($w = 1/3$) the source term reads

$$f_i(\mathbf{k},t') = \frac{2}{(2\pi)^3}\sqrt{\kappa}a_{\mathrm{in}}a(t'){\mathcal{P}_i}^r(\hat{\mathbf{k}})\times\left[\int \mathrm{d}^3q\dot{\Sigma}_{rm}^{(1)}(\mathbf{q},t')B_{(1)}^m(\mathbf{k}-\mathbf{q},t') - 2\mathcal{H}(t')\int \mathrm{d}^3q\Sigma_{rm}^{(1)}(\mathbf{q},t')B_{(1)}^m(\mathbf{k}-\mathbf{q},t')\right]$$
$$= \frac{2}{(2\pi)^3}\sqrt{\kappa}\frac{a_{\mathrm{in}}^2}{a^2(t')}{\mathcal{P}_i}^r(\hat{\mathbf{k}})\times\left[\int \mathrm{d}^3q\Sigma_{rm}^{(1)\,\mathrm{in}}(\mathbf{q})\dot{T}(q,t')B_m^{(1)\,\mathrm{in}}(\mathbf{k}-\mathbf{q}) - 2\mathcal{H}(t')\int \mathrm{d}^3q\Sigma_{rm}^{(1)\,\mathrm{in}}(\mathbf{q})T(q,t')B_m^{(1)\,\mathrm{in}}(\mathbf{k}-\mathbf{q})\right]\ , \qquad (72)$$

and equivalently for $f_j^*(\mathbf{p},t'')$. To determine the power spectrum of f_i we assume that the magnetic field $B_{(1)}$ and gravity waves $\sigma_{(1)}$ are uncorrelated, so that

$$\langle f_i(\mathbf{k},t')f_j^*(\mathbf{p},t'')\rangle = \frac{16\kappa}{(2\pi)^6}\mathcal{H}(t')\mathcal{H}(t'')\left[\frac{a(t')a(t'')}{a_{\mathrm{in}}^2}\right]^2{\mathcal{P}_i}^r(\hat{\mathbf{k}}){\mathcal{P}_j}^n(\hat{\mathbf{p}})\times\int \mathrm{d}^3q\int \mathrm{d}^3s\langle\Sigma_{rm}^{(1)\,\mathrm{in}}(\mathbf{q})\Sigma_{nl}^{*(1)\,\mathrm{in}}(\mathbf{s})\rangle\times\langle B_m^{(1)\,\mathrm{in}}(\mathbf{k}-\mathbf{q})B_l^{*(1)\,\mathrm{in}}(\mathbf{p}-\mathbf{s})\rangle \equiv (2\pi)^3\delta^3(\mathbf{k}-\mathbf{p})\mathcal{P}_{ij}(\hat{\mathbf{k}})h(t',t'',k)\ . \qquad (73)$$

The function $h(t',t'',k)$ is given by [1]

$$h(t',t'',k) = \frac{8\kappa}{(2\pi)^3}\mathcal{H}(t')\mathcal{H}(t'')\left[\frac{a(t')a(t'')}{a_{\mathrm{in}}^2}\right]^2 I(k)\ ,$$
$$I(k) \equiv \int \mathrm{d}^3q(1+\gamma^2)(1+\alpha^2)\mathcal{P}_{\Sigma\,\mathrm{in}}^{(1)}(q)\mathcal{P}_{B\,\mathrm{in}}^{(1)}(|\mathbf{k}-\mathbf{q}|)\ , \qquad (74)$$

where $\alpha \equiv \hat{k}\cdot\widehat{(k-q)}$ and $\gamma \equiv \hat{k}\cdot\hat{q}$. We neglect the angular dependence of $(1+\gamma^2)$ and $(1+\alpha^2)$ and simply set

$$(1+\gamma^2)(1+\alpha^2) \simeq 1\ .$$

We then have to solve the following integral,

$$I(k) = 4\pi\Sigma^2_{(1)\,\mathrm{in}}B^2_{(1)\,\mathrm{in}}\lambda^{A+M+6}\int_0^{1/\max(t',t'')}\mathrm{d}q q^{A+2} \times \int_{-1}^{1}\mathrm{d}\mu(k^2+q^2-2\mu kq)^{M/2} \ .$$

Here we evaluate the integral only up to the scale q which enters the horizon at the later of the two times. All scales $q < 1/\max(t',t'')$ are super-horizon from t_{in} to $\max(t',t'')$. A soon as q enters the horizon, the gravity wave transfer function begins to oscillate and the contribution to the integral becomes negligible. The integral over μ can be evaluated; for $M \neq -2$ it yields

$$I(k) = \frac{8\pi}{2+M}\Sigma^2_{(1)\,\mathrm{in}}B^2_{(1)\,\mathrm{in}}\lambda^{A+M+6} \times \int_0^{1/\max(t',t'')}\frac{\mathrm{d}q\ q^{A+2}}{kq}\left(|k+q|^{M+2}-|k-q|^{M+2}\right) \ .$$

We shall not treat the case $M = -2$, where the angular integral introduces a logarithmic dependence on q, separately. This corresponds to approximating $\log(k/q) \sim 1$. We approximate these integrals by their dominant contribution.

- If the spectra are sufficiently red such that $A + M + 3 < 0$, the result is dominated by the region $k < 1/\max(t',t'')$ and we obtain

$$I(k) \simeq 16\pi\Sigma^2_{(1)\,\mathrm{in}}B^2_{(1)\,\mathrm{in}}\lambda^3 \times (\lambda k)^{A+M+3}\left(\frac{1}{A+3}-\frac{1}{A+M+3}\right) \ .$$

- On the other hand, if the spectra are blue such that $A+M+3 > 0$, the integral is dominated by its value at the upper boundary,

$$I \simeq 16\pi\Sigma^2_{(1)\,\mathrm{in}}B^2_{(1)\,\mathrm{in}}\lambda^3\frac{1}{A+M+3} \times \left[\frac{\lambda}{\max(t',t'')}\right]^{A+M+3} \ .$$

If, as in the previous sub-section, we can write the function $h(t',t'',k)$ in the form

$$h(t',t'',k) \simeq F(k)g(t')g(t'') \ , \qquad (75)$$

we can proceed as we did before to obtain the results (58) and (61). A source where the time dependence of the unequal time correlator factorizes is called “totally coherent”. In the totally coherent case, the power spectrum is simply the square of the solution which has as its source the square root of the power spectrum of the source [23]. In most cases, the unequal time correlator is more complicated than this, but the totally coherent approximation is often quite reasonable [23]. If the source is totally coherent, the square root of the power spectrum $\mathcal{P}^{(2)}_{\mathcal{B}}$ simply satisfies the same evolution equation as $\mathcal{B}_{(2)}$ with source term $\sqrt{F}g$.

- If $A + M + 3 < 0$, we can write

$$F(k) = \frac{128\pi\kappa}{(2\pi)^3}(k\lambda)^{A+M+3}\lambda^3 \times \left(\frac{1}{A+3}-\frac{1}{A+M+3}\right) \ ,$$
$$g(t') = \frac{B_{(1)\,\mathrm{in}}\Sigma_{(1)\,\mathrm{in}}}{a^2_{\mathrm{in}}}\mathcal{H}(t')a^2(t') \ .$$

- For $A + M + 3 > 0$, we set

$$F(k) = \frac{128\pi\kappa}{(2\pi)^3}\frac{1}{A+M+3}\lambda^{A+M+6} \ ,$$
$$g(t') = \frac{B_{(1)\,\mathrm{in}}\Sigma_{(1)\,\mathrm{in}}}{a^2_{\mathrm{in}}}\mathcal{H}(t')a^2(t')\left(\frac{1}{t'}\right)^{(A+M+3)/2} \ .$$

  This corresponds to replacing

$$\left[\frac{1}{\max(t',t'')}\right]^{(A+M+3)} \quad \text{by} \quad \left(\frac{1}{t't''}\right)^{(A+M+3)/2}$$

  which is of course not entirely correct and we expect this to over estimate the true result somewhat. However, within the accuracy of our approximations this is sufficient. To obtain a more accurate result we would have to expand the function $h(k,t',t'')$ in eigenfunctions with respect to convolution in time, as it is done in Ref. [23].

Within this totally coherent approximation we can now solve the problem like in the previous sub-section. In the case $A+M+3 < 0$ we find on super-horizon scales, where the source is active

$$k^3\mathcal{P}^{(2)}_{\mathcal{B}}(k,t) \simeq \frac{32\pi\kappa}{(2\pi)^3}\frac{\left[k^3\mathcal{P}^{(1)}_{B\,\mathrm{in}}(k)\right]}{H^2_{\mathrm{in}}}\left[k^3\mathcal{P}^{(1)}_{\Sigma\,\mathrm{in}}(k)\right] \times \left(\frac{a}{a_{\mathrm{in}}}\right)^6 \ , \quad kt \ll 1 \ . \qquad (76)$$

On sub-horizon scales, performing the matching at horizon crossing, we obtain

$$k^3\mathcal{P}^{(2)}_{\mathcal{B}}(k,t) \simeq \frac{16\pi\kappa}{(2\pi)^3}\frac{\left[k^3\mathcal{P}^{(1)}_{B\,\mathrm{in}}(k)\right]}{H^2_{\mathrm{in}}}\left[k^3\mathcal{P}^{(1)}_{\Sigma\,\mathrm{in}}(k)\right] \times \left(\frac{a}{a_{\mathrm{in}}}\right)^2\frac{1}{(kt_{\mathrm{in}})^4} \ , \quad kt \gg 1 \ . \qquad (77)$$

If $A + M + 3 > 0$, we analyze in in more detail only the case $A \simeq -3$ and $M = 2$. The spectral index $A = -3$ correspond to a scale invariant gravity wave power spectrum as it is obtained in slow-roll inflation [24]. The index $M = 2$ characterizes a causal magnetic field $B_{(1)}$. In this case, we have to solve the differential equation,

$$P'' - \frac{2}{z}P' + \left(1+\frac{2}{z^2}\right)P = \frac{\alpha}{k^2} \ , \qquad (78)$$
$$\alpha \equiv (a_{\mathrm{in}}H_{\mathrm{in}})^2B^{(1)}_{\mathrm{in}}\Sigma^{(1)}_{\mathrm{in}}\sqrt{\frac{64\pi}{(2\pi)^3}\kappa\lambda^5} \ ,$$

where the source is constant in time. Detailed comments about the initial conditions chosen for the solution of the above equation can be found in Appendix A. Finally, we can write the solution for $P(z)$ in the case where $z = kt \ll 1$ as

$$P(z) \simeq \frac{\alpha}{k^2} z^2 \log\left(\frac{z}{z_{\rm in}}\right) , \qquad z \ll 1 .$$

The power spectrum of $\mathcal{B}^{(2)}$ on super-horizon scales is therefore given by

$$k^3 \mathcal{P}_{\mathcal{B}}^{(2)}(k,t) \simeq \frac{16\pi\kappa}{(2\pi)^3} \frac{\left[k^3 \mathcal{P}_{B\,\rm in}^{(1)}(k)\right]}{H_{\rm in}^2} \left[k^3 \mathcal{P}_{\Sigma\,\rm in}^{(1)}(k)\right] \times \left(\frac{a}{a_{\rm in}}\right)^4 \frac{1}{(kt_{\rm in})^2} \log^2\left(\frac{a}{a_{\rm in}}\right) , \qquad kt \ll 1 . \quad (79)$$

On sub-horizon scales, $z = kt \gg 1$, we match the super-horizon solution at horizon crossing with the homogeneous solution of Eq. (78), as we did above, obtaining

$$k^3 \mathcal{P}_{\mathcal{B}}^{(2)}(k,t) \simeq \frac{32\pi\kappa}{(2\pi)^3} \frac{\left[k^3 \mathcal{P}_{B\,\rm in}^{(1)}(k)\right]}{H_{\rm in}^2} \left[k^3 \mathcal{P}_{\Sigma\,\rm in}^{(1)}(k)\right] \times \left(\frac{a}{a_{\rm in}}\right)^2 \frac{1}{(kt_{\rm in})^4} \log^2(kt_{\rm in}) , \qquad kt \gg 1 . \quad (80)$$

1. Density parameter

Using Eq. (62), we find the following expressions for the energy density of the stochastic second order magnetic field. If $A + M + 3 < 0$, we have on super-horizon scales

$$\frac{{\rm d}\rho_B^{(2)}(k,t)}{{\rm d}\log k} \equiv \frac{1}{(2\pi)^3} k^3 \mathcal{P}_B^{(2)}(k,\eta) \left(\frac{a_{\rm in}}{a}\right)^2 \simeq \frac{288\pi}{(2\pi)^6} \left[k^3 \mathcal{P}_{B\,\rm in}^{(1)}(k)\right] \left[k^3 \mathcal{P}_{\Sigma\,\rm in}^{(1)}(k)\right] . \quad (81)$$

This results in a density parameter for $B_{(2)}$ given by

$$\frac{{\rm d}\Omega_B^{(2)}(k,t)}{{\rm d}\log k} \equiv \frac{1}{\rho_c} \frac{d\rho_B^{(2)}(k,\eta)}{d\log k} \simeq 6 \frac{{\rm d}\Omega_{B\,\rm in}^{(1)}(k)}{{\rm d}\log k} \frac{{\rm d}\Omega_{GW}^{(1)}(k,t)}{{\rm d}\log k} , \quad kt \ll 1 . \quad (82)$$

Inside the horizon we obtain for the second order magnetic field density parameter

$$\frac{{\rm d}\Omega_B^{(2)}(k,t)}{{\rm d}\log k} \simeq 6 \frac{{\rm d}\Omega_{B\,\rm in}^{(1)}(k)}{{\rm d}\log k} \frac{d\Omega_{GW}^{(1)}(k)}{d\log k} , \quad kt \gg 1 . \quad (83)$$

The gravity wave density parameter, $\left[{\rm d}\Omega_{GW}^{(1)}(k,t)/{\rm d}\log k\right]$ is given by Eqs. (37) and (40) respectively. This corresponds, as in the previous section for a constant magnetic field, to the naively expected result, $\Omega_B^{(2)} \sim \Omega_{\rm GW}^{(1)} \Omega_B^{(1)}$.

For blue spectra, $A + M + 3 > 0$, the second order magnetic field density parameter reads in the interesting case $A \simeq -3$ and $M = 2$ on super-horizon scales

$$\frac{{\rm d}\Omega_B^{(2)}(k,t)}{{\rm d}\log k} = \frac{12}{(kt)^2} \frac{{\rm d}\Omega_{B\,\rm in}^{(1)}(k)}{{\rm d}\log k} \frac{{\rm d}\Omega_{\rm GW}^{(1)}(k,t)}{{\rm d}\log k} \log^2\left(\frac{a}{a_{\rm in}}\right) = 12 \frac{{\rm d}\Omega_{\rm GW}^{(1)}(k,t)}{{\rm d}\log k} \left.\frac{{\rm d}\Omega_{B\,\rm in}^{(1)}(k)}{{\rm d}\log k}\right|_{k=1/t} (kt)^3 \times \log^2\left(\frac{a}{a_{\rm in}}\right) , \quad kt \ll 1 . \quad (84)$$

Note that the value of $\left[{\rm d}\Omega_B^{(2)}(k,t)/{\rm d}\log k\right]$ on super-Hubble scales is affected by $\left[{\rm d}\Omega_B^{(1)}(k_t)/{\rm d}\log k_t\right]$ at horizon crossing, $k_t = 1/t$ which may well be larger than $\left[{\rm d}\Omega_B^{(1)}(k)/{\rm d}\log k\right]$ but of course has also to be much smaller than 1.

This expression grows only logarithmically faster than $\left[{\rm d}\Omega_{GW}^{(1)}(k,t)/{\rm d}\log k\right]$. The growth stops at horizon entry where the second order magnetic field density parameter has acquired a factor $\log^2(kt_{\rm in})$. Inside the horizon we obtain a density parameter of

$$\frac{{\rm d}\Omega_B^{(2)}(k,t)}{{\rm d}\log k} = 12 \frac{{\rm d}\Omega_{B\,\rm in}^{(1)}(k)}{{\rm d}\log k} \frac{{\rm d}\Omega_{\rm GW}^{(1)}(k)}{{\rm d}\log k} \log^2(kt_{\rm in}) , \quad kt \gg 1 . \quad (85)$$

Up to the logarithmic correction, this corresponds to the result for red spectra above.

2. Reheating and matter dominated epochs

In order to make contact with Refs. [2, 15], we now repeat the calculation in a matter dominated background ($w = 0$). We want to point out that the results we obtain are mathematically the same as the ones found in [15]. The only difference lies in the interpretation. In the previous paragraph we have seen that, even though

$$\frac{{\rm d}\Omega_{GW}^{(1)}(k)}{{\rm d}\log k} \sim \left(\frac{1}{kt_{\rm in}}\right)^4 \left[k^3 \mathcal{P}_{\Sigma\,\rm in}^{(1)}(k)\right] ,$$

and even though $(kt_{\rm in})^{-4}$ can become very large, this product is never larger than about 10^{-10}. We believe that this point has been missed in Ref. [15].
If $w = 0$, the scale factor grows like $a \propto t^2$ so that $\mathcal{H} = 2/t$. As mentioned before, for the super horizon amplification the question whether the conductivity is high or low is not relevant.

¿From the first order perturbations, we obtain the same behaviour for the magnetic field $B^{(1)}$ in terms of the scale factor, therefore the density parameter is then given by

$$\frac{\mathrm{d}\Omega_B^{(1)}(k,t)}{\mathrm{d}\log k} = \frac{8\pi G}{3(2\pi)^3}\frac{\left[k^3\mathcal{P}_{B\,\mathrm{in}}^{(1)}(k)\right]}{H_{\mathrm{in}}^2}\frac{a_{\mathrm{in}}}{a}\ . \quad (86)$$

The first order gravity waves on super-horizon scales now behaves as

$$\Sigma_{ij}^{(1)}(\mathbf{k},t) = \Sigma_{ij\,\mathrm{in}}^{(1)}(\mathbf{k})\left(\frac{a}{a_{\mathrm{in}}}\right)^3\ . \quad (87)$$

Once the gravitational waves enter the horizon, they start oscillating and the energy density decays as radiation. Therefore in this case the relative density parameters for the first order gravity waves is

$$\frac{\mathrm{d}\Omega_{\mathrm{GW}}^{(1)}(k,t)}{\mathrm{d}\log k} = \frac{48\pi}{(2\pi)^3}\left[k^3\mathcal{P}_{\Sigma\,\mathrm{in}}^{(1)}(k)\right]\left(\frac{a}{a_{\mathrm{in}}}\right)^2\ , \quad kt\ll 1\ . \quad (88)$$

On sub-horizon scales we obtain

$$\frac{\mathrm{d}\Omega_{\mathrm{GW}}^{(1)}(k,t)}{\mathrm{d}\log k} = \frac{24\pi}{(2\pi)^3}\left[k^3\mathcal{P}_{\Sigma\,\mathrm{in}}^{(1)}(k)\right]\left(\frac{a_{\mathrm{in}}}{a}\right)\frac{1}{(kt_{\mathrm{in}})^6}\ , \quad kt\gg 1\ . \quad (89)$$

Computing finally the induced second order magnetic field density parameter, we obtain the naively expected result on super-horizon scales

$$\frac{\mathrm{d}\Omega_B^{(2)}(k,t)}{\mathrm{d}\log k} \simeq \begin{cases} \frac{\mathrm{d}\Omega_B^{(1)}(k,t)}{\mathrm{d}\log k}\frac{\mathrm{d}\Omega_{\mathrm{GW}}^{(1)}(k,t)}{\mathrm{d}\log k}\ , & \text{for } A+M+3<0 \\ (kt)^3\left[\frac{\mathrm{d}\Omega_B^{(1)}(k,t)}{\mathrm{d}\log k}\right]_{k=1/t}\times \frac{\mathrm{d}\Omega_{\mathrm{GW}}^{(1)}(k,t)}{\mathrm{d}\log k}\log^2\frac{a}{a_{\mathrm{in}}}\ , & \text{for } A+M+3>0 \end{cases} \quad kt\ll 1\ . \quad (90)$$

On sub-horizon scales the density parameter turns out to be given by

$$\frac{\mathrm{d}\Omega_B^{(2)}(k,t)}{\mathrm{d}\log k} \simeq \frac{\mathrm{d}\Omega_B^{(1)}(k,t)}{\mathrm{d}\log k}\frac{\mathrm{d}\Omega_{\mathrm{GW}}^{(1)}(k,t_k)}{\mathrm{d}\log k} \simeq \frac{\mathrm{d}\Omega_B^{(1)}(k,t)}{\mathrm{d}\log k}\left(\frac{H_{\mathrm{inf}}}{M_{\mathrm{P}}}\right)^2,\ kt\gg 1\ , \quad (91)$$

for both cases $A+M+3<0$ and $A\simeq -3$, $M=2$, up to logarithmic corrections. Here t_k stands for the horizon crossing time, $t_k=1/k$, and in the last $\simeq$ sign we have used that $\left[\mathrm{d}\Omega_{\mathrm{GW}}^{(1)}(k,t_k)/\mathrm{d}\log k\right]\simeq (H_{\mathrm{inf}}/M_{\mathrm{P}})^2$ is the gravity waves density parameter at horizon crossing, which is smaller than 10^{-10}. This means that the second order magnetic field does not grow larger the the first order one. Inside the horizon they decrease both like $\propto a^{-1}$. $\Omega_B^{(2)}$ stays always much smaller than $\Omega_B^{(1)}$, as we have found in the case of a radiation dominated background.

C. Second order gravity waves

Starting from Eq. (46), we can write the evolution equation for $\sigma_{ij}^{(2)}$ in real space $(\mathbf{x},t)$ as follows:

$$\ddot{\sigma}_{ij}^{(2)} - a^2D^2\sigma_{ij}^{(2)} - \frac{3}{2}\mathcal{H}^2(1+w)\sigma_{ij}^{(2)} = -2\kappa a\mathcal{H}\Pi_{ij}^{(1)} + \left[a\mathcal{H}\sigma_{\langle i(1)}{}^{n}\sigma_{j\rangle n}^{(1)} + 2a\sigma_{\langle i(1)}{}^{n}\dot{\sigma}_{j\rangle n}^{(1)} - a\dot{\sigma}_{\langle i(1)}^{n}\sigma_{j\rangle n}^{(1)}\right]\frac{1}{a^2}\ . \quad (92)$$

The factor $1/a^2$ in the source part of the above equation comes from the fact that in Eq. (46) we had to add factors $a^2(t)$ in order to lower or rise indices. On the other hand, now we deal with purely spatial tensors such that $\sigma_{ij}=\sigma^{ij}$ and also $\dot{\sigma}_{ij}=\dot{\sigma}^{ij}$.

Introducing again the dimensionless expansion-normalized variable $\Sigma_{ij}^{(2)}$, the previous equation can be written as

$$\ddot{\Sigma}_{ij}^{(2)} - 3(1+w)\mathcal{H}\dot{\Sigma}_{ij}^{(2)} + 3\mathcal{H}^2\left(\frac{3}{2}w^2+2w+\frac{1}{2}\right)\Sigma_{ij}^{(2)} - a^2D^2\Sigma_{ij}^{(2)} = -\frac{2}{3}\kappa\frac{a^2}{a_{\mathrm{in}}^2}\Pi_{ij}^{(1)} + \left[-\frac{3}{2}(1+3w)\mathcal{H}^2\Sigma_{\langle i(1)}{}^{n}\Sigma_{j\rangle n}^{(1)} + 6\mathcal{H}\Sigma_{\langle i(1)}{}^{n}\dot{\Sigma}_{j\rangle n}^{(1)} - 3\mathcal{H}\dot{\Sigma}_{\langle i(1)}^{n}\Sigma_{j\rangle n}^{(1)}\right]\left(\frac{a_{\mathrm{in}}}{a}\right)^2\ . \quad (93)$$

As for $B^{(2)}$, the source is given by the first order perturbations magnetic field $[\Pi_{ij}^{(1)}]$ and the first order gravity waves and does *e.g.* not couple to the second order magnetic field. Since we assume the first order magnetic field and gravity wave fluctuations to be independent, we can add the power spectra for the solutions of the individual source terms,

$$k^3\mathcal{P}_\Sigma^{(2)}(k,t) = k^3\mathcal{P}_\Sigma^{(2)\,\Pi}(k,t) + k^3\mathcal{P}_\Sigma^{(2)\,\mathrm{GW}}(k,t)\ .$$

where $\mathcal{P}_\Sigma^{(2)\,\Pi}(k,t)$ is the power spectrum of the solution of Eq. (93) with source term $\Pi^{(1)}$ only and $\mathcal{P}_\Sigma^{(2)\,\mathrm{GW}}(k,t)$ comes from the source terms containing $\Sigma^{(1)}$.

1. Magnetic field part of the source $\left[k^3\mathcal{P}_\Sigma^{(2)\,\Pi}(k,t)\right]$

Considering first the magnetic field part of the source, we have to solve the following differential equation in the momentum space $(\mathbf{k},t)$

$$\ddot{\Sigma}_{ij}^{(2)} - 3(1+w)\mathcal{H}\dot{\Sigma}_{ij}^{(2)} + \left[k^2 + 3\mathcal{H}^2\left(\frac{3}{2}w^2+2w+\frac{1}{2}\right)\right]\Sigma_{ij}^{(2)} = f_{ij}\ , \quad (94)$$

where the source is given by

$$f_{ij}(\mathbf{k},t) \equiv -\frac{2}{3}\kappa\frac{a^2}{a_{\rm in}^2}\Pi_{ij}^{(1)}(\mathbf{k},t)\ . \qquad (95)$$

As before, we have to compute the unequal time correlator:

$$\langle \Pi_{ij}^{(1)}(\mathbf{k},t')\Pi_{rn}^{*(1)}(\mathbf{p},t'')\rangle = (2\pi)^3\delta^3(\mathbf{k}-\mathbf{p})\mathcal{M}_{ijrn}(\hat{\mathbf{k}})h(k,t',t'')\ , \qquad (96)$$

where the anisotropic stresses are given by

$$\Pi_{ij}^{(1)}(\mathbf{k},t') = -\frac{1}{16\pi(2\pi)^3}\mathcal{M}_{ij}{}^{ls}(\hat{\mathbf{k}})\int d^3q B_l^{(1)}(\mathbf{q},t')\times B_s^{(1)}(\mathbf{k}-\mathbf{q},t')\quad .$$

$(1/2)\mathcal{M}_{ij}{}^{ls}(\hat{\mathbf{k}})$ is the projector on the tensor modes. We have neglected a trace contribution to the magnetic field stress tensor since, once we project with $\mathcal{M}_{ij}{}^{ls}$, the trace vanishes.

After some computation [1], we find for the function $h(k,t',t'')$ the following expression:

$$h(k,t',t'') = \frac{1}{(8\pi)^2}\frac{1}{4(2\pi)^3}I(k)\left[\frac{a_{\rm in}^2}{a(t')a(t'')}\right]^2, \qquad (97)$$

$$I(k) = \int d^3q(1+\gamma^2)(1+\alpha^2)\mathcal{P}_{B\,{\rm in}}^{(1)}(q)\times \mathcal{P}_{B\,{\rm in}}^{(1)}(|\mathbf{k}-\mathbf{q}|)\ . \qquad (98)$$

where $\alpha \equiv \hat{k}\cdot\widehat{(k-q)}$ and $\gamma \equiv \hat{k}\cdot\hat{q}$. As before, we approximate $(1+\gamma^2)(1+\alpha^2)\simeq 1$. With this, we obtain the following expression for the expectation value of the source term:

$$\langle f_{ij}(\mathbf{k},t')f_{rn}^*(\mathbf{p},t'')\rangle = \frac{4}{9}\kappa^2\frac{a^2(t')a^2(t'')}{a_{\rm in}^4}\times \langle \Pi_{ij}^{(1)}(\mathbf{k},t')\Pi_{rn}^{*(1)}(\mathbf{p},t'')\rangle\ . \qquad (99)$$

The expectation value of the stochastic variable $\Sigma_{ij}^{(2)}$ can be written as

$$\langle \Sigma_{ij}^{(2)}(\mathbf{k},t)\Sigma_{rn}^{*(2)}(\mathbf{p},t)\rangle = (2\pi)^3\delta^3(\mathbf{k}-\mathbf{p})\mathcal{M}_{ijrn}(\hat{\mathbf{k}})\mathcal{P}_{\Sigma}^{(2)}(k,t)\ . \qquad (100)$$

If $\langle \Pi_{ij}^{(1)}(\mathbf{k},t')\Pi_{rn}^{*(1)}(\mathbf{p},t'')\rangle$ can be written as a product of a function of (k,t) and (k,t''), this source is totally coherent and we can write the function $h(k,t',t'')$ of Eq. (97) in the form

$$\frac{4}{9}\kappa^2\frac{a^2(t')a^2(t'')}{a_{\rm in}^4}h(k,t',t'') = F(k)g(t')g(t'')\ ,$$

where we introduced the pre-factor of h since we finally need an expression for the unequal time correlator of the source, as in Eq. (99), while the function h alone is only part of the correlator of the anisotropic stress, Eq. (96). The square root of the power spectrum is then a solution of the differential equation (94) with source term $\sqrt{F(k)}g(t)$. Written as differential equation for the variable $z=kt$ and setting $w=1/3$, this becomes

$$\left[\sqrt{\mathcal{P}_{\Sigma}^{(2)\,\Pi}(k,z)}\right]'' - \frac{4}{z}\left[\sqrt{\mathcal{P}_{\Sigma}^{(2)\,\Pi}(k,z)}\right]' + \left(1+\frac{4}{z^2}\right)\left[\sqrt{\mathcal{P}_{\Sigma}^{(2)\,\Pi}(k,z)}\right] = \sqrt{F(k)}\frac{g(z/k)}{k^2} \qquad (101)$$

As for the second order magnetic field, we distinguish between two cases. First we consider $2M+3>0$. The integral I is then dominated by the upper cutoff. The magnetic field is not oscillating and we therefore take damping scale k_d as the upper cutoff. We neglect the slow time dependence of this scale. Using Eq. (65) for the magnetic field power spectrum, I can be approximated by

$$I \simeq \frac{8\pi}{2M+3}\left[B_{\rm in}^{(1)4}\lambda^3\right](\lambda k_d)^{2M+3}\ .$$

Hence the functions $F(k)$, $g(t')$ are given by

$$F(k) = \frac{\kappa^2}{36(2\pi)^4}\frac{1}{2M+3}(\lambda k_d)^{2M+3}\left[B_{\rm in}^{(1)4}\lambda^3\right]\ ,$$
$$g(t') = 1\ .$$

In the case $2M+3<0$, we obtain

$$I \simeq 8\pi\left[B_{\rm in}^{(1)4}\lambda^3\right](\lambda k)^{2M+3}\left(\frac{1}{M+3}-\frac{1}{2M+3}\right)\ .$$

This case is totally coherent and we can set

$$F(k) = \frac{\kappa^2}{36(2\pi)^4}\left[B_{\rm in}^{(1)4}\lambda^3\right]\times \left(\frac{1}{M+3}-\frac{1}{2M+3}\right)(\lambda k)^{2M+3}\ ,$$
$$g(t') = 1\,.$$

We now solve Eq. (101) for the two different source terms.

- In the case $2M+3>0$, we can write Eq. (101) in the form

$$P'' - \frac{4}{z}P' + \left(1+\frac{4}{z^2}\right)P = \frac{\alpha}{k^2}\ ,$$
$$z\equiv kt\ ,\qquad |P|\equiv\sqrt{\mathcal{P}_{\Sigma}^{(2)\,\Pi}(k,t)}\ ,$$
$$\alpha\equiv\sqrt{F(k)}\ .$$

Solving the above equation on super-horizon scales and following the considerations for the choice of initial conditions explained in Appendix A, we find

$$P(z)\simeq -\frac{\alpha}{2k^2}z^2\ ,\qquad z\ll 1\ ,$$

this gives the second order power spectrum

$$k^3\mathcal{P}_\Sigma^{(2)\,\Pi}(k,t) \simeq \frac{\kappa^2}{36(2\pi)^4(2M+3)}\left[\frac{k^3\mathcal{P}_{B\,\rm in}^{(1)}(k)}{H_{\rm in}^2}\right]^2 \times \left(\frac{a}{a_{\rm in}}\right)^4\left(\frac{k_d}{k}\right)^{2M+3} , \quad kt \ll 1 \ . \quad (102)$$

This is equivalent to a density parameter for $\Sigma^{(2)}$ given by

$$\frac{{\rm d}\Omega_{\rm GW}^{(2)\,\Pi}(k,t)}{{\rm d}\log k} \simeq \left[\frac{{\rm d}\Omega_{B\,\rm in}^{(1)}(k)}{{\rm d}\log k}\right]^2\left(\frac{k_d}{k}\right)^{2M+3} \simeq \left[\frac{{\rm d}\Omega_{B\,\rm in}^{(1)}(k_d)}{{\rm d}\log k}\right]^2\left(\frac{k}{k_d}\right)^3 , \quad kt \ll 1 \ . \quad (103)$$

Inside the horizon, the Green function oscillates and we can neglect the contribution from the source. The solution for the power spectrum is then given by

$$k^3\mathcal{P}_\Sigma^{(2)\,\Pi}(k,t) \simeq \frac{\kappa^2}{36(2\pi)^4(2M+3)} \times \left[\frac{k^3\mathcal{P}_{B\,\rm in}^{(1)}(k)}{H_{\rm in}^2}\right]^2\left(\frac{a}{a_{\rm in}}\right)^4\left(\frac{k_d}{k}\right)^{2M+3} , \quad kt \gg 1 \ . \quad (104)$$

Therefore, the second order density parameter is given by the same expression,

$$\frac{{\rm d}\Omega_{\rm GW}^{(2)\,\Pi}(k,t)}{{\rm d}\log k} \simeq \left[\frac{{\rm d}\Omega_{B\,\rm in}^{(1)}(k_d)}{{\rm d}\log k}\right]^2\left(\frac{k}{k_d}\right)^3 , \quad kt \gg 1 \ . \quad (105)$$

Up to logarithmic factors this result agrees with the findings of Ref. [1].

- In the case $2M+3<0$ we have again to solve the equation

$$P'' - \frac{4}{z}P' + \left(1+\frac{4}{z^2}\right)P = \frac{\alpha}{k^2} , \quad (106)$$

Hence

$$P(z) \simeq -\frac{\alpha}{2k^2}z^2 , \quad z \ll 1 \, .$$

But now

$$\alpha \equiv \frac{\kappa}{6(2\pi)^2}\sqrt{\frac{1}{2}\left(\frac{1}{M+3}-\frac{1}{2M+3}\right)}\,k^{2M+3} \times \left[B_{\rm in}^{(1)2}\lambda^3\right]\lambda^M ,$$

so that

$$k^3\mathcal{P}_\Sigma^{(2)\,\Pi}(k,t) \simeq \frac{\kappa^2}{144(2\pi)^4}\left[\frac{k^3\mathcal{P}_{B\,\rm in}^{(1)}(k)}{H_{\rm in}^2}\right]^2 \times \left(\frac{a}{a_{\rm in}}\right)^4 , \quad kt \ll 1 \ . \quad (107)$$

As in the first case, the density parameter is the same for $kt<1$ and $kt>1$,

$$\frac{d\Omega_{\rm GW}^{(2)\,\Pi}(k,t)}{d\log k} \simeq \left[\frac{d\Omega_{B\,\rm in}^{(1)}(k)}{d\log k}\right]^2 . \quad (108)$$

2. *Gravity waves part of the source* $\left[k^3\mathcal{P}_\Sigma^{(2)\,\rm GW}(k,t)\right]$

Let us finally consider the part of the source given by first order gravity waves. In this case, we can write the source f_{ij} as:

$$f_{ij}(\mathbf{x},t) = \Big[-\frac{3}{2}(1+3w)\mathcal{H}^2\Sigma^{(1)\,n}_{\langle i}\Sigma^{(1)}_{j\rangle n} + 6\mathcal{H}\Sigma^{(1)\,n}_{\langle i}\dot\Sigma^{(1)}_{j\rangle n} - 3\mathcal{H}\dot\Sigma^{(1)n}_{\langle i}\Sigma^{(1)}_{j\rangle n}\Big]\left(\frac{a_{\rm in}}{a}\right)^2 . (109)$$

As before, we ignore the traces that are present in the above products, once we evaluate them in the momentum space, since we project them out with $(1/2)\mathcal{M}_{ij}{}^{lm}$ afterwards. Remembering that $\Sigma_{ij}=\Sigma^{ij}$, we have on super-horizon scales, where the transfer function is given by Eq. (67):

$$\left[\Sigma_{\langle i(1)}{}^n\Sigma^{(1)}_{j\rangle n}\right](\mathbf{k},t) = \frac{1}{2(2\pi)^3}\left(\frac{a}{a_{\rm in}}\right)^8 \times \mathcal{M}_{ij}{}^{lm}(\hat{\mathbf{k}})\int {\rm d}^3q\,\Sigma^{(1)\,\rm in}_{ln}(\mathbf{q})\Sigma^{(1)\,\rm in}_{nm}(\mathbf{k}-\mathbf{q}) ,$$

$$\left[\Sigma_{\langle i(1)}{}^n\dot\Sigma^{(1)}_{j\rangle n}\right](\mathbf{k},t) = \left[\dot\Sigma^n_{\langle i(1)}\Sigma^{(1)}_{j\rangle n}\right](\mathbf{k},t) = \frac{2\mathcal{H}}{(2\pi)^3}\left(\frac{a}{a_{\rm in}}\right)^8\mathcal{M}_{ij}{}^{lm}(\hat{\mathbf{k}}) \times \int {\rm d}^3q\,\Sigma^{(1)\,\rm in}_{ln}(\mathbf{q})\Sigma^{(1)\,\rm in}_{nm}(\mathbf{k}-\mathbf{q}) \, .$$

These equations are strictly true only on super-horizon scales where $\Sigma\propto 1/a^4$. However, since inside the horizon Σ oscillates and the contribution from the source is negligible, we can use this approximation. Setting $w=1/3$ we can finally write the source in the form

$$f_{ij}(\mathbf{k},t) = \frac{9}{2(2\pi)^3}\mathcal{H}^2\left(\frac{a}{a_{\rm in}}\right)^6 \times \mathcal{M}_{ij}{}^{lm}(\hat{\mathbf{k}})\int {\rm d}^3p\,\Sigma^{(1)\,\rm in}_{ln}(\mathbf{p})\Sigma^{(1)\,\rm in}_{nm}(\mathbf{k}-\mathbf{p}) , \quad (110)$$

and the two-point correlation function of the source part reads:

$$\langle f_{ij}(\mathbf{k},t')f^*_{rc}(\mathbf{q},t'')\rangle = (2\pi)^3\delta^3(\mathbf{k}-\mathbf{q})\mathcal{M}_{ijrc}(\hat{\mathbf{k}}) \times h(k,t',t'') ,$$
$$h(k,t',t'') = \frac{1}{8(2\pi)^3}U(t',t'')I(k) ,$$
$$U(t',t'') = \frac{81}{4}\mathcal{H}^2(t')\mathcal{H}^2(t'')\left[\frac{a(t')}{a_{\rm in}}\right]^6\left[\frac{a(t'')}{a_{\rm in}}\right]^6 ,$$
$$I(k) = \mathcal{M}_{bdlm}(\hat{\mathbf{k}})\int \mathrm{d}^3p\left[\mathcal{M}_{lnbf}(\hat{\mathbf{p}})\mathcal{M}_{mndf}(\widehat{\mathbf{k}-\mathbf{p}})+ \mathcal{M}_{lndf}(\hat{\mathbf{p}})\mathcal{M}_{mndf}(\widehat{\mathbf{k}-\mathbf{p}})\right]\mathcal{P}^{(1)}_{\Sigma\,\rm in}(p)\mathcal{P}^{(1)}_{\Sigma\,\rm in}(|\mathbf{k}-\mathbf{p}|) .$$

More details about the computation of $h(k,t't'')$ and of the four point correlation function of the gravity waves can be found in Appendix B.

Using the tensor calculus package "xAct" for Mathematica [25], we can compute the above products of the three projectors,

$$\mathcal{M}_{bdlm}(\hat{\mathbf{k}})\left[\mathcal{M}_{lnbf}(\hat{\mathbf{p}})\mathcal{M}_{mndf}(\widehat{\mathbf{k}-\mathbf{p}})+ \mathcal{M}_{lndf}(\hat{\mathbf{p}})\mathcal{M}_{mnbf}(\widehat{\mathbf{k}-\mathbf{p}})\right] = 2(1+\alpha^2+\beta^2+\alpha^2\beta^2-8\alpha\beta\gamma+ \gamma^2+\alpha^2\gamma^2+\beta^2\gamma^2+\alpha^2\beta^2\gamma^2)\simeq 2 . \tag{111}$$

where $\alpha\equiv\hat{k}\cdot(\widehat{k-p})$, $\beta\equiv\hat{p}\cdot(\widehat{k-p})$ and $\gamma\equiv\hat{k}\cdot\hat{p}$. Again we have approximated this angular dependence by a constant to simplify the calculations. This approximation is well justified within our accuracy. In order to write the function $h(k,t',t'')\simeq F(k)g(t')g(t'')$, we have to evaluate the integral I as before. We first consider the most interesting case of a scale invariant spectrum, $A\simeq -3$. Up to an infrared log-divergence which we neglect as usual (this divergence can be avoided if we choose $A=-2.99$ instead of $A=-3$), we have

$$F(k)\simeq\frac{81\pi}{2(2\pi)^3}\left[k^3\mathcal{P}^{(1)}_{\Sigma\,\rm in}(k)\right]^2\frac{1}{k^3} ,$$
$$g(t')\simeq\mathcal{H}^2(t')\left[\frac{a(t')}{a_{\rm in}}\right]^6 .$$

Therefore, the equation for $\sqrt{\mathcal{P}^{(2)\,\rm GW}_{\Sigma}(k,t)}$ in the radiation dominated era becomes

$$P''(z)-\frac{4}{z}P'+\left(1+\frac{4}{z^2}\right)P=\frac{\alpha}{k^6}z^4 ,$$
$$z\equiv kt , \qquad |P|\equiv\sqrt{\mathcal{P}^{(2)\,\rm GW}_{\Sigma}(k,t)} ,$$
$$\alpha\equiv\frac{1}{t^6_{\rm in}}\sqrt{\frac{81\pi}{2(2\pi)^3}\frac{\left[k^3\mathcal{P}^{(1)}_{\Sigma\,\rm in}(k)\right]^2}{k^3}} .$$

The super-horizon solution, evaluated always with the help of the Wronskian method and keeping only the non-homogeneous part as explained in the Appendix A, is the given by

$$P(z)\simeq\frac{1}{10}\frac{\alpha}{k^6}z^6 , \qquad z\ll 1 ,$$

that yields a contribution to the gravity wave power spectrum given by

$$k^3\mathcal{P}^{(2)\,\rm GW}_{\Sigma}(k,t) \simeq 0.4\frac{\pi}{(2\pi)^3}\left[k^3\mathcal{P}^{(1)}_{\Sigma\,\rm in}(k)\right]^2\left(\frac{a}{a_{\rm in}}\right)^{12} , \quad kt\ll 1 . \tag{112}$$

For the density parameter on super-horizon scales this yields

$$\frac{\mathrm{d}\Omega^{(2)\,\Sigma}_{\rm GW}(k,t)}{\mathrm{d}\log k} \simeq 0.01\left[\frac{\mathrm{d}\Omega^{(1)}_{\rm GW}(k,t)}{\mathrm{d}\log k}\right]^2 , \quad kt\ll 1 . \tag{113}$$

Considering now the sub-horizon limit, we obtain for the power spectrum the following expression:

$$k^3\mathcal{P}^{(2)\,\rm GW}_{\Sigma}(k,t)\simeq 0.1\frac{\pi}{(2\pi)^3}\left[k^3\mathcal{P}^{(1)}_{\Sigma\,\rm in}(k)\right]^2\times \frac{1}{(kt_{\rm in})^8}\left(\frac{a}{a_{\rm in}}\right)^4 , \quad kt\gg 1 , \tag{114}$$

and the density parameter becomes

$$\frac{\mathrm{d}\Omega^{(2)\,\Sigma}_{\rm GW}(k,t)}{\mathrm{d}\log k}\simeq 0.02\left[\frac{\mathrm{d}\Omega^{(1)}_{\rm GW}(k)}{\mathrm{d}\log k}\right]^2 , \quad kt\gg 1 . \tag{115}$$

On the other hand, when $2A+3>0$ we have

$$F(k)\simeq\frac{81\pi}{2(2\pi)^3}\Sigma^4_{(1)\,\rm in}\lambda^{6+2A}\frac{1}{2A+3} ,$$
$$g(t')\simeq\mathcal{H}^2(t')\left[\frac{a(t')}{a_{\rm in}}\right]^6\left(\frac{1}{t'}\right)^{(2A+3)/2} .$$

In the radiation epoch the equation for $\sqrt{\mathcal{P}^{(2)\,\rm GW}_{\Sigma}(k,t)}$ reads:

$$P''-\frac{4}{z}P'+\left(1+\frac{4}{z^2}\right)P=\alpha z^{(5/2-A)} ,$$
$$\alpha\equiv\sqrt{\frac{81\pi}{2(2\pi)^3}\frac{1}{2A+3}}k^{3/2}\mathcal{P}^{(1)}_{\Sigma\,\rm in}(k)\frac{1}{(kt_{\rm in})^6} .$$

Solving the above equation in the long wavelengths limit, we find:

$$P(z)\simeq\frac{\alpha}{2}z^{9/2-A} , \qquad z\ll 1 ,$$

where the exact pre-factor depends weakly on the value of A. For the power spectrum this results in

$$k^3\mathcal{P}_\Sigma^{(2)\,\mathrm{GW}}(k,t) \simeq \frac{81\pi}{8(2\pi)^3}\left[k^3\mathcal{P}_{\Sigma\,\mathrm{in}}^{(1)}(k)\right]^2\left(\frac{a}{a_{\mathrm{in}}}\right)^{12}\times (kt)^{-2A-3}\,, \qquad kt \ll 1\,. \tag{116}$$

and inside the horizon this reads

$$k^3\mathcal{P}_\Sigma^{(2)\,\mathrm{GW}}(k,t) \simeq \frac{81\pi}{16(2\pi)^3}\left[k^3\mathcal{P}_{\Sigma\,\mathrm{in}}^{(1)}(k)\right]^2\left(\frac{a}{a_{\mathrm{in}}}\right)^{4}\times \frac{1}{(kt_{\mathrm{in}})^8}\,, \qquad kt \gg 1\,. \tag{117}$$

Translating this to the density parameter as above, we obtain

$$\begin{aligned}\frac{\mathrm{d}\Omega_{\mathrm{GW}}^{(2)\,\Sigma}(k,t)}{\mathrm{d}\log k} &\simeq 0.2\left[\frac{\mathrm{d}\Omega_{\mathrm{GW}}^{(1)}(k)}{\mathrm{d}\log k}\right]^2 (kt)^{-2A-3}\,,\\ &\simeq 0.2\left[\left.\frac{\mathrm{d}\Omega_{\mathrm{GW}}^{(1)}(k)}{\mathrm{d}\log k}\right|_{k=1/t}\right]^2 (kt)^3\,,\\ &\quad kt \ll 1\,,\\ \frac{\mathrm{d}\Omega_{\mathrm{GW}}^{(2)\,\Sigma}(k,t)}{\mathrm{d}\log k} &\simeq 0.1\left[\frac{\mathrm{d}\Omega_{\mathrm{GW}}^{(1)}(k)}{\mathrm{d}\log k}\right]^2\,, \quad kt \gg 1\,.\end{aligned} \tag{118}$$

V. SUMMARY AND CONCLUSIONS

In this work we have studied the evolution of stochastic cosmic magnetic fields and gravity waves up to second order in the perturbations. We have especially calculated the density parameters of the generated second order perturbations. We start with density parameters $\left[\mathrm{d}\Omega_{\mathrm{B}}^{(1)}(k,t)/\mathrm{d}\log k\right]$ and $\left[\mathrm{d}\Omega_{\mathrm{GW}}^{(1)}(k,t)/\mathrm{d}\log k\right]$ which are related to the first order magnetic field and gravitational wave power spectra in Section III C. Since tensor perturbations grow on super-horizon scales, the gravity wave density parameter grows on super-Hubble scales and only becomes constant once the perturbations enter the horizon. For perturbation theory to be valid, we have of course to require that these density parameters are much smaller than unity. As we have seen in Section III C, to require that $\left[\mathrm{d}\Omega_{\mathrm{GW}}^{(1)}(k,t)/\mathrm{d}\log k\right]$ is smaller than one also on sub-Hubble scales, is equivalent to

$$\frac{\mathrm{d}\Omega_{\mathrm{GW}}^{(1)}(k,t_{\mathrm{in}})}{\mathrm{d}\log k}\frac{1}{(kt_{\mathrm{in}})^4} \simeq \left(\frac{H_{\mathrm{inf}}}{M_{\mathrm{P}}}\right)^2 \ll 1\,. \tag{119}$$

Here we summarize the new results on the density parameters for second order perturbation on sub-horizon scales. For magnetic fields, we obtain

$$\frac{\mathrm{d}\Omega_{\mathrm{B}}^{(2)}(k)}{\mathrm{d}\log k} \simeq \frac{\mathrm{d}\Omega_{\mathrm{GW}}^{(1)}(k)}{\mathrm{d}\log k}\frac{\mathrm{d}\Omega_{\mathrm{B}}^{(1)}(k)}{\mathrm{d}\log k}\,, \quad tk \gg 1\,, \tag{120}$$

up to numerical constants and logarithms which are beyond the accuracy of our approximation. Hence, it is not correct that the presence of gravity waves resonantly enhances a first order magnetic field. The second order density parameter is quite what we would naively expect and it is much smaller than the first order perturbations as long as the latter are small. Also on super-horizon scales, the second order magnetic field density parameter is always much smaller than the first order one, see Eqs.(82) and (84).

Since the growth comes from super horizon scales, conductivity is not relevant for this result. We have shown that also in a matter dominated background we obtain

$$\begin{aligned}\frac{\mathrm{d}\Omega_{\mathrm{B}}^{(2)}(k)}{\mathrm{d}\log k} &\simeq \left.\frac{\mathrm{d}\Omega_{\mathrm{GW}}^{(1)}(k)}{\mathrm{d}\log k}\right|_{kt=1}\frac{\mathrm{d}\Omega_{\mathrm{B}}^{(1)}(k)}{\mathrm{d}\log k}\\ &\simeq \left(\frac{H_{\mathrm{inf}}}{M_{\mathrm{P}}}\right)^2\frac{\mathrm{d}\Omega_{\mathrm{B}}^{(1)}(k)}{\mathrm{d}\log k} \qquad (121)\\ &\ll \frac{\mathrm{d}\Omega_{\mathrm{B}}^{(1)}(k)}{\mathrm{d}\log k}\,, \qquad (122)\end{aligned}$$

hence no significant amplification.

Second order gravity waves are induced on the one hand by the anisotropic stresses of the first order magnetic fields and on the other hand by the quadratic terms in the evolution equation for σ_{ij} which are, e.g., of the form $\sigma_{im}\dot\sigma_j^m$ and similar expressions. In Section IV C 1 we have shown that the second order contribution from anisotropic stresses on sub-Hubble scales is of the order of

$$\frac{\mathrm{d}\Omega_{\mathrm{GW}}^{(2)\,\Pi}(k,t)}{\mathrm{d}\log k} \simeq \begin{cases}\left[\frac{\mathrm{d}\Omega_B^{(1)}(k_d)}{\mathrm{d}\log k_d}\right]^2\left(\frac{k}{k_d}\right)^3, & \text{if } 2M+3>0\\ \left[\frac{\mathrm{d}\Omega_B^{(1)}(k)}{\mathrm{d}\log k}\right]^2, & \text{if } 2M+3<0\,,\end{cases} \tag{123}$$

both on super- and sub-horizon scales. Note that the above expression is continuous at $2M+3=0$, where both expressions scale like $(k\lambda)^3$ and are independent of k_d. One should point out that we neglected the slow time dependence of the damping scale. Correctly one has to choose the value of the damping scale at horizon crossing, $k_d(t_k)$ with $t_k = 1/k$. Depending on the magnetic field spectrum, the resulting gravity waves come mainly from the small scale magnetic field, if its spectrum is blue $2M+3>0$. In this case the gravity waves power spectrum is always proportional to k^3. In our case of a simple power law magnetic field spectrum, this behavior is maintained for all $k < k_d$. If the magnetic field spectrum is red, $2M+3<0$, gravity waves depend on the field at scale k and their spectrum is the square of the B-field spectrum. In the first case, the non-linearity leads to a 'sweeping' of magnetic field power on small scales to gravitational wave power on larger scales. This can be regarded as an 'inverse cascade' of small scale magnetic field power into large scale gravity waves. But in no case can the gravity wave density parameter become larger than the one of the magnetic field, which has to be much smaller than one, for perturbation theory to be valid.

A similar result was already obtained in Ref. [1]. Contrary to this reference we have no logarithmic build-up of gravity waves. This comes from our different treatment; we directly calculate the shear σ_{ij} and not the tensor perturbation of the metric, h_{ij}. In this way we loose the log term which corresponds to the homogeneous h_{ij} = constant solution on super-horizon scales to which we are not sensitive. However, in our more qualitative work, we do not want to insist on log terms which we neglect in this work also in other places.

The second order gravity wave density parameter induced by first order gravity waves is given by

$$\frac{\mathrm{d}\Omega^{(2)\,\Sigma}_{\rm GW}(k,t)}{\mathrm{d}\log k} \simeq \left[\frac{\mathrm{d}\Omega^{(1)}_{\rm GW}(k,t)}{\mathrm{d}\log k}\right]^2 , \qquad kt \gg 1 \ , \tag{124}$$

on sub-horizon scales.
Adding both contributions we find

$$\frac{\mathrm{d}\Omega^{(2)}_{\rm GW}(k,t)}{\mathrm{d}\log k} \simeq \begin{cases} \left[\frac{\mathrm{d}\Omega^{(1)}_B(k_d)}{\mathrm{d}\log k_d}\right]^2 \left(\frac{k}{k_d}\right)^3 + \left[\frac{\mathrm{d}\Omega^{(1)}_{\rm GW}(k,t)}{\mathrm{d}\log k}\right]^2 , \\ \quad \text{if } 2M+3>0\,, \quad kt \gg 1 \\ \\ \left[\frac{\mathrm{d}\Omega^{(1)}_B(k)}{\mathrm{d}\log k}\right]^2 + \left[\frac{\mathrm{d}\Omega^{(1)}_{\rm GW}(k,t)}{\mathrm{d}\log k}\right]^2 , \\ \quad \text{if } 2M+3<0\,, \quad kt \gg 1 \ . \end{cases} \tag{125}$$

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APPENDIX A: GENERAL SOLUTION OF A DIFFERENTIAL EQUATION WITH THE WRONSKIAN METHOD

Here we discuss in detail the Wronskian method with which we find the solution of the differential equations in this paper. If we have a inhomogeneous linear second order equation with inhomogeneity $S(z)$, its most general solution is of the form

$$P(z) = c_1(z)P_1(z) + c_2(z)P_2(z) + a_1P_1(z) + a_2P_2(z)\,,$$

where $P_1(z)$ and $P_2(z)$ are two (linearly independent) homogeneous solutions which we suppose to be known, $W(z) = P_1P_2' - P_1'P_2$ is their Wronskian, and

$$c_1(z) \ = \ -\int_{z_{\rm in}}^{z} dx \frac{S(x)}{W(x)} P_2(x) \ ,$$
$$c_2(z) \ = \ \int_{z_{\rm in}}^{z} dx \frac{S(x)}{W(x)} P_1(x) \ .$$

The particular solution given by the first two terms is such that $P_{\rm inh}(z) = c_1(z)P_1(z) + c_2(z)P_2(z)$ vanishes at $z = z_{\rm in}$ and also $P'_{\rm inh}(z_{\rm in}) = 0$. The general solution is obtained by adding a homogeneous solution, $P_{\rm hom}(z) = a_1P_1(z) + a_2P_2(z)$ with arbitrary constants a_1 and a_2.

Let us first consider the example given in Eq. (106),

$$P'' - \frac{4}{z}P' + \left(1 + \frac{4}{z^2}\right) P = \frac{\alpha}{k^2} \ ,$$

where α/k^2 is a constant source term. The homogeneous solutions are given by $P_1(z) = z^3 j_1(z)$ and $P_2(z) = z^3 y_1(z)$ and the Wronskian determinant reads

$$W(z) = z^3 \ .$$

In the regime $z \ll 1$ we can approximate the spherical Bessel functions by powers and we find the following general expression for $P(z)$:

$$P(z) = -\frac{\alpha}{3k^2}\left(\frac{z^2}{2} - \frac{z^4}{2z_{\rm in}^2} + z^2 - zz_{\rm in}\right) + a_1z^4 + a_2z \ . \tag{A1}$$

where we have used the fact that, when $z \ll 1$, we can approximate $P_1(z) \simeq z^4$ and $P_2(z) \simeq -z$. Now it is important to notice that the second and the fourth terms of the inhomogeneous solution (A1) have the same functional behavior as homogeneous solutions and we can always choose a_1 and a_2 such that the homogeneous part cancels them. This is actually always true for the contributions from the lower boundary of the inhomogeneous solution. This may sound pedantic, but it is very important in this specific case as the second term in (A1) dominates if it is present. In our analysis we have always subtracted such "homogeneous contributions" and only kept the "minimal part", which in this case is

$$P(z) \simeq -\frac{\alpha}{2k^2}z^2 \ , \quad z \ll 1\,. \tag{A2}$$

This procedure is important and it is responsible for the results which we have obtained. We justify it also by the fact that the first order solution has exactly the the same time evolution as the homogeneous term and therefore a term $\propto z^4$ present at early times, should be included in the first order perturbations. Once the wave number has entered the horizon, $z \gg 1$, the Green function starts to oscillate and the additional contribution to the integral can be neglected. We then can match the inhomogeneous solution at horizon crossing to the homogenous one at later times. Up to matching details which we have not considered, this yields

$$P(z) \simeq \frac{\alpha}{2k^2}z^2\cos z \ , \quad z \gg 1 \ . \tag{A3}$$

In the same way, we deal with Eq. (78)

$$P'' - \frac{4}{z}P' + \left(1 + \frac{4}{z^2}\right) + P = \frac{\alpha}{k^2}\ . \tag{A4}$$

The homogeneous solutions are $P_1(z) = z^2 j_0(z) \simeq z^2$ and $P_2(z) = z^2 y_0(z) \simeq -z$. These approximations are valid for $z \ll 1$. Using again the Wronskian method, we obtain the following general solution on super-Hubble scales, $z \ll 1$:

$$P(z) = \frac{\alpha}{k^2}\left(z^2 \log\left(\frac{z}{z_{\rm in}}\right) - z^2 + z z_{\rm in}\right) + a_1 z^2 + a_2 z\ , \quad z \ll 1\ , \tag{A5}$$

Here, the homogeneous solution parts are $-z^2$ and $zz_{\rm in}$, therefore we can identify the solution due to the presence of the source again as

$$P(z) \simeq \frac{\alpha}{k^2} z^2 \log\left(\frac{z}{z_{\rm in}}\right)\ , \quad z \ll 1\ . \tag{A6}$$

On sub-horizon scales this becomes, up to matching details which only modify the phase and have an irrelevant effect on the pre-factors,

$$P(z) \simeq \frac{\alpha}{k^2} \log(k t_{\rm in}) z \cos z\ , \quad z \gg 1\ . \tag{A7}$$

If the source term depends on z, the details of the calculation as well as the results change somewhat, but the basic argumentation remains the same. We therefore do not repeat the z-dependent examples which arise in this work here.

APPENDIX B: THE FOUR-POINT CORRELATOR OF GRAVITY WAVES

Starting from Eq. (110), we compute the two-point correlation function of the source term $\langle f_{ij}(\mathbf{k},t') f^*_{rn}(\mathbf{p},t'')\rangle$, which is given by

$$\langle f_{ij}(\mathbf{k},t') f^*_{rc}(\mathbf{q},t'')\rangle = \frac{1}{(2\pi)^6} U(t',t'') \mathcal{M}_{ij}{}^{lm}(\hat{\mathbf{k}}) \times \mathcal{M}_{rc}{}^{bd}(\hat{\mathbf{q}}) \int d^3p \int d^3s \langle \Sigma^{(1)\,{\rm in}}_{ln}(\mathbf{p}) \Sigma^{(1)\,{\rm in}}_{nm}(\mathbf{k}-\mathbf{p}) \times \Sigma^{*(1)\,{\rm in}}_{bf}(\mathbf{s}) \Sigma^{*(1)\,{\rm in}}_{fd}(\mathbf{q}-\mathbf{s})\rangle\ , \tag{B1}$$

where the function $U(t',t'')$ contains the all time-dependence of the above expression:

$$U(t',t'') = \frac{81}{4}\mathcal{H}^2(t')\mathcal{H}^2(t'') \left[\frac{a(t')}{a_{\rm in}}\right]^6 \left[\frac{a(t'')}{a_{\rm in}}\right]^6\ . \tag{B2}$$

To compute the four-point correlator, we assume that the random variables that describe gravity waves are Gaussian, therefore we can apply Wick's theorem. The we can write the products of four gravity waves $\Sigma^{(1)}$ as

$$\begin{aligned}&\langle \Sigma^{(1)\,{\rm in}}_{ln}(\mathbf{p}) \Sigma^{(1)\,{\rm in}}_{nm}(\mathbf{k}-\mathbf{p}) \Sigma^{*(1)\,{\rm in}}_{bf}(\mathbf{s}) \Sigma^{*(1)\,{\rm in}}_{fd}(\mathbf{q}-\mathbf{s})\rangle = \\ &\langle \Sigma^{(1)\,{\rm in}}_{ln}(\mathbf{p}) \Sigma^{*(1)\,{\rm in}}_{bf}(\mathbf{s})\rangle \langle \Sigma^{(1)\,{\rm in}}_{nm}(\mathbf{k}-\mathbf{p}) \Sigma^{*(1)\,{\rm in}}_{fd}(\mathbf{q}-\mathbf{s})\rangle + \\ &\langle \Sigma^{(1)\,{\rm in}}_{ln}(\mathbf{p}) \Sigma^{*(1)\,{\rm in}}_{fd}(\mathbf{q}-\mathbf{s})\rangle \langle \Sigma^{(1)\,{\rm in}}_{nm}(\mathbf{k}-\mathbf{p}) \Sigma^{*(1)\,{\rm in}}_{bf}(\mathbf{s})\rangle + \\ &\langle \Sigma^{(1)\,{\rm in}}_{ln}(\mathbf{p}) \Sigma^{(1)\,{\rm in}}_{nm}(\mathbf{k}-\mathbf{p})\rangle \langle \Sigma^{*(1)\,{\rm in}}_{bf}(\mathbf{s}) \Sigma^{*(1)\,{\rm in}}_{fd}(\mathbf{q}-\mathbf{s})\rangle\ .\end{aligned} \tag{B3}$$

Once the double integration is performed, the last term contributes a constant $\propto \delta^3(\mathbf{k})$ which can be disregarded (a background term). Integrating the remaining two terms over d^3s, we can eliminate one of the two δ-functions which come from the expression of the two point gravity wave correlator. Using the reality condition, $\Sigma^*_{ij}(\mathbf{k}) = \Sigma_{ij}(-\mathbf{k})$, and the expression for the two-point correlation function of gravity waves given in Eq. (66), we then obtain

$$\begin{aligned}&\int d^3p \int d^3s \langle \Sigma^{(1)\,{\rm in}}_{ln}(\mathbf{p}) \Sigma^{(1)\,{\rm in}}_{nm}(\mathbf{k}-\mathbf{p}) \Sigma^{*(1)\,{\rm in}}_{bf}(\mathbf{s}) \times \\ &\Sigma^{*(1)\,{\rm in}}_{fd}(\mathbf{q}-\mathbf{s})\rangle = (2\pi)^6 \delta^3(\mathbf{k}-\mathbf{q}) \int d^3p \mathcal{P}^{(1)}_{\Sigma\,{\rm in}}(p) \times \\ &\mathcal{P}^{(1)}_{\Sigma\,{\rm in}}(|\mathbf{k}-\mathbf{p}|) \Big[\mathcal{M}_{lnbf}(\hat{\mathbf{p}}) \mathcal{M}_{mndf}(\widehat{\mathbf{k}-\mathbf{p}}) \\ &+\mathcal{M}_{lndf}(\hat{\mathbf{p}}) \mathcal{M}_{mnbf}(\widehat{\mathbf{k}-\mathbf{p}})\Big]\ .\end{aligned} \tag{B4}$$

The above equation is symmetric in $\mathbf{k}$ and $\mathbf{q}$, as well as under the exchange of the first and second pairs of indices. Moreover, it is symmetric under the exchange of the first index with the second and the third with the fourth. This suggests us to write the two point correlation function of the source term as

$$\langle f_{ij}(\mathbf{k},t') f^*_{rc}(\mathbf{q},t'')\rangle = (2\pi)^3 \delta^3(\mathbf{k}-\mathbf{q}) \mathcal{M}_{ijrc}(\hat{\mathbf{k}}) \times h(k,t',t'')\ , \tag{B5}$$

since the tensor $\mathcal{M}_{ijrc}$ has the same symmetries.

To obtain an expression for the function $h(k,t',t'')$, it is sufficient to calculate the trace of the above two point correlator. We hence should multiply the r.h.s. of the above equation and of Eq. (B1) by $\mathcal{M}^{ijrc}(\hat{\mathbf{k}})$. Then, setting them to be equal and remembering that $\mathcal{M}^{ijrc}\mathcal{M}_{ijrc} = 8$ [20], we obtain

$$\begin{aligned}&8(2\pi)^3 \delta^3(\mathbf{k}-\mathbf{q}) h(k,t',t'') = U(t',t'') \delta^3(\mathbf{k}-\mathbf{q}) \times \\ &\mathcal{M}^{crlm}(\hat{\mathbf{k}}) \mathcal{M}^{bd}{}_{rc}(\hat{\mathbf{q}}) \int d^3p \Big[\mathcal{M}_{lnbf}(\hat{\mathbf{p}}) \mathcal{M}_{mndf}(\widehat{\mathbf{k}-\mathbf{p}}) + \\ &\mathcal{M}_{lndf}(\hat{\mathbf{p}}) \mathcal{M}_{mnbf}(\widehat{\mathbf{k}-\mathbf{p}})\Big] \mathcal{P}^{(1)}_{\Sigma\,{\rm in}}(p) \mathcal{P}^{(1)}_{\Sigma\,{\rm in}}(|\mathbf{k}-\mathbf{p}|)\end{aligned} \tag{B6}$$

with

$$h(k,t',t'') = \frac{1}{8(2\pi)^3} U(t',t'') \mathcal{M}_{bdlm}(\hat{\mathbf{k}}) \times$$
$$\int \mathrm{d}^3 p \mathcal{P}^{(1)}_{\Sigma\,\mathrm{in}}(p) \mathcal{P}^{(1)}_{\Sigma\,\mathrm{in}}(|\mathbf{k}-\mathbf{p}|) \times$$
$$\left[\mathcal{M}_{lnbf}(\hat{\mathbf{p}}) \mathcal{M}_{mndf}(\widehat{\mathbf{k}-\mathbf{p}}) + \mathcal{M}_{lndf}(\hat{\mathbf{p}}) \mathcal{M}_{mnbf}(\widehat{\mathbf{k}-\mathbf{p}})\right] . \tag{B7}$$

Finally, we have to perform the above product of three polarization tensors, defined as in Eq. (32). To achieve this aim, we use the free source package "xAct" for Mathematica [25]: it is sufficient to define a three dimensional flat metric and the projection tensor $\mathcal{P}_{ij}(\hat{\mathbf{k}}) = \delta_{ij} - k^{-2} k_i k_j$ onto the plane normal to $\mathbf{k}$. Then, we can express $\mathcal{M}_{ijlm}(\hat{\mathbf{k}})$ in terms of this projector as

$$\mathcal{M}_{ijlm} \equiv \mathcal{P}_{il}\mathcal{P}_{jm} + \mathcal{P}_{im}\mathcal{P}_{jl} - \mathcal{P}_{ij}\mathcal{P}_{lm} \, .$$

Defining the angles between the three directions as $\alpha \equiv \hat{k} \cdot (\widehat{k-p})$, $\beta \equiv \hat{p} \cdot (\widehat{k-p})$ and $\gamma \equiv \hat{k} \cdot \hat{p}$, we obtain the expression given in Eq. (111).

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