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Vectors and translations in mathematics and physics

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Abstract: In mathematics, students learn about vectors and translation, in physics they model forces, speed, acceleration, etc. with vectors and study movements of translation. Do they make the connection between these concepts introduced in different disciplines or do they put things in separate boxes? In this paper we will start with some partial considerations on the history of vectors and we will give some references. Then we will show some examples of naïve illustrations of vectors from physics in mathematics textbooks. We will then present a non-conventional example and the difficulties it created for both mathematics and physics teachers. Finally we will develop an example of a possible collaboration between teachers of both disciplines in relation to movement of translation.

Introduction

Our purpose in this paper is to see, with the example of vectors, how mathematics can be actually connected to physics and to give propositions to make this connection more efficient for the benefit of both mathematics and physics. This talk is based on the supervision of Ba's doctorate (Ba 2007) and some common publications (Ba & Dorier 2006, 2007, 2010, 2011) and our own work on vectors and linear algebra (Dorier 1995, 1996, 1998, 2000).

Mathematics is often seen as a very specific subject, either by students, parents, media or even mathematicians themselves. In many contexts, mathematics is feared and seen as a subject for selection, disconnected from interesting applications in real life. Moreover, the structure of teaching institutions, in many cases, makes the collaboration between teachers from different disciplines very difficult. At the same time, mathematics is more and more invisible in everyday life, since high technology tends to hide the mathematics necessary for its creation in sophisticated black boxes. As a result, it is quite a challenge to give an adequate answer to those who, legitimately, wonder what mathematics is useful for.

Our study reveals that, most often, teachers know very little about other subjects, even in relation to their own subject. Mathematics teachers do not want to get involved in too specialised applications while physics teachers send their students back to their mathematics teacher for explanations on the use of mathematics in their field. As a result, students are used to seeing mathematics and physics as disconnected. This is reinforced by cultural differences, especially visible in the use of vocabulary or recipes that create artificial gaps between different disciplines.

In this paper we will start with some partial considerations on the history of vectors and we will give some references. Then we will show some examples of naïve illustrations of vectors from physics in mathematics textbooks. We will then present a non-conventional example and the difficulties it created for both mathematics and physics teachers. Finally we will develop an example of a possible collaboration between teachers of both disciplines in relation with movement of translation.

History of vectors: some comments

Like we argued in our history of linear algebra, the links between vectors and traditional geometry on the one side and the emergence of modern linear algebra on the other are not as obvious as one may think:

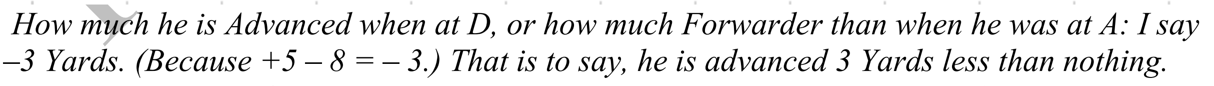
One of the myths (supported by traditional teaching) about the natural link between geometry and linear algebra comes from the extensive use of common vocabulary in the two fields. For instance, the fact that the linear structure is called a 'vector space' automatically certifies the geometrical origins of the theory. Another reason for assuming a natural link comes from the use of geometrical representation: e.g. the sum of two vectors can be represented as the diagonal of a parallelogram (as in the parallelogram of velocities and forces, a very ancient type of representation used in physics to symbolize the combined action of two velocities or forces applied at a same point). However, there is a big gap between the traditional use of this representation and the modern interpretation of the algebraic sum of vectors (see Crowe 1967, p.2) (Dorier 2000, pp.12-13).

The real starting point in the history of vectors lies in the invention of analytical geometry, independently by René Descartes (1596–1650) and Pierre de Fermat (160?–1665) and the criticism made by Gottfried Wilhelm Leibniz (1646–1716) as expressed in a letter to Christian Huyghens, dated 8th September 1679:

I am still not satisfied with Algebra, because it does not give the shortest methods or the most beautiful constructions in Geometry. That is why I believe that we need still another analysis which is distinctly geometrical or linear (see footnote above), and which will express situation directly, as Algebra expresses magnitude directly. And I believe that I have found the way and that we could represent figures and even machines and movements by characters, as algebra represents numbers or magnitudes. (Translation by Crowe (1967, p. 3) from the original in French published in (Leibniz 1850, vol.1, p.382)).

Leibniz tried to invent a geometrical calculus on the basis of his criticisms towards analytic geometry but never managed to take into account the idea of direction of a line. A specificity of vectors lies in their double nature as geometrical entities with algebraic properties. In this sense, Leibniz failed to introduce the idea of negative in geometry. Moreover, negative quantities remained problematic for many mathematicians until the 19th century. For instance, in his *Algebra* published in 1673, John Wallis (1616–1703) claims: "But it is also impossible that any Quantity can be *Negative*. Since it is not possible that any *Magnitude* can be *Less* than *Nothing*, or any *Number Fewer than None*." (vol. II, p. 264). To this vision of a quantity less than nothing, Wallis opposes the idea of a magnitude in an opposite direction: "Yet when it comes to Physical Application, it denotes as Real a Quantity as if the Sign were +; but to be interpreted in a contrary sense." (Ibid., p. 265).

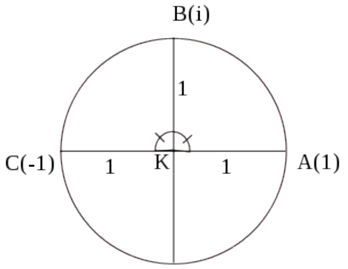
To illustrate this idea, he imagines a person who goes 5 yards forward from A to C and then 8 yards backward from C to D.



How much he is Advanced when at D, or how much Forwarder than when he was at A: I say –3 Yards. (Because +5 – 8 = – 3.) That is to say, he is advanced 3 Yards less than nothing. Which in property of speech, cannot be, (since there cannot be less than nothing.) And therefore as to the Line AB Forward, the case is Impossible.

But if (contrary to the Supposition) the Line from A, be continued Backward, we shall find D, 3 Yards Behind A. (Which was presumed to be Before it). (Ibid., p.265)

The difficulty here is due to the dominant model of the negative quantities being less than nothing, under zero (thermometer, lift, underwater levels…). The obvious, yet revolutionary idea is to see the zero not as a pushing point but as an articulation point, where one can turn back and face the other direction. Wallis' text is one of the first, in which the two lines AB and BA are seen as opposite to each other and the negative is associated with a change of sense on a right line (a half turn). This represents the first conceptualisation of one-dimensional vectors. This idea of directed line segments will be developed further when representing imaginary quantities geometrically in the 2-dimensional plane. One of the pioneer works in this sense was published in 1806 by Jean-Robert Argand (1768-1822) an amateur mathematician from Geneva. In his treatise, like Wallis, he explains that the negative quantities are on a symmetrical line of the positive from the origin 0. He then interprets the quantity (–1) as the product (+1).(–1) and the square root √(-1) as the geometrical means of this product, therefore a quantity with magnitude 1 and with direction perpendicular to the line of real numbers, either in one sense or the other.



We can see here that the point K which is the origin, the zero, is an articulation point and that with the half turn from positive to negative, the line opens on the whole plane (or even space) (see Châtelet 1993, p.128). In this sense the representation of imaginary quantities is a crucial step towards the creation of vectors. Yet, it would take another 50 years to reach some decisive discovery.

Hermann Grassmman (1809–1877) published in 1844 his first version of *Die lineale Ausdehnungslehre*, a revolutionary treatise in which he invented, in a total disconnection to mathematics of his time, not only vectors but linear and multi-linear algebra. This book was ignored and never understood in its time (see Schubring 1996). In his introduction, Grassmann points out very clearly the inspiration he got from the introduction of negatives in geometry.

The initial incentive was provided by the consideration of negatives in geometry; I was used to regarding the displacements AB and BA as opposite magnitudes. From this, it follows that: if A,B,C are points of a straight line, then AB+BC=AC is always true, whether AB and BC are directed similarly or oppositely; that is, even if C lies between A and B. In the latter case, AB and BC are not interpreted merely as lengths, but rather their directions are simultaneously retained as well, according to which they are precisely oppositely oriented. Thus the distinction can be drawn between the sum of lengths and the sum of such displacements, in which the directions were taken into account. From this there followed the demand to establish this latter concept of a sum, not only for the case that the displacements were similarly or oppositely directed, but also for all other cases. This can most easily be accomplished if the law AB+BC=AC is imposed even when A, B, C do not lie on a single straight line.

While I was pursuing the concept of the product in geometry as it had been established by my father, I concluded that not only rectangles, but also parallelograms in general, may be regarded as products of an adjacent pair of their sides, provided one again interprets the product, not as the product of their lengths, but as that of the two displacements with their directions taken into account. (Grassmann 1844, translation from Kannenberg 1995, p. 9)

I cannot give more information in this text about the history of vectors. One can see (Crowe 1967) or Dorier (1995, 1996, 1998 and 2000, part I) for more details.

However, I would like to stress two points for what is going to follow:

1. Vectors are, by nature, hybrid objects with a dual geometrical/algebraic feature, which is an essential component in their learning.

2. The history of vectors is largely independent from the history of linear algebra (which is linked with infinite dimensional function spaces). Vectors are linked with the history of physics (Maxwell equations).

In Ba & Dorier (2006, 2010 and 2011) we have shown by analysing the history of the teaching of vectors in France that in the early XX^{e} century a didactical object “the geometrical vector” has been artificially created and cut from its roots in physics. Then modern mathematics made vectors the prototype of linear algebra for ideological reasons. Since the 80s vectors are struggling to find a correct place at the junction between physics and mathematics. That is what we are going to analyse briefly now.

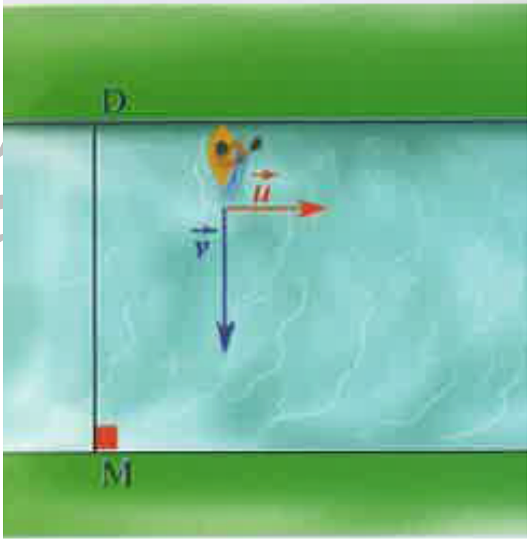
Naïve situations from physics in mathematics teaching of vectors

It is well known that when situations from extra-mathematical contexts are used in mathematics it is often very naïve.

In his work, Ba (2007) analysed several mathematics textbooks[1] and showed that the situations from physics contexts used in the teaching of vectors are not very numerous and usually very naïve.

I will give only two examples.

First naïve example

An Indian in a canoe wants to cross a river whose sides are parallel. The canoe is subject to two forces: the stream force represented by vector $\overrightarrow{u}$and the force exerted by the rower represented by vector $\vec{v}$. One considers that the canoe starts at point D and moves in the direction of vector $\vec{w}$ defined by the equality $\vec{u} + \vec{v} = \vec{w}$. In the whole exercise, it is assumed that the length of $\vec{v}$ is the double of the length of $\vec{u}$. 1. Reproduce the schema above and represent vector $\vec{w}$ and the trajectory of the canoe. One calls A the arrival point of the canoe and assumes that the river is 35m wide 2. Calculate the angle $\widehat{MDA}$ and the length DA 3. Make a drawing at scale 1/1000 and verify the results on the drawing.	

Here it is quite clear that the context is purely a pretext to develop some basic geometrical skill; there is no motivation to make use of the sum of vectors, since the vector equation is given. Moreover physicists can argue that the use of the term “force” to designate the effects of both the stream of the river and the rowing could be subject to a debate… which is clearly ignored here. However, at least, the directions of the forces are discussed, which not always the case. Indeed, various studies have shown that, although a force is characterised by a magnitude and a direction, tasks given in physics focus on the magnitude only (Genin et al. 1987 and Lounis 1989).

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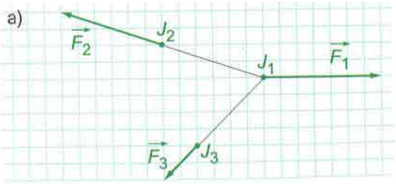
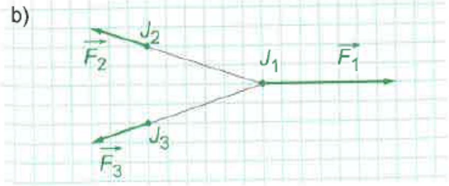
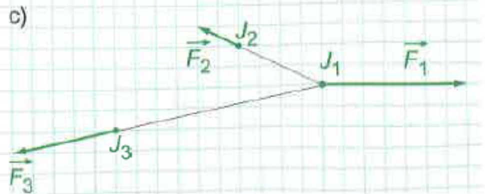
[1] He analysed textbooks of last year of lower secondary and first year of upper secondary (age 15-16 years old) which is when vectors are introduced in the mathematics curriculum in France., while forces and velocity are introduced only in the two first years of upper secondary school.

Second naïve example

In a rugby training, one player J_1 is challenged by two other players J_2 and J_3 attached with two ropes. In each case, is player J_1 going to move forward or backward?	
	

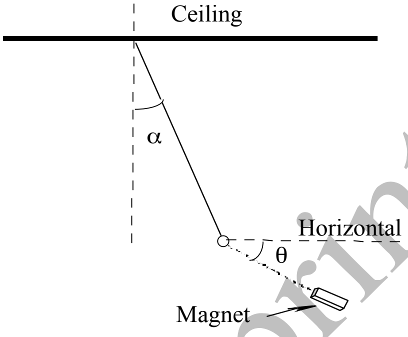
In the three cases the sum $\overrightarrow{F_2} + \overrightarrow{F_3}$ has exactly the opposite direction of $\overrightarrow{F_1}$ (with magnitude respectively bigger, equal and smaller!!!): too nice to be true… one can imagine the type of discussion if the directions had been totally different, while in this simplified cases, the main argument is one dimensional, therefore insufficiently representative of the use of vectors and suspicious regarding the credibility of the context.

In order to make the use of vectors more substantial in a physics context, we created our own exercise… and faced other difficulties…

A non conventional example

This specific situation concerns a variation from a very classical problem in dynamics: the pendulum. We designed this version in order to make the determination of the direction of a force essential for its solution.

Here is the text of the problem:

	*An iron small ball (comparable to a point M) with mass m is hung to the ceiling by a thread (whose mass will be neglected).* *A magnet attracts the ball, the direction of the force makes an angle θ under the horizontal line (see drawing) and its magnitude is F.* *When in equilibrium, the thread makes an angle α with the vertical (see drawing)* *The only forces are: the weight of the ball, the attraction of the magnet and the tension of the thread.*

Data: $m=200g$, $\theta=30°$, $F=2N$, take $g=10N/g$.

1. Write the equilibrium equation.

2. Represent with the scale (1cm=1N) the forces in action.

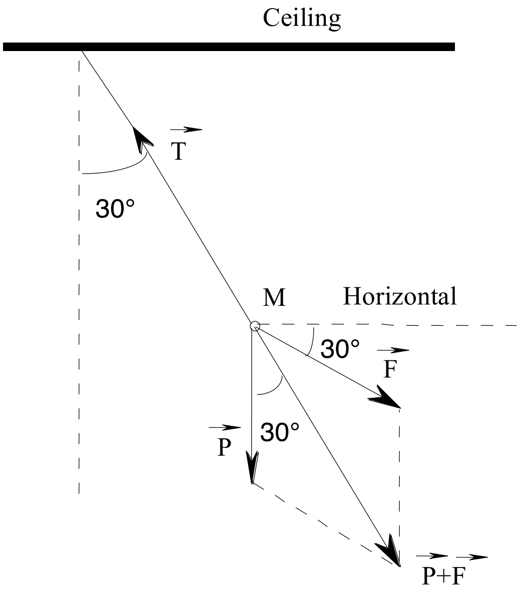
3. What are the characteristics of the tension of the thread?

We now give the answers:

1. The equilibrium equation is given by the first fundamental law of dynamics:

$$\vec{P} + \vec{F} + \vec{T} = \vec{0}$$

2.



3. With use of relations in an isosceles triangle, it is easy to see that $\vec{T}$ makes an angle of 30° with the vertical and has a magnitude of $2\sqrt{2}$ N.

The interesting point in this problem is that in question 2, one has to draw $\vec{F}$ and $\vec{P}$ first in order to draw $\vec{T}$ (since this the opposite of their sum). Then the direction of $\vec{T}$ gives the direction of the thread. Therefore in order to draw the thread, one has to use the sum of two vectors, which is the essential key to the problem.

However, this task is problematic in the context of physics. Indeed, the construction required in question 2 has to take place in a mathematical model, which is not reality. Moreover, in this model the point where the thread is attached to the ceiling can only be determined at the end of the process. Once this theoretical construction is made, one can come back to the drawing representing the reality and use the results of question 3 to represent the situation starting with the fact that the thread makes an angle of 30° with the vertical.

In his work, Ba (2007) submitted this problem both to students and teachers in *Première S* (second scientific class of upper secondary school, *Lycée*, in France, students aged 16) both in mathematics and in physics lessons. The students, tested during physics lessons, did not have any problem with question 1. But they met real difficulties in question 2. They could not transfer the problem into the mathematical model. As a matter of fact, they did not see that there were two levels in the representation of the situation (reality and model) and were blocked because of the missing thread. On the other hand, different studies show that students at this level have acquired sufficient knowledge about vectors to be able to draw the sum of two vectors and to answer questions like question 3, when given in a purely geometrical setting. This shows that students have sufficient knowledge of mathematics but are not able to mobilise it, when required, in physics. Moreover, they do not identify the mathematics at stake in a physics problem. The difficulty here is typical of modelling situations.

Furthermore we interviewed physics teachers and asked them if they would give such a problem to their students, and if so, what difficulty they think would appear. Massively, they admitted that this problem was close to a typical situation of dynamics, but at the same time they felt uncomfortable with the formulation. They did not believe that their students would handle the geometrical construction. For the solving of question 3, they also massively prefer a solution using projections on two orthogonal axes, a technique widely used in physics.

Mathematics teachers, on the other hand, would not be ready to give such a problem to their students because they do not consider it as part of mathematics. Moreover, the physics notions at stake are only taught one or two years after the sum of vectors is studied in mathematics, so there is a problem of priority.

This problem appears to be typical of the difficulty in building a bridge between mathematics and physics even when two notions are naturally related like vectors and forces. Teachers of both disciplines do not want to take the burden of linking the two, and students cannot transfer their knowledge from one to the other. Only a joint effort from teachers of both disciplines can solve the problem. Moreover a real change in culture is necessary to fight against the compartmentalization of disciplines

In order to show some possible collaboration between mathematics and physics teachers, we are now going to explore the connection between translation in mathematics and movement of translation in physics.

Movement of translation and translation: an impossible dialogue between mathematics and physics?

The question now is quite different from the previous case of vectors and forces. Indeed, here, the relation between physics and mathematics seems more obvious, since the same terms are used but, on the other hand, it is more mysterious. Indeed, it is well known that geometrical transformations are cognitively attached to dynamical representations. A mathematical transformation only takes into account an initial and a final state (i.e. an element and its image), but one, often implicitly, attaches an idea of movement between those two states. In this sense, the effect of a rotation on a geometrical object can be seen as a movement of rotation of the object. This representation of a geometrical rotation is coherent with the concept of movement of rotation in physics. However, it is quite different with translation, since the dynamical representation of the translation of a geometrical object is attached to the special case of rectilinear movement of translation only and does not take into account all the other types of movement of translation studied in physics.

Indeed, in physics an object is said to have a movement of translation when any segment attached to the solid remains parallel to itself during the movement (def.1).

Therefore, the trajectory of the object can be non-rectilinear, but follow any type of curve:

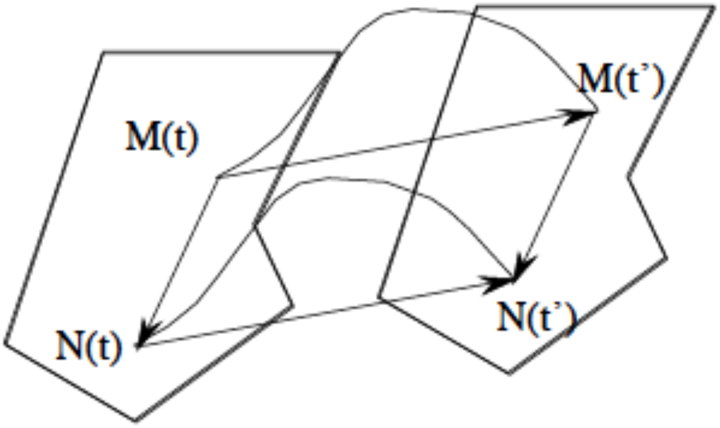


Figure 1. The general case of a movement of translation

Figure 2. *Ferris Wheel:* the prototypical example of a circular movement of translation, often confused with a movement of rotation

Experiments have been made involving mathematics and physics teachers about their representation of a movement of translation, and it shows that most mathematics teachers only think of rectilinear movement of translation and are totally puzzled when physics teachers try to explain what is a movement of translation by showing a movement with their hand following a non-rectilinear trajectory, yet with the hand remaining parallel to itself (Gasser 1996).

Another puzzling question is that most French physics textbooks (at the level of Première S), in the chapter introducing the definition of a movement of translation as given above, also give illustrations with objects on which vectors are drawn (like on the figure above), although the definition only mentions segments. Indeed, the objects are always supposed not to change their shape during the movement, therefore a segment [M,N] on the solid cannot change its length. In a movement of translation any segment remains parallel to itself. So vector $\overrightarrow{MN}$ can have only two opposite directions, and cannot change length. Thus, according to a basic continuity principle, it is clear that vector $\overrightarrow{MN}$ cannot change its direction (because it would have to go from one direction to the opposite without being able to have any intermediary positions in between). Having the same direction and the same length it therefore remains identical.

In other words, a movement of translation can be characterised by the fact that every vector on the solid remains identical (def.2).

One can wonder why such a formulation is never used in physics, while vectors appear in practically all drawings. Certainly, the fear of being too abstract is the main reason. This is characteristic of the distance separating physics and mathematics.

Let us now see what the connection between movement of translations and mathematical translation can be and why this is neither explicit in physics nor in mathematics teaching.

Let us introduce the time in the notation, what physics teachers usually do not do at this level in order to avoid abstraction and formal notations. For each value *t* of [0,T] (the duration of the movement) and any point *M* of the solid S, one calls *M*(t) the position of the point *M* at time *t*. Then the definition of a movement of translation becomes:

S has a movement of translation if, for any *t, t'* of [0,T] and *M, N* of S:

$$\overrightarrow{M(t)N(t)} = \overrightarrow{M(t')N(t')}. \text{ (def3)}$$

In terms of translation, the condition can be expressed by:

S has a movement of translation if, for any *M, N* of S, there is a translation τ_{MN} (independent of the time) such that for any *t* of [0,T] : τ_{MN} (*M*(*t*)) = *N*(*t*). (def.4)

Of course τ_{MN} is the translation of vector $\overrightarrow{MN}$. This is a first characterisation of a movement of translation using the mathematical notion of translation.

Moreover, if one applies what is sometimes known as the parallelogram rule: $\overrightarrow{M(t)N(t)} = \overrightarrow{M(t')N(t')}$ is equivalent to : $\overrightarrow{M(t)M(t')} = \overrightarrow{N(t)N(t')}$, one gets another characterisation of a movement of translation using the mathematical notion of translation:

S has a movement of translation if, for any *t, t'* of [0,T] there exists a translation $\tau_{tt'}$ (independent of the point) such that for any *M* of S : $\tau_{tt'}$(*M*(t)) = *M*(t'). (def.5)

The difficulty here is that this translation does not give any information about the trajectory followed by the solid S between *t* and *t*'.

Finally, if, for distinct *t* and *t*', one divides the preceding equality by (*t' – t*), one gets:

$$\frac{\overrightarrow{M(t)M(t')}}{t'-t} = \frac{\overrightarrow{N(t)N(t')}}{t'-t}$$

Which becomes, when t' tends to t: $\overrightarrow{v_M(t)}$== $v_N(t)$, which means that at any time during the movement all points have the same velocity.

Reciprocally, by integrating between *t* and *t*' the equality of velocity, one gets that: $\overrightarrow{M(t)M(t')} = \overrightarrow{N(t)N(t')}$.

This gives another characterisation of a movement of translation that students see in physics without any proof:

S has a movement of translation if, at any time, all points have the same velocity. (def.6)

In the teaching of physics in France, only definitions 1 and 6 are given to the students without any proof of the fact that they are equivalent and no attempt to connect them to mathematical translations, either in books or by teachers (according to a questionnaire sent to a large number of teachers).

Moreover, physics teachers do not care about this connection or simply believe that translation and movement of translation are the same thing, while most mathematics teachers reduce movement of translation to the rectilinear case.

Most students are used to not trying to make bridges between physics and mathematics and therefore use the same word in two different disciplines without trying to find a connection. However, they have difficulties with movement of translation. They often get confused, for instance, between circular movement of translation and movement of rotation (like for the Ferry wheel). They also have difficulties in non-“classical” examples in making their definition operational when trying to prove that a given movement is of translation, while they have the mathematical skills at hand (Ba, 2007).

This situation is not satisfactory. Especially since students have all the necessary knowledge at hand to be able to understand, with a minimum of time and work, the different connections we have briefly established above. Again, the question is to know who, among the mathematics teachers and the physics teachers, should be in charge of making the connection explicit. Moreover such clarification would benefit to physics teaching, of course, since it enriches the notion of movement of translation, but also to mathematics teaching, since it provides a use of vectors and translations in a rich context, with a challenging use of notations. For these reasons, it seems that this should be a joint effort of both teachers, either in parallel, in the mathematics class and the physics class, or even better, in a common session with both teachers. In his work Ba (2007) experimented the second solution in Senegal with a joint teaching involving the mathematics and physics teachers on the basis of the previous analysis. This was a relatively positive experiment, although it would need to be repeated in different contexts, in order to evaluate the real impact.

Conclusion

Nowadays, the teaching of mathematics is subject to a social pressure that requires more applications and raises issues about modelling. The outside world forces mathematics to come out of its ivory tower. This is true for all levels of education in any context. However, it is even more essential for students whose major interest lies outside mathematics. It is not possible anymore for mathematicians to remain isolated, away from applications, in a position of superiority. This is the best thing that could have happened to mathematics, which needs to make itself more visible.

Our belief is that mathematics will not sell its soul by getting more interested in other disciplines. We hope to have shown with the example of vector and translation that by connecting itself to outside contexts, mathematics can be taught in a richer way, without reducing the value of its concepts. As we have shown here but also in other works in relation with economics (Dorier 2005) the connection with other disciplines is also a way of making the formal aspect of mathematics accessible. Making the connection with another discipline is not only a question of psychological motivation, but also an epistemological challenge. Indeed, using an example from another discipline is not only a (fashionable) way to motivate students, but it is also a way to present a richer context where issues on the meaning of mathematics will automatically be addressed and questioned. This is not just an abdication of supremacy, but also a humble recognition of the power of mathematics as a provider of models to other disciplines, which has always been an essential part of its history.

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