Tropicalization in Poisson Geometry and Lie Theory

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Tropicalization in Poisson Geometry and Lie Theory

THÈSE

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Résumé

Cette thèse est consacré à l'étude de la géométrie de Poisson et la théorie des représentations en utilisant l'approche de la géometrisation et tropicalisation. Le but est de trouver des relations entre quelques nouveaux résultats dans le domaine. La motivation est de deux sortes:

(1) la géometrisation de la base canonique dans [16];

(2) la relation entre le systeèmes de Gelfand-Zeitlin et la tropicalisation d'un dual d'un groupe de Lie-Poisson du groupe unitaire dans [4].

Les résultats principeaux sont les suivantes:

- Nous présentons des variétés avec une ptentiel fibré sur un sous groupe de Cartan H d'un groupe réductif. Ces variétés sont connus comme des multiplicités géometriques. Elles forment catégorie monoïdiale et nous construisons un foncteur monoïdiale de cette catégorie à la catégorie des réprésentations d'un dual d'un groupe de Langlands $G^\vee$ de G. En utilisant ce foncteur, nous retrouvons et généralisons le calcul des multiplicités du produit tensoriel de [21].
- Nous présentons la notion d'une tropicalisation partielle d'une variété Poisson positive muni un potentiel. Le dual d'un groupe de Lie-Poisson G^* d'un groupe de Lie réductif complexe G est naturellement muni avec une structure positive et un potentiel. Comme cela, on assigne à G^* un système intégrable sur $\mathrm{PT}(G^*)$, lequel est un produit du cône des cordes prolongé et d'un tore compacte de dimension $\frac{1}{2}(\dim G - \operatorname{rank} G)$.
- Nous trouvons une relations entre deux constructions naturelles de dualié d'un groupe algebraique semi-simple G: son dual d'un groupe de Langlands $G^\vee$ et son dual d'un groupe de Lie-Poisson G^*. Ça veut dire: le cône intégrale défini par le potentiel de Berenstein-Kazhdan sur un sous-groupe de Borel, $B_-^\vee \subset G^\vee$ est isomorphique au cône intégrale de Bohr-Sommerfeld défini par le structure de Poisson sur la tropicalisation partielle $\mathrm{PT}(G^*)$.
- En utilisant la tropicalisation partielles et l'isomorphisme de Ginzburg-Weinstein, nous construisons une saturation pour chaque orbite coadjointe d'un groupe compacte semi-simple par des plongements symplectiques des domaines toriques.

Abstract

In this thesis, we study Poisson geometry and representation theory by using the approach of geometrization and tropicalization. The goal is to find relations between some new achievements in these directions. This is motivated by

(1) the geometrization of canonical basis in [16];

(2) the relation between Gelfand-Zeitlin systems and the tropicalization of dual Poisson-Lie group of unitary group in [4].

The main results are as follows:

- We introduce geometric multiplicities, which are positive varieties with potential fibered over the Cartan subgroup H of a reductive group G. They form a monoidal category, and we construct a monoidal functor from this category to the representations of the Langlands dual group $G^\vee$ of G. Using this, we recover the computation of tensor product multiplicities from [21] and generalize them in several directions.

- We introduce a notion of partial tropicalization of a positive Poisson variety with potential. The dual Poisson-Lie group G^* of a reductive complex Lie group G carries a natural positive structure and potential. This procedure assigns G^* an integrable system on $\mathrm{PT}(G^*)$, which is the product of the extended string cone and the compact torus of dimension $\frac{1}{2}(\dim G - \operatorname{rank} G)$.

- We find a relation between the two natural duality constructions of a semisimple algebraic group G: its Langlands dual group $G^\vee$ and its Poisson-Lie dual group G^*. That is, the integral cone defined by the Berenstein-Kazhdan potential on Borel subgroup $B_-^\vee \subset G^\vee$ is isomorphic to the integral Bohr-Sommerfeld cone defined by the Poisson structure on the partial tropicalization $\mathrm{PT}(G^*)$.

- For each regular coadjoint orbit of a semisimple compact group, we construct an exhaustion by symplectic embeddings of toric domains by using partial tropicalization and Ginzburg-Weinstein isomorphisms.

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Contents

1 | Introduction

Overview

The main goal of my thesis is to search for new connections between Poisson geometry and representation theory using the approach of tropicalization and geometrization respectively. The approach of the "geometrization" of combinatorial data dates back to the work of Lusztig, who constructed a birational isomorphism from an affine space to the unipotent radical of a Borel subgroup of a reductive Lie group from the combinatorial data of the canonical basis. The approach of "tropicalization" in our setting, which we use in this project, was first introduced by Berenstein and Kazhdan. It consists of a set of formal rules to pass from birational to piecewise-linear isomorphisms.

This project lies at the crossroads of many branches of mathematics and physics. The main research object, positive varieties with potentials, has drawn the attention of many mathematicians and physicists due to its surprisingly significant roles in many different areas. To list a few of the most important related works: positive varieties with potentials find applications in the study of cluster algebra, super-potential (Gross-Hacking-Keel-Kontsevich [40]), canonical basis and mirror symmetry (Goncharov-Shen [44]) and moduli spaces of local systems (Goncharov-Shen [45]).

Poisson geometry originates in solving the problems of classical mechanics, and later was connected with a number of areas. The motivation and starting point for this project are the connection involving Poisson geometry and cluster theory, which was studied by Kogan-Zelevinsky [64], Gekhtman-Shapiro-Vainshtein [41, 42] and Alekseev-Davydenkova [4]. In particular, in [4], the latter studied the mysterious relation between dual Poisson-Lie groups and Gelfand-Zeitlin integrable system. This project also builds upon the work of the (*global*) linearization of Poisson-Lie groups, which was originally studied by Ginzburg-Weinstein [43]. Other proofs appeared in different perspective in Alekseev [1], Boalch [23], and Enriquez-Etingof-Marshall [30]. This project therefore lies at the cross-section of many areas in mathematical physics.

What lies at the center of this project is the construction of *global* action-angle coordinates on coadjoint orbits of a compact Lie group. Co-adjoint orbits play an important role in symplectic and Poisson geometry. Action-angle coordinates yield a canonical model for Liouville integrable systems in the neighborhood of a Liouville torus. The construction of global action-angle coordinates, however, remains an open problem for many important spaces, such as coadjoint orbits, multiplicity spaces, and moduli spaces of flat connections.

On positivity theory and representation theory

Let us start from positivity theory and representation theory. A positive variety with potential is an irreducible algebraic variety X and a rational function Φ on X, together with an open embedding of a split algebraic torus to X so as the potential pulls back to a subtraction-free rational function. Assume for simplicity that the potential can be written as a Laurent polynomial

$$\Phi(x_1, \ldots, x_n) = \sum_{\boldsymbol{m} \in \mathbb{Z}^n \setminus \{0\}} c_{\boldsymbol{m}} x_1^{m_1} \cdots x_n^{m_n}, \quad c_{\boldsymbol{m}} \geqslant 0,$$

where $\boldsymbol{m} = (m_1, \ldots, m_n)$ and $c_{\boldsymbol{m}} \neq 0$ holds for finite number of $\boldsymbol{m}$'s. To such a potential one can assign a piece-wise linear function $\Phi^t \colon \mathbb{Z}^n \to \mathbb{Z}$ by

$$\Phi^t(\xi_1, \ldots, \xi_n) = \min_{\boldsymbol{m}; c_{\boldsymbol{m}} > 0} \sum_{k=1}^{n} m_k \xi_k$$

and a rational convex cone $(X, \Phi)^t \subset \mathbb{Z}^m$ by

$$(X, \Phi)^t := \left\{ (\xi_1, \ldots, \xi_n) \in \mathbb{Z}^n \mid \Phi^t(\xi_1, \ldots, \xi_n) \geqslant 0 \right\}.$$

Examples of positive varieties with potentials are complete and partial flag varieties for semisimple algebraic groups [16], especially the Borel subgroup $B_- \subset G$. For $G = \mathrm{SL}_2$, B_- is the set of lower triangular matrices

$$\theta \colon \mathbb{G}_{\mathbf{m}}^2 \to B_- \ : \ (a, b) \mapsto x := \begin{bmatrix} a & 0 \\ b & a^{-1} \end{bmatrix}.$$

In this case, the so-called *Berenstein-Kazhdan potential* [16] on B_- is given by

$$\Phi_{BK}(x) = ab^{-1} + a^{-1}b^{-1}.$$

For general semisimple algebraic group G, let w_0 be the longest element in the Weyl group W of G. Then each reduced word $\mathbf{i}$ of w_0 determines an open embedding (*cluster chart*)

$$\theta_{\mathbf{i}} \colon \mathbb{G}_{\mathbf{m}}^{m+r} \to B_-, \quad \text{where } r = \mathrm{rank}(G), m = \ell(w_0).$$

Together with the potential Φ_{BK}, we get a polyhedral cone $(B_-, \Phi_{BK}, \theta_{\mathbf{i}})^t$, which we call a *BK cone* of G. What is interesting and important is that the polyhedral cone $(B_-, \Phi_{BK}, \theta_{\mathbf{i}})^t$ admits a structure of Kashiwara crystal and parametrizes the canonical basis [16, 21]. Besides, the cone $(B_-, \Phi_{BK}, \theta_{\mathbf{i}})^t$ has another structure map hw^t, the so-called *highest weight* map, to the set of dominant integral weights $X_*^+(H)$ of Langlands dual group $G^\vee$. Denote by $\mathrm{hw}^{-t}(\lambda^\vee)$ the pre-image of hw^t.

Theorem 1 (Theorem 3.5.9)**.** *[16, Main Theorem 6.15] There is a direct decomposition of Kashiwara crystals:*

$$(B_-, \Phi_{BK}, \theta_{\mathbf{i}})^t = \bigsqcup_{\lambda^\vee \in X_*^+(H)} \mathrm{hw}^{-t}(\lambda^\vee).$$

Moreover, $\mathrm{hw}^{-t}(\lambda^\vee) \cong B_{\lambda^\vee}$ *as Kashiwara crystals, where* $B_{\lambda^\vee}$ *is the crystal associated with the irreducible* $G^\vee$*-module with highest weight* $\lambda^\vee$*.*

Similar story goes along with $G^\vee$. Actually, one can compare the BK cones of G and $G^\vee$. Let $\boldsymbol{I} := \{1, \ldots, r\}$ be the indexes of the Dynkin digram of $\mathfrak{g}$, $\{\alpha_1, \ldots, \alpha_r\} \subset \mathfrak{h}^*$ be the set of simple roots and $\{\alpha_1^\vee, \ldots, \alpha_r^\vee\} \subset \mathfrak{h}$ be the set of simple coroots. Fix a symmetrizer $\{d_1, \ldots, d_r\}$ of the Cartan matrix of $\mathfrak{g}$, which in turn determine an isomorphism

$$\psi \colon \mathfrak{h} \to \mathfrak{h}^* \ : \ \alpha_i^\vee \to d_i \alpha_i.$$

Note that the group G and $G^\vee$ have the same Weyl group. There is a *dual* chart $\theta_{\mathbf{i}}^\vee$ (in the context of *dual cluster variety*) for $B_-^\vee \subset G^\vee$

$$\theta_{\mathbf{i}}^\vee : \mathbb{G}_{\mathbf{m}}^{m+r} \to B_-^\vee,$$

and a (positive) rational map $\Psi_{\mathbf{i}} \colon B_- \to B_-^\vee$. The following theorem finds a relation between the BK cones of G and $G^\vee$:

Theorem 2 (Theorem 8.3.1 and Theorem 8.3.3)**.** *The tropicalization $\psi_{\mathbf{i}} := \Psi_{\mathbf{i}}^t$, with respect to charts $\theta_{\mathbf{i}}$ and $\theta_{\mathbf{i}}^\vee$, is injective*

$$\psi_{\mathbf{i}} \colon (B_-, \Phi_{BK}, \theta_{\mathbf{i}})^t \to (B_-^\vee, \Phi_{BK}^\vee, \theta_{\mathbf{i}}^\vee)^t,$$

and it extends to an isomorphism of real BK cones. Moreover, for any $i \in \boldsymbol{I}$,

$$\psi_{\mathbf{i}} \circ \tilde{e}_i = \tilde{e}_i^{d_i} \circ \psi_{\mathbf{i}}, \quad \psi_{\mathbf{i}} \circ \tilde{f}_i = \tilde{f}_i^{d_i} \circ \psi_{\mathbf{i}}, \quad (\mathrm{hw}^\vee)^t \circ \psi_{\mathbf{i}} = \psi \circ \mathrm{hw}^t,$$

where we write $\tilde{e}_i, \tilde{f}_i$ for the crystal operators on both $(B_-, \Phi_{BK}, \theta_{\mathbf{i}})^t$ and $(B_-^\vee, \Phi_{BK}^\vee, \theta_{\mathbf{i}}^\vee)^t$.

Our theorem provides a different interpretation and a new perspective on a result of Kashiwara [58] and Frenkel-Hernandez [39].

Following [57], Kashiwara crystals have natural tensor product $B_{\lambda^\vee} \otimes B_{\nu^\vee}$. Since there is geometric interpretation of $B_{\lambda^\vee}$, one natural question to ask is that if there exists a "geometric object" such that its tropicalization counts the tensor multiplicity of $G^\vee$ modules. Following the idea of "multiplicity geometrization" program, originated in [15, 16, 20, 18, 21], we introduce the notion of *geometric multiplicity* to answer this question. A geometric multiplicity is a positive variety with potential (M, Φ_M) fibered over the Cartan subgroup H of a reductive group G and additionally fibered over some split torus S. They form a category, which we denote it by $\mathbf{Mult}_G$ (see Definition 6.3.1 for more details).

Theorem 3 (Theorem 6.3.5 and Theorem 6.4.1)**.** *The category $\mathbf{Mult}_G$ is a non-strict unitless monoidal category with product $M_1 \star M_2$ given by $M_1 \star M_2 := M_1 \times M_2 \times U$. Let $M_{\lambda^\vee}^t$ be the tropical fiber over $\lambda^\vee$ of $(M, \Phi_M)^t$. Then the assignments $M \mapsto \mathcal{V}(M)$*

$$\mathcal{V}(M) := \bigoplus_{\lambda^\vee \in P_+^\vee} \mathbb{C}[M_{\lambda^\vee}^t] \otimes V_{\lambda^\vee},$$

define a monoidal functor from $\mathbf{Mult}_G$ to $\mathbf{Mod}_{G^\vee}$, the category of $G^\vee$-modules, where $\mathbb{C}[\cdot]$ is the linearization of the set.

The extra fibration over S as a part of the structure of geometric multiplicities is introduced to resolve the problem of possibly having infinite multiplicities, which happens in the following case. Note that the basic object in $\mathbf{Mult}_G$ is H with 0 potential. We define the multiplication in

$\mathbf{Mult}_G$ such that $H \star H = H^2 \times U$ has a non-trivial potential $\overline{\Delta}_2$. This potential is originated in the so-called central charge Δ_2, which is defined on the space $B_- \times B_-$ as

$$\Delta_2 := \Phi_{BK}(g_1) + \Phi_{BK}(g_2) - \Phi_{BK}(g_1 g_2).$$

By applying Theorem 3 to $H \star H$, one gets infinite multiplicities since

$$\mathcal{V}(H) = \bigoplus_{\lambda^\vee \in P_+^\vee} V_{\lambda^\vee}.$$

The product $\star$ is defined so that if M_i is additionally fibered over S_i for $i = 1, 2$, then $M_1 \star M_2$ is additionally fibered over $S_1 \times S_2 \times H$. Now for a geometric multiplicity M additionally fibered over S, its tropicalization $(M, \Phi_M)^t$ is naturally fibered over the direct product of $P_+^\vee$ and cocharacter lattice $X_*(S) := \mathrm{Hom}(\mathbb{G}_{\mathbf{m}}, S)$. Then for every cocharacter $\xi \in X_*(S)$ we define $\mathcal{V}_\xi(M)$ by

$$\mathcal{V}_\xi(M) := \bigoplus_{\lambda \in P_+^\vee} \mathbb{C}[M^t_{\lambda^\vee, \xi}] \otimes V_{\lambda^\vee}.$$

Theorem 4 (Theorem 6.4.2)**.** *Given geometric multiplicities M_i additionally fibered over S_i for $i = 1, 2$, one has the following natural isomorphism of $G^\vee$-modules*

$$\mathcal{V}_{\xi_1, \xi_2, \lambda^\vee, \nu^\vee}(M_1 \star M_2) \cong I_{\lambda^\vee}\left(\mathcal{V}_{\xi_1}(M_1)\right) \otimes I_{\nu^\vee}\left(\mathcal{V}_{\xi_2}(M_2)\right), \tag{1.1}$$

where $I_{\mu^\vee}(V)$ denotes the $\mu^\vee$-th isotypic component of a $G^\vee$-module V.

This indeed fixes the "infinite multiplicity" issue for $\mathcal{V}(H \star H)$ since (1.1) boils down to an isomorphism

$$\mathcal{V}_{\lambda^\vee, \nu^\vee}(H \star H) \cong V_{\lambda^\vee} \otimes V_{\nu^\vee}.$$

Thus the geometric multiplicity $H \star H$ (fibered over H^3) computes tensor product multiplicities $\dim \mathrm{Hom}_{G^\vee}(V_{\mu^\vee}, V_{\lambda^\vee} \otimes V_{\nu^\vee})$.

On Poisson geometry and integrable systems

Now let us dive into the world of Poisson geometry. Recall that the dual vector space $\mathfrak{k}^*$ of the Lie algebra $\mathfrak{k}$ of a compact Lie group K carries a natural *linear* Poisson structure $\pi_{\mathfrak{k}^*}$ (which is known in the literature as the Kirillov-Kostant-Souriau Poisson bracket, or Lie-Poisson structure). The coadjoint orbits $\mathcal{O}_\xi$, which are parameterized by elements ξ of the positive Weyl chamber, are the symplectic leaves of this Poisson manifold. For the unitary group U_n, Guillemin-Sternberg [50] gave a beautiful construction of global action-angle coordinates on $\mathrm{Lie}(\mathrm{U}_n)^*$, the so-called *Gelfand-Zeitlin system*. The natural inequalities of action variables defines a polyhedral cone $\mathcal{C}_{GZ}$ in $\mathbb{R}^m$ for $m = \frac{1}{2}n(n+1)$. One approach to the generalization of this construction is the method of toric degenerations. Toric degenerations were used by Harada-Kaveh in [51] to construct dense embeddings for three families of projective Kähler manifolds: generalized flag manifolds, spherical varieties, and weight varieties. If ξ is an integral weight, then $\mathcal{O}_\xi$ is projective Kähler. Toric degenerations of $\mathcal{O}_\xi$ for ξ integral were first constructed by Caldero [24]. It follows from [51] that for ξ integral, there is a global action-angle coordinates on the integral coadjoint orbits.

We are suggesting a new approach to this problem, which involves Poisson-Lie groups, positivity theory and cluster algebras. A *Poisson-Lie group* is a group object in the category of

Poisson manifolds. Poisson-Lie groups were first introduced by Drinfel'd [29] and Semenov-Tian-Shansky [80]. The results in [42, 64] show that the natural multiplicative Poisson structure π_G on $G := K^{\mathbb{C}}$ is *log-canonical* with respect to cluster coordinates, *i.e.*, the Poisson bracket with respect to cluster coordinates $\{\Delta_i\}$ takes the form: $\{\Delta_i, \Delta_j\} = \pi_{ij}\Delta_i\Delta_j$ for $\pi_{ij} \in \mathbb{Q}$. The Poisson structure π_G induces a Poisson structure π_K on K such that (K, π_K) is Poisson-Lie group. Poisson-Lie groups have natural duals, which are again Poisson-Lie groups. Note that the Poisson space $\mathfrak{k}^*$ is an abelian Poisson-Lie group, which is the Poisson-Lie dual of $(K, 0)$. Denote by (K^*, π_{K^*}) the Poisson-Lie dual of (K, π_K). What is interesting is that these two (dual) Poisson-Lie groups are isomorphic as Poisson manifolds. In [43], the authors show that there exists a Poisson isomorphism, the so-called *Ginzburg-Weinstein isomorphism*:

$$\mathrm{GW}\colon (\mathfrak{k}^*, \pi_{k^*}) \xrightarrow{\sim} (K^*, \pi_{K^*}).$$

In [4], the authors considered the dual Poisson-Lie group U_n^* of U_n and showed for the dual Poisson-Lie group U_n^*, the dual Poisson bracket has the form

$$\{\Delta_i, \Delta_j\} = \Delta_i\Delta_j(\pi_{ij} + f_{ij}),$$

where $\pi_{ij} \in \mathbb{Q}$ and f_{ij} are functions on U_n^*. They showed that the tropicalization f_{ij}^t's of f_{ij}'s define a polyhedral cone which is isomorphic to the Gelfand-Zeitlin cone:

$$\bigcap \mathcal{C}_{f_{ij}} \cong \mathcal{C}_{GZ}.$$

We generalize this results to any semisimple compact Lie group K by using the techniques from cluster theory and by the procedure of partial tropicalization that we introduced. In more details, let $G = K^{\mathbb{C}}$ be the complexification of K. Note that the group G admits an Iwasawa decomposition $G = U_-AK$ and one can identify the dual Poisson Lie group K^* of K with $U_-A \subset B_-$. Denote by Δ_k for $k \in [-r, -1] \cup [1, m]$ the natural coordinates of open embedding $\theta_{\mathbf{i}}$. In other words, we have:

$$\Delta_{\mathbf{i}}\colon B_- \to (\mathbb{C}^\times)^{m+r} \ : \ b \mapsto (\Delta_{-r}(b), \ldots, \Delta_m(b)).$$

Denote by $\mathfrak{C}_s$, for $s < 0$, the following change of coordinates: $\Delta_k = \exp(s\xi_k + \mathbb{i}\varphi_k)$. Note that Δ_k for $k \in [-r, -1]$ is real-valued on K^*, the phases φ_k vanish for such k. In summary, we have the following chain of birational isomorphisms

$$B_- \supset K^* \xrightarrow{\Delta_{\mathbf{i}}} \mathbb{R}^r \times (\mathbb{C}^\times)^m \xrightarrow{\mathfrak{C}_s} \mathbb{R}^{m+r} \times (S^1)^m.$$

The pushforward of the scaled Poisson bi-vector $s\pi_{K^*}$ through $\mathfrak{C}_s \circ \Delta_{\mathbf{i}}$ gives a Poisson bi-vector $\pi_s^{\mathbf{i}}$ on $\mathbb{R}^{m+r} \times (S^1)^m$. Denote by $(\mathfrak{t}_+^*)^\circ$ the interior of the positive Weyl chamber and identify

$$\eta := -\mathbb{i}\psi\colon \mathfrak{t}^* \xrightarrow{\sim} X_*(H) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Denote by $\mathcal{C}_{\mathbf{i}}$ the topological interior of the real extension $(B_-, \Phi_{BK}, \theta_{\mathbf{i}}) \otimes_{\mathbb{Z}_+} \mathbb{R}_+$ of a BK cone.

Theorem 5 (Theorem 7.6.2, Proposition 7.7.2 and Theorem 8.5.3)**.** *The following holds true:*

- *The limit* $\pi_\infty^{\mathbf{i}} := \lim_{s \to -\infty} \pi_s^{\mathbf{i}}\big|_{\mathcal{C}_{\mathbf{i}} \times (S^1)^m}$ *exists, and defines a constant bi-vector.*
- *For* $\xi \in (\mathfrak{t}_+^*)^\circ$*, the symplectic leaves of* $\pi_\infty^{\mathbf{i}}$ *are of form* $\mathcal{P}_\xi := \Delta_\xi \times (S^1)^m$*, where* $\Delta_\xi := \mathrm{hw}^{-t}(\eta(\xi))$ *is a polytope. Denote by* ω_∞^ξ *the symplectic form on* $\mathcal{P}_\xi$*.*

- *For* $\xi \in (\mathfrak{t}_+^*)^\circ$*, the symplectic volume of* $\mathcal{P}_\xi$ *is given by* $\mathrm{Vol}(\mathcal{P}_\xi, \omega_\infty^\xi) = \mathrm{Vol}(\mathcal{O}_\xi, \omega_\xi)$.

Combining Theorem 5 with the scaled Ginzburg-Weinstein diffeomorphism for $s < 0$

$$\mathrm{GW}_s \colon (\mathfrak{k}^*, \pi_{\mathfrak{k}^*}) \xrightarrow{\sim} (K^*, s\pi_{K^*}),$$

we have the following

Conjecture 6 (Conjecture 9.2.3)**.** *There exists a scaled Ginzburg-Weinstein map such that*

$$\lim_{s \to -\infty} \mathfrak{E}_s \circ \Delta_{\mathbf{i}} \circ GW_s$$

exists on an open dense subset of $\mathfrak{k}^*$ *and defines a Poisson diffeomorphism.*

For $K = \mathrm{U}_n$, the conjecture was proven in [9], and it recovers the Gelfand-Zeitlin system. Following the notation in Theorem 5, denote by Δ_ξ^δ the δ-interior of polytope Δ_ξ and $\mathcal{P}_\xi^\delta := \Delta_\xi^\delta \times (S^1)^m$. The best result towards this conjecture we get so far is as follows:

Theorem 7 (Theorem 9.5.1)**.** *For any* $\varepsilon > 0$*, there exist* $\delta > 0$ *and a symplectic embedding* $\mathcal{P}_\xi^\delta \hookrightarrow \mathcal{O}_\xi$ *such that*

$$\mathrm{Vol}(\mathcal{O}_\xi, \omega_\xi) \geqslant \mathrm{Vol}(\mathcal{P}_\xi^\delta, \omega_\infty^\xi) \geqslant \mathrm{Vol}(\mathcal{O}_\xi, \omega_\xi) - \varepsilon.$$

In other words, we have proven that on an open chart of $\mathcal{O}_\xi$, which exhausts almost all of the symplectic volume, there exist action-angle coordinates.

Organization of the thesis

- In Chapter 2, we recall the general terminology and notation for semisimple algebraic groups: root datum, SL_2-triples and generalized minors. Then we give a detailed discussion on twist maps, evaluation of generalized minors and the factorization problem.

- In Chapter 3, we recall the notion of affine tropical varieties, positive varieties with potentials and tropicalization functor. As an example of positive varieties with potential, we focus on Borel subgroup B_- of an reductive algebraic group G with the Berenstein-Kazhdan potential Φ_{BK}.

- In Chapter 4, we recall basic definitions on (homogeneous) cluster varieties and their dual cluster varieties. Then we focus on the double Bruhat cell $G^{w_0,e}$, which is homogeneous and admits a natural dual.

- In Chapter 5, we recall the notion of Poisson-Lie groups and its real forms. Then we recall the Ginzburg-Weinstein isomorphisms between the Poisson space $\mathfrak{k}^*$ and K^*.

- Chapter 6 is based on a joint work [17] with A. Berenstein. We introduce the notion of geometric multiplicities of a reductive group G, which form a monoidal category. We then construct a functor from this category to the representation of the Langlands dual group $G^\vee$. Using this, we manage to compute various multiplicities of $G^\vee$ modules in many ways.

- Chapter 7 is based on a joint work [5] with A. Alekseev, A. Berenstein, and B. Hoffman. We define a positive structure and potential on G^* and show that the natural Poisson-Lie structure on G^* is weakly log-canonical with respect to this positive structure and potential. Using this construction, we assign to the real form $K^* \subset G^*$ an integrable system on $\mathrm{PT}(G^*)$, which is a product of the decorated string cone and the compact torus of dimension $\frac{1}{2}(\dim G - \operatorname{rank} G)$.

- Chapter 8 is based on a joint work [6] with A. Alekseev, A. Berenstein, and B. Hoffman. For a semisimple algebraic group G, we explain a relation between its Langlands dual group $G^\vee$ and its Poisson-Lie dual group G^*. That is, the integral cone defined by the Berenstein-Kazhdan potential on Borel subgroup $B_-^\vee \subset G^\vee$ is isomorphic to the integral Bohr-Sommerfeld cone defined by the Poisson structure on the partial tropicalization $\mathrm{PT}(G^*)$.

- Chapter 9 is based on a joint work [9] with A. Alekseev and J. Lane and joint works [7, 8] with A. Alekseev, B. Hoffman and J. Lane. We first show that one can recover the Gelfand-Zeitlin system of $\mathfrak{su}_n^*$ by Ginzburg-Weinstein isomorphism and partial tropicalization. Then we manage to construct *big* action-angle variables for coadjoint orbits on the dual space of any semisimple compact Lie algebra $\mathfrak{k}$.

2 | Preliminaries on Lie Theory

In this chapter, we recall the general terminology and notation for semisimple algebraic groups. We start by considering the root datum and SL_2-triples. Then we introduce the notion of generalized minors and discuss in more details about twist maps, evaluation of generalized minors and the factorization problem. Most material are based on [21, 37].

2.1 Root datum of semisimple algebraic groups

Let $A = [a_{ij}]_{i,j\in \boldsymbol{I}}$ be a Cartan matrix for a index set $\boldsymbol{I} = \{1, \dots, r\}$, *i.e.*, $a_{ii} = 2$ and $a_{ij} \in \mathbb{Z}_{\leqslant 0}$ for $i \neq j$, and there exists a sequence of positive integers $\boldsymbol{d} = \{d_1, \dots, d_r\}$ called a symmetrizer so that $a_{ij}d_j = a_{ji}d_i$. Let $D := \mathrm{diag}(d_1, \dots, d_r)$, then the matrix AD is positive-definite, and $(AD)^T = AD$.

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the semisimple Lie algebra over $\mathbb{Q}$ corresponding to the Cartan matrix A. Recall that $\mathfrak{g}$ is generated by $\{E_i, F_i\}_{i=1}^r$ subject to the Serre relations [54]. Denote by $\alpha_i^\vee = [E_i, F_i]$ the i th simple coroot and by $\mathfrak{h}$ the span of all simple coroots. Let $\mathfrak{h}^*$ be the linear dual space and choose a basis of simple roots $\alpha_1, \dots, \alpha_r \in \mathfrak{h}^*$ such that

$$\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}. \tag{2.1}$$

Using this definition and a chosen symmetrizer $\boldsymbol{d}$, define a symmetric bilinear form on $\mathfrak{h}$ such that $(\alpha_i^\vee, \alpha_j^\vee) := a_{ij}d_j$. This form uniquely extends to a $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$, and induces a symmetric bilinear form on $\mathfrak{h}^*$:

$$(\alpha_i, \alpha_j) = d_i^{-1} a_{ij},$$

as well as an isomorphism $\psi \colon \mathfrak{h} \to \mathfrak{h}^*$ such that $\psi(\alpha_i^\vee) = d_i \alpha_i$. The formulas above imply the following standard identities:

$$d_i = \frac{a_{ii}}{(\alpha_i, \alpha_i)} = \frac{2}{(\alpha_i, \alpha_i)}, \quad a_{ij} = d_i(\alpha_i, \alpha_j) = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}, \quad \psi(\alpha_i^\vee) = \frac{2\alpha_i}{(\alpha_i, \alpha_i)}.$$

Fix a positive integer d such that each d_i divides d (we can choose $d = \mathrm{lcm}\{d_1, \dots, d_i\}$ for instance). Note that $\boldsymbol{d}^\vee := \{d_i^\vee := d/d_i\}$ defines a symmetrizer for the transposed Cartan matrix $A^\vee = A^T = [a_{ji}]$. Indeed,

$$A^\vee D^\vee = dA^T D^{-1} = d(D^{-1}AD)D^{-1} = dD^{-1}A = (A^\vee D^\vee)^T.$$

Define the dual Lie algebra $\mathfrak{g}^\vee = \mathfrak{g}(A^\vee)$ with generators $E_i^\vee, F_i^\vee$, and choose the standard identification $\mathfrak{h}^\vee = \mathfrak{h}^*$ via

$$[E_i^\vee, F_i^\vee] = \alpha_i.$$

The symmetrizer $\boldsymbol{d}^\vee$ defines new symmetric bilinear forms $(\cdot,\cdot)^\vee$ on $\mathfrak{h} = (\mathfrak{h}^\vee)^*$ and $\mathfrak{h}^* = \mathfrak{h}^\vee$ as well as a map $\psi^\vee : \mathfrak{h}^* \to \mathfrak{h}$. It is easy to check that

$$(\cdot,\cdot)^\vee_{\mathfrak{h}} = d^{-1}(\cdot,\cdot)_{\mathfrak{h}}, \quad (\cdot,\cdot)^\vee_{\mathfrak{h}^*} = d(\cdot,\cdot)_{\mathfrak{h}^*}, \quad \psi^\vee = d\psi^{-1}.$$

The fundamental weights $\omega_i \in \mathfrak{h}^*$ associated to the given simple coroots are defined by

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}. \tag{2.2}$$

The lattice generated by $\{\omega_i\}$ is the weight lattice of $\mathfrak{g}$, which we denote by P. By (2.1) and (2.2), one has

$$(\alpha_1,\ldots,\alpha_r) = (\omega_1,\ldots,\omega_r)A, \qquad \textit{i.e.}, \ \alpha_i = \sum_{j=1}^r a_{ji}\omega_j. \tag{2.3}$$

Let Q be the root lattice and $P^\vee = \mathrm{Hom}(Q,\mathbb{Z}) \subset \mathfrak{h}$ be the dual lattice of Q with dual basis $\{\omega_i^\vee\}$. Thus

$$(\alpha_i^\vee, \omega_j^\vee) = \langle \alpha_i^\vee, \psi(\omega_j^\vee)\rangle = d_j\delta_{ij}, \quad (\alpha_1^\vee,\ldots,\alpha_r^\vee) = (\omega_1^\vee,\ldots,\omega_r^\vee)A^T.$$

Let $Q^\vee = \mathrm{Hom}(P,\mathbb{Z}) \subset \mathfrak{h}$ be the dual lattice of P, which is just the coroot lattice.

Now let us recall the notion of character and cocharacter lattice. Let $\mathbb{G}_{\mathbf{m}}$ be the multiplicative group defined over $\mathbb{Q}$. Let G be a semisimple algebraic group defined over $\mathbb{Q}$ with Lie algebra $\mathfrak{g}$. Let H be the maximal torus of G and $X^*(H) = \mathrm{Hom}(H,\mathbb{G}_{\mathbf{m}})$ the character lattice of H. For any $\gamma \in X^*(H)$, denote the multiplicative character by $\gamma\colon h \mapsto h^\gamma$. Let $X_*(H) = \mathrm{Hom}(\mathbb{G}_{\mathbf{m}}, H)$ be the cocharacter lattice of H. Define the subset

$$X^*_+ = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee\rangle \in \mathbb{Z}_{\geqslant 0} \text{ for all } i \in \boldsymbol{I}\} \subset X^*(H),$$

which is the set of dominant weights of G.

In summary, we have the following lattices:

$$Q \subset X^*(H) \subset P; \quad Q^\vee \subset X_*(H) \subset P^\vee.$$

Example 2.1.1. Let $G = \mathrm{SL}_2$ and H be the subgroup of diagonal matrices. The roots of $\mathfrak{sl}_2$ give the following characters of H

$$\alpha\colon \begin{bmatrix} a & 0 \\ 0 & a^{-1}\end{bmatrix} \mapsto a^2, \ -\alpha\colon \begin{bmatrix} a & 0 \\ 0 & a^{-1}\end{bmatrix} \mapsto a^{-2}.$$

Therefore $X^*(H) = \frac{1}{2}\mathbb{Z}\alpha = \mathbb{Z}\omega$, where $\omega = \frac{1}{2}\alpha$ is the only fundamental weight. The cocharacter lattice is $X_*(H) = \mathbb{Z}\alpha^\vee$, where $\alpha^\vee$ is the simple coroot of the root α. The dual of the weight lattice is $\mathbb{Z}\alpha^\vee$. Thus we know:

$$Q(\mathfrak{sl}_2) \subset X^*(H) = P(\mathfrak{sl}_2); \quad Q^\vee(\mathfrak{sl}_2) = X_*(H) \subset P^\vee(\mathfrak{sl}_2).$$

Definition 2.1.2. The quadruple of $(X^*, Q; X_*, Q^\vee)$ is called *root datum* of G, and the dual root datum $(X_*, Q^\vee; X^*, Q)$ is defined by switching characters with cocharacters, and roots with coroots.

The Langlands dual group $G^\vee$ is the connected semisimple group whose root datum is dual to that of G. Let $H^\vee$ be the maximal torus of $G^\vee$. If G is semisimple, the map ψ restricts to cocharacter lattice:

Proposition 2.1.3. *There exists a symmetrizer $\boldsymbol{d}$ such that the isomorphism ψ restricts to a lattice (abelian group) homomorphism*

$$\psi\colon X_*(H) \to X^*(H) = X_*(H^\vee),$$

which induces a group homomorphism $\Psi^H\colon H \to H^\vee$.

Proof. Since $X_*(H) \subset P^\vee$ and $Q \subset X^*(H)$, it suffices to show that $\boldsymbol{d}$ can be chosen so that $\psi(P^\vee) \subset Q$. Considering (2.3) for the Lie algebra $\mathfrak{g}^\vee$ gives

$$(\omega_1^\vee, \dots, \omega_r^\vee) = (\alpha_1^\vee, \dots, \alpha_r^\vee) A^{-T}, \tag{2.4}$$

where we write $A^{-T} = (A^T)^{-1}$. Applying $\psi : \mathfrak{h} \to \mathfrak{h}^*$ to both sides of (2.4), one finds

$$(\psi(\omega_1^\vee), \dots, \psi(\omega_r^\vee)) = (\psi(\alpha_1^\vee), \dots, \psi(\alpha_r^\vee)) A^{-T} = (\alpha_1, \dots, \alpha_r) D A^{-T}.$$

It is enough then to choose $\boldsymbol{d}$ so that DA^{-T} is an integer matrix; since A is invertible over $\mathbb{Q}$, this is always possible. $\diamondsuit$

Note that if G is simply connected, any symmetrizer $\boldsymbol{d}$ satisfies Proposition 2.1.3. In the remainder of the paper, we fix a symmetrizer $\boldsymbol{d}$ as in Proposition 2.1.3.

Example 2.1.4. Here we list some examples of Langlands dual groups:

$$\mathrm{SL}_n^\vee = \mathrm{PSL}_n,\ \mathrm{SO}_{2n+1}^\vee = \mathrm{Sp}_{2n},\ \mathrm{Spin}_{2n}^\vee = \mathrm{SO}_{2n}/\{\pm 1\},\ \mathrm{SO}_{2n}^\vee = \mathrm{SO}_{2n}.$$

2.2 SL_2-Triples

Let G be a semisimple algebraic group as before. Fix a pair of opposite Borel subgroups B, B_- of G containing H. Denote by U and U_- the corresponding unipotent radicals of B and B_-. Each triple $\alpha_i^\vee, E_i, F_i$ determines a group homomorphism $\phi_i\colon \mathrm{SL}_2 \to G$ given by

$$\phi_i \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} = \exp(aF_i) \subset U_-,\ \phi_i \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \exp(aE_i) \subset U,\ \phi_i \begin{bmatrix} c & 0 \\ 0 & c^{-1} \end{bmatrix} = \alpha_i^\vee(c) \subset H$$

for $a \in \mathbb{G}_{\mathbf{a}}$ and $c \in \mathbb{G}_{\mathbf{m}}$. Let $W = N(H)/H$ be the Weyl group of G and $s_i \in W$ be the simple reflection generated by simple root α_i. Let w_0 be the longest element in W with length

$$m = \ell(w_0).$$

The action of W on H gives rise to the action of W on the character lattice $X^*(H)$, *i.e.*,

$$h^{w(\gamma)} = (w^{-1} h w)^\gamma, \quad \gamma \in X^*(H),\ h \in H.$$

Using the SL_2 homomorphisms $\{\phi_i\}$, define for $i \in \boldsymbol{I}$,

$$\overline{s_i} := \phi_i \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ x_i(t) := \phi_i \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix},\ y_i(t) := \phi_i \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix},\ x_{-i}(t) := \phi_i \begin{bmatrix} t^{-1} & 0 \\ 1 & t \end{bmatrix}. \tag{2.5}$$

Since $\overline{s}_i$'s satisfy the Coxeter relations of W, any reduced expression of $w \in W$ gives the same lift $\overline{w} \in G$. For $i \in \boldsymbol{I}$, define the elementary (additive) character χ_i of U by

$$\chi_i(x_j(t)) = \delta_{ij} \cdot t, \quad \text{for } t \in \mathbb{G}_{\mathbf{a}}.$$

Denote by $\chi^{\text{st}} = \sum \chi_i$ the standard character of U.

Let $x \mapsto x^\iota$ be the group antiautomorphism of G given by

$$\alpha_i^\vee(c)^\iota = \alpha_i^\vee(-c), \quad x_i(t)^\iota = x_i(t), \quad y_i(t)^\iota = y_i(t), \ i \in \boldsymbol{I}.$$

Thus we know $x_{-i}(t)^\iota = x_{-i}(t^{-1})$ for $i \in \boldsymbol{I}$. Similarly, let $x \mapsto x^T$ be the group antiautomorphism of G given by

$$\alpha_i^\vee(c)^T = \alpha_i^\vee(c), \quad x_i(t)^T = y_i(t), \quad y_i(t)^T = x_i(t), \ i \in \boldsymbol{I}.$$

2.3 Generalized minors

Let $G_0 = U_- H U \subset G$ be the *Gaussian decomposable* elements of G. Thus $x \in G_0$ has a unique decomposition $x = [x]_-[x]_0[x]_+$ for $[x]_- \in U_-$, $[x]_0 \in H$, and $[x]_+ \in U$. We abbreviate

$$[x]_{\leqslant 0} := [x]_-[x]_0, \quad [x]_{\geqslant 0} := [x]_0[x]_+.$$

For a dominant weight $\mu \in X_+^*(H)$, the *principal minor* $\Delta_\mu \in \mathbb{Q}[G]$ is uniquely determined by

$$\Delta_\mu(u_- a u) := \mu(a), \text{ for any } u_- \in U_-, a \in H, u \in U.$$

For two weights γ and δ in W orbit of μ, *i.e.*, $\gamma = w\mu$ and $\delta = v\mu$ for $w, v \in W$, the *generalized minor* $\Delta_{w\mu, v\mu} \in \mathbb{Q}[G]$ is given by

$$\Delta_{\gamma,\delta}(g) = \Delta_{w\mu,v\mu}(g) := \Delta_\mu(\overline{w}^{-1} g \overline{v}), \text{ for all } g \in G.$$

Proposition 2.3.1. *[37, (2.25) and Lemma 2.25] For any generalized minor $\Delta_{\gamma,\delta}$ and $x \in G$:*

$$\Delta_{\gamma,\delta}(x) = \Delta_{-\delta,-\gamma}(x^\iota) = \Delta_{\delta,\gamma}(x^T); \quad \Delta_{\gamma,\delta}(a_1 x a_2) = a_1^\gamma a_2^\delta \Delta_{\gamma,\delta}(x), \textit{ for } a_1, a_2 \in H. \tag{2.6}$$

We conclude this section by recalling how generalized minors appear in representations of G. Recall that the coordinate algebra $\mathbb{Q}[G]$ can be realized as certain subalgebra of $U(\mathfrak{g})^* := \mathrm{Hom}_{\mathbb{Q}}(U(\mathfrak{g}), \mathbb{Q})$ such that the evaluation pairing $(f, x) \to f(x)$ for $f \in \mathbb{Q}[G]$ and $x \in U(\mathfrak{g})$ is non-degenerate. This turns $\mathbb{Q}[G]$ into a $U(\mathfrak{g}) \otimes U(\mathfrak{g})$-module as left actions in the natural way. In particular, for $x \in \mathfrak{n} \oplus \mathfrak{n}_-$,

$$(x \cdot f)(g) := \frac{d}{dt}\Big|_{t=0} f\left(\exp(-tx) g\right), \quad (f \cdot x)(g) := \frac{d}{dt}\Big|_{t=0} f\left(g \exp(tx)\right).$$

Denote by $U^L(\mathfrak{g})$ (resp. $U^R(\mathfrak{g})$) for the action of $U(\mathfrak{g}) \otimes \mathbb{1}$ (resp. $\mathbb{1} \otimes U(\mathfrak{g})$). By algebraic Peter-Weyl Theorem, we have the following $U(\mathfrak{g}) \otimes U(\mathfrak{g})$-modules isomorphism

$$\mathbb{Q}[G] \cong \bigoplus_{\lambda \in X_+^*(H)} V_\lambda \otimes V'_\lambda,$$

where V_λ (resp. V'_λ) is the irreducible $U^L(\mathfrak{g})$ (resp. $U^R(\mathfrak{g})$) module with highest weight λ.

An element $v \otimes v' \in V_\lambda \otimes V'_\lambda$, as a function on G, evaluates at $g \in G$ by

$$v \otimes v'(g) = \langle v, g.v' \rangle,$$

where $g.v'$ is the action of g on v', and $\langle \cdot, \cdot \rangle$ is the unique paring such that $\langle v_{w_0\lambda}, v_\lambda \rangle = 1$ and $\langle v, g.v' \rangle = \langle (g^{-1}).v, v' \rangle$.

Then as functions, we have $v_{w_0\lambda} \otimes v_\lambda = \Delta_{w_0\lambda,\lambda}$. Extend $v_\lambda, v_{w_0\lambda}$ to a weight basis $v_1, \ldots, v_n$ of V_λ such that $v_1 = v_\lambda$ and $v_n = w_0\lambda$.

Moreover, the function $v_j \otimes v_k$ is a linear combination of terms of the form $F_{\mathbf{j}} \cdot \Delta_\mu \cdot F_{\mathbf{k}}$, which satisfy the condition

$$h \cdot (F_{\mathbf{j}} \cdot \Delta_\mu \cdot F_{\mathbf{k}}) \cdot h' = h^{-\operatorname{wt}(v_j)} (h')^{\operatorname{wt}(v_k)} (F_{\mathbf{j}} \cdot \Delta_{w_0\mu,\mu} \cdot F_{\mathbf{k}}), \quad \text{for } h, h' \in H. \tag{2.7}$$

where $F_{\mathbf{j}} = F_{j_1} F_{j_2} \cdots F_{j_n} \in U(\mathfrak{g})$ for a sequence of indices $\mathbf{j} = (j_1, \ldots, j_n)$ in $\boldsymbol{I}$.

2.4 Double Bruhat cells and their factorization parameters

In this section, we introduce the notion of (reduced) double Bruhat cells and discuss their basic properties. For a pair of Weyl group elements (u, v), a *double Bruhat cell* is defined by:

$$G^{u,v} := BuB \cap B_- v B_-.$$

Meanwhile, we introduce the so-called *reduced* double Bruhat cell associated to (u, v):

$$L^{u,v} := U\overline{u}U \cap B_- v B_-.$$

Note that multiplication in G induces a biregular isomorphism $H \times L^{u,v} \cong G^{u,v}$.

Let $\widehat{L}^{u,v}$ be the reduced double Bruhat cell of the universal cover $\widehat{G}$ of G and let

$$p \colon \widehat{G} \to G$$

be the covering map. The cell $\widehat{L}^{u,v}$ can be characterized by the following

Proposition 2.4.1. *[21, Proposition 4.3] An element $x \in \widehat{G}^{u,v}$ belongs to $\widehat{L}^{u,v}$ if and only if*

$$\Delta_{u\omega_i,\omega_i}(x) = 1, \quad \forall i \in \boldsymbol{I}.$$

Corollary 2.4.2. *The restriction of p to $\widehat{L}^{u,v}$ is a biregular isomorphism $\widehat{L}^{u,v} \to L^{u,v}$.*

Proof. For $h \in \widehat{H}$, we know $h = \mathrm{Id}$ if and only if $h^{\omega_i} = 1$ for all $i \in \boldsymbol{I}$. Let $x \in L^{u,v}$ and consider some $\widehat{x}, \widehat{x}' \in p^{-1}(x) \subset \widehat{L}^{u,v}$. Then $\widehat{x}' = \widehat{x}h$ for some $h \in \widehat{H}$. By Proposition 2.4.1 and (2.3.1) we have $h^{\omega_i} = 1$ for all i, which implies there is a unique lift of x. ♢

Therefore the generalized minors $\Delta_{u\omega_i,v\omega_i}$ can be viewed as well defined functions on $L^{u,v}$ under the isomorphism p. By abuse of notation, we write $\Delta_{u\omega_i,v\omega_i}(z)$ for $z \in L^{u,v}$ instead of $\Delta_{u\omega_i,v\omega_i}(p^{-1}z)$.

Next, we introduce the factorization parameter of (reduced) double Bruhat cells.

A *double reduced word* $\mathbf{i} = (i_1, \ldots, i_n)$ for (u, v) is a shuffle of a reduced word for u, written in the alphabet $\{-r, \ldots, -1\}$, and a reduced word for v, written in the alphabet $\{1, \ldots, r\}$, where $n = \ell(u) + \ell(v)$. Denote by $R(u, v)$ the set of double reduced word for (u, v). Given $\mathbf{i} = (i_1, \ldots, i_n) \in R(u, v)$, denote by

$$\mathbf{i}^{\mathrm{op}} := (-i_n, -i_{n-1} \ldots, -i_1) \in R(v^{-1}, u^{-1}), \quad -\mathbf{i} := (-i_1, -i_2 \ldots, -i_n) \in R(v, u).$$

Remark 2.4.3. The minus signs on the letters of the reduced word for u are occasionally troublesome, we make the following abbreviations. For $i, j \in [-r, -1] \cup [1, r]$, let

$$d_i = d_{|i|},\ a_{i,j} = a_{|i|,|j|},\ \omega_i = \omega_{|i|}, s_i = s_{|i|}$$

extending the notation for the skew-symmetrizer, Cartan matrix, and fundamental weights, and simple reflection respectively. Our notation is set up in this way to agree with that of [14].

Proposition 2.4.4. *[21, Proposition 4.5] Given a double reduced word* $\mathbf{i} = (i_1, \ldots, i_n)$ *of* $(u, v) \in W \times W$*, the following map*

$$x_{\mathbf{i}} \colon \mathbb{G}_{\mathbf{m}}^n \xrightarrow{\sim} \widehat{L}^{u,v} \xrightarrow{p} L^{u,v} \ :\ (t_1, \ldots, t_n) \mapsto x_{i_1}(t_1) \cdots x_{i_n}(t_n), \tag{2.8}$$

is an open embedding of $L^{u,v}$.

Thus factoring $G^{u,v}$ as $H \times L^{u,v}$ gives open embedding of $G^{u,v}$:

$$x_{\mathbf{i}} \colon H \times \mathbb{G}_{\mathbf{m}}^n \xrightarrow{\sim} G^{u,v} = H \times L^{u,v} \ :\ (h, t_1, \ldots, t_n) \mapsto h x_{i_1}(t_1) \cdots x_{i_n}(t_n). \tag{2.9}$$

We have overloaded the notation $x_{\mathbf{i}}$ here but the meaning will be clear from context.

By the definition of x_i for $i < 0$, we have the following

Proposition 2.4.5. *[21, Lemma 6.1] For any double word* $(j_1, \ldots, j_n)$*, we have:*

$$x_{j_1}(t_1) \cdots x_{j_n}(t_n) = z_{j_1}(t'_1) \cdots z_{j_n}(t'_n) \cdot \prod_{j_l < 0} t_l^{-\alpha^\vee_{j_l}}, \quad \textit{where } t'_l = t_l \prod_{k<l, j_k<0} t_i^{\operatorname{sgn}(-i_l) a_{j_k, j_l}},$$

where $z_i := x_i$ *if* $i > 0$ *and* $z_i := y_i$ *if* $i < 0$.

Proof. First of all, write $x_{-i}(t) = y_i(t)\alpha_i^\vee(t^{-1})$ for $i > 0$. Then use the fact that $hx_i(t) = x_i(h^{\alpha_i} t)h$ and $hy_i(t) = y_i(h^{-\alpha_i} t)h$ for any $h \in H$ and $i > 0$. ♢

2.5 Transition maps for d-moves

In this section, we describe the transition maps for d-moves, which were computed in[21, Theorem 3.1]. For the convenience of the reader, we list these results here. The tuples $t_1, \ldots, t_d$ and $p_1, \ldots, p_d$ are related by

$$x_i(t_1)x_j(t_2)x_i(t_3) \cdots = x_j(p_1)x_i(p_2)x_j(p_3) \cdots\ . \tag{2.10}$$

Proposition 2.5.1. *For* $i, j \in [1, r]$*, denote by* d *the order of* $s_i s_j$ *in* W*. Then the transition map in* (2.10) *is given as follows:*

(1) *Type* $A_1 \times A_1$*: if* $a_{ij} = a_{ji} = 0$ *then* $d = 2$*, and*

$$p_1 = t_2,\ p_2 = t_1.$$

(2) *Type* A_2*: if* $a_{ij} = a_{ji} = -1$ *then* $d = 3$*, and*

$$p_1 = \frac{t_2 t_3}{t_1 + t_3},\ p_2 = t_1 + t_3,\ p_3 = \frac{t_1 t_2}{t_1 + t_3}\ .$$

(3) *Type B_2: if $a_{ij} = -2$, $a_{ji} = -1$ then $d = 4$, and*

$$p_1 = t_2 t_3^2 t_4 \pi_2^{-1}, \quad , p_2 = \pi_2 \pi_1^{-1}, \quad p_3 = \pi_1^2 \pi_2^{-1}, \quad p_4 = t_1 t_2 t_3 \pi_1^{-1},$$

where $\pi_1 = t_1 t_2 + (t_1 + t_3) t_4$ and $\pi_2 = t_1^2 t_2 + (t_1 + t_3)^2 t_4$.

(4) *Type G_2: if $a_{ij} = -3$, $a_{ji} = -1$ then $d = 6$, and*

$$\begin{aligned} p_1 &= t_2 t_3^3 t_4^2 t_5^3 t_6 \pi_3^{-1}, & p_2 &= \pi_3 \pi_2^{-1}, & p_3 &= \pi_2^3 \pi_3^{-1} \pi_4^{-1}, \\ p_4 &= \pi_4 \pi_1^{-1} \pi_2^{-1}, & p_5 &= \pi_1^3 \pi_4^{-1}, & p_6 &= t_1 t_2 t_3^2 t_4 t_5 \pi_1^{-1}, \end{aligned}$$

where

$$\begin{aligned} \pi_1 &= t_1 t_2 t_3^2 t_4 + t_1 t_2 (t_3 + t_5)^2 t_6 + (t_1 + t_3) t_4 t_5^2 t_6 \ , \\ \pi_2 &= t_1^2 t_2^2 t_3^3 t_4 + t_1^2 t_2^2 (t_3 + t_5)^3 t_6 + (t_1 + t_3)^2 t_4^2 t_5^3 t_6 + t_1 t_2 t_4 t_5^2 t_6 (3 t_1 t_3 + 2 t_3^2 + 2 t_3 t_5 + 2 t_1 t_5) \\ \pi_3 &= t_1^3 t_2^2 t_3^3 t_4 + t_1^3 t_2^2 (t_3 + t_5)^3 t_6 + (t_1 + t_3)^3 t_4^2 t_5^3 t_6 + t_1^2 t_2 t_4 t_5^2 t_6 (3 t_1 t_3 + 3 t_3^2 + 3 t_3 t_5 + 2 t_1 t_5) \\ \pi_4 &= t_1^2 t_2^2 t_3^3 t_4 \big(t_1 t_2 t_3^3 t_4 + 2 t_1 t_2 (t_3 + t_5)^3 t_6 + (3 t_1 t_3 + 3 t_3^2 + 3 t_3 t_5 + 2 t_1 t_5) t_4 t_5^2 t_6 \big) \\ &\quad + t_6^2 \big(t_1 t_2 (t_3 + t_5)^2 + (t_1 + t_3) t_4 t_5^2 \big)^3 \ . \end{aligned}$$

(5) *In each of the cases* (1)–(4) *above, interchanging a_{ij} with a_{ji}, the corresponding transition map in* (2.10) *is obtained from the given one by sending $p_k \to p_{d+1-k}$, $t_k \to t_{d+1-k}$.*

Following [37, (2.5), (2.11)], the transition maps for mixed 2-moves are given by:

Proposition 2.5.2. *For any $i, j \in [1, r]$, we have $x_j(t_1) x_{-i}(t_2) = x_{-i}(p_1) x_j(p_2)$, where*

$$\begin{cases} p_1 = t_2, \ p_2 = t_1 t_2^{a_{ij}}, & \text{for } i \neq j, \\ p_1^{-1} = t_1 + \dfrac{1}{t_2}, \quad p_2^{-1} = \dfrac{1}{t_2} \left(1 + \dfrac{1}{t_1 t_2} \right), & \text{for } i = j. \end{cases}$$

Finally, the transition maps for negative d-moves are given as follows.

Proposition 2.5.3. *For $i, j \in [-1, -r]$, denote by d be the order of $s_i s_j$ in W. Then the transition map in* (2.10) *is given as follows:*

(1) *Type $A_1 \times A_1$: if $a_{i,j} = a_{j,i} = 0$ then $d = 2$, and*

$$p_1 = t_2, \quad p_2 = t_1.$$

(2) *Type A_2: if $a_{i,j} = a_{j,i} = -1$ then $d = 3$, and*

$$p_1^{-1} = \frac{1}{t_3} + \frac{t_1}{t_2}, \quad p_2 = t_1 t_3, \quad p_3 = t_1 + \frac{t_2}{t_3}.$$

(3) *Type B_2: if $a_{i,j} = -1, a_{j,i} = -2$ then $d = 4$, and*

$$\frac{1}{p_1} = \frac{t_1}{t_2} + \frac{t_2}{t_3} + \frac{1}{t_4}, \quad \frac{1}{p_2} = \frac{1}{t_1} \left(\frac{t_2}{t_3} + \frac{1}{t_4} \right)^2 + \frac{1}{t_3},$$

$$p_3 = t_2 + t_1 t_4 + \frac{t_2^2 t_4}{t_3}, \quad p_4 = t_1 + t_3 \left(\frac{t_2}{t_3} + \frac{1}{t_4} \right)^2.$$

(4) *Type G_2: if $a_{i,j} = -1$, $a_{j,i} = -3$ then $d = 6$, and*

$$\frac{1}{p_1} = \frac{t_1}{t_2} + t_3\left(\frac{t_2}{t_3} + \frac{1}{t_4}\right)^2 + \frac{t_4}{t_5} + \frac{1}{t_6},$$

$$\frac{1}{p_2} = \frac{t_1}{t_3} + 2t_3\left(\frac{t_2}{t_3} + \frac{1}{t_4}\right)^3 + \frac{1}{t_1}\left(t_3\left(\frac{t_2}{t_3} + \frac{1}{t_4}\right)^2 + \frac{t_4}{t_5} + \frac{1}{t_6}\right)^3$$
$$+ \frac{3t_2t_4}{t_3t_5} + \frac{3t_2}{t_3t_6} + \frac{3}{t_4t_6} + \frac{2}{t_5},$$

$$p_5 = t_1t_6 + t_3^2t_6\left(\frac{t_2}{t_3} + \frac{1}{t_4}\right)^3 + t_4t_6\left(\frac{t_4}{t_5} + \frac{1}{t_6}\right)^2 + 2t_2 + \frac{2t_3}{t_4} + \frac{3t_2t_4t_6}{t_5} + \frac{2t_3t_6}{t_5},$$

$$p_6 = t_1 + t_3^2\left(\frac{t_2}{t_3} + \frac{1}{t_4}\right)^3 + t_5\left(\frac{t_4}{t_5} + \frac{1}{t_6}\right)^3 + \frac{3t_2t_4}{t_5} + \frac{3t_2}{t_6} + \frac{3t_3}{t_4t_6} + \frac{2t_3}{t_5}\ ;$$

the two middle components p_3 and p_4 are determined from two additional relations

$$p_1p_3p_5 = t_2t_4t_6\ , \quad p_2p_4p_6 = t_1t_3t_5.$$

(5) *In each of the cases* (1)–(4) *above, interchanging $a_{i,j}$ with $a_{j,i}$, the corresponding map in* (2.10) *is obtained from the given one by transformation $p_k \to 1/p_{d+1-k}$, $t_k \to 1/t_{d+1-k}$.*

2.6 Twist maps and their decompositions

In this section we introduce the so-called “twist maps” in [21, 37].

Definition 2.6.1. For any $u, v \in W$, we have the following twist maps:

$$\zeta^{u,v}\colon G^{u,v} \to G^{u^{-1},v^{-1}}\ :\ x \mapsto \left([\overline{u}^{-1}x]_{\geqslant 0}x^{-1}[x\overline{v^{-1}}]_{\leqslant 0}\right)^{\iota\circ T}\ ;$$
$$\psi^{u,v}\colon L^{u,v} \to L^{v,u}\ :\ x \mapsto [(\overline{v}x^{\iota})^{-1}]_+\overline{v}([\overline{u}^{-1}x]_+)^{\iota}.$$

Remark 2.6.2. Note that the map $\zeta^{u,v}$ does not send $L^{u,v}$ to $L^{u^{-1},v^{-1}}$. However, the maps $\zeta^{u,v}$ and $\psi^{u,v}$ are related by the following formula:

$$\psi^{u,v}(x) = (\zeta^{u,v}(x))^T\ , \quad \forall x \in L^{u,v}.$$

Theorem 2.6.3. *[37, Theorem 1.6] [21, Theorem 4.6] The map $\zeta^{u,v}$ is a biregular isomorphism between $G^{u,v}$ and $G^{u^{-1},v^{-1}}$, whose inverse is $\zeta^{u^{-1},v^{-1}}$. The $\psi^{u,v}$ is a biregular isomorphism between $L^{u,v}$ and $L^{v,u}$, whose inverse is $\psi^{v,u}$.*

Using the twist maps, we have the following embeddings of reduced double Bruhat cells:

Theorem 2.6.4. *Let $u, v \in W$ such that $\ell(u) + \ell(v) = \ell(u^{-1}v)$, the following map*

$$\xi^{u,v}\colon L^{u,v} \hookrightarrow L^{e,u^{-1}v}\ :\ x \to [\overline{u}^{-1}x]_+$$

is an open embedding. For $u, v \in W$ such that $\ell(u) + \ell(v) = \ell(uv^{-1})$, the following map

$$\xi_{u,v}\colon L^{u,v} \hookrightarrow L^{uv^{-1},e}\ :\ x \to [x\overline{v^{-1}}]_{\leqslant 0}$$

is an open embedding.

Proof. We only show the statement for $\xi^{u,v}$. First of all, we show that the map $\xi^{u,v}$ is well-defined. Since $x \in U\overline{u}U$, we have

$$\overline{u}^{-1}x \in \overline{u}^{-1}U\overline{u}U \subset B_-U, \tag{2.11}$$

which shows that $[\overline{u}^{-1}x]_+$ is well-defined. Since $x \in B_-vB_-$,

$$\overline{u}^{-1}x \in u^{-1}B_-vB_- \subset B_-u^{-1}B_-vB_- = B_-u^{-1}vB^- = B_-u^{-1}vB_-.$$

Here we use that $\ell(u)+\ell(v)=\ell(u^{-1}v)$. Thus $[\overline{u}^{-1}x]_+ \in B_-u^{-1}vB_-$. The unique factorization (2.11) implies injectivity of $\xi^{u,v}$. What remains is to show that $\xi^{u,v}$ maps an open subset $L^{u,e}\cdot L^{e,v} \subset L^{u,v}$ onto an open subset $L^{e,u^{-1}}\cdot L^{e,v}$ of $L^{e,u^{-1}v}$. Denote by $x = x_-\cdot x_+ \in L^{u,e}\cdot L^{e,v}$, then we have

$$\xi^{u,v}(x) = [\overline{u}^{-1}x_-x_+]_+ = [\overline{u}^{-1}x_-]_+x_+ = \iota\circ\psi^{u,e}(x_-)x_+,$$

where $\psi^{u,e}\colon L^{u,e}\to L^{e,u}$ is the twist map. Thus $\xi^{u,v}$ is an open embedding. $\diamondsuit$

Next, we decompose some of the twist maps as a sequence of "elementary moves" at least on some open dense chart. Let us introduce:

Definition 2.6.5. Given a pair of Weyl group elements (u,v) with $\ell(u)=p$ and $\ell(v)=q$, a double reduced word $\mathbf{i}=(i_1,\dots,i_p,j_1,\dots,j_q)$ for (u,v) is *separated* if $i_1,\dots,i_p\in[-r,-1]$ and $j_1,\dots,j_q\in[1,r]$.

For a separated double reduced word $\mathbf{i}$ for (u,v), define

$$\mathbf{i}_- := (-i_1,i_2,\dots,i_p,j_1,\dots,j_q), \quad \mathbf{i}_+ := (i_1,\dots,i_p,j_q,\dots,j_2,-j_1).$$

Here we use $-$ (resp. $+$) to indicate the decreasing (resp. increasing) of negative indexes in the new words. Note that

$$\mathbf{i}_- \in R(s_{i_1}u, s_{i_1}v), \text{if } \ell(s_{i_1}v)=\ell(v)+1; \quad \mathbf{i}_+ \in R(us_{j_1}, vs_{j_1}), \text{if } \ell(us_{i_1})=\ell(u)+1.$$

We then define the following birational map in terms of open embedding $x_{\mathbf{i}}, x_{\mathbf{i}_+}$ and $x_{\mathbf{i}_-}$:

$$P_{\mathbf{i}}\colon L^{u,v}\to L^{s_{i_1}u,s_{i_1}v} \;:\; x_{\mathbf{i}}(t_1,\dots,t_n)\mapsto x_{\mathbf{i}_-}(t_1,\dots,t_n),\ \text{if } \ell(s_{i_1}v)=\ell(v)+1;$$

$$Q_{\mathbf{i}}\colon L^{u,v}\to L^{us_{j_1},vs_{j_1}} \;:\; x_{\mathbf{i}}(t_1,\dots,t_n)\mapsto x_{\mathbf{i}_+}(t_1,\dots,t_n^{-1}),\ \text{if } \ell(us_{j_1})=\ell(u)+1.$$

Lemma 2.6.6. *For a separated double reduced word $(i_1,\dots,i_p,j_q,\dots,j_1)$ for (u,v), denote*

$$\mathbf{i}_k := (i_{k+1},\dots,i_p,-i_k,\dots,-i_1,j_q,\dots,j_1),$$
$$\mathbf{j}_k := (i_1,\dots,i_p,-j_1,\dots,-j_k,j_q,\dots,j_{k+1}).$$

We have $\xi^{u,v}=P_{\mathbf{i}_p}\circ P_{\mathbf{i}_{p-1}}\circ\cdots\circ P_{\mathbf{i}_1}$ if $\ell(u)+\ell(v)=\ell(u^{-1}v)$, and $\xi_{u,v}=Q_{\mathbf{j}_q}\circ Q_{\mathbf{j}_{q-1}}\circ\cdots\circ Q_{\mathbf{j}_1}$ if $\ell(u)+\ell(v)=\ell(uv^{-1})$. .

Proof. We only show it for $\xi_{u,v}$. Just need to show on the chart $x_{\mathbf{j}_0}$. For $k\in[0,q]$, write $v_k = s_{j_{k+1}}\cdots s_{j_q}$. Note that $v_0=v^{-1}$. For $i>0$, we have $x_i(t)\overline{s_i}=x_{-i}(t^{-1})x_i(-t^{-1})$, then

$$\begin{aligned} x_{\mathbf{j}_0}(t'_1,\dots,t'_p;t_q,\dots,t_1)\overline{v_0} &= x_{i_1}(t'_1)\cdots x_{i_p}(t'_p)x_{j_q}(t_q)\cdots x_{j_1}(t_1)\overline{s_{j_1}}\cdot\overline{v}_1 \\ &= x_{i_1}(t'_1)\cdots x_{i_p}(t'_p)x_{j_q}(t_q)\cdots x_{j_2}(t_2)\cdot x_{-j_1}(t_1^{-1})x_{j_1}(-t_1^{-1})\cdot\overline{v}_1 \\ &= \left(Q_{\mathbf{j}_1}(x_{\mathbf{j}_0}(t'_1,\dots,t'_p;t_1,\dots,t_q))\overline{v}_1\right)\cdot\left(\overline{v}_1^{-1}x_{j_1}(-t_1^{-1})\overline{v}_1\right). \end{aligned}$$

In the last line, note that $\overline{v}_1^{-1}x_{i_1}(-t_1^{-1})\overline{v}_1 \in U$; this follows from well-known results on the Weyl group, as in Section 10.2 of [53]. Using the mixed d-moves, write

$$Q_{\mathbf{j}_1}(x_{\mathbf{j}_0}(t'_1,\dots,t'_p;t_1,\dots,t_q)) = x_{\mathbf{j}_1}(t'_1,\dots,t'_p;t''_1,\dots,t''_q),$$

then we can repeat the argument from above q times. Then by taking $[\cdot]_{\leqslant 0}$ on both sides, we get the desired formula. ♢

Note that ι restrict to a biregular isomorphism $L^{u,v} \to L^{u^{-1},v^{-1}}$. Then one can check that $\xi^{u,e} = \iota \circ \psi^{u,e}$ and $\xi_{e,v} = \iota \circ \psi^{e,v}$. Now we are ready to decompose some twist maps. In more details, we have

Proposition 2.6.7. *For $u, v \in W$ such that $\ell(u) + \ell(v) = \ell(v^{-1}u)$, the following diagram commutes.*

$$\begin{array}{ccc} L^{u,v} & \xrightarrow{\xi^{u,v}} & L^{e,u^{-1}v} \\ {\scriptstyle \iota\circ\psi^{u,v}}\big\downarrow & & \big\downarrow{\scriptstyle \xi_{e,u^{-1}v}} \\ L^{v^{-1},u^{-1}} & \xrightarrow{\xi_{v^{-1},u^{-1}}} & L^{v^{-1}u,e} \end{array}$$

Proof. Note that both composition maps from $L^{u,v}$ to $L^{v^{-1}u,e}$ are open embeddings. We only need to show the statement on an open chart of $L^{u,v}$. Denote by $x = x_- \cdot x_+ \in L^{u,e} \cdot L^{e,v} \subset L^{u,v}$. First, the composition of the top and right arrows sends x to

$$\left[[\overline{u^{-1}}x_-]_+ x_+ \overline{v^{-1}u}\right]_{\leqslant 0}$$

and the composition of left and bottom arrows sends x to

$$\left[[\overline{u^{-1}}x_-]_+ x_+ \left[(\overline{v}x^{\iota})^{-1}\right]_+^{\iota} \overline{u}\right]_{\leqslant 0}.$$

Note $[x]_+^{-1} = [x^{-\iota}]_+^{\iota}$ and $\overline{v}^{\iota} = \overline{v^{-1}}$ implies

$$\left[[\overline{u^{-1}}x_-]_+ x_+ \left[(\overline{v}x^{\iota})^{-1}\right]_+^{\iota} \overline{u}\right]_{\leqslant 0} = \left[[\overline{u^{-1}}x_-]_+ [x_+\overline{v^{-1}}]_{\leqslant 0}\overline{u}\right]_{\leqslant 0}$$

Next, we need to show

$$\overline{u}^{-1} \cdot [x_+\overline{v^{-1}}]_+ \cdot \overline{u} \in U.$$

Given double reduced word $(i_1,\dots,i_n)$ of (e,v), the statement for $x_+ = x_{i_1}(t_1)\cdots x_{i_n}(t_n)$ follows from the same argument as in previous proof. ♢

2.7 Evaluation of generalized minors, factorization problem

The evaluation of generalized minors can be described using the following

Definition 2.7.1. Let V be a finite-dimensional $\mathfrak{g}$-module, γ and δ two weights in $P(V)$, and $\mathbf{i} = (i_1,\dots,i_n)$ a sequence of indices in $\boldsymbol{I}$. An $\mathbf{i}$-trail from γ to δ in V is a sequence of weights $\pi = (\gamma = \gamma_0, \gamma_1, \dots, \gamma_n = \delta)$ such that $\gamma_{k-1} - \gamma_k = c_k\alpha_{i_k}$ for $c_k \in \mathbb{Z}_{\geqslant 0}$ and $e_{i_1}^{c_1}\cdots e_{i_n}^{c_n}$ is a non-zero linear map from $V(\delta)$ to $V(\gamma)$.

For every $\mathbf{i}$-trail $\pi = (\gamma_0, \ldots, \gamma_n)$ in V, denote by

$$d_k = d_k(\pi) = \frac{\gamma_{k-1} + \gamma_k}{2}(\alpha_{i_k}^\vee) \,. \tag{2.12}$$

Theorem 2.7.2. *[21, Theorem 5.8] Let γ and δ be two weights in the W-orbit of the same fundamental weight ω_i of $\mathfrak{g}$, and let $\mathbf{i} = (i_1, \ldots, i_n)$ be any sequence of indices in $\boldsymbol{I}$. Then*

(1) the evaluation of $\Delta_{\gamma,\delta}$ at $x_{\mathbf{i}}(t_1, \ldots, t_n)$ is a positive integral linear combination of monomials $t_1^{c_1(\pi)} \cdots t_n^{c_n(\pi)}$ for all $\mathbf{i}$-trails π from γ to δ in V_{ω_i}.

(2) the evaluation of $\Delta_{\gamma,\delta}$ at $x_{-\mathbf{i}}(t_1, \ldots, t_n)$ is a positive integer linear combination of monomials $t_1^{d_1(\pi)} \cdots t_n^{d_n(\pi)}$ for all $\mathbf{i}$-trails π from $-\gamma$ to $-\delta$ in $V_{\omega_{i*}}$.

Next, we recall the following *factorization problem* for $L^{u,v}$, which is to find explicit formulas for the inverse birational isomorphism $x_{\mathbf{i}}^{-1}$ between $L^{u,v}$ and $\mathbb{G}_{\mathbf{m}}^n$.

For $\mathbf{i} \in R(u, v)$ and $k \in [1, n]$, let

$$k^- = \max\{l \mid l < k, |i_l| = |i_k|\} \quad k^+ = \min\{l \mid l > k, |i_l| = |i_k|\}, \tag{2.13}$$

so that k^- (resp. k^+) is the previous (resp. next) occurrence of an index $\pm i_k$ in $\mathbf{i}$; if k is the first (resp. last) occurrence of $\pm i_k$ in $\mathbf{i}$ then we set $k^- = 0$ (resp. $k^+ = m + 1$). An index k is $\mathbf{i}$-*exchangeable* if $k^+ \in [1, m]$. Let $\boldsymbol{e}(\mathbf{i})$ denote the set of all $\mathbf{i}$-exchangeable indices. Note that for $k \in [-r, -1] \cup \boldsymbol{e}(\mathbf{i})$, we have $k^+ \in [1, n]$.

For $k \in [-r, -1]$, denote by $u_k = e$ and $v_k = v^{-1}$. For $k \in [1, m]$, denote by

$$u_k = \prod_{\substack{l=1,\ldots,k \\ i_l<0}} s_{i_l}, \quad v_k = \prod_{\substack{l=n,\ldots,k+1 \\ i_l>0}} s_{i_l},$$

where the index is increasing in the product on the left, and decreasing in the product on the right. For $k \in [-r, -1]$, denote by

$$u^k = u^{-1}u_k; \quad v^k = vv_k.$$

Extend the word $\mathbf{i}$ to $(i_{-r}, \ldots, i_{-1}; i_1, \ldots, i_n)$, where $i_{-r} = -r$. For $k \in [-r, -1] \cup [1, n]$, define a regular function $M_k = M_{k,\mathbf{i}}$ on $L^{u,v}$ by

$$M_k := \Delta_{v^k\omega_{i_k}, u^k\omega_{i_k}}(\psi^{u,v}(x)).$$

Note that $M_k = 1$ for $k \in [1, n] \setminus \boldsymbol{e}(\mathbf{i})$. Then the solution to the factorization problem is

Theorem 2.7.3. *[21, Theorem 4.8] For $\mathbf{i} = (i_1, \ldots, i_n) \in R(u, v)$, and an elements x in $L^{u,v}$ which can be factored $x = x_{i_1}(t_1) \cdots x_{i_n}(t_n)$ with all $t_k \in \mathbb{G}_{\mathbf{m}}$, for $k \in [-r, -1] \cup \boldsymbol{e}(\mathbf{i})$, the factorization parameters t_{k^+} are determined by the following formulas:*

$$t_{k^+} = \begin{cases} M_k(x)/M_{k^+}(x), & i_{k^+} < 0 \\ \dfrac{1}{M_k(x)M_{k^+}(x)} \displaystyle\prod_{l:\ l^- < k^+ < l} M_l(x)^{-a_{i_l,i_k}}, & i_{k^+} > 0 \end{cases}$$

Remark 2.7.4. Note that the notation we use here is different from the one in [21]. The one we use here aligns the notation for twisted minors and the one for cluster variables. See more details in Section 4.4.

Moreover, we have the following special cases:

Proposition 2.7.5. *[21, Proposition 4.11.(i)] For* $\mathbf{i} = (i_1, \dots, i_m) \in R(w_0, e)$ *and* $x \in H \times L^{w_0,e}$ *admitting factorization* $x = x_{\mathbf{i}}(h; t_1, \dots, t_m)$*, we have*

$$\frac{F_{i_1} \cdot \Delta_{w_0\omega_{i_1^*},\omega_{i_1^*}}}{\Delta_{w_0\omega_{i_1^*},\omega_{i_1^*}}}(x) = \frac{h^{\alpha_i}}{t_1}; \quad \frac{\Delta_{w_0\omega_{i_m},\omega_{i_m}} \cdot F_{i_m}}{\Delta_{w_0\omega_{i_m},\omega_{i_m}}}(x) = t_m.$$

3 | Preliminaries on Positivity Theory

In this chapter, we recall the notion of affine tropical variety, positive varieties with potential and tropicalization functor. As an example of positive varieties with potential, we focus on Borel subgroup B_- of an reductive algebraic group G. The Berenstein-Kazhdan potential Φ_{BK} on B_- is introduced in the context of unipotent bicrystals. At the end, we briefly discribe the relation between the tropicalization of (B_-, Φ_{BK}) and the parametrization of canonical basis. More detailed discussion can be found in [16, 21].

3.1 Affine tropical varieties

In this section, we introduce the notion of *affine tropical variety*, which is an analog of affine variety in the "tropical word".

For any subring R of $\mathbb{R}$, denote by $R_+ := R \cap \mathbb{R}_+$ the semi-subring. Given subsets C and D of free R-modus V and V' respectively, a map $\phi\colon C \to D$ is *piecewise R-linear* if there is a piecewise R-linear R-module homomorphism $\widetilde{\phi}\colon V \to V'$ such that $\widetilde{\phi}\big|_C = \phi$. For a free $\mathbb{R}$-module V and a subring $R \subset \mathbb{R}$, denote by $R[V]$ the set of piecewise R-linear functions on V. Note that $R[V]$ is an algebra with multiplication and addition given by

$$(f \odot g)(v) := f(v) + g(v), \quad (f \oplus g)(v) := \min\{f(v), g(v)\}.$$

Note that the multiplication unit is 0 and the addition unit is $+\infty$.

Definition 3.1.1. Fix a subring R of $\mathbb{R}$. A m-dimensional *affine tropical variety* $\mathcal{C}$ over R is a family of pairs $\{(C_\theta, j_\theta) \mid \theta \in \Theta\}$ together with a m-dimensional free $\mathbb{R}$-module V, where C_θ is a set with an R_+-action and $j_\theta\colon C_\theta \to V$ is an injective map, called a *tropical chart*, s.t.

(i) The map j_θ commutes with the R_+-action;

(ii) There exists a piecewise R-linear bijection from $j_\theta(C_\theta)$ to $j_{\theta'}(C_{\theta'})$, denote by $j_{\theta'}^{-1} \circ j_\theta\colon C_\theta \to C_{\theta'}$ the induced map for simplicity.

An affine tropical variety $\mathcal{C}$ is *convex* if there exists a $\theta \in \Theta$ such that the subset $j_\theta(C_\theta)$ of V is a R_+-submodule of the free module V. The *coordinate algebra* $R[\mathcal{C}]$ of $\mathcal{C}$ is the pull-back of the algebra $R[V]$ along the injective map j_θ. Since $j_\theta^{-1} \circ j_{\theta'}$ is piecewise R-linear, the coordinate algebra $R[\mathcal{C}]$ is independent of charts.

Similarly to algebraic varieties, for an affine tropical variety $\mathcal{C}$ over R, tensoring with $R' \supset R$, one obtains its R'-points

$$\mathcal{C}(R') := \{(C_\theta \underset{R_+}{\otimes} R'_+, j_\theta) \mid \theta \in \Theta\}.$$

A morphism $\boldsymbol{f}$ of affine tropical varieties $\mathcal{C} = \{(C_\theta, j_\theta) \mid \theta \in \Theta\}$ and $\mathcal{D} = \{(D_\vartheta, k_\vartheta) \mid \vartheta \in \Theta'\}$ over R is a family of piecewise R_+-equivariant maps $f_{\theta,\vartheta}$, such that the following diagram is commutative.

$$\begin{array}{ccc} C_\theta & \xrightarrow{j_\theta^{-1}\circ j_{\theta'}} & C_{\theta'} \\ {\scriptstyle f_{\theta,\vartheta}}\downarrow & & \downarrow{\scriptstyle f_{\theta',\vartheta'}} \\ D_\vartheta & \xrightarrow{k_\vartheta^{-1}\circ k_{\vartheta'}} & D_{\vartheta'} \end{array} \tag{3.1}$$

Affine tropical varieties over R form a category $\mathbf{AffTropVar}(R)$.

Definition 3.1.2. Let $\boldsymbol{f}$ be a morphism of affine tropical varieties $\mathcal{C}$ and $\mathcal{D}$ over $\mathbb{Z}$. For $\xi \in D_\vartheta$, define the *multiplicity* of ξ over $\boldsymbol{f}$ as

$$\dim \mathbb{C}[f_{\theta,\vartheta}^{-1}(\xi)],$$

where $\mathbb{C}[X]$ is the linearization of the set X. Note that the multiplicity of ξ over $\boldsymbol{f}$ doesn't depend on the charts since by (3.1), we have:

$$\dim \mathbb{C}[f_{\theta,\vartheta}^{-1}(\xi)] = \dim \mathbb{C}[f_{\theta',\vartheta'}^{-1}(k_\vartheta^{-1} \circ k_{\vartheta'}(\xi))].$$

The morphism $\boldsymbol{f}$ is *finite* if every $\xi \in D_\vartheta$ has finite multiplicity.

Example 3.1.3. Define an affine tropical variety over R as $\mathcal{I}_R := \{(R_+, j\colon R_+ \hookrightarrow R)\}$, which we refer to as the *trivial* affine tropical variety over R. Given affine tropical varieties $\mathcal{C}$ and $\mathcal{D}$ over R, the product

$$\mathcal{C} \times \mathcal{D} := \big\{(C_\theta \times D_\vartheta, j_\theta \times k_\vartheta) \,\big|\, (\theta, \vartheta) \in \Theta \times \Theta'\big\}$$

is an affine tropical variety over R. For a morphism $\boldsymbol{f}$ of $\mathcal{C}$ and $\mathcal{D}$, one can show that both $\boldsymbol{f}(\mathcal{C})$ and $\boldsymbol{f}^{-1}(\mathcal{D})$ are affine tropical varieties.

3.2 Positive varieties and tropicalization

In this section, we first briefly recall basic definitions in positivity theory and then realize tropicalization as a functor from the category of positive varieties with potential to the category of affine tropical varieties.

Consider a split algebraic torus $S \cong \mathbb{G}_{\mathbf{m}}^n$. Denote the character lattice of S by $S_t = \operatorname{Hom}(S, \mathbb{G}_{\mathbf{m}})$ and the cocharacter lattice by $S^t = \operatorname{Hom}(\mathbb{G}_{\mathbf{m}}, S)$. The lattices S_t and S^t are naturally in duality and denote by $\langle \cdot, \cdot \rangle\colon S_t \times S^t \to \mathbb{Z}$ this canonical pairing. The coordinate algebra $\mathbb{Q}[S]$ is the group algebra (over $\mathbb{Q}$) of the lattice S_t, that is, each $f \in \mathbb{Q}[S]$ can be written as

$$f = \sum_{\chi \in S_t} c_\chi \chi, \tag{3.2}$$

where only a finite number of coefficients c_χ are non-zero. Following [16], to each positive rational map $\phi\colon S \to S'$, we associate a *tropicalized map* $\Phi^t\colon S^t \to (S')^t$ as follows:

Case 1. If ϕ is positive regular on S, *i.e.*, $\phi = \sum_{\chi \in S_t} c_\chi \chi$ with all $c_\chi \geqslant 0$, define ϕ^t by

$$\phi^t\colon S^t \to \mathbb{G}_{\mathbf{m}}^t = \mathbb{Z} \;:\; \xi \mapsto \min_{\chi;\, c_\chi > 0} \langle \chi, \xi \rangle.$$

Case 2. If ϕ is positive rational on S, *i.e.*, $\phi = f/g$ with f, g positive regular functions, then

$$\phi^t := f^t - g^t.$$

Example 3.2.1. **a)** Consider $S = \mathbb{G}_{\mathbf{m}}$. Take $\phi = x^2 - x + 1 = (x^3+1)/(x+1)$. Note that ϕ is positive rational but not positive regular. Then $\phi^t(\xi) = \min(3\xi, 0) - \min(\xi, 0) = 2\min(\xi, 0)$. Note that for any $a, b, c, d \in \mathbb{Q}_{>0}$ the function $\phi' = (ax^3+b)/(cx+d)$ has the same tropicalization $(\phi')^t = \phi^t$.

b) Consider $S = \mathbb{G}_{\mathbf{m}}^2$. Take $\phi_1 = (x_1+x_2)^2$ and $\phi_2 = x_1^2 + x_2^2$. Then we know $\phi_1^t = \phi_2^t = 2\min\{\xi_1, \xi_2\}$

Case 3. For $\phi\colon S \to S'$ a positive rational map, define $\phi^t\colon S^t \to (S')^t$ as the unique map such that for every character $\chi \in S'_t$ and for every cocharacter $\xi \in S^t$ we have

$$\langle \chi, \phi^t(\xi)\rangle = (\chi \circ \phi)^t(\xi).$$

A more concrete description is as follows. Let $\phi_1, \dots, \phi_n$ be the components of ϕ given by the splitting $S' \cong \mathbb{G}_{\mathbf{m}}^n$. Then, in the induced coordinates on $(S')^t$, we have

$$\phi^t = (\phi_1^t, \dots, \phi_n^t).$$

Example 3.2.2. Consider the following positive rational map:

$$\phi\colon \mathbb{G}_{\mathbf{m}}^3 \to \mathbb{G}_{\mathbf{m}}^3 \;:\; (x_1, x_2, x_3) \mapsto \left(\frac{x_2x_3}{x_1+x_3}, x_1+x_3, \frac{x_1x_2}{x_1+x_3}\right)$$

which has tropicalization

$$\begin{aligned}\phi^t\colon &(\mathbb{G}_{\mathbf{m}}^3)^t \cong \mathbb{Z}^3 \to (\mathbb{G}_{\mathbf{m}}^3)^t \cong \mathbb{Z}^3;\\ &(\xi_1, \xi_2, \xi_3) \mapsto (\xi_2 + \xi_3 - \min\{\xi_1, \xi_3\}, \min\{\xi_1, \xi_3\}, \xi_1 + \xi_2 - \min\{\xi_1, \xi_3\}).\end{aligned}$$

Definition 3.2.3. Let (X, Φ) be an irreducible variety over $\mathbb{Q}$ with a rational function Φ on X. A *rational chart* of X is a birational isomorphism $\theta\colon S \to X$ from a split algebraic torus S to X. A rational chart θ is *toric* if θ is an open embedding. A chart $\theta\colon S \to X$ is *positive* with respect to Φ if $\Phi \circ \theta$ is a positive rational function on S. Two charts $\theta_1\colon S_1 \to X$ and $\theta_2\colon S_2 \to X$ are called *positively equivalent* if $\theta_1^{-1} \circ \theta_2\colon S_2 \to S_1$ and $\theta_2^{-1} \circ \theta_1\colon S_1 \to S_2$ are positive rational maps. A *positive variety with potential* is a triple (X, Φ, Θ_X), where Θ_X is a set of positive equivalent charts who are positive with respect to Φ. A *positive variety* (X, Θ_X) is a positive variety with potential $\Phi = 0$. Given a positive chart $\theta\colon S \to X$ of (X, Φ, Θ_X), denote by

$$(X, \Phi, \theta)^t := \left\{\xi \in \operatorname{Hom}(\mathbb{G}_{\mathbf{m}}, S) \mid \Phi^t(\xi) \geqslant 0\right\},$$

the tropicalization of (X, Φ, θ). For convenience, we define $0^t := +\infty$. If θ is toric and Φ is regular, the set $(X, \Phi, \theta)^t$ is a convex cone in $\operatorname{Hom}(\mathbb{G}_{\mathbf{m}}, S)$.

Example 3.2.4. Here we give an example arising from Lie theory. Let $X = U$ be the unipotent radical of a semisimple algebraic group over $\mathbb{Q}$, and let $\Phi = \chi^{\mathrm{st}}$ on U. Given a double reduced word $\mathbf{i} = (i_1, \dots, i_m)$ of (e, w_0), where w_0 is the longest element in Weyl group W, the following is an open embedding by Proposition 2.4.4

$$x_{\mathbf{i}}\colon \mathbb{G}_{\mathbf{m}}^m \to U \;:\; (t_1, \dots, t_m) \to x_{i_1}(t_1)\cdots x_{i_m}(t_m) \in L^{e,w_0} \subset U,$$

and it is positive with respect to χ^{st} since

$$\chi^{\mathrm{st}}(x_{\mathbf{i}}(t_1, \dots, t_m)) = t_1 + \cdots + t_m.$$

By Proposition 2.5.1, we conclude that x_{i} and $x_{\mathbf{i}'}$ are positive equivalent toric chart for (U, χ^{st}), where both $\mathbf{i}$ and $\mathbf{i}'$ are double reduced word for (e, w_0).

A morphism $f\colon (X,\Phi,\Theta_X)\to(Y,\Phi',\Theta_Y)$ of positive varieties with potential is a rational map $f\colon X\to Y$ such that the rational function $\Phi - f^*\Phi'$ is positive, and for some (equivalently any) $\theta_X\in\Theta_X$ and $\theta_Y\in\Theta_Y$, the rational map $\theta_Y^{-1}\circ f\circ\theta_X\colon S\to S'$ is positive.

Denote by $\mathbf{PosVarPot}(\mathbb{Q})$ the category of the positive varieties with potential over $\mathbb{Q}$.

Proposition 3.2.5. *Let (X,Φ,Θ) be a positive variety with potential. Fix a splitting of S. Define*

$$X^t_\Phi := \Big\{X_\theta := (X,\theta,\Phi)^t, j_\theta\colon X_\theta\hookrightarrow \mathrm{Hom}(\mathbb{G}_{\mathbf{m}},S)\xrightarrow{\sim}\mathbb{Z}^m \mid \theta\in\Theta\Big\}.$$

Then X^t_Φ is an affine tropical variety over $\mathbb{Z}$. If X has a positive toric chart with Φ regular, the affine tropical variety X^t_Φ is convex. In summary, tropicalization defines a functor from $\mathbf{PosVarPot}(\mathbb{Q})$ *to* $\mathbf{AffTropVar}(\mathbb{Z})$.

Note that $(\mathbb{Q},\mathrm{Id},\theta\colon\mathbb{Q}_{\mathbf{m}}\to\mathbb{Q})$ is a positive variety with potential and $\mathbb{Q}^t_{\mathrm{Id}} = \mathcal{I}_{\mathbb{Z}}$. Thus potential Φ_X on (X,Θ) can be viewed as a morphism of positive varieties with potential:

$$\Phi_X\colon (X,\Phi_X,\Theta)\to(\mathbb{Q},\mathrm{Id},,j\colon\mathbb{Q}_{\mathbf{m}}\to\mathbb{Q}).$$

Let f be a morphism of two positive varieties with potential (X,Φ_X,Θ_X) and (Y,Φ_Y,Θ_Y). Denote by

$$f^{-t}_{\theta_X,\theta_Y}(\xi) := (f^t_{\theta_X,\theta_Y})^{-1}(\xi)\subset (X,\Phi_X,\theta_X)^t$$

the pre-image of $\xi\in(Y,\Phi_Y,\theta_Y)^t$ of the tropical function $f^t_{\theta_X,\theta_Y}$. We sometimes write $f^{-t}(\xi)$ instead if the positive chart we choose is clear from the context. Note that $f^{-t}(\xi)$ is not an affine tropical variety in general.

We are also interested in the real points of the tropicalization of $(X,\Phi,\theta)^t$. Denote by

$$\Phi^t_{\mathbb{R}}\colon\ \mathrm{Hom}(\mathbb{G}_{\mathbf{m}},S)\otimes_{\mathbb{Z}}\mathbb{R}\to\mathbb{R}$$

the real extension of Φ^t. Then let us introduce the following notation

$$(X,\Phi,\theta)^t_{\mathbb{R}} = \{x\in\mathrm{Hom}(\mathbb{G}_{\mathbf{m}},S)\otimes_{\mathbb{Z}}\mathbb{R}\mid \Phi^t_{\mathbb{R}}(x)\geqslant 0\}. \tag{3.3}$$

For any $\delta\geqslant 0$, define the *δ-interior* of the tropicalization as

$$(X,\Phi,\theta)^t_{\mathbb{R}}(\delta) = \{x\in\mathrm{Hom}(\mathbb{G}_{\mathbf{m}},S)\otimes_{\mathbb{Z}}\mathbb{R}\mid \Phi^t_{\mathbb{R}}(x)>\delta\}. \tag{3.4}$$

Besides, the morphism $f^t\colon (A,\Phi_A,\theta_A)^t\to(B,\Phi_B,\theta_B)^t$ has a piecewise linear real extension $f^t_{\mathbb{R}}\colon (A,\Phi_A,\theta_A)^t_{\mathbb{R}}\to(B,\Phi_B,\theta_B)^t_{\mathbb{R}}$.

At the end of this section, we would like to describe the tropicalization as a limiting procedure. First of all, for $\mathbb{G}_{\mathbf{m}}=\mathbb{R}^\times$, tropicalization of a positive function f over $(\mathbb{R}^\times)^n$ can be interpreted as follows. By substituting $x_i = e^{s\xi_i}$, one finds

$$f^t(\xi_1,\dots,\xi_n) = \lim_{s\to-\infty}\frac{1}{s}\ln\Big(f(e^{s\xi_1},\dots,e^{s\xi_n})\Big).$$

For $\mathbb{G}_{\mathbf{m}}=\mathbb{C}^\times$, we have the following

Proposition 3.2.6. *Let* $f\colon (\mathbb{C}^\times)^n \to \mathbb{C}^\times$ *be a positive rational function, and* $\mathrm{C} \subset ((\mathbb{C}^\times)^n)^t$ *be an* open *linearity chamber of* f^t*. Then by substituting* $z_i = e^{s\xi_i + \mathbb{i}\varphi_i}$*, the following equalities hold on* $\mathrm{C} \times (S^1)^n$*:*

$$\lim_{s\to-\infty} \frac{1}{s} \ln \left| f(e^{s\xi_1+\mathbb{i}\varphi_1}, \ldots, e^{s\xi_n+\mathbb{i}\varphi_n}) \right| = f^t_{\mathbb{R}}(\xi_1, \ldots, \xi_n);$$
$$\lim_{s\to-\infty} \arg f(e^{s\xi_1+\mathbb{i}\varphi_1}, \ldots, e^{s\xi_n+\mathbb{i}\varphi_n}) = f^t_{\mathbb{R}}(\varphi_1, \ldots, \varphi_n).$$

Proof. First of all, the statement is clearly true for a Laurent monomial f.

Now suppose f is a positive Laurent polynomial in $z_1, \ldots, z_n$ and write $f = \sum f_i$, where f_i's are Laurent monomials. Write $\boldsymbol{\xi} := (\xi_1, \ldots, \xi_n)$. Without loss of generality, let us assume that $(f_0^t)_{\mathbb{R}}(\boldsymbol{\xi}) < (f_i^t)_{\mathbb{R}}(\boldsymbol{\xi})$ on C for $i \neq 0$. Write f as

$$f = f_0 \left(1 + \sum_{i\neq 0} f_i/f_0 \right).$$

Note that $1/s \ln |f_i/f_0|$ tends to 0 as $s \to -\infty$. Thus by triangle inequality, we get the claim.

At the last, let $f = A/B$ for A and B are positive Laurent polynomials in $z_1, \ldots, z_n$. Note that $\ln |A/B| = \ln |A| - \ln |B|$ and $\arg f = \arg A - \arg B$. ♢

Remark 3.2.7. Consider the function $f = z_1 + z_2$ on $(\mathbb{C}^\times)^2$. So $f = 0$ on the subset $|z_1| = |z_2|$ and $\arg(z_1) = \pi + \arg(z_2)$. This is one example that one can not extend the result to the whole space $((\mathbb{C}^\times)^n)^t$.

3.3 Domination by potentials

Definition 3.3.1. Let (A, Φ_A, Θ_A) be a positive variety with potential. A rational function f on A is *dominated by* Φ_A, which we denote by $f \prec \Phi_A$, if there exists positive (with respect to Θ_A) rational functions f^+ and f^- and polynomial p with coefficients in $\mathbb{R}_+$ such that: $f = f^+ - f^-$ and both

$$p(\Phi_A) - f^+, \quad \text{and} \quad p(\Phi_A) - f^-$$

are positive with respect to Θ.

What follows is immediate and we omit the proof here.

Lemma 3.3.2. *Let* (A, Φ_A, Θ_A) *be a positive variety with potential. The set of rational functions on* A *that are dominated by* Φ_A *forms a subring of the coordinate ring of* A*.*

Let G be a reductive algebraic group and $\mathfrak{g}$ its Lie algebra. Recall that U_- is the unipotent radical of the Borel B_-. A $U_- \times U_-$*-variety* is an affine variety A equipped with an action of the algebraic group $U_- \times U_-$, where the first factor acts on the left and the second factor acts on the right.

Since U_- is unipotent, the exponential map $\exp\colon \mathfrak{n}_- \to U_-$ is algebraic. Thus for a fixed $(F, F') \in \mathfrak{n}_- \times \mathfrak{n}_-$, the map

$$\mathbb{G}_{\mathbf{m}} \times A \to A, \; : \; (t, a) \mapsto \exp(-tF) \cdot a \cdot \exp(tF') \tag{3.5}$$

is algebraic. The action $(F, F') \in \mathfrak{n}_- \times \mathfrak{n}_-$ on $f \in \mathbb{Q}[A]$ given by

$$(F \cdot f \cdot F')(a) = \frac{d}{dt}\bigg|_{t=0} f\left(\exp(-tF) \cdot a \cdot \exp(tF')\right)$$

is algebraic since the map (3.5) is algebraic.

Proposition 3.3.3. *Let A be a $U_- \times U_-$-variety and (A, Φ, Θ) be a positive variety with potential. Let $\{a_i\}_1^n$ be a set of positive functions on (A, Θ), and let $(F, F') \in \mathfrak{n}_- \times \mathfrak{n}_-$. If*

$$\frac{F \cdot a_i \cdot F'}{a_i} \prec \Phi, \quad \forall i \in [1, n],$$

then $(F \cdot f \cdot F')/f \prec \Phi$ for any subtraction free Laurent polynomial $f := f(a_1, \ldots, a_n)$ in functions a_i.

Proof. First of all, since the Lie algebra $\mathfrak{n}_- \times \mathfrak{n}_-$ acts by derivations, for a Laurent monomial $a_1^{m_1} \cdots a_n^{m_n}$ and any $c \in \mathbb{R}_+$, we have:

$$\frac{F \cdot (c a_1^{m_1} \cdots a_n^{m_n}) \cdot F'}{c a_1^{m_1} \cdots a_n^{m_n}} = \sum_i m_i \frac{F \cdot a_i \cdot F'}{a_i} \prec \Phi.$$

Next, denote by $f = f_1 + \cdots + f_m$ a $\mathbb{R}_+$-linear combination of Laurent monomials in the functions a_i. By the first step, we know $(F \cdot f_i \cdot F')/f_i \prec \Phi$. In other words, one can write

$$\frac{F \cdot f_i \cdot F'}{f_i} = f_i^+ - f_i^-$$

where $p(\Phi) - f_i^+$ and $p(\Phi) - f_i^-$ are positive with respect to Θ. Then we have:

$$\frac{F \cdot f \cdot F'}{f} = \sum_i \frac{f_i}{f} \cdot \frac{F \cdot f_i \cdot F'}{f_i} = \sum_i \frac{f_i}{f} \cdot f_i^+ - \sum_i \frac{f_i}{f} \cdot f_i^-.$$

Then one can choose p such that

$$p(\Phi) - \sum \frac{f_i}{f} \cdot f_i^+ = \sum \frac{f_i}{f} \left(p(\Phi) - f_i^+\right), \quad p(\Phi) - \sum \frac{f_i}{f} \cdot f_i^- = \sum \frac{f_i}{f} \left(p(\Phi) - f_i^-\right)$$

are positive with respect to Θ. Thus $(F \cdot f \cdot F')/f \prec \Phi$. ♢

3.4 Unipotent χ-linear bicrystals

Let G be a reductive algebraic group. In this section, we briefly recall the basic definitions about $U \times U$ varieties, unipotent χ-linear bicrystals in [16] and then introduce the notion of trivializable unipotent bicrystals. Note that χ-linear functions for unipotent bicrystals play the role of potential for positive varieties.

Definition 3.4.1. A *$U \times U$-variety* $\boldsymbol{X}$ is a pair (X, α), where X is an irreducible affine variety over $\mathbb{Q}$ and $\alpha \colon U \times X \times U \to X$ is a $U \times U$-action on X, such that group U acts (both action) freely on X. The *convolution product* $*$ of $U \times U$-varieties $\boldsymbol{X} = (X, \alpha)$ and $\boldsymbol{Y} = (Y, \alpha')$ is $\boldsymbol{X} * \boldsymbol{Y} := (X * Y, \beta)$, where the variety $X * Y$ is the quotient of $X \times Y$ by the following left action of U on $X \times Y$:

$$u(x, y) = (xu^{-1}, uy).$$

And the action $\beta \colon U \times X * Y \times U \to X * Y$ is defined by $u(x * y)u' = (ux) * (yu')$.

Example 3.4.2. It is clear that the group G itself is a $U \times U$-variety with left and right multiplication as $U \times U$ action. Denote by

$$G^{(n)} := G * \cdots * G, \quad \text{for } n \geqslant 2$$

the convolution product of n-copies of G's.

Definition 3.4.3. For a $U \times U$-variety $\boldsymbol{X}$ and $\chi\colon U \to \mathbb{A}^1$ a character, a function Φ on X is *χ-linear* if

$$\Phi(u \cdot x \cdot u') = \chi(u) + \Phi(x) + \chi(u'), \forall x \in X, u, u' \in U. \tag{3.6}$$

A $(U \times U, \chi)$*-bicrystal* is a triple $(\boldsymbol{X}, \boldsymbol{p}, \Phi)$, where $\boldsymbol{X}$ is a $U \times U$-variety, and $\boldsymbol{p}\colon X \to G$ is a $U \times U$-equivariant morphism, and Φ is χ- linear function. We refer to the pair $(\boldsymbol{X}, \boldsymbol{p})$ as *unipotent bicrystal*. The convolution product is defined by

$$(\boldsymbol{X}, \boldsymbol{p}, \Phi_X) * (\boldsymbol{Y}, \boldsymbol{p}', \Phi_Y) := (\boldsymbol{X} * \boldsymbol{Y}, \boldsymbol{p}'', \Phi_{X*Y}),$$

where $\boldsymbol{p}''\colon X*Y \to G$ is defined by $\boldsymbol{p}''(x*y) = \boldsymbol{p}(x)\boldsymbol{p}'(y)$ and $\Phi_{X*Y}(x*y) = \Phi_X(x)+\Phi_Y(y)$.

On Bw_0B, we have the following regular function (*Berenstein-Kazhdan potential*):

Definition 3.4.4. On the Bruhat cell $G^{w_0} = Bw_0B$, the BK potential Φ_{BK} is

$$\Phi_{BK}(uh\overline{w_0}u') := \chi^{\text{st}}(u) + \chi^{\text{st}}(u'), \quad \text{for } uh\overline{w_0}u' \in G^{w_0}.$$

Since $Bw_0B \cap B_- \hookrightarrow B_-$, so the potential restrict to open dense subset of B_-. The *highest weight map* hw of G is the following $U \times U$-invariant rational morphism

$$\text{hw}\colon Bw_0B \to H \ : \ uh\overline{w_0}u' \mapsto h. \tag{3.7}$$

Example 3.4.5. By Definition 3.4.4, the BK potential Φ_{BK} is a χ^{st}-linear function on the $U\times U$ variety G. Therefore, the $U \times U$ variety $G^{(n)}$ is a $(U \times U, \chi^{\text{st}})$-bicrystal with $\boldsymbol{p}\colon G^{(n)} \to G$ by sending $g_1 * \cdots * g_n$ to $g_1 \cdots g_n$, and the χ^{st}-linear function, or potential, is given by

$$\Phi_{G^{(n)}}(g_1 * \cdots * g_n) := \sum \Phi_{BK}(g_i).$$

To a $(U\times U, \chi^{\text{st}})$-bicrystal $(\boldsymbol{X}, \boldsymbol{p}, \Phi)$, the *central charge* of $(\boldsymbol{X}, \boldsymbol{p}, \Phi)$ is the $U\times U$-invariant function:

$$\Delta_X(x) := \Phi(x) - \Phi_{BK}(\boldsymbol{p}(x)), \quad \forall x \in X.$$

Assume that $U \setminus X/U$ is an affine variety in what follows. Since both Δ_X and $\text{hw}_X := \text{hw} \circ \boldsymbol{p}$ are $U \times U$-invariant, they descents to functions $\overline{\Delta}_X$ and $\overline{\text{hw}}_X$ on $U \setminus X/U$ respectively. Now consider the affine variety:

$$\widetilde{X} := U \setminus X/U \times_H G,$$

where the fiber product is over $\overline{\text{hw}}_X$ and hw. The variety $\widetilde{X}$ gets an $U \times U$ action on G by:

$$u \cdot (\overline{x}, g) \cdot u' \mapsto (\overline{x}, ugu').$$

Define a χ^{st}-linear function $\widetilde{\Phi}$ on $\widetilde{X}$ by

$$\widetilde{\Phi}(\overline{x}, g) := \overline{\Delta}_X(\overline{x}) + \Phi_{BK}(g), \quad \text{for } (\overline{x}, g) \in U \setminus X/U \times_H G.$$

Denote by p_2 is the projection $\widetilde{X}$ to the second factor G. All these make the triple $(U\setminus X/U \times_H G, \widetilde{\Phi}, p_2)$ into a unipotent bicrystal.

Definition 3.4.6. A $(U \times U, \chi^{\mathrm{st}})$-bicrystal $(\boldsymbol{X}, \boldsymbol{p}, \Phi)$ is *trivializable* if the following map is a birational isomorphism of $(U \times U, \chi^{\mathrm{st}})$-bicrystals

$$\varphi\colon X \xrightarrow{\sim} U \setminus X/U \times_H G \ :\ x \mapsto (\overline{x}, \boldsymbol{p}(x)). \tag{3.8}$$

Denote by $\mathbf{TriUB}_G$ the category of trivializable $(U \times U, \chi^{\mathrm{st}})$-bicrystals over G.

The following proposition shows that $U \setminus X/U$ can be realized a subvariety of X:

Proposition 3.4.7. *For a trivializable $(U \times U, \chi^{\mathrm{st}})$-bicrystal $(\boldsymbol{X}, \boldsymbol{p}, \Phi)$, the following natural map is a birational isomorphism of varieties*

$$Y := \boldsymbol{p}^{-1}(\phi(H)) \to U \setminus X/U,$$

where $\phi\colon H \to G$ is the natural rational lift of $\mathrm{hw}\colon G \to H$ given by $\phi(h) = h\overline{w_0} \in Bw_0B \subset G$. Moreover, we have $X \cong Y \times_H G \cong U \setminus X/U \times_H G$.

Proof. Note that each $U \times U$-orbit in X intersects Y at exactly one point. Thus we have the following commuting diagram

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ H & \longrightarrow & G \end{array},$$

which implies $Y \xrightarrow{\sim} U \setminus X/U$. Then $X \cong Y \times_H G \cong U \setminus X/U \times_H G$. ♢

Following the spirit of [16], we say $(U \times U, \chi^{\mathrm{st}})$-bicrystal $(\boldsymbol{X}, \boldsymbol{p}, \Phi)$ is *positive trivializable* if there exist positive structures for $X^- := \boldsymbol{p}^{-1}(B_-)$ and $U \setminus X/U$ respectively, such that the map φ in (3.8) and its inverse φ^{-1} restrict to positive birational isomorphisms of unipotent bicrystals:

$$\varphi_-\colon X^- \overset{\sim}{\leftrightarrows} U \setminus X/U \times_H B_- : (\varphi^{-1})_-. \tag{3.9}$$

In [16], the authors constructed a functor

$$\mathcal{B}\colon \mathbf{UB}_G^+ \to \mathbf{Mod}_{G^\vee}$$

from the category $\mathbf{UB}_G^+$ of positive unipotent bicrystals [16, Definition 3.29] to the category $\mathbf{Mod}_{G^\vee}$ of $G^\vee$ module by passing through the geometric crystals and Kashiwara crystals [16, Claim 6.9, 6.10, 6.12, Theorem 6.15]. Here we briefly recall some properties of the functor $\mathcal{B}$.

Let $(\boldsymbol{X}, \boldsymbol{p}, \Phi)$ be a positive unipotent bicrystals. Denote by $\mathrm{hw}_X := \mathrm{hw} \circ \boldsymbol{p}\colon X \to H$ the highest weight map of X. In what follows, we write $\boldsymbol{X}$ for $(\boldsymbol{X}, \boldsymbol{p}, \Phi)$ for simplicity. Then

(1) $\mathcal{B}(\boldsymbol{X} * \boldsymbol{X}') \cong \mathcal{B}(\boldsymbol{X}) \otimes \mathcal{B}(\boldsymbol{X}')$ and $\mathcal{B}$ is monoidal.

Denote by $\pi_X\colon X \to S$ an $U \times U$-invariant positive map to a torus S. Then the $G^\vee$-module $\mathcal{B}(\boldsymbol{X})$ can be parametrized over $\xi \in X_*(S)$ as direct sums of $G^\vee$-submodules, *i.e.*,

(2) $\mathcal{B}(\boldsymbol{X}) = \bigoplus_{\xi \in X_*(S)} \mathcal{B}_\xi(\boldsymbol{X})$.

Moreover, the typical components respect the convolution product $*$:

(1′) $\mathcal{B}_{\xi_1,\xi_2}(\boldsymbol{X}_1 * \boldsymbol{X}_2) \cong \mathcal{B}_{\xi_1}(\boldsymbol{X}_1) \otimes \mathcal{B}_{\xi_2}(\boldsymbol{X}_2)$.

The unipotent bicrystal $\boldsymbol{G} := (G, \mathrm{Id}_G, \Phi_{BK})$ is positive and we have:

(3) For $\lambda^\vee \in X_*^+(H)$, one has $\mathcal{B}_{\lambda^\vee}(\boldsymbol{G}) \cong V_{\lambda^\vee}$, where $V_{\lambda^\vee}$ is the irreducible $G^\vee$ module with highest weight $\lambda^\vee$; for $\lambda^\vee \notin X_*^+(H)$, one has $\mathcal{B}_{\lambda^\vee}(\boldsymbol{G}) = \emptyset$.

Let (M, Φ_M) be positive variety with potential (positive) fibered over torus $H \times S$. For unipotent bicrystal $(\boldsymbol{X} = (X, \alpha), \boldsymbol{p}, \Phi)$, denote by $\boldsymbol{X}_M := (M \times_H X, \alpha')$, where $\alpha'(u, (m, x), u) = (m, \alpha(u, x, u'))$. Thus $(\boldsymbol{X}_M, \boldsymbol{p}, \Phi_M + \Phi)$ is a positive unipotent bicrystal (positive) fibered over $H \times S$, then we have for $(\lambda^\vee, \xi) \in X_*(H) \times X_*(S)$

(4) $\mathcal{B}_{\lambda^\vee,\xi}(\boldsymbol{X}_M) \cong \mathbb{C}[M^t_{\lambda^\vee,\xi}] \otimes \mathcal{B}_{\lambda^\vee}(\boldsymbol{X})$, where $M^t_{\lambda^\vee,\xi}$ is the tropical fiber of M over $(\lambda^\vee, \xi)$.

3.5 Double Bruhat cells as positive varieties with potential

In this section, we introduce a positive structure for (B_-, Φ_{BK}), which are various factorization charts. We start by

Definition 3.5.1. Given an element $w \in W$, write $u \prec w$ if $\ell(uw) = \ell(w) - \ell(u)$.

By Theorem 2.6.4, for $u \prec w$, we have a open embedding $L^{u,uw} \to L^{e,w}$. Pre-compose it with the toric chart for $L^{u,uw}$, we have

Proposition 3.5.2. *For any $u \prec w$, denote by $\mathbf{i}(u) = (i_1, \ldots, i_n)$ a double reduced word for (u, uw), the following maps*

$$\xi_{\mathbf{i}(u)} \colon \mathbb{G}_{\mathbf{m}}^n \to L^{e,w} \ : \ (t_1, \ldots, t_n) \mapsto [\overline{u}^{-1} x_{i_1}(t_1) \cdots x_{i_n}(t_n))]_+;$$
$$\xi^{\mathbf{i}(u)} \colon \mathbb{G}_{\mathbf{m}}^n \to L^{w^{-1},e} \ : \ (t_1, \ldots, t_n) \mapsto [x_{-i_n}(t_1) \cdots x_{-i_1}(t_n))\overline{u^{-1}}]_{\leqslant 0}.$$

are open embeddings. Moreover, for $u, u' \prec w$, and any $\mathbf{i}(u)$ and $\mathbf{i}'(u')$, , $\xi_{\mathbf{i}(u)}$ (resp. $\xi^{\mathbf{i}(u)}$) and $\xi_{\mathbf{i}'(u')}$ (resp. $\xi^{\mathbf{i}'(u')}$) are positively equivalent.

Proof. All we need to show hare is that $\xi_{\mathbf{i}(u)}$ and $\xi_{\mathbf{i}'(u')}$ are positively equivalent. Using the commutating relation of Proposition 2.5.2, any associated double reduced word $\mathbf{i}(u)$ is positive equivalent to a separated word of (u, uw) by commutating all x_{j_i} for $j_i < 0$ to the left one by one. For a separated word $\mathbf{i}(u)$ for (u, uw), the map $\xi_{\mathbf{i}(u)}$ is actually the composition:

$$\mathbb{G}_{\mathbf{m}}^{\ell(u)} \times \mathbb{G}_{\mathbf{m}}^{\ell(uw)} \to L^{u,e} \times L^{e,uw} \xrightarrow{\xi^{u,e} \times \mathrm{Id}} L^{e,u^{-1}} \times L^{e,uw} \to L^{e,w}.$$

By Proposition 2.5.2 and Lemma 2.6.6, this is positive equivalent to $x_{\mathbf{i}}$ for some double reduced word $\mathbf{i}$ for (e, w). ♢

Example 3.5.3. Let $w = w_0 = s_1 s_2 s_3 s_1 s_2 s_1$. Then we know $u = s_3 s_2 s_2 \prec w = w_0$. Choose the double reduced word for (u, uw_0) as $(-3, -2, 1, 2, -1, 1)$. The map $\xi_{\mathbf{i}(u)}$ is:

$$(t_1, \ldots, t_6) \mapsto [\overline{s_3 s_2 s_1}^{-1} x_{-3}(t_1) x_{-2}(t_2) x_1(t_3) x_2(t_4) x_{-1}(t_5) x_1(t_6)]_+,$$

and the matrix on the right hand side is:

$$\begin{bmatrix} 1 & 2t_6 & t_2 + t_4 & t_1 \\ 0 & 1 & t_2 t_3 + t_2 t_6^{-1} + t_4 t_6^{-1} & t_1 t_3 + t_1 t_6^{-1} \\ 0 & 0 & 1 & t_1 t_2^{-1} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Recall that $H \times L^{w_0,e} \to B_-$ and $H \times (L^{e,w_0})^T \to B_-$ are all open. Now Proposition 3.5.2 gives various toric charts on $L^{w_0,e}$, B_- and U, which we refer to them as *factorization charts*. We denote the sets of these toric charts by

$$\Theta_L, \quad \Theta_{B_-}, \quad \Theta_U \tag{3.10}$$

for $L^{w_0,e}$, B_- and U respectively.

Theorem 3.5.4. *The triple $(B_-, \Phi_{BK}, \Theta_{B_-})$ is a positive variety with potential.*

To show that Φ_{BK} is positive with respect to Θ_{B_-}, we first write the potential Φ_{BK} using generalized minors. To begin with, let $F = \sum F_i$ be the sum of the negative root vectors associated with the simple roots. Let i^* be the index of the simple root $\alpha_{i^*} := -w_0\alpha_i$, then F_{i*} is the negative root vector corresponding to the root $\alpha_{i^*} := -w_0\alpha_i$. Let $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha = \sum \omega_i$ be the Weyl vector. Then:

Proposition 3.5.5. *[16, Corollary 1.25] The BK potential Φ_{BK} is a regular function on $G^{w_0,e}$ and it has the following expressions:*

$$\Phi_{BK} = \sum_{i\in \boldsymbol{I}} \frac{\Delta_{w_0\omega_i,s_i\omega_i} + \Delta_{w_0s_i\omega_i,\omega_i}}{\Delta_{w_0\omega_i,\omega_i}} \tag{3.11}$$

$$= \sum_{i\in \boldsymbol{I}} \frac{F_{i^*}\cdot\Delta_{w_0\omega_i,\omega_i} + \Delta_{w_0\omega_i,\omega_i}\cdot F_i}{\Delta_{w_0\omega_i,\omega_i}} = \frac{F\cdot\Delta_{w_0\rho,\rho} + \Delta_{w_0\rho,\rho}\cdot F}{\Delta_{w_0\rho,\rho}}. \tag{3.12}$$

Proof. The expression (3.11) is just [16, Corollary 1.25]. We need to show the rest here. Since $\Delta_{w_0\omega_i,\omega_i}\cdot F_i = \Delta_{w_0\omega_i,s_i\omega_i}$ for the right action, and $F_{i^*}\cdot\Delta_{w_0\omega_i,\omega_i} = \Delta_{w_0s_i\omega_i,\omega_i}$ for the left action, we get the second equality. To show the last equality, one uses:

$$\Delta_{w_0\rho,\rho}(g) = \Delta_\rho(\overline{w_0}^{-1}g) = ([\overline{w_0}^{-1}g]_0)^\rho = \prod_{i\in \boldsymbol{I}}([\overline{w_0}^{-1}g]_0)^{\omega_i} = \prod_{i\in \boldsymbol{I}}\Delta_{w_0\omega_i,\omega_i}(g),$$

as well as $\Delta_{w_0\omega_i,\omega_i}\cdot F_j = 0$ $(j\neq i)$ for the right action, and $F_{j^*}\cdot\Delta_{w_0\omega_i,\omega_i} = 0$ $(j^*\neq i^*)$ for the left action. ♢

Proof of Theorem 3.5.4. Combine Theorem 2.7.2 and Proposition 3.5.5. ♢

Remark 3.5.6. If G is not simply connected, generalized minors of the form $\Delta_{u\omega_i,v\omega_i}$ are not in general functions on G. However, Φ_{BK} is: Suppose $\widehat{G}$ is the universal cover of G with $p\colon \widehat{G}\to G$ the covering map. Then the right-hand side of (3.11) is well defined on $\widehat{G}$ and is invariant under the action of any element belonging to $\ker p$. Thus Φ_{BK} descends to a function on G.

Proposition 3.5.7. *For $(h,z)\in H\times L^{w_0,e}$, the BK potential has the form:*

$$\Phi_{BK}(hz) = \sum_{i\in \boldsymbol{I}}\left(\Delta_{w_0\omega_i,s_i\omega_i}(z) + h^{-w_0\alpha_i}\Delta_{w_0s_i\omega_i,\omega_i}(z)\right). \tag{3.13}$$

Proof. We only need to consider the case when G is simply connected. By (2.6), we have

$$\Delta_{w_0\omega_i,s_i\omega_i}(hz) = h^{w_0\omega_i}\Delta_{w_0\omega_i,s_i\omega_i}(z), \quad \Delta_{w_0s_i\omega_i,\omega_i}(hz) = h^{w_0s_i\omega_i}\Delta_{w_0s_i\omega_i,\omega_i}(z),$$

$$\Delta_{w_0\omega_i,\omega_i}(hz) = h^{w_0\omega_i}\Delta_{w_0\omega_i,\omega_i}(z).$$

Since $h^{s_i\omega_i-\omega_i} = h^{-\alpha_i}$ and $\Delta_{w_0\omega_i,\omega_i}(z) = 1$ for $z\in L^{w_0,e}$, we get the desired form. ♢

Example 3.5.8. Let $G = \mathrm{SL}_2$ and H be the subgroup of diagonal matrices as in Example 2.1.1. We have the following factorization and potential:

$$x = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \begin{bmatrix} t^{-1} & 0 \\ 1 & t \end{bmatrix} = \begin{bmatrix} at^{-1} & 0 \\ a^{-1} & a^{-1}t \end{bmatrix}; \quad \Phi_{BK} = t + \frac{a^2}{t}.$$

Recall $X_*(H) = \mathbb{Z}\alpha^\vee$. Then the BK cone is cut out by the following inequalities:

$$\min\{\langle e_1^*, \xi_1 e_1\rangle, \langle e_1^*, -\xi_1 e_1\rangle + \langle x\alpha^\vee, \alpha\rangle\} \geqslant 0,$$

where e_1^* is the dual of e_1. In other words,

$$(B_-, \Phi_{BK}, x_{\mathbf{i}})^t = \{(x\alpha^\vee, \xi_1 e_1) \in X_*(H) \times \mathbb{Z} \mid 2x \geqslant \xi_1 \geqslant 0\}.$$

Note $X^*(H) = \mathbb{Z}\omega$. Now, for $G^\vee = \mathrm{PSL}_2$, the BK cone is given by the inequalities:

$$\min\{\langle e_1^*, \xi_1 e_1\rangle, \langle e_1^*, -\xi_1 e_1\rangle + \langle x\omega, \alpha^\vee\rangle\} \geqslant 0.$$

Therefore,

$$(B_-^\vee, \Phi_{BK}^\vee, x_{\mathbf{i}}^\vee)^t = \{(x\omega, \xi_1 e_1) \in X^*(H) \times \mathbb{Z} \mid x \geqslant \xi_1 \geqslant 0\}.$$

The lattice cones $(B_-, \Phi_{BK}, x_{\mathbf{i}})^t$ and $(B_-^\vee, \Phi_{BK}^\vee, x_{\mathbf{i}}^\vee)^t$ are depicted in Figure 3.1.

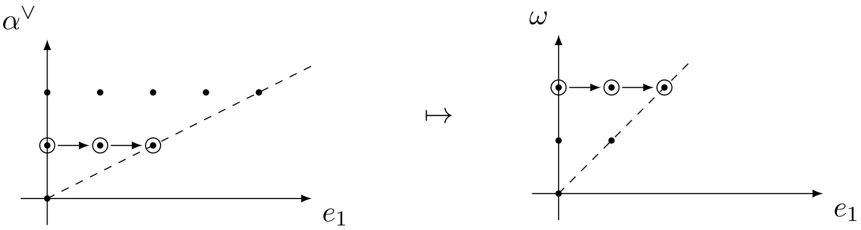


Figure 3.1: Comparison of the lattice cones for $G = \mathrm{SL}_2$ and $G^\vee = \mathrm{PSL}_2$.

For our purpose, we would like to state the construction $\mathcal{B}(\boldsymbol{G})$ in the following form, where recall $\boldsymbol{G} := (G, \mathrm{Id}_G, \Phi_{BK})$ is a positive unipotent crystal.

Theorem 3.5.9. *[16, Main Theorem 6.15] Consider the positive variety $(B_-, \Phi_{BK}, \Theta_{B_-})$. Given a toric chart $\theta \in \Theta_{B_-}$, then $(B_-, \Phi_{BK}, \theta)^t$ carries a structure of Kashiwara crystal, the image of* hw^t *lies in set of the dominant weights $X_*^+(H)$, and there is a direct decomposition as Kashiwara crystals:*

$$(B_-, \Phi_{BK}, \theta)^t = \bigsqcup_{\lambda^\vee \in X_*^+(H)} \mathrm{hw}^{-t}(\lambda^\vee).$$

Moreover, $\mathrm{hw}^{-t}(\lambda^\vee) \cong B_{\lambda^\vee}$ *as Kashiwara crystals, where $B_{\lambda^\vee}$ is the crystal associated with the irreducible $G^\vee$-module with highest weight $\lambda^\vee$.*

Definition 3.5.10. For any $\theta \in \Theta_{B_-}$, we refer to $(B_-, \Phi_{BK}, \theta)^t$ as a *BK cone*.

From [78, Lemma 3.10], the operators $\tilde{e}_i$ and $\tilde{f}_i$ on the crystal $(B_-, \Phi_{BK}, x_{\mathbf{i}})^t$, where $x_{\mathbf{i}}$ is the factorization chart as in (2.9), can be written explicitly as follows. Let $v_1, \dots, v_m$ be the standard basis of $\mathbb{Z}^m$. Let

$$x := \left(\lambda^\vee, \sum \xi_j v_j\right) \in (B_-, \Phi_{BK}, x_{\mathbf{i}})^t \subset X_*(H) \times \mathbb{Z}^m.$$

Then the crystal operators on $(B_-, \Phi_{BK}, x_{\mathbf{i}})^t$ are given by

$$\tilde{f}_i\left(\lambda^\vee, \sum \xi_j v_j\right) = \begin{cases} \left(\lambda^\vee, \sum \xi_j v_j + v_{n_f}\right) & \text{if } \left(\lambda^\vee, \sum \xi_j v_j + v_{n_f}\right) \in (B_-, \Phi_{BK}, x_{\mathbf{i}})^t, \\ \oslash & \text{else;} \end{cases}$$
$$\tilde{e}_i\left(\lambda^\vee, \sum \xi_j v_j\right) = \begin{cases} \left(\lambda^\vee, \sum \xi_j v_j - v_{n_e}\right) & \text{if } \left(\lambda^\vee, \sum \xi_j v_j - v_{n_e}\right) \in (B_-, \Phi_{BK}, x_{\mathbf{i}})^t, \\ \oslash & \text{else.} \end{cases}$$

The indices $n_f = n_f(x, i)$ and $n_e = n_e(x, i)$ are given by:

$$n_f := \min\left\{l \,\middle|\, 1 \leqslant l \leqslant m,\ i_l = i, X_l = \min_{l'}\{X_{l'} \mid i_{l'} = i\}\right\}; \tag{3.14}$$
$$n_e := \max\left\{l \,\middle|\, 1 \leqslant l \leqslant m,\ i_l = i, X_l = \min_{l'}\{X_{l'} \mid i_{l'} = i\}\right\},$$

where, for an index l,

$$X_l(x, i) = \sum_{k=1}^{l} a_{i_k, i}\xi_k.$$

Observe that, if $x, \tilde{e}_i x \in (B_-, \Phi_{BK}, x_{\mathbf{i}})^t$, then

$$n_e(x, i) = n_e(\tilde{e}_i x, i). \tag{3.15}$$

3.6 Polyhedral parametrizations of canonical bases

Let $\mathbf{i}$ be a double reduced word for (w_0, e). In [21], the authors introduce the so-called string cone $\mathcal{C}_{\mathbf{i}}$, which is a polyhedral cone and its integral points parametrize the (dual) canonical bases of the quantized universal enveloping algebra $U_q(\mathfrak{n})$.

The name "string cone" comes from the interpretation of points of the cone as strings of operators on $U_q(\mathfrak{n})$. The cone $\mathcal{C}_{\mathbf{i}}$ is equal to $(L^{w_0,e}, \Phi_L, \theta)^t$, for a specific $\theta \in \Theta_L$. Extending this terminology slightly, in Definition 3.6.2 we introduce the name *string cone* for any cone of the form $(L^{w_0,e}, \Phi_L, \theta)^t$. In this section, we recall this construction and describe the relation with the BK cone we constructed in the previous section.

Proposition 3.6.1. *The triple $(L^{w_0,e}, \Phi_L, \Theta_L)$ is a positive variety with potential, where*

$$\Phi_L = \sum_{i \in \boldsymbol{I}} \Delta_{w_0\omega_i, s_i\omega_i}, \tag{3.16}$$

is a regular function on $L^{w_0,e}$ and Θ_L is the factorization charts as in (3.10). *Moreover, the projection* $\mathrm{pr}\colon B_- \supset H \times L^{w_0,e} \to L^{w_0,e}$ *is a morphism of positive varieties with potential from $(B_-, \Phi_L, \Theta_{B_-})$ to $(L^{w_0,e}, \Phi_L, \Theta_L)$. For any $\theta \in \Theta_L$, it induces a surjective map*

$$\mathrm{pr}^t\colon (B_-, \Phi_L, \mathrm{Id}_H \times \theta)^t \twoheadrightarrow (L^{w_0,e}, \Phi_L, \theta)^t.$$

Proof. Choose a double reduced word $\mathbf{i}$ for (w_0, e). Consider charts $x_{\mathbf{i}}\colon \mathbb{G}_{\mathbf{m}}^m \to L^{w_0,e}$ and

$$\widetilde{x}_{\mathbf{i}}\colon H \times \mathbb{G}_{\mathbf{m}}^m \to B_- \;:\; (h, t_1, \ldots, t_m) \mapsto h x_{\mathbf{i}}(t_1, \ldots, t_m).$$

By Theorem 2.7.2, the evaluation of generalized minor $\Delta_{\gamma,\delta}$ at $x_{\mathbf{i}}$ is

$$\Delta_{\gamma,\delta}(x_{\mathbf{i}}(t_1,\ldots,t_m)) = \sum_{\pi} N_\pi t_1^{d_1(\pi)} \cdots t_m^{d_m(\pi)}, \tag{3.17}$$

where π's are certain $\mathbf{i}$-trails and $N_\pi \in \mathbb{R}_{>0}$, and $d_k(\pi) \in \mathbb{Z}$. Thus the first statement follows immediately. To show that pr is a morphism of positive varieties with potential, it is enough to show that

$$\Phi_{BK} - \Phi_L \circ \mathrm{pr}$$

is positive with respect to $\widetilde{x}_{\mathbf{i}}$ and $x_{\mathbf{i}}$. By Proposition 3.5.7, for any $(h,x) \in H \times L^{w_0,e}$, one has

$$\left(\Phi_{BK} - \Phi_L \circ \mathrm{pr}\right)(hx) = \Phi_{BK}(hx) - \Phi_L(x) = \sum_{i\in \boldsymbol{I}} h^{-w_0\alpha_i} \Delta_{w_0 s_i\omega_i,\omega_i}(x), \tag{3.18}$$

and so $\Phi_{BK} - \Phi_{BZ} \circ \mathrm{pr}$ is positive.

It remains to show that $\mathrm{pr}^t\colon (B_-,\Phi_L,\Theta_{B_-})^t \to (L^{w_0,e},\Phi_L,\Theta_L)^t$ is surjective. In other words, for $(\xi_1,\ldots,\xi_m) \in (L^{w_0,e},\Phi_L,x_{\mathbf{i}})^t$, we must find $\lambda^\vee \in X_*(H)$ such that

$$(\lambda^\vee,\xi_1,\ldots,\xi_m) \in (B_-,\Phi_{BK},\widetilde{x}_{\mathbf{i}})^t.$$

By (3.18), one has $(\lambda^\vee,\xi_1,\ldots,\xi_m) \in (B_-,\Phi_{BK},\widetilde{x}_{\mathbf{i}})^t$ if and only if

$$\sum_{k=1}^{m} d_k(\pi)\xi_k - \langle w_0\alpha_i,\lambda^\vee\rangle + \Phi_L^t(\xi_1,\ldots,\xi_m) \geqslant 0, \tag{3.19}$$

for all $i \in \boldsymbol{I}$ and the corresponding $\mathbf{i}$-trails. Let $\lambda^\vee = \sum n_i\omega_i^\vee$; by picking the n_i sufficiently large one ensures the inequalities (3.19) all hold. $\diamondsuit$

Definition 3.6.2. For any $\theta \in \Theta_L$, we refer to $(L^{w_0,e},\Phi_L,\theta)^t$ as a *string cone* and $\triangle_\lambda := \mathrm{hw}^{-t}(\lambda^\vee)$ a *string polytope*.

Remark 3.6.3. The string cone $(L^{w_0,e},\Phi_L,\theta)^t$ naturally carries a Kashiwara crystal structure and we have $(L^{w_0,e},\Phi_L,\theta)^t \cong B_\infty$ as crystals, where B_∞ is the e Kashiwara crystal for a $G^\vee$ Verma module of highest weight 0.

4 | Preliminaries on Cluster Varieties

In this chapter, we recall some basic definitions of (homogeneous) cluster varieties. Following [35], we introduce the notion of dual cluster algebra and a family of comparison maps. Then we focus on the double Bruhat cell $G^{w_0,e}$, which is homogeneous and admits a natural dual. Most of the material of this chapter follows from [14, 36, 42].

4.1 Cluster varieties

Definition 4.1.1. A *seed* $\sigma = (I, J, M)$ consists of a finite set I, a subset $J \subset I$ and an integer matrix $M = [M_{ij}]_{i,j\in I}$ which is skew-symmetrizable, *i.e.*, there exists a sequence of positive integers $\boldsymbol{d} = \{d_i\}_{i\in I}$ called a skew-symmetrizer such that $M_{ij}d_j = -M_{ji}d_i$. The *principal part* of M is given by $M_0 = [M_{ij}]_{i,j\in J}$.

Remark 4.1.2. As with the symmetrizable matrix A in Section 2.1, the existence of a skew-symmetrizer for M easily implies that M is skew-symmetrizable in the usual sense. Note that the submatrix $\widetilde{B} = [M_{ij}]_{i\in I,\, j\in J}$ is called an *exchange matrix* and usually mutations of seeds are defined in terms of $\widetilde{B}$, however the seed matrix M is more convenient for our purposes.

We associate a split algebraic torus to a given seed σ:

$$\mathcal{A}_\sigma := \mathbb{G}_{\mathbf{m}}^{|I|},$$

and write $\{a_i\}_{i\in I}$, which refer to as *cluster variables*, for the natural coordinates on $\mathcal{A}_\sigma$. Recall that the matrix mutation of any matrix M in direction k is defined as:

$$\mu_k(M)_{ij} = \begin{cases} -M_{ij}, & \text{if } k \in \{i,j\}; \\ M_{ij} + \dfrac{1}{2}\Big(|M_{ik}|M_{kj} + M_{ik}|M_{kj}|\Big), & \text{otherwise .} \end{cases}$$

If M is skew-symmetrizable, one can easily show that $\mu_k(M)$ is skew-symmetrizable with the same skew-symmetrizer. A *mutation of a seed* σ *in direction* $k \in J$ is the seed $\sigma_k = (I_k, J_k, \mu_k(M))$, where $I_k = I$, $J_k = J$, together with a birational map of tori $\mu_k : \mathcal{A}_\sigma \to \mathcal{A}_{\sigma_k}$ given in terms of their coordinate algebras by:

$$\mu_k^*(a_i) = \begin{cases} a_i, & \text{if } i \neq k; \\ a_k^{-1}\left(\displaystyle\prod_{M_{jk}>0} a_j^{M_{jk}} + \prod_{M_{jk}<0} a_j^{-M_{jk}}\right), & \text{if } i = k. \end{cases} \tag{4.1}$$

Two seeds are *mutation equivalent* if they are related by a sequence of mutations. The equivalence class of a seed σ is denoted by $|\sigma|$.

Definition 4.1.3. Given a seed σ, the cluster variety $\mathcal{A} \equiv \mathcal{A}_{|\sigma|}$ is the scheme obtained by gluing the $\mathcal{A}_\sigma$ for all $\sigma \in |\sigma|$ using the birational mutation maps. The seed σ is called the *initial seed* of $\mathcal{A}$. Each natural coordinate of $\mathcal{A}_{\sigma'}$ for any $\sigma' \in |\sigma|$ is called a cluster variable, and the set of natural coordinates of $\mathcal{A}_{\sigma'}$ is called a cluster for $\mathcal{A}$.

Abusing notation, each seed σ gives a toric chart $\sigma\colon \mathcal{A}_\sigma \hookrightarrow \mathcal{A}$, which will be called a *cluster chart*. Mutation equivalent seeds σ and σ' give positively equivalent charts for $\mathcal{A}$. Denote $[\sigma]$ the class of positively equivalent charts given by the equivalence class $|\sigma|$ of the seed σ.

Remark 4.1.4. Note that a new scheme can be obtained by gluing existing schemes through gluing maps. Even through the scheme $\mathcal{A}$ we get is not an affine variety in general, we still call it a cluster *variety*. Denote by $\mathbb{Q}[\mathcal{A}]$ the algebra of regular functions on $\mathcal{A}$. Then $\mathbb{Q}[\mathcal{A}]$ coincides with the *upper cluster algebra* generated by the seed σ; see [14]. The algebra homomorphism $\sigma^*\colon \mathbb{Q}[\mathcal{A}] \to \mathbb{Q}[\mathcal{A}_\sigma]$ is an injection, for any cluster chart σ.

Example 4.1.5. (Stasheff pentagon) To the very top pentagon in Figure 4.1, we associate a seed

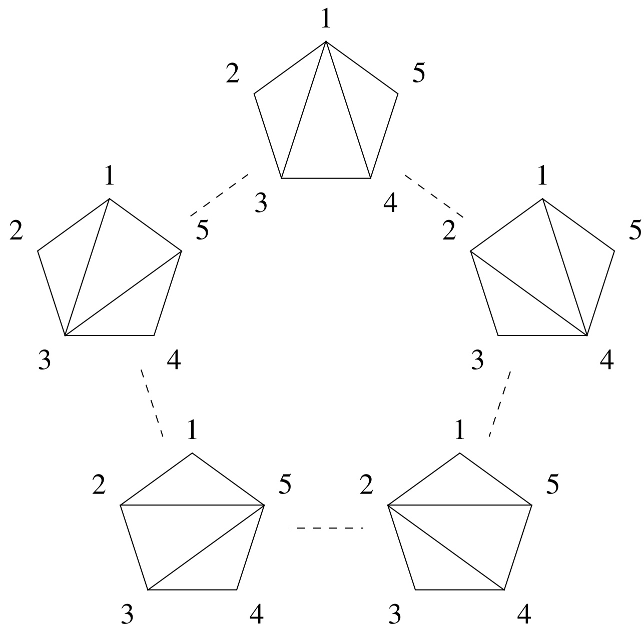


Figure 4.1: Stasheff pentagon

(I, J, M), where $I = \{1, \dots, 7\}$, $J = \{1, 2\}$ and M is given by

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & 0 \end{bmatrix}, \text{ where } M_{11} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, M_{12} = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 \end{bmatrix}.$$

To each edge (m, n) in the top pentagon we associate a variable b_{mn}, which will be a coordinate on the seed torus $\mathcal{A}_{(I,J,M)}$. By ordering the variables b_{mn} in the following way, we index them by I:

$$b_{13}, b_{14}, b_{12}, b_{23}, b_{34}, b_{45}, b_{15}.$$

By the definition of mutation in direction 2, we get

$$\mu_2(b_{14}) = \frac{b_{13}b_{45} + b_{15}b_{34}}{b_{14}}.$$

This mutation can be presented by the *Whitehead move* from edge $(1,4)$ to edge $(3,5)$ and the *Plücker relation*: Let $b_{35} := \mu_2(b_{14})$ be the variable corresponding to edge $(3,5)$, then

$$b_{14}b_{35} = b_{13}b_{45} + b_{15}b_{34}.$$

In fact, each dashed line in Figure 4.1 is a Whitehead move and gives a cluster mutation. The algebra generated by $\{b_{ij}\}$ with all Plücker relations is the homogeneous coordinate ring of the Grassmannian $\mathcal{G}_2(5)$ of 2-dimensional planes in the 5-dimensional space. Note that the principal part of M is M_{11}. More details can be found in [42].

At the end of this section, we state the famous

Theorem 4.1.6. *(Laurent phenomenon) For a cluster variety $\mathcal{A}_{|\sigma|}$ with initial seed σ, each cluster variable can be expressed as a Laurent polynomial with integer coefficients in the elements of the initial (or any other) cluster variables.*

4.2 Homogeneous cluster varieties

In this section, we introduce the notion of homogeneous cluster varieties.

Definition 4.2.1. A cluster variety $\mathcal{A}$ with initial seed σ is *graded* by an abelian group $\mathcal{G}$ if the algebra $\mathbb{Q}[\mathcal{A}_\sigma]$ is graded by $\mathcal{G}$ and the initial cluster variables a_i are homogeneous for $i \in I$.

Denote by $|\cdot|$ the degree of homogeneous elements in $\mathbb{Q}[\mathcal{A}_\sigma]$. By Laurent phenomenon [42, Theorem 3.14], we know any cluster variable can be written as a Laurent polynomial in initial cluster variables. Thus we give

Definition 4.2.2. A graded cluster variety $\mathcal{A}$ is *homogeneous* if all cluster variables are homogeneous with respect to the grading.

Proposition 4.2.3. *A $\mathcal{G}$-graded cluster variety $\mathcal{A}_{|\sigma|}$ with initial seed $\sigma = (I, J, M)$ is homogeneous if and only if*

$$\sum_{i\in I} |a_i| M_{ij} = 0, \quad \forall j \in J. \tag{4.2}$$

Proof. If $\mathcal{A}_{|\sigma|}$ is homogeneous, the equation (4.2) follows from the fact that the cluster variable a'_k of seed $\sigma' = \mu_k(\sigma)$ is homogeneous. To be more precise, the variable a'_k is homogeneous if and only if the monomials in (4.1) have the same degree. Then we have:

$$\sum_{M_{jk}>0} |a_j| M_{jk} = -\sum_{M_{jk}<0} |a_j| M_{jk},$$

which is equivalent to (4.2).

For the other direction, by induction, all we need to show is

$$\sum_{i\in I} |\mu_k^*(a_i)| \mu_k(M)_{ij} = 0, \quad \forall k, j \in J. \tag{4.3}$$

First all, note that $\mu_k^*(a_k)$ has degree:

$$|\mu_k^*(a_k)| = -|a_k| + \frac{1}{2}\sum_{i\in I} |a_i||M_{ik}|.$$

Then, for $j \neq k$, we have: (note that $M_{kk} = 0$)

$$2\sum_{i\in I} |\mu_k^*(a_i)|\mu_k(M)_{ij} = 2\sum_{i\neq k} |a_i|\mu_k(M)_{ij} + 2|\mu_k^*(a_k)|\mu_k(M)_{kj}$$

$$= \sum_{i\neq k} |a_i| \left(2M_{ij} + |M_{ik}|M_{kj} + M_{ik}|M_{kj}|\right) - \left(\sum_{i\in I} |a_i||M_{ik}| - 2|a_k|\right) M_{kj}$$

$$= \sum_{i\neq k} |a_i|M_{ik}|M_{kj}| = |M_{kj}|\sum_{i\in I} |a_i|M_{ik} = 0.$$

For $j = k$, we have

$$\sum_{i\in I} |\mu_k^*(a_i)|\mu_k(M)_{ik} = -\sum_{i\neq k} |a_i|M_{ik} = 0.$$

Thus we get the (4.3). ♢

4.3 Dual cluster varieties

Definition 4.3.1. [35] The *(Langlands) dual seed* of a seed σ is $\sigma^\vee := (I, J, -M^T)$. For the skew-symmetrizer $\boldsymbol{d}$ of M, fix an integer d such that each d_i divides d for all $i \in I$. Then $\boldsymbol{d}^\vee := \{d_i^\vee := d/d_i\}$ is a skew-symmetrizer of $-M^T$. For a seed σ, denote the torus associated to the dual seed $\sigma^\vee$ by $\mathcal{A}_\sigma^\vee \equiv \mathcal{A}_{\sigma^\vee}$.

It is not hard to check that

$$\mu_k(-M^T) = -\mu_k(M)^T.$$

In other words, we have $\mu_k(\sigma)^\vee = \mu_k(\sigma^\vee)$. Therefore, the tori $\mathcal{A}_\sigma^\vee$ assemble to a *dual cluster variety* $\mathcal{A}^\vee$. That is, $\mathcal{A}^\vee = \mathcal{A}_{|\sigma^\vee|} = \mathcal{A}_{|(\sigma')^\vee|}$ for any $\sigma, \sigma' \in |\sigma|$.

Definition 4.3.2. The quadruple $(\mathcal{A}, \mathcal{A}^\vee; \boldsymbol{d}, d)$ is called a *double cluster variety*. We write $(\mathcal{A}, \mathcal{A}^\vee)$ for short if is the choice of $\boldsymbol{d}$ and d is clear from context.

Given a seed σ, there is a natural morphism of tori associated to the skew-symmetrizer $\boldsymbol{d}$:

$$\Psi_\sigma\colon \mathcal{A}_\sigma \to \mathcal{A}_\sigma^\vee \ : \ (x_{i_1}, \dots, x_{i_{|I|}}) \mapsto (x_{i_1}^{d_{i_1}}, \dots, x_{i_{|I|}}^{d_{i_{|I|}}}). \tag{4.4}$$

On the coordinate algebra, we have the algebra homomorphism

$$\Psi_\sigma^*\colon \mathbb{Q}[\mathcal{A}_\sigma^\vee] \to \mathbb{Q}[\mathcal{A}_\sigma] \ : \ a_i^\vee \mapsto a_i^{d_i}, \quad i \in I.$$

Since $\mu_k(M)$ is skew-symmetrized by $\boldsymbol{d}$ as well, for any $\sigma' \in |\sigma|$, there is another map of tori:

$$\Psi_{\sigma'}\colon \mathcal{A}_{\sigma'} \to \mathcal{A}_{\sigma'}^\vee \ : \ (x'_{i_1}, \dots, x'_{i_{|I|}}) \mapsto ({x'}_{i_1}^{d_{i_1}}, \dots, {x'}_{i_{|I|}}^{d_{i_{|I|}}}).$$

So for each seed $\sigma' \in |\sigma|$, there is a rational comparison map $\Psi_{\sigma'}\colon \mathcal{A}_{|\sigma|} \to \mathcal{A}_{|\sigma|}$.

Note that $(\mathcal{A}^\vee, \mathcal{A}; \boldsymbol{d}^\vee, d)$ is also double cluster variety. Therefore, similar to (4.4), we have map $\Psi_{\sigma^\vee}\colon \mathcal{A}_\sigma^\vee \to \mathcal{A}_\sigma$. Direct computation shows $\Psi_{\sigma^\vee} \circ \Psi_\sigma\colon \mathcal{A}_\sigma \to \mathcal{A}_\sigma$ is the map which simply raises each coordinate a_i to the same power d. The cluster variety $\mathcal{A}$ and its dual $\mathcal{A}^\vee$ therefore play symmetric roles in the double cluster variety $(\mathcal{A}, \mathcal{A}^\vee; \boldsymbol{d}, d)$.

In what follows, write $\psi_\sigma := \Psi_\sigma^t$, where tropicalization is taken with respect to toric charts that are positively equivalent to σ and $\sigma^\vee$. We shall discuss the comparison map $\psi_\sigma = \Psi_\sigma^t$ in more detail. Let us look at an example first.

Example 4.3.3. We follow the notation in Example 4.1.5. Since the matrix M is skew-symmetric, the dual of $\mathcal{G}_2(5)$ is itself by identifying $b^\vee_{ij}$ and b_{ij}. The skew-symmetrizer $\boldsymbol{d}$ can be chosen as $\mathrm{diag}(d,\dots,d)$ for $d\in\mathbb{Z}_+$. Then on each seed σ, we have:

$$\Psi_\sigma\colon \mathcal{A}_\sigma \to \mathcal{A}^\vee_\sigma \cong \mathcal{A}_\sigma \text{ s.t. } \Psi^*_\sigma(b^\vee_{ij}) = b^d_{ij}.$$

So on the seed σ containing edges $(1,3)$ and $(1,4)$, one computes:

$$\Psi^*_\sigma(b^\vee_{35}) = \frac{b^d_{13}b^d_{45}+b^d_{15}b^d_{34}}{b^d_{14}} \tag{4.5}$$

On the seed σ' containing edges $(3,1)$ and $(3,5)$, one has:

$$\Psi^*_{\sigma'}(b^\vee_{35}) = b^d_{35} = \left(\frac{b_{13}b_{45}+b_{15}b_{34}}{b_{14}}\right)^d. \tag{4.6}$$

Note that right hand sides of (4.5) and (4.6) are equal after tropicalization:

$$\left(\frac{b^d_{13}b^d_{45}+b^d_{15}b^d_{34}}{b^d_{14}}\right)^t = \min\{d\xi_{13}+d\xi_{45}, d\xi_{15}+d\xi_{34}\} - d\xi_{14} = \left(\frac{(b_{13}b_{45}+b_{15}b_{34})^d}{b^d_{14}}\right)^t,$$

where $\xi_{mn} = b^t_{mn}$ is the tropicalization of b_{mn}.

Next we want to generalize what happened in the previous example. Recall that for positively equivalent charts $\theta,\theta'\colon \mathbb{G}^n_{\mathbf{m}} \to X$ on X, the tropical changing of coordinates $\mathrm{Id}^t\colon (X,\theta)^t \to (X,\theta')^t$ is defined as $(\theta'\circ\theta^{-1})^t\colon \mathbb{G}^n_{\mathbf{m}} \to \mathbb{G}^n_{\mathbf{m}}$.

Proposition 4.3.4. *The tropical maps ψ_σ agree for all σ. More precisely, let σ be a seed, and μ be a sequence of mutations of σ. Then the following diagram commutes.*

$$\begin{array}{ccc}
(\mathcal{A},\sigma)^t & \xrightarrow{\ \mathrm{Id}^t\ } & (\mathcal{A},\mu(\sigma))^t \\
\Big\downarrow{\scriptstyle \psi_\sigma} & & \Big\downarrow{\scriptstyle \psi_{\mu(\sigma)}} \\
(\mathcal{A}^\vee,\sigma^\vee)^t & \xrightarrow{\ (\mathrm{Id}^\vee)^t\ } & (\mathcal{A}^\vee,\mu(\sigma^\vee))^t
\end{array}$$

Here we abbreviate $\mathrm{Id} = \mathrm{Id}_{\mathcal{A}}$ *and* $\mathrm{Id}^\vee = \mathrm{Id}_{\mathcal{A}^\vee}$.

Proof. In fact, we only need to show the proposition for $\mu=\mu_k$ a single mutation. Let $\sigma_k := \mu_k(\sigma)$. Let $\{a_i \mid i\in I\}$ be the coordinates on $\mathcal{A}_\sigma$, and $\{a_{i'} \mid i'\in I'\}$ be the coordinates on $\mathcal{A}_{\sigma_k}$. And let $\{a^\vee_i \mid i\in I\}$ be the coordinates on $\mathcal{A}^\vee_\sigma$, and $\{a^\vee_{i'} \mid i'\in I'\}$ be the coordinates on $\mathcal{A}^\vee_{\sigma_k}$. On one hand, by definition:

$$\Psi^*_{\sigma_k}(a^\vee_{k'}) = a^{d_k}_{k'} = \mu^*_k(a_k)^{d_k} = a_k^{-d_k}\left(\prod_{M_{ik}>0} a_i^{M_{ik}} + \prod_{M_{ik}<0} a_i^{-M_{ik}}\right)^{d_k}.$$

On the other hand, using the formula for mutation, we get:

$$\begin{aligned}\Psi_\sigma^*(a_{k'}^\vee) = \Psi_\sigma^*(\mu_k^*(a_k^\vee)) &= a_k^{-d_k}\Psi_\sigma^*\left(\prod_{M_{ki}>0}(a_i^\vee)^{M_{ki}} + \prod_{M_{ki}<0}(a_i^\vee)^{-M_{ki}}\right)\\ &= a_k^{-d_k}\left(\prod_{M_{ki}>0} a_i^{d_iM_{ki}} + \prod_{M_{ki}<0} a_i^{-d_iM_{ki}}\right)\\ &= a_k^{-d_k}\left(\prod_{M_{ik}<0}(a_i^{-M_{ik}})^{d_k} + \prod_{M_{ik}>0}(a_i^{M_{ik}})^{d_k}\right).\end{aligned}$$

Then the tropicalization gives

$$(\Psi_{\sigma_k}\circ\mu_k)^t\colon \begin{cases}\xi_k^\vee \mapsto d_k\min\Big\{\sum_{M_{ik}<0} -M_{ik}\xi_i^\vee, \sum_{M_{ik}>0} M_{ik}\xi_i^\vee\Big\} - d_k\xi_{k'}^\vee;\\ \xi_i^\vee \mapsto d_i\xi_{i'}, & \text{for } i\neq k,\end{cases}$$

and,

$$(\mu_k\circ\Psi_\sigma)^t\colon \begin{cases}\xi_k^\vee \mapsto \min\Big\{\sum_{M_{ik}<0} -d_kM_{ik}\xi_i^\vee, \sum_{M_{ik}>0} d_kM_{ik}\xi_i^\vee\Big\} - d_k\xi_{k'}^\vee;\\ \xi_i^\vee \mapsto d_i\xi_{i'}, & \text{for } i\neq k.\end{cases}$$

where $\{\xi_i\}_{i\in I}$ is the natural basis of $\mathcal{A}_\sigma^t = \mathrm{Hom}(\mathbb{G}_\mathbf{m}, \mathcal{A}_\sigma)$, and similarly for $\mathcal{A}_\sigma^{\vee t}$, $\mathcal{A}_{\sigma'}^t$, and $\mathcal{A}_{\sigma'}^{\vee t}$. Thus

$$(\mu\circ\Psi_\sigma)^t = (\Psi_{\mu(\sigma)}\circ\mu)^t. \qquad \diamondsuit$$

Note that our tropical map ψ_σ is in general an injection (but not a bijection) of the lattice $\mathcal{A}_\sigma^t$ into the lattice $\mathcal{A}_\sigma^{\vee t}$.

At the end of this section, we would like to extend the cluster variety $\mathcal{A}$ with initial seed $\sigma = (I, J, M)$ by a split torus H of rank r. Denote $\widetilde{\mathcal{A}} = H\times\mathcal{A}$ the *extension* of $\mathcal{A}$ by H. Any choice of isomorphism of tori $H\cong\mathbb{G}_\mathbf{m}^r$ gives an isomorphism of $\widetilde{\mathcal{A}}$ and the cluster variety $\mathcal{A}_{|\widetilde{\sigma}|}$ generated by the seed

$$\widetilde{\sigma} := (I\cup\{1,\ldots,r\}, J, \mathrm{diag}(M,0)).$$

The variety $\widetilde{\mathcal{A}}$ is called a *decorated* cluster variety, or cluster variety if the decoration H is clear from the context.

Note that $H = X_*(H)\otimes_\mathbb{Z}\mathbb{G}_\mathbf{m}$ and consider the group $H^\vee = X^*(H)\otimes_\mathbb{Z}\mathbb{G}_\mathbf{m}$. Then $H^\vee$ is the Langlands dual group of H (in a slightly more general sense than was recalled in Section 2.1). Define the (Langlands) dual of $\widetilde{\mathcal{A}}$ as $\widetilde{\mathcal{A}}^\vee := H^\vee\times\mathcal{A}^\vee$. Given a double cluster variety $(\mathcal{A}, \mathcal{A}^\vee, \mathbf{d}, d)$, choose homomorphisms of tori $\Psi^H\colon H\to H^\vee$ and $\Psi^{H^\vee}\colon H^\vee\to H$ such that $\Psi^{H^\vee}\circ\Psi^H$ simply raises each coordinate to the d power. Then the tuple $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{A}}^\vee, \mathbf{d}, d, \Psi^H, \Psi^{H^\vee})$ is a *decorated double* cluster variety. We often write $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{A}}^\vee)$ for short.

On each seed of a decorated double cluster variety $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{A}}^\vee)$, the comparison maps extends:

$$\Psi^H\times\Psi_\sigma\colon \widetilde{\mathcal{A}}_\sigma\to\widetilde{\mathcal{A}}_\sigma^\vee \quad\text{and}\quad \Psi^{H^\vee}\times\Psi_{\sigma^\vee}\colon \widetilde{\mathcal{A}}_\sigma^\vee\to\widetilde{\mathcal{A}}_\sigma.$$

Let $\psi^H = (\Psi^H)^t\colon (H)^t\to(H^\vee)^t$. By Proposition 4.3.4, the maps $\psi^H\times\psi_\sigma$ agree for all seeds σ in the sense of Proposition 4.3.4.

4.4 Double Bruhat cells as homogeneous cluster varieties

Let G be a semisimple algebraic group. In this section, we recall how to make $G^{u,v}$ into a cluster variety, for any pair $(u, v) \in W \times W$. We begin by working with $L^{u,v}$. By decomposing $G^{u,v} = H \times L^{u,v}$, we then get a decorated cluster variety $G^{u,v}$ by extending $L^{u,v}$ to $H \times L^{u,v}$.

Recall that for $\mathbf{i} = (i_1, \ldots, i_n) \in R(u, v)$ and $k \in [1, n]$, we denote

$$k^+ = \min\{j \mid j > k, |i_j| = |i_k|\}, \quad k^- = \max\{j \mid j < k, |i_j| = |i_k|\}, \tag{4.7}$$

so that k^- (resp. k^+) is the previous (resp. next) occurrence of an index $\pm i_k$ in $\mathbf{i}$; for k the first (resp. last) occurrence of $\pm i_k$ in $\mathbf{i}$, we set $k^- = 0$ (resp. $k^+ = n + 1$). An index k is $\mathbf{i}$-exchangeable if $k^+ \in [1, m]$. Let $\boldsymbol{e}(\mathbf{i})$ denote the set of all $\mathbf{i}$-exchangeable indices.

Extend the word $\mathbf{i} \in R(u, v)$ to $(i_{-r}, \ldots, i_{-1}; i_1, \ldots, i_n)$, where $i_{-r} = -r$. Let

$$I = [-r, -1] \cup [1, n], \quad J = \boldsymbol{e}(\mathbf{i}), \quad L := [-r, -1] \cup \boldsymbol{e}(\mathbf{i}).$$

Construct a $I \times I$ matrix $\widetilde{M}$ as in [14, Remark 2.4]: For $k, l \in I$, set $p = \max\{k, l\}$ and $q = \min\{k^+, l^+\}$, and let $\epsilon(k)$ be the sign of k. Let

$$\widetilde{M}(\mathbf{i})_{kl} = \begin{cases} -\epsilon(k-l)\epsilon(i_p) \cdot A_{|i_k|,|i_l|}, & \text{if } p < q \text{ and } \epsilon(i_p)\epsilon(i_q)(k-l)(k^+ - l^+) > 0; \\ -\epsilon(k-l)\epsilon(i_p), & \text{if } p = q; \\ 0, & \text{otherwise .} \end{cases} \tag{4.8}$$

where we recall that A is the Cartan matrix of $\mathfrak{g}$. Denote by $M(\mathbf{i}) := [\widetilde{M}(\mathbf{i})_{kl}]_{k,l \in L}$ the $L \times L$ submatrix of $\widetilde{M}(\mathbf{i})$. Let

$$\boldsymbol{d}_{\mathbf{i}} = \{d_{-i_r}, \ldots, d_{-i_r}, d_{i_1}, \ldots, d_{i_m}\}, \tag{4.9}$$

where the sequence $\boldsymbol{d} = \{d_1, \ldots, d_r\}$ is the fixed symmetrizer of A. It is easy to see that $\boldsymbol{d}_{\mathbf{i}}$ is a skew-symmetrizer of $\widetilde{M}(\mathbf{i})$. Define the following seeds:

$$\widetilde{\sigma}(\mathbf{i}) := (I, J, \widetilde{M}(\mathbf{i})); \quad \sigma(\mathbf{i}) := (L, J, M(\mathbf{i})),$$

As before, we fix a positive integer d such that each d_i divides d.

Recall that for $\mathbf{i} \in R(u, v)$ and $k \in [1, n]$, we denote by

$$u_k = \prod_{\substack{l=1,\ldots,k \\ i_l < 0}} s_{i_l}, \quad v_k = \prod_{\substack{l=n,\ldots,k+1 \\ i_l > 0}} s_{i_l},$$

and for $k \in [-r, -1]$, denote by $u_k = e$ and $v_k = v^{-1}$. Define the generalized minors

$$\Delta_k(\mathbf{i}) := \Delta_k := \Delta_{u_k \omega_{i_k}, v_k \omega_{i_k}} \text{ for } k \in [-r, -1] \cup [1, n]. \tag{4.10}$$

Theorem 4.4.1. *[14, Theorem 2.10] For every $\mathbf{i} \in R(u, v)$, let $\mathscr{A}_{|\sigma(\mathbf{i})|}$ be the cluster variety generated by the seed $\sigma(\mathbf{i})$. Then the map given by*

$$\varphi_{\mathbf{i}}^* : \mathbb{Q}[\mathscr{A}_{|\sigma(\mathbf{i})|}] \to \mathbb{Q}[L^{u,v}] \; : \; a_k \mapsto \Delta_k, \; \textit{for} \; k \in L = [-r, -1] \cup \boldsymbol{e}(\mathbf{i})$$

is an isomorphism of algebras. If G is simply connected, the map

$$\widetilde{\varphi}_{\mathbf{i}}^* : \mathbb{Q}[\mathscr{A}_{|\widetilde{\sigma}(\mathbf{i})|}] \to \mathbb{C}[G^{u,v}], \quad a_k \mapsto \Delta_k, \; \textit{for} \; k \in I = [-r, -1] \cup [1, n]$$

is an isomorphism of algebras.

Proof. By [14, Eq (2.11)], the set of cluster variables on the chart $\sigma(\mathbf{i})$ of the double Bruhat cell $\widehat{G}^{u,v}$ for simply connected $\widehat{G}$ is $\{\Delta_k \mid k \in [-r,-1] \cup [1,n]\}$. Recall that for $x \in \widehat{L}^{u,v} \subset \widehat{G}^{u,v}$, we have $\Delta_{u\omega_i,\omega_i}(x) = 1$. Thus Theorem 4.4.1 follows from [14, Theorem 2.10] by applying $\Delta_{u\omega_i,\omega_i} = 1$ and identifying $\mathbb{Q}[L^{u,v}]$ and $\mathbb{Q}[\widehat{L}^{u,v}]$. ♢

Remark 4.4.2. Given $\mathbf{i} \in R(u,v)$, the set of functions $\{\Delta_k(\mathbf{i}) \mid k \in L\}$ is called (initial) cluster variables for cluster variety $L^{u,v}$ with initial seed $\sigma(\mathbf{i})$.

Remark 4.4.3. Note that the twisted minors on $L^{u,v}$ for $\mathbf{i} \in R(u,v)$ we introduced in Section 2.7 are exactly the composition of twist map $\psi^{u,v}$ with the labeled cluster variables for $L^{v,u}$ for the double reduced word $\mathbf{i}^{\mathrm{op}} \in R(v,u)$, where $\mathbf{i}^{\mathrm{op}} = (-i_m, -i_{m-1}, \ldots, -i_1)$.

Since the Weyl groups of G and $G^\vee$ are isomorphic, the reduced word $\mathbf{i}$ also gives the reduced double Bruhat cell $L^{\vee;u,v}$ for $G^\vee$ the structure of a cluster variety. Moreover, we have:

Corollary 4.4.4. *Fix $(u,v) \in W \times W$. Let $\boldsymbol{d}_{\mathbf{i}}$ be as in (4.9) and let d be the integer fixed in Section 2.1. Then the quadruple $(L^{u,v}, L^{\vee;u,v}; \boldsymbol{d}_{\mathbf{i}}, d)$ is a double cluster variety.*

Proof. What we need to show actually is $(L^{u,v})^\vee \cong L^{\vee;u,v}$, where $(L^{u,v})^\vee$ is the dual cluster variety of $L^{u,v}$. Let $(I, J, M^\vee(\mathbf{i}))$ be the initial seed of $L^{\vee;w_0,e}$. Following the definitions, one obtains

$$(L, J, M(\mathbf{i}))^\vee = \left(L, J, -M(\mathbf{i})^T\right) = \left(I, J, M^\vee(\mathbf{i})\right). \qquad ♢$$

For a seed $\sigma \in |\sigma(\mathbf{i})|$, denote by $\Psi^L_\sigma \colon L^{u,v} \to L^{\vee;u,v}$ the comparison map for the double cluster variety $(L^{u,v}, L^{\vee;u,v})$. Extending the cluster variety $L^{u,v}$ by H, we get the decorated cluster variety $G^{u,v} \cong H \times L^{u,v}$. For any seed σ on $L^{u,v}$, the following map gives a positive chart on $G^{u,v}$:

$$\mathrm{Id} \times \sigma \colon H \times \mathbb{G}_{\mathbf{m}}^n \to H \times L^{u,v} = G^{u,v}, \tag{4.11}$$

which is denoted by σ as well if there is no ambiguity. Combining with $\mathit{\Psi}^H \colon H \to H^\vee$ as in Proposition 2.1.3, we have the following comparison map on the decorated cluster variety:

$$\mathit{\Psi}_\sigma := \mathit{\Psi}^H \times \Psi^L_\sigma \colon G^{u,v} = H \times L^{w_0,e} \to G^{\vee;u,v} = H^\vee \times L^{\vee;u,v}. \tag{4.12}$$

The tuple $(G^{u,v}, G^{\vee;u,v}, \boldsymbol{d}_{\mathbf{i}}, d, \mathit{\Psi}^H, \mathit{\Psi}^{H^\vee})$ is then a decorated double cluster variety.

Next, we would like to give a grading to the the cluster variety $L^{u,v}$.

First of all, note that $H \times H$ acts on $\mathbb{Q}[G]$ in the natural way: $(a_1, a_2) \cdot f(x) = f(a_1^{-1} x a_2)$ for $f \in \mathbb{Q}[G]$ and $(a_1, a_2) \in H \times H$. Then $\mathbb{Q}[G^{u,v}]$ has a natural $P \times P$-grading and the $P \times P$-homogeneous elements are $H \times H$-eigenvectors in $\mathbb{Q}[G^{u,v}]$. Then any generalized minor $\Delta_{\gamma,\delta}$ on G has degree $(-\gamma, \delta)$ by Proposition 2.3.1.

Secondly, split double Bruhat cell $G^{u,v} \cong H \times L^{u,v}$ and denote the natural projections by:

$$\mathrm{pr}_1 \colon G^{u,v} \cong H \times L^{u,v} \to H; \quad \mathrm{pr}_2 \colon G^{u,v} \cong H \times L^{u,v} \to L^{u,v}.$$

Proposition 4.4.5. *For $\mathbf{i} \in R(u,v)$, the function $\Delta_k \circ \mathrm{pr}_2$ for $k \in [-r,-1] \cup \boldsymbol{e}(\mathbf{i})$ on G is homogeneous of degree $(0, -u^{-1}u_k\omega_{i_k} + v_k\omega_{i_k})$. The function $\Delta_{w\omega_i} \circ \mathrm{pr}_1$ is homogeneous of degree $(-w\omega_i, u^{-1}w\omega_i)$. Thus the cluster variety $L^{u,v}$ with initial seed $\sigma(\mathbf{i})$ is $P \times P$-graded.*

Proof. For $y \in G^{u,v}$, denote by $h = \mathrm{pr}_1(y)$ and $x = \mathrm{pr}_2(y)$. In other words, $y = hx$. For $a \in H$, it is clear that $\mathrm{pr}_2(a \cdot y) = x$. Since $B_-uB_- \cdot a = B_-uB_-$ and $U\overline{u}U \cdot a = (\overline{u}a\overline{u}^{-1}) \cdot U\overline{u}U$, for $x \in L^{u,v}$, we have

$$x' := (\overline{u}a^{-1}\overline{u}^{-1}) \cdot x \cdot a \in L^{u,v}.$$

Thus we have $\mathrm{pr}_2(y \cdot a) = x'$ since the factorization $G^{u,v} \cong H \times L^{u,v}$ is unique. Then we know:

$$\Delta_k \circ \mathrm{pr}_2(a_1 \cdot y \cdot a_2) = a_2^{-u^{-1}u_k\omega_{i_k} + v_k\omega_{i_k}} \Delta_k \circ \mathrm{pr}_2(y).$$

Thus we conclude that $\Delta_k \circ \mathrm{pr}_2$ is of degree $(0, -u^{-1}u_k\omega_{i_k} + v_k\omega_{i_k})$. ♢

Remark 4.4.6. Note that on $G^{u,v}$, we have for $k \in [-r, -1] \cup [1, n]$

$$\Delta_{u_k\omega_k, v_k\omega_{i_k}} = \Delta_{u_k\omega_{i_k}} \circ \mathrm{pr}_1 \cdot \Delta_{u_k\omega_{i_k}, v_k\omega_{i_k}} \circ \mathrm{pr}_2,$$

which also justifies that $\Delta_{u_k\omega_k, v_k\omega_{i_k}}$ has degree $(-u_k\omega_k, v_k\omega_{i_k})$.

To show that the cluster algebra on $L^{u,v}$ is homogeneous, we need

Proposition 4.4.7. *[42, Lemma 4.22] For any $k \in J = \boldsymbol{e}(\mathbf{i})$, we have:*

$$\sum u_k\omega_{i_k}\widetilde{M}(\mathbf{i})_{kl} = 0, \quad \sum v_k\omega_{i_k}\widetilde{M}(\mathbf{i})_{kl} = 0.$$

Then what follows is immediate:

Proposition 4.4.8. *For $\mathbf{i} \in R(u, v)$, the $P \times P$-graded cluster variety $L^{u,v}$ with initial seed $\sigma(\mathbf{i})$ is homogeneous. If G is simply connected, the $P \times P$-graded cluster variety $G^{u,v}$ with initial seed $\widetilde{\sigma}(\mathbf{i})$ is homogeneous.*

Proof. By Proposition 4.4.7, we have

$$\sum u^{-1}u_k\omega_{i_k}\widetilde{M}(\mathbf{i})_{kl} = 0, \quad \sum v_k\omega_{i_k}\widetilde{M}(\mathbf{i})_{kl} = 0.$$

Thus $\sum(-u^{-1}u_k\omega_{i_k} + v_k\omega_{i_k})\widetilde{M}(\mathbf{i})_{kl} = 0$. Note $-u^{-1}u_k\omega_{i_k} + v_k\omega_{i_k} = 0$ for $k \in I \setminus L$, since $u_k = u$ and $v_k = e$ in this case. Thus we get $\sum(-u^{-1}u_k\omega_{i_k} + v_k\omega_{i_k})M(\mathbf{i})_{kl} = 0$. ♢

Next, we show that the twist map is homogeneous:

Proposition 4.4.9. *Given a double reduced word $\mathbf{i}$ for (u, v), the function $M_k \circ \mathrm{pr}_2$ is homogeneous of degree $(0, -v^{-1}v^k\omega_{i_k} + u^k\omega_{i_k})$ for $k \in [-r, -1] \cup \boldsymbol{e}(\mathbf{i})$.*

Proof. By the uniqueness of the Gauss decomposition, we have for all $g \in G_0$ and $h \in H$:

$$[hg]_+ = [g]_+, \text{ since } hg = [hg]_{\leqslant 0}[hg]_+ = h[g]_{\leqslant 0} \cdot [g]_+;$$
$$[gh]_+ = h^{-1}[g]_+h, \text{ since } gh = [gh]_{\leqslant 0}[gh]_+ = [g]_{\leqslant 0}h \cdot h^{-1}[g]_+h.$$

Thus one computes for $x \in L^{u,v}$ and $h \in H$,

$$\psi^{u,v}(hx) = \psi^{u,v}(x); \quad \psi^{u,v}(xh) = (\overline{v}h^{-1}\overline{v}^{-1}) \cdot \psi^{u,v}(x) \cdot h$$

Thus the twisted minors M_k has degree $(0, -v^{-1}v^k\omega_{i_k} + u^k\omega_{i_k})$. ♢

4.5 Double Bruhat cells as positive varieties revisited

Recall that we have introduced the positive varieties (B_-, Θ_{B_-}) and $(L^{w_0,e}, \Theta_L)$. In this section, we would like to expand the set Θ_{B_-} and Θ_L.

For $\mathbf{i} \in R(u,v)$ and for $k \in [-r,-1] \cup e(\mathbf{i})$, the factorization parameters t_{k^+}'s of $x_{\mathbf{i}}\colon \mathbb{G}_{\mathbf{m}}^n \to L^{u,v}$ are Laurent monomials of twisted minors by Theorem 2.7.3, thus they are homogeneous. Actually by the $H \times H$ action on G, one can show the degree of t_k is

$$|t_k| = \begin{cases} (0, -u^k \alpha_{i_k}), & \text{if } i_k < 0; \\ \left(0, -\alpha_{i_k} + \sum_{l\colon l>k, i_l<0} a_{i_l,i_k}\alpha_{i_l}\right), & \text{if } i_k > 0 \end{cases} \tag{4.13}$$

by using $x_i(t)h = hx_i(th^{-\alpha_i})$ and $x_{-i}(t)h = h\alpha_i^\vee(h^{-\alpha_i})x_{-i}(th^{-\alpha_i})$ for $i \in \boldsymbol{I}$.

Definition 4.5.1. A toric chart $\theta\colon S \to X$ for a irreducible variety X is *graded* by an ablian group $\mathcal{G}$ if the $\mathbb{Q}[S]$ is a $\mathcal{G}$-graded algebra and the natural coordinates on S are homogeneous. Two toric charts $\theta_1, \theta_2\colon S \to X$ are *homogeneous equivalent* if both $(\theta_1^{-1} \circ \theta_2)^*$ and $(\theta_2^{-1} \circ \theta_1)^*$ send natural coordinates to homogeneous elements. Write $\theta_1 \overset{\mathrm{h}}{\sim} \theta_2$ if θ_1 and θ_2 are positive and homogeneous equivalent to each other.

Proposition 4.5.2. *Given a positive variety (X, Θ_X) with graded toric charts $\theta_i\colon \mathbb{G}_{\mathbf{m}}^n \to X$ for $i = 1, 2$. Denote by $\{a_i\}_{i=1}^n$ (resp. $\{b_i\}_{i=1}^n$) the natural coordinates for θ_1 (resp. θ_2). Suppose $\theta_1 \overset{\mathrm{h}}{\sim} \theta_2$, then then there exists a unimodular matrix M such that*

$$(|a_1|, \ldots, |a_n|) = (|b_1|, \ldots, |b_n|)M,$$

where $|f|$ is the degree of f.

Proof. Without loss of generality, let assume a_i's are homogeneous positive rational functions in b_i's. Write $a_i = f_i(b_1, \ldots, b_n)$. Let C be a linearity chamber for all f_i^t's, which is a chamber C such that $f_i^t|_C$'s are all liner. Thus on C, we have

$$(a_1^t, \ldots, a_n^t) = (b_1^t, \ldots, b_n^t)M, \tag{4.14}$$

where M is the coefficient matrix of $f_i^t|_C$'s. Since $\theta_1 \overset{\mathrm{h}}{\sim} \theta_2$, the matrix M is unimodular since it is an isomorphism of lattice $X^*(\mathbb{G}_{\mathbf{m}}^n)$. Note that (4.14) implies the claim we need to show. ♢

For $\mathbf{i} \in R(u,v)$, we have a toric chart $x_{\mathbf{i}}\colon \mathbb{G}_{\mathbf{m}}^n \to L^{u,v}$. Let $\Theta_{(u,v)}$ be set of all charts that are positive equivalent and homogeneous equivalent as well to $x_{\mathbf{i}}$. Denote by

$$\begin{aligned} \widetilde{\Theta}_{u,v} := \quad & \{\sigma \mid \sigma \in |\sigma(\mathbf{i})| \text{ for all } \mathbf{i} \in R(u,v)\} \cup \{\psi^{u,v} \circ \sigma \mid \sigma \in |\sigma(\mathbf{i})| \text{ for all } \mathbf{i} \in R(v,u)\} \\ & \cup \{x_{\mathbf{i}}, \psi^{u,v} \circ x_{\mathbf{i}^{\mathrm{op}}} \mid \mathbf{i}(u) \in R(u,v)\}. \end{aligned}$$

Proposition 4.5.3. *For $u, v \in W$, the pair $(L^{u,v}, \Theta_{u,v})$ is a positive variety. Moreover, the set $\widetilde{\Theta}_{u,v}$ is a subset of $\Theta_{u,v}$.*

Proof. The positivity of $\theta \in \widetilde{\Theta}_L$ is clear. The homogeneity of θ follows from that the cluster variety $L^{w_0,e}$ is homogeneous and all the transition maps in Section 2.5 are homogeneous. ♢

Remark 4.5.4. Recall for any $u \prec w \in W$, we have open embeddings $\xi_{\mathbf{i}(u)} \colon \mathbb{G}_{\mathbf{m}}^n \to L^{e,w}$ and $\xi^{\mathbf{i}(u)} \colon \mathbb{G}_{\mathbf{m}}^m \to L^{w^{-1},e}$ by Proposition 3.5.2. For $w = w_0$, the following set is a subset of $\Theta_{w_0,e}$ as well:

$$\left\{\xi^{\mathbf{i}(u)}, \psi^{e,w_0} \circ \xi_{\mathbf{i}(u)} \mid u \in W, \mathbf{i}(u) \in R(u, uw_0)\right\}.$$

Definition 4.5.5. (Notation) In the rest of the paper, denote by $\Theta_U := \Theta_{e,w_0}$ and $\Theta_{B_-} := \Theta_H \times \Theta_{w_0,e}$.

To achieve different properties, we normally have different preferred charts. Then the positivity and homogeneity will help us to transfer the properties in charts that are more accessible to the problem to the preferred charts.

5 | Preliminaries on Poisson-Lie Groups

First introduced by Drinfel'd [29] as semi-classical limits of quantum groups, a *Poisson-Lie group* is a group object in the category of Poisson varieties. A another motivation for the introduction of Poisson-Lie groups was the study of integrable systems associated to infinite-dimensional Lie algebras, see Semenov-Tian-Shansky[80]. Most of the material in this chapter is based on [31, 75].

5.1 Poisson-Lie groups

Definition 5.1.1. A *Poisson-Lie* group is a Lie group G endowed with a *multiplicative* Poisson structure π_G, *i.e.*, the group multiplication $G \times G \to G$ is a Poisson map, where $G \times G$ has the product Poisson structures.

Example 5.1.2. [64, Example 2.1] For $G = \mathrm{SL}_2$, we can define a family of Poisson structure on G parametrized by $d \in \mathbb{C}$

$$\{x_{12}, x_{11}\} = \frac{d}{2}x_{11}x_{12}, \qquad \{x_{21}, x_{11}\} = \frac{d}{2}x_{11}x_{21}, \qquad \{x_{22}, x_{11}\} = dx_{12}x_{21},$$
$$\{x_{12}, x_{21}\} = 0, \qquad \{x_{22}, x_{12}\} = \frac{d}{2}x_{12}x_{22}, \qquad \{x_{22}, x_{21}\} = \frac{d}{2}x_{21}x_{22},$$

where x_{ij} is the (i, j) entry of a matrix $x \in \mathrm{SL}_2$. Denote by $\mathrm{SL}_2^{(d)}$ the Poisson-Lie group with the Poisson structure defined above.

Example 5.1.3. Recall that a bivector $r \in \mathfrak{g} \otimes \mathfrak{g}$ is called classical r*-matrix* for $\mathfrak{g}$ if $r + r^{21} \in S^2(\mathfrak{g})^{\mathfrak{g}}$ and r satisfies the classical Yang-Baxter equation:

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0.$$

For any r-matrix $r \in \mathfrak{g} \otimes \mathfrak{g}$, the following bivector field π_G on G defined by

$$\pi_G := r^{\lambda} - r^{\rho}$$

is Poisson and (G, π_G) is a Poisson-Lie group, where r^{λ} (resp. r^{ρ}) is the left (resp. right) invariant 2-tensor fields on G.

Note that the bivector field π_G necessarily vanishes at the group identity e and therefore leading to a linear Poisson structure on $T_eG = \mathfrak{g}$, hence a Lie bracket $[\cdot, \cdot]_{\mathfrak{g}^*}$ on the dual vector space $\mathfrak{g}^*$. Moreover, the tuple $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, [\cdot,\cdot]_{\mathfrak{g}^*})$ is a *Lie bialgebra*, *i.e.*, the transpose $\delta\colon \mathfrak{g} \to \wedge^2\mathfrak{g}$ of $[\cdot,\cdot]_{\mathfrak{g}^*}\colon \wedge^2 \mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie algebra 1-cocycle. Then the tuple $(\mathfrak{g}^*, [\cdot,\cdot]_{\mathfrak{g}^*}, [\cdot,\cdot]_{\mathfrak{g}})$ is a Lie bialgebra as well, which is called the *dual Lie bialgebra* of $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, [\cdot,\cdot]_{\mathfrak{g}^*})$.

Denote by G^* the simply connected Lie group with the Lie algebra $(\mathfrak{g}^*, [\cdot,\cdot]_{\mathfrak{g}^*})$. Then by the theory of Poisson-Lie groups, there exists a unique Poisson structure π_{G^*} on G^* such that (G^*, π_{G^*}) is a Poisson-Lie group with Lie bialgebra $(\mathfrak{g}^*, [\cdot,\cdot]_{\mathfrak{g}^*}, [\cdot,\cdot]_{\mathfrak{g}})$. In the rest, we say two Poisson-Lie groups are *dual* to each other if their Lie bialgebras are dual to each other.

As an application of the previous example, let G be a semisimple Lie group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$ with a fixed bilinear form $(\cdot,\cdot)$ as in Section 2.1. For positive roots $\alpha \in R^+$, choose root vectors $E_\alpha \in \mathfrak{g}_\alpha$ and $F_\alpha \in \mathfrak{g}_{-\alpha}$ so that $(E_\alpha, F_\alpha) = 1$. Let X_i be an orthonormal basis for $\mathfrak{h}$ under the fixed bilinear form. Then the following 2-tensor is called the classical r-matrix

$$r_G := \frac{1}{2}\sum_{i\in \boldsymbol{I}} X_i \otimes X_i + \sum_{\alpha\in R^+} E_\alpha \otimes F_\alpha.$$

Note that the Borel subgroups B and B_- are Poisson-Lie subgroups of G. Recall that we have the symmetrizer $\boldsymbol{d}$ for the Cartan matrix A for $\mathfrak{g}$ and the canonical homomorphism $\phi_i\colon \mathrm{SL}_2 \to G$. Then the Poisson structure π_G is the unique the Poisson structure such that $\phi_i\colon \mathrm{SL}_2^{(d_i)} \to G$ is Poisson.

In this case, the simply connected dual Poisson-Lie group of G is the Lie subgroup of $G\times G$

$$G^* = \{(b, b_-) \in B \times B_- \mid [b]_0[b_-]_0 = 1\} \subset G \times G \tag{5.1}$$

with the Poisson bracket described by

Proposition 5.1.4. *, For $f \in \mathbb{C}[B]$ and $g \in \mathbb{C}[B_-]$, denote $f_1 := f \circ \mathrm{pr}_1 \in \mathbb{C}[G^*]$ and $g_2 := g \circ \mathrm{pr}_2 \in \mathbb{C}[G^*]$, where $\mathrm{pr}_1\colon G^* \to B$ and $\mathrm{pr}_2\colon G^* \to B_-$ are the natural projections. Then the projections pr_1 and pr_2 are anti-Poisson and the mixed bracket is given by*

$$\begin{aligned}\{f_1, g_2\}_{G^*} = &\ \frac{1}{2}\sum_{i\in \boldsymbol{I}}(X_i\cdot f)_1(X_i\cdot g)_2 - (f\cdot X_i)_1(g\cdot X_i)_2\\ &+ \sum_{\alpha\in R^+}(E_\alpha\cdot f)_1(F_\alpha\cdot g)_2 - (f\cdot E_\alpha)_1(g\cdot F_\alpha)_2.\end{aligned}$$

5.2 Log-canonical coordinates and twist maps

In this section, we recall the symplectic leaves of the Poisson-Lie group (G, π_G) for semisimple Lie group G and describe the log-canonical coordinates for it. Most of the material is based on [42, 64].

For $w \in W$ and $h \in H$, denote by $h^w := w^{-1}hw$. For $u, v \in W$, let $H^{u,v}$ be the subtorus of H formed by $(h^u)^{-1}h^v$ for $h \in H$.

On double Bruhat cell $G^{u,v}$, we introduce the following maps

$$\begin{aligned} &p_u\colon BuB \to H \ :\ g \to h, && \text{if } g \text{ has form } n_1\overline{u}hn_2 \text{ for } n_i \in U;\\ &q_v\colon B_-vB_- \to H \ :\ g \to h, && \text{if } g \text{ has form } n_1\overline{v}hn_2 \text{ for } n_i \in U_-.\end{aligned}$$

Note that in the factorization $g = n_1\overline{u}hn_2$, h is unique even through that n_i is not.

Proposition 5.2.1. *The symplectic leaves of π_G in G are the connected components of the set:*

$$\left\{g \in G^{u,v} \mid p_u(g) \cdot q_v(g) \in hH^{u,v}\right\}$$

for some $u, v \in W$ and $h \in H$, which has dimension $\ell(u) + \ell(v) + \dim H^{u,v}$.

Assume that G is simply connected, denote by $I(u, v)$ the set of all indices i such that $u\omega_i = v\omega_i = \omega_i$, where ω_i is the fundamental weight of G. Then

Theorem 5.2.2. *[64, Theorem 2.3] For G simply connected, the following set:*

$$S^{u,v} := \left\{x \in G^{u,v} \mid [\overline{u}^{-1}x]_0 \cdot ([x\overline{v^{-1}}]_0)^v \in H^{u,v},\ [\overline{u}^{-1}x]_0^{\omega_i} = 1 \forall i \in I(u,v)\right\}.$$

is a symplectic leaf in G. Every symplectic leaf in G is of the form $S^{u,v} \cdot h$ for some $u, v \in W$ and $h \in H$.

Next we would like to describe a log-canonical coordinates on G. First of all, we have

Theorem 5.2.3. *[42, Theorem 3.1] The double Bruhat cell $G^{u,v}$ is a Poisson submanifold of (G, π_G). The twist map $\xi^{u,v} \colon G^{u,v} \to G^{u^{-1},v^{-1}}$ defined in 2.6.1 is an anti-Poisson map.*

Assume that G is simply connected. Recall that on the $G^{u,v}$, for each $\mathbf{i} \in R(u, v)$, we have a collection of cluster variables $\Delta_k := \Delta_k(\mathbf{i})$ for $k \in [-r, -1] \cup [1, n]$. Together with the [64, Theorem 2.6], we have

Theorem 5.2.4. *[42, Corollary 4.12] On the double Bruhat cell $G^{u,v}$, the standard Poisson bracket between generalized minors is given by*

$$\{\Delta_k, \Delta_l\} = \frac{1}{2}\left((u_k\omega_{i_k}, u_l\omega_{i_l}) - (v_k\omega_{i_k}, v_l\omega_{i_l})\right)\Delta_k\Delta_l$$

for $k < l \in [-r, -1] \cup [1, n]$.

Recall that by assuming that G is simply connected, we know $G^{u,v}$ is a homogeneous cluster variety. Define a skew-symmetric paring $\langle \cdot, \cdot \rangle$ by requiring

$$2\langle |\Delta_k|, |\Delta_l| \rangle := \left((u_k\omega_{i_k}, u_l\omega_{i_l}) - (v_k\omega_{i_k}, v_l\omega_{i_l})\right), \text{ if } k < l.$$

Then we have the following corollary:

Corollary 5.2.5. *For $\mathbf{i} \in R(u, v)$, denote by z_i's the cluster variables for seed $\sigma \in |\sigma(\mathbf{i})|$, the standard Poisson bracket $\{z_k, z_l\}$ is given by*

$$\{z_k, z_l\} = \langle |z_k|, |z_l| \rangle z_k z_l.$$

Proof. Let f and g be monomials in $\{\Delta_i\}$, thus direct computation shows:

$$\{f, g\} = \langle |f|, |g| \rangle fg.$$

Now suppose that $f = \sum f_i$ and $g = \sum g_i$ are homogeneous Laurent polynomials in $\{\Delta_i\}$, which means that $|f| = |f_1| = \cdots = |f_m|$ and $|g| = |g_1| = \cdots = |g_n|$. Thus by the previous computation, we have:

$$\begin{aligned}\{f, g\} &= \sum_{i,j}\{f_i, g_i\} = \sum_{i,j}\langle |f_i|, |g_j| \rangle f_i g_j \\ &= \sum_{i,j}\langle |f|, |g| \rangle f_i g_j = \langle |f|, |g| \rangle \sum_{i,j} f_i g_j \\ &= \langle |f|, |g| \rangle fg.\end{aligned}$$

Then by Laurent phenomenon and homogeneity of z_i, we get the conclusion. ♢

5.3 Poisson involutions

In this section, we shall talk about Poisson involutions on a Poisson variety and its fix locus. As an application, we discuss holomorphic Poisson structures and their real forms. Most material can be fond in [86].

Definition 5.3.1. A Poisson involution on a Poisson variety (X, π) is a Poisson diffeomorphism $\tau\colon X \to X$ such that $\tau^2 = \mathrm{Id}$.

Proposition 5.3.2. *[86, Proposition 4.2] Let X^τ be the stable locus of a Poisson involution $\tau\colon X \to X$. Assume that the Poisson tensor π on P is $\pi = \sum_i A_i \wedge B_i$, where A_i and B_i are vector fields on X. Then the tensor $\pi^\tau := \sum A_i^+ \wedge B_i^+\Big|_{X^\tau}$ is a Poisson tensor on X^τ, where*

$$A^+ = \frac{1}{2}(A + \tau_* A)$$

for any vector field A on X.

Without using local coordinates, the Poisson tensor π^τ can be defined as follows. For any $p \in X^\tau$, decompose T_pX as $T_pX = (T_pX)^\tau + (T_pX)^{-\tau}$, where

$$(T_pX)^\tau = \{v \in T_pX \mid \tau(v) = v\}, \quad (T_pX)^{-\tau} = \{v \in T_pX \mid \tau(v) = -v\}.$$

Then we know π can be write as $\pi(p) = \pi_+(p) + \pi_\pm(p) + \pi_-(p)$, where

$$\pi_+(p) \in \wedge^2(T_pX)^\tau, \quad \pi_\pm(p) \in (T_pX)^\tau \otimes (T_pX)^{-\tau}, \quad \pi_-(p) \in \wedge^2(T_pX)^{-\tau}.$$

Then $\pi^\tau(p) := \pi_+(p)$.

Example 5.3.3. Let $G = \mathrm{SL}_n$ with the standard r-matrix r_G. Let $(b, b_-) \in G^* \subset B \times B_-$ be an element in its Poisson-Lie dual. Let τ be the following involution on G^*:

$$\mathcal{T}\colon G^* \to G^* \ : \ (b, b_-) \mapsto (b_-^T, b^T).$$

Thus the fix locus of $\mathcal{T}$ is U, *i.e.*, upper triangular matrices with all diagonal entries equal to 1. For $n = 3$, the Poisson structure induced as in Proposition 5.3.2 can be described as

$$\{x, y\} = xy - 2z;\ \{y, z\} = yz - 2x;\ \{z, x\} = zx - 2y, \qquad \text{where } u = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \in U_+.$$

This Poisson structure was also obtained independently by [27] and by [82] in the general case in connection with the study of Frobenius manifolds. In [23], Boalch also realized that this Poisson structure on U coincides with the induced Poisson structure on G^*.

Next we would like to talk about the real forms of holomorphic Poisson structures. Let (X, π) be a complex manifold with holomorphic Poisson structure $\pi \in \wedge^2(T^{1,0}X)$. Denote by

$$\mathbb{i} := \sqrt{-1}.$$

Let $\pi = \pi_R + \mathbb{i}\pi_I$ be the decomposition of π into real and imaginary parts; it is well-known that $\pi_R, \pi_I \in \Gamma(\wedge^2(TX))$ are (real) Poisson bivectors.

Definition 5.3.4. Let X be a complex manifold with holomorphic Poisson structure π. A *real form* of (X, π) is an anti-holomorphic involution $\tau\colon X \to X$ of X, which satisfies $\tau(\pi_R) = \pi_R$.

Remark 5.3.5. Equivalently, $\tau(\pi_R) = \pi_R$ if and only if $\tau(\pi_I) = -\pi_I$. Extending τ conjugate-linearly to $TX \otimes \mathbb{C}$, this is equivalent to the condition $\tau(\pi) = \pi$.

Let Y be a (real) open submanifold of the fixed locus of τ. For any $p \in Y$, decompose T_pX as $T_pX = (T_pX)^\tau + (T_pX)^{-\tau}$, where

$$(T_pX)^\tau = \{v \in T_pX \mid \tau(v) = v\}, \quad (T_pX)^{-\tau} = \{v \in T_pX \mid \tau(v) = -v\}.$$

As shown in [86], π_R can be decomposed as $\pi_R(p) = \pi_R^\tau(p) + \pi_R^{-\tau}(p)$, where

$$\pi_R^\tau(p) \in \Lambda^2(T_pX)^\tau, \quad \pi_R^{-\tau}(p) \in \Lambda^2(T_pX)^{-\tau}.$$

Lemma 5.3.6 ([86])**.** *Using the notation above, π_R^τ is a Poisson bivector on $Y \subset X^\tau$.*

Example 5.3.7. Consider any holomorphic Poisson structure $\pi = \sum_{i,j} \pi_{ij}(z)\partial_{z_i} \wedge \partial_{z_j}$ on $\mathbb{C}^n$, where $\pi_{ij}(z)$ are holomorphic functions. Let τ be the anti-holomorphic involution of $\mathbb{C}^n$ given by $\tau(z) = \overline{z}$. Then the set of fixed points of τ is $Y = \mathbb{R}^n \subset \mathbb{C}^n$. Thus $\tau(\pi_R) = \pi_R$ if and only if $\pi_{ij}(\overline{z}) = \overline{\pi_{ij}(z)}$. Write $\partial_{z_i} = 1/2(\partial_{x_i} - \mathbb{i}\partial_{y_i})$, and direct calculation shows

$$\pi_R^\tau = \frac{1}{4}\sum_{i,j} \pi_{ij}(x)\partial_{x_i} \wedge \partial_{x_j}. \tag{5.2}$$

Lemma 5.3.8. *Let X be a complex manifold with holomorphic Poisson structure π. Let τ be real form of (X, π) and Y be a (real) open submanifold of X^τ. Then we have*

$$\{f_1|_Y, f_2|_Y\}_{\pi_R^\tau} = \frac{1}{4}\{f_1, f_2\}_\pi|_Y,$$

where the f_i are holomorphic functions on an open subset $U \subset X$ satisfying $f_i(\tau(z)) = \overline{f_i(z)}$.

Proof. We only need to show the lemma in a neighborhood of each fixed point of τ. Choose holomorphic local coordinates $z_1, \ldots, z_n$ such that τ is given by $\tau(z) = \overline{z}$. Then in these coordinates, the f_i satisfy $f_i(\overline{z}) = \overline{f_i(z)}$. Set $z_j = x_j + \mathbb{i}y_j$ and let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Write $f_i(z) = u_i(x, y) + \mathbb{i}v_i(x, y)$, where $u_i, v_i \in C^\infty(X)$ are smooth real-valued functions on X. Since $f_i(\overline{z}) = \overline{f_i(z)}$, we know $u_i(x, y) = u_i(x, -y)$ and $v_i(x, y) = -v_i(x, -y)$. Thus $\partial_{y_j} u_i|_{y=0} = 0$. Then by the Cauchy-Riemann equations, we have

$$\begin{aligned}\Big(\partial_{z_j} f_i(z)\Big)\Big|_Y &= \frac{1}{2}\Big(\partial_{x_j} u_i + \partial_{y_j} v_i - \mathbb{i}\partial_{y_j} u_i + \mathbb{i}\partial_{x_j} v_i\Big)\Big|_{y=0} \\ &= \frac{1}{2}\Big(\partial_{x_j} u_i + \partial_{y_j} v_i\Big)\Big|_{y=0} = \partial_{x_j} f_i(x)\end{aligned}$$

By Equation (5.2), we get the conclusion. ♢

Proposition 5.3.9. *Let $X = (\mathbb{C}^\times)^n$ with holomorphic Poisson bivector π and τ be a real form of X. Let f_1, f_2 be holomorphic functions on an open subset $U \subset X$. Then*

$$\{f_1|_Y, f_2|_Y\}_{\pi_R^\tau} = \frac{1}{4}\{f_1, f_2\}_\pi|_Y.$$

Proof. Let $\overline{\tau} = \tau \circ \overline{(\cdot)}$ and $g_i := f_i + f_i \circ \overline{\tau}$ and $h_i := f_i - f_i \circ \overline{\tau}$. Note that $f_i(\overline{z}) = \overline{f_i(z)}$. Then g_i and $\mathbb{i}h_i$ satisfy the condition from Lemma 5.3.8:

$$g_i(\tau(z)) = \overline{g_i(z)}, \qquad \mathbb{i}h_i(\tau(z)) = \overline{\mathbb{i}h_i(z)}.$$

We then compute $4\{f_1|_Y, f_2|_Y\}_{\pi_R^\tau}$ by

$$\begin{aligned}
&\{g_1|_Y + h_1|_Y, g_2|_Y + h_2|_Y\}_{\pi_R^\tau} \\
=& \Big(\{g_1|_Y, g_2|_Y\}_{\pi_R^\tau} - \mathbb{i}\{g_1|_Y, \mathbb{i}h_2|_Y\}_{\pi_R^\tau} - \mathbb{i}\{\mathbb{i}h_1|_Y, g_2|_Y\}_{\pi_R^\tau} - \{\mathbb{i}h_1|_Y, \mathbb{i}h_2|_Y\}_{\pi_R^\tau}\Big) \\
=& \frac{1}{4}\Big(\{g_1, g_2\}_\pi|_Y - \mathbb{i}\{g_1, \mathbb{i}h_2\}_\pi|_Y - \mathbb{i}\{\mathbb{i}h_1, g_2\}_\pi|_Y - \{\mathbb{i}h_1, \mathbb{i}h_2\}_\pi|_Y\Big) \\
=& \frac{1}{4}\{g_1 + h_1, g_2 + h_2\}_\pi|_Y = \{f_1, f_2\}_\pi|_Y.
\end{aligned}$$

◇

At the end, let us discuss when we have two involutions:

Proposition 5.3.10. *Let (X, π) be a complex Poisson variety with a holomorphic Poisson involution τ_1 and a real form τ_2. Suppose that $\tau_1 \circ \tau_2 = \tau_2 \circ \tau_1$, then τ_2 is a real from of $(X^{\tau_1}, \pi^{\tau_1})$.*

Proof. Since $\tau_1 \circ \tau_2 = \tau_2 \circ \tau_1$, τ_2 restrict to an anti-holomorphic involution on X^{τ_1}. We need to show τ_2 is a Poisson map on X^{τ_1}. For any $p \in X^{\tau_1}$, decompose T_pX as $T_pX = (T_pX)^{\tau_1} + (T_pX)^{-\tau_1}$, where

$$(T_pX)^{\tau_1} = \{v \in T_pX \mid \tau_1(v) = v\}, \quad (T_pX)^{-\tau_1} = \{v \in T_pX \mid \tau_1(v) = -v\}.$$

Since $\tau_1 \circ \tau_2 = \tau_2 \circ \tau_1$, we know $\tau_2(T_pX)^{\tau_1} \subset (T_pX)^{\tau_1}$ and $\tau_2(T_pX)^{-\tau_1} \subset (T_pX)^{-\tau_1}$. Recall π can be write as $\pi(p) = \pi_+(p) + \pi_\pm(p) + \pi_-(p)$, where

$$\pi_+(p) \in \wedge^2(T_pX)^{\tau_1}, \quad \pi_\pm(p) \in (T_pX)^{\tau_1} \otimes (T_pX)^{-\tau_1}, \quad \pi_-(p) \in \wedge^2(T_pX)^{-\tau_1}.$$

Then $\pi^\tau(p) = \pi_+(p)$. Thus we know that $\tau_2(\pi^{\tau_1}) = \pi^{\tau_1}$. ◇

5.4 Ginzburg-Weinstein isomorphisms

In this section, we discuss a special Poisson isomorphism, which is the so-called Ginzburg-Weinstein (GW) isomorphism arising from Poisson-Lie theory.

First of all, we recall what is real forms of complex Poisson-Lie groups.

Definition 5.4.1. Let (G, π_G) be a connected complex Poisson Lie group, *i.e.*, a complex Lie group G together with a multiplicative holomorphic Poisson structure π_G. A *real form* of (G, π_G) is an anti-holomorphic involution τ on G such that τ is a group automorphism of G and $\tau(\pi_G) = \pi_G$.

Proposition 5.4.2. *For any real form τ of a connected complex Poisson Lie group (G, π_G), the pair $(G^\tau, 4(\pi_G)_R^\tau)$ is a real Poisson-Lie group.*

Let G be a semisimple complex Lie group with the standard Poisson-Lie structure π_G. Let (G^*, π_{G^*}) be the Poisson-Lie dual. Denote by K the compact real form of G and $G \cong AU_-K$ its Iwasawa decomposition. Then:

Theorem 5.4.3. *The involution* $\tau := \overline{\tau} \circ \overline{(\cdot)} \colon G^* \to G^*$ *is a real form of* $(G^*, \mathbb{i}\pi_{G^*})$*, where*

$$\overline{\tau} \colon (b, b_-) \mapsto \left(b_-^{-T}, b^{-T}\right).$$

The map $\mathrm{pr}_2 \colon (G^*)^\tau \to B_-$ *is an isomorphism to its image, and* $\mathrm{Im}(\mathrm{pr}_2|_{(G^*)^\tau}) = K^* := AU_-$.

In the rest, write $\pi_{K^*} := (\mathrm{pr}_2)_*(4(\mathbb{i}\pi_{G^*})^\tau_R)$. Thus we have a Poisson-Lie group (K^*, π_{K^*}).

Next we recall the natural K actions on $\mathfrak{k}^*$ and K^*. Denote by $T = K \cap H$. We make the standard identifications

$$X^*(H) \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{i}\mathfrak{t}^*, \qquad X^*(H) \otimes_{\mathbb{Z}} \mathbb{C} = \mathfrak{h}^*, \qquad X_*(H) \otimes_{\mathbb{Z}} \mathbb{C} = \mathfrak{h}.$$

The positive Weyl chamber $\mathfrak{t}^*_+ \subset \mathfrak{k}^*$ is

$$\mathfrak{t}^*_+ = \{\xi \in \mathfrak{t}^* \mid \langle \mathbb{i}\xi, \alpha_i^\vee \rangle \geqslant 0, \ \forall i = 1, \ldots, r\}.$$

Recall that the *coadjoint action* of K on $\mathfrak{k}^*$ is defined in terms of the adjoint action by the equation

$$\langle \mathrm{Ad}_k^* \xi, x \rangle = \langle \xi, \mathrm{Ad}_{k^{-1}} x \rangle, \qquad k \in K, \ \xi \in \mathfrak{k}^*, \ \text{and } x \in \mathfrak{k}.$$

The Lie-Poisson structure $\pi_{\mathfrak{k}^*}$ is preserved by coadjoint action, and the symplectic leaves of $\pi_{\mathfrak{k}^*}$ are the coadjoint orbits. The action of K on K^* is the so-called *dressing action*, which is defined by re-factorizing $kb \in G$ according to the Iwasawa decomposition $G = AU_-K$ for $k \in K$ and $b \in K^*$. If

$$kb = b'k' \in AU_-K, \qquad k, k' \in K, \ b, b' \in K^* = AU_-,$$

then the dressing action of k on b is defined as ${}^k b = b'$. The symplectic leaves of π_{K^*} are the dressing orbits, which are the joint level sets of the Casimir functions [75],

$$C_i(b)^2 := \mathrm{Tr}\left(\rho_i\left(bb^*\right)\right), \qquad b \in K^*, \tag{5.3}$$

where ρ_i is the fundamental representation of G with highest weight $\omega_i \in P_+$. The map $\varphi \colon b \mapsto bb^*$ is a diffeomorphism of K^* onto the set $S = \{g \in G \mid g^* = g\}$.

There is a family of diffeomorphisms $\mathfrak{F}_s \colon \mathfrak{k}^* \to K^*$ parameterized by $s \neq 0$ [34]. Let $\psi \colon \mathfrak{k}^* \to \mathfrak{k}$ be the K-equivariant isomorphism given by the fixed bilinear form on $\mathfrak{g}$. Then,

$$\mathfrak{F}_s \colon \mathfrak{k}^* \xrightarrow{\ \psi\ } \mathfrak{k} \xrightarrow{\exp\left(2s\sqrt{-1}\cdot\right)} S \xrightarrow{\ \varphi^{-1}\ } K^* = AU_-. \tag{5.4}$$

The map $\mathfrak{F}_s$ is equivariant with respect to the coadjoint and dressing actions of K. Let $\mathcal{O}_\xi$ be the coadjoint orbit through $\xi \in \mathfrak{t}^*_+$. Denote by $\mathcal{D}_{s\xi}$ the dressing orbit through $\mathfrak{F}_s(\xi) = \exp\left(-s\sqrt{-1}\psi(\xi)\right)$. Since $\mathfrak{F}_s$ is K-equivariant, $\mathfrak{F}_s(\mathcal{O}_\xi) = \mathcal{D}_{s\xi}$.

However, $\mathfrak{F}_1$ is not a Poisson map. But the following theorem address

Theorem 5.4.4. *[43] There exists a Poisson diffeomorphism from Poisson manifold* $(\mathfrak{k}^*, \pi_{\mathfrak{k}^*})$ *to* (K^*, π_{K^*}).

Such Poisson isomorphisms are called *Ginzburg-Weinstein isomorphisms/diffeomorphisms*. Such maps is denoted by GW_s through out the paper. Note that $\mathfrak{k}^*$ is abelian, whereas K^* is not, so Ginzburg-Weinstein diffeomorphisms can not be group homomorphisms, and $\mathfrak{k}^*$ and K^* can not be isomorphic as Poisson-Lie groups.

In particular, for each $s \neq 0$ the map GW_s restricts to a symplectomorphism between the coadjoint orbit $\mathcal{O}_\xi \subset \mathfrak{k}^*$ and the dressing orbit $\mathcal{D}_{s\xi} \subset K^*$. Therefore, we may study $\mathcal{D}_{s\xi}$ for arbitrary $s \in \mathbb{R}^\times$ instead of $\mathcal{O}_\xi$.

There are several proofs of the Ginzburg-Weinstein Theorem in the literature: the original proof [43] is an existence proof using a cohomology calculation, the proof in [1] gives Ginzburg-Weinstein diffeomorphisms as flows of certain Moser vector fields, the proof in [30] is by integration of a non-linear PDE of a classical dynamical r-matrix, and the proof in [23] uses the Stokes data of an ODE on a disc with an irregular singular point in the center.

Example 5.4.5. For $G = \mathrm{SL}_2$, by identifing $\mathfrak{su}(2)^*$ with Hermitian 2×2 matrices

$$A = \begin{bmatrix} x & z \\ \overline{z} & -x \end{bmatrix}, \text{ where } x \in \mathbb{R},\ z \in \mathbb{C}.$$

The following map is a Ginzburg-Weinstein diffeomorphism

$$\mathrm{GW}(A) = \begin{bmatrix} e^{x/2} & e^{i\theta}\sqrt{e^r + e^{-r} - e^x - e^{-x}} \\ 0 & e^{-x/2} \end{bmatrix},$$

where $r = \sqrt{x^2 + |z|^2}$, $z = \rho e^{i\theta}$.

5.5 More involutions on $\mathrm{SL}_n(\mathbb{C})^*$

In this section, we would like to consider more involutions on $G^* = \mathrm{SL}_n(\mathbb{C})^*$. Not like the real form of G^*, the fix locus we get is a Poisson variety, rather than a Poisson-Lie group. At the end of this section, we explain why we want to consider these involutions.

Recall on $\mathrm{SL}_n(\mathbb{C})^*$ in Example 5.3.3, we have the following Poisson involution

$$\mathcal{T} : (b_-, b) \mapsto (b^T, b_-^T).$$

Denote by

$$P_n = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix}, \quad P = \begin{bmatrix} 0 & P_n \\ -P_n & 0 \end{bmatrix}; \quad J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

And we introduce the following Poisson involution on $\mathrm{SL}_{2n}(\mathbb{C})^*$:

$$\tau_1 : \mathcal{P} : (b_-, b) \to (P b_-^{-T} P, P b^{-T} P).$$

Recall that we have a real form τ on G^*, it easy to check that τ commutes with $\mathcal{T}$ and $\mathcal{P}$. By Proposition 5.3.10, τ give rise to real forms on $(G^*)^{\mathcal{T}}$ and $(G^*)^{\mathcal{P}}$ respectively. The identity component of fix locus of $\mathcal{T}$ can be identified with U via pr_1. and the real form on U is

$$U \to U \ : \ u \mapsto \overline{u}^{-1}.$$

Let us explain why we care about these involutions. From the Stokes phenomenon in [23], there is a local GW isomorphism for $G = \mathrm{SL}_n(\mathbb{C})$:

$$\mathcal{S}\colon \mathfrak{g} \cong \mathfrak{g}^* \to G^*,$$

which intertwines the involution ρ_1 on $\mathfrak{g}$ and $\mathcal{T}$ on G^*, involution ρ_2 on $\mathfrak{g}$ and $\mathcal{P}$ on G^*, and the involution ρ_3 on $\mathfrak{g}$ and τ on G^*, where

$$\rho_1\colon \mathfrak{g} \to \mathfrak{g} \;:\; a \to -a^T; \quad \rho_2\colon \mathfrak{g} \to \mathfrak{g} \;:\; a \to -J^{-1}a^T J.$$

$$\rho_3\colon \mathfrak{g} \to \mathfrak{g} \;:\; a \to -\overline{a}^T.$$

Note that ρ_3 commutes ρ_1 and ρ_2. Thus taking the intersection of fix points of involutions ρ_1 and ρ_3 (resp. ρ_2 and ρ_3) on $\mathfrak{g}$ and the intersection of fix points of involutions $\mathcal{T}$ and τ (resp. $\mathcal{P}$ and τ) of SL_n^* and restricting the map $\mathcal{S}$, we get a (globe) Poisson isomorphism:

$$\mathfrak{so}_n(\mathbb{R}) \to U^{\tau_1}; \quad \mathfrak{sp}_n(\mathbb{R}) \to (B_-)^{\mathcal{P}}$$

Note that U^{τ_1} and $(B_-)^{\mathcal{P}}$ are just Poisson spaces, rather than a Poisson-Lie group. These give us a different view of GW isomorphism in the classical type.

6 Tensor Multiplicities via Potential

6.1 Overview

The goal of this chapter is to continue and, to some extent, complete the "multiplicity geometrization" program, originated in [15, 16, 20, 18, 21]. Then by using our geometric multiplicities, one can recover all known and obtain many new formulas for such classical multiplicities as tensor product multiplicities, weight multiplicities, etc emerging in representation of complex reductive groups.

In our approach, a *geometric multiplicity* is a positive variety with potential fibered over the Cartan subgroup H of a reductive group G and additionally fibered over some extra split torus S. They form a category, which we denote it by $\mathbf{Mult}_G$ (see Definition 6.3.1 for more details).

Theorem 6.1.1 (Theorem 6.3.5)**.** *The category* $\mathbf{Mult}_G$ *is a non-strict unitless monoidal category with product* $M_1 \star M_2$ *given by*

$$M_1 \star M_2 := M_1 \times M_2 \times U.$$

(Here U *is the maximal unipotent subgroup of* G *normalized by* H*.)*

The associator for $\mathbf{Mult}_G$ is extremely non-trivial. We construct it via embedding $\mathbf{Mult}_G$ into the monoidal category of decorated $U \times U$-bicrystals, see Section 6.3. In fact, even though the category $\mathbf{Mult}_G$ has no unit, it has a natural tropicalization functor $\mathcal{T}$ (see Proposition 3.2.5) to what we call *affine tropical varieties* (see Definition 3.1.1). The latter category is an "honest" monoidal category with the natural unit 0 (thus, in what follows, we will ignore this minor deficiency of $\mathbf{Mult}_G$).

Using this tropicalization functor $\mathcal{T}$, we can recover many interesting representation of the Langlands dual group $G^\vee$ of G and various multiplicities in them out of geometric multiplicities over G as follows.

First, to any $M \in \mathbf{Mult}_G$ we assign an affine tropical variety $M^t := \mathcal{T}(M)$ fibered over the dominant coweight monoid $P_+^\vee$ of G. Then assign a $G^\vee$-module $\mathcal{V}(M)$ to M via

$$\mathcal{V}(M) = \bigoplus_{\lambda^\vee \in P_+^\vee} \mathbb{C}[M^t_{\lambda^\vee}] \otimes V_{\lambda^\vee},$$

where $M^t_{\lambda^\vee}$ is the tropical fiber over $\lambda^\vee$, $\mathbb{C}[\cdot]$ is the linearization of the set, and $V_{\lambda^\vee}$ is the irreducible representation of $G^\vee$ with highest weight $\lambda^\vee$ (thus we passed from the geometric multiplicities to the algebraic ones).

This chapter is based on a joint work [17] with A. Berenstein.

Theorem 6.1.2 (Theorem 6.4.1)**.** *The assignments $M \mapsto \mathcal{V}(M)$ define a monoidal functor from $\mathbf{Mult}_G$ to $\mathbf{Mod}_{G^\vee}$, the category of locally finite $G^\vee$-modules.*

Note that a geometric multiplicity is a additionally fiber over a split torus. This structure is introduced to resolve the problem of possibly having infinite multiplicities, which, in particular, happens for the module $\mathcal{V}(H \star H) = \mathcal{V}(H) \otimes \mathcal{V}(H)$ because

$$\mathcal{V}(H) = \bigoplus_{\lambda^\vee \in P_+^\vee} V_{\lambda^\vee}. \tag{6.1}$$

We define the multiplication in $\mathbf{Mult}_G$ in such a way that if M_i is additionally fibered over S_i for $i = 1, 2$, then $M_1 \star M_2$ is additionally fibered over $S_1 \times S_2 \times H$. This fixes the issue with $H \star H$ because now it is fibered over H^3. And the finiteness of the multiplicities is restored as follows. Given a geometric multiplicity M additionally fibered over S, its tropicalization M^t is naturally fibered over the direct product of $P_+^\vee$ and cocharacter lattice $X_*(S) := \mathrm{Hom}(\mathbb{G}_\mathbf{m}, S)$ so that for every cocharacter $\xi \in X_*(S)$ we define $\mathcal{V}_\xi(M)$ by

$$\mathcal{V}_\xi(M) := \bigoplus_{\lambda \in P_+^\vee} \mathbb{C}[M^t_{\lambda^\vee, \xi}] \otimes V_{\lambda^\vee}.$$

Theorem 6.1.3 (Theorem 6.4.2)**.** *Given geometric multiplicities M_i additionally fibered over S_i for $i = 1, 2$, one has the following natural isomorphism of $G^\vee$-modules*

$$\mathcal{V}_{\xi_1, \xi_2, \lambda^\vee, \nu^\vee}(M_1 \star M_2) \cong I_{\lambda^\vee}\left(\mathcal{V}_{\xi_1}(M_1)\right) \otimes I_{\nu^\vee}\left(\mathcal{V}_{\xi_2}(M_2)\right), \tag{6.2}$$

where $I_{\mu^\vee}(V)$ denotes the $\mu^\vee$-th isotypic component of a $G^\vee$-module V.

This indeed fixes the "infinite multiplicity" issue for $\mathcal{V}(H \star H)$ since (6.2) boils down to an isomorphism

$$\mathcal{V}_{\lambda^\vee, \nu^\vee}(H \star H) \cong V_{\lambda^\vee} \otimes V_{\nu^\vee}.$$

In turn, by applying this argument repeatedly to the geometric multiplicities:

$$H^{\star n} := \underbrace{H \star \cdots \star H}_{n},$$

we get an isomorphism of $G^\vee$-modules

$$\mathcal{V}_{\lambda_1^\vee, \ldots, \lambda_n^\vee}(H^{\star n}) \cong V_{\lambda_1^\vee} \otimes \cdots \otimes V_{\lambda_n^\vee}, \quad \forall \lambda_1^\vee, \ldots, \lambda_n^\vee \in P_+^\vee,\ n \geqslant 2.$$

Thus the geometric multiplicities $H^{\star n}$ (fibered over H^{n+1}) compute all tensor product multiplicities $c^{\mu^\vee}_{\lambda_1^\vee, \ldots, \lambda_n^\vee} := \dim \mathrm{Hom}_{G^\vee}(V_{\mu^\vee}, V_{\lambda_1^\vee} \otimes \cdots \otimes V_{\lambda_n^\vee})$.

Remark 6.1.4. Our construction of $H^{\star n}$ bears resemblance with the approach by Goncharov-Shen in [44, 45]. In particular, they related their configuration space Conf($\mathcal{A}^{n+1}, \mathcal{B}$) to geometric crystals in [44, Appendix B] and [45, Section 7.1].

6.2 The model space $M^{(2)}$

In this section, we describe a "trivialization" of $G^{(2)}$ and extend it to $G^{(n)}$, where we recall that the group G itself is a $(U \times U, \chi^{\mathrm{st}})$-bicrystal and $G^{(n)} := G * \cdots * G$ is the convolution product of n-copies of G's. Also denote by $M^{(n)} := U^{n-1} \times H^n$ for $n \geqslant 2$.

Recall to any $(U \times U, \chi^{\mathrm{st}})$-bicrystal $(\boldsymbol{X}, \boldsymbol{p}, \Phi)$, the central charge of $(\boldsymbol{X}, \boldsymbol{p}, \Phi)$ is the $U \times U$-invariant function:

$$\Delta_X(x) := \Phi(x) - \Phi_{BK}(\boldsymbol{p}(x)), \forall x \in X.$$

Thus on $G^{(n)}$, we have the central charge

$$\Delta_n(g_1, \ldots, g_n) := \sum \Phi_{BK}(g_i) - \Phi_{BK}(g_1 \cdots g_n).$$

Define a rational map π on $G^{(2)}$ as

$$\pi \colon G^{(2)} \to U \ : \ (g_1, g_2) \mapsto v_1 u_2, \quad \text{where } g_1 = u_1 h_1 \overline{w_0} v_1, \quad g_2 = u_2 h_2 \overline{w_0} v_2. \tag{6.3}$$

Proposition 6.2.1. *[16, Proposition 2.42] The map F defined as follows is a birational isomorphism of varieties:*

$$F \colon G^{(2)} \to M^{(2)} \times_H G \ : \ (g_1, g_2) \mapsto (\pi(g_1, g_2), \mathrm{hw}(g_1), \mathrm{hw}(g_2); g_1 g_2), \tag{6.4}$$

where the fiber product $M^{(2)} \times_H G = (U \times H^2) \times_H G$ is over

$$\mathrm{hw}_2 \colon (u, h_1, h_2) \mapsto \mathrm{hw}(\overline{w_0} u \overline{w_0}) h_1 h_2, \text{ and } \mathrm{hw} \colon g \mapsto \mathrm{hw}(g).$$

Since F is an $U \times U$-invariant isomorphism, we get an isomorphism of affine varieties

$$\overline{F} \colon U \setminus G^{(2)} / U \to M^{(2)}. \tag{6.5}$$

Thus the central charge Δ_2 on $G^{(2)}$ descends to a function on $M^{(2)}$:

$$\overline{\Delta}_2 := \Delta_2 \circ \overline{F}^{-1}. \tag{6.6}$$

The following corollary is clear:

Corollary 6.2.2. *The map F defined by* (6.4) *is an isomorphism of varieties with potential:*

$$F \colon \left(G^{(2)}, \Phi_{G^{(2)}}\right) \to \left(M^{(2)} \times_H G, \overline{\Delta}_2 + \Phi_{BK}\right).$$

From now on, we refer to (6.4) as a trivialization of $G^{(2)}$.

Note that by the usage of (6.4), for each vertex a of associahedron K_n for any $n \geqslant 3$, one can get a trivialization $F_a \colon G^{(n)} \to M^{(n)} \times_H G$. For example, in case of $n = 3$, we have

$$F_{12,3} \colon (G * G) * G \xrightarrow{F \times \mathrm{Id}} \left(M^{(2)} \times_H G\right) * G \xrightarrow{\mathrm{Id} \times F} M^{(2)} \times_H \left(M^{(2)} \times_H G\right) \xrightarrow{\sim} M^{(3)} \times_H G,$$

and

$$F_{1,23} \colon G * (G * G) \xrightarrow{\mathrm{Id} \times F} G * \left(G \times_H M^{(2)}\right) \xrightarrow{F \times \mathrm{Id}} \left(G \times_H M^{(2)}\right) \times_H M^{(2)} \xrightarrow{\sim} M^{(3)} \times_H G,$$

where the fiber product $M^{(3)} \times_H G = (U^2 \times H^3) \times_H G$ is over

$$\mathrm{hw}_3\colon (u_1, u_2, h_1, h_2, h_3) \mapsto \mathrm{hw}(\overline{w_0} u_1 \overline{w_0})\,\mathrm{hw}(\overline{w_0} u_2 \overline{w_0}) h_1 h_2 h_3, \text{ and } \mathrm{hw}\colon g \mapsto \mathrm{hw}(g).$$

Hence we get the a non-trivial automorphism of $U \times U$-varieties

$$F_{1,23} \circ F_{12,3}^{-1}\colon M^{(3)} \times_H G \xrightarrow{\sim} M^{(3)} \times_H G.$$

Denote by $\mathrm{pr}_{M^{(3)}}$ and pr_G the natural projection of $M^{(3)} \times_H G$ to $M^{(3)}$ and G respectively. Then we see immediately that $\mathrm{pr}_G \circ F_{1,23} \circ F_{12,3}^{-1}(m, g) = g$. Note that $F_{12,3}$ and $F_{1,23}$ are $U \times U$-equivariant, thus they descents to maps $\overline{F}_{1,23}$ and $\overline{F}_{12,3}$ on the quotient space respectively. By Proposition 3.4.7, we have the following birational isomorphism:

$$\iota_{M^{(3)}}\colon M^{(3)} \to M^{(3)} \times_H U \setminus G/U \;:\; m \mapsto (m, U\,\mathrm{hw}_3(m)\overline{w_0} U)\,.$$

Assembling all these components, we get:

$$\Psi\colon M^{(3)} \xrightarrow{\iota_{M^{(3)}}} M^{(3)} \times_H U\backslash G/U \xrightarrow{\overline{F}_{12,3}^{-1}} U\backslash G^{(3)}/U \xrightarrow{\overline{F}_{1,23}} M^{(3)} \times_H U\backslash G/U \xrightarrow{\mathrm{pr}_{M^{(3)}}} M^{(3)}.$$

Proposition 6.2.3. *The map Ψ defined above is an automorphism of variety $U^2 \times H^3$:*

$$\Psi = \mathrm{pr}_{M^{(3)}} \circ \overline{F}_{1,23} \circ \overline{F}_{12,3}^{-1} \circ \iota_{M^{(3)}}\colon U^2 \times H^3 \to U^2 \times H^3.$$

Example 6.2.4. Here we work out the example of the map Ψ for $G = \mathrm{GL}_2$. Write

$$x_i := \begin{bmatrix} a_i & 0 \\ b_i & c_i \end{bmatrix}$$

as coordinates for B_-. Suppose that $b_i \in \mathbb{G}_\mathbf{m}$, then we have

$$\begin{bmatrix} a_i & 0 \\ b_i & c_i \end{bmatrix} = \begin{bmatrix} 1 & \frac{a_i}{b_i} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{a_i c_i}{b_i} & 0 \\ 0 & b_i \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{c_i}{b_i} \\ 0 & 1 \end{bmatrix}.$$

Note the map $F_{12,3}$ restrict to $(B_- \times B_-) \times B_- \xrightarrow{\sim} M^{(3)} \times_H B_-$, which sends (x_1, x_2, x_3) to

$$\left(\begin{bmatrix} 1 & \frac{1}{X_{12} b_1 b_2} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \frac{X_{12}}{b_3 Y} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{a_1 c_1}{b_1} & 0 \\ 0 & b_1 \end{bmatrix}, \begin{bmatrix} \frac{a_2 c_2}{b_2} & 0 \\ 0 & b_2 \end{bmatrix}, \begin{bmatrix} \frac{a_3 c_3}{b_3} & 0 \\ 0 & b_3 \end{bmatrix}, x_1 x_2 x_3 \right),$$

where $X_{ij}^{-1} = b_i a_j + c_i b_j$ and $Y^{-1} = b_1 a_2 a_3 + c_1 b_2 a_3 + c_1 c_2 b_3$.

For elements in $M^{(3)} \times_H B_-$, we use coordinates as follows

$$\left(\begin{bmatrix} 1 & u_1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & u_2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} e_1 & 0 \\ 0 & f_1 \end{bmatrix}, \begin{bmatrix} e_2 & 0 \\ 0 & f_2 \end{bmatrix}, \begin{bmatrix} e_3 & 0 \\ 0 & f_3 \end{bmatrix}, \begin{bmatrix} p & 0 \\ u_1 u_2 \prod f_i & p^{-1} \prod e_i \prod f_i \end{bmatrix} \right)$$

as coordinates. Thus the inverse map of $F_{12,3}$ is given by

$$c_3 = \frac{u_1 e_3 + p^{-1} f_3 \prod e_i}{u_1 u_2}; \qquad a_3 = e_3 f_3 c_3^{-1}; \qquad b_3 = f_3;$$

$$c_2 = \frac{e_2}{u_1}(1 + p^{-1} e_1 f_2 a_3); \qquad a_2 = e_2 f_2 c_2^{-1}; \qquad b_2 = f_2;$$

$$a_1 = \frac{e_1}{u_1}(1 + p e_1^{-1} f_2^{-1} a_3^{-1}); \qquad c_1 = e_1 f_1 a_1^{-1}; \qquad b_1 = f_1.$$

Then applying $F_{1,23}$, we get

$$\left(\begin{bmatrix}1 & \dfrac{e_2}{u_1 f_2}+u_2\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & \dfrac{u_1u_2f_2}{u_1^{-1}e_2+f_2u_2}\\ 0 & 1\end{bmatrix}, \begin{bmatrix}e_1 & 0\\ 0 & f_1\end{bmatrix}, \begin{bmatrix}e_2 & 0\\ 0 & f_2\end{bmatrix}, \begin{bmatrix}e_3 & 0\\ 0 & f_3\end{bmatrix}, x_1x_2x_3\right).$$

Thus the map Ψ is the map sending

$$(u_1, u_2, e_i, f_i) \mapsto \left(\frac{e_2}{u_1 f_2}+u_2, \frac{u_1u_2f_2}{u_1^{-1}e_2+f_2u_2}, e_i, f_i\right).$$

6.3 Geometric multiplicities

Now let us define the category $\mathbf{Mult}_H$ of geometric multiplicities for any abelian reductive group H.

Definition 6.3.1. For any abelian reductive group H, the category $\mathbf{Mult}_H$ of geometric multiplicities is:

- The object in $\mathbf{Mult}_H$ is a quadruple $\boldsymbol{M} = (M, \Phi_M, \mathrm{hw}_M, \pi_M)$, which consists of an irreducible affine variety M with potential Φ_M, a rational map $\mathrm{hw}_M\colon M \to H$ and a rational map $\pi_M\colon M \to S_M$ from M to a split torus S_M;
- A morphism $\boldsymbol{f}\colon \boldsymbol{M} \to \boldsymbol{N}$ is a triple of rational maps $f_1\colon M \to N$, and $f_2\colon H \to H$ and $f_3\colon S_M \to S_N$ s.t. $\mathrm{hw}_N \circ f_1 = f_2 \circ \mathrm{hw}_M$ and $\pi_N \circ f_1 = f_3 \circ \pi_M$.

If moreover that the abelian reductive group H is the Cartan subgroup of certain reductive group G, we can equip the category $\mathbf{Mult}_H$ with a binary operator as follows

Definition 6.3.2. For a reductive group G with the Cartan subgroup H, denote by

$$\mathbf{Mult}_{H,G} := (\mathbf{Mult}_H, \star_G)$$

the category of geometric multiplicities $\mathbf{Mult}_H$ with a binary operation $\star_G$ as follows:

$$\boldsymbol{M} \star_G \boldsymbol{N} := (M \star_G N, \Phi_{M\star_G N}, \mathrm{hw}_{M\star_G N}, \pi_{M\star_G N}),$$

where each component is defined as follows:

- $M \star_G N := (M \times N) \times_{H^2} M^{(2)}$, where $M^{(2)} := U \times H^2$, the fiber product is over $\mathrm{hw}_M \times \mathrm{hw}_N$ and the natural projection $\mathrm{pr}_{H^2}\colon M^{(2)} \to H^2$;
- $\Phi_{M\star_G N}(m, n; u, h_1, h_2) = \Phi_M(m) + \Phi_N(n) + \overline{\Delta}_2(u, h_1, h_2)$;
- $\mathrm{hw}_{M\star_G N}\colon M \star_G N \to H \ :\ (m, n, u) \mapsto \mathrm{hw}_M(m)\,\mathrm{hw}_N(n)\,\mathrm{hw}(\overline{w_0}u\overline{w_0})$;
- $S_{M\star_G N} = S_M \times S_N \times H^2$ and

  $\pi_{M\star_G N}\colon M\star_G N \to S_M\times S_N\times H^2 \ :\ (m, n, u) \mapsto (\pi_M(m), \pi_N(n), \mathrm{hw}_M(m), \mathrm{hw}_N(n)).$

Here the category $\mathbf{Mult}_H$ plays certain universal role in this game. This category $\mathbf{Mult}_H$ has different product $\star_G$ when group H is considered as the Cartan for different groups. One important example is that for $H \subset L \subset G$, where L is a Levi subgroup of G.

Example 6.3.3. The first example of objects in $\mathbf{Mult}_{G,H}$ is just $\boldsymbol{H} := (H, 0, \mathrm{Id}, 0)$. Then $\boldsymbol{H} \star_G \boldsymbol{H}$ is

$$(M^{(2)}, \overline{\Delta}_2, \mathrm{hw}_2, \pi_2),$$

where $\mathrm{hw}_2(u, h_1, h_2) = h_1 h_2 \,\mathrm{hw}(\overline{w_0} u \overline{w_0})$ and $\pi_2(u, h_1, h_2) = (h_1, h_2)$.

Remark 6.3.4. For simplicities, we write $\star$ rather than $\star_G$ if the group G is clear from the context.

From the definition, one natural question to ask is whether the binary operation $\star$ is associative. First, by defintion, the triple product $(M_1 \star M_2) \star M_3$ isomorphic to $(M_1 \times M_2 \times M_3) \times_{H^3} M^{(3)}$ in the natural way, so is $M_1 \star (M_2 \star M_3)$. By Proposition 6.2.3, we have the following non-trivial isomorphism $\widetilde{\Psi}_{M_1,M_2,M_3} \colon (M_1 \star M_2) \star M_3 \to M_1 \star (M_2 \star M_3)$:

$$\begin{array}{ccc} (M_1 \star M_2) \star M_3 & \xrightarrow{\sim} & (M_1 \times M_2 \times M_3) \times_{H^3} M^{(3)} \\ & & \Big\downarrow {\scriptstyle \mathrm{Id} \times \Psi} \\ & & (M_1 \times M_2 \times M_3) \times_{H^3} M^{(3)} \xrightarrow{\sim} M_1 \star (M_2 \star M_3) \end{array} \tag{6.7}$$

Theorem 6.3.5. *For a reductive group G, the category $\mathbf{Mult}_{G,H}$ is monoidal with product $\boldsymbol{M}_1 \star_G \boldsymbol{M}_2$ given by Definition 6.3.1, and associator given by the formula* (6.7).

Proof. We first show that the equivalence of category $\mathbf{TriUB}_G$ and $\mathbf{Mult}_{G,H}$, where $\mathbf{TriUB}_G$ is the category of trivializable $(U \times U, \chi^{\mathrm{st}})$-bicrystals over G, which is defined in Definition 3.4.6. Define the following functors:

$$\mathcal{F} \colon \mathbf{TriUB}_G \to \mathbf{Mult}_{G,H} \ : \ (X \cong U \setminus X/U \times_H G, \Phi_X) \to (U \setminus X/U, \overline{\Delta}_X);$$
$$\mathcal{G} \colon \mathbf{Mult}_{G,H} \to \mathbf{TriUB}_G \ : \ (M, \Phi_M) \to (M \times_H G, \Phi_M + \Phi_{BK}).$$

One can check that $\mathcal{F}\mathcal{G}$ (resp. $\mathcal{G}\mathcal{F}$) is natural isomorphic to the identity functor for $\mathbf{Mult}_{G,H}$ (resp. $\mathbf{TriUB}_G$). Moreover, by definition we have $\mathcal{G}(M \star N) \cong \mathcal{G}(M) * \mathcal{G}(N)$ and the following commuting diagram:

$$\begin{array}{ccc} \mathcal{G}((M_1 \star M_2) \star M_3) & \xrightarrow{\mathcal{G}(\widetilde{\Psi}_{M_1,M_2,M_3})} & \mathcal{G}(M_1 \star (M_2 \star M_3)) \\ \sim \Big\downarrow & & \Big\downarrow \sim \\ (\mathcal{G}(M_1) * \mathcal{G}(M_2)) * \mathcal{G}(M_3) & \xrightarrow{\sim} & \mathcal{G}(M_1) * (\mathcal{G}(M_2) * \mathcal{G}(M_3)) \end{array} , \tag{6.8}$$

where all the $\sim$'s are natural isomorphisms. Since $\mathbf{TriUB}_G$ is a monoidal category with trivial associator, $\mathbf{Mult}_{G,H}$ is a monoidal category with associator given by the formula (6.7). $\diamondsuit$

Next, we add positive structure to the category $\mathbf{Mult}_H$.

Definition 6.3.6. A geometric multiplicity $M \in \mathbf{Mult}_H$ is positive if there exists a positive structure Θ_M on M s.t. (M, Φ_M, Θ_M) is a positive variety with potential, and hw_M (resp. π_M) is a (Θ_M, Θ_H) (resp. (Θ_M, Θ_S)) positive map. For simplicity, we denote by $\boldsymbol{M}$ the quintuple $(M, \Phi_M, \Theta_M, \mathrm{hw}_M, \pi_M)$ as well. A morphism $\boldsymbol{f} = (f_1, f_2, f_3)$ of positive geometric multiplicities $\boldsymbol{M}$ and $\boldsymbol{N}$ is morphism of geometric multiplicities s.t. $f_1 \colon (M, \Phi_M, \Theta_M) \to (N, \Phi_N \Theta_M)$ is morphism of positive varieties, f_2 and f_3 are positive maps of tori. Denote by $\mathbf{Mult}_H^+$ the subcategory of $\mathbf{Mult}_H$ consists of positive geometric multiplicities.

Proposition 6.3.7. *For any reductive group G with Cartan subgroup H, denote by $\Theta_{M^{(2)}} := \Theta_U \times \Theta_H \times \Theta_H$ the positive structure on $M^{(2)} := U \times H^2$. Then*

$$\boldsymbol{H} \star_G \boldsymbol{H} = \left(M^{(2)}, \overline{\Delta}_2, \mathrm{hw}_2, \pi_2\right) \in \mathbf{Mult}_H^+$$

is a positive geometric multiplicity.

Proof. Let us consider the toric chart $\Theta_{M^{(2)}} := \Theta_U \times \Theta_H \times \Theta_H$ for $M^{(2)}$. What we need to show actually is that $(M^{(2)}, \overline{\Delta}_2, \Theta_{M^{(2)}})$ is a positive variety with potential. Denote by $g_1 = u_1 h_1 \overline{w_0} v_1, g_2 = u_2 h_2 \overline{w_0} v_2$.

For $(u, h_1, h_2) \in U \times H^2$, choose a lift in $G \times G$ as $(h_1 \overline{w_0} u, h_2 \overline{w_0})$. Then one has:

$$\begin{aligned}\overline{\Delta}_2(u, h_1, h_2) &= \Delta_2 \circ \overline{F}^{-1}(u, h_1, h_2) \\ &= \Phi_{BK}(h_1 \overline{w_0} u) + \Phi_{BK}(h_2 \overline{w_0}) - \Phi_{BK}(h_1 \overline{w_0} u h_2 \overline{w_0}) \\ &= \chi^{\mathrm{st}}(u) - \Phi_{BK}(h_1 \overline{w_0} u h_2 \overline{w_0}) \\ &= \chi^{\mathrm{st}}(u) + \Phi_{BK}(h_2 u^T h_1^{w_0}),\end{aligned}$$

where $h_1^{w_0}$ is short for $\overline{w_0}^{-1} h_1 \overline{w_0}$ and the last equality is true because that for $g = u \overline{w_0} h v$

$$\Phi_{BK}(g) = \chi^{\mathrm{st}}(u) + \chi^{\mathrm{st}}(v) = -\chi^{\mathrm{st}}\left(\left(u^T\right)^{w_0}\right) - \chi^{\mathrm{st}}\left(\left(v^T\right)^{w_0}\right) = -\Phi_{BK}\left(\left(g^T\right)^{w_0}\right).$$

Thus the function $\overline{\Delta}_2$ is positive with respect to the positive structure $\Theta_{M^{(2)}}$. ♢

Example 6.3.8. Denote by

$$(M^{(n)}, \overline{\Delta}_n, \mathrm{hw}_n, \pi_n) := (\cdots((\boldsymbol{H} \star \boldsymbol{H}) \star \boldsymbol{H}) \star \cdots \star \boldsymbol{H}),$$

where the $\star$ products of n copies of $\boldsymbol{H}$ is in the canonical order. Write an element in $M^{(n)}$ as $\boldsymbol{u} := (u_1, \ldots, u_{n-1}, h_1, \ldots, h_n)$. Then the potential $\overline{\Delta}_n$ is given by

$$\overline{\Delta}_n(\boldsymbol{u}) = \sum_{i=1}^{n-1} \chi^{\mathrm{st}}(u_i) + \sum \Phi_{BK}(h_{i+1} u_i p_i^{w_0}(\boldsymbol{u})),$$

where $p_i(\boldsymbol{u}) = h_i \prod_{j=1}^{i} \mathrm{hw}(\overline{w_0} u_i \overline{w_0}) h_j \in H$ and h^{w_0} is short for $\overline{w_0}^{-1} h \overline{w_0}$ for $h \in H$. Note that this potential can be interpreted as the decent of the central charge Δ_n of $G^{(n)}$ under the canonical trivialization $G^{(n)} \cong M^{(n)} \times_H G$. We leave it as an excise for the readers to write down explicit formulas for hw_n and π_n. Then we know $(M^{(n)}, \overline{\Delta}_n, \mathrm{hw}_n, \pi_n)$ is a positive geometric multiplicity.

Example 6.3.9. Here we work out the potential of geometric multiplicities $\boldsymbol{N} := \boldsymbol{H} \star (\boldsymbol{H} \star \boldsymbol{H})$ as a comparison to the previous example. By definition

$$\boldsymbol{H} \star (\boldsymbol{H} \star \boldsymbol{H}) = (H \times M^{(2)}) \times_{H^2} M^{(2)} = M^{(2)} \times_H M^{(2)},$$

where the last fibration is given by $h_3 = h_1 h_2 \,\mathrm{hw}(\overline{w_0} u_1 \overline{w_0})$. Then the potential is

$$\chi^{\mathrm{st}}(u_1) + \chi^{\mathrm{st}}(u_2) + \Phi_{BK}(h_2 u_1^T h_1^{w_0}) + \Phi_{BK}(h_4 u_2^T h_3^{w_0}).$$

As a corollary of this proposition, the binary product $\star$ is well defined in the category $\mathbf{Mult}_{G,H}^+ := (\mathbf{Mult}_H^+, \star_G)$. Moreover:

Theorem 6.3.10. *For any reductive group G, the category $\mathbf{Mult}^+_{G,H}$ is monoidal with product $\boldsymbol{M}_1 \star_G \boldsymbol{M}_2$ given by Definition 6.3.1, and associator given by the formula* (6.7).

Remark 6.3.11. Because of Theorem 6.3.5 and Proposition 6.3.7, what Theorem 6.3.10 says is that the associator $\widetilde{\Psi}_{M_1,M_2,M_3}$ in (6.7) is a positive isomorphism of positive varieties.

Proof of Theorem 6.3.10. Because of Theorem 6.3.5 and Remark 6.3.11, what we need to show is that the following map F and its inverse F^{-1} are isomorphisms of positive varieties with potential:

$$F\colon B_- \times B_- \to (U \times H^2) \times_H B_- \ : \ (g_1, g_2) \mapsto (\pi(g_1, g_2), \mathrm{hw}(g_1), \mathrm{hw}(g_2), g_1 g_2)\,. \quad (6.9)$$

By [16, Claim 3.41], we know that F is positive. What left is to show F^{-1} is a positive isomorphism.

In what follows, we first give explicit formulas (6.10)-(6.12) for the inverse of F. Then show that (6.10)-(6.12) are positive with respect to the positive structures given by Lemma 3.5.2.

Note that L^{e,w_0} is open dense in U and $G^{w_0,e}$ is open dense in B_-. Let $(u, h_1, h_2, y) \in (L^{e,w_0} \times H^2) \times_H G^{w_0,e}$. We need to find the expression of g_i in terms of u, y and h_i. Since

$$\overline{\mathrm{hw}}_{M^{(2)}}(u, h_1, h_2) = \mathrm{hw}(y),$$

there exists a unique pair $(u_1, v_2) \in U \times U$ such that $y = u_1 \overline{w_0} h_1^{w_0} u h_2 \overline{w_0} v_2$, where h^{w_0} is short for $\overline{w_0}^{-1} h \overline{w_0}$. Denote by $x = h_1^{w_0} u h_2$ for simplicity. Then by taking $[\cdot]_+$ part of $\overline{w_0}^{-1} y$, we have

$$[\overline{w_0}^{-1} y]_+ = [\overline{w_0}^{-1} u_1 \overline{w_0} x \overline{w_0} v_2]_+ = [x \overline{w_0}]_+ v_2$$

since $\overline{w_0}^{-1} u_1 \overline{w_0} \in B_-$ and $v_2 \in U$. Therefore we have

$$v_2 = [\overline{w_0}^{-1} y]_+ [x \overline{w_0}]_+^{-1}. \quad (6.10)$$

Now let's applying ι to y, we have $y^\iota = v_2^\iota \overline{w_0} x^\iota \overline{w_0} u_1^\iota$. Then using (6.10), we get

$$u_1^\iota = [\overline{w_0}^{-1} y^\iota]_+ [x^\iota \overline{w_0}]_+^{-1}. \quad (6.11)$$

In order to write g_i as $u_i h_i \overline{w_0} v_i$, we just need to define

$$v_1 = [u_1 h_1 \overline{w_0}]_+^{-1}, \quad \text{and} \quad u_2^\iota = [v_2^\iota \overline{w_0} h_2^{-1}]_+^{-1}. \quad (6.12)$$

Now one can easily check that (6.12) does give the inverse of F.

What's next is to show the positivity. Note that the restriction of ι on $G^{w_0,e}$ is positive with respect to the positive structure. Thus the positivity of (6.10) implies the positivity of (6.11). What we actually show is that the two factors of (6.10)

$$\eta\colon G^{w_0,e} \to U \ : \ g \mapsto [\overline{w_0}^{-1} g]_+ \quad \text{and} \quad \zeta\colon G^{e,w_0} \to U \ : \ g \mapsto [g \overline{w_0}]_+^{-1}$$

are positive. Then the positivity of (6.12) follows from the positivity of the map ζ.

First, by [16, Claim 3.25,(3.6)] and the fact that ι is positive, one knows η is positive.

Second, write $b := [g\overline{w_0}]_- [g\overline{w_0}]_0$, then one has $\zeta(g) = \overline{w_0}^{-1} g^{-1} b$. Applying $(\cdot)^\iota$ to both side of equation $\zeta(g) = \overline{w_0}^{-1} g^{-1} b$, we get

$$\zeta(g)^\iota = b^\iota \overline{w_0}^{-1} \sigma(g), \quad (6.13)$$

where $\sigma(g) = \overline{w_0} g^{-\iota} \overline{w_0}^{-1}$. Thus we get $\zeta(g)^\iota = [\overline{w_0}^{-1} \sigma(g)^T]_+$. Since $\sigma(g)^T$ is positive by [21, Eq (4.6)], the map ζ is positive. $\diamondsuit$

6.4 From geometric multiplicities to tensor product multiplicities

In this section, we explain how to pass from the geometric multiplicities to tensor multiplicities by the usage of tropicalization.

For $\boldsymbol{M} = (M, \Phi_M, \Theta_M, \mathrm{hw}_M, \pi_M) \in \mathbf{Mult}_H^+$, the tropicalization of the positive map $\pi_M \times \mathrm{hw}_M$ gives a morphism of affine tropical varieties:

$$(\pi_M \times \mathrm{hw}_M)^t \colon (M, \Phi_M)^t \to S^t \times H^t = X_*(S) \times X_*(H),$$

where $X_*(S)$ (resp. $X_*(H)$) is the cocharacter lattice of the torus S (resp. H). Note that $X_*(H)$ is isomorphic naturally to the character lattice $X^*(H^\vee)$ of $G^\vee$. For $(\xi, \lambda^\vee) \in X_*(S) \times X_*(H)$, denote by

$$M^t_{\xi,\lambda^\vee} := (\pi_M \times \mathrm{hw}_M)^{-t}(\xi, \lambda^\vee)$$

the tropical fiber of $(M, \Phi_M)^t$. We say the positive geometric multiplicity $\boldsymbol{M}$ is *finite* if the morphism $(\pi_M \times \mathrm{hw}_M)^t$ is finite as in Definition 3.1.2.

Given a reductive group $G^\vee$ contains $H^\vee$ as its Cartan subgroup, to each positive geometric multiplicity $\boldsymbol{M}$, we assign a $G^\vee$-module $\mathcal{V}_{H,G^\vee}(\boldsymbol{M})$ to $\boldsymbol{M}$ via

$$\mathcal{V}_{H,G^\vee}(\boldsymbol{M}) = \bigoplus_{(\xi,\lambda^\vee)\in X_*(S)\times X_*^+(H;G^\vee)} \mathbb{C}[M^t_{\xi,\lambda^\vee}] \otimes V_{\lambda^\vee},$$

where $X_*^+(H)$ is the set of dominant integral weights of $G^\vee$ and $V_{\lambda^\vee}$ is the irreducible representation of $G^\vee$ with highest weight $\lambda^\vee$. Denote by $\mathbf{Mod}_{G^\vee}$ the category of $G^\vee$-modules.

Theorem 6.4.1. *For a reductive group $G^\vee$ contains $H^\vee$ as its Cartan subgroup, the assignments $\boldsymbol{M} \mapsto \mathcal{V}_{H,G^\vee}(\boldsymbol{M})$ is a well-defined functor from $\mathbf{Mult}_H^+$ to $\mathbf{Mod}_{G^\vee}$. Moreover, if G contains H as its Cartan subgroup, the assignments $\boldsymbol{M} \mapsto \mathcal{V}_{H,G^\vee}(\boldsymbol{M})$ is a monoidal functor from $\mathbf{Mult}_{G,H}^+$ to $\mathbf{Mod}_{G^\vee}$.*

For $\xi \in X_*(S)$, define the typical ξ-component $\mathcal{V}^\xi_{H,G^\vee}(\boldsymbol{M})$ by

$$\mathcal{V}^\xi_{H,G^\vee}(\boldsymbol{M}) := \bigoplus_{\lambda^\vee\in X_*^+(H;G^\vee)} \mathbb{C}[M^t_{\xi,\lambda^\vee}] \otimes V_{\lambda^\vee}. \tag{6.14}$$

Then we have a similar statement as Theorem 6.4.1 for the typical components:

Theorem 6.4.2. *Let G be a reductive group containing H as its Cartan subgroup, given positive geometric multiplicities $\boldsymbol{M}_i \in \mathbf{Mult}_{H,G}^+$ for $i = 1, 2$, the following $G^\vee$-modules are isomorphic:*

$$\mathcal{V}^{\xi_1,\xi_2,\lambda^\vee,\nu^\vee}_{H,G^\vee}(\boldsymbol{M}_1 \star_G \boldsymbol{M}_2) \cong I_{\lambda^\vee}\left(\mathcal{V}^{\xi_1}_{H,G^\vee}(\boldsymbol{M}_1)\right) \otimes I_{\nu^\vee}\left(\mathcal{V}^{\xi_2}_{H,G^\vee}(\boldsymbol{M}_2)\right), \tag{6.15}$$

where $I_{\lambda^\vee}(V)$ denotes the $\lambda^\vee$-th isotypic component of a $G^\vee$-module V.

One of the fundamental problems of the representation of G is to determine the tensor product multiplicity $c^\mu_{\lambda,\nu}$ of V_μ in $V_\lambda \otimes V_\nu$. Now we can find a solution to this problem by using Theorem 6.4.2:

Theorem 6.4.3. *Let G be a reductive group containing H as its Cartan subgroup. The positive geometric multiplicity $(M^{(2)}, \overline{\Delta}_2, \mathrm{hw}_2, \pi_2) \in \mathbf{Mult}_H^+$ is finite and the tensor multiplicity $c_{\lambda^\vee,\nu^\vee}^{\mu^\vee}$ is the multiplicity of $(\lambda^\vee, \nu^\vee, \mu^\vee)$ over $(\pi_2 \times \mathrm{hw}_2)^t$*, i.e.,

$$c_{\lambda^\vee,\nu^\vee}^{\mu^\vee} = \dim \mathbb{C}\left[\left(M^{(2)}\right)^t_{\lambda^\vee,\nu^\vee,\mu^\vee}\right].$$

Proof. Let us fix G. Note that $\boldsymbol{H} := (H, 0, \mathrm{Id}, 0)$ is an object in the category $\mathbf{Mult}_H^+$. By definition,

$$\mathcal{V}_{H,G^\vee}(\boldsymbol{H}) = \bigoplus_{\lambda^\vee \in X_*^+(H;G^\vee)} V_{\lambda^\vee}.$$

By Proposition 6.3.7, the geometric multiplicity $\boldsymbol{H} \star_G \boldsymbol{H} = (M^{(2)}, \overline{\Delta}_2, \mathrm{hw}_2, \pi_2)$ is positive. Applying Theorem 6.4.2 to $\boldsymbol{H} \star_G \boldsymbol{H}$:

$$\mathcal{V}_{H,G^\vee}^{\lambda^\vee,\nu^\vee}(\boldsymbol{H} \star \boldsymbol{H}) \cong \mathcal{V}_{H,G^\vee}^{\lambda^\vee}(\boldsymbol{H}) \otimes \mathcal{V}_{H,G^\vee}^{\nu^\vee}(\boldsymbol{H}) = V_{\lambda^\vee} \otimes V_{\nu^\vee}.$$

Together with the definition (6.14) of $\mathcal{V}_{H,G^\vee}^{\lambda^\vee,\nu^\vee}(\boldsymbol{H} \star_G \boldsymbol{H})$, one gets:

$$V_{\lambda^\vee} \otimes V_{\nu^\vee} \cong \mathcal{V}_{H,G^\vee}^{\lambda^\vee,\nu^\vee}(\boldsymbol{H} \star_G \boldsymbol{H}) = \bigoplus_{\lambda \in X_*^+(H;G^\vee)} \mathbb{C}\left[\left(M^{(2)}\right)^t_{\lambda^\vee,\nu^\vee,\mu^\vee}\right] \otimes V_{\mu^\vee},$$

which gives the statement we need. ♢

Similarly, for $n \geqslant 2$, denote by $c_{\lambda_1,\ldots,\lambda_n}^{\mu}$ the higher tensor multiplicities:

$$\bigotimes_{i=1}^{n} V_{\lambda_i} = \bigoplus_{\mu} c_{\lambda_1,\ldots,\lambda_n}^{\mu} V_\mu.$$

Theorem 6.4.4. *Let G be a reductive group containing H as its Cartan subgroup. For $n \geqslant 2$, the positive geometric multiplicity $(M^{(n)}, \overline{\Delta}_n, \mathrm{hw}_n, \pi_n)$ is finite and the tensor multiplicity $c_{\lambda_1^\vee,\ldots,\lambda_n^\vee}^{\mu^\vee}$ the multiplicity of $(\lambda_1^\vee, \ldots, \lambda_n^\vee, \mu^\vee)$ over $(\pi_n \times \mathrm{hw}_n)^t$*, i.e.,

$$c_{\lambda_1^\vee,\ldots,\lambda_n^\vee}^{\mu^\vee} = \dim \mathbb{C}\left[\left(M^{(n)}\right)^t_{\lambda_1^\vee,\ldots,\lambda_n^\vee,\mu^\vee}\right].$$

6.5 Isomorphism of geometric multiplicities: one example

In this section, we show one example of isomorphism of geometric multiplicities arising from Howe duality without giving all details. We show that our geometric multiplicity $M^{(n)}$ is naturally isomorphic to a subvariety of $\mathbb{C}_{\mathbf{m}}^{2n}$, which carries a geometric multiplicity structure arising from geometric crystals. This section is motivated by Howe $(\mathrm{GL}_2, \mathrm{GL}_n)$-duality [52].

Let us recall the result from Howe $(\mathrm{GL}_2, \mathrm{GL}_n)$-duality. Let λ be a Young diagram of depth $\leqslant n$ and V_λ^n be the polynomial representation of GL_n parametrized by λ. The following is clear:

$$\bigoplus_{(p_1,\ldots,p_n)} S^{p_1}(\mathbb{C}^2) \otimes \cdots \otimes S^{p_l}(\mathbb{C}^2) \cong \mathbb{C}\begin{bmatrix} x_n & x_{n-1} & \ldots & x_1 \\ y_n & y_{n-1} & \ldots & y_1 \end{bmatrix} \cong \bigoplus_{\text{depth}(\lambda) \leqslant 2} V_\lambda^2 \otimes V_\lambda^n,$$

where $S^l(\mathbb{C}^m)$ is the symmetric polynomial of degree l. Moreover we have

$$S^{p_1}(\mathbb{C}^2) \otimes \cdots \otimes S^{p_n}(\mathbb{C}^2) \cong \bigoplus V^l_\lambda(p_1, \ldots, p_n) \otimes V^2_\lambda. \tag{6.16}$$

Thus for the tensor multiplicities, we have

$$\left[V^2_\lambda \ : \ S^{p_1}(\mathbb{C}^2) \otimes \cdots \otimes S^{p_l}(\mathbb{C}^2)\right] = \dim V^l_\lambda(p_1, \ldots, p_l).$$

Note here the counting of left hand side is described by the tropicalization of tensor product geometric multiplicities. In what follows, we introduce a "geometrization" of the right hand side from the theory of geometric crystals. Inspired by this equation, we show at the end, we get an isomorphism of geometric multiplicities.

We consider *geometric (pre)-crystals* of GL_2:

$$\boldsymbol{X} := \left(\mathbb{G}^2_{\mathbf{m}}, \gamma, \varphi, \varepsilon, e^{\bullet}\right),$$

where for $(x, y) \in \mathbb{G}^2_{\mathbf{m}}$,

$$\gamma(x, y) = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}, \ \varphi(x, y) = x^{-1}, \ \varepsilon(x, y) = y^{-1}, \ e^c(x, y) = (cx, c^{-1}y).$$

Note that $\boldsymbol{X}$ admits a natural potential $\Phi(x, y) = x + y$. By [16, Definition 2.15], we consider the n copies of $\boldsymbol{X}$. Write the basis variety of $\boldsymbol{X}^n := \left(\mathbb{G}^{2n}_{\mathbf{m}}, \gamma_n, \varphi_n, \varepsilon_n, e^{\bullet}_n\right)$ as $2 \times n$ matrix with the following coordinates:

$$\boldsymbol{x} := \begin{bmatrix} x_n & x_{n-1} & \cdots & x_1 \\ y_n & y_{n-1} & \cdots & y_1 \end{bmatrix}$$

with potential $\Phi_n = \sum_1^n (x_i + y_i)$. Direct computation shows that

$$\varepsilon_n(\boldsymbol{x}) = \sum_{i=0}^{n-1} \frac{x_1 \cdots x_i}{y_1 \cdots y_{i+1}} = \frac{1}{y_1}\left(1 + \sum_{i=1}^{n-1} \frac{x_1 \cdots x_i}{y_2 \cdots y_{i+1}}\right).$$

Let $H := \{\mathrm{diag}(x, y) \mid x, y \in \mathbb{G}_{\mathbf{m}}\}$ be the Cartan subgroup of GL_2 and

$$T := \{\mathrm{diag}(x, 1) \mid x \in \mathbb{G}_{\mathbf{m}}\} \subset H$$

be a subgroup of H. Consider the subvariety Y of $\mathbb{G}^{2n}_{\mathbf{m}}$ given by $y_1 = 1$. Then we introduce the following positive geometric multiplicity:

$$\boldsymbol{M}^n_2 := (Y, \Phi_n + \varepsilon_n, \mathrm{Id}_Y, \gamma_n, p_n),$$

where $p_n \colon \boldsymbol{M}^n_2 \to T^n$ by sending $\boldsymbol{x}$ to $\mathrm{diag}(x_1, 1) \times \mathrm{diag}(x_n y_n, 1)$, and Id_Y is the natural positive structure on Y.

To compare with our geometric tensor multiplicities , we restrict the positive geometric multiplicity $(U^{n-1} \times H^n, \overline{\Delta}_n, \mathrm{hw}_n, \pi_n)$ for GL_2 to

$$\boldsymbol{U}^n_2 := (U^{n-1} \times T^n, \overline{\Delta}_n, \Theta^{n-1}_U \times \theta^n_T, \mathrm{hw}_n, \pi_n).$$

Denote by $\boldsymbol{u} := (u_1, \ldots, u_{n-1}; e_1, \ldots, e_n)$ the natural coordinates on $\boldsymbol{U}^n_2$. Then

Theorem 6.5.1. *The following map* $\Psi\colon \boldsymbol{M}_2^n \to \boldsymbol{U}_2^n$ *is a morphism of positive geometric multiplicities:*

$$u_i \circ \Psi = y_{i+1}; \quad e_i \circ \Psi = x_i y_i,$$

i.e.*, we have* $\mathrm{hw}_n \circ \Psi = \gamma_n$*,* $\pi_n \circ \Psi = p_n$ *and* $\Phi_n + \varepsilon_n - \overline{\Delta}_n \circ \Psi$ *is positive. Moreover,*

$$\dim \mathbb{C}\left[(\gamma_n, p_n)^{-t}(\mu; \lambda_1, \dots, \lambda_n)\right] = \dim \mathbb{C}\left[(\mathrm{hw}_n, \pi_n)^{-t}(\mu; \lambda_1, \dots, \lambda_n)\right], \tag{6.17}$$

where $\mu \in X_*(H)$ *and* $\lambda_i \in X_*(T)$.

Proof. Note that $\mathrm{hw}_n \circ \Psi = \gamma_n$ and $\pi_n \circ \Psi = p_n$ is clear from the definition. Recall that $\overline{\Delta}_n$ is given by

$$\overline{\Delta}_n(\boldsymbol{u}) = \sum_{i=1}^{n-1} \left(u_i + \frac{e_{i+1}}{u_i} + \frac{e_1 \cdots e_i}{(u_1 \cdots u_{i-1})^2} \cdot \frac{1}{u_i} \right).$$

Direct computation shows

$$\frac{e_{i+1}}{u_i} = x_{i+1}; \quad \frac{e_1 \cdots e_i}{(t_1 \cdots u_{i-1})^2} \cdot \frac{1}{u_i} = \frac{x_1 \cdots x_i}{y_2 \cdots y_{i+1}}.$$

Thus $\Phi_n + \varepsilon_n - \overline{\Delta}_n \circ \Psi = x_1 + 2$. It is easy to see that

$$(\Phi_n + \varepsilon_n)^t = (\Phi_n + \varepsilon_n - x_1 - 2)^t.$$

Thus we get the comparison (6.17). ♢

We hope to generalize the result here to Howe $(\mathrm{GL}_m, \mathrm{GL}_n)$-duality in future works.

7 | Partial tropicalization

7.1 Overview

A Poisson structure is a bivector which induces a Poisson bracket on the ring of regular functions on the variety. The Poisson bracket on a positive variety is called *log-canonical* if

$$\{x_i, x_j\} = c_{ij} x_i x_j,$$

where $x_1, \ldots, x_m$ are toric coordinates (defined by the positive structure) and c_{ij} is a constant matrix. Important examples of Poisson varieties with log-canonical Poisson structures are cluster varieties, as in Section 5.2.

The condition of a Poisson structure to be log-canonical is very restrictive. On a positive variety with potential (X, Φ), we can generalize it to a notion of *weakly log-canonical* Poisson structures, *i.e.*, the Poisson bracket is given by the formulas

$$\{x_i, x_j\} = x_i x_j (c_{ij} + f_{ij}(x)),$$

where $f_{ij}(x)$ are functions dominated by the potential Φ.

Given a weakly log-canonical Poisson structure π_X on a smooth complex variety X, consider the real form $(Y, \pi_Y) \subset (X, \pi_X)$ defined by the equations $x_i \in \mathbb{R}$ in the toric chart. Then, to such a structure we assign a constant Poisson bracket on the space

$$\mathcal{C} \times \mathbb{T}, \tag{7.1}$$

where $\mathcal{C}$ is a subcone of $(X, \Phi)^t_{\mathbb{R}}$ and $\mathbb{T} \cong (S^1)^r$ is a compact torus of dimension r, where $2r$ is the maximal rank of π_K. This Poisson bracket has the form

$$\{\xi_i, \xi_j\} = 0, \quad \{\phi_i, \phi_j\} = 0, \quad \{\xi_i, \phi_j\} = d_{ij}, \tag{7.2}$$

where $d_{ij} \in \mathbb{R}$ is determined by the log-canonical part c_{ij} of the bracket π_X. Here ξ_i's are coordinates on the cone $\mathcal{C}$ and ϕ_i's are coordinates on the torus $\mathbb{T}$. We refer to the space (7.1) together with the Poisson Bracket (7.2) as a *partial tropicalization* of the Poisson variety (X, π_X, Φ). Up to a change of variables, the Poisson bracket (7.2) defines an integrable system on the partial tropicalization.

Our prime example is the dual Poisson-Lie group G^* of a semisimple complex Lie group G endowed with the standard Poisson-Lie structure. In this chapter, we generalize the following example to any G^* for G semisimple.

This chapter is based on a joint work [5] with A. Alekseev, A. Berenstein and B. Hoffman.

Example 7.1.1. Recall that For $G = \mathrm{SL}_2(\mathbb{C})$, the group G^* is of the form $G^* = \{(x, x_-) \in B \times B_- \mid [x]_0[x_-]_0 = 1\}$. One can assign a potential

$$\Phi_{G^*} = b^{-1}(a + a^{-1}) + c^{-1}(a + a^{-1})$$

under the positive structure

$$x = \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}, \quad x_-^{-1} = \begin{bmatrix} a & 0 \\ c & a^{-1} \end{bmatrix}.$$

The real form of G^* is defined by equations $a \in \mathbb{R}_{>0}$ and $b = \overline{c}$. The canonical real Poisson bracket on K^* is given by [75, 80]:

$$\{a, b\} = \mathbb{i}ab, \quad \{a, c\} = -\mathbb{i}ac, \quad \{b, c\} = \mathbb{i}(a^2 - a^{-2}).$$

Note that the first two expressions are log-canonical on the nose, whereas the third expression has no log-canonical part and the corresponding function $f(x)$ is of the form

$$\mathbb{i}\left(b^{-1}a \cdot c^{-1}a - b^{-1}a^{-1} \cdot c^{-1}a^{-1}\right).$$

This expression is dominated by the potential Φ_{G^*}. The corresponding partial tropicalization is the product of the Gelfand-Zeitlin cone and the circle S^1:

$$\{(\xi_a, \xi_b) \in \mathbb{R}^2 \mid -\xi_b > \xi_a > \xi_b\} \times S^1$$

with Poisson bracket

$$\{\xi_a, \xi_b\} = 0, \quad \{\xi_a, \phi\} = 1, \quad \{\xi_b, \phi\} = 0,$$

which is the $n = 2$ Gelfand-Zeitlin integrable system.

7.2 Positive Poisson varieties

In this section, we add "positive structures" to Poisson varieties.

Definition 7.2.1. A *positive Poisson variety* is a quadruple (X, π, Φ, Θ), such that

(1) (X, π) is an irreducible Poisson variety over $\mathbb{G}_\mathbf{m}$;

(2) (X, Φ, Θ) is a positive variety with potential;

(3) for any $\theta \in \Theta$ with standard coordinates $(z_1, \ldots, z_n)$, the bracket is of the form

$$\{z_i, z_j\} = z_i z_j(\pi_{ij} + f_{ij}), \tag{7.3}$$

where $\pi_{ij} \in \mathbb{G}_\mathbf{a}$ is constant and f_{ij} is a rational function satisfying $f_{ij} \prec \Phi$.

Remark 7.2.2. The constant π_{ij} is called the *log-canonical part* of Poisson structure π under coordinates $\{z_i\}$. Sometimes we write $\{z_i, z_j\}_{\log} := z_i z_j \pi_{ij}$.

Definition 7.2.3. A *positive Poisson map* of positive Poisson varieties

$$\phi : (X_1, \pi_1, \Theta_1, \Phi_1) \to (X_2, \pi_2, \Theta_2, \Phi_2)$$

is a Poisson map $\phi : (X_1, \pi_1) \to (X_2, \pi_2)$, which is also a map of positive varieties with potential. Denote by $\mathbf{PosPoiss}$ the category of positive Poisson varieties.

Example 7.2.4. In [42], the authors define the *cluster manifold* $X(A)$ associated to a cluster algebra A. The cluster algebra structure on A gives $X(A)$ a Poisson structure π and a family of positively equivalent toric charts θ on X which are log-canonical for π. Cluster manifolds are then examples of positive Poisson varieties, with potential $\Phi = 0$.

We record the following observation for future reference.

Proposition 7.2.5. *Let (X, π, Θ, Φ) be a positive Poisson variety. Given $\theta \in \Theta$, let M and N be Laurent monomials in the standard coordinates z_j of θ, then the bracket $\{M, N\}$ is weakly log-canonical,* i.e.,

$$\{M, N\} = MN(\pi_{MN} + f_{MN}),$$

where $\pi_{MN} \in \mathbb{G}_{\mathbf{a}}$ is constant and f_{MN} is a rational function satisfying $f_{MN} \prec \Phi$.

Proof. Assume M, N, and L are Laurent monomials in $z_1, \ldots, z_n$ and that

$$\{M, N\} = MN(\pi_{MN} + f_{MN}) \text{ and } \{M, L\} = ML(\pi_{ML} + f_{ML})$$

are weakly log-canonical. The proposition follows by induction, using the following two facts. First,

$$\{M, N^{-1}\} = -N^{-2}\{M, N\} = N^{-2}MN(-\pi_{MN} - f_{MN}) = MN^{-1}(-\pi_{MN} - f_{MN})$$

is weakly log-canonical. Second,

$$\{M, NL\} = \{M, N\}L + N\{M, L\} = MNL(\pi_{MN} + f_{MN}) + MNL(\pi_{ML} + f_{ML})$$

is weakly log-canonical. ♢

7.3 Domination by BK potential on Double Bruhat Cells

In this section, we estimate the action of $U_{\mathfrak{g}}$ on generalized minors by potential Φ_{BK}. To be more precise, we start with the key

Lemma 7.3.1. *Consider the positive variety with potential $(B_-, \Phi_{BK}, \Theta_{B_-})$ and a sequence of indices $(j_1, \ldots, j_n)$ in $\boldsymbol{I}$ such that $\Delta_{w\omega_i,\omega_i} \cdot F_{j_2} \cdots F_{j_n} \neq 0$, we have*

$$\frac{F_{j_1} \cdots F_{j_n} \Delta_{w\omega_i,\omega_i}}{F_{j_2} \cdots F_{j_n} \Delta_{w\omega_i,\omega_i}} \prec \Phi_{BK}; \quad \frac{\Delta_{w\omega_i,\omega_i} F_{j_1} \cdots F_{j_n}}{\Delta_{w\omega_i,\omega_i} F_{j_2} \cdots F_{j_n}} \prec \Phi_{BK}.$$

Proof. We only show the first domination here, which can be done by in a chosen chart. Denote by $\mathbf{i} = (i_1, \ldots, i_m)$ a double reduced word for (e, w_0) such that $i_1 = j_1$ and consider the factorization chart:

$$x_{\mathbf{i}} \colon \mathbb{G}_{\mathbf{i}}^m \times H \hookrightarrow (L^{e,w_0})^T \times H \cong G^{w_0,e} \hookrightarrow B_- \ : \ (t_1, \ldots, t_m; h) \to y_{i_1}(t_1) \cdots y_{i_m}(t_m)h.$$

Denote by $\mathbf{j}' := (j_2, \ldots, j_n)$. For a sequence of indices $\mathbf{j} = (j_1, \ldots, j_n)$, let $\partial_{\mathbf{j}} := \frac{d}{dq_1}\Big|_0 \cdots \frac{d}{dq_n}\Big|_0$ be the differential at zero. By Theorem 2.7.2, we get

$$\begin{aligned}
&(-1)^{n-1} F_{j_2} \cdots F_{j_n} \Delta_{w\omega_i,\omega_i} (y_{i_1}(t_1) \cdots y_{i_m}(t_m)h) \\
&= \partial_{\mathbf{j}'} \Delta_{w\omega_i,\omega_i} (y_{j_n}(q_n) \cdots y_{j_2}(q_2) y_{i_1}(t_1) \cdots y_{i_m}(t_m)h) \\
&= h^{\omega_i} \sum_{k=0} f_k(t_2, \ldots, t_m) t_1^k,
\end{aligned}$$

where f_k's are positive polynomials in $t_2, \dots, t_m$ and $k \in \mathbb{Z}_{\geqslant 0}$. Similarly, since $j_1 = i_1$,

$$\begin{aligned}
&(-1)^n F_{j_1} \cdots F_{j_n} \Delta_{w\omega_i,\omega_i}(y_{i_1}(t_1)\cdots y_{i_m}(t_m)h) \\
&= \partial_{\mathbf{j}} \Delta_{w\omega_i,\omega_i}(y_{j_n}(q_n)\cdots y_{i_2}(q_2) y_{i_1}(t_1+q_1)\cdots y_{i_m}(t_m)h) \\
&= h^{\omega_i} \frac{d}{dq_1}\Big|_0 \sum f_k(t_2,\dots,t_m)(t_1+q_1)^k, \\
&= h^{\omega_i} \sum k f_k(t_2,\dots,t_m) t_1^{k-1},.
\end{aligned}$$

By Proposition 2.7.5, we know

$$\left(\frac{F_{i_1} \cdot \Delta_{w_0\omega_{i_1^*},\omega_{i_1^*}}}{\Delta_{w_0\omega_{i_1^*},\omega_{i_1^*}}}\right)(y_{i_1}(t_1)\cdots y_{i_m}(t_m)h) = \frac{1}{t_1}.$$

Direct calculation gives:

$$\left(\frac{a}{t_1} - \frac{(-1)^n F_{j_1}\cdots F_{j_n}\Delta_{w\omega_i,\omega_i}}{(-1)^{n-1}F_{j_2}\cdots F_{j_n}\Delta_{w\omega_i,\omega_i}}\right)(y_{i_1}(t_1)\cdots y_{i_m}(t_m)h) = \frac{af_0 + \sum_{k=1}(a-k)f_k t_1^k}{t_1\sum_{k=0} f_k t_1^k}.$$

Let a be a sufficiently large positive integer, then the right hand side is a positive function. ♢

Theorem 7.3.2. *Consider the positive variety with potential $(B_-, \Phi_{BK}, \Theta_{B_-})$. For $F, F' \in \mathfrak{n}_-$, there exist positive $n \in \mathbb{Z}$ depends on F, F' such that*

$$\frac{F \cdot \Delta_{w\omega_i,\omega_i} \cdot F'}{\Delta_{w\omega_i,\omega_i}} \prec \Phi_{BK}, \quad \textit{for } i \in \boldsymbol{I}.$$

Proof. Without loss of generality, we represent F as a nested commutator of simple roots

$$F = [F_{k_1}[\cdots[F_{k_{n-1}}, F_{k_n}]\cdots]].$$

Note that for a sequence of indices $\mathbf{j} = (j_1, \dots, j_n)$ in $\boldsymbol{I}$, following from Lemma 7.3.1, we have:

$$\frac{F_{j_1}\cdots F_{j_n}\Delta_{\omega_i,w\omega_i}}{\Delta_{\omega_i,w\omega_i}} = \frac{F_{j_1}\cdots F_{j_n}\Delta_{\omega_i,w\omega_i}}{F_{j_2}\cdots F_{j_n}\Delta_{\omega_i,w\omega_i}} \cdots \frac{F_{j_n}\Delta_{\omega_i,w\omega_i}}{\Delta_{\omega_i,w\omega_i}} \prec \Phi_{BK}^n.$$

One can generalize this result to actions on both side. ♢

Corollary 7.3.3. *Let G be a connected semisimple algebraic group and $\theta \in \Theta_{B_-}$ be a toric chart as in Definition 4.5.5. Then for any coordinate z of θ, we have*

$$\frac{F \cdot z \cdot F'}{z} \prec \Phi_{BK},$$

for all $(F, F') \in \mathfrak{n}_- \times \mathfrak{n}_-$.

Proof. Note by Theorem 7.3.2 and Proposition 3.3.3, the corollary is true for any cluster chart of any double reduced word of (w_0, e).

Consider the minors Δ_k on L^{e,w_0}. By Theorem 2.7.2, the minor Δ_k can be written as a subtraction free polynomial in "factorization parameters" t_k's. In turn, by Theorem 2.7.3, these t_k can be written as Laurent monomials in twisted minors M_k's on L^{e,w_0}. Note that the $\Delta_k \circ \psi^{w_0,e}$ is a twisted minor on $L^{w_0,e}$, and $M_k \circ \psi^{w_0,e}$ is a minor on L^{e,w_0} since $\psi^{e,w_0} \circ \psi^{w_0,e} =$ Id. Therefore the twisted minors on $L^{w_0,e}$ is a subtraction free Laurent polynomial in the minors on $L^{w_0,e}$, and the corollary for twisted minors follows immediately by Proposition 3.3.3.

Any other charts in Θ_{B_-} are positive Laurent polynomials in either minors or twisted minors of $L^{w_0,e}$, then the corollary follows by Proposition 3.3.3. ♢

7.4 Dual Poisson-Lie groups as positive Poisson varieties

In this section, we endow Poisson variety G^* with a positive structure, which is compatible with certain involution.

Recall that we have a group involution $\overline{\tau}$ on G^* defined by

$$\overline{\tau}\colon (b, b_-) \mapsto \left(b_-^{-T}, b^{-T}\right).$$

First of all, let us introduce a potential Φ_{G^*} on G^*. Define the potential Φ_{G^*} as

$$\Phi_{G^*} := \Phi_{BK} \circ \mathrm{pr}_2 + \Phi_{BK} \circ \mathrm{pr}_2 \circ \overline{\tau}.$$

In other words, we have $\Phi_{G^*}(b, b_-) = \Phi_{BK}(b_-) + \Phi_{BK}(b^{-T})$. Next, let us introduce the following identification $H \times (U_-)^2 \cong G^*$:

$$\eta\colon H \times (U_-)^2 \xrightarrow{\sim} G^* \;:\; (h, u_1, u_2) \to (h^{-1}u_1^{-T}, u_2 h). \tag{7.4}$$

Thus the potential $\Phi_{G^*} \circ \eta(h, u_1, u_2) = \Phi_{BK}(u_2 h) + \Phi_{BK}(h u_1)$ is positive with respect to the positive structures of $H \times (U_-)^2$. We consider the potential Φ_{G^*} for the following reason. Note that the fix locus of $\overline{\tau}$ is identified with B_- via pr_2. Thus $\Phi_{G^*}|_{(G^*)^{\overline{\tau}}} = 2\Phi_{BK} \circ \mathrm{pr}_2$. Now denote by

$$\Theta_{G^*} := \left\{\eta \circ \theta \mid \theta = \theta_0 \times \theta_1 \times \theta_1 \in \Theta_H \times \Theta_{U_-} \times \Theta_{U_-}\right\}.$$

Note that we only consider the toric charts for $H \times (U_-)^2$ who are using the same toric charts on the two copies of U_-.

Theorem 7.4.1. *The quadruple $(G^*, \pi_{G^*}, \Theta_{G^*}, \Phi_{G^*})$ is a positive Poisson variety.*

Proof. All we need to verify is the item (3) in Definition 7.2.1. Let $\mathbf{i}$ be a double reduced word of (w_0, e). Let Δ, Δ' be minors on $G^{w_0,e}$. We must consider three types of brackets. Denote by $\Delta_i := \Delta \circ \mathrm{pr}_i$.

(a) Bracket of type $\{\Delta_2, \Delta'_2\}_{G^*}$. By Proposition 5.1.4,

$$\{\Delta_2, \Delta'_2\}_{G^*} = -\{\Delta, \Delta'\}_B \circ \mathrm{pr}_2 .$$

By Theorem 5.2.4, the bracket $\{\Delta, \Delta'\}_{B_-}$ is log-canonical on the open subset $G^{w_0,e}$.

(b) Bracket of type $\{(\Delta \circ \overline{\tau})_1, (\Delta' \circ \overline{\tau})_1\}_{G^*}$. Note that $\overline{\tau}\colon B \to B_-$ that sends b to b^{-T} is anti-Poisson by Theorem 5.4.3, the argument is the same as the previous case.

(c) Bracket of type $\{\Delta_2, (\Delta' \circ \overline{\tau})_1\}_{G^*}$. By the definition of τ and (5.1.4), we have

$$\begin{aligned}\{\Delta_2, (\Delta' \circ \overline{\tau})_1\}_{G^*} = {}& \frac{1}{2}\sum_{i=1}^{r}(X_i \cdot \Delta)_2((X_i \cdot \Delta') \circ \overline{\tau})_1 - (\Delta \cdot X_i)_2((\Delta' \cdot X_i) \circ \overline{\tau})_1 \\ &+ \sum_{\alpha \in R^+} (F_\alpha \cdot \Delta)_2((F_\alpha \cdot \Delta') \circ \overline{\tau}))_1 - (\Delta \cdot F_\alpha)_2((\Delta' \cdot F_\alpha) \circ \overline{\tau})_1.\end{aligned}$$

Write $\Delta = \Delta_{u\omega_j,\omega_j}$ and $\Delta' = \Delta_{v\omega_k,\omega_k}$. By Proposition 2.3.1, the first sum is $c\Delta_2(\Delta' \circ \overline{\tau})_1$ for

$$2c = \sum_{i=1}^{r} u\omega_j(X_i) v\omega_k(X_i) - \omega_j(X_i)\omega_k(X_i) = (u\omega_j, v\omega_k) - (\omega_j, \omega_k) \tag{7.5}$$

is log-canonical. Then write $\{\Delta_2, (\Delta' \circ \tau)_1\}_{G^*} = \Delta_2(\Delta' \circ \overline{\tau})_1\big(c + f\big)$, and

$$f = \sum_{\alpha \in R^+} \frac{(\Delta \cdot F_\alpha)_2}{\Delta_2}\frac{((\Delta' \cdot F_\alpha) \circ \overline{\tau})_1}{(\Delta' \circ \overline{\tau})_1} - \frac{(F_\alpha \cdot \Delta)_2}{\Delta_2}\frac{((F_\alpha \cdot \Delta') \circ \overline{\tau})_1}{(\Delta' \circ \overline{\tau})_1}.$$

By Theorem 7.3.2, we have $f \prec \Phi_{G^*}$. ♢

7.5 Involutions on positive Poisson varieties

In this section, we discuss involutions on positive Poisson varieties. Then we explain what is a real form of a positive Poisson variety, and give as an example the real form K^* of the positive Poisson variety G^*.

Definition 7.5.1. An *involution* of a positive Poisson variety with potential (X, π, Φ, Θ) is an algebraic Poisson involution τ of (X, π) such that

(1) the involution τ is a positive map of (X, Φ, Θ);

(2) there exists a positive chart $\theta\colon \mathbb{G}_{\mathbf{m}}^n \to X$ in Θ such that τ maps $\theta(\mathbb{G}_{\mathbf{m}}^n)$ to $\theta(\mathbb{G}_{\mathbf{m}}^n)$ and induces a group isomorphism on $\mathbb{G}_{\mathbf{m}}^n$. We call θ a τ-*compatible* chart.

Remark 7.5.2. If the Poisson structure $\pi = 0$ on X as in the previous definition, we simply say τ is a involution of positive variety (X, Φ, Θ).

Since τ induces a group isomorphism on $\mathbb{G}_{\mathbf{m}}^n$, the fix points set is a subvariety, thus the irreducible component containing identity $(\mathbb{G}_{\mathbf{m}}^n)^\tau$ is a subtorus. Denote by X^τ the irreducible component that containing $\theta\left((\mathbb{G}_{\mathbf{m}}^n)^\tau\right)$ of the fix locus of X. Then the restriction

$$\theta^\tau := \theta|_{(\mathbb{G}_{\mathbf{m}}^n)^\tau}\colon \mathbb{G}_{\mathbf{m}}^{n_\tau} \cong (\mathbb{G}_{\mathbf{m}}^n)^\tau \to X^\tau$$

is a toric chart, where $n_\tau := \dim(\mathbb{G}_{\mathbf{m}}^n)^\tau$ and we fix the isomorphism $\mathbb{G}_{\mathbf{m}}^{n_\tau} \cong (\mathbb{G}_{\mathbf{m}}^n)^\tau$ once for all. Thus $(X^\tau, \Phi_X^\tau, \theta^\tau)$ is a positive variety with potential, where $\Phi_X^\tau := \Phi_X|_{X^\tau}$. Denote by π^τ the natural Poisson bracket induced by π as in Proposition 5.3.2. Then

Proposition 7.5.3. *For an involution τ of a positive Poisson variety with potential (X, π, Φ, Θ), the quadruple $(X^\tau, \Phi^\tau, \pi^\tau, \theta^\tau)$ is a positive Poisson variety with potential.*

Proof. All we need to show is item (3) in Definition 7.2.1. Denote by z_i's the natural coordinates of $\theta\colon \mathbb{G}_{\mathbf{m}}^n \to X$. Then $a_i := z_i + z_i \circ \tau$'s are τ-invariant functions on X. To show item (3) in Definition 7.2.1 for $(X^\tau, \Phi^\tau, \pi^\tau, \theta^\tau)$, we just need to show it for the set of functions $\{b_i := a_i|_{X^\tau}\}$. By Proposition 5.3.2, we have:

$$\{b_i, b_j\}_{\pi^\tau} = \{a_i, a_j\}_\pi|_{X^\tau}.$$

Then the proposition follows from the fact that $f|_{X^\tau} \prec \Phi^\tau$ if $f \prec \Phi$ ♢

Example 7.5.4. Following the notation of Example 5.3.3. One can assign a positive structure to G^* in a different way. In more details, consider the identification

$$\eta\colon H \times U^2 \to G^* \;:\; (h, u_1, u_2) \mapsto (hu_1, u_2^T h^{-1}).$$

Define the potential $\Phi'_{G^*}(b, b_-) = \Phi_{BK}(b_-) + \Phi_{BK}(b^T)$, which is positive with respect to $\eta \circ \theta$ for a toric chart $\theta = \mathrm{Id}_H \times \theta_1 \times \theta_1 \in \Theta_H \times \Theta_U \times \Theta_U$. Similar to the previous section, one can show that $(G^*, \Phi'_{G^*}, , \Theta'_{G^*})$ is a positive Poisson variety. And $\mathcal{T}$ is an the involution on $(G^*, \Theta'_{G^*}, \Phi'_{G^*})$. Thus we get an other positive Poisson variety $(U_+, \pi_{U^+}, (\Phi'_{G^*})^{\mathcal{T}})$.

Next, we pass to the real forms of complex positive Poisson varieties.

Let (X, Θ) be a complex positive variety with toric chart $\theta\colon (\mathbb{C}^\times)^n \to X$. Then complex conjugation $\overline{(\cdot)}\colon (\mathbb{C}^\times)^n \to (\mathbb{C}^\times)^n$ defines an anti-holomorphic involution on the open subvariety $\theta((\mathbb{C}^\times)^n) \subset X$. Since the transition maps between charts in Θ are positive, they commute with complex conjugation of $(\mathbb{C}^\times)^n$. Thus $\overline{(\cdot)}$ extends to the open subvariety $\bigcup_{\theta\in\Theta} \theta((\mathbb{C}^\times)^n)$. In particular, if the charts in Θ cover X, the involution $\overline{(\cdot)}$ is defined on all of X. In the rest, we always assume that $\overline{(\cdot)}$ on X is defined charts by charts.

Definition 7.5.5. A *real form* of a complex positive Poisson variety (X, π, Θ, Φ) is an anti-holomorphic involution of X, such that

(1) τ is a real from of (X, π);

(2) $\overline{\tau} := \tau \circ \overline{(\cdot)}$ is an involution of positive variety (X, Θ, Φ);

(3) there exists a toric chart $\theta \in \Theta$ such that the log-canonical part π_{ij} of the bracket of coordinate functions $\{z_i, z_j\}$ is pure imaginary.

Theorem 7.5.6. *Let $(G^*, \mathbb{i}\pi_{G^*}, \Theta_{G^*}, \Phi_{G^*})$ be the positive Poisson variety over $\mathbb{C}$ as in Theorem 7.4.1. Then the real form τ of $(G^*, \mathbb{i}\pi_{G^*})$ introduced in Theorem 5.4.3*

$$\tau\colon G^* \to G^* \;:\; (b, b_-) \mapsto \left(\overline{b_-}^{-T}, \overline{b}^{-T}\right).$$

is a real form of $(G^*, \mathbb{i}\pi_{G^*}, \Theta_{G^*}, \Phi_{G^*})$.

Proof. Item (1) is just Theorem 5.4.3. For item (2), consider that isomorphism $\eta\colon H \times (U_-)^2 \to G^*$ as in previous section. Thus the fix points set is given by the relation $u_2 = h^{-1}u_1 h$. Note that the toric charts on G^* are chosen to be same on the two copies on U_-. Thus for any cluster charts or factorization charts on U_-, one can show that $\overline{\tau}$ is always a monomial transformation of the chosen toric chart. Item (3) follows from the proof of Theorem 7.4.1, since the weakly log-canonical part of the bracket π_{G^*} is a rational number. ♢

We next want to specify a canonical choice of submanifold Y inside the fix locus of positive complex Poisson variety (X, π, Φ, Θ) which carries the induced Poisson structure $4\pi_R^\tau$.

Let (X, π, Θ, Φ) be a positive complex Poisson variety with real form τ. Then we know that $\overline{\tau}$ is an involution for the positive Poisson variety (X, π, Θ, Φ). Take $\theta \in \Theta$ a $\overline{\tau}$-compatible chart. Recall that $(X)^t_{\mathbb{R}} := (X, \theta)^t_{\mathbb{R}}$ is the extension by scalars of the tropicalization $(X, \theta)^t$ of a positive variety (X, θ) as in Eq (3.3). Recall that we have a limit explanation of tropicalization as in Proposition 3.2.6. Thus, for $s < 0$ a real parameter, define the *detropicalization* map

$$\begin{aligned} \mathfrak{E}_{X,\theta,s}\colon\; (X, \theta)^t_{\mathbb{R}} \times \left(S^1\right)^n &\to \left(\mathbb{C}^\times\right)^n \xrightarrow{\theta} X; \\ \left(\xi_1, \dots, \xi_n, e^{\mathbb{i}\varphi_1}, \dots, e^{\mathbb{i}\varphi_n}\right) &\mapsto \theta\left(e^{s\xi_1 + \mathbb{i}\varphi_1}, \dots, e^{s\xi_n + \mathbb{i}\varphi_n}\right). \end{aligned} \tag{7.6}$$

Since τ gives rise to an anti-holomorphic involution on $(\mathbb{C}^\times)^n$, it induces involutions on real and imaginary parts of $(\mathbb{C}^\times)^n$, thus induces involutions on $(X, \theta)^t_{\mathbb{R}}$ and $\left(S^1\right)^n$, through $\mathfrak{E}_{X,\theta,s}$ respectively. The fixed locus of $(X, \theta)^t_{\mathbb{R}}$ is the submanifold $\left(X^{\overline{\tau}}, \theta^{\overline{\tau}}\right)^t_{\mathbb{R}}$, in which we have submanifold $\left(X^{\overline{\tau}}, \Phi^{\overline{\tau}}, \theta^{\overline{\tau}}\right)^t_{\mathbb{R}}(0)$. The fixed locus of $\left(S^1\right)^n$ is

$$\left\{g \in (S^1)^n \subset (\mathbb{C}^\times)^n \mid \tau(g) = g\right\}$$

and let $\mathbb{T}_X^\tau$ be the identity component of these fixed points. Denote by

$$\mathrm{PT}(X,\Phi,\theta,\tau) := \left(X^{\overline{\tau}},\Phi^{\overline{\tau}},\theta^{\overline{\tau}}\right)^t_{\mathbb{R}}(0)\times\mathbb{T}_X^\tau;\ \mathrm{PT}(X,\Phi,\theta,\tau)^\delta := \left(X^{\overline{\tau}},\Phi^{\overline{\tau}},\theta^{\overline{\tau}}\right)^t_{\mathbb{R}}(\delta)\times\mathbb{T}_X^\tau.$$

The following proposition is straightforward.

Proposition 7.5.7. *Let (X,π,Θ,Φ) be a positive complex Poisson variety with real form τ. Let*

$$Y_{\tau,s} := \theta\circ\mathfrak{C}_{X,\theta,s}\left(\mathrm{PT}(X,\Phi,\theta,\tau)\right)\subset X.$$

Then $Y_{\tau,s}$ is independent of $s<0$, which we denote by $Y_\tau = Y_{\tau,s}$, and $Y_\tau\subset\mathrm{Fix}(\tau)$. Thus the pair $(Y_\tau, 4\pi_R^\tau)$ is a real Poisson manifold.

Remark 7.5.8. For the complex Poisson variety $(G^*, \mathbb{i}\pi_{G^*})$ with real forms as in Theorem 5.4.3, we have a Poisson variety (K^*,π_{K^*}). By Theorem 7.5.6, we know $(G^*,\mathbb{i}\pi_{G^*},\Theta_{G^*},\Phi_{G^*})$ is a positive complex Poisson variety with real form τ. We then think of $(Y_\tau,4\pi_R^\tau)$ as a coordinate neighborhood on (K^*,π_{K^*}).

We leave the proof of the following proposition to the reader:

Proposition 7.5.9. *Given positive complex Poisson varieties $(X_i,\pi_i,\Theta_i,\Phi_i)$ with real forms τ_i, let $f\colon(X_1,\pi_1,\Theta_1,\Phi_1)\to(X_2,\pi_2,\Theta_2,\Phi_2)$ be a positive Poisson map which intertwines τ_1 and τ_2. Then f restricts to a Poisson map from $(Y_{\tau_1},4(\pi_1)_R^\tau)$ to $(Y_{\tau_2},4(\pi_2)_R^\tau)$.*

7.6 Partial tropicalization

In the previous section, we get a real Poisson manifold $(Y_\tau,4\pi_R^\tau)$ out of positive complex Poisson variety (X,π,Φ,Θ) with real form τ. Now we would like to take certain limit of $(Y_\tau,4s\pi_R^\tau)$, which we call *partial tropicalization* of (X,π,Θ,Φ,τ) . We use the same notation as in the previous section.

Given a positive complex Poisson variety (X,π,Φ,Θ) with involution τ, let $\theta\in\Theta$ be $\overline{\tau}$-compatible chart with the standard coordinate functions $z_1,\ldots,z_n$.

Now let us introduce two Poisson brackets on the manifold $\mathrm{PT}(X,\theta,\tau)$. First, one can get a constant Poisson structure on $(X,\theta)^t_{\mathbb{R}}\times\left(S^1\right)^n$ in the following way. Since $\overline{\tau}$ induced a group homomorphism on $(\mathbb{C}^\times)^n$, the bracket $\{z_i, z_j\circ\overline{\tau}\}$ is weakly log-canonical by Proposition 7.2.5. Let ϖ_{ij} be the log-canonical part of the bracket, which is pure imaginary by Definition 7.5.5 and Proposition 7.2.5. Now define a constant real skew-symmetric bracket on $(X,\theta)^t_{\mathbb{R}}\times\left(S^1\right)^n$ as follows:

$$\{\xi_i,\varphi_j\} = \mathbb{i}(\varpi_{ij}-\pi_{ij}),\quad \{\xi_i,\xi_j\}=\{\varphi_i,\varphi_j\}=0.$$

Set π_∞^θ to be the restriction of this bracket to $\mathrm{PT}(X,\Phi,\theta,\tau)$.

Definition 7.6.1. The manifold $\mathrm{PT}(X,\Phi,\theta,\tau)$ together with the constant Poisson structure π_∞^θ is called the *partial tropicalization* of a positive complex Poisson variety (X,π,Φ,Θ) with real form τ.

We sometimes write $\mathrm{PT}(X,\theta)=\mathrm{PT}(X,\Phi)=\mathrm{PT}(X^\tau)$ for $\mathrm{PT}(X,\Phi,\theta,\tau)$ if the other structures are clear from the context.

Second, denote by the π_s^θ the bivector on $\mathrm{PT}(X,\theta)$ by pulling back the scaled Poisson bivector $4s\pi_R^\tau$ along the isomorphism $\mathfrak{C}_{X,\theta,s}$ from $\mathrm{PT}(X,\theta)$ to $Y_{\tau},$. Thus we have Poisson manifold $(\mathrm{PT}(X,\theta),\pi_s^\theta)$.

The bracket π_∞^θ is the limit of π_s^θ in the following sense:

Theorem 7.6.2. *For $s \ll 0$, on the Poisson manifold $(\mathrm{PT}(X, \Phi, \theta, \tau)^{\delta}, \pi_s^{\theta})$, we have*

$$\pi_s^{\theta} = \pi_{\infty}^{\theta} + O(e^{s\delta}),$$

where the term $O(e^{s\delta})$ indicates a bivector field whose component functions, written in terms of the coordinates λ_i and φ_j, are $O(e^{s\delta})$.

Before the proof of the theorem, we need the following

Lemma 7.6.3. *Given a positive complex variety (X, Θ, Φ), suppose that $f \prec \Phi$ and there exists a toric chart $\theta \in \Theta$ such that $f \circ \theta$ is regular. Then for every $\delta > 0$, $f \circ \mathfrak{E}_{X,\theta,s} = O(e^{s\delta})$ on $(X, \Phi, \theta)_{\mathbb{R}}^{t}(\delta) \times \left(S^1\right)^n$.*

Proof. By assumption $f \circ \theta$ is regular and so by the triangle inequality we may assume $\varphi_i = 0$ for all i. Write $\widetilde{f}_s(\xi_1, \ldots, \xi_n) := (f \circ \mathfrak{E}_{X,\theta,s})(\xi_1, \ldots, \xi_n, 1, \ldots, 1)$. Then one gets

$$\lim_{s \to -\infty} \frac{1}{s} \ln \widetilde{f}_s(\xi_1, \ldots, \xi_n) = f_{\mathbb{R}}^{t}(\xi_1, \ldots, \xi_n).$$

Thus for $(\xi_1, \ldots, \xi_n) \in (X, \theta, \Phi)_{\mathbb{R}}^{t}(\delta)$, $f_{\mathbb{R}}^{t}(\xi_1, \ldots, \xi_n) > \delta$ since $f \prec \Phi$. Then for $s \ll 0$,

$$\frac{1}{s} \ln \widetilde{f}_s(\xi_1, \ldots, \xi_n) > \delta > 0.$$

Since logarithm is monotonic, $f \circ \mathfrak{E}_{X,\theta,s} = O(e^{s\delta})$ on $(X, \Phi, \theta)_{\mathbb{R}}^{t}(\delta) \times \left(S^1\right)^n$. ♢

Proof of Theorem 7.6.2. On the toric chart θ, We compute:

$$\{z_i, z_j\}_s / (z_i z_j) = s(\pi_{ij} + f_{ij}), \tag{7.7}$$

where we know $f_{ij} \prec \Phi$. Since $\{\cdot, \cdot\}_s$ is a biderivation, so

$$\frac{\{e^{s\xi_i + \mathbb{i}\varphi_i}, e^{s\xi_j + \mathbb{i}\varphi_j}\}_s}{e^{s\xi_i + \mathbb{i}\varphi_i} e^{s\xi_j + \mathbb{i}\varphi_j}} = s^2\{\xi_i, \xi_j\}_s + \mathbb{i}s(\{\xi_i, \varphi_j\}_s + \{\varphi_i, \xi_j\}_s) - \{\varphi_i, \varphi_j\}. \tag{7.8}$$

Combining (7.7) and (7.8) gives

$$s^2\{\xi_i, \xi_j\}_s + \mathbb{i}s(\{\xi_i, \varphi_j\}_s + \{\varphi_i, \xi_j\}_s) - \{\varphi_i, \varphi_j\}_s = s(\pi_{ij} + f_{ij}). \tag{7.9}$$

Recall log-canonical part π_{ij} of the bracket $\{z_i, z_j\}$ is pure imaginary by definition. Thus

$$s^2\{\xi_i, \xi_j\}_s - \{\varphi_i, \varphi_j\}_s = s\Re f_{ij}, \tag{7.10}$$

$$\{\xi_i, \varphi_j\}_s + \{\varphi_i, \xi_j\}_s = -\mathbb{i}\pi_{ij} + \Im f_{ij}. \tag{7.11}$$

Restricting to the fixed locus of τ, for $z_j \circ \tau$, there exists a k such that $\overline{z}_j = z_k \circ \tau$ by definition. Therefore, on $\mathrm{PT}(X, \theta, \tau)$ with respect to the bracket $4\pi_R^{\tau}$,

$$\{z_i, \overline{z}_j\} = z_i \overline{z}_j (\varpi_{ij} + g_{ij}) \tag{7.12}$$

with $g_{ij} \prec \Phi$. Repeating calculations similar to those before (7.10) and (7.11) gives

$$s^2\{\xi_i, \xi_j\}_s + \{\varphi_i, \varphi_j\}_s = s\Re g_{ij}, \tag{7.13}$$

$$-\{\xi_i, \varphi_j\}_s + \{\varphi_i, \xi_j\}_s = -\mathbb{i}\varpi_{ij} + \Im g_{ij}. \tag{7.14}$$

Combining (7.10), (7.11), (7.13), and (7.14), and applying Lemma 7.6.3 gives the result. ♢

7.7 Partial tropicalization of G^* with real form τ

In this section, we would like to apply the construction in previous section to the positive complex Poisson variety $(G^*, \mathbb{i}\pi_{G^*}, \Theta_{G^*}, \Phi_{G^*})$ with real form τ.

Recall that we have the real form τ on G^* is given by

$$\tau\colon G^* \to G^* \;:\; (b, b_-) \mapsto \left(\overline{b_-}^{-T}, \overline{b}^{-T}\right).$$

and we know that $((G^*)^{\overline{\tau}}, \Phi^{\overline{\tau}}_{G^*}, \Theta^{\overline{\tau}}_{G^*})$ is isomorphic to $(B_-, \Phi_{BK}, \Theta_{B_-})$ as positive variety via the projection pr_2. Let $\vartheta = \theta_0 \times \theta_1 \colon \mathbb{G}^r_{\mathbf{m}} \times \mathbb{G}^m_{\mathbf{m}} \to H \times U_- \cong B_-$ be a toric chart of B_-, which extend to a $\overline{\tau}$-compatible chart $\widetilde{\theta} := \eta \circ (\theta_0 \times \theta_1 \times \theta_1)$ for G^*, where η is given by Eq (7.4). Note that $\mathrm{pr}_2 \circ \theta^{\overline{\tau}} = \theta_0 \times \theta_1$. Also note that $(G^*)^{\overline{\tau}}$ is given by $\left\{(b, b_-) \in G^* \mid b_- = \overline{b}^{-T}\right\}$, which is isomorphic to $AU_- \subset B_-$ via pr_2. Thus $\mathrm{PT}(G^*, \widetilde{\theta}, \Phi_{G^*}, \tau) \cong (B_-, \Phi_{BK}, \vartheta)^t_{\mathbb{R}}(0) \times (S^1)^m$ via pr_2 and we get the following detropicalization map:

$$\begin{aligned} \mathfrak{E}_{\vartheta,s} \colon \mathbb{R}^{r+m} \times (S^1)^m &\to AU_- \\ (\lambda_{-r}, \ldots, \lambda_m, \varphi_1, \ldots, \varphi_m) &\mapsto \vartheta\left(e^{s\lambda_{-r}}, \ldots, e^{s\lambda_{-1}}, e^{s\lambda_1 + \mathbb{i}\varphi_1}, \ldots, e^{s\lambda_m + \mathbb{i}\varphi_m}\right). \end{aligned}$$

Next we compute the constant Poisson bracket π^{θ}_{∞} of $\mathrm{PT}(G^*, \theta)$ for a specific toric chart θ. Let $\mathbf{i} \in R(w_0, e)$ be a double reduced word for (w_0, e). Recall that for the seed $\sigma(\mathbf{i})$, we have the following cluster variables on $G^{w_0,e} \subset B_-$ by (4.10):

$$\Delta_k := \Delta_{u_k\omega_{i_k}, \omega_{i_k}}, \text{ for } k \in [-r, -1] \cup [1, m],$$

where $u_k = s_{i_1} \cdots s_{i_k}$ for $k \in [1, m]$ and $u_k = e$ for $k \in [-r, -1]$. Note that Δ_k's determine a birational isomorphism

$$G^{w_0,e} \to (\mathbb{C}^\times)^{m+r} \;:\; g \mapsto (\Delta_{-r}(g), \ldots, \Delta_m(g)).$$

whose inverse gives a toric chart of B_-

$$\sigma(\mathbf{i}) \colon (\mathbb{C}^\times)^{m+r} \to G^{w_0,e} \hookrightarrow B_-. \tag{7.15}$$

Denote by $\theta(\mathbf{i})$ the $\overline{\tau}$-compatible chart for G^* extending $\sigma(\mathbf{i})$ as in previous discussion. Then one compute:

Theorem 7.7.1. *The Poisson bivector $\pi^{\theta(\mathbf{i})}_{\infty}$ on $\mathrm{PT}(G^*, \mathbb{i}\pi_{G^*}, \Phi_{G^*}, \theta(\mathbf{i}), \tau)$ is given by*

$$\{\lambda_k, \varphi_p\} = 0, \qquad \textit{for } k \geqslant p; \tag{7.16}$$

$$\{\lambda_k, \varphi_p\} = (u_k\omega_{i_k}, u_p\omega_{i_p}) - (\omega_{i_k}, \omega_{i_p}), \qquad \textit{for } k < p; \tag{7.17}$$

$$\{\lambda_k, \lambda_p\} = \{\varphi_k, \varphi_p\} = 0. \qquad \textit{for all } k, p.$$

Proof. Denote by $\{\cdot,\cdot\}_{\log}$ the log-canonical part of $\mathbb{i}\pi_{G^*}$. By Eq (7.5), the number $2\mathbb{i}\varpi_{k,p}$ is

$$2\mathbb{i}\frac{\{(\Delta_k)_2, (\Delta_p \circ \tau)_1\}_{\log}}{(\Delta_k)_2(\Delta_p \circ \tau)_1} = (\omega_{i_k}, \omega_{i_p}) - (u_k\omega_{i_k}, u_p\omega_{i_p}).$$

By Theorem 5.2.4 and the fact that pr_2 is anti-Poisson, we know $2\mathbb{i}\pi_{k,p}$ is

$$2\mathbb{i}\frac{\{(\Delta_k)_2, (\Delta_p)_2\}_{\log}}{(\Delta_k)_2(\Delta_p)_2} = (u_k\omega_{i_k}, u_p\omega_{i_p}) - (\omega_{i_k}, \omega_{i_p}), \quad \text{where } k \leqslant p.$$

Take this into the definition, one gets desired formulas. ♢

Proposition 7.7.2. *For the Poisson manifold* $\mathrm{PT}(G^*) = \mathrm{PT}(G^*, \mathrm{i}\pi_{G^*}, \Phi_{G^*}, \theta(\mathbf{i}), \tau)$ *with the Poisson structure* $\pi_\infty^{\theta(\mathbf{i})}$*, we have*

(1) The functions λ_k *are Casimirs for* $k \in [1, m] \setminus \boldsymbol{e}(\mathbf{i})$*;*

(2) The matrix $B = [\{\lambda_{k^-}, \varphi_l\}]_m$*, where* k, l *is indexed by* $(1, \dots, m)$ *and* k^- *is defined by Eq 4.7, is of the form* $B = DB'$ *for*

$$D = \mathrm{diag}\left((\alpha_{i_1}, \omega_{i_1}), \dots, (\alpha_{i_m}, \omega_{i_m})\right) = (1/d_{i_1}, \dots, 1/d_{i_m}),$$

and B' *is an upper triangular positive integer matrix with diagonal elements being* 1*;*

(3) Let $\mathrm{hw}\colon B_- \to H$ *be the highest weight map. Then the symplectic leaves of* $\mathrm{PT}(G^*)$ *are of the form*

$$\mathscr{P}_{\lambda^\vee} := \mathrm{hw}_{\mathbb{R}}^{-t}(\lambda^\vee) \times (S^1)^m,$$

where $\lambda^\vee \in X_*^+(H) \otimes_{\mathbb{Z}_+} \mathbb{R}_+$.

Proof. For item (1), let $k \in [1, m] \setminus \boldsymbol{e}(\mathbf{i})$. By (7.16), we have $\{\lambda_k, \varphi_p\} = 0$ for $p \leqslant k$. All we need to show is $\{\lambda_k, \varphi_p\} = 0$ for $p > k$. Since $k \in [1, m] \setminus \boldsymbol{e}(\mathbf{i})$, we know $p \in [1, m] \setminus \boldsymbol{e}(\mathbf{i})$ as well. Therefore, by (7.17),

$$\{\lambda_k, \varphi_p\} = (\omega_{i_k}, \omega_{i_p}) - (w_0\omega_{i_k}, w_0\omega_{i_p}) = 0,$$

because the bilinear form is W-invariant.

For item (2), recall that s_i is the simple reflection generated by α_i. Note that $s_i\omega_i = \omega_i - \alpha_i$, and $s_j\omega_i = \omega_i$ for $j \neq i$. By (4.7), we have $\omega_{i_k} = \omega_{i_{k^-}}$. Thus by (7.17): for $k = l$

$$\begin{aligned}\{\lambda_{k^-}, \varphi_k\} &= (\omega_{i_{k^-}}, \omega_{i_k}) - (\omega_{i_{k^-}}, s_{i_{k^-+1}} \cdots s_{i_k}\omega_{i_k}) \\ &= (\omega_{i_k}, \omega_{i_k}) - (\omega_{i_k}, s_{i_k}\omega_{i_k}) = (\alpha_{i_k}, \omega_{i_k}) > 0.\end{aligned}$$

For $k > l$, there are two possible cases. For $k > k^- \geqslant l$, we have $\{\lambda_{k^-}, \varphi_l\} = 0$ by by (7.16). For $k > l > k^-$, we have:

$$\{\lambda_{k^-}, \varphi_l\} = (\omega_{i_{k^-}}, \omega_{i_l}) - (\omega_{i_{k^-}}, s_{i_{k^-+1}} \cdots s_{i_l}\omega_{i_l}) = (\omega_{i_k}, \omega_{i_l}) - (\omega_{i_k}, \omega_{i_l}) = 0.$$

The equalities follows from the fact $i_{k^-+1}, \dots, i_l$ are all different from i_k and $\omega_{i_k} = \omega_{i_{k^-}}$.

For $k < l$, we have $\{\lambda_{k^-}, \varphi_l\} = \left(\omega_{i_k}, \omega_{i_l} - u_{k^-}^{-1}u_l\omega_{i_l}\right) = c(\omega_{i_k}, \alpha_{i_k})$ for $c \in \mathbb{Z}_{\geqslant 0}$, because $\omega_i - v\omega_i$ is a non-negative integer linear combination of α_j's and $(\alpha_i, \omega_j) = 0$ for $i \neq j$.

For item (3), recall that

$$(\mathrm{hw}(g))^{w_0\omega_i} = \Delta_{w_0\omega_i, \omega_i}(g), \quad \forall g \in G^{w_0, e}, i \in \boldsymbol{I}.$$

The tropicalization of hw with respect to the chart $\sigma(\mathbf{i})$ can then be written as a linear combination of the cocharacters $w_0\alpha_i^\vee \in X_*(H)$, with the tropical functions $\Delta_{w_0\omega_i, \omega_i}$ as coefficients:

$$\mathrm{hw}_{\mathbb{R}}^t = \sum_i \Delta_{w_0\omega_i, \omega_i}^t \cdot (w_0\alpha_i^\vee).$$

Together with the nondegeneracy of the matrix B in previous item, this proves the claim. $\diamondsuit$

8 | Poisson-Lie duality vs Langlands duality

8.1 Overview

Let K be a compact connected semisimple Lie group. There are two very interesting duality constructions which involve K. First, one can associate to it the Langlands dual group $K^\vee$ corresponding to the root system dual to the one of K. Second, the group K carries the standard Poisson-Lie structure π_K. As a Poisson-Lie group, it admits the dual Poisson-Lie group K^*.

The group $K^\vee$ is a compact connected semisimple Lie group while the group K^* is solvable. Despite this fact, they share some common features. Let $T \subset K$ be a maximal torus of K and $\mathfrak{t} = \mathrm{Lie}(T)$ be its Lie algebra. The Lie algebra $\mathfrak{t}^\vee = \mathrm{Lie}(T^\vee)$ of the maximal torus $T^\vee \subset K^\vee$ is in a natural duality with $\mathfrak{t}$:

$$\mathfrak{t}^\vee \cong \mathrm{Hom}(S^1, T)^* \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathrm{Hom}(T, S^1) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathfrak{t}^*.$$

The Lie algebra $\mathbb{i}\mathfrak{t} \cong \mathfrak{t}^*$ plays a role analogous to the one of Cartan subalgebra for the group K^*. The isomorphism above is induced by the invariant scalar product on $\mathfrak{k} = \mathrm{Lie}(K)$ used to define the standard Poisson structures on K and K^*.

Furthermore, both the Langlands dual group and the Poisson-Lie dual group can be used to parametrize representations of K (or finite dimensional representations of $G = K^{\mathbb{C}}$). On one hand, by the Borel-Weil-Bott Theorem, geometric quantization of coadjoint orbits passing through dominant integral weights in $\mathfrak{t}^*$ yields all irreducible representations of K. By the Ginzburg-Weinstein Theorem [43], the Poisson spaces K^* and $\mathfrak{k}^*$ are isomorphic to each other and we can extend the Borel-Weil-Bott result to K^*, where for a dominant integral weight $\lambda \in \mathfrak{t}^* \cong \mathbb{i}\mathfrak{t}$ we consider the K-dressing orbit in K^* passing through $\exp(\lambda)$.

On the other hand, as we discussed in Chapter 3, from $B_-^\vee \subset G^\vee = (K^\vee)^{\mathbb{C}}$, Berenstein-Kazhdan [16] constructed an integral polyhedral cone $(B_-^\vee, \Phi_{BK}^\vee)^t$ by tropicalization, together with a highest weight map

$$(\mathrm{hw}^\vee)^t \colon (B_-^\vee, \Phi_{BK}^\vee)^t \to X_*(H^\vee) = X^*(H).$$

The fiber $(\mathrm{hw}^\vee)^{-t}(\lambda)$ carries a Kashiwara crystal structure, which isomorphisc to the the one of irreducible representations of G with highest weight λ.

This chapter is based on a joint work [6] with A. Alekseev, A. Berenstein and B. Hoffman.

It is the goal of this paper to establish a relation between the two duality constructions described above.

There are several tools that we are using to this effect. First, note that that of the double Bruhat cells $G^{w_0,e} \subset B_-$ and $G^{\vee;w_0,e} \subset B_-^\vee$ is a pair of cluster varieties dual to each other. In this case, the relationship between tropicalizations can be further improved: We show that our comparison map ψ maps one of these cones into the other, and preserves their Kashiwara crystal structure (up to some scaling). This gives a new perspective on a result of Kashiwara [58] and Frenkel-Hernandez [39].

For the discussion of the Poisson-Lie dual K^*, we use the tool of partial tropicalization that we introduced in Chapter 7. Recall that $\mathrm{PT}(K^*)$ comes equipped with a constant Poisson structure which induces integral affine structures on symplectic leaves. Together with the structure of the weight lattice of K, they define a natural Bohr-Sommerfeld lattice $\Lambda \subset (B_-, \Phi_{BK})^t$.

We show that

$$\psi(\Lambda) = (B_-^\vee, \Phi_{BK}^\vee)^t.$$

That is, the integral Bohr-Sommerfeld cone Λ defined by the Poisson-Lie data on K^* is isomorphic to the integral cone $(B_-^\vee, \Phi_{BK}^\vee)^t$ defined by the potential $\Phi_{BK}^\vee$. The isomorphism is given by the tropical duality map of the double cluster variety.

In more details, the cone $(B_-^\vee, \Phi_{BK}^\vee)^t$ parametrizes canonical bases of irreducible G-modules. For the representation with highest weight λ, the canonical basis in V_λ is parametrized by the points of $(\mathrm{hw}^\vee)^{-t}(\lambda)$. The preimage of this set under the duality map ψ is exactly the set of points of Λ which belongs to the integral symplectic leaf of $\mathrm{PT}(K^*)$ corresponding to the weight λ. The relations are depicted in Figure 8.1.

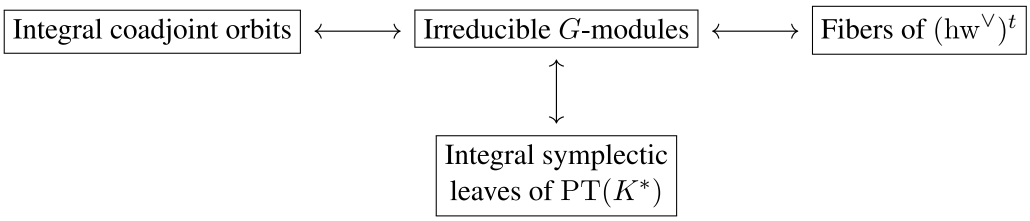


Figure 8.1

8.2 Comparison maps

Let G be a semisimple algebraic group over $\mathbb{Q}$ as before, whose Cartan matrix is A with symmetrizer $\boldsymbol{d} = \{d_1, \ldots, d_r\}$. For a pair of elements (u, v) in the Weyl group, we have a positive variety $(G^{u,v}, \Theta_{u,v})$.

In this section, we want to compare the positive variety $(G^{u,v}, \Theta_{u,v})$ and $(G^{\vee;u,v}, \Theta_{u,v}^\vee)$. For $\mathbf{i} \in R(u, v)$, let $\Psi_{\mathbf{i}} \colon G^{u,v} \to G^{\vee;u,v}$ be the positive rational map which is given in terms of the toric charts $x_{\mathbf{i}}$ and $x_{\mathbf{i}}^\vee$ by

$$x_{\mathbf{i}}(h, t_1, \ldots, t_n) \mapsto x_{\mathbf{i}}^\vee(\Psi^H(h), t_1^{d_{i_1}}, \ldots, t_n^{d_{i_n}}). \tag{8.1}$$

Write $\psi_{\mathbf{i}} = \Psi_{\mathbf{i}}^t$ for the tropicalized comparison map

$$(G^{u,v}, x_{\mathbf{i}})^t \to (G^{\vee;u,v}, x_{\mathbf{i}}^{\vee})^t$$

Then the comparison maps $\psi_{\mathbf{i}}$ for $\mathbf{i} \in R(u,v)$ all agree after tropicalization. To be more precise,

Proposition 8.2.1. *For* $\mathbf{i}, \mathbf{i}' \in R(u,v)$*, the following diagram commutes,*

$$\begin{array}{ccc} (G^{u,v}, x_{\mathbf{i}})^t & \xrightarrow{\mathrm{Id}^t} & (G^{u,v}, x_{\mathbf{i}'})^t \\ \downarrow{\scriptstyle \psi_{\mathbf{i}}} & & \downarrow{\scriptstyle \psi_{\mathbf{i}'}} \\ (G^{\vee;u,v}, x_{\mathbf{i}}^{\vee})^t & \xrightarrow{(\mathrm{Id}^{\vee})^t} & (G^{\vee;u,v}, x_{\mathbf{i}'}^{\vee})^t \end{array}$$

where we recall that $\mathrm{Id}^t = \mathrm{Id}^t_{G^{u,v}} = (x_{\mathbf{i}'}^{-1} \circ x_{\mathbf{i}})^t$ *by definition, and we abbreviate* $\mathrm{Id}^{\vee} = \mathrm{Id}_{G^{\vee;u,v}}$.

Proof. By the previous discussion it is enough to assume that $\mathbf{i}$ and $\mathbf{i}'$ are related by a single move. Then the proposition follows by direct computation; we only give the proof for one type of move and leave the rest to the reader.

Say $i, j \in \{-r, \dots, -1\}$ with $a_{i,j} = -1$ and $a_{j,i} = -2$. Without loss of generality assume $d_i = 1$ and $d_j = 2$. Let

$$\mathbf{i} = (i_1, \dots, i_k, i, j, i, j, i_{k+5}, \dots, i_n), \quad \mathbf{i}' = (i_1, \dots, i_k, j, i, j, i, i_{k+5}, \dots, i_n).$$

Then by [21, Proposition 7.3],

$$\begin{aligned} \Psi_{\mathbf{i}'} \circ x_{\mathbf{i}'}^{-1} \circ x_{\mathbf{i}}(h, t_1, \dots, t_m) &= \Psi_{\mathbf{i}'}(h, p_1, \dots, p_m) \\ &= (\Psi^H(h), p_1^{d_{i_1}}, \dots, p_{k+1}^2, p_{k+2}, p_{k+3}^2, p_{k+4}, \dots, p_m^{d_{i_m}}), \end{aligned}$$

where

$$p_{k+1} = \left(\frac{t_{k+1}}{t_{k+2}} + \frac{t_{k+2}}{t_{k+3}} + \frac{1}{t_{k+4}} \right)^{-1} \qquad p_{k+2} = \left(\frac{1}{t_{k+1}} \left(\frac{t_{k+2}}{t_{k+3}} + \frac{1}{t_{k+4}} \right)^2 + \frac{1}{t_{k+3}} \right)^{-1}$$

$$p_{k+3} = t_{k+2} + t_{k+1} t_{k+4} + \frac{t_{k+2}^2 t_{k+4}}{t_{k+3}} \qquad p_{k+4} = t_{k+1} + t_{k+3} \left(\frac{t_{k+2}}{t_{k+3}} + \frac{1}{t_{k+4}} \right)^2$$

and $p_i = t_i$ otherwise. On the other hand, again using [21, Proposition 7.3] one finds

$$\begin{aligned} (x_{\mathbf{i}'}^{\vee})^{-1} \circ x_{\mathbf{i}}^{\vee} \circ \Psi_{\mathbf{i}}(h, t_1, \dots, t_m) &= (x_{\mathbf{i}'}^{\vee})^{-1} \circ x_{\mathbf{i}}^{\vee}(\Psi^H(h), t_1^{d_{i_1}}, \dots, t_m^{d_{i_m}}) \\ &= (\Psi^H(h), P_1, \dots, P_m), \end{aligned}$$

where

$$P_{k+1} = \left(\frac{1}{t_{k+4}^2} + \frac{1}{t_{k+2}^2} \left(\frac{t_{k+2}^2}{t_{k+3}} + t_{k+1} \right)^2 \right)^{-1} \qquad P_{k+2} = \left(\frac{1}{t_{k+3}} + \frac{1}{t_{k+4}^2 t_{k+1}} + \frac{t_{k+2}^2}{t_{k+3}^2 t_{k+1}} \right)^{-1}$$

$$P_{k+3} = t_{k+4}^2 \left(\frac{t_{k+2}^2}{t_{k+3}} + t_{k+1} \right)^2 + t_{k+2}^2 \qquad P_{k+4} = \frac{t_{k+3}}{t_{k+4}^2} + \frac{t_{k+2}^2}{t_{k+3}} + t_{k+1}$$

and $P_i = t_i^{d_i}$ otherwise. Then it is easy to verify that $\Psi_{\mathbf{i}}$ and $\Psi_{\mathbf{i}'}$ agree after tropicalization.

The proofs for the other types of move (described in Propositions 7.1, 7.2, and 7.3 of [21]) are along the same lines. The computation for the two types of moves associated to type G_2 are slightly more involved. One must show that some terms in the expressions for the p_i's and P_i's do not contribute after tropicalization; this can be done using the fact that the tropicalization of $(A+B)^k$ is the same as the tropicalization of $A^k + B^k$, for positive functions A and B and positive integers k. This verification is straightforward and tedious. $\diamondsuit$

Recall we should that $(G^{u,v} = H \times L^{u,v}, G^{\vee;u,v} = H^\vee \times L^{\vee;u,v}$ is a (decorated) double cluster variety. Thus we know the tropical maps ψ_σ agree for all $\sigma \in |\sigma(\mathbf{i})|$. Next, we compare the comparison map ψ_σ and $\psi_{\mathbf{i}}$:

Theorem 8.2.2. *For $u, v \in W$, suppose that $\ell(u) + \ell(v) = \ell(u^{-1}v)$. For $\mathbf{i} \in R(u,v)$, denote by let $x_{\mathbf{i}}$ and $\sigma \in |\sigma(\mathbf{i})|$ be the factorization chart and cluster charts for $G^{u,v}$ respectively. Then the following diagram commutes,*

$$\begin{array}{ccc}
(G^{u,v}, x_{\mathbf{i}})^t & \xrightarrow{\mathrm{Id}^t} & (G^{u,v}, \sigma)^t \\
\downarrow \psi_{\mathbf{i}} & & \downarrow \psi_\sigma \\
(G^{\vee;u,v}, x_{\mathbf{i}}^\vee)^t & \xrightarrow{(\mathrm{Id}^\vee)^t} & (G^{\vee;u,v}, \sigma^\vee)^t
\end{array}$$

where we recall that $\mathrm{Id}^t = (\sigma^{-1} \circ x_{\mathbf{i}})^t$ by definition.

We split the proof into the following Lemmas:

Lemma 8.2.3. *For u, v satisfying $\ell(u) + \ell(v) = \ell(u^{-1}v)$ and $\mathbf{i} \in R(u,v)$the following diagram commutes.*

$$\begin{array}{ccc}
(L^{u,v}, x_{\mathbf{i}})^t & \xrightarrow{(\psi^{u,v})^t} & (L^{v,u}, x_{-\mathbf{i}})^t \\
\downarrow \psi_{\mathbf{i}} & & \downarrow \psi_{-\mathbf{i}} \\
(L^{\vee;u,v}, x_{\mathbf{i}}^\vee)^t & \xrightarrow{(\psi^{\vee;u,v})^t} & (L^{\vee;v,u}, x_{-\mathbf{i}}^\vee)^t
\end{array}$$

Proof. Recall that $\psi^{u,v}$ can be decomposed as a sequence of "elementary moves" as in Proposition 2.6.7. Let $(i_1, \ldots, i_p, j_q, \ldots, j_1)$ be a separated reduced word for (u,v). Denote by $\mathbf{j} = (i_1, \ldots, i_p, -j_1, j_q, \ldots, j_2)$. For elementary move $Q_{\mathbf{i}}$, the following diagram commutes:

$$\begin{array}{ccccc}
(L^{u,v}, x_{\mathbf{i}})^t & \xrightarrow{Q_{\mathbf{i}}^t} & (L^{us_{j_1}, vs_{j_1}}, x_{\mathbf{i}_+})^t & \xrightarrow{\mathrm{Id}^t} & (L^{us_{j_1}, vs_{j_1}}, x_{\mathbf{j}})^t \\
\downarrow \psi_{\mathbf{i}} & & \downarrow \psi_{\mathbf{i}_+} & & \downarrow \psi_{\mathbf{j}} \\
(L^{\vee;u,v}, x_{\mathbf{i}}^\vee)^t & \xrightarrow{(Q_{\mathbf{i}}^\vee)^t} & (L^{\vee;us_{j_1}, vs_{j_1}}, x_{\mathbf{i}_+}^\vee)^t & \xrightarrow{(\mathrm{Id}^\vee)^t} & (L^{\vee;us_{j_1}, vs_{j_1}}, x_{\mathbf{j}}^\vee)^t
\end{array},$$

which follows from the definition of $Q_{\mathbf{i}}$ and $\psi_{\mathbf{i}}$ and of the right square is just Proposition 8.2.1 for "mixed" moves. $\diamondsuit$

Lemma 8.2.4. *For $u, v \in W$, let $\sigma \in |\sigma(-\mathbf{i})|$ be a cluster chart for $L^{v,u}$. Then the following diagram commutes.*

$$\begin{array}{ccc} L^{u,v} & \xrightarrow{\psi^{u,v}} & L^{v,u} \\ \downarrow{\Psi_{\mathbf{i}}} & & \downarrow{\Psi_{\sigma}} \\ L^{\vee;u,v} & \xrightarrow{\psi^{\vee;u,v}} & L^{\vee;v,u} \end{array} \tag{8.2}$$

Proof. Recall that the factorization parameters can be written as monomial of the twist minors as in 2.7.3. Also note that $a_{ij}d_j = a_{ji}d_i$. Direct computation will give the proof of the lemma 8.2.4. ♢

proof of Theorem 8.2.2. Consider the following diagram.

$$\begin{array}{ccccc} (L^{v,u}, x_{-\mathbf{i}})^t & \xrightarrow{(\psi^{v,u})^t} & (L^{u,v}, x_{\mathbf{i}})^t & \xrightarrow{\mathrm{Id}^t} & (L^{u,v}, \sigma)^t \\ \downarrow{\psi_{-\mathbf{i}}} & & \downarrow{\psi_{\mathbf{i}}} & & \downarrow{\psi_{\sigma}} \\ (L^{\vee;v,u}, x^{\vee}_{-\mathbf{i}})^t & \xrightarrow{(\psi^{\vee;v,u})^t} & (L^{\vee;u,v}, x^{\vee}_{\mathbf{i}})^t & \xrightarrow{\mathrm{Id}^{\vee t}} & (L^{\vee;u,v}, \sigma)^t \end{array} \tag{8.3}$$

We must show that the square on the right commutes. From Lemmas 8.2.3, the square on the left commutes. The outer square commutes by Lemma . ♢

As a consequence of Proposition 4.3.4, Proposition 8.2.1 and Theorem 8.2.2, we immediately obtain the following. Recall that, for any double reduced words $\mathbf{i}, \mathbf{i}'$ for (w_0, e), the toric charts $x_{\mathbf{i}}, x_{\mathbf{i}'}, \sigma(\mathbf{i}), \sigma(\mathbf{i}')$ on $G^{w_0,e}$ are all positively equivalent.

Corollary 8.2.5. *Let $\mathbf{i}, \mathbf{i}'$ be double reduced words for (w_0, e), and let $\sigma \in |\sigma(\mathbf{i})|$ and $\sigma' \in |\sigma(\mathbf{i}')|$. Consider the (rational) comparison maps*

$$\Psi_{\mathbf{i}}, \Psi_{\mathbf{i}'}, \Psi_{\sigma}, \Psi_{\sigma'} \colon G^{w_0,e} \to G^{\vee;w_0,e}.$$

For any toric charts $\theta \in [x_{\mathbf{i}}]$ and $\theta^{\vee} \in [x^{\vee}_{\mathbf{i}}]$ on $G^{w_0,e}$ and $G^{\vee;w_0,e}$, respectively, the tropicalized maps

$$\Psi^t_{\mathbf{i}}, \Psi^t_{\mathbf{i}'}, \Psi^t_{\sigma}, \Psi^t_{\sigma'} \colon (G^{w_0,e}, \theta)^t \to (G^{\vee;w_0,e}, \theta^{\vee})^t$$

are all equal.

8.3 Comparison of BK cones

In this section, we focus on the positive varieties with potential $(B_-, \Phi_{BK}, \Theta_{B_-})$ and its "dual" $(B^{\vee}_-, \Phi^{\vee}_{BK}, \Theta^{\vee}_{B_-})$.

Theorem 8.3.1. *For $\mathbf{i} \in R(w_0, e)$, the natural real extension $(\psi_{\mathbf{i}})_{\mathbb{R}}$ of $\psi_{\mathbf{i}}$ restricts to an isomorphism of real BK cones*

$$(B_-, \Phi_{BK}, x_{\mathbf{i}})^t_{\mathbb{R}} \to (B^{\vee}_-, \Phi^{\vee}_{BK}, x^{\vee}_{\mathbf{i}})^t_{\mathbb{R}}.$$

Moreover, the map $\psi_{\mathbf{i}}$ restricts to an injective map of integral BK cones.

Proof. We introduce the following notation for simplicity:

$$p_i := \Delta_{w_0\omega_i, s_i\omega_i}, \quad q_i := \Delta_{w_0 s_i\omega_i, \omega_i} \in \mathbb{Q}[L^{w_0,e}];$$
$$p_i^\vee := \Delta_{w_0\omega_i^\vee, s_i\omega_i^\vee}, \quad q_i^\vee := \Delta_{w_0 s_i\omega_i^\vee, \omega_i^\vee} \in \mathbb{Q}[L^{\vee; w_0,e}].$$

For each index k, one can choose a double reduced word $\mathbf{i}_k = (i_1, \dots, i_m)$ such that $i_m = -k$, and $\mathbf{i}_{k^*} = (i_1, \dots, i_m)$ such that $i_1 = k^*$. Let α_{k^*} denote the function which sends h to $h^{-w_0\alpha_k}$, and let

$$(\alpha_{k^*} \cdot q_k)(h, z) = \alpha_{k^*}(h) q_k(z).$$

Then by Proposition 2.7.5, the function

$$\left((\alpha_{k^*}^\vee \cdot q_k^\vee) \circ \Psi_{\mathbf{i}_{k^*}}\right)^t : (B_-, x_{\mathbf{i}_{k^*}})^t \to \mathbb{Z}$$

can be written

$$\left(\sum a_i \omega_i^\vee, \sum \xi_i e_i\right) \mapsto d_{k^*}\left(a_{k^*} - \xi_1\right).$$

From Proposition 2.7.5 one also has:

$$d_k \left(\alpha_{k^*} \cdot q_k\right)^t : (B_-, x_{\mathbf{i}_{k^*}})^t \to \mathbb{Z} \; : \; \left(\sum a_i \omega_i^\vee, \sum \xi_i e_i\right) \mapsto d_{k^*}\left(a_{k^*} - \xi_1\right).$$

Thus we get:

$$\left((\alpha_{k^*}^\vee \cdot q_k^\vee) \circ \Psi_{\mathbf{i}_{k^*}}\right)^t = d_{k^*}\left(\alpha_{k^*} \cdot q_k\right)^t.$$

Similarly, for the other terms, we get $(p_k^\vee \circ \Psi_{\mathbf{i}_k})^t = d_k p_k^t$, where we write $p_k(h, z) = p_k(z)$.

Then by Corollary 8.2.5, we have

$$(p_k^\vee \circ \Psi_\sigma)^t = d_k p_k^t; \quad \left((\alpha_{k^*}^\vee \cdot q_k^\vee) \circ \Psi_\sigma\right)^t = d_{k^*}\left(\alpha_{k^*} \cdot q_k\right)^t, \tag{8.4}$$

where $\sigma \in |\sigma(\mathbf{i}')|$, for any double reduced word $\mathbf{i}'$ for (w_0, e). From (8.4), a point $x = (h, z)$ satisfying $\Psi_\sigma^t(x) \in C_\sigma^{G^\vee}(\mathbb{R})$ if and only if

$$d_k p_k^t(z) \geqslant 0 \text{ and } d_{k^*}\left(\alpha_{k^*} \cdot q_k\right)^t(h, z) \geqslant 0, \quad \forall k \in \boldsymbol{I}.$$

Dividing both sides of each equation by d_k, this is equivalent to the condition that

$$x \in (B_-, \Phi_{BK}, \sigma)(\mathbb{R}).$$

Again by Corollary 8.2.5, we can replace ψ_σ with $\psi_{\mathbf{i}}$. In particular, restricting to the integral cone $C_{\mathbf{i}}^G$, the map is an injection of cones. ♢

Remark 8.3.2. We give an direct computation for the comparison of the BK cones for $\mathrm{SO}(5)$ and $\mathrm{Sp}(4)$ in Section 8.6.

Theorem 8.3.3. *Consider the map $\psi_{\mathbf{i}}$ as in Theorem 8.3.1. Then for any $i \in \boldsymbol{I}$,*

$$\psi_{\mathbf{i}} \circ \tilde{e}_i = \tilde{e}_i^{d_i} \circ \psi_{\mathbf{i}}, \quad \psi_{\mathbf{i}} \circ \tilde{f}_i = \tilde{f}_i^{d_i} \circ \psi_{\mathbf{i}}, \quad (\mathrm{hw}^\vee) \circ \psi_{\mathbf{i}} = \psi \circ \mathrm{hw}^t$$

where we write $\tilde{e}_i, \tilde{f}_i$ for the crystal operators in both $(B_-, \Phi_{BK}, x_{\mathbf{i}})^t$ and $(B_-^\vee, \Phi_{BK}^\vee, x_{\mathbf{i}}^\vee)^t$ as described at the end of section 3.5.

Proof. Write $\mathcal{C}_{\mathbf{i}} := (B_-, \Phi_{BK}, x_{\mathbf{i}})^t$ and $\mathcal{C}_{\mathbf{i}}^\vee := (B_-^\vee, \Phi_{BK}^\vee, x_{\mathbf{i}}^\vee)^t$ for simplicity. We prove the statement for $\tilde{e}_i$; the one for $\tilde{f}_i$ follows immediately from the crystal axioms. Assume $x, \tilde{e}_i x \in \mathcal{C}_{\mathbf{i}}$. By Theorem 8.3.1, we have $\psi_{\mathbf{i}}(x), \psi_{\mathbf{i}}(\tilde{e}_i x) \in \mathcal{C}_{\mathbf{i}}^\vee$.

From $i_{n_e(x,i)} = i$, one sees immediately that

$$\psi_{\mathbf{i}}\left(\tilde{e}_i\left(h, \sum \xi_j v_j\right)\right) = \left(\psi(h), \sum d_{i_j}\xi_j v_j - d_i v_{n_e(x,i)}\right).$$

By convexity of $\mathcal{C}_{\mathbf{i}}^\vee$, the lattice points between $\psi_{\mathbf{i}}(x)$ and $\psi_{\mathbf{i}}(\tilde{e}_i x)$ are contained in $\mathcal{C}_{\mathbf{i}}^\vee$ as well. We will show that these are exactly the points obtained by repeatedly applying the operator $\tilde{e}_i$ in $\mathcal{C}_{\mathbf{i}}^\vee$.

First, by the description above

$$\tilde{e}_i\left(\psi_{\mathbf{i}}\left(h, \sum \xi_j v_j\right)\right) = \left(\psi(h), \sum d_{i_j}\xi_j v_j - v_{n_e(\psi_{\mathbf{i}}(x),i)}\right) \in \mathcal{C}_{\mathbf{i}}^\vee.$$

Assume that $n_e(x,i) = n_e(\psi_{\mathbf{i}}(x), i)$. From (3.15) applied to the crystal $\mathcal{C}_{\mathbf{i}}^\vee$, one gets that

$$n_e(\tilde{e}_i^k \psi_{\mathbf{i}}(x), i) = n_e(\psi_{\mathbf{i}}(x), i) = n_e(x,i),$$

for $0 \leqslant k < d_i$. So, applying $\tilde{e}_i$ repeatedly gives

$$\psi_{\mathbf{i}}(\tilde{e}_i x) = \tilde{e}_i^{d_i}\psi_{\mathbf{i}}(x).$$

It remains to show that $n_e(x,i) = n_e(\psi_{\mathbf{i}}(x), i)$. Indeed,

$$X_l(\psi_{\mathbf{i}}(x), i) = \sum_{k=1}^{l} (a^T)_{i_k,i} d_{i_k}\xi_k = \sum_{k=1}^{l} a_{i,i_k} d_{i_k}\xi_k = \sum_{k=1}^{l} d_i a_{i_k,i}\xi_k = d_i X_l(x,i).$$

So,

$$\begin{aligned} n_e(x,i) &= \max\left\{ l \,\middle|\, 1 \leqslant l \leqslant m,\ i_l = i, X_l(x,i) = \min_{l'}\{X_{l'}(x,i) \mid i_{l'} = i\}\right\} \\ &= \max\left\{ l \,\middle|\, 1 \leqslant l \leqslant m,\ i_l = i, d_i X_l(x,i) = \min_{l'}\{d_i X_{l'}(x,i) \mid i_{l'} = i\}\right\} \\ &= \max\left\{ l \,\middle|\, 1 \leqslant l \leqslant m,\ i_l = i, X_l(\psi_{\mathbf{i}}(x),i) = \min_{l'}\{X_{l'}(\psi_{\mathbf{i}}(x),i) \mid i_{l'} = i\}\right\} \\ &= n_e(\psi_{\mathbf{i}}(x), i). \end{aligned}$$

This proves the claim. ♢

Remark 8.3.4. Restricting to $\mathrm{hw}^{-t}(\lambda^\vee)$ and identifying $\mathrm{hw}^{-t}(\lambda^\vee)$ with $B_{\lambda^\vee}$, Theorem 8.3.3 is a special case of Kashiwara's theorem as in [58, Theorem 5.1]. Note that Theorem 2.6 in [39] is also a special case of Kashiwara's theorem, as indicated by the authors.

8.4 Comparison of lattices

Let (K^*, π_{K^*}, T) be the dual Poisson-Lie group together with the dressing action of T on K. In this section, we use our results to compare two lattices on $(B_-, \Phi_{BK}, \sigma(\mathbf{i}))_{\mathbb{R}}^t$. The first lattice comes from the crystal structure on $(B_-^\vee, \Phi_{BK}^\vee, \sigma^\vee(\mathbf{i}))^t$. The integrable system on $\mathrm{PT}(K^*, \sigma(\mathbf{i}))$ gives us the second lattice Λ, which we call *Bohr-Sommerfeld* lattice.

The lattice Λ is built out of a lattice in $\mathfrak{h}$ and a lattice in each integral symplectic leaf of $\mathrm{PT}(K^*)$. The lattice in $\mathfrak{h}$ is $\psi^{-1}(X^*(H))$; recall that the T action on K^* determines the lattice $X^*(H)$ in $\mathfrak{h}^*$ and that the Poisson-Lie structure π_K on K depends on a choice of an invariant inner product on $\mathfrak{g}$ (which, in turn, determines ψ).

Next, we describe the lattice in the symplectic leaves of $\mathrm{PT}(K^*)$. The Bohr-Sommerfeld quantization defines a lattice (integral affine structure) in the tangent spaces to leaves as follows. Assume $\lambda^\vee \in \mathfrak{h}$ is a regular dominant weight of G such that

$$\lambda := \psi(\lambda^\vee) \in X_+^*(H) \subset \mathfrak{h}^\vee.$$

Let hw^t be the tropicalization of $\mathrm{hw}\colon G^{w_0,e} \to H$ relative to the chart $\sigma(\mathbf{i})$, and let $\mathrm{hw}_{\mathbb{R}}^t$ be the real extension of hw^t. Recall we have symplectic leaf

$$\mathscr{P}_{\lambda^\vee} := \mathrm{hw}_{\mathbb{R}}^{-t}(\lambda^\vee) \times (S^1)^m \subset \mathrm{PT}(K^*, \sigma(\mathbf{i}))$$

together with the symplectic form $\omega_\infty^{\lambda^\vee} := (\pi_\infty)^{-1}$. Let $\boldsymbol{\lambda}_0 \in \mathrm{hw}^{-t}(\lambda^\vee)$ be the unique point in $(B_-, \Phi_{BK}, \sigma(\mathbf{i}))_{\mathbb{R}}$ such that $\mathrm{wt}_{\mathbb{R}}^t(\boldsymbol{\lambda}_0) = \mathrm{hw}_{\mathbb{R}}^t(\boldsymbol{\lambda}_0) = \lambda^\vee$. And denote by

$$\Lambda_0 := \left\{\boldsymbol{\lambda} \mid \psi \circ \mathrm{hw}_{\mathbb{R}}^t(\xi) = \psi \circ \mathrm{wt}_{\mathbb{R}}^t(\xi) = \lambda \in X_+^*(H)\right\}.$$

Consider the lattice $X_*(S^1)^m \subset T_1(S^1)^m$ of cocharacters of $(S^1)^m$; this lattice is generated by

$$\left\{2\pi \frac{d}{d\varphi_k} \;\middle|\; k = 1, \ldots, m\right\}.$$

Thus the following

$$\left\{v \in T_{\boldsymbol{\lambda}_0} \mathrm{hw}_{\mathbb{R}}^{-t}(\lambda^\vee) \mid \omega_{\lambda^\vee}(v, X_*(S^1)^m) \subset 2\pi\mathbb{Z}\right\} \tag{8.5}$$

is a lattice in $T_{\boldsymbol{\lambda}_0} \mathrm{hw}_{\mathbb{R}}^{-t}(\lambda^\vee)$. The natural identification of $\mathrm{hw}_{\mathbb{R}}^{-t}(\lambda^\vee)$ with a subset of $T_{\boldsymbol{\lambda}_0} \mathrm{hw}_{\mathbb{R}}^{-t}(\lambda^\vee)$ determines the lattice $\widetilde{\Lambda}$ on $\mathrm{hw}_{\mathbb{R}}^{-t}(\lambda^\vee)$. Alternatively, think of the points of the set (8.5) as elements of a (scaled) dual basis to $X_*(S^1)^m$, under the pairing given by the symplectic form. In our choice of coordinates, the symplectic form is described by the matrix B in Proposition 7.7.2. So another description of the lattice $\widetilde{\Lambda}$ is

$$\widetilde{\Lambda} = \left(\boldsymbol{\lambda}_0 + B\,(\mathbb{Z}, \ldots, \mathbb{Z})^T\right) \cap (B_-, \Phi_{BK}, \sigma(\mathbf{i}))_{\mathbb{R}}. \tag{8.6}$$

Then the lattice Λ_0 and the lattices $\widetilde{\Lambda}$ on the integral symplectic leaves determine the *Bohr-Sommerfeld lattice*

$$\Lambda := B(\mathbb{Z}, \ldots, \mathbb{Z})^T + \Lambda_0. \tag{8.7}$$

Remark 8.4.1. Actually, the Bohr-Sommerfeld lattice on $\mathrm{PT}(K^*)$ is independent of the choice of toric chart $\sigma(\mathbf{i})$, as a consequence of Lemma B2 of [6] and Theorem 6.23 of [5]. We omit the detail of the proof here.

Theorem 8.4.2. *The comparison map $\psi_{\sigma(\mathbf{i})}$ sends the Bohr-Sommerfeld lattice to the integral points of $(B_-^\vee, \Phi_{BK}^\vee, \sigma^\vee(\mathbf{i}))^t$, i.e., $(\psi_{\sigma(\mathbf{i})})_{\mathbb{R}}(\Lambda) = (B_-^\vee, \Phi_{BK}^\vee, \sigma^\vee(\mathbf{i}))^t$.*

Proof. From (8.6) and Theorem 7.7.2, we have

$$\widetilde{\Lambda} = \left(\boldsymbol{\lambda}_0 + \left(0, \ldots, 0, \frac{1}{d_{i_1}}\mathbb{Z}, \frac{1}{d_{i_2}}\mathbb{Z}, \ldots, \frac{1}{d_{i_m}}\mathbb{Z}\right)\right) \cap (B_-, \Phi_{BK}, \sigma(\mathbf{i}))_{\mathbb{R}}.$$

From Theorem 8.3.1, it is then clear that $(\psi_{\sigma(\mathbf{i})})_{\mathbb{R}}(\widetilde{\Lambda}) = (\mathrm{hw}^\vee)^{-t}(\lambda)$. ♢

8.5 Comparison of volumes

As an application of the results of the previous section, we compare the symplectic volume of symplectic leaves of $\mathrm{PT}(K^*)$ with that of symplectic leaves of $\mathfrak{k}^*$.

Definition 8.5.1 (Notation)**.** Let L be a lattice in $\mathbb{R}^n$. Then L induces a natural translation-invariant measure μ_L on $\mathbb{R}^n$. For a compact domain $U \subset \mathbb{R}^n$, let $\mathrm{Vol}(U, L)$ be the volume of U with respect to L. For a symplectic form ω on U, let $\mathrm{Vol}(U, \omega)$ be the volume of U with respect to the Liouville form.

Proposition 8.5.2. *Using notation just defined, we have*

$$\frac{1}{(2\pi)^m}\mathrm{Vol}(\mathscr{P}_{\lambda^\vee}, \omega_\infty^{\lambda^\vee}) = \mathrm{Vol}(\mathrm{hw}_{\mathbb{R}}^{-t}(\lambda^\vee), \Lambda). \tag{8.8}$$

Proof. The Liouville measure of $\omega_\infty^{\lambda^\vee}$ is a product of the translation-invariant measure μ_Λ on $\mathrm{hw}_{\mathbb{R}}^{-t}(\lambda^\vee)$ and $(2\pi)^m$ times the normalized Haar measure on $(S^1)^m$. The proposition follows immediately by Fubini theorem. ♢

Recall the standard Lie-Poisson structure $\pi_{\mathfrak{k}^*}$ on $\mathfrak{k}^*$. Let $\xi := -\mathbb{i}\psi(\lambda^\vee) \in X_+^*(H)$ be an regular element in the positive Weyl chamber $\mathfrak{t}_+^*$. Denote by $\mathcal{O}_\xi$ the coadjoint orbit through ξ. Let ω_ξ be the corresponding symplectic form.

Theorem 8.5.3. *Let $\xi := -\mathbb{i}\psi(\lambda^\vee) \in X_+^*(H)$ be an regular element in the positive Weyl chamber. The symplectic volume of the symplectic leaf $\mathscr{P}_{\lambda^\vee} \subset \mathrm{PT}(K^*, \sigma(\mathbf{i}))$ is equal to the symplectic volume of $\mathcal{O}_\xi \subset \mathfrak{k}^*$.*

Proof. Let $\lambda := \psi(\lambda^\vee)$. Let V_λ be the irreducible G-module with highest weight λ. Recall from Theorem 3.5.9 that $\dim(V_\lambda) = \#(\mathrm{hw}^\vee)^{-t}(\lambda)$, the number of lattice points in $(\mathrm{hw}^\vee)^{-t}(\lambda)$. Recall also that Weyl dimension formula is:

$$\dim(V_\lambda) = \prod_{\alpha>0}\frac{(\lambda+\rho, \alpha)}{(\rho, \alpha)},$$

where ρ is the half-sum of positive roots of G. Let N be a positive integer. Then

$$\lim_{N\to\infty}\frac{\#(\mathrm{hw}^\vee)^{-t}(N\lambda)}{\mathrm{Vol}\left((\mathrm{hw}^\vee)_{\mathbb{R}}^{-t}(N\lambda), (\mathrm{hw}^\vee)^{-t}(N\lambda)\right)} = 1.$$

Also, $\mathrm{Vol}\left((\mathrm{hw}^\vee)_{\mathbb{R}}^{-t}(N\lambda), (\mathrm{hw}^\vee)^{-t}(N\lambda)\right) = N^m \,\mathrm{Vol}\left((\mathrm{hw}^\vee)_{\mathbb{R}}^{-t}(\lambda), (\mathrm{hw}^\vee)^{-t}(\lambda)\right)$. Therefore,

$$\begin{aligned}\mathrm{Vol}\left((\mathrm{hw}^\vee)_{\mathbb{R}}^{-t}(\lambda), (\mathrm{hw}^\vee)^{-t}(\lambda)\right) &= \lim_{N\to\infty}\frac{1}{N^m}\prod_{\alpha>0}\frac{(N\lambda+\rho, \alpha)}{(\rho, \alpha)} \\ &= \prod_{\alpha>0}\lim_{N\to\infty}\left(\frac{(\lambda, \alpha)}{(\rho, \alpha)} + \frac{1}{N}\right) \\ &= \prod_{\alpha>0}\frac{(\lambda, \alpha)}{(\rho, \alpha)}.\end{aligned}$$

It is well known that

$$\mathrm{Vol}(\mathcal{O}_\xi, \omega_\xi) = (2\pi)^m\prod_{\alpha>0}\frac{(\lambda, \alpha)}{(\rho, \alpha)}, \tag{8.9}$$

see for instance Section 3.5 of [62]. Combining the fact

$$\mathrm{Vol}\left(\mathrm{hw}_{\mathbb{R}}^{-t}\left(\psi^{-1}(\lambda)\right), \Lambda\right) = \mathrm{Vol}\left((\mathrm{hw}^\vee)_{\mathbb{R}}^{-t}(\lambda), (\mathrm{hw}^\vee)^{-t}(\lambda)\right).$$

an (8.8), (8.9), we get the result. ♢

Corollary 8.5.4. *For all $\lambda^\vee \in \psi^{-1}(X_+^*(H) + \rho)$, one has*

$$\dim V_{\lambda-\rho} = \left(\prod_{\alpha>0} d_\alpha\right) \cdot \dim V_{\lambda^\vee - \rho^\vee},$$

where $d_\alpha = \frac{2}{(\alpha,\alpha)}$ and $\lambda = \psi(\lambda^\vee)$.

Proof. Given a reduced word $\mathbf{i} = (i_1, \ldots, i_m)$ of the longest element w_0, positive roots can be written in the following order:

$$\alpha_{i_1},\ s_{i_1}\alpha_{i_2},\ \ldots,\ s_{i_1}\cdots s_{i_{m-1}}\alpha_{i_m}.$$

Since the bilinear form is W-invariant, for positive root $\alpha = s_{i_1}\cdots s_{i_{j-1}}\alpha_{i_j}$, we get:

$$(\alpha, \alpha) = (\alpha_{i_j}, \alpha_{i_j}).$$

Then one has $\prod_{\alpha>0} d_\alpha = \prod_{j=1}^m d_{i_j}$. By Theorem 8.5.3 and its proof, we get

$$\dim V_{\lambda-\rho} = \mathrm{Vol}\left((\mathrm{hw}^\vee)_{\mathbb{R}}^{-t}(\lambda), (\mathrm{hw}^\vee)^{-t}(\lambda)\right).$$

Taking the determinant of $(\psi_{\sigma(\mathbf{i})})_{\mathbb{R}}$, one finds

$$\begin{aligned}\mathrm{Vol}\left((\mathrm{hw}^\vee)_{\mathbb{R}}^{-t}(\lambda), (\mathrm{hw}^\vee)^{-t}(\lambda)\right) &= \left(\prod_{\alpha>0} d_\alpha\right) \cdot \mathrm{Vol}\left(\mathrm{hw}_{\mathbb{R}}^{-t}(\lambda), \mathrm{hw}^{-t}(\lambda)\right) \\ &= \left(\prod_{\alpha>0} d_\alpha\right) \cdot \dim V_{\lambda^\vee-\rho^\vee}.\end{aligned}$$

♢

In the following, we present a direct computation of Corollary 8.5.4. Let $\psi(\lambda^\vee) = \lambda \in X_+^*(H)$ and denote $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ and $\rho^\vee = \frac{1}{2}\sum_{\alpha>0}\alpha^\vee$ as before. Note ψ preserves the bilinear forms on $\mathfrak{h}$ and $\mathfrak{h}^*$ and commutes with the W-action.

Lemma 8.5.5. *For each complex semisimple $\mathfrak{g}$ one has for a formal parameter q,*

$$\prod_{\alpha>0}(q^{\frac{1}{2}\langle\rho^\vee,\alpha\rangle} - q^{-\frac{1}{2}\langle\rho^\vee,\alpha\rangle}) = \prod_{\alpha^\vee>0}(q^{\frac{1}{2}\langle\alpha^\vee,\rho\rangle} - q^{-\frac{1}{2}\langle\alpha^\vee,\rho\rangle}).$$

In particular, $\prod\limits_{\alpha>0}\langle\rho^\vee,\alpha\rangle = \prod\limits_{\alpha^\vee>0}\langle\alpha^\vee,\rho\rangle$.

Proof. Note we have the following Weyl denominator Formula:

$$e^\rho\prod_{\alpha>0}(1-e^{-\alpha}) = \sum_{w\in W}(-1)^{\ell(w)}e^{w\rho}.$$

Let Q be the root lattice and $Q^\vee$ be the coroot lattice of $\mathfrak{g}$. Applying the ring homomorphisms

$$\eta\colon \mathbb{Z}[\frac{1}{2}Q] \to \mathbb{Z}[q^{\pm\frac{1}{2}}] \;:\; e^{\beta} \mapsto q^{\langle\rho^\vee,\beta\rangle}; \quad \eta^\vee\colon \mathbb{Z}[\frac{1}{2}Q^\vee] \to \mathbb{Z}[q^{\pm\frac{1}{2}}] \;:\; e^{\beta^\vee} = q^{\langle\beta^\vee,\rho\rangle}$$

to the Weyl denominator formula for Q and $Q^\vee$, we obtain

$$\prod_{\alpha>0}(q^{\frac{1}{2}\langle\rho^\vee,\alpha\rangle} - q^{-\frac{1}{2}\langle\rho^\vee,\alpha\rangle}) = \sum_{w\in W}(-1)^{\ell(w)}e^{\langle\rho^\vee,w\rho\rangle} = \prod_{\alpha^\vee>0}(q^{\frac{1}{2}\langle\alpha^\vee,\rho\rangle} - q^{-\frac{1}{2}\langle\alpha^\vee,\rho\rangle}).$$

The second assertion follows by dividing both sides with the appropriate power of $q^{\frac{1}{2}} - q^{-\frac{1}{2}}$ and taking the limit as $q \mapsto 1$. ♢

For $\lambda^\vee \in \mathfrak{h}$ and $\lambda \in \mathfrak{h}^*$, we rewrite the Weyl dimension formula: If $\lambda \in X_+^*(H) + \rho$, then

$$\dim V_{\lambda-\rho} = \prod_{\alpha>0}\frac{(\lambda,\alpha)}{(\rho,\alpha)} = \prod_{\alpha^\vee>0}\frac{\langle\alpha^\vee,\lambda\rangle}{\langle\alpha^\vee,\rho\rangle},$$

where $V_{\lambda-\rho}$ is the irreducible highest weight module with highest weight $\lambda - \rho$. Then:

Proof of Corollary 8.5.4. Indeed, Lemma 8.5.5 implies that

$$\dim V_{\psi\lambda^\vee-\rho} = \prod_{\alpha^\vee>0}\frac{\langle\alpha^\vee,\psi\lambda^\vee\rangle}{\langle\alpha^\vee,\rho\rangle} = \prod_{\alpha^\vee>0}\frac{\langle\psi\alpha^\vee,\lambda^\vee\rangle}{\langle\alpha,\rho^\vee\rangle} = \prod_{\alpha>0}\frac{d_\alpha\langle\alpha,\lambda^\vee\rangle}{\langle\alpha,\rho^\vee\rangle} = \prod_{\alpha>0}d_\alpha\cdot\dim V_{\lambda^\vee-\rho^\vee}.$$

The corollary is proved. ♢

8.6 Example: duality between B_2 and C_2

Note $\mathrm{SO}_{2n+1}^\vee = \mathrm{Sp}_{2n}$. Let us focus on the case $n = 2$. Here we use an alternative description of SO_5. Denote

$$J_n = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}.$$

The group SO_5 is isomorphic to

$$G = \{X \in \mathrm{GL}_5 \mid XJ_5X^T = J_5\},$$

with Lie algebra:

$$\mathfrak{g} = \{x \in \mathfrak{gl}(5) \mid x + J_5x^TJ_5 = 0\}.$$

Cartan subalgebra:

$$\mathfrak{h} = \{\mathrm{diag}(x_1, x_2, 0, -x_2, -x_1)\}.$$

Borel subalgebra:

$$\mathfrak{b} = \mathfrak{g} \cap \{\text{upper-triangular matrices}\}.$$

Cartan matrix and a symmetrizer:

$$A = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & \\ & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & \\ & 2 \end{bmatrix},$$

Orthonormal basis in $\mathfrak{h}^*$:

$$\zeta_i : \operatorname{diag}(x_1, x_2, 0, -x_2, -x_1) \mapsto x_i.$$

Simple roots:

$$\alpha_1 = \zeta_1 - \zeta_2, \quad \alpha_2 = \zeta_2.$$

Positive roots:

$$\alpha_1, \quad \alpha_2, \quad \alpha_3 := \alpha_1 + \alpha_2, \quad \alpha_4 := \alpha_1 + 2\alpha_2.$$

Simple coroots:

$$\alpha_1^\vee = \operatorname{diag}(1, -1, 0, 1, -1); \quad \alpha_2^\vee = \operatorname{diag}(0, 2, 0, -2, 0).$$

Simple root vectors:

$$F_1 = E_{21} - E_{54}; \quad F_2 = E_{32} - E_{43}.$$

Fundamental weights:

$$\omega_1 = \alpha_3, \quad \omega_2 = \frac{1}{2}\alpha_4.$$

Fundamental coweights:

$$\omega_1^\vee = \alpha_1^\vee + \frac{1}{2}\alpha_2^\vee, \quad \omega_2^\vee = \alpha_1^\vee + \alpha_2^\vee.$$

Character lattice of the maximal torus:

$$X^*(H) = \mathbb{Z}\{\alpha_1, \alpha_2\}.$$

Cocharacter lattice of the maximal torus:

$$X_*(H) = \mathbb{Z}\{\omega_1^\vee, \omega_2^\vee\}.$$

Weyl group:

$$W = S_2 \ltimes \mathbb{Z}_2, \text{ with generator } s_1, s_2 \text{ satisfying } (s_1 s_2)^4 = 1.$$

The longest element:

$$w_0 = (s_1 s_2)^2 = (s_2 s_1)^2.$$

Now let us compute the BK potential and the BK cone. Note that the lift of s_i to G is given by:

$$\overline{s_1} = P_1 P_4; \quad \overline{s_2} = P_2 P_3 P_2; \quad \text{where } P_i = E_{i,i+1} - E_{i+1,i}.$$

And note $(\overline{s_1 s_2})^2 = P_1P_2P_3P_4P_1P_2P_3P_1P_2P_1$. Let

$$\boldsymbol{x} = \exp\left(\ln(x_1)\omega_1^\vee + \ln(x_2)\omega_2^\vee\right)$$

and

$$x_{-1}(t) = \begin{bmatrix} t^{-1} & & & & \\ 1 & t & & & \\ & & 1 & & \\ & & & t^{-1} & \\ & & & -1 & t \end{bmatrix}; \quad x_{-2}(t) = \begin{bmatrix} 1 & & & & \\ & t^{-2} & & & \\ & t^{-1} & 1 & & \\ & -\frac{1}{2} & -t & t^2 & \\ & & & & 1 \end{bmatrix}.$$

Then for the longest word $(s_1s_2)^2$, generic elements of the double Bruhat cell $G^{w_0,e}$ can be written:

$$\boldsymbol{x}x_{-1}(t_1)x_{-2}(t_2)x_{-1}(t_3)x_{-2}(t_4) \in G^{w_0,e}.$$

This element is equal to

$$\boldsymbol{x}\begin{bmatrix} \frac{1}{t_1t_3} & & & & \\ \frac{t_1}{t_2^2}+\frac{1}{t_3} & \frac{t_1t_3}{t_2^2t_4^2} & & & \\ \frac{1}{t_2} & \frac{t_3}{t_2t_4^2}+\frac{1}{t_4} & 1 & & \\ -\frac{1}{2t_1} & -\frac{(t_3+t_2t_4)^2}{2t_1t_3t_4^2} & -\frac{t_2(t_3+t_2t_4)}{t_1t_3} & \frac{t_2^2t_4^2}{t_1t_3} & \\ \frac{1}{2} & \frac{1}{2}\left(\frac{(t_3+t_2t_4)^2}{t_3t_4^2}+t_1\right) & \frac{t_4t_2^2}{t_3}+t_2+t_1t_4 & -\frac{(t_2^2+t_1t_3)\,t_4^2}{t_3} & t_1t_3 \end{bmatrix}.$$

Thus the potential is

$$\left(t_1+\frac{(t_3+t_2t_4)^2}{t_3t_4^2}\right)+t_4+x_1\cdot\frac{1}{t_1}+x_2\left(\frac{1}{t_4}+\frac{t_2^2+t_1t_3}{t_2t_3}\right),$$

which gives us the cone cut out by the inequalities:

$$\begin{aligned} x_1 &\geqslant t_1 \geqslant 0;\\ x_2 &\geqslant t_4 \geqslant 0;\\ 2t_2 \geqslant t_3 &\geqslant 2t_4 \geqslant 0;\\ x_2 &\geqslant t_2 - t_1;\\ x_2 &\geqslant t_3 - t_2. \end{aligned} \tag{8.10}$$

Now let us describe Sp_4 as the dual of SO_5. Denote

$$J'_{2n} = \begin{bmatrix} & J_n \\ -J_n & \end{bmatrix}.$$

The group Sp_4 is isomorphic to

$$G^\vee = \{X \in \mathrm{GL}_4 \mid XJ'_4X^T = J'_4\}$$

with Lie algebra:

$$\mathfrak{g}^\vee = \{x \in \mathfrak{gl}(4) \mid x - J'x^TJ' = 0\}.$$

Cartan subalgebra:

$$\mathfrak{h} = \{\mathrm{diag}(x_1, x_2, -x_2, -x_1)\}.$$

The Borel subalgebra:

$$\mathfrak{b}^\vee = \mathfrak{g} \cap \{\text{upper-triangular matrices}\}.$$

Orthonormal basis in $(\mathfrak{h}^\vee)^*$:

$$\zeta_i^\vee : \mathrm{diag}(x_1, x_2, -x_2, -x_1) \mapsto x_i.$$

Simple roots:

$$\beta_1 = \zeta_1^\vee - \zeta_2^\vee, \quad \beta_2 = 2\zeta_2^\vee.$$

Positive roots:

$$\beta_1, \quad \beta_2, \quad \beta_3 := 2\beta_1 + \beta_2, \quad \beta_4 := \beta_1 + \beta_2.$$

Simple coroots of $\mathfrak{g}^\vee$ are given by:

$$\beta_1^\vee = \operatorname{diag}(1, -1, 1, -1); \quad \beta_2^\vee = \operatorname{diag}(0, 1, -1, 0);$$

Simple root vectors:

$$F_1 = E_{21} - E_{43}; \quad F_2 = E_{32}.$$

Fundamental weights:

$$\kappa_1 = \frac{1}{2}\beta_3, \quad \kappa_2 = \beta_4.$$

Fundamental coweights:

$$\kappa_1^\vee = \beta_1^\vee + \beta_2^\vee, \quad \kappa_2^\vee = \frac{1}{2}\beta_1^\vee + \beta_2^\vee.$$

Character lattice of the maximal torus:

$$X^*(H^\vee) = \mathbb{Z}\{\kappa_1, \kappa_2\}.$$

Cocharacter lattice of the maximal torus:

$$X_*(H) = \mathbb{Z}\{\beta_1^\vee, \beta_2^\vee\}.$$

To calculate the potential for $G^\vee$, we need the lift of s_i to $G^\vee$:

$$\overline{s_1} = P_1 P_3; \quad \overline{s_2} = P_2; \qquad \text{where } P_i = E_{i,i+1} - E_{i+1,i}.$$

Note $(\overline{s_1 s_2})^2 = P_1 P_2 P_3 P_1 P_2 P_1$. Let

$$\boldsymbol{y}^\vee = \exp\left(\ln(y_1)\beta_1^\vee + \ln(y_2)\beta_2^\vee\right)$$

and

$$x_{-1}^\vee(t) = \begin{bmatrix} t^{-1} & & & \\ 1 & t & & \\ & & t^{-1} & \\ & & -1 & t \end{bmatrix}; \quad x_{-2}^\vee(t) = \begin{bmatrix} 1 & & & \\ & t^{-1} & & \\ & 1 & t & \\ & & & 1 \end{bmatrix}.$$

Then for the longest word $(s_1 s_2)^2$, generic elements of the double Bruhat cell $G^{\vee;w_0,e}$ can be written:

$$\boldsymbol{y}^\vee x_{-1}^\vee(t_1) x_{-2}^\vee(t_2) x_{-1}^\vee(t_3) x_{-2}^\vee(t_4) \in G^{\vee;w_0,e}.$$

This element is equal to

$$\boldsymbol{y}^\vee \begin{bmatrix} \frac{1}{t_1 t_3} & & & \\ \frac{t_1}{t_2} + \frac{1}{t_3} & \frac{t_1 t_3}{t_2 t_4} & & \\ \frac{1}{t_1} & \frac{t_2}{t_1 t_3} + \frac{t_3}{t_1 t_4} & \frac{t_2 t_4}{t_1 t_3} & \\ -1 & -t_1 - \frac{t_2}{t_3} - \frac{t_3}{t_4} & \left(-t_1 - \frac{t_2}{t_3}\right) t_4 & t_1 t_3 \end{bmatrix}.$$

Thus the potential is

$$\left(t_1 + \frac{t_2}{t_3} + \frac{t_3}{t_4}\right) + t_4 + \frac{y_1^2}{y_2} \cdot \frac{1}{t_1} + \frac{y_2^2}{y_1^2}\left(\frac{(t_1 t_3 + t_2)^2}{t_2 t_3^2} + \frac{1}{t_4}\right),$$

which gives us the cone cut out by the following inequalities:

$$\begin{aligned} 2y_1 - y_2 \geqslant t_1 \geqslant 0; \\ 2y_2 - 2y_1 \geqslant t_4 \geqslant 0; \\ t_2 \geqslant t_3 \geqslant t_4 \geqslant 0; \\ 2y_2 - 2y_1 \geqslant t_2 - 2t_1; \\ 2y_2 - 2y_1 \geqslant 2t_3 - t_2. \end{aligned} \tag{8.11}$$

Recall that $\psi : X_*(H) \to X^*(H)$ is given by:

$$x_1\omega_1^\vee + x_2\omega_2^\vee \mapsto (x_1 + x_2)\alpha_1 + (x_1 + 2x_2)\alpha_2.$$

Then the map $\psi_{\mathbf{i}} : \mathcal{L} \to \mathcal{L}^\vee$ is given by:

$$(x_1, x_2; t_1, t_2, t_3, t_4) \mapsto (x_1 + x_2, x_1 + 2x_2; t_1, 2t_2, t_3, 2t_4).$$

Thus it easy to see, after replacing $(y_1, y_2; t_1, t_2, t_3, t_4)$ by $(x_1 + x_2, x_1 + 2x_2; t_1, 2t_2, t_3, 2t_4)$, that the real cone defined by (8.11) is the real cone defined by (8.10).

9 | Action-angle Variables for Coadjoint Orbits

9.1 Overview

There is a dichotomy in symplectic geometry between local and global coordinates. Whereas Darboux's theorem tells us that symplectic manifolds have no local invariants, the problem of finding large coordinate charts often relates to subtle properties of symplectic manifolds. Most famously, Gromov's non-squeezing theorem demonstrates that the volume of certain coordinate charts on a symplectic manifold may have an upper bound strictly less than the total volume of the symplectic manifold [48].

Action-angle coordinates are a type of coordinate chart on symplectic manifolds that originate from the study of commutative completely integrable systems in classical mechanics. The domains of action-angle coordinates are products of the form $\mathcal{U} \times (S^1)^n$, where $\mathcal{U}$ is an open subset of $\mathbb{R}^n$. Such domains carry a canonical symplectic form,

$$\omega_{\mathrm{std}} := \sum_{i=1}^{n} d\lambda_i \wedge d\varphi_i, \tag{9.1}$$

where λ_i are coordinates on $\mathbb{R}^n$ and φ_i are coordinates on $(S^1)^n$. The Liouville-Arnold theorem guarantees existence of local action-angle coordinates in a neighborhood of compact regular fibers of commutative completely integrable systems [2]. A compact toric manifold of dimension $2n$ with Delzant polytope $\triangle$ has a dense subset symplectomorphic to $(\mathring{\triangle} \times (S^1)^n, \omega_{\mathrm{std}})$, where $\mathring{\triangle}$ denotes the interior of $\triangle$. However, there are also many interesting examples of action-angle coordinates on dense subsets that do not arise from a toric structure, such as Gelfand-Zeitlin systems [44], Goldman systems on moduli spaces of flat connections [47, 83], bending flow systems on moduli spaces of polygons [55], and integrable systems constructed by toric degeneration on smooth projective varieties [51, 59, 60].

The main result of this chapter is a construction of action-angle coordinates on large subsets of a regular coadjoint orbit for a compact semisimple Lie group K. Recall that regular coadjoint orbits are parameterized by elements ξ in the interior of the positive Weyl chamber $(\mathfrak{t}^*_+)^\circ$ of K. Denote by T a fix maximal torus of K. The coadjoint orbit parameterized by ξ along with its Lie-Poisson form is denoted $(\mathcal{O}_\xi, \omega_\xi)$. The main result is stated as follows.

Theorem 9.1.1 (Theorem 9.5.1)**.** *For any $\delta > 0$, and $\xi \in (\mathfrak{t}^*_+)^\circ$, there is a convex polytope $\triangle_\xi$ of dimension $m = \frac{1}{2}(\dim K - \dim T)$ such that there exists a symplectic embedding*

$$(\triangle_\xi(\delta) \times (S^1)^m, \omega_{\mathrm{std}}) \hookrightarrow (\mathcal{O}_\xi, \omega_\xi),$$

where $\triangle_\xi(\delta)$ denotes the set of points in $\triangle_\xi$ that have distance more than δ from the boundary in Euclidean space $\mathbb{R}^m$. Moreover, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\mathrm{Vol}(\triangle_\xi(\delta) \times (S^1)^n, \omega_{\mathrm{std}}) > \mathrm{Vol}(\mathcal{O}_\xi, \omega_\xi) - \varepsilon.$$

Theorems 9.1.1 have a limitation: it does not yield action-angle coordinates on a dense subset. In particular, these action-angle charts do not currently have an interpretation as action-angle coordinates for a globally defined commutative integrable system. On the other hand, these theorems illustrate that there are no non-trivial obstructions to the volume of action-angle coordinates on regular coadjoint orbits. Note that such integrable system is known when either K is of type A, B, or D and λ by using Gelfand-Zeitlin systems in [79], or when K is arbitrary type and ξ is a positive scalar multiple of a dominant integral weight by using toric degenerations by [33].

The method we use combines previous results in Chapter 7 and Chapter 8. The main idea is that to each coadjoint orbit $\mathcal{O}_\xi$ one can associate a family of dressing orbits $D_{\exp(s\xi)}$ in in Poisson-Lie group K^*. The dressing orbits are symplectomorphic to $\mathcal{O}_\xi$ for all values of the parameter $s < 0$. For s small, $D_{\exp(s\xi)}$ resembles of $\mathcal{O}_\xi$, and there is a natural way to include $\mathcal{O}_\xi$ in the family at $s = 0$. For $s \ll 0$ large, there are coordinates on $D_{\exp(s\xi)}$ coming from cluster variety theory which make its symplectic structure (exponentially) close to the constant one. Using this, one may construct toric charts on $D_{\exp(s\xi)}$, and hence on $\mathcal{O}_\xi$, which exhaust the symplectic volume as $s \to -\infty$.

9.2 Gelfand-Zeitlin as a tropical limit

First of all, let us recall the Gelfand-Zeitlin system for $\mathfrak{su}_n^*$ with the Lie-Poisson structure, where SU_n is the special unitary group and $\mathfrak{su}_n = \mathrm{Lie}(\mathrm{SU}_n)$.

Denote by $\mathcal{H}$ the set of $n \times n$ traceless Hermitian matrices, which one can identify with $\mathfrak{su}(n)^*$ via the non-degenerate bilinear form $(X, Y) = \mathrm{tr}(XY)$. Under this identification, the *Gelfand-Zeitlin functions* on $\mathfrak{su}(n)^*$ are defined as follows. For $k \in [1, n]$, let $A^{(k)}$ be the $k \times k$ principal submatrix sitting in the bottom-right corner of A. Let $\lambda_i^{(k)} \colon \mathcal{H} \to \mathbb{R}$, $1 \leqslant i \leqslant k \leqslant n$ be the ordered eigenvalues of $A^{(k)}$:

$$\lambda_1^{(k)}(A) \geqslant \cdots \geqslant \lambda_k^{(k)}(A).$$

Note that $\lambda_n^{(n)} = -\sum_{k=1}^{n-1} \lambda_k^{(n)}$ since A is traceless. The Gelfand-Zeitlin functions satisfy "interlacing inequalities",

$$\lambda_i^{(k)} \geqslant \lambda_i^{(k-1)} \geqslant \lambda_{i+1}^{(k)}, \text{ for all } 1 \leqslant i < k \leqslant n, \tag{9.2}$$

and the image of map $F \colon \mathcal{H} \to \mathbb{R}^{(n^2+n-1)/2}$, defined by the Gelfand-Zeitlin functions, is the polyhedral cone defined by the inequalities (9.2), called the *Gelfand-Zeitlin cone*.

Let $\mathcal{H}_0$ denote the open dense subset of $\mathcal{H}$, where all the inequalities (9.2) are strict. The k-torus $\mathrm{T}_k \subset \mathrm{U}_k \subset \mathrm{U}_n$ acts on $\mathcal{H}_0$ as follows:

$$t \bullet A := \mathrm{Ad}_{U^{-1}tU} A, \quad \text{for } t \in \mathrm{T}_k, A \in \mathcal{H}_0, U \in \mathrm{U}_k.$$

This section is based on a joint work [9] with A. Alekseev and J. Lane.

where U is chosen such that $\mathrm{Ad}_U\, A^{(k)} = \mathrm{diag}(\lambda_1^{(k)}, \dots, \lambda_k^{(k)})$. The actions of T_k and T_l commute for $k \neq l$, hence define an action of

$$\mathrm{T}_{n-1} \times \cdots \times \mathrm{T}_1 \cong (S_1)^m, \text{ where } m := n(n-1)/2.$$

The Gelfand-Zeitlin functions are smooth on $\mathcal{H}_0$ and define global action coordinates for a completely integrable system: the functions $\lambda_1^{(n)}, \dots, \lambda_{n-1}^{(n)}$ are a complete set of Casimir functions. The torus action T_k is Hamiltonian with moment map $\{\lambda_1^{(k)}, \dots, \lambda_k^{(k)}\}$. Angle coordinates on $\mathcal{H}_0$ corresponding to the global action coordinates $\lambda_i^{(k)}$ are defined by choosing a Lagrangian section σ of the Gelfand-Zeitlin map and defining $\psi_i^{(k)}(p) = 0$ for all $p \in \mathrm{Im}(\sigma)$.

Recall that for $K = \mathrm{SU}_n$, as a Poisson-Lie group with the standard Poisson structure π_K, has a Poisson-Lie dual $(K^* = AU_-, \pi_{K^*})$, where A is diagonal matrices with positive real entries and U_- is lower triangular unipotent matrices.

Denote by $\mathcal{H}^+$ the set of positive definite $n \times n$ Hermitian matrices with determinate being 1. Then the map

$$\phi\colon K^* \to \mathcal{H}^+ \;:\; b \mapsto bb^*, \tag{9.3}$$

is a diffeomorphism. Observed by [34], the functions $\ln(\lambda_i^{(k)})$, define a completely integrable system on $\mathcal{H}^+$ (equipped with the Poisson structure $\phi_*\pi_{K^*}$). Let $\mathrm{Sym}(n)$ be the set of symmetric $n \times n$ matrices and $\mathrm{Sym}_0(n) := \mathrm{Sym}(n) \cap \mathcal{H}_0$. This system was related to the Gelfand-Zeitlin system on $\mathcal{H}$ by the following

Theorem 9.2.1. *[10] There is exists a unique Poisson isomorphism $\gamma\colon \mathcal{H} \to \mathcal{H}^+$ such that*

- *γ intertwines the Gelfand-Zeitlin functions*

$$\lambda_i^{(k)}(A) = \ln\left(\lambda_i^{(k)}(\gamma(A))\right), \; \forall\, 1 \leqslant i \leqslant k \leqslant n. \tag{9.4}$$

- *γ intertwines the Gelfand-Zeitlin torus actions on $\mathcal{H}_0$ and $\mathcal{H}_0^+$.*
- *For any connected component $C \subseteq \mathrm{Sym}_0(n)$, $C \subseteq \mathcal{S}$.*

Moreover, the map γ is equivariant with respect to the conjugation action of $\mathrm{T}_n \subseteq \mathrm{U}_n$ and

$$\gamma(A + uI) = e^u \gamma(A), \quad \gamma(\overline{A}) = \overline{\gamma(A)}.$$

Note that the map $\phi^{-1} \circ \gamma$ is a Ginzburg-Weinstein diffeomorphism. Now let us define a family of Ginzburg-Weinstein diffeomorphisms

$$\mathrm{GW}_s\colon \mathcal{H} \to AU_-, \;:\; A \mapsto \phi^{-1}(\gamma(sA)), \tag{9.5}$$

For all $s < 0$, we have $(\mathrm{GW}_s)_*\pi_{\mathfrak{k}^*} = (\phi^{-1} \circ \gamma)_*(s\pi_{\mathfrak{k}^*}) = s\pi_{K^*}$. Let $\mathbf{i}_0$ be the *standard* double reduced word for (w_0, e), which is

$$\mathbf{i}_0 = (-1, \dots, -(n-1), -1, \dots, -(n-2), \dots, -1, -2, -1).$$

Recall that for $\mathbf{i}_0$, we have a seed $\sigma(\mathbf{i}_0)$ for $G^{w_0,e}$. Denote by Δ_k the cluster variables of seed $\sigma(\mathbf{i}_0)$, which are rational functions on B_-. Then we have:

Theorem 9.2.2. *For any fixed $A \in \mathscr{H}_0$ and $k \in [-(n-1), -1] \cup [1, m]$, the following limits exist*

$$\lambda_k(A) := \lim_{s \to -\infty} \frac{1}{s} \ln \left|\Delta_k(\mathrm{GW}_s(A))\right|; \quad \varphi_l(A) := \lim_{s \to -\infty} \arg\left(\Delta_l(\mathrm{GW}_s(A))\right),$$

hence are well defined functions on $\mathscr{H}_0$. Moreover, functions λ_k's (resp. φ_l) are linear combinations of $\lambda_p^{(q)}$ (resp. $\psi_p^{(q)}$), and the transformation matrices are unimodular.

Inspired by this theorem and the theory of partial tropicalization, we come up with the following

Conjecture 9.2.3. *Let K be a compact semisimple Lie group of rank r with standard Poisson structure π_K. Denote by $(K^* = AU_-, \pi_{K^*})$ its Poisson-Lie dual. For a given double reduced word $\mathbf{i}$ of (w_0, e), denote by Δ_k, for $k \in [-r, -1] \cup [1, \ell(w_0)]$, the cluster variables for some seed $\sigma \in |\sigma(\mathbf{i})|$. Then there exists a Ginzburg-Weinstein diffeomorphism $\mathrm{GW}_s \colon \mathfrak{k}^* \to K^*$ such that the following limits exist*

$$\lambda_k(A) := \lim_{s \to -\infty} \frac{1}{s} \ln \left|\Delta_k(\mathrm{GW}_s(A))\right|; \quad \varphi_l(A) := \lim_{s \to -\infty} \arg\left(\Delta_l(\mathrm{GW}_s(A))\right),$$

for A in an open dense subset $\mathcal{U}$ of $\mathfrak{k}^*$. Moreover, the functions λ_k's and φ_l's give a action-angle coordinates for $(\mathcal{U}, \pi_{\mathfrak{k}^*})$.

9.3 Symplectic leaves of π_∞

In this section, we shall take a closer look at the symplectic leaves of $(\mathrm{PT}(K^*), \pi_\infty)$ by combining the result in Chapter 7 and Chapter 8, where recall that

$$\mathrm{PT}(K^*) = \mathrm{PT}(G^*, \tau) = \left((B_-, \Phi_{BK}, \sigma(\mathbf{i}))^t_{\mathbb{R}}(0) \times (S^1)^m, \pi_\infty\right)$$

with Poisson bracket π_∞ described by (Proposition 7.7.2)

(1) The functions λ_k are Casimirs for $k \in [1, m] \setminus \boldsymbol{e}(\mathbf{i})$;

(2) $\{\lambda_k, \lambda_p\} = \{\varphi_k, \varphi_p\} = 0$, for all $k, p \in [-r, -1] \cup [1, m]$;

(3) The matrix $B = [\{\lambda_{k^-}, \varphi_l\}]_m$ is of the form $B = DB'$ for $B' \in U(\mathrm{GL}_m(\mathbb{Z}))$ and

$$D = \mathrm{diag}\left((\alpha_{i_1}, \omega_{i_1}), \ldots, (\alpha_{i_m}, \omega_{i_m})\right) = (1/d_{i_1}, \ldots, 1/d_{i_m}).$$

Recall that for (dual) charts $\sigma(\mathbf{i})$ of B_- and $\sigma^\vee(\mathbf{i})$ of $B_-^\vee$, we have an isomorphism of real BK cones (Theorem 8.3.1):

$$\psi_{\mathbf{i}} \colon (B_-, \Phi_{BK}, \sigma(\mathbf{i}))^t_{\mathbb{R}} \to (B_-^\vee, \Phi_{BK}^\vee, \sigma^\vee(\mathbf{i}))^t_{\mathbb{R}},$$

such that $(\mathrm{hw}^\vee)^t_{\mathbb{R}} \circ (\psi_{\mathbf{i}})_{\mathbb{R}} = \psi_{\mathbb{R}} \circ \mathrm{hw}^t_{\mathbb{R}}$ (Theorem 8.3.3), where $\psi \colon X_*(H) \to X_*(H^\vee)$ is the group homomorphism as in Proposition 2.1.3.

Now let us equip a *standard* Poisson structure on $(B_-^\vee, \Phi_{BK}^\vee, \sigma^\vee(\mathbf{i}))^t_{\mathbb{R}}(0) \times (S^1)^m$ as

$$\pi_{\mathrm{std}}^\vee := \sum_{k=1}^m \frac{\partial}{\partial \lambda_{k^-}^\vee} \wedge \frac{\partial}{\partial \varphi_k^\vee}.$$

Proposition 9.3.1. *There is a group automorphism η on $(S_1)^m$, such that*

$$\psi_{\mathbf{i}} \times \eta\colon \left((B_-, \Phi_{BK}, \sigma(\mathbf{i}))^t_{\mathbb{R}}(0) \times (S^1)^m, \pi_\infty\right) \to \left((B_-^\vee, \Phi_{BK}^\vee, \sigma^\vee(\mathbf{i}))^t_{\mathbb{R}}(0) \times (S^1)^m, \pi^\vee_{\mathrm{std}}\right)$$

is a Poisson isomorphism.

Proof. Note that $[\{\lambda_{k^-}, \varphi_l\}]_m = DB'$ and B' is unimodular. Thus B' defines a group automorphism η on $(S_1)^m$ such that

$$\pi_\infty := \sum_{k=1}^m \frac{1}{d_{i_k}} \frac{\partial}{\partial \lambda_{k^-}} \wedge \frac{\partial}{\partial \eta(\varphi_k)}.$$

Then we get the conclusion by the definition of $\psi_{\mathbf{i}}$. ♢

Let $\omega_\infty^{\lambda^\vee}$ be the constant symplectic structure on the symplectic leaf $\mathscr{P}_{\lambda^\vee} = \mathrm{hw}^{-t}(\lambda^\vee) \times (S^1)^m$. Denote by $\lambda := \psi(\lambda^\vee)$. As an immediate corollary, we have a symplectomorphism:

$$\left(\mathscr{P}_{\lambda^\vee}, \omega_\infty^{\lambda^\vee}\right) \cong \left(\triangle_\lambda \times (S^1)^m, \omega_{\mathrm{std}}\right)$$

where recall $\triangle_\lambda$ is the string polytope.

Recall ψ induces an isomorphism

$$-\mathbb{i}\psi\colon X_*^+(H) \otimes_{\mathbb{Z}} \mathbb{R} \mapsto \mathfrak{t}^*.$$

Then our purpose now is constructing action-angle coordinates on coadjoint. Let us align the notation with coadjoint orbit. For example, denote by $\mathscr{P}_\xi := \mathscr{P}_{(-\mathbb{i}\psi)^{-1}(\xi)}$ for $\xi \in \mathfrak{t}_+^*$.

Next, we discuss how the leaves $\mathscr{P}_\xi$ are related to the generic symplectic leaves of $K^* = AU_-$. Recall that the symplectic leaves on K^* are the level sets of the following Casimir functions [75]:

$$C_i^2(b) := \mathrm{Tr}\left(\rho_i\left(bb^*\right)\right), \text{ for } b \in \mathrm{Im}(K^* \hookrightarrow B_- \subset G) \text{ and } i \in \boldsymbol{I},$$

where ρ_i is the fundamental G-representation with highest weight ω_i and $b^* = \overline{b}^{-T}$.

Recall that in Section 7.7, we have the following detropicalization map:

$$\begin{aligned} \mathfrak{E}_{\vartheta,s}\colon \mathbb{R}^{r+m} \times (S^1)^m &\to K^* = AU_- \\ (\lambda_{-r}, \dots, \lambda_m, \varphi_1, \dots, \varphi_m) &\mapsto \vartheta\left(e^{s\lambda_{-r}}, \dots, e^{s\lambda_{-1}}, e^{s\lambda_1 + \mathbb{i}\varphi_1}, \dots, e^{s\lambda_m + \mathbb{i}\varphi_m}\right). \end{aligned}$$

where $\vartheta = \theta_0 \times \theta_1\colon \mathbb{G}_{\mathbf{m}}^r \times \mathbb{G}_{\mathbf{m}}^m \to H \times U_- \cong B_-$ is a toric chart of B_-. In the rest of this chapter, we fix a double reduced word $\mathbf{i}$ of (w_0, e) and let the toric chart ϑ be the cluster chart $\sigma(\mathbf{i})$. Thus we just write $\mathfrak{E}_s = \mathfrak{E}_{\vartheta,s}$ for simplicity. Direct computation shows

$$\begin{aligned} C_i^2(b) &= \sum_{j,k} |(\rho_i(b)_{jk}|^2 = \sum_{j,k} \sum_{\mathbf{j},\mathbf{k}} c_{\mathbf{j},\mathbf{k}} \left| \left(F_{\mathbf{j}} \cdot \Delta_{w_0\omega_i,\omega_i} \cdot F_{\mathbf{k}}\right)(b) \right|^2 \\ &= \Delta^2_{w_0\omega_i,\omega_i}(b) \left(1 + \sum_{j,k} \sum_{\mathbf{j},\mathbf{k}} c_{\mathbf{j},\mathbf{k}} \left| \frac{\left(F_{\mathbf{j}} \cdot \Delta_{w_0\omega_i,\omega_i} \cdot F_{\mathbf{k}}\right)(b)}{\Delta_{w_0\omega_i,\omega_i}(b)} \right|^2 \right), \end{aligned} \tag{9.6}$$

where $(\rho_i(b))_{jk}$ is entry of the matrix $\rho_i(b)$ at (j,k). The second sum is over some non-zero sequences of indices $\mathbf{j} = (j_1, \dots, j_p)$ and $\mathbf{k} = (k_1, \dots, k_q)$, and $F_{\mathbf{j}}$ is shorthand for $F_{j_i} \dots F_{j_p}$. Here we use the standard left and right action of $\mathfrak{g}$ on $\mathbb{C}[G]$.

Proposition 9.3.2. *For* $(\boldsymbol{\lambda}, \boldsymbol{\varphi}) \in \mathrm{PT}(K^*)$, *each term*

$$\left| \sum_{\mathbf{i},\mathbf{j}} c_{\mathbf{i},\mathbf{j}} \frac{F_{\mathbf{i}} \Delta_{w_0\omega_i,\omega_i} F_{\mathbf{j}}}{\Delta_{w_0\omega_i,\omega_i}} \left(\mathfrak{E}_{G^*,\theta,s}(\boldsymbol{\lambda}, \boldsymbol{\varphi})\right) \right| = O(e^{s\delta}).$$

Thus for $\xi \in \mathfrak{t}^*_+$ *and* $(\boldsymbol{\lambda}, \boldsymbol{\varphi}) \in \mathcal{P}_\xi$, *and for each* $i = 1, \ldots, r$,

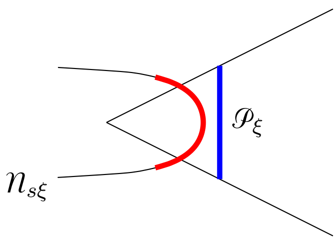
$$\lim_{s \to -\infty} \frac{1}{s} \ln \circ C_i \circ \mathfrak{E}_{G^*,\theta,s}(\boldsymbol{\lambda}, \boldsymbol{\varphi}) = (w_0\omega_i, \mathbb{i}\xi).$$

9.4 Symplectic leaves of π_s

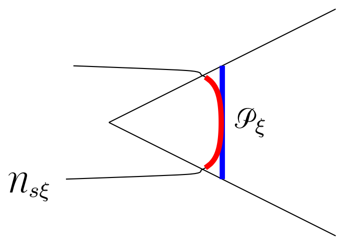
Following the notation from the previous section. Now we study the symplectic leaves of the Poisson bivector

$$\pi_s := (\mathfrak{E}_s)^*(s\pi_{K^*}).$$

on the space $\mathbb{R}^{r+m} \times (S^1)^m$. Roughly, for $s \ll 0$ each of these leaves has a piece closing to the corresponding leaf of $\mathrm{PT}(K^*)$. For $s \ll 0$, the volume of the symplectic leaves concentrate there, see Figure 9.1.



(a) $s = -1$



(b) $s = -2$

Figure 9.1: Volume of the symplectic leaves $\mathcal{N}_{s\xi}$ of π_s concentrates on the part of $\mathcal{N}_{s\xi}$ that is close to the corresponding tropical leaf $\mathcal{P}_\xi$.

First, recall that the symplectic leaves of (K^*, π_{K^*}) is the dressing orbit $\mathcal{D}_\xi$ for $\xi \in \mathfrak{t}^*_+$. Then the symplectic leaves of π_s is the preimage under $\mathfrak{E}_s$ of a dressing orbit. Denote the leaf and its symplectic form by

$$\mathcal{N}_{s\xi} := \mathfrak{E}_s^{-1}(\mathcal{D}_{s\xi}), \qquad \omega_s^\xi := (\pi_s)^{-1}.$$

Let $\mathcal{P}_\xi^\delta := \mathcal{P}_\xi \cap \mathrm{PT}(K^*)^\delta$ be the δ-interior of $\mathcal{P}_\xi$. The relation between leaves $\mathcal{P}_\xi$ and $\mathcal{N}_{s\xi}$ for $s \ll 0$ are stated by the following

Theorem 9.4.1. *For* $\delta > 0$, *there exists* s_δ, *such that for any* $s \leqslant s_\delta$, *there is a map of form*

$$\mathfrak{C}_s \colon \mathcal{P}_\xi^\delta \to \mathcal{N}_{s\xi} \ : \ (\boldsymbol{\lambda}, \boldsymbol{\varphi}) \mapsto (\mathfrak{c}_s(\boldsymbol{\lambda}, \boldsymbol{\varphi}), \boldsymbol{\varphi})$$

where $\boldsymbol{\lambda} := (\lambda_{-r}, \ldots, \lambda_m)$ *and* $\boldsymbol{\varphi} := (\varphi_{-1}, \ldots, \varphi_m)$, *satisfying:*

This section is based on a joint work [7] with A. Alekseev, B. Hoffman and J. Lane.

- $\mathfrak{C}_s$ *is a diffeomorphism to its image* $\mathcal{N}^{\delta}_{s\xi} := \mathrm{Im}(\mathfrak{C}_s)$*;*
- *at points in* $\mathcal{P}^{\delta}_{\xi}$*, we have* $(\mathfrak{C}_s)^*\omega^{\xi}_s = \omega^{\xi}_{\infty} + O(e^{s\delta})$*;*
- *the symplectic volume of* $\mathcal{N}^{\delta}_{s\xi}$ *satisfies*

$$\mathrm{Vol}(\mathcal{N}_{s\xi}, \omega^{\xi}_s) \geqslant \mathrm{Vol}(\mathcal{N}^{\delta}_{s\xi}, \omega^{\xi}_s) = \mathrm{Vol}(\mathcal{N}_{s\xi}, \omega^{\xi}_s) - \mathrm{Vol}(\mathcal{P}_{\xi} \setminus \mathcal{P}^{\delta}_{\xi}, \omega^{\xi}_{\infty}) + O(e^{\delta s}).$$

Before the proof of the theorem, we need some preparations. As discussed in previous section, the symplectic leaf $\mathcal{N}_{s\xi}$ can be described as the level set at $(w_0\omega_k, \mathrm{i}\xi)$ of function

$$f_k(\boldsymbol{\lambda}, \boldsymbol{\varphi}) := \frac{1}{s} \ln \circ C_k \circ \mathfrak{C}_s(\boldsymbol{\lambda}, \boldsymbol{\varphi}), \quad \forall k \in \boldsymbol{I}. \tag{9.7}$$

Denote by $J = \{j_1, \ldots, j_r\} = [1, m] \setminus \boldsymbol{e}(\mathbf{i})$ such that λ_{j_k} has weight ω_k. Then we compute

Lemma 9.4.2. *For all* $(\boldsymbol{\lambda}, \boldsymbol{\varphi}) \in \mathrm{PT}(K^*)^{\delta}$*, and* $k \in \boldsymbol{I}$ *the derivatives*

$$\partial_{\lambda_{j_k}} f_k = 1 + O(e^{2s\delta}); \quad \partial_{\lambda_j} f_k = O(e^{2s\delta}) \textit{ for } j \neq j_k; \quad \partial_{\varphi_j} f_k = O(e^{2s\delta}).$$

Proof. Taking Eq (9.6) into f_k, differentiating it gives

$$\frac{\partial f_k}{\partial \lambda_j} = e^{2s(\lambda_{j_k} - f_k)} \left(\delta_{j,j_k} + \sum_{l,j} \left(\frac{\partial L_{l,j}}{\partial \lambda_j} + \delta_{j,i_k} \right) c_{l,j} e^{2sL_{l,j}} \right);$$

$$\frac{\partial f_k}{\partial \varphi_j} = e^{2s(\lambda_{j_k} - f_k)} \sum_{l,j} \frac{\partial L_{l,j}}{\partial \varphi_j} c_{l,j} e^{2sL_{l,j}}.$$

where δ_{j,j_k} is the Kronecker-delta function, $c_{l,k}$'s are constants, and some linear combinations $L_{j,k}(\boldsymbol{\lambda}, \boldsymbol{\varphi})$. By Proposition 9.3.2, for $(\boldsymbol{\lambda}, \boldsymbol{\varphi}) \in \mathcal{C}^{\delta} \times \mathbb{T}^m$,

$$e^{2s(\lambda_{j_k} - f_k)} = 1 + O(e^{2s\delta}); \quad e^{2sL_{j,k}} = O(e^{2s\delta}),$$

which completes the proof. ♢

Next we recall an elementary result from calculus:

Lemma 9.4.3. *Consider a smooth family of maps* $\mathfrak{F}_s \colon \mathbb{R}^n \to \mathbb{R}^n$ *for* $s < 0$*. Denote by* $f^i_s \colon \mathbb{R}^n \to \mathbb{R}$ *the ith component of* F_s*. Suppose that there is convex open subset* $\mathcal{U}$ *such that for each* $i \in [1, n]$*,*

$$\partial_{x_i} f^i_s = 1 + O(e^{2s}); \quad \partial_{x_j} f^i_s = O(e^{2s\delta}) \quad \textit{for } j \neq i.$$

Then there exists a $s_0 < 0$ *such that for any* $s < s_0$*,* $\mathfrak{F}_s|_{\mathcal{U}}$ *is a diffeomorphism to its image.*

Proof. Fix s for a moment. Then only thing we need to show is that $\mathfrak{F}_s|_{\mathcal{U}}$ is injective. If not, let $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ be two points in $\mathcal{U}$ such that $\mathfrak{F}_s(\boldsymbol{a}) = \mathfrak{F}_s(\boldsymbol{b})$. For $0 \leqslant t \leqslant 1$, consider the line $t\boldsymbol{a} + (1-t)\boldsymbol{b}$ in $\mathcal{U}$ since $\mathcal{U}$ is convex. Let $g_i(t) = f^i_s(t\boldsymbol{a} + (1-t)\boldsymbol{b})$. Thus for each i, we have $g_i(0) = g_i(1)$. Then by Rolle's Theorem, there exists a t_i such that $g'_i(t_i) = 0$. Denote by $\boldsymbol{x}_i = t_i\boldsymbol{a} + (1-t_i)\boldsymbol{b}$. Then we know:

$$g'_i(t_i) = \sum_j (a_j - b_j) \partial_{x_j} f^i_s(\boldsymbol{x}_i) = 0. \tag{9.8}$$

Denote by M a matrix with the entry $M_{ji} = \partial_{x_j} f^i_s(\boldsymbol{x}_i)$. Thus $(a_1 - b_1, \ldots, a_n - b_n)M = 0$ by Eq (9.8). By the assumption of $\partial_{x_j} f^i_s$, one can choose $s \ll 0$ such that $\det(M) \neq 0$, which implies $\boldsymbol{a} = \boldsymbol{b}$. ♢

Denote by $I = [-r, -1] \cup \boldsymbol{e}(\mathbf{i})$, and split $\mathbb{R}^{r+m} = \mathbb{R}^J \times \mathbb{R}^I$. Write $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_J, \boldsymbol{\lambda}_I)$. For fixed $\boldsymbol{\lambda}_I$ and $\boldsymbol{\varphi}$, the intersection $\mathbb{R}^J \times \{\boldsymbol{\lambda}_I\} \times \{\boldsymbol{\varphi}\} \cap \mathcal{P}_\xi^\delta$ is either a point or empty, where the point is $(\lambda_{j_1}, \ldots, \lambda_{j_r}, \boldsymbol{\lambda}_I, \boldsymbol{\varphi})$ with $\lambda_{j_k} := (w_0\omega_k, \mathbb{i}\xi)$. Denote by $\boldsymbol{p} := (\boldsymbol{\lambda}_I, \boldsymbol{\varphi})$ and by $\mathcal{U}^\xi := \mathcal{U}^\xi(\delta) \subset \mathbb{R}^I \times (S^1)^m$ such that for any point $\boldsymbol{p} \in \mathcal{U}^\xi$, the intersection $\big(\mathbb{R}^J \times \{\boldsymbol{p}\}\big) \cap \mathcal{P}_\xi^\delta \neq \emptyset$. Then

$$\mathcal{P}_\xi^\delta = \Big(\mathbb{R}^J \times U^\xi(\delta)\Big) \cap \mathcal{P}_\xi.$$

Denote by $\mathcal{U}_{\boldsymbol{p}}^\xi := \mathbb{R}^J \times \{\boldsymbol{p}\}$ for $\boldsymbol{p} \in \mathcal{U}^\xi$. It is clear that $\mathcal{U}^\xi = \mathcal{U}' \times (S^1)^m$ for some convex open set $\mathcal{U}'$ in $\mathbb{R}^I$. The intersection $\mathcal{U}_{\boldsymbol{p}}^\xi \cap \mathcal{N}_{s\xi}$ is stated in the following Lemma.

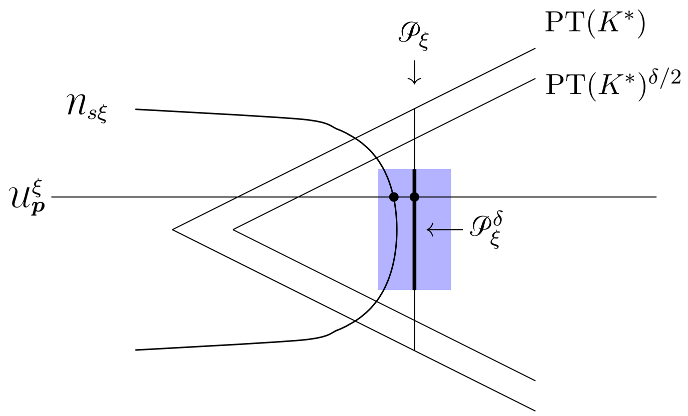


Figure 9.2: The intersection described in Lemma 9.4.4.

Lemma 9.4.4. *There exists $s_\delta < 0$ such that for all $s < s_\delta$, $\mathcal{N}_{s\xi} \cap \mathcal{U}_{\boldsymbol{p}}^\xi = \mathrm{pt}$ for any $\boldsymbol{p} \in \mathcal{U}^\xi$.*

Proof. First, we show $\mathcal{N}_{s\xi} \cap \mathcal{U}_{\boldsymbol{p}}^\xi \neq \emptyset$. Fix ε sufficiently small such that for any $\boldsymbol{p} \in \mathcal{U}^\xi$,

$$\mathcal{B} := [\lambda_{j_1} - \varepsilon, \lambda_{j_1} + \varepsilon] \times \cdots \times [\lambda_{j_r} - \varepsilon, \lambda_{j_r} + \varepsilon] \times \{\boldsymbol{p}\} \subset \mathrm{PT}(K^*)^{\delta/2};$$

where we recall that $\lambda_{j_k} := (w_0\omega_k, \mathbb{i}\xi)$. By Proposition 9.6, we have,

$$\lim_{s \to -\infty} f_k(\lambda_{j_1}, \ldots, \lambda_{j_k} \pm \varepsilon, \ldots, \lambda_{j_r}, \boldsymbol{p}) = \lambda_{j_k} \pm \varepsilon,$$

Thus there exists a s_δ, such that for $s < s_\delta$ the collection of functions $(f_1 - \lambda_{j_1}, \ldots, f_r - \lambda_{j_r})$ satisfies the assumptions of the Poincaré-Miranda Theorem on the box $\mathcal{B}$ for any $\boldsymbol{p} \in \mathcal{U}^\xi$. Thus there is a point in $\mathcal{U}_{\boldsymbol{p}}^\xi$ such that $(f_1 - \lambda_{j_1}, \ldots, f_r - \lambda_{j_r}) = 0$, which means $\mathcal{N}_{s\xi} \cap \mathcal{U}_{\boldsymbol{p}}^\xi \neq \emptyset$. If there are more than one points in $\mathcal{N}_{s\xi} \cap \mathcal{U}_{\boldsymbol{p}}^\xi \neq \emptyset$, say $(\boldsymbol{\lambda}_J, \boldsymbol{p})$ and $(\boldsymbol{\lambda}'_J, \boldsymbol{p})$ for example, thus we know that

$$(f_1, \ldots, f_r)(\boldsymbol{\lambda}'_J, \boldsymbol{p}) = (f_1, \ldots, f_r)(\boldsymbol{\lambda}_J, \boldsymbol{p}).$$

By previous lemma, we know this can not happen. ♢

Proof of Theorem 9.4.1. The first bullet is proved by applying the implicit function theorem to $\mathfrak{F}_s := (f_1, \ldots, f_r) \colon \mathbb{R}^J \times \mathbb{R}^I \times (S^1)^m \to \mathbb{R}^J$ for local result and gluing them together by Lemma 9.4.4. Thus, we know $\mathfrak{C}_s$ is of form

$$\mathfrak{C}_s \colon \mathcal{P}_\xi^\delta \to \mathcal{N}_{s\xi} \ : \ (\boldsymbol{\lambda}, \boldsymbol{\varphi}) \mapsto (\mathfrak{c}_s(\boldsymbol{\lambda}, \boldsymbol{\varphi}), \boldsymbol{\lambda}^I, \boldsymbol{\varphi})$$

where $\mathfrak{c}_s(\boldsymbol{\lambda}, \boldsymbol{\varphi}) \in \mathbb{R}^J$.

Next we prove the second bullet. For $(X,Y) \in T_{(\boldsymbol{\lambda},\boldsymbol{\varphi})}\mathcal{P}_\xi^\delta \cong \mathbb{R}^m \times \mathbb{R}^m$ with $(\boldsymbol{\lambda},\boldsymbol{\varphi}) \in \mathcal{P}_\xi^\delta$, by the implicit function theorem,

$$D_{(\boldsymbol{\lambda},\boldsymbol{\varphi})}\mathfrak{C}_s(X,Y) = \left(-(D_{\boldsymbol{\lambda}^J}\mathfrak{F}_s)^{-1}(D_{\boldsymbol{\lambda}^I}\mathfrak{F}_s X + D_{\boldsymbol{\varphi}}\mathfrak{F}_s Y), X, Y\right).$$

On the one hand, since the constant bivector π_∞ has form

$$\pi_\infty = \sum_k X_k \wedge Y_k, \quad \text{for some } \; X_k, Y_k \in T_{(\boldsymbol{\lambda},\boldsymbol{\varphi})}\mathcal{P}_\xi^\delta,$$

we find $(\mathfrak{C}_s)_*\pi_\infty = \pi_\infty + O(e^{s\delta})$ by Lemma 9.4.2. For the 2-form, we know

$$(\mathfrak{C}_s)_*\omega_\infty^\xi = ((\mathfrak{C}_s)_*\pi_\infty)^{-1} = \pi_\infty^{-1} + O(e^{s\delta}).$$

On the other hand, by Theorem 7.6.2, at $\mathfrak{C}_s(\boldsymbol{\lambda},\boldsymbol{\varphi}) \in \mathrm{PT}(K^*)^{\delta/2}$,

$$\omega_s^\xi = (\pi_s)^{-1} = \left(\pi_\infty + O(e^{s\delta})\right)^{-1} = \pi_\infty^{-1} + O(e^{s\delta}).$$

Let us show the last bullet now. The first inequality is clear since volume is monotonic. By the first two bullet

$$\mathrm{Vol}(\mathcal{N}_{s\xi}^\delta, \omega_s^\xi) = \mathrm{Vol}(\mathcal{P}_\xi^\delta, \omega_\infty^\xi) + O(e^{s\delta}).$$

Note that $\mathrm{Vol}(\mathcal{P}_\xi^\delta, \omega_\infty^\xi) = \mathrm{Vol}(\mathcal{P}_\xi, \omega_\infty^\xi) - \mathrm{Vol}(\mathcal{P}_\xi \setminus \mathcal{P}_\xi^\delta, \omega_\infty^\xi)$ since $\mathcal{P}_\xi^\delta = \mathcal{P}_\xi \cap \mathrm{PT}(K^*)^\delta$ by definition. Finally, by Theorem 8.5.3, we have $\mathrm{Vol}(\mathcal{P}_\xi, \omega_\infty^\xi) = \mathrm{Vol}(\mathcal{N}_{s\xi}, \omega_s^\xi)$. ♢

9.5 Construction of symplectic embeddings

Let K be a compact semisimple Lie group with a fix maximal torus T as before. Denote by $\mathcal{O}_\xi$ the regular coadjoint orbits labeled by an element ξ in the interior of the positive Weyl chamber $(\mathfrak{t}_+^*)^\circ$ of K. The coadjoint orbit $\mathcal{O}_\xi$ carries the Lie-Poisson form ω_ξ. The goal of this section is to complete the proof of

Theorem 9.5.1. *For any $\varepsilon > 0$ and $\xi \in (\mathfrak{t}_+^*)^\circ$, there exists $\delta > 0$ and a symplectic embedding*

$$\left(\mathcal{P}_\xi^\delta, \omega_\infty^\xi\right) \cong \left(\triangle_\xi(\delta) \times (S^1)^m, \omega_{\mathrm{std}}\right) \hookrightarrow (\mathcal{O}_\xi, \omega_\xi),$$

such that

$$\mathrm{Vol}(\mathcal{O}_\xi, \omega_\xi) > \mathrm{Vol}(\triangle_\xi(\delta) \times (S^1)^n, \omega_{\mathrm{std}}) > \mathrm{Vol}(\mathcal{O}_\xi, \omega_\xi) - \varepsilon.$$

Fix $\delta > 0$ and $\xi \in \mathfrak{t}_+^*$ as before, recall from last section, we have a diffeomorphism:

$$\mathfrak{C}_s \colon \mathcal{P}_\xi^\delta \to \mathcal{N}_{s\xi}^\delta.$$

Denote by $\kappa_s^\xi := (\mathfrak{C}_s)^*\omega_s^\xi$. The construction of symplectic embedding as in Theorem 9.5.1 will be accomplished by construction of the first arrow of

$$\left(\mathcal{P}_\xi^\delta, \omega_\infty^\xi\right) \xrightarrow{\mathfrak{L}_s} \left(\mathcal{P}_\xi^{\delta/2}, \kappa_s^\xi\right) \xrightarrow{\mathfrak{C}_s} \left(\mathcal{N}_{s\xi}^{\delta/2}, \omega_s^\xi\right) \xhookrightarrow{\mathfrak{E}_s} \left(\mathcal{D}_{s\xi}, (s\pi_{K^*})^{-1}\right) \xrightarrow{\mathrm{GW}_s} \left(\mathcal{O}_\xi, (\pi_{\mathfrak{k}^*})^{-1}\right).$$

This section is based on a joint work [8] with A. Alekseev, B. Hoffman and J. Lane.

In the other words, the goal is to construct a Moser flow that deforms κ_s^ξ to ω_∞^ξ. In the rest of this section, we omit ξ from the notation of symplectic forms for simplicity. For $s < 0$, define a closed 2-form ω_s^t on $\mathscr{P}_\xi^\delta$ by the equation

$$\omega_s^t = (1-t)\omega_\infty + t\kappa_s, \quad t \in [0,1].$$

Lemma 9.5.2. *For $\delta > 0$, $t \in [0,1]$, and $s \ll 0$, the form ω_s^τ is non-degenerate on $\mathscr{P}_\xi^\delta$.*

Proof. By Theorem 9.4.1, we know $\omega_s^t = (1-t)\omega_\infty + t(\omega_\infty + O(e^{s\delta})) = \omega_\infty + O(e^{s\delta})$ is non-degenerate for $s \ll 0$. ♢

Lemma 9.5.3. *The two form $\alpha_s := \kappa_s - \omega_\infty$ is exact on $\mathscr{P}_\xi^\delta$.*

Proof. The form ω_∞ is exact, and so it suffices to show that ω_s on $\mathcal{N}_{s\xi}$ is exact.

Recall that complex conjugation $\overline{(\cdot)}\colon K^* \to K^*$ is an anti-Poisson automorphism on K^*, which induces an anti-Poisson automorphism on $\mathbb{R}^{m+r} \times (S^1)^n$ via map $\mathfrak{E}_s$:

$$\tau\colon (\boldsymbol{\lambda}, \boldsymbol{\varphi}) \mapsto (\boldsymbol{\lambda}, -\boldsymbol{\varphi}).$$

Since Casimirs are invariant under τ, we have $\tau^*\omega_s = -\omega_s$. While, $H^2(\mathscr{P}_\xi^\delta) \cong H^2\left((S^1)^m\right)$ since $\triangle_\xi(\delta) \subset \mathbb{R}^m$ is contractible. Therefore $\tau^*[\omega] = [\omega]$ for any $[\omega] \in H^2(\mathscr{P}_\xi^\delta)$.

Put these together, the class $[\omega_s] \in H^2(\mathscr{P}_\xi^\delta)$ is 0 and hence ω_s is exact. ♢

Lemma 9.5.4. *Let $\sigma_s \in \Omega^l\left((S^1)^n\right)$ be a family of exact smooth forms, parametrized by $s < 0$ such that $\sigma_s = O(e^{s\delta})$ for some $\delta > 0$. Then, there exists a family $\gamma_s \in \Omega^{l-1}\left((S^1)^n\right)$ of smooth forms such that $d\gamma_s = \sigma_s$ and $\gamma_s = O(e^{s\delta})$.*

Proof. We would like to use the Fourier mode of the differential forms. Let us first introduce some standard notation from Fourier analysis on torus. Denote by $J = (j_1, \ldots, j_l)$ the multi-indices. Let $\boldsymbol{\varphi} := (\varphi_1, \ldots, \varphi_n)$ be the natural coordinates on $(S^1)^n$. Denote by $d\boldsymbol{\varphi}_J = d\varphi_{j_1} \wedge \cdots \wedge d\varphi_{j_l}$ a l-form. Thus any $\omega \in \Omega^l\left((S^1)^n\right)$ can be written as $\omega = \sum_J \omega_J d\boldsymbol{\varphi}_J$. For $\boldsymbol{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, let $\boldsymbol{m} \cdot \boldsymbol{\varphi} := m_1\varphi_1 + \cdots + m_n\varphi_n$. Then for any smooth function f on $\left((S^1)^n\right)$, let

$$\widehat{f}(\boldsymbol{m}) = \int_{(S^1)^n} f(\boldsymbol{\varphi}) e^{-2\pi \mathrm{i} \boldsymbol{m}\cdot\boldsymbol{\varphi}} d\boldsymbol{\varphi}$$

be the $\boldsymbol{m}^{\text{th}}$ Fourier coefficient of f. Then the Fourier expansion of f at $\boldsymbol{\varphi}$ is given by

$$f(\boldsymbol{\varphi}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^n} \widehat{f}(m) e^{2\pi \mathrm{i} \boldsymbol{m}\cdot\boldsymbol{\varphi}}.$$

Now one can write $\omega = \sum_{\boldsymbol{m}\in\mathbb{Z}^n} \omega_{\boldsymbol{m}}$, where $\omega_{\boldsymbol{m}} = e^{2\pi \mathrm{i} \boldsymbol{m}\cdot\boldsymbol{\varphi}} \sum_J \widehat{\omega_J}(\boldsymbol{m}) d\boldsymbol{\varphi}_J$, where the sum on the right converges uniformly because ω is smooth. If ω is exact, say $\omega = d\gamma$, we have

$$\omega_{\boldsymbol{m}} = d\gamma_{\boldsymbol{m}} \tag{9.9}$$

Note than $\omega_{\boldsymbol{0}}$ is exact if and only if $\omega_{\boldsymbol{0}} = 0$. Thus let $\gamma_{\boldsymbol{0}} = 0$. To find a primitive $\gamma_{\boldsymbol{m}}$ for $\boldsymbol{m} \neq \boldsymbol{0}$, one can make use of Cartan's magic formula, since the Lie derivative acts on $\omega_{\boldsymbol{m}}$ by a scale. To be more precise, for each $0 \neq \boldsymbol{m} = (m_1, \ldots, m_n)$, choose $j(\boldsymbol{m}) \in \{1, \ldots, n\}$ so that $m_{j(\boldsymbol{m})} \neq 0$. Then the following form satisfying Eq (9.9):

$$\gamma_{\boldsymbol{m}} := \frac{1}{2\pi \mathbb{i} m_{j(\boldsymbol{m})}} \iota_{X_{j(\boldsymbol{m})}} \omega_{\boldsymbol{m}}, \qquad \text{where } X_j := \partial/\partial\varphi_j.$$

Let K be a sequence of indices of length $l-1$. Because the Fourier series of ω converges uniformly, we have

$$2\pi \sum_{\boldsymbol{m}} |\widehat{\gamma_K}(\boldsymbol{m})| \leqslant \sum_{\boldsymbol{m}\neq \boldsymbol{0}} \frac{1}{m_{j(\boldsymbol{m})}} \sum_J |\widehat{\sigma_J}(\boldsymbol{m})| = \sum_J \sum_{\boldsymbol{m}\neq \boldsymbol{0}} \frac{|\widehat{\sigma_J}(\boldsymbol{m})|}{m_{j(\boldsymbol{m})}} < \sum_J \sum_{m\neq 0} |\widehat{\sigma_J}(\boldsymbol{m})| < \infty$$

and hence $\gamma := \sum_{\boldsymbol{m}} \gamma_{\boldsymbol{m}}$ is a well-defined smooth $l-1$ form.

Now suppose $\omega = \sigma_s$. Then by the construction we have a family of $l-1$ form γ_s. Let us control the size of γ_s. By Plancherel's identity and Parseval's relation, we have

$$||\gamma_K||^2 = \sum_{\boldsymbol{m}} |\widehat{\gamma_K}(\boldsymbol{m})|^2 \leqslant \sum_J \sum_{\boldsymbol{m}} |\hat{\sigma_J}(\boldsymbol{m})|^2 = \sum_J \int_{(S^1)n} ||\sigma_J||^2.$$

Since σ_s is $O(e^{s\delta})$, we conclude $\sum_J \int_{(S^1)n} ||\sigma_J||^2$ is $O(e^{2s\delta})$. Hence γ_s is $O(e^{s\delta})$. ♢

Lemma 9.5.5. *There exists a* 1*-form* $\beta_s \in \Omega^1(\mathscr{P}_\xi^\delta)$ *such that* $d\beta_s = \alpha_s$ *and* β_s *is* $O(e^{s\delta})$.

Proof. Fix a point $\boldsymbol{\lambda}_0 \in \triangle_\xi(\delta)$ and define a straight line retract from $\mathscr{P}_\xi^\delta$ to $\{\boldsymbol{\lambda}_0\} \times (S^1)^n$ as

$$\mathfrak{Q}\colon [0,1] \times \mathscr{P}_\xi^\delta \to \mathscr{P}_\xi^\delta \ : \ (t, \boldsymbol{\lambda}, \boldsymbol{\varphi}) \mapsto \mathfrak{Q}_t(\boldsymbol{\lambda}, \boldsymbol{\varphi}) := (\boldsymbol{\lambda}_0 + t(\boldsymbol{\lambda} - \boldsymbol{\lambda}_0), \boldsymbol{\varphi}).$$

Let $\mathfrak{q}\colon \Omega^\bullet(\mathscr{P}_\xi^\delta) \to \Omega^{\bullet-1}(\mathscr{P}_\xi^\delta)$ be the homotopy operator associated with $\mathfrak{Q}$, so

$$\gamma_s := \mathfrak{q}\alpha_s = \int_0^1 \left(\iota_{\frac{\partial}{\partial t}} \mathfrak{Q}^* \alpha_s\right) dt \in \Omega^1(\mathscr{P}_\xi^\delta).$$

Since α_s is $O(e^{s\delta})$ and $\triangle_\xi$ is bounded, the form γ_s is $O(e^{s\delta})$. Since $\mathfrak{q}$ is a homotopy operator and since α_s is closed, one has

$$\alpha_s = d\gamma_s + \mathfrak{Q}_0^* \alpha_s.$$

By Lemma 9.5.3, the form $\mathfrak{Q}_0^*\alpha_s$ is exact and is $O(e^{s\delta})$. So by Lemma 9.5.4, there is a form $\gamma_s' \in \Omega^1(\{\boldsymbol{\lambda}_0\} \times (S^1)^m)$ of $\mathcal{O}(e^{s\delta})$ satisfying $d\gamma_s' = \mathfrak{Q}_0^*\alpha_s$. Then $\beta_s := \gamma_s + \gamma_s'$ has the desired property. ♢

With all these preparations, we are now ready to run Moser's trick. Let us summaries what we have now (replace δ by $\delta/2$): for $s \ll 0$, the 2-form ω_s^t is closed and non-degenerate for all $t \in [0,1]$; and $\kappa_s - \omega_\infty = d\beta_s$ and β_s is $O(e^{s\delta/2})$. For any such s, Moser's equation

$$\iota_{X_s^t} \omega_s^t = -\beta_s \tag{9.10}$$

defines a t-dependent vector field X_s^t for $t \in [0,1]$ on $\mathscr{P}_\xi^{\delta/2}$. Denote by ϕ_s^t the flow of X_s^t for all points in $\mathscr{P}_\xi^{\delta/2}$ and all $t \in [0,1]$ for which it is defined.

Lemma 9.5.6. *The vector field* X_s^t *is* $O(e^{s\delta/2})$ *for all* t.

Proof. By (9.10), we have $\iota_{X_s^t}\omega_s^t = \iota_{X_s^t}\omega_\infty + t\iota_{X_s^t}\alpha_s = -\beta_s = O(e^{s\delta/2})$. Since α_s is $O(e^{s\delta/2})$ and ω_∞ is constant and none-degenerated, the vector field X_s^t has to be $O(e^{s\delta/2})$. $\diamondsuit$

Lemma 9.5.7. *For $\delta > 0$, there exists s_δ such that for any $s < s_\delta$,*

- *the flow $\phi_s^t|_{\mathscr{P}_\xi^\delta} : \mathscr{P}_\xi^\delta \to \mathscr{P}_\xi^{\delta/2}$ is defined for all $t \in [0,1]$;*
- *the time 1 flow ϕ_s^1 satisfies $(\phi_s^1)^*(\omega_s) = \omega_\infty$.*

Proof. Fix a metric on $(S^1)^m$ and equip $\mathscr{P}_\xi$ with the product metric. Since X_s^t is $O(e^{s\delta/2})$, we may choose $s \ll 0$ such that $||X_s^\tau|| < \delta/2$ at all points of $\mathscr{P}_\xi^\delta$. Then for all $(\boldsymbol{\lambda}, \boldsymbol{\varphi}) \in \mathscr{P}_\xi^\delta$ and all $t \in [0,1]$, the distance from $\phi_s^t(\boldsymbol{\lambda}, \boldsymbol{\varphi})$ to $(\boldsymbol{\lambda}, \boldsymbol{\varphi})$ is less than $\delta/2$. Therefore the flow of X_s^t, restricted to $\mathscr{P}_\xi^\delta$, does not escape $\mathscr{P}_\xi^{\delta/2}$ for $t \in [0,1]$. This establishes the first claim.

The second claim is due to the standard Moser argument: By (9.10), one has

$$d\iota_{X_s^t}\omega_s^t + d\beta_s = L_{X_s^t}\omega_s^t + \frac{\partial \omega_s^t}{\partial t} = 0,$$

therefore $(\phi_s^t)^*\omega_s^t = \omega_s^0 = \omega_\infty$ wherever the flow is defined. $\diamondsuit$

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