Geometry and topology of refined structures on the Hilbert scheme of points on the plane

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Geometry and topology of refined structures on the Hilbert scheme of points on the plane

THÈSE

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走的是道路，走的是海獭獭

獻給海獭獭與 AT

To my family and Cesco.
To Fredy, we will always miss you.

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Résumé

L'étude du schéma de Hilbert des points sur le plan complexe $\mathrm{Hilb}^{[n]}$ a été un sujet de recherche capital, avec d'abondants résultats et applications dans diverses branches des mathématiques. Plusieurs variétés connexes ont joué un rôle essentiel dans la géométrie énumérative, la théorie des singularités et la théorie des représentations. Par exemple les travaux de L. Göttsche, H. Nakajima, I. Grojnowski, G. Ellingsrud et S. A. Strømme associent la variété d'incidence à la représentation en dimension infinie de l'algèbre de Heisenberg sur les groupes d'homologie de $\mathrm{Hilb}^{[n]}$.

Le but de cette thèse est d'explorer la géométrie et la topologie du schéma de Hilbert raffiné

$$\mathrm{Hilb}^{[n,n+r]} := \left\{ (I,J) \in \mathrm{Hilb}^{[n]} \times \mathrm{Hilb}^{[n+r]} \,\middle|\, I \supset J \supseteq \langle x,y \rangle I \right\},$$

une variété liée au $\mathrm{Hilb}^{[n]}$ qui fait son apparition dans les travaux de H. Nakajima et K. Yoshioka sur l'espace de modules de certains faisceaux cohérents.

Dans la première partie de la thèse, nous révisons quelques propriétés géométriques et topologiques de $\mathrm{Hilb}^{[n,n+r]}$. Nous construisons deux types de correspondances sur ces espaces: le type de Nakajima et le nouveau type. Nous obtenons des résultats sur les opérateurs sur $\oplus_n H_*(\mathrm{Hilb}^{[n,n+r]})$ définits par ces correspondances; nous conjecturons que $\oplus_n H_*(\mathrm{Hilb}^{[n,n+r]})$ est une représentation irréductible de dimension infinie de cette algèbre d'opérateurs.

Nous introduisons aussi une description matricielle de $\mathrm{Hilb}^{[n,n+r]}$, analogue au cas de $\mathrm{Hilb}^{[n]}$. Considérant une décomposition cellulaire de $\mathrm{Hilb}^{[n,n+r]}$, nous construisons les coordonnées locales d'une cellule de dimension supérieure de $\mathrm{Hilb}^{[n,n+r]}$ dans cette description par matrices, fournissant un outil d'étude des structures près d'un point fixe du tore. Pour $r = 2$, inspirée par les exemples d'étude de cette question par ces coordonnées locales, nous identifions le fibré en droite correspondant au diviseur $\mathrm{Hilb}^{[n,n+2]}_{s\geq 2}$ consistant en les éléments (I,J) tel que la multiplicité de I en $(0,0)$ est au moins 2.

Dans la deuxième partie, nous présentons une formule pour les E-polynômes des strates de $\mathrm{Hilb}^{[n]}$ associées à un nombre fixe de générateurs en $(0,0)$. La preuve utilise une structure de fibré Grassmannien sur $\mathrm{Hilb}^{[n,n+r]}$ ainsi que la propriété motivique du E-polynôme. Ces formules suggèrent des connexions avec des formules modulaires et des formules de points fixes.

Abstract

The study of the Hilbert scheme of points on the plane $\mathrm{Hilb}^{[n]}$ has been one of the central research subjects with abundant results and applications across various branches of mathematics. Several related varieties have shown to play essential roles in enumerative geometry, singularity theory and representation theory by H. Nakajima, L. Göttsche. For instance, works of L. Göttsche, H. Nakajima, I. Grojnowski, G. Ellingsrud and S. A. Strømme relate the incidence variety to the infinite dimensional representation of the Heisenberg algebra on the homology groups of $\mathrm{Hilb}^{[n]}$.

This thesis aims to explore the geometry and topology of the refined Hilbert scheme

$$\mathrm{Hilb}^{[n,n+r]} := \left\{ (I,J) \in \mathrm{Hilb}^{[n]} \times \mathrm{Hilb}^{[n+r]} \,\middle|\, I \supset J \supseteq \langle x,y \rangle I \right\}$$

related to the Hilbert scheme of points on the plane $\mathrm{Hilb}^{[n]}$.

In the first part of the thesis, we review some geometric and topological properties of $\mathrm{Hilb}^{[n,n+r]}$. We obtain results on the operators on the direct sum $\oplus_n H_*(\mathrm{Hilb}^{[n,n+r]})$ induced by these correspondences; we conjecture that the space $\oplus_n H_*(\mathrm{Hilb}^{[n,n+r]})$ is an infinite-dimensional irreducible representation of this algebra of operators.

We also introduce a matrix description of $\mathrm{Hilb}^{[n,n+r]}$ which is analogous to the case of $\mathrm{Hilb}^{[n]}$. Considering a cell decomposition of $\mathrm{Hilb}^{[n,n+r]}$, we construct local coordinates of a top-dimensional cell of $\mathrm{Hilb}^{[n,n+r]}$ in this matrix description, which provides a tool for studying structures near a torus fixed point. For $r = 2$, inspired by examples of study this question through local coordinates, we identify the corresponding line bundle of the divisor $\mathrm{Hilb}^{[n,n+2]}_{s\geq 2}$ consisting of elements (I,J) such that the multiplicity of I at $(0,0)$ is at least 2.

In the second part, we present a formula for the E-polynomials of the strata of $\mathrm{Hilb}^{[n]}$ associated with a fixed number of generators. The proof uses a Grassmannian bundle structure on $\mathrm{Hilb}^{[n,n+r]}$ together with the motivic property of the E-polynomial. These formulas suggest connections with modular forms and fixed points formulas.

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Chapter 0

Introduction

Originally arising from algebraic geometry, the Hilbert scheme of points connects several fields in mathematics: enumerative geometry, singularity theory, combinatorics of 2-dimensional partitions, moduli spaces of sheaves, representation theory of infinite dimensional Lie-algebras and symmetric functions, as well as theoretical physics (e.g [13, 18, 20, 24, 30, 36]). In this thesis, we focus on the Hilbert scheme of points on the complex plane and investigate the topology and geometry of various related varieties. The first is the refined Hilbert scheme, an incidence variety of the Hilbert scheme of points that enjoys many similar properties to the usual Hilbert scheme. For this space, we have partial results of a structure of representation of Lie algebra on the direct sum of the homology groups as well as a matrix description, where the latter is used in examples of analyzing local structure of the refined Hilbert scheme. The second is the stratum of the Hilbert scheme of points related to the function of the minimum number of generators, where we establish a formula for the generating function of its E-polynomials.

The Hilbert scheme of the points on $\mathbb{C}^2$

Denote by $\mathrm{Hilb}^{[n]}$, *the Hilbert scheme of n points on* $\mathbb{C}^2$. Set-theoretically, $\mathrm{Hilb}^{[n]}$ can be described as the space of ideals of codimension n of the polynomial ring $R := \mathbb{C}[x, y]$:

$$\mathrm{Hilb}^{[n]} = \{I \trianglelefteq R \,\big|\, \dim_{\mathbb{C}} (R/I) = n\}.$$

The space $\mathrm{Hilb}^{[n]}$ is naturally endowed with the structure of a smooth $2n$-dimensional complex variety ([16]) and a universal sequence of R-modules

$$I \to R \to R/I$$

over $I \in \mathrm{Hilb}^{[n]}$. In addition, $\mathrm{Hilb}^{[n]}$ comes with a surjective morphism $\rho : \mathrm{Hilb}^{[n]} \to \mathrm{Sym}^n\mathbb{C}^2$ of $\mathrm{Hilb}^{[n]}$ to the symmetric product $\mathrm{Sym}^n\mathbb{C}^2$ of $\mathbb{C}^2$. Counted with multiplicity, the map ρ sends an ideal $I \in \mathrm{Hilb}^{[n]}$ to the underlying support of n points. We will write the elements of $\mathrm{Sym}^n\mathbb{C}^2$ as formal sums of points in $\mathbb{C}^2$.

The *Briançon variety* or *the punctual Hilbert scheme* $Br^{[n]}$ is the subvariety of $\mathrm{Hilb}^{[n]}$, which collects all ideals supported at $(0, 0)$:

$$Br^{[n]} = \{I \trianglelefteq R \,\big|\, \dim_{\mathbb{C}} (R/I) = n,\ \rho(I) = n\,[(0, 0)]\}.$$

The variety $Br^{[n]}$ is a deformation retract of $\mathrm{Hilb}^{[n]}$. It is $(n-1)$-dimensional, compact and irreducible. However, $Br^{[n]}$ is not smooth: $Br^{[n]}$ is non-singular only for $n = 1, 2$ (cf. [5,6]). The Hilbert schemes $\mathrm{Hilb}^{[n]}$ and $Br^{[n]}$ are two of the central objects of study of modern geometry; their geometry and topology have been intensively studied.

Like other moduli spaces, $\mathrm{Hilb}^{[n]}$ inherits structures from the base space $\mathbb{C}^2$. In fact, more structures appear when we consider all $\mathrm{Hilb}^{[1]}, \ldots, \mathrm{Hilb}^{[n]}, \ldots$ together. An example of this phenomenon is Göttche's formula for the Poincaré polynomials of $\mathrm{Hilb}^{[n]}$:

$$\sum_{n=0}^{\infty} P\left(\mathrm{Hilb}^{[n]}; t\right) q^n = \prod_{d=1}^{\infty} \frac{1}{(1 - t^{2d-2} q^d)} \tag{1}$$

where each polynomial $P\left(\mathrm{Hilb}^{[n]}; t\right)$ has no simple expression. This is linked to a representation of an infinite-dimensional Heisenberg algebra on the direct sum of homology groups of $\mathrm{Hilb}^{[n]}$ $\mathbb{H} := \oplus_n H_* \left(\mathrm{Hilb}^{[n]}\right)$ by I. Grojnowski ([20]) and H. Nakajima ([31]). It has a beautiful geometrical interpretation through the correspondences: convolutions on the Borel-Moore homology groups by incidence varieties in $\mathrm{Hilb}^{[n]} \times \mathrm{Hilb}^{[n+k]}$, $k \in \mathbb{Z}$:

$$\left\{(I, J) \in \mathrm{Hilb}^{[n]} \times \mathrm{Hilb}^{[n+k]} \,\middle|\, I \supset J,\ \rho\,(I/J) = k\,[p]\right\}$$

for some fixed $p \in \mathbb{C}^2$. The generating function in the formula (1) has an interpretation of a character of this algebra action on $\mathbb{H}$. Furthermore, the self-intersection numbers of $Br^{[n]}$ play an important role in the proof.

The refined Hilbert scheme

Among various related varieties, we are particularly interested in *the refined Hilbert scheme* $\mathrm{Hilb}^{[n,n+r]}$, a refined incidence variety of $\mathrm{Hilb}^{[n]} \times \mathrm{Hilb}^{[n+r]}$ that parameterize ideals (I, J) satisfying the inclusion $I \supset J$ and an extra condition: the quotient I/J is an r-dimensional $\mathbb{C}$ vector space with a trivial R-modules structure. Denoted by $\mathfrak{m} := \langle x, y \rangle$ the maximal ideal of R at $(0, 0)$, the *refined Hilbert scheme* is defined by

$$\mathrm{Hilb}^{[n,n+r]} := \left\{(I, J) \in \mathrm{Hilb}^{[n]} \times \mathrm{Hilb}^{[n+r]} \,\middle|\, I \supset J \supseteq \mathfrak{m}I\right\}.$$

It follows from the definition that if $(I, J) \in \mathrm{Hilb}^{[n,n+r]}$, then I/J is supported at $(0, 0)$.

The space $\mathrm{Hilb}^{[n,n+r]}$ is an example of the moduli spaces of perverse coherent sheaves on the blow-up of $\mathbb{P}^2$ studied by H. Nakajima and K. Yoshioka ([34]). In particular, they show the following results.

Theorem (H. Nakajima, K. Yoshioka, [34]). *The refined Hilbert scheme* $\mathrm{Hilb}^{[n,n+r]}$ *is smooth, irreducible and has complex dimension* $2n - r(r-1)$.

We shall mention that the refined Hilbert schemes have appeared in different settings. In [35] and [36], the same type of incidence varieties of the Hilbert scheme of points of a plane curve singularity C plays an important rule in studying connections of the HOMFLY polynomial with the link of C.

The algebraic torus $T \simeq (\mathbb{C}^*)^2$ acts on $\mathbb{C}^2$ by

$$(t_1, t_2) \cdot (x, y) \mapsto (t_1 x, t_2 y)$$

for $(t_1,t_2)\in T$, $(x,y)\in\mathbb{C}^2$. This induces an action on the refined Hilbert schemes $\mathrm{Hilb}^{[n,n+r]}$. For a generic choice of one-parameter subgroup $(t_1,t_2)\mapsto(t,t^N),\ N\gg 1$, the fixed points of the $\mathbb{C}^*$-action and T-action are the same. Moreover, we have an affine cell decomposition of $\mathrm{Hilb}^{[n,n+r]}$, where each T-fixed point represents an affine cell. As in the case of $\mathrm{Hilb}^{[n]}$, H. Nakajima and Yoshioka show that each affine cell is associated to a torus fixed point (which corresponds to a pair of monomial ideals) and the dimension of the affine cell is given by the number of positive weights of the isotropy representation of T on the tangent space of the fixed point. Using this fact, they determine the generating function of the Poincaré polynomials of $\mathrm{Hilb}^{[n,n+r]}$ which has the form

$$\sum_{n=\binom{r}{2}}^{\infty} P\left(\mathrm{Hilb}^{[n,n+r]};t\right)q^n = q^{\binom{r}{2}}\left(\prod_{d=1}^{\infty}\frac{1}{1-t^{2(d-1)}q^d}\right)\left(\prod_{d=1}^{r}\frac{1}{1-t^{2d}q^d}\right). \tag{2}$$

Operators on $\bigoplus_n H_*\left(\mathrm{Hilb}^{[n,n+r]}\right)$

The previous formula (2) for the generating function of $P\left(\mathrm{Hilb}^{[n,n+r]};t\right)$ is very suggestive, as it indicates an action of an infinite dimensional algebra on the direct sum of homology groups of $\mathrm{Hilb}^{[n,n+r]}$

$$\widetilde{\mathbb{H}} := \bigoplus_n H_*\left(\mathrm{Hilb}^{[n,n+r]}\right) \tag{3}$$

might exist. And there could be two "types" of operators: operators of Nakajima's type of bidgree $(d-1,d)$, and r operators of new type of bidgree (d,d) from the term $\prod_{d=1}^{r}\frac{1}{1-t^{2d}q^d}$ in the above formula. The Nakajima type operators are constructed through the following incidence varieties:

$$Z(r,k)_n := \left\{((I,J),(K,L),p)\in \mathrm{Hilb}^{[n,n+r]}\times\mathrm{Hilb}^{[n+k,n+k+r]}\times\mathbb{C}^2\backslash\{(0,0)\}\,\middle|\, I\supset K,\ \rho\left(I/K\right)=k\left[p\right]\right\}.$$

Denote by π_i, $i=1,2,3$, the projection from $\mathrm{Hilb}^{[n,n+r]}\times\mathrm{Hilb}^{[n+k,n+k+r]}\times\mathbb{C}^2\backslash\{(0,0)\}$ to its i-th factor. If $\alpha\in H_0\left(\mathbb{C}^2\right),\beta\in H_0^{BM}\left(\mathbb{C}^2\right)\simeq H^4\left(\mathbb{C}^2\right)$ and $u\in H_i\left(\mathrm{Hilb}^{[n]}\right)$, we define the operators $\beta_k(\cdot)$ of adding k points and the operators $\alpha_k(-)$ of subtracting k points:

$$\alpha_k(u) := \pi_{2*}\left(\left(\pi_1^*(\mathrm{PD}^{-1}u)\cup\pi_3^*(\mathrm{PD}^{-1}\alpha)\right)\cap\left[\overline{Z(r,k)_n}\right]\right)\in H_{i+(k-1)}\left(\mathrm{Hilb}^{[n+k,n+k+r]}\right) \tag{4}$$

$$\beta_k(u) := \pi_{1*}\left(\left(\pi_2^*(\mathrm{PD}^{-1}u)\cup\pi_3^*(\beta)\right)\cap\left[\overline{Z(r,k)_n}\right]\right)\in H_*\left(\mathrm{Hilb}^{[n-k,n-k+r]}\right). \tag{5}$$

Proposition. *Let $\alpha\in H_0^{BM}\left(\mathbb{C}^2\right),\beta\in H_0\left(\mathbb{C}^2\right)$. The operators α_k,β_k satisfy the Heisenberg relation*

$$[\alpha_i,\beta_j] = i\delta_{i+j,0}\langle\alpha,\beta\rangle\mathrm{id}_{\widetilde{\mathbb{H}}_r}$$

where the pairing $\langle\alpha,\beta\rangle$ stands for the push-forward $f_*(\alpha\cap\beta)$ of the unique morphism $f\colon\mathbb{C}^2\to\{pt\}$.

We note that the only nontrivial choice of such α and β is $\alpha=[pt],\beta=[\mathbb{C}^2]$.

Now, for the operators of new type w_j^r, the situation becomes more complicated. We define correspondences of new type:

Conjecture 1. *For $1 \leq j \leq r$, Let $w \in H_*(\{(0,0)\}) \simeq H_*(\{pt\})$ and $0 < j \leq r$. The new type operator $w_j^r : H_*\left(\mathrm{Hilb}^{[n,n+r]}\right) \to H_*\left(\mathrm{Hilb}^{[n+j,n+j+r]}\right)$ is given by the correspondence*

$$Q_{n,j}^r := \left\{ \begin{array}{c} ((I,J),(K,L)) \\ \in \mathrm{Hilb}^{[n,n+r]} \times \mathrm{Hilb}^{[n+j,n+j+r]} \end{array} \middle| \begin{array}{c} I \supseteq K \supseteq J \supseteq L \\ x, y \text{ acts trivially on } I/K, K/J, J/L \end{array} \right\},$$

where

$$w_j^r(y) := \pi_{2*}\left((\pi_1^*(\mathrm{PD}^{-1}y) \cup \pi_3^*(\mathrm{PD}^{-1}w)) \cap [Q_{n,j}^r] \right). \tag{6}$$

An important motivation for this conjecture is the following proposition, which implies that the operator w_j^r has homological degree $2j$ as expected.

Proposition 0.0.1. *The incidence variety $Q_{n,j}^r$ has complex dimension $2n + j - r(r-1)$.*

However, the Heisenberg commutation relations no longer hold and their algebra structure still needs to be identified. In particular, for the case of $r = 1$, W.-P. Li and Z. Qin used a similar construction of these operators for the nested Hilbert scheme of points of a smooth projective surface S in [28, 37], where the new operator does not satisfy the Heisenberg commutation relation. They showed in their case, that the old type operators together with the new operator generated $\oplus_n H^*\left(S^{[n]}\right)$.

We formulate the following conjecture:

Conjecture. *Let $v_0 \in \mathrm{Hilb}^{[N_0,N_0+r]}$, where $N_0 = \binom{r}{2}$. The operators α_k, w_j^r for $\alpha \in H_0\left(\mathbb{C}^2\right)$, $k \in \mathbb{N}, j = 1, \cdots, r$ act on $v_0 \in \mathrm{Hilb}^{[N_0,N_0+r]}$ spans $\widetilde{\mathbb{H}} = \bigoplus_n H_*\left(\mathrm{Hilb}^{[n,n+r]}\right)$ and the space $\widetilde{\mathbb{H}}$ is an infinite-dimensional irreducible representation of this algebra of operators.*

The refined Brainçon variety

Following from the fact that $(I,J) \in \mathrm{Hilb}^{[n,n+r]}$, I/J is supported at $(0,0)$, we define the refined Briançon variety:

Theorem-Definition. The refined Briançon variety

$$Br^{[n,n+r]} := \left\{ (I,J) \in \mathrm{Hilb}^{[n]} \times \mathrm{Hilb}^{[n+r]} \,\middle|\, I \supset J \supseteq \mathfrak{m}I,\ I/J \simeq \mathbb{C}^r,\ \rho(I) = [(0,0)] \right\}$$

is of complex dimension $n - \binom{r}{2}$ and admits an affine cell decomposition associated to fixed points of $\mathbb{C}^*$-action. Furthermore, the number of top-dimensional components of $Br^{[n,n+r]}$ is equal to the number of Young diagrams with $n + r$ boxes with r columns.

We shall show that the refined Briançon variety may be identified with the zero locus of a section of a vector bundle. Let $B \to \mathrm{Hilb}^{[n,n+r]}$ be the tautological bundle whose fiber at (I,J) is the quotient $B_{(I,J)} = R/I$. We define bundle map

$$\Sigma : B \to \mathcal{O}_{\mathrm{Hilb}^{[n,n+r]}},\ h \mapsto \sum_{p \in \mathrm{Supp}(I)} m_{p,I} h(p),$$

and the subundle $B' := \ker \Sigma$. Let $\gamma : B \to \mathcal{O}_{\mathrm{Hilb}^{[n,n+r]}}$, $h \mapsto h(0,0)$ be evaluation at $(0,0)$. This map is well-define since the support of the ideal I contains $(0,0)$ for any $(I,J) \in \mathrm{Hilb}^{[n,n+r]}$.

Theorem 3.3.4. *The refined Briançon variety $Br^{[n,n+r]}$ is the zero locus of the section γ^* of the bundle $(B^{'})^*$.*

A matrix description of the refined Hilbert scheme

Next, we shall show the refined Hilbert scheme admits a matrix description:

Theorem 3.2. *The refined Hilbert scheme* $\mathrm{Hilb}^{[n,n+r]}$ *is isomorphic to* $\widetilde{\mathcal{M}}/\mathrm{P}_{r,n}$ *where*

$$\widetilde{\mathcal{M}} := \left\{ (X,Y,i) \;\middle|\; \begin{array}{ll} (1) & \textit{The entries of } r \textit{ first columns of } X \textit{ and } Y \textit{ are 0.} \\ (2) & [X,Y]=0. \\ (3) & \textit{Stability condition: there exists no subspace } S \subsetneq V = \mathbb{C}^{n+r} \\ & \quad \textit{such that } X(S)\subset S,\ Y(S)\subset S \textit{ and } \mathrm{Im}(i)\subset S. \end{array} \right\},$$

for $X,Y \in \mathrm{Mat}_{n+r}(\mathbb{C})$ *and* $i \in \mathrm{Hom}(\mathbb{C},\mathbb{C}^n)$*, acting by the parabolic subgroup*

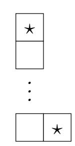
$$\mathrm{P}_{r,n} := \left\{ \left(\begin{array}{c|c} \mathbf{P_1} & \mathbf{P_2} \\ \hline \mathbf{0} & \mathbf{P_4} \end{array} \right) \in \mathrm{GL}_{n+r}(\mathbb{C}^2) \;\middle|\; P_1 \in \mathrm{GL}_r(\mathbb{C}), P_3 \in \mathrm{GL}_n(\mathbb{C}), P_2 \in \mathrm{Mat}_{r\times n}(\mathbb{C}) \right\}$$

by $g\cdot(X,Y,i) \mapsto \left(gXg^{-1}, gYg^{-1}, gi\right)$.

This description of $\mathrm{Hilb}^{[n,n+r]}$ as an orbit space of matrices gives us an idea of a way to study $\mathrm{Hilb}^{[n,n+r]}$ locally using reduction by $\mathrm{P}_{r,n}$. We get a representative (X,Y) of the orbits $(X,Y,i)\mathrm{P}_{r,n}$, which is parametrized by $2n-r(r-1)$ variables. We use this technique to calculate the examples $\mathrm{Hilb}^{[2,4]}$ and $\mathrm{Hilb}^{[3,5]}$.

Following the analysis in these examples, we investigate the possibility of constructing local coordinates for top-dimensional cells of $\mathrm{Hilb}^{[n,n+2]}$. These top-dimensional cells of $\mathrm{Hilb}^{[n,n+2]}$ correspond to Young diagrams $\triangle$ with two columns, where the associating partitions are (h_1,h_2) for $1 \le h_2 \le \frac{n+2}{2}+1$ and $h_1+h_2 = n+2$.

We get a result for the fixed point corresponds to the Young diagram $\triangle_{(I,J)} = (n+1,1)$.

Proposition 3.2.3. *If* $(I,J) \in \mathrm{Hilb}^{[n,n+2]}$ *is a torus fixed point of type* $\triangle_{(I,J)} =$  *, then local coordinates of the top dimensional cell associate to* (I,J) *in a small neighborhood is given by the formula:*

$$\mathbb{C}^{2n-2} \longrightarrow \widetilde{\mathcal{M}} \to \mathrm{Hilb}^{[n,n+2]} \tag{7}$$

$$(z_1,\dots,z_{n-1},w_1,\dots,w_{n-1}) \mapsto \left(\begin{pmatrix} 0 & 0 & z_1 & \dots & z_{n-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & * & \dots & * & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & w_1 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & w_{n-1} & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right), \tag{8}$$

where $* \in \mathbb{C}[z_1,\dots,z_{n-1},v_1,\dots v_{n-1}]$.

As the torus action on $\mathbb{C}^2$ lifts to $\widetilde{\mathcal{M}}/\mathrm{P}_{r,n}$, each entry of a representative (X,Y) has a T-weight.

We check that the list of the weights of local coordinates z_i, v_j in the proposition and the weights of the isotropy representation on $T_{\triangle_{(I,J)}}$ are indeed the same. Moreover, we develop a rule of collecting entries of X and Y with correct weights using the Young diagram $\triangle_{(I,J)}$ in Subsection 3.1.1. Based on these observations, we establish a conjecture for an arbitrary h_2:

Conjecture. *Let $(I,J) \in \mathrm{Hilb}^{[n,n+2]}$ be fixed point presenting a top dimensional cell with the corresponding partition (h_1,h_2), where h_2 satisfies $1 \leq h_2 \leq \frac{n+2}{2}+1$.*

We choose a numbering of the diagram $\triangle_{(I,J)}$ as follows: Counting from the highest box, the h_2 boxes of the first column are numbered by $1,3,\dots,1+2(h_2-1)$ and the h_2 boxes of the second column are numbered by $2,4,\dots,2+2(h_2-1)$.

Then the following entries of X and Y

$$\begin{cases} X_{1i}\ ,\ Y_{i3} & \text{for } i=3,\dots,1+2(h-1),\ 2h+1,\ \dots,\ 2h+n+2-3h = n+2-h \\ X_{2j}\ ,\ Y_{j4} & \text{for } j=4,\dots,\ 2+2(h-1) \\ X_{1k}\ ,\ Y_{k3} & \text{for } k=4,\dots,\ 2+2(h-1) \end{cases}$$

form local coordinates from $\mathbb{C}^{2(n-1)} \to \mathrm{Hilb}^{[n,n+2]}$ in a neighborhood of this fixed point.

Another result motivated by the previous local analysis of $\mathrm{Hilb}^{[2,4]}$, $\mathrm{Hilb}^{[3,5]}$, is a study of a special divisor $\mathrm{Hilb}^{[n,n+2]}_{s\geq 2}$ of $\mathrm{Hilb}^{[n,n+2]}$, which consists of element $(I,J) \in \mathrm{Hilb}^{[n,n+2]}$ such that the multiplicity of I at $(0,0)$ is at least two. The divisor $\mathrm{Hilb}^{[n,n+2]}_{s\geq 2}$ is in fact the set-theoretical support of the image $\pi_{n+2}\left(Q^2_{n,j=1}\right)$. In general, we have

Proposition 3.3.1. *The support of the image of the projection $\mathrm{Hilb}^{[n,n+r]} \times \mathrm{Hilb}^{[n+1,n+r+1]} \to \mathrm{Hilb}^{[n+1,n+r+1]}$ of the correspondence $Q^r_{n,j=1}$ of adding a point at $(0,0)$ is equal to the divisor $\mathrm{Hilb}^{[n+1,n+r+1]}_{s\geq\binom{r}{2}+1}$.*

Let F be the vector bundle over $\mathrm{Hilb}^{[n,n+2]}$, where the fibers are given by

$$F_{(I,J)} = I/J,$$

and let $M = \mathrm{Hilb}^{[n,n+2]} \times \left(\mathfrak{m}/\mathfrak{m}^2\right)$ be a trivial (but equivariantly non-trivial) vector bundle.

We identify the corresponding line bundle of $\mathrm{Hilb}^{[n,n+2]}_{s\geq 2}$:

Theorem 3.3.3. $\mathrm{Hilb}^{[n,n+2]}_{s\geq 2} \subset \mathrm{Hilb}^{[n,n+2]}$ *is, set-theoretically, the zero locus given by a section of $\left(\wedge^{top} F\right)^* \otimes \wedge^{top} M$.*

The E-polynomials of some strata of the Hilbert scheme

There is an extra structure that appears, which links $\mathrm{Hilb}^{[n,n+r]}$ and some refined strata of the usual Hilbert scheme $\mathrm{Hilb}^{[n]}$ when we consider the projection $\mathrm{Hilb}^{[n,n+r]} \to \mathrm{Hilb}^{[n]}$. Let $I \in \mathrm{Hilb}^{[n]}$. The *minimum number of generators of I at $(0,0)$*:

$$\mu(I) := \dim_{\mathbb{C}}(I/\mathfrak{m}I)$$

is a classical invariant that has been studied by A. Iarrobino in [26] for the case of Briançon varieties. Using the function μ as the grading, we define the strata of $\mathrm{Hilb}^{[n]}$ and $Br^{[n]}$:

$$\mathrm{Hilb}^{[n]}_m := \left\{ I \in \mathrm{Hilb}^{[n]} \,\middle|\, \mu(I) = m \right\} \ , \ Br^{[n]}_m := \left\{ I \in Br^{[n]} \,\middle|\, \mu(I) = m \right\}.$$

Let $\mathrm{Hilb}^{[n,n+r]}_m := \left\{ (I,J) \in \mathrm{Hilb}^{[n,n+r]} \,\middle|\, \mu(I) = m \right\}$. The fiber of the projection $\mathrm{Hilb}^{[n,n+r]}_m \to \mathrm{Hilb}^{[n]}_m$ at an ideal I having minimum $\mu(I) = m$ number of generators is isomorphic to the Grassmannian of r-dimensional subspace of $\mathbb{C}^m$. This makes $\mathrm{Hilb}^{[n,n+r]}_m \to \mathrm{Hilb}^{[n]}_m$ a $Gr_r(\mathbb{C}^m)$ Grassmannian bundle.

Using the knowledge of this bundle structure, we compute the *E-polynomial* (or the Hodge-Deligne polynomial) of the refined strata $Br^{[n]}_m$ and $\mathrm{Hilb}^{[n]}_m$.

The E-polynomial or the Hodge-Deligne polynomial of a complex algebraic variety Z is defined to be the compactly supported mixed Hodge polynomial specialized at $s = -1$,

$$E(Z; u, v) := \sum_{p,q;j} h_c^{p,q;j}(Z)(-1)^j u^p v^q,$$

where $h_c^{p,q;j}(Z)$ are the mixed Hodge numbers of Z. The E-polynomial is *motivic*. By this, we mean that $E(Z; u, v)$ is additive over stratifications and multiplicative for fibrations. In fact, mixed Hodge numbers of $\mathrm{Hilb}^{[n]}$ and $\mathrm{Hilb}^{[n,n+r]}$ vanish except when $p = q$. Their E-polynomials are completely determined by the monomial uv. For this reason, we adopt the simplified notation $E(Z; t)$ for the E-polynomial, where t is a variable of degree 2:

$$E(Z; t) := E\left(Z; \sqrt{t}, \sqrt{t}\right). \tag{9}$$

Using this Grassmannian bundle structure and the motivic properties of the E-polynomial, we compute a formula of the generating function for $E\left(Br^{[n]}_m; t\right)$ and $E\left(\mathrm{Hilb}^{[n]}_m; t\right)$.

Theorem 4.2.1. *The generating function of the E-polynomials of $Br^{[n]}_m$ has the form*

$$\sum_{n=0}^{\infty} E\left(Br^{[n]}_m; t\right) q^n = \prod_{i=1}^{m-1} \frac{1}{1 - t^{i+1}} \sum_{a=1}^{m} (-1)^{a+1} t^{\binom{a}{2}+m-1} \begin{bmatrix} m \\ a \end{bmatrix}_t \left(\prod_{k=0}^{\infty} \frac{1 - q^k t^{k-a}}{1 - q^k t^{k-1}} \right).$$

Theorem 4.2.3. *The generating function of the E-polynomials of* $\mathrm{Hilb}^{[n]}_m$ *has the form*

$$\sum_{n=0}^{\infty} E\left(\mathrm{Hilb}^{[n]}_m; t\right) q^n = \prod_{i=1}^{m-1} \frac{1}{1 - t^{i+1}} \sum_{a=1}^{m} (-1)^{a} t^{\binom{a}{2}+m} \begin{bmatrix} m \\ a \end{bmatrix}_t \left(\prod_{k=0}^{\infty} \frac{1 - q^k t^{k-a}}{1 - q^k t^{k+1}} \right).$$

We also found a formula for the generating function of the Euler characteristics $\chi\left(Br^{[n]}_m\right)$:

$$\sum_{n=0}^{\infty} \chi\left(B^{[n]}_m\right) q^n = \left(\prod_{d=1}^{\infty} \frac{1}{1 - q^d} \right) \sum_{k=0}^{\infty} \frac{(-1)^{k-m} q^{\binom{k}{2}}}{(q)_k} \binom{k}{m}. \tag{10}$$

The structure of the thesis:

- In Chapter 1, we review some classical results on the geometry and topology for the Hilbert scheme on the plane $\mathrm{Hilb}^{[n]}$ and the Briançon variety $Br^{[n]}$.

- In Chapter 2, we introduce the refined Hilbert schemes $\mathrm{Hilb}^{[n,n+r]}$ and study their geometry and topology with examples of $\mathrm{Hilb}^{[1,3]}$, $\mathrm{Hilb}^{[2,4]}$, $\mathrm{Hilb}^{[3,5]}$. We explain the algebraic torus action/cell decompositions. In the last part of this chapter, we focus on the theory of operators on the direct sum of homology groups of $\mathrm{Hilb}^{[n,n+r]}$ through correspondences. We give a conjecture associated with these operators.

- In Chapter 3, we first give a matrix description for $\mathrm{Hilb}^{[n,n+r]}$. Using this matrix description, we build examples of local coordinates of a top-dimensional cell of $\mathrm{Hilb}^{[n,n+r]}$. With this tool at hand, we first illustrate examples of analyzing the local structure of $\mathrm{Hilb}^{[2,4]}$ and $\mathrm{Hilb}^{[3,5]}$ using local coordinates through our matrix description. We propose a general formula for top dimension cells of $\mathrm{Hilb}^{[n,n+2]}$. After that, we study the divisor $\mathrm{Hilb}^{[n,n+2]}_{s\geq 2}$, motivated by a particular case of the correspondence $Q^r_{1,n}$ that we conjectured in Chapter 2.

- In Chapter 4, we investigate the relation between the strata $Br^{[n]}_m$, $\mathrm{Hilb}^{[n]}_m$ and the refined Hilbert scheme. Particularly, we illustrate that the projection $\mathrm{Hilb}^{[n,n+r]} \to \mathrm{Hilb}^{[n]}$ admits the structure of Grassmannian bundle $\mathrm{Gr}_r(\mathbb{C}^m)$ over the strata $\mathrm{Hilb}^{[n]}_m$. Using this property and the motivic property of the E-polynomial, we proceed to calculate the formula for the generating function of the E-polynomials. Lastly, we provide a formula of the Euler characteristics of $Br^{[n]}_m$. We list a table of examples of $E\left(Br^{[n]}_m; t\right)$ in Appendix A.

Chapter 1

Hilbert scheme of points on the affine plane

In this chapter, we review some results of the Hilbert scheme of points on the complex plane and the Briançon variety. These results include the statements of their geometric and topological properties, as well as the torus action and induced affine cool decomposition. The literature of the Hilbert scheme of points is vast. We finish the chapter with a geometric construction of operators on the direct sum of homology groups of Hilbert scheme.

1.1 The Hilbert scheme of points on the affine plane

The Hilbert schemes were first introduced, in more general context (not necessarily of points, nor on a surface), by A. Grothendieck in EGA [21] as the functor

$$h_P : (schemes)^\circ \longrightarrow (sets)$$

that associates to any scheme B over the base scheme S the set of subschemes $Y \subset P$ that are flat over B whose fiber over points of B have Hilbert polynomial P. He has shown in an important theorem that the functor h_P is representable by a projective scheme. In this thesis, we are only interested in the case where the base scheme is the complex plane $\mathbb{C}^2$. However, we should remark that many of the properties of the Hilbert scheme of points on the plane are valid as well for the general smooth quasi-projective surfaces. In fact for the two-dimensional case, the theory of Hilbert schemes of points on a smooth projective surface exploded once Fogarty proved following theorem.

Theorem 1.1.1 (Fogarty J. [16]). *The Hilbert scheme of points on a two-dimensional, smooth, quasi-projective scheme is smooth, irreducible and* $2n$*-dimensional.*

Denoted by $\mathrm{Hilb}^{[n]}$, the Hilbert scheme of points on the affine plane $\mathbb{C}^2$ is the moduli space parametrized 0-dimensional subschemes of $\mathbb{C}^2$. Generically, such a subscheme is a set of n distinct points on $\mathbb{C}^2$. The Hilbert scheme $\mathrm{Hilb}^{[n]}$ has a natural scheme structure with a universal sequence of R

$$I \to R \to R/I$$

over $I \in \mathrm{Hilb}^{[n]}$.

The $\mathrm{Hilb}^{[n]}$ as a set

Despite the fact that its original definition arises in algebraic geometry, the Hilbert scheme $\mathrm{Hilb}^{[n]}$ has a rather simple set-theoretical description and is very useful. Let $R := \mathbb{C}[x, y]$ be the ring of polynomials in variables x and y.

Definition 1.1.2. The Hilbert scheme of points on $\mathbb{C}^2$ $\mathrm{Hilb}^{[n]}$ is defined as the set of ideals of codimension n,

$$\mathrm{Hilb}^{[n]} := \{I \trianglelefteq R \,|\, \dim_{\mathbb{C}} (R/I) = n\}.$$

Example 1.1.3. There are two types of ideals in the Hilbert scheme of two points $\mathrm{Hilb}^{[2]}$. One is an ideal given by two distinct points (a, b), (a', b') on $\mathbb{C}^2$ with the corresponding ideal

$$I = \langle x - a, y - b\rangle \cap \langle x - a', y - b'\rangle.$$

When these points coincide $(a, b) = (a', b')$ and we have a "fat" point, it gives the second type an ideal of the form

$$I = \left\{f \in R \,\middle|\, f(a, b) = 0, df_{(a,b)}(v) = 0\right\}$$

for some $(a, b) \in \mathbb{C}^2$ and $v \neq 0 \in T_{(a,b)}\mathbb{C}^2$.

Example 1.1.4. A **monomial ideal** is an ideal $I \trianglelefteq R$ generated by monomials. They form an important set of examples of ideals in $\mathrm{Hilb}^{[n]}$. If I be a monomial ideal, then the quotient R/I is a vector space span by the monomials that does not lie in I. Thus I defines an element of $\mathbb{C}^{2^{[\dim_{\mathbb{C}}(R/I)]}}$.

Following from the Grothendieck's results, the Hilbert scheme $\mathrm{Hilb}^{[n]}$ has a universal family F_n:

Definition 1.1.5. The universal family $F_n = \{(I, p) \,|\, p \in \mathrm{Supp}(I)\}$

$$\begin{array}{c} F_n \subseteq \mathrm{Hilb}^{[n]} \times \mathbb{C}^2 \\ \big\downarrow \\ \mathrm{Hilb}^{[n]} \end{array}$$

is the unique subscheme of $\mathrm{Hilb}^{[n]} \times \mathbb{C}^2$ such that the projection $F_n \to \mathrm{Hilb}^{[n]}$ is a flat with fiber at $I \in \mathrm{Hilb}^{[n]}$ is the underlying subscheme of points.

We first prove Fogarty's theorem 1.1.1 for the case of $\mathbb{C}^2$.

Theorem 1.1.1 for $\mathbb{C}^2$. *The Hilbert scheme of n points on $\mathbb{C}^2$ is nonsingular, irreducible and of complex dimension $2n$.*

Proof by Nakajaima, [30]. Let $I_Z \in \mathrm{Hilb}^{[n]}$ and denote by Z the underlying subscheme. The structure sheaf of Z is $\mathcal{O}_Z = R/I_Z \subset \mathcal{O}_{\mathbb{C}^2} = R$. The Zariski tangent space $\mathrm{Hom}\left(I_Z, I/I_Z^2\right)$ of $\mathrm{Hilb}^{[n]}$ at I_Z is isomorphic to $\mathrm{Hom}\,(I_Z, \mathcal{O}_Z)$. To prove the smoothness of $\mathrm{Hilb}^{[n]}$, it is sufficient to claim that $\dim_{\mathbb{C}} T_{I_Z}\mathrm{Hilb}^{[n]} = \dim_{\mathbb{C}} \mathrm{Hom}\,(I_Z, \mathcal{O}_Z)$ is $2n$ and is independent of choice of $I_Z \in \mathrm{Hilb}^{[n]}$.

To show this, we apply the contravairant functor $\mathrm{Hom}\,(-, \mathcal{O}_Z)$ to the exact sequence

$$0 \to I_Z \to \mathcal{O}_{\mathbb{C}^2} \to \mathcal{O}_Z \to 0,$$

and observe the long exact sequence

$$\begin{aligned}(\star) : 0 \to &\operatorname{Hom}(\mathcal{O}_Z, \mathcal{O}_Z) \to \operatorname{Hom}(\mathcal{O}_{\mathbb{C}^2}, \mathcal{O}_Z) \to \underline{\operatorname{Hom}(I_Z, \mathcal{O}_Z)} \\ \to &\operatorname{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) \to \operatorname{Ext}^1(\mathcal{O}_{\mathbb{C}^2}, \mathcal{O}_Z) \to \operatorname{Ext}^1(I_Z, \mathcal{O}_Z) \\ \to &\operatorname{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) \to \operatorname{Ext}^2(\mathcal{O}_{\mathbb{C}^2}, \mathcal{O}_Z) \to \operatorname{Ext}^2(I_Z, \mathcal{O}_Z) \to 0.\end{aligned}$$

For $i \geq 0$ and $\mathcal{F}$ an $\mathcal{O}_{\mathbb{C}^2}$-module, there are isomorphisms $\operatorname{Ext}^i(\mathcal{O}_{\mathbb{C}^2}, \mathcal{F}) \simeq H^i(X, \mathcal{F})$. Since X in affine, $H^i(X, \mathcal{O}_Z) = 0$ for all $i \geq 1$. We have $\operatorname{Ext}^1(\mathcal{O}_{\mathbb{C}^2}, \mathcal{O}_Z) \simeq H^i(X, \mathcal{O}_Z) = 0$ for $i = 1, 2$. Then the long exact sequence $(\star)$ becomes

$$\begin{aligned}0 \to \operatorname{Hom}(\mathcal{O}_Z, \mathcal{O}_Z) \to \operatorname{Hom}(\mathcal{O}_{\mathbb{C}^2}, \mathcal{O}_Z) \to \underline{\operatorname{Hom}(I_Z, \mathcal{O}_Z)} \to \operatorname{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) \to 0 \to \operatorname{Ext}^1(I_Z, \mathcal{O}_Z) \\ \to \operatorname{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) \to 0 \to \operatorname{Ext}^2(I, \mathcal{O}_Z) \to 0,\end{aligned}$$

Since $\mathbb{C}^n \simeq \operatorname{Hom}(\mathcal{O}_Z, \mathcal{O}_Z) \to \operatorname{Hom}(\mathcal{O}_{\mathbb{C}^2}, \mathcal{O}_Z) \simeq \mathcal{O}_Z \simeq \mathbb{C}^n$ is injective, we have

$$\operatorname{Hom}(\mathcal{O}_Z, \mathcal{O}_Z) \simeq \operatorname{Hom}(\mathcal{O}_{\mathbb{C}^2}, \mathcal{O}_Z).$$

These imply the following

- $\operatorname{Hom}(\mathcal{O}_Z, \mathcal{O}_Z) \simeq \operatorname{Hom}(\mathcal{O}_{\mathbb{C}^2}, \mathcal{O}_Z) \simeq \mathbb{C}^n$,
- $\operatorname{Hom}(I_Z, \mathcal{O}_Z) \simeq \operatorname{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z)$,
- $\operatorname{Ext}^1(I, \mathcal{O}_Z) \simeq \operatorname{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z)$,
- $\operatorname{Ext}^2(I, \mathcal{O}_Z) = 0$.

Since X is quasi-projective (it is an open subset of $\mathbb{P}^2$), we have the duality

$$\operatorname{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) \xrightarrow{\sim} \operatorname{Hom}(\mathcal{O}_{\mathbb{C}^2}, \mathcal{O}_Z \otimes K_{\mathbb{C}^2})^\vee \simeq \operatorname{Hom}(\mathcal{O}_{\mathbb{C}^2}, \mathcal{O}_Z)^\vee \simeq \mathbb{C}^n.$$

Now the sum $\chi(\mathcal{O}_Z, \mathcal{O}_Z) = \sum_{i=0}^2 (-1)^i \operatorname{Ext}^i(\mathcal{O}_Z, \mathcal{O}_Z)$ is the Euler characteristic, which is independent of choice of I by flatness.

We claim that $\chi(\mathcal{O}_Z, \mathcal{O}_Z) = 0$. $\mathcal{O}_Z$ admits a projective resolution $P_\bullet \to \mathcal{O}_Z \to 0$. We have that

$$\chi(\mathcal{O}_Z, \mathcal{O}_Z) = \sum_i (-1)^i \dim_{\mathbb{C}} \operatorname{Hom}(P_i, \mathcal{O}_Z) = \sum_i (-1)^i n \cdot \operatorname{rk}(P_i)$$

Then the dimension

$$\begin{aligned}\dim_{\mathbb{C}} T_I \operatorname{Hilb}^{[n]} &= \dim_{\mathbb{C}} \operatorname{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) \\ &= -\chi(\mathcal{O}_Z, \mathcal{O}_Z) + \dim_{\mathbb{C}} \operatorname{Hom}(\mathcal{O}_Z, \mathcal{O}_Z) + \dim_{\mathbb{C}} \operatorname{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) \\ &= -\chi(\mathcal{O}_Z, \mathcal{O}_Z) + n + n = 2n.\end{aligned}$$

is independent of the element $I_Z \in \operatorname{Hilb}^{[n]}$. We conclude that $\operatorname{Hilb}^{[n]}$ is smooth and of dimension $2n$.

■

The $\mathrm{Hilb}^{[n]}$ **as a scheme: an open affine cover and local coordinates**

We describe the scheme structure of $\mathrm{Hilb}^{[n]}$ through local coordinates on open affine subsets by M. Haiman ([22]) indexed by monomial ideals: ideals generated by the monomials. We first explain how monomial ideals of codimension n relate to Young diagrams with n box and to partitions of n.

Definition 1.1.6. A *Young diagram* $\triangle$, also called Ferrers diagram, is a finite collection of boxes that are arranged in left-justified columns such that the lengths of column increasing from the left to the right.

Remark 1.1.7. There are two conventions of representing the diagrams. The English notation uses matrix-like indices, and the French notation uses Cartesian coordinate-like indices for the boxes $b_{i,j}$ in the diagram. Here, we adopt the French notation.

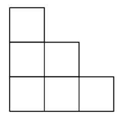


Figure 1.1: Example: Young diagram of the partition $3 \geq 2 \geq 1$ in French notation.

On one hand, there is a one-to-one correspondence between monomial ideals and Young diagrams: We first observe that the monomials $x^k y^h$ are in bijection with points $(h,k) \in Z^2_{\geq 0}$ by

$$x^k y^h \leftrightarrow (h,k).$$

Given a monomial ideal $I \in \mathrm{Hilb}^{[n]}$, for each monomial $x^k y^h$ not in I, we assign a box at the corresponding index $(h,k) \in \mathbb{Z}^2_{\geq 0}$ and get a diagram with n boxes. Since $\mathfrak{m}I \subset I$, the resulting diagram is a Young diagram.

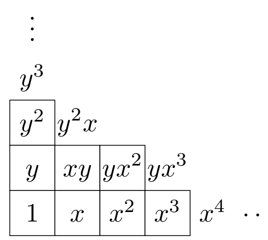


Figure 1.2: The diagram correspondence to the ideal $\langle y^3, y^2x, yx^3, x^3\rangle$.

Next, we explain how the Young diagram represent graphically an integer partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s)$.

Definition 1.1.8. A *partition* λ *of a positive integer number* n is an increasing sequence of positive integers $\lambda = (\lambda_1 \geq \lambda_2 \cdots \geq \lambda_{s-1} \geq \lambda_s)$ such that $\sum_{i=1}^{s} \lambda_i = n$.

The notation "$\lambda \vdash n$" means that λ is a partition of n and we say that $|\lambda| := \sum_{i=1}^{s} \lambda_i$ is the size of the partition λ.

If λ is a partition of n, then the Young diagram $\triangle_\lambda$ has n boxes. To be more precise about

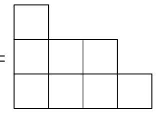
$\lambda = (3, 2, 2, 1) \Leftrightarrow \triangle_\lambda =$ 

Figure 1.3: An example of the correspondence between the partition and the Young diagram.

representing a partition by Young diagram, we establish a bijection between Young diagrams with n boxes and integer partitions of n: Given a partition of n, $\lambda = (\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_s)$, we assign the Young diagram such that i-th column (counting from the left to the right) has λ_i boxes (Figure **??**).

Accordingly, we have the correspondence between monomial ideals, partition and Young diagrams:

$$I_\lambda \Leftrightarrow \triangle_{\lambda_I} \Leftrightarrow \lambda = (\lambda_0 \geq \lambda_2 \cdots \geq \lambda_{s-1} \geq \lambda_s)\,, \tag{1.1}$$

and without confusions, we write an element $(h, k) \in \lambda$ as the corresponding index of box in $\triangle_\lambda$ and $x^h y^k$ the associated monomial in R/I.

We now describe Haiman's local coordinates. First, we assign a subset of monomials to a partition λ of n:

$$\mathcal{B}_\lambda = \{x^h y^k \,\big|\, (h, k) \in \lambda\}, \ \lambda \vdash n. \tag{1.2}$$

We construct open affine sets U_λ around each fixed point so that the coordinate rings on U_λ are algebraic functions of the ideals $I \in U_\lambda$.

For each partition λ of n, we define

$$U_\lambda := \left\{ I \in \mathrm{Hilb}^{[n]} \,\middle|\, \mathcal{B}_\lambda \text{ forms a vector space basis of } R/I \right\}. \tag{1.3}$$

Let $I \in \mathrm{Hilb}^{[n]}$ and assume that $R/I \simeq \mathcal{B}_\lambda$. For each $(i, j) \notin \lambda \subset \mathbb{N}^2$, $(h, k) \in \lambda$, there exists well-defined constants $C^{ij}_{hk} \in \mathbb{C}$ such that

$$x^i y^j \equiv \sum_{(h,k)\in\lambda} C^{i,j}_{h,k} x^h y^k \mod I. \tag{1.4}$$

The fact that I is an ideal, it requires $\langle x, y\rangle I \subset I$ which indicates relations among $C^{i,j}_{h,k}$. Multiplying $x^i y^j - \sum_{(h,k)\in\lambda} C^{i,j}_{h,k} x^h y^k$ by monomial $x^a y^b$ yields

$$= x^{i+a} y^{j+b} - \sum_{(h',k')} C^{i,j}_{h,k} C^{h+a,k+b}_{h',k'} x^{h'} y^{k'} \mod I.$$

Comparing with the coefficients of $x^{i+a} y^{j+b} - \sum_{(h,k)\in\lambda} C^{i+a,j+b}_{h,k} x^h y^k$, we get relations

$$C^{i+a,j+b}_{h',k'} = \sum_{(h,k)} C^{i,j}_{h,k} C^{h+a,k+b}_{h',k'}.$$

The condition $\langle x, y\rangle I \subset I$ is then equivalent to $C^{i,j}_{h,k}$ satisfying relations:

$$\begin{cases} C^{i+1,j}_{h',k'} = \displaystyle\sum_{(h,k)\in\lambda} C^{i,j}_{h,k} C^{h+1,k}_{h,k'} \\ C^{i,j+1}_{h',k'} = \displaystyle\sum_{(h,k)\in\lambda} C^{i,j}_{h,k} C^{h,k+1}_{h,k} \end{cases}. \tag{1.5}$$

Conversely, if $C_{hk}^{ij} \in \mathbb{C}$ satisfy the equations in 1.5, then I, as a vector space, is spanned by well-defined elements

$$x^i y^j - \sum_{(h,k)\in\lambda} C_{h,k}^{i,j} x^h y^k \mod I. \tag{1.6}$$

is indeed an ideal. More importantly, we have the following proposition regarding U_λ:

Proposition 1.1.9 ([1, 22]). *The sets U_λ are open affine covering of $\mathrm{Hilb}^{[n]} = \cup_{\lambda\vdash n} U_\lambda$. The coordinates ring $\mathcal{O}_{U_\lambda}$ is generated by the function $C_{h,k}^{i,j}$, $(h,k) \in \lambda$ and $(i,j) \in \mathbb{N}^2$, that is*

$$U_\lambda = \mathrm{Spec}\mathbb{C}\left[\left(C_{h,k}^{i,j}\right)_{(h,k)\in\lambda}\right] \Big/ (\textit{equations 1.5}). \tag{1.7}$$

The Hilbert-Chow morphism

Let $\mathrm{Sym}^n\mathbb{C}^2$ be the n-th symmetric product of $\mathbb{C}^2$. It is given by the quotient

$$\mathrm{Sym}^n\mathbb{C}^2 := \left(\mathbb{C}^2\right)^n \Big/ S_n$$

where the symmetric group S_n acts by permuting the points in $\left(\mathbb{C}^2\right)^n$. It can be viewed as a compactification of the space parametrizes unordered n-tuple of points in $\mathbb{C}^2$. $\mathrm{Sym}^n\mathbb{C}^2$ is not smooth. Namely, it has singularities along the diagonal where some of the points coincide.

The symmetric product has a natural stratification indexed by partitions λ of n:

$$\mathrm{Sym}^n\mathbb{C}^2 = \bigcup_{\lambda\vdash n} \mathrm{Sym}^n_\lambda\mathbb{C}^2,$$

where $\mathrm{Sym}^n_\lambda\mathbb{C}^2 := \left\{\sum_k \lambda_k[z_k] \,\middle|\, z_k \in \mathbb{C}^2, z_i \neq z_j \text{ if } i \neq j\right\}$. For example, if $\lambda = (1,\dots,1) = (1^n)$, then $\mathrm{Sym}^n_{(1^n)}\mathbb{C}^2$ is a non-singular locus of $\mathrm{Sym}^n\mathbb{C}^2$: the open set where n points are all different.

The Hilbert scheme $\mathrm{Hilb}^{[n]}$ is closely related to the symmetric product. There is the Hilbert-Chow morphism from $\mathrm{Hilb}^{[n]}$ to $\mathrm{Sym}^n\mathbb{C}^2$ by associate each ideal $I \in \mathrm{Hilb}^{[n]}$, an unordered n-tuple

$$\begin{aligned}\rho : \mathrm{Hilb}^{[n]} &\to \mathrm{Sym}^n\mathbb{C}^2\\ I &\mapsto \sum_{p\in\mathrm{Supp}(I)} m_{p,I}[p]\end{aligned}$$

where $m_{p,I}$ is the multiplicity of I at the point p which is the length of the local ring $\mathcal{O}_{I,p} = (R/I)_p$. We say that I has the support

$$\mathrm{Supp}(I) := \left\{p \in \mathbb{C}^2 \,\middle|\, m_{I,p} \neq 0\right\} = \left\{p \in \mathbb{C}^2 \,\middle|\, f(p) = 0, \forall f \in I\right\}. \tag{1.8}$$

Proposition 1.1.10 ([16]). *The Hilbert-Chow morphism $\rho : \mathrm{Hilb}^{[n]} \to \mathrm{Sym}^n\mathbb{C}^2$ is a resolution of singularities of $\mathrm{Sym}^n\mathbb{C}^2$.*

Example 1.1.11. There are two types of ideals in the Hilbert scheme of two points $\mathrm{Hilb}^{[2]}$. First type corresponds to ideals given by two distinct points $(a,b),(a',b')$ of $\mathbb{C}^2$

$$I = \langle x-a, y-b\rangle \cap \langle x-a', y-b'\rangle.$$

Second type consists of ideals of the form

$$I = \left\{ f \in R \,\middle|\, f(a,b) = 0, df_{(a,b)}(v) = 0 \right\}$$

for some $(a,b) \in \mathbb{C}^2$ and $v \neq 0 \in T_{(a,b)}\mathbb{C}^2$. The ideals of second type are limit of the ideals of the first type when (a',b') approaches to the point (a,b) in which the information of the approaching direction is remembered in I. However, in $\mathrm{Sym}^2\mathbb{C}^2$ the limit is $2(a,b)$ and this information is missing.

A matrix description of $\mathrm{Hilb}^{[n]}$

The Hilbert scheme of points on the plane has a particular description as orbits space of matrix. This provides us a model to study it.

Theorem 1.1.12 ([30]). *The Hilbert scheme of affine plane* $\mathrm{Hilb}^{[n]}$ *is isomorphic to the quotient space*

$$\mathrm{Hilb}^{[n]} \simeq \mathcal{M}/\mathrm{GL}_n(\mathbb{C})$$

where

$$\mathcal{M} := \left\{ \begin{array}{c} (X,Y,i) \\ \in \mathrm{End}(\mathbb{C}^n) \times \mathrm{End}(\mathbb{C}^n) \times \mathrm{Hom}(\mathbb{C},\mathbb{C}^n) \\ \end{array} \middle| \begin{array}{ll} (1) & [X,Y] = 0 \\ (2) & \textit{(Stability condition) There exists no subspace } S \subsetneq \mathbb{C}^n \\ & \textit{such that } X(S) \subset S,\ Y(S) \subset S \textit{ and } \mathrm{Im}(i) \subset S. \end{array} \right\} \tag{1.9}$$

and the action of $\mathrm{GL}_n(\mathbb{C})$ *is given by:*

$$g \cdot (X,Y,i) \mapsto (gXg^{-1}, gYg^{-1}, gi), \quad g \in \mathrm{GL}_n(\mathbb{C}).$$

In other words, the products of X and Y act on i span $\mathbb{C}^n$. Here, we say that i is a *cyclic vector*.

Proof of Nakajima, [30]. We construct a map between $\mathcal{M}/\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{Hilb}^{[n]}$. Let $I \in \mathrm{Hilb}^{[n]}$, then $V = R/I$ is a n-dimensional $\mathbb{C}$-vector space.

We claim that the corresponding triple is given by (X,Y,i) where X,Y are operators of multiplications by x and y in $\mathrm{End}(V)$:

$$X(f \mod I) := xf \mod I \tag{1.10}$$
$$Y(f \mod I) := yf \mod I, \tag{1.11}$$
$$i(1) := 1_V \mod I. \tag{1.12}$$

Then we have $[X,Y] = 0$ by the fact that the multiplication by x and y commute. The stability condition holds since 1 multiplied by the monomials $x^k y^l$, $k,l \in \mathbb{N}$ generate whole R.

Conversely, if a vector space V and $(X,Y,i) \in \mathcal{M}$ is given, we can define the ideal I as the kernel of the surjective morphism

$$\begin{aligned} \phi \colon R &\to \mathbb{C}^n \\ f(x,y) &\mapsto f(X,Y)i. \end{aligned}$$

Here, $f(X,Y)$ makes sense because $[X,Y] = 0$. Since any power of $M = X$ or Y, we have $g \cdot M^s = \left(gMg^{-1}\right)^s = gM^sg^{-1}$. This implies that ϕ is $\mathrm{GL}_n(\mathbb{C})$-equivariant:

$$f(g \cdot X, g \cdot Y)\, gi = f\left(gXg^{-1}, gYg^{-1}\right) g\dot{i} = g \cdot f(X,Y)\, i.$$

Since $\mathrm{Im}\phi$ is invariant under X and Y and contains $\mathrm{Im}\,(i)$, then by the stability condition, $\mathrm{Im}\phi$ has to be the whole space $\mathbb{C}^n$. Hence, ϕ is surjective and $\ker\phi$ is an ideal with $\dim_{\mathbb{C}}(R/I) = n$. ■

In addition, the Hilbert scheme $\mathrm{Hilb}^{[n]}$ can be understood as a special case of $M(r,n)$ the framed moduli space of torsion free sheaves on $\mathbb{P}^2$ with rank $r = 1$ and $c_2 = n$ (cf.[30]). The later space has been showed to be isomorphic to the space

$$\left\{ (X,Y,i,j) \;\middle|\; \begin{array}{ll} (1) & [X,Y] + ij = 0 \\ (2) & \text{(Stability) There exists no subspace } S \subsetneq \mathbb{C}^n \\ & \quad \text{such that } X(S) \subset S,\ Y(S) \subset S \text{ and } \mathrm{Im}\,(i) \subset S. \end{array} \right\} \Big/ \mathrm{GL}_n(\mathbb{C}), \tag{1.13}$$

where $X, Y \in \mathrm{End}(\mathbb{C}^n)$, $i \in \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)$, $j \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^r)$ with the action given by

$$g \cdot (X,Y,i,j) \mapsto (gXg^{-1}, gYg^{-1}, gi, jg^{-1}), \quad g \in \mathrm{GL}_n(\mathbb{C}).$$

Proposition 1.1.13 ([34])**.** *The space $M(1,n)/\,\mathrm{GL}_n(\mathbb{C})$ is smooth and has complex dimension $2n$.*

Proof. We consider the following map

$$\begin{aligned} \theta : \mathrm{End}(\mathbb{C}^n) \times \mathrm{End}(\mathbb{C}^n) \times \mathbb{C}^n \times (\mathbb{C}^n)^* &\to \mathrm{End}(\mathbb{C}^n) \\ (X,Y,i,\mu) &\mapsto [X,Y] + ij. \end{aligned}$$

Then $M(1,n)/\,\mathrm{GL}_n(\mathbb{C})$ is an open subset of $\theta^{-1}(0)/\,\mathrm{GL}_n(\mathbb{C})$. We first show that the differential of θ

$$d\theta : (\delta X, \delta Y, \delta i, \delta\mu) \mapsto [\delta X, Y] + [X, \delta Y] + \delta i\ j + i\ \delta j$$

is surjective. The kernel is a subspace with $\dim_{\mathbb{C}} = 2n^2 + 2n - n^2 = n^2 + 2n$.

To prove that θ is surjective, it is sufficient to show the following statement:
If $\mathrm{tr}\,(A d\theta(\delta X, \delta Y, \delta i, \delta\mu)) = 0$ for all $(\delta X, \delta Y, \delta i, \delta\mu)$, then $A = 0$. Since the trace map is linear and invariant under cyclic permutations, we have

$$\begin{aligned} \mathrm{tr}\,(A d\theta) &= \mathrm{tr}\,(A[\delta X, Y]) + \mathrm{tr}\,(A[X, \delta Y]) + \mathrm{tr}\,(A\delta i\ j) + \mathrm{tr}\,(Ai\ \delta j) \\ &= \mathrm{tr}\,(A\delta X Y - AY\delta X) + \mathrm{tr}\,(AX\delta Y - A\delta Y X) + \mathrm{tr}\,(A\delta i\ j) + \mathrm{tr}\,(Ai\delta j) \\ &= \mathrm{tr}\,(YA\delta X - AY\delta X) + \mathrm{tr}\,(AX\delta Y - XA\delta Y) + \mathrm{tr}\,(jA\delta i) + \mathrm{tr}\,(Ai\delta j) \\ &= \mathrm{tr}\,([Y,A]\delta X) + \mathrm{tr}\,([A,X]\delta Y) + \mathrm{tr}\,(jA\delta i) + \mathrm{tr}\,(Ai\delta j)\,. \end{aligned}$$

Moreover, the trace map is non-degenerated and $\mathrm{tr}\,(A d\theta(\delta X, \delta Y, \delta i, \delta\mu)) = 0$ for all $(\delta X, \delta Y, \delta i, \delta\mu))$, implies $[Y,A]$, $[A,X]$, jA and Ai vanish. Since X and Y commute with A, the subspace spanned by A are invariant under the multiplications by X, Y. But i is a cyclic vector, then by the stability condition, it cannot happen. Therefore we have $A = 0$.

Next, we show that the $\mathrm{GL}_n(\mathbb{C})$ acts on $M(1,n)$ freely. Assume that there are (X,Y,i) and $g \in \mathrm{GL}_n(\mathbb{C})$ such that $g \cdot (X,Y,i) = \left(gXg^{-1}, gYg^{-1}, gi\right) = (X,Y,i)$. Let $id_n \in \mathrm{Mat}_n$ be the identity matrix. Then $\ker(g - id_n) \subseteq \mathbb{C}^n$ is a subspace invariant under X and Y. Clearly $\ker(g - id_n)$ contains i and again by the stability condition, $\ker(g - id_n)$ is the entire space $\mathbb{C}^n$. Thus $g \in \mathrm{GL}_n(\mathbb{C})$ is the identity matrix. ■

The description in (1.9) and (1.13) should be the same. The difference is the appearance of j. As in the case of $\mathrm{Hilb}^{[n]}$, we have $j = 0$.

Proposition 1.1.14 ([30], Proposition 2.7)**.** *Assume that $r = 1$ and $(X, Y, i, j) \in M(1, r)$ satisfy conditions (1) and (2) in 1.13, then $j = 0$.*

Proof. We construct a subspace S of $\mathbb{C}^n$ such that the restriction of j vanishes. Let $S := \sum(\text{product of } X\text{'s and } Y\text{'s})\, i(\mathbb{C}) \subset \mathbb{C}^n$. We claim, by the induction k, that for any

$$\hat{A} = A_1 A_2 \cdots A_k,\ A_l \in \{X, Y\}$$

implies $j\hat{A}i = 0$. We start with $k = 0$, then $\tilde{A} = 1$ and from the definition of the trace

$$j1i = ji = \mathrm{tr}(ji) = \mathrm{tr}(ij) = -\mathrm{tr}([X, Y]) = 0.$$

Suppose that the statement is true for $k \leq q - 1$. Given $\hat{A}$ with the form $A_1 \cdots YXY \cdots A_q$, then

$$\begin{aligned} j\hat{A} &= jA_1 \cdots YX \cdots A_q \\ &= jA_1 \cdots ([Y, X] + XY) \cdots A_q \\ &= jA_1 \cdots (ij + XY) \cdots A_q && \text{(since } [Y, X] = -[X, Y]) \\ &= j\underbrace{(A_1 \cdots)}_{\leq q-1} ij \cdots A_q + jA_1 \ldots XY \cdots A_q \\ &= jA_1 \ldots XY \cdots A_q && \text{(by the hypothesis).} \end{aligned}$$

This implies that $j\hat{A} = jX^\alpha Y^\beta$ where α and β are numbers of X and Y in $\hat{A}$, $\alpha + \beta = q$. Thus to prove the hypothesis, it is sufficient to consider the case of $\hat{A} = X^\alpha Y^\beta$. We have

$$j\hat{A}i = \mathrm{tr}(\hat{A}ij) = -\mathrm{tr}\left(\hat{A}[X, Y]\right) = -\mathrm{tr}(X^\alpha Y^\beta)$$

Finally, we have $j\hat{A}i = -\beta\left(j\hat{A}i\right)$ and $jX^\alpha Y^\beta i = 0$. By the induction, $j\hat{A}i = 0$ for every $\hat{A} = A_1 \ldots A_k$ and hence the restriction of j on the subspace S vanishes. Now if (X, Y, i, j) satisfies the stability condition. Note that the subspace S is invariant under X and Y containing $\mathrm{Im}\,(i)$, then by the stability condition, S has to be $\mathbb{C}^n$. Thus $j|_S = j = 0$. ∎

We formulate the Hilbert-Chow morphism $\rho : \mathrm{Hilb}^{[n]} \to \mathrm{Sym}^n\mathbb{C}^2$ in this context. For a triple $(X, Y, i) \in \mathrm{Hilb}^{[n]}$, since commuting matrices are simultaneously triangularizable, we can make X and Y into upper triangular matrices

$$X = \begin{pmatrix} \alpha_1 & * & \ldots & * \\ 0 & \alpha_2 & \ldots & * \\ \vdots & & \ddots & \vdots \\ 0 & & \ldots & \alpha_n \end{pmatrix},\ Y = \begin{pmatrix} \beta_1 & * & \ldots & * \\ 0 & \beta_2 & \ldots & * \\ \vdots & & \ddots & \vdots \\ 0 & & \ldots & \beta_n \end{pmatrix}.$$

Then the Hilbert-Chow morphism is given by

$$\rho : (X, Y, i) \mapsto (\alpha_1, \beta_1) + \cdots + (\alpha_n, \beta_n).$$

If (X, Y, i) corresponds to an ideal given by distinct n points (α_k, β_k), then X and Y are simultaneously diagonalizable, and the eigenvalues are the given points. Thus ρ is an isomorphism on an open set consisting of ideals given by n distinct points.

The Briançon variety

Recall that the Hilbert-Chow morphism $\rho : \mathrm{Hilb}^{[n]} \to \mathrm{Sym}^{\mathrm{n}}\mathbb{C}^2$ is the morphism of resolution of singularities sending an element of $\mathrm{Hilb}^{[n]}$ to the unerderlying cycle $I \mapsto \sum_{p \in \mathrm{Supp}(I)} m_p\,[p]$. The preimage of a fat point $n\,[p] \in \mathrm{Sym}^n\mathbb{C}^2$ forms a special class of subvarieties of $\mathrm{Hilb}^{[n]}$ that are closely related to the topology and the geometry of $\mathrm{Hilb}^{[n]}$.

Definition 1.1.15. The *n-th Briançon variety* or the *punctual Hilbert scheme of n points* is the zero fiber $\rho^{-1}\left(n\,[(0,0)]\right)$:

$$Br^{[n]} := \left\{ I \in R \,|\, \dim_{\mathbb{C}}\left(R/I\right) = n,\ \rho(I) = n\,[(0,0)] \right\}.$$

Definition 1.1.16. Let $v = \{v_1 = (a_1, b_1), \cdots, v_{n-1} = (a_{n-1}, b_{n-1})\}$ be a collection of $n-1$ vectors in $\mathbb{C}^2$. To a collection v, we associate a curve $\gamma_v : \mathbb{C} \to \mathbb{C}^2$, $t \mapsto \left(\sum_i a_i t^i, \sum_i b_i t^i\right)$. An ideal $I \in Br^{[n]}$ is *curvelinear* if there exists a collection v such that $I = I_\gamma = \{f \in R \mid f \circ \gamma = 0 \mod t^n\}$.

We denote by $Br^{[n]}_{curve}$ the subspace of $Br^{[n]}$ containing curvelinear element. Different curves might define same ideals: $I_\gamma = I_{\gamma\circ\phi}$ for any $\phi(s) = \sum_i \alpha_i t^i$, $\alpha_i \in \mathbb{C}$. The dimension of this space that gives same ideals is of $n-1$ dimensional. We conclude that $\dim_C Br^{[n]}_{curve} = 2(n-1)-(n-1) = n-1$.

J. Briaçon has shown that the generic component of $Br^{[n]}$ is $Br^{[n]}_{curve}$ in the proof of the following theorem.

Theorem 1.1.17 ([6])**.** *The Briançon varieties $Br^{[n]}$ are irreducible of complex dimension $n-1$.*

Despite being singular, the varieties $Br^{[n]}$ has an important property: they are deformation retracts to the Hilbert scheme $\mathrm{Hilb}^{[n]}$.

M. Haiman has identified the structure sheaf of $Br^{[n]}$, which provides a resolution of the structure sheaf of $Br^{[n]}$.

Proposition 1.1.18 ([22, 23])**.** *There is a locally $\mathcal{O}_{\mathrm{Hilb}^{[n]}}$-free resolution of the structure sheaf of $Br^{[n]}$*

$$\cdots \to B \otimes \wedge^{i-1}\left(B' \oplus \mathcal{O}_{t_1} \oplus \mathcal{O}_{t_2}\right) \to \cdots \to B \otimes \left(B' \oplus \mathcal{O}_{t_1} \oplus \mathcal{O}_{t_2}\right) \to B \to \mathcal{O}_{Br^{[n]}} \to 0 \tag{1.14}$$

where B is a free module with basis $\mathcal{B}_\lambda = \{x^h y^k \mid (h,k) \in \lambda\}$, $\lambda \vdash n$, B' is same with the basis $\mathcal{B}_\lambda \setminus \left\{x^0 y^0\right\}$, and $\mathcal{O}_{t_1} = \mathcal{O}_{\mathrm{Hilb}^{[n]}} \otimes \mathbb{C}_{t_1}$, $\mathcal{O}_{t_2} = \mathcal{O}_{\mathrm{Hilb}^{[n]}} \otimes \mathbb{C}_{t_2}$ are the twisting by one-dimensional representations of T in which τ_{t_1,t_2} acts as t_1 or t_2, respectively.

1.2 Topology of the Hilbert schemes $\mathrm{Hilb}^{[n]}$

1.2.1 Torus action, Bialynicki-Birula cell decomposition and homology of $\mathrm{Hilb}^{[n]}$

Let $\mathbb{C}^* := \mathbb{C} \setminus \{(0,0)\}$ and denote by $T \simeq (C^*)^2$ the two-dimensional algebraic torus. It acts on $\mathbb{C}^2$ diagonally by

$$t \cdot x = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \cdot x = t_1 x, \quad t \cdot y = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \cdot y = t_2 y.$$

This induces an action on ideals I of R:

$$t \cdot I = \left\{ f(t_1^{-1}x, t_2^{-1}y) \,\middle|\, f(x,y) \in I \right\},$$

which preserves the degrees of the polynomials. Thus, this T-action is codimension preserving and hence lifts to $\mathrm{Hilb}^{[n]}$.

We should identify the fixed points of torus action. Note that an ideal I is invariant under T-action if it is homogeneous in both variables x and y. Thus I is a T-fixed point if it is a monomial ideal. We associate to a monomial $x^\alpha y^\beta$ the coordinate $(\alpha, \beta) \in \mathbb{Z}^2_{\geq 0}$ that corresponds to the

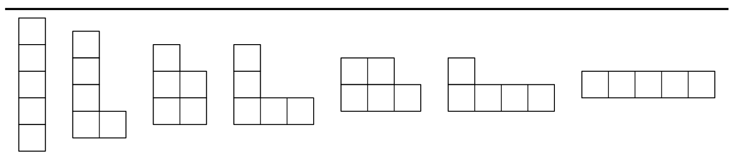


Figure 1.4: Fixed points of $\mathrm{Hilb}^{[5]}$

one-dimensional weight space $\mathbb{C}x^\alpha y^\beta$ of weight (t_1^α, t_2^β). We are interested in the weights of the isotropy representation on the tangent space of a fixed point.

We recall some definitions in the representation theory. A *representation* of a group G on a complex vector space V over is a group homomorphism $\phi\colon G \to \mathrm{GL}(V)$, where V ia said to be the representation space and the dimension of the representation is $\dim_{\mathbb{C}} V$. We define the *character* of ϕ to be the function $\chi_\phi(g)\colon G \to \mathbb{C}$, $g \mapsto \mathrm{tr}\,(\phi(g))$.

Let $\triangle_\lambda$ be a Young diagram with n boxes corresponding to the partition λ of n. To a box $\boxed{\bullet}$ of $\triangle_\lambda$ with coordinate (h,k), we define the following values:

- *the arm-length* $a(\boxed{\bullet}) = \lambda_h - (k+1)$, that is the number of the boxes above $\boxed{\bullet}$.

- *the leg-length* $l(\boxed{\bullet})$ as the number of the boxes on right of $\boxed{\bullet}$.

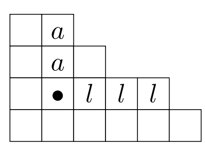


Figure 1.5: Example: $a\left(\boxed{\bullet}\right) = 2$ and $l\left(\boxed{\bullet}\right) = 3$.

Proposition 1.2.1 ([30]). *Let I_λ be a torus fixed point of $\mathrm{Hilb}^{[n]}$ and $\triangle_\lambda$ be the corresponding Young diagram. Then the character of $T_{I_\lambda}\mathrm{Hilb}^{[n]}$ is given by the formula:*

$$\mathrm{ch} T_{I_\lambda} \mathrm{Hilb}^{[n]} = \sum_{\square : \text{box of } \triangle_\lambda} \left(t_1^{-l(\square)} t_2^{a(\square)+1} + t_1^{l(\square)+1} t_2^{-a(\square)} \right). \tag{1.15}$$

Given a topological space Z which has finitely generated homology, the Poincaré polynomial of Z is defined as the generating function of its Betti numbers: $P\,(Z;t) := \sum_j \dim_{\mathbb{C}} H_i(Z)$.

Corollary 1.2.2 ([13,19,30])**.** *The Poincaré polynomial of* $\mathrm{Hilb}^{[n]}$ *is given by*

$$P\left(\mathrm{Hilb}^{[n]};t\right)=\sum_{\lambda\,\vdash n}t^{2(n-w(\lambda))}, \tag{1.16}$$

where $w(\lambda)$ *is the length of the partition* λ*.*

Therefore, the generating function of $P\left(\mathrm{Hilb}^{[n]};t\right)$ *has the form:*

$$\sum_{n=0}^{\infty}P\left(\mathrm{Hilb}^{[n]};t\right)q^n=\prod_{d=1}^{\infty}\frac{1}{(1-t^{2d-2}q^d)}. \tag{1.17}$$

Next, we wish to show the existence a cell decomposition of $\mathrm{Hilb}^{[n]}$. The following part of this subsection mainly comes from [32, Section 4.3]. Before discussing the case of $\mathrm{Hilb}^{[n]}$, we first state a general result about a $\mathbb{C}^*$-action on a projective manifold Bialynicki-Birula (cf. [2,3]). Assume that $\mathbb{C}^*$ acts on a projective manifold Z algebraically. Then by Borel fixed-point theorem (cf. [4]), the fixed point set of the action is non-empty. We consider the case when fixed points are isolated and are given by $p_1,\cdots,p_k\in Z$ for some $k\in\mathbb{N}$. We define the $(\pm)$-attracting sets:

$$S_i:=\left\{p\in Z\,\middle|\,\lim_{t\to 0}t\cdot p=p_i\right\}$$
$$U_i:=\left\{p\in Z\,\middle|\,\lim_{t\to \infty}t\cdot p=p_i\right\},$$

which are locally closed submanifolds of Z. Each S_i (respectively U_i) is isomorphic to an affine space whose dimension is equal to the dimension of the positive (respectively negative) weight space in $T_{p_i}Z$ with respect to the $\mathbb{C}^*$-action. One can order the fixed point set p_i such that $\dim_{\mathbb{C}}S_i\geq\dim_{\mathbb{C}}S_j$ for $i<j$. Then the union $\bigcup_{i\leq i_0}S_i$ (respectively $\bigcup_{i\leq i_0}U_i$) is open (respectively closed) for each $1\geq i_0\geq r$. More importantly, we have the theorem:

Theorem 1.2.3 ([2])**.** *The odd homology group* $H_{odd}(M)$ *vanishes, and the fundamental classes of closures of* S_i *(or* U_i*) give a base of* $H_{even}(M)$*.*

Back to our case, the Hilbert scheme $\mathrm{Hilb}^{[n]}$ is not projective but quasi-projective. We can still apply the theory to $\mathrm{Hilb}^{[n]}$ by the following argument: We consider the Hilbert scheme of points on the projective plane $\left(\mathbb{P}^2\right)^{[n]}$ with a $\mathbb{C}^*$-action and $\mathbb{C}^2$ as an invariant open subset of $\mathbb{P}^2$. Then we apply the theory to $\left(\mathbb{P}^2\right)^{[n]}$, and check that $(\pm)$-attracting sets are contained in the open subset $\mathrm{Hilb}^{[n]}$.

We choose a generic one-parameter subgroup $\mathbb{C}^*\to T$

$$t\mapsto(t,t^N)=(t_1,t_2)\ ,\text{ where }N\gg 1$$

such that the $\mathbb{C}^*$-fixed point set and T-fixed point set coincides. Let $I_\lambda\mathrm{Hilb}^{[n]}$ be the fixed point corresponding to the partition $\lambda\vdash n$. The $(\pm)$-attracting set with respect to the one-parameter subgroup are given by

$$S_\lambda:=\left\{I\in\mathrm{Hilb}^{[n]}\,\middle|\,\lim_{t\to 0}t\cdot I\in I_\lambda\right\},$$
$$U_\lambda:=\left\{I\in\mathrm{Hilb}^{[n]}\,\middle|\,\lim_{t\to \infty}t\cdot I\in I_\lambda\right\}.$$

We have:

Lemma 1.2.4 ([30])**.**

$$\biguplus_{\lambda\vdash n} S_\lambda = \mathrm{Hilb}^{[n]} \text{ and } \biguplus_{\lambda\vdash n} U_\lambda = Br^{[n]}.$$

Proof. Consider the $\mathbb{C}^*$-action on $\mathrm{Sym}^n\mathbb{C}^2$. Then, for any $z \in \mathrm{Sym}^n\mathbb{C}^2$, $t\cdot z \in \mathrm{Hilb}^{[n]} \to n\,[(0,0)]$ in $\mathrm{Sym}^n\mathbb{C}^2$ as $t \to 0$. Then the limit of $t\cdot I$ exists if and only if it stays in a compact subset. Thus, $\lim_{t\to 0} t\cdot I$ exists for any $I \in \mathrm{Hilb}^{[n]}$.

On the other hand, when $t \to \infty$, the limit $\lim_{t\to\infty} t\cdot z \in \mathrm{Sym}^n\mathbb{C}^2$ exists if and only if it is $n(0,0) \in \mathrm{Sym}^n\mathbb{C}^2$. Since ρ is proper, the limits in $\mathrm{Hilb}^{[n]}$ $\lim_{t\to\infty} t\cdot I$ exists if and only if $\rho(I) = n\cdot(0,0)$. ■

An immediate consequence of the above lemma is the Euler number of $\mathrm{Hilb}^{[n]}$ equals to the total numbers of partitions of n. Now, we have an open cover of $\mathrm{Hilb}^{[n]}$ by open sets S_λ. And each S_λ contains a single fixed point. They are smooth by the following theorem:

Theorem 1.2.5. *Let Z be a normal variety with a torus action by T. If all the points in the locus of T-fixed points are smooth, then the variety Z is smooth.*

Proof. The singular locus of Z is T invariant and closed. Therefore it consists of closures of T-orbits. The closure of every torus orbit contains a torus fixed point, and so the singular locus of Z must contain a torus fixed point if it is non-empty. If the torus fixed points are all smooth, then the singular locus must be empty. ■

We know also the dimension of S_λ and U_λ.

Lemma 1.2.6. *We have $\dim_{\mathbb{C}} S_\lambda = n + w(\lambda)$ and $\dim_{\mathbb{C}} U_\lambda = n - w(\lambda)$, where $w(\lambda)$ is the number of columns of the Young diagram associated to λ.*

Proof. We know that $\dim_{\mathbb{C}} S_\lambda$ is equal to the dimension of the positive weight space of the tangent space at I. From formula (1.15) and our choice of N, this is equal to the sum of dimensions of the weight spaces that satisfy either

$$-l(s) + N\,(a(s)+1) > 0$$

which always positive, or

$$l(s) + 1 - Na(s) > 0$$

which is positive if and only if $a(s) = 0$. These are boxes on the top of each column and there are length(λ) of them.

Now, for U_λ, we count the number of negative weights. This is equal to $2n$ minus the number of positive weights. Therefore, we have $\dim_{\mathbb{C}} U_\lambda = n - l(\lambda)$. ■

Remark 1.2.7. The open set U_λ in (1.3) is T-invariant and induces action on Haiman's local coordinates $C^{i,j}_{h,k}$ by

$$C^{i,j}_{h,k}\,(t\cdot I) = t_1^{h-i} t_2^{k-j} C^{i,j}_{h,k}.$$

An alternative proof of Proposition 1.2.1 use $C^{i,j}_{h,k}$ to express the local coordinates of the cotangent space $\mathcal{M}/\mathcal{M}^2$ at the fixed point λ .

1.2.2 Operators on the direct sum of the homology groups

In this subsection, we explain the construction of a representation of the Heisenberg algebra on the homology group of $\mathrm{Hilb}^{[n]}$ which makes the homology group of $\mathrm{Hilb}^{[n]}$ a Fock space. The idea is to construct *correspondences* in $\mathrm{Hilb}^{[n]} \times \mathrm{Hilb}^{[n+k]}$ for arbitrary $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ and the associated representation on the direct sum of homology groups of all $\mathrm{Hilb}^{[n]}$

$$\mathbb{H} := \bigoplus_n H_* \left(\mathrm{Hilb}^{[n]}\right).$$

These correspondences define operators on the homology group and give generators of the Heisenberg algebra. In this subsection, we explain Nakajima's construction ([30, 31]).

We will first state a general construction of correspondences in Borel-Moore homology groups and give the definition of Heisenberg algebra. After that, we illustrate the construction of the representation of a infinite dimension Heisenberg algebra on $\mathbb{H}$.

Convolution products and correspondences in Borel-Moore homology groups

Existence of fundamental class in Borel-Moore homology groups:

We recall some facts about the homology and cohomology of a topological space X. The cohomology $H^*(X)$ has a cup product structure

$$\cup\colon H^i(X) \times H^j(X) \to H^{i+j}(X),$$
$$(\alpha, \beta) \mapsto \alpha \cup \beta$$

(sometimes also written with a dot ' "·") which makes $H^*(X)$ a ring. Unlike $H^*(X)$, the homology $H_*(X)$ does not have such a natural product structure, however, it is a module over the cohomology through the cap product

$$\cap : H^i(X) \otimes H_j(X) \to H_{j-i}(X)$$
$$\alpha \otimes \beta \mapsto \alpha \cap \beta.$$

The homology $H_*(-)$ and $H^*(-)$ have following properties:

- Push-forward and pullback of a continuous map:
  If $f\colon X \to Y$ is a continuous map. Then it determines *push-forward* on homology

  $$f_*\colon H_i(X) \to H_i(Y)$$

  and
  $emphpull - back$ on cohomology

  $$f^*\colon H^i(X) \to H^i(Y).$$

- Poincaré duality:
  A complex n-dimensional projective non-singular variety X is a compact oriented 2n-dimensional real manifold. Thus, the top homology $H_{2n}(X)$ is isomorphic to $\mathbb{Z}$. The image of the generator

$1 \in \mathbb{Z}$, denoted by $[X] \in H_{2n}(X)$, is the fundamental class of X. We define the Poincaré duality map $H^i(X) \to H_{2n-i}(X)$ by capping with the fundamental class of X:

$$\begin{aligned} \mathrm{PD}\colon H^i(X) &\to H_{2n-i}(X) \\ \alpha &\mapsto \alpha \cap [X]. \end{aligned}$$

as PD is an isomorphism for compact oriented manifolds.

In general, for X a smooth space, but not necessarily compact, the homology groups are Poincaré dual with compactly supported cohomology

$$\mathrm{PD}\colon H^i_{cpt}(X) \to H_{2n-i}(X).$$

In this case, we obtain a push-forward homomorphism of $f\colon X^n \to Y^m$ on cohomology

$$f_*\colon H^i(X) \overset{\mathrm{PD}}{\simeq} H_{2n-i}(X) \to H_{2n-i}(Y) \overset{\mathrm{PD}}{\simeq} H^{2m-(2n-i)}(Y) = H^{2(m-n)+i}(Y).$$

Existence of fundamental class of a closed subvariety inside of a non-singular projective variety:
If X is not compact and smooth, it is not clear if the fundamental $[X]$ exists. However, the fundamental class can be defined for a closed subvariety a non-singular projective variety as a refined class in BM homology.

We define, for a topological space X embedded as a closed subspace of a oriented manifold M, a version of *Borel-Moore homology*:

$$H_i^{\mathrm{BM}}(X) := H^{\dim X - i}(M, M \setminus X).$$

Intersection pairing:

If $X \subset M, Y \subset N$ are closed subsets of oriented manifolds M, N of dimension m, n and if $f\colon X \to Y$ is a smooth map with $f^{-1}(Y)$ contained in X. We consider the composite on relative cohomology:

$$\begin{aligned} H_k^{\mathrm{BM}}(Y) = H^{n-k}(N, N \setminus Y) &\xrightarrow{f^*} H^{n-k}(M, M \setminus f^{-1}(Y)) \\ \to H^{m-(k-n+m)}\,(M, (M \setminus X) \cup (N \setminus Y)) &= H^{\mathrm{BM}}_{k-n+m}(X \cap Y). \end{aligned}$$

This gives rise to the pullback on BM homology groups:

$$f_*\colon H_k^{\mathrm{BM}}(Y) \to H^{\mathrm{BM}}_{k+(m-n)}(X).$$

Lemma 1.2.8. *If V is a k-dimensional algebraic subset of a non-singular variety. Then $H_i^{\mathrm{BM}}(V)$ vanishes for $i > 2k$, and $H_{2k}^{\mathrm{BM}}(V)$ is a free Abelian group with a generator for each k-dimensional component of V.*

Poincaré duality:

The cap product $H^j(X) \otimes H_i^{\mathrm{BM}}(X) \to H_{i-j}^{\mathrm{BM}}(X)$ gives rise to a non-degenerate pairing

$$H^{\dim X - i} \simeq H_i^{\mathrm{BM}}(X).$$

Convolution products on Borel-Moore homology groups

Let M, N be oriented smooth varieties of dimensions m, n and $Z \subset M \times N$ a subvariety such that the projection map $Z \to N$ is proper. From the previous statement, we know that the fundamental class $[Z]$ of Z exists as an element in $H_{2dim_{\mathbb{C}}Z}(M \times N)$. A convolution product is an operator on the homology groups of M, N

$$\begin{aligned} c_Z \colon H_i(M) &\to H^{\mathrm{BM}}_{i+2\dim_{\mathbb{C}} Z - 2\dim_{\mathbb{C}} M}(N) \\ y &\mapsto c_Z(y) = p_{N_*}(p_M^*(\mathrm{PD}^{-1}(y)) \cap [Z]). \end{aligned}$$

Or equivalently on the cohomology groups

$$\begin{aligned} c_Z \colon H^i(M) &\to H^{i+2\dim_{\mathbb{C}} N - 2\dim_{\mathbb{C}} Z}(N) \\ \alpha &\mapsto c_Z(\alpha) = \mathrm{PD}^{-1}(p_{N_*}(p_M^*(\alpha) \cap [Z])). \end{aligned}$$

We sometimes also refer these constructions of operators as *correspondences*.

Let $y \in H_i(M)$, then $\mathrm{PD}(y) \in H^{2\dim_{\mathbb{C}} M - i}$. Since the pullpack of a cohomology class leaves the degree invariant, $p_M^*(PD(y)) \cap [Z]$ is an element in $H_{2\dim_{\mathbb{C}} Z - (2\dim_{\mathbb{C}} M - i)}(M \times N)$. Then $c_Z(y)$ is a class of degree

$$2\dim_{\mathbb{C}} Z - (2\dim_{\mathbb{C}} M - i) = i + 2(\dim_{\mathbb{C}} Z - \dim_{\mathbb{C}} M). \tag{1.18}$$

That is c_Z increases the homological degree by $2(\dim_{\mathbb{C}} Z - \dim_{\mathbb{C}} M)$. The cohomological degree is computed in the same way, and we get the degree $2\dim_{\mathbb{C}} N - 2\dim_{\mathbb{C}} Z$.

Let M_1, M_2, M_3 be oriented C^∞-manifolds of dimension m_1, m_2, m_3 and $p_{ij} \colon M_1 \times M_2 \times M_3 \to M_i \times M_j$ be the projection to (i, j) factor. Suppose that $Z_{12}^{d_{12}} \subset M_1 \times M_2$ and $Z_{23}^{d_{23}} \subset M_2 \times M_3$ are two correspondences induced by $[Z_{12}] \in H_{d_{12}}(M_1 \times M_2)$ and $[Z_{23}] \in H_{d_{23}}(M_2 \times M_3)$ respectively. We ask in addition that the projection p_{13} is proper when restricts to $p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \subset M_1 \times M_2 \times M_3$.

Then the composition of convolutions equals to

$$[Z_{12}] \circ [Z_{23}] = (p_{13})_*\big(p_{12}^*([Z_{12}]) \cap p_{23}^*([Z_{23}])\big) \in H_{d_{12}+d_{23}-m_2}(M_1 \times M_3)$$

defines a correspondence $(p_{M_3})_*(p_{M_1}^*(\cdot) \cap [Z_{12}] \circ [Z_{23}]) \colon H_i(M_1) \to H_{i+d_{12}+d_{23}-m_1-m_2}(M_3)$ of degree $d_{12} + d_{23} - m_1 - m_2$.

The infinite dimensional Heisenberg algebra

Definition 1.2.9. The infinite dimensional Heisenberg algebra $\mathfrak{s}$ is a Lie algebra spanned by $\{p[k] \mid k \in \mathbb{Z}\} \cup \{K\}$ such that they satisfy the "commutation relations"

$$[p[k], p[l]] = k\delta_{k+l,0} \text{ and } [p[k], K] = 0. \tag{1.19}$$

For every $c \neq 0 \in \mathbb{C}$, the algebra $\mathfrak{s}$ has a representation as an algebra of operators on $\mathbb{C}[q_1, q_2, \cdots]$, on which the $\mathfrak{s}$-module structure is given as follows:

$$\begin{aligned} P[k] &\mapsto \begin{cases} q_{-k} & \text{if } k < 0 \\ 0 & \text{if } \text{ k=0} \\ c\ k\frac{\partial}{\partial q_k} & \text{if } k > 0 \end{cases} \\ K &\mapsto c \cdot \mathrm{id} \end{aligned}$$

This representation has a highest weight vector 1. Moreover $\mathbb{C}[q_1, q_2, \dots]$ are spanned by $q_1^{j_1} q_2^{j_2} \cdots q_k^{j_k} = P[1]^{j_1} P[2]^{j_2} \cdots P[k]^{j_k} \cdot 1$ We extend $\mathfrak{s}$ by defining a derivation d_0 by

$$[d_0, p[k]] = -kp[k].$$

And above representation can be extended by

$$d_0 \mapsto \sum_k kq_k \frac{\partial}{\partial q_k}$$

with the character formula

$$\sum_i \dim \{v \in \mathbb{C}[q_1, q_2, \dots] \mid d_0 v = iv\} \ t^i = \prod_{i=1}^{\infty} \frac{1}{1 - t^i}. \tag{1.20}$$

Construction of correspondence on $\oplus_n H_*(\mathrm{Hilb}^{[n]})$

We introduce H.Nakajima's contruction of correspondences on $\oplus_n H_*(\mathrm{Hilb}^{[n]})$ (cf. [12, 29, 30]). He originally constructed these correspondences for smooth quasi-projective surfaces. We introduce an incidence variety for $k > 0$:

$$P[k] := \coprod_n \left\{ (I, J, p) \in \mathrm{Hilb}^{[n]} \times \mathrm{Hilb}^{[n+k]} \times \mathbb{C}^2 \,\middle|\, I \supset J \ , \ \mathrm{Supp}(I \setminus J) = \{p\} \right\}$$

and $k < 0$ by switching I and J.

We show that $P[k]$ has complex dimension $2n - k + 1$. Assume that $k > 0$, we may decompose $P[k]$ into disjoint union of strata $P[k]_0 \cup P[k]_1 \cup \dots \cup P[k]_k$ where

$$P[k]_l = \left\{ (I, J, x) \in P[k] \,\middle|\, I \in \mathrm{Hilb}^{[l]} \right\}.$$

We define the Briançon variety on the whole space $\mathbb{C}^2$:

$$Br_{\mathbb{C}^2}^{[n]} := \left\{ I \trianglelefteq R \,\middle|\, \rho(I) = n\,[p] \text{, for some } p \in \mathbb{C}^2 \right\}. \tag{1.21}$$

If $J \in \mathrm{Hilb}^{[n+k]}$ is obtained by adding a point p that is disjoint from the support of I, then $J = I \cup B$ for some ideal $B \in Br_{\mathbb{C}^2}^{[k]}$. Hence we have a birational map

$$P[k]_0 \to \mathrm{Hilb}^{[n]} \times Br_{\mathbb{C}^2}^{[k]}$$

and therefore $\dim_{\mathbb{C}} P[k]_0 = 2n + k + 1$. Similarly, for each $i \in \mathbb{N}$, $0 < l \leq k$ we have birational maps

$$P[k]_l \to \mathrm{Hilb}^{[n-l]} \times Br_{\mathbb{C}^2}^{[k+l]}.$$

But these strata have dimension

$$\dim_{\mathbb{C}} P[k]_l = 2n - 2l + k + l = 2n - l + k + 1,$$

which are strictly less than $2n + k + 1$ for $l > 0$. A Similar argument can be applied to the case of $k < 0$ by exchanging I and J. Let $u \in H_*(\mathbb{C}^2)$ and $v \in H_*^{BM}(\mathbb{C}^2)$ be homology classes of the ordinary homology and Borel-Moore homology of $\mathbb{C}^2$ respectively. We define, for $k > 0$, $P_u[k] := u \cap [P_u[k]] \in H_*\left(\mathrm{Hilb}^{[n+k]}\right)$ and $P_v[-k] := v \cap [P_v[-k]] \in H_*\left(\mathrm{Hilb}^{[n-k]}\right)$. Let π_n, π_{n+k} and $\pi_{\mathbb{C}^2}$ be the projection of $\mathrm{Hilb}^{[n]} \times \mathrm{Hilb}^{[n+k]} \times \mathbb{C}^2$ to $\mathrm{Hilb}^{[n]}$, $\mathrm{Hilb}^{[n+k]}$ and $\mathbb{C}^2$ respectively, $i = 1, 2, 3$. Then $P_u[k]$, $P_v[-k](z)$ define operators on $\mathbb{H} = \oplus_n H_*(\mathrm{Hilb}^{[n]})$

$$P_u[k](z) := (\pi_{n+k})_* \left(p^*_{\mathbb{C}^2}(u) \cup \pi^*_n(z) \cap [P[k]]\right) \in H_*\left(\mathrm{Hilb}^{[n+k]}\right),$$
$$P_v[-k](z) := (\pi_{n-k})_* \left(p^*_{\mathbb{C}^2}(v) \cup \pi^*_n(z) \cap [P[-k]]\right) \in H_*\left(\mathrm{Hilb}^{[n-k]}\right).$$

Most importantly, next theorem state that they satisfy the commutation relation and thus $\mathbb{H}$ becomes a representation space of Heisenberg algebra.

Theorem 1.2.10 ([12, 20, 29])**.** *For all $u, v \in H_*(\mathbb{C}^2)$, $k, l \in \mathbb{Z}$, we have*

$$[P_u[k], P_v[l]] = k\delta_{k+l,0}\langle u, v\rangle \mathrm{id}_{\mathbb{H}}. \tag{1.22}$$

An important part of proof involves a calculation of the intersection number:

Theorem 1.2.11 ([15, 30])**.**

$$\langle [Br^{[i]}_p], [Br^{[i]}_{\mathbb{C}^2}]\rangle = (-1)^{i-1} i. \tag{1.23}$$

Chapter 2

Refined Hilbert scheme of points on the plane

2.1 The refined Hilbert scheme $\mathrm{Hilb}^{[n,n+r]}$

We start with the definition of the refined Hilbert scheme $\mathrm{Hilb}^{[n,n+r]}$. Denoted by $\mathfrak{m} = \langle x, y\rangle$ the maximal ideal of R at $(0,0)$.

Definition 2.1.1. The *refined Hilbert scheme of points on* $\mathbb{C}^2$ is the subvariety $\mathrm{Hilb}^{[n,n+r]} \subset \mathrm{Hilb}^{[n]} \times \mathrm{Hilb}^{[n+r]}$ defined by

$$\mathrm{Hilb}^{[n,n+r]} = \left\{ (I, J) \in \mathrm{Hilb}^{[n]} \times \mathrm{Hilb}^{[n+r]} \,\middle|\, I \supset J \supset \mathfrak{m}I \right\}.$$

Here, the condition $I \supset J \supset \mathfrak{m}I$ is equivalent to the requirement that the quotient space I/J lies in the kernel of the multiplications by x and y. In other words, the quotient space $I/J \simeq \mathbb{C}^r$ carries a trivial R-module structure. The space $\mathrm{Hilb}^{[n,n+r]}$ was studied in a different context by Nakajima and Yoshioka [34] as a special case of the moduli spaces of perverse coherent sheaves on the blow-up of $\mathbb{P}^2$.

It follows from the definition that, for an element $(I, J) \in \mathrm{Hilb}^{[n,n+r]}$, the quotient I/J is supported at $(0,0)$. We introduce the refined Briançon variety as well:

Definition 2.1.2. The *refined Briançon variety* is defined by

$$Br^{[n,n+r]} := \left\{ (I, J) \in \mathrm{Hilb}^{[n]} \times \mathrm{Hilb}^{[n+r]} \,\middle|\, I \supset J \supset \mathfrak{m}I,\ \rho(I) = n\,[(0,0)] \right\}.$$

Unlike the ordinary Briançon variety $Br^{[n]}$, which is irreducible, we will see later that $Br^{[n,n+r]}$ may have several top-dimensional components.

We note that when $r = 1$, the two conditions $\mathrm{Supp}(I/J) = (0,0)$ and $I \supset J \supset \mathfrak{m}I$ are equivalent. It is a subvariety of the *Nested Hilbert scheme* parametrizes pairs of ideals $(I, J) \in \mathrm{Hilb}^{[n]} \times \mathrm{Hilb}^{[n+1]}, I \supset J$, which has been studied in various context [7, 9, 28].

We recall that the action of the two-dimensional algebraic torus T on $\mathbb{C}^2$ induces an action on the Hilbert scheme $\mathrm{Hilb}^{[n]}$ (cf. Section 1.2.1). Moreover, since the inclusion relations $I \supset J \supset \mathfrak{m}I$

are invariant under the induced action $t \cdot (I, J) \mapsto (t \cdot I, t \cdot J)$, the action of T lifts to $\mathrm{Hilb}^{[n,n+r]}$ equally.

Next, we want to identify the fixed points in $\mathrm{Hilb}^{[n,n+r]}$ of the T-action, which are the elements $(I, J) \in \mathrm{Hilb}^{[n,n+r]}$ such that

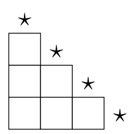
$$\begin{cases} t \cdot I = I \\ t \cdot J = J \end{cases}.$$

For this reason, I and J must be both T-fixed points of $\mathrm{Hilb}^{[n]}$ and $\mathrm{Hilb}^{[n+r]}$, which are monomial ideals. Thus, a T-fixed point of $\mathrm{Hilb}^{[n,n+r]}$ is a pair of monomial ideals (I, J) satisfying $I/J \simeq \mathbb{C}^r$ as a trivial R-module.

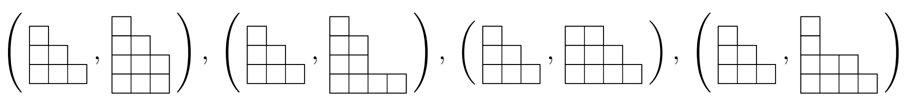
These fixed points are easier to describe with the help of the corresponding presentation using Young diagram.

Example 2.1.3 (Monomial ideals)**.** Let $(I, J) \in \mathrm{Hilb}^{[n,n+r]}$ be a T-fixed point. In fact, for each choice of a set of r generators of I, it gives rise to an ideal J satisfying $I \supset J \supset \mathfrak{m}I$.

For example, let $I = \langle y^3, y^2x, x^2y, x^3 \rangle \in \mathrm{Hilb}^{[6]}$ and consider the corresponding Young diagram

$\triangle_I =$  :$\star$'s are (monomial) generators of I.

Then each choice of three $\star$'s of $\triangle_I$ and the diagram $\triangle_I$ gives an element of $\mathrm{Hilb}^{[6,9]}$. There are four of them:

$(\triangle_I, \triangle_J) =$ 

To describe a T-fixed point $(I, J) \in \mathrm{Hilb}^{[n,n+r]}$ in terms of the Young diagram, we denote by $\triangle_I$ and $\triangle_J$ the corresponding Young diagrams of I and J. The inclusion $I \supset J$ means that $\triangle_I$ is a subdiagram of $\triangle_J$ and I/J is a subset of r *removable boxes* in $\triangle_J$. Here, we say a box of a Young diagram is *removable* if it does not have adjacent boxes above and on the right (See following example). In other words, a removable box represents a monomial in R/J on which x and y act

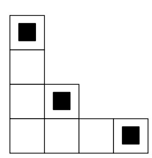
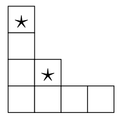


Table 2.1: Young diagram of (4,2,1,1). ■ are removable boxes.

trivially.

Moreover, with the following proposition, we can identify $\left(\mathrm{Hilb}^{[n,n+r]}\right)^T$ with a certain subset of marked Young diagrams with $n + r$ boxes.

Lemma 2.1.4. *The T-fixed points of* $\mathrm{Hilb}^{[n,n+r]}$ *are pairs of monomial ideals (I,J) satisfying $I \supset J \supset \mathfrak{m}I$ and are in bijection with Young diagrams of $n+r$ boxes with r marked (by $\star$) removable boxes (Table 2.2).*



$$\Leftrightarrow \begin{array}{l}\left(\langle x^4, xy, y^3\rangle,\right.\\ \left.\langle x^4, x^2y, xy^2, y^4\rangle\right)\end{array} \in \mathrm{Hilb}^{[6,8]}$$

Table 2.2: A fixed point of $\mathrm{Hilb}^{[6,8]}$ represented by a marked Young diagram.

Proof. Given a Young diagram $\triangle$ of size $n+r$ with r marked boxes, we denote by $\triangle'$ the subdiagram $\triangle$ with these r marked boxes removed. We define J the ideal generated by monomials $x^h y^k$, $(h,k) \in \mathbb{N}^2 \backslash \triangle$ and I the ideal generated by monomials $x^h y^k$, $(h,k) \in \mathbb{N}^2 \backslash \triangle'$. This pair of ideals (I,J) is unique for a fixed diagram $\triangle$. Thus the map $\triangle \mapsto (I,J) \in \left(\mathrm{Hilb}^{[n,n+r]}\right)^T$ is well-defined. ∎

We introduce $\mathbb{N}$-valued functions σ and μ on the Hilbert schemes. Given an ideal $I \in \mathrm{Hilb}^{[n]}$, there exist distinct points $p_1, \ldots, p_k \in \mathbb{C}^2$ such that I may be expressed as the intersection of ideals $I = I_{p_1} \cap \cdots \cap I_{p_k}$, where I_{p_i} is supported at $\{p_i\}$ for each $i = 1, \ldots, k$. We will write I_0 for I_p with $p = (0,0)$. We define the function σ of the multiplicity of $I \in \mathrm{Hilb}^{[n]}$ at the point $(0,0)$ by

$$\sigma(I) := \dim_{\mathbb{C}}(R/I_0),$$

and the function μ of the minimal number of generators of I_0:

$$\mu(I) := \dim_{\mathbb{C}}(I/\mathfrak{m}I) = \dim_{\mathbb{C}}(I/\mathfrak{m}I_0).$$

By abusing of the notation, we denote the $\mathbb{N}$-valued functions σ_n and μ_n on $\mathrm{Hilb}^{[n,n+r]}$:

$$\sigma_n(I,J) := \sigma(I) = \dim_{\mathbb{C}}(R/I_0). \tag{2.1}$$

$$\mu_n(I,J) := \mu(I) = \dim_{\mathbb{C}}(I/\mathfrak{m}I) \tag{2.2}$$

which depend only on I.

Proposition 2.1.5. *If $I \in \mathrm{Hilb}^{[n]}$ has $\mu(I) = k$, then $n \geq \frac{k(k-1)}{2} = \binom{k}{2}$. Moreover, the equality holds if and only if $I = \mathfrak{m}^{k-1}$.*

Proof. We look for the maximal value of $\mu(I)$ for $I \in \mathrm{Hilb}^{[n]}$. We first note that if $I_\lambda \in (\mathrm{Hilb}^{[n]})^T$ and $\triangle_\lambda$ is the corresponding Young diagram with n boxes, then $\mu(I_\lambda)$ is equal to the number of "elbows" of $\triangle_\lambda$.

Lemma 2.1.6. *We have*

$$\max\left\{\mu(I) \,\middle|\, I \in \mathrm{Hilb}^{[n]}\right\} = \max\left\{\mu(I) \,\middle|\, I \in \left(\mathrm{Hilb}^{[n]}\right)^T\right\}.$$

Proof of Lemma 2.1.6. Consider the action of the one-parameter subgroup $\phi : \mathbb{C}^* \to T,\ t \mapsto (t, t^N)$ on $\mathrm{Hilb}^{[n]}$: $t \cdot f(x,y) = f(t^{-1}x, t^{-N}y)$. We choose $N \in \mathbb{N}$ sufficiently large so that the fixed points of the $\mathbb{C}^*$-action and the T-action are the same. For each $I \in \mathrm{Hilb}^{[n]}$, the limit $t \to 0$ of the action $\phi(t) \cdot (x,y) = (t^{-1}x, t^{-N}y)$ is a T-fixed point I_λ (cf. [30]). Let $U_\lambda := \left\{ I \in \mathrm{Hilb}^{[n]} \,\middle|\, \lim_{t\to 0} \phi(t) \cdot I = I_\lambda \right\}$. We claim that $\mu(I_\lambda) \geq \mu(I)$ for $I \in U_\lambda$. Taking the limit $t \to 0$ of the action induces the monomial ordering "$y \succ x$" and the limit $\lim_{t\to 0} \phi(t) \cdot I = I_\lambda$ is, in fact, the initial ideal of I with respect to this ordering $\succ$. Now, if $I \in U_\lambda$ and G_I is a reduced Göbner basis of I, then by definition, $\mu(I_\lambda) = |G_I|$. Then we have $\mu(I) \leq |G_I| = \mu(I_\lambda)$ by the Buchberger Algorithm construction of the reduced Göbner basis. We conclude that the function $\mu(I), I \in \mathrm{Hilb}^{[n]}$ reaches its maximum value in $\left(\mathrm{Hilb}^{[n]}\right)^T$. ■

Lemma 2.1.6 implies that, for a fixed integer n, $I \in \mathrm{Hilb}^{[n]}$, the value $\mu(I)$ is at most the maximal number of "elbows" that a Young diagram with n boxes can have. We observe that, the Young diagram $\triangle_\lambda$, $\lambda = k-1 \geq k-2 \geq \cdots \geq 1$, is the Young diagram having k elbows and with $\binom{k}{2}$ boxes, which is the least number of boxes among all Young diagrams satisfying this property. Any other Young diagram with k elbows contains $\triangle_\lambda$ as a subdiagram. Since the number of boxes is equal to the codimension of the corresponding monomial ideal, we have $n \geq \binom{k}{2}$. Moreover, if I is an ideal with $\dim_{\mathbb{C}} R/I = \binom{k}{2}$ and $\mu(I) = k$, then it can only be $\mathfrak{m}^{k-1} = \langle y^{k-1}, y^{k-2}x, \ldots, yx^{k-2}, x^{k-1} \rangle$, which is the monomial ideal corresponding to $\triangle_\lambda$. ■

An immediate consequence of Proposition 2.1.5 is that the refined Hilbert scheme $\mathrm{Hilb}^{[n,n+r]}$ is empty if $n < \binom{r}{2}$ and $\mathrm{Hilb}^{[n,n+r]}$ consists of a single point $\left(\mathfrak{m}^{r-1}, \mathfrak{m}^r\right)$ if $n = \binom{r}{2}$, since the condition $I \supset J \supset \mathfrak{m}I$ requires that $\mu_n(I)$ must be at least r. This also implies that $\sigma_n(I) \geq \binom{r}{2}$.

Fix $n \in \mathbb{N}$, the number $\max\left\{ m \,\middle|\, n \geq \binom{m}{2} \right\}$ is $\left\lfloor \left(\frac{-1+\sqrt{1+8n}}{2} \right) + 1 \right\rfloor$, where $\lfloor z \rfloor = \max\{k \in \mathbb{Z} \mid k \geq z\}$ is the floor function.

To sum up, we have the following statement:

Lemma 2.1.7. *Let $(I,J) \in \mathrm{Hilb}^{[n,n+r]}$, then σ_n and μ_n satisfy the following inequalities:*

$$\binom{r}{2} \leq \sigma_n(I,J) \leq n, \tag{2.3}$$

$$r \leq \mu_n(I,J) \leq \left\lfloor \left(\frac{-1+\sqrt{1+8n}}{2} \right) + 1 \right\rfloor. \tag{2.4}$$

2.1.1 Examples of $\mathrm{Hilb}^{[n,n+r]}$ for $r = 2$, $n = 1, 2$ and 3

First, we would like to give examples of $\mathrm{Hilb}^{[n,n+r]}$ to get a better feeling for these spaces.

Example: $\mathrm{Hilb}^{[1,3]}$

If $(I,J) \in \mathrm{Hilb}^{[1,3]}$, then the quotient $I/J \simeq \mathbb{C}^2$ is a subspace of $I/\mathfrak{m}I \simeq \mathbb{C}^{\mu(I)}$. So $\mu(I)$ must be at least 2. The only ideal $I \in \mathrm{Hilb}^{[1]}$ of codimension 1 with $\mu(I) \geq 2$ is the maximal ideal $\langle x, y \rangle = \mathfrak{m}$ of the local ring $\mathcal{O}_{(0,0)}$. Moreover, from the fact that $\dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2 = 2$, J is the ideal $\mathfrak{m}^2$. We see that the Hilbert scheme $\mathrm{Hilb}^{[1,3]} = \left\{ \left(\mathfrak{m}, \mathfrak{m}^2\right) \right\} = Br^{[1,3]}$ is a point.

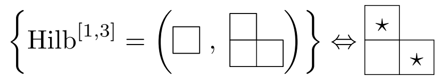


Table 2.3: $\mathrm{Hilb}^{[1,3]}$.

Example: $\mathrm{Hilb}^{[2,4]}$

We show that $\mathrm{Hilb}^{[2,4]}$ is isomorphic to the blow-up of $\mathbb{C}^2$ at $(0,0)$: Recall that $\sigma(I)$ is the multiplicity of $I \in \mathrm{Hilb}^{[n]}$ at the point $(0,0)$. We define strata $\mathrm{Hilb}^{[n]}_{s\geq k}$ with respect to σ:

$$\mathrm{Hilb}^{[n]}_{s\geq k} = \left\{ I \in \mathrm{Hilb}^{[n]} \mid \sigma(I) = s \geq k \right\}.$$

Each fiber of the projection map $\pi : \mathrm{Hilb}^{[2,4]} \to \mathrm{Hilb}^{[2]}_{s\geq 1}$ is a single point: For every ideal $I \in \pi\left(\mathrm{Hilb}^{[2,4]}\right) \subset \mathrm{Hilb}^{[2]}$ $\dim_C I/\mathfrak{m}I = 2$. This is because the image $\pi\left(\mathrm{Hilb}^{[2,4]}\right) \simeq \bigcup_{i=1}^{2}\left(Br^{[i]} \times \mathrm{Hilb}^{[2-i]}\right)$. Since $\mu(I)$ is less or equal to the minimum number of generators of to the fixed point ideal that attract I, elements of $Br^{[1]}$ and $Br^{[2]}$ are ideals with minimum two generators. Then the only possible $J \in \mathrm{Hilb}^{[4]}$ such that $I \supset J \supset \mathfrak{m}I$ is the ideal $\mathfrak{m}I = J$.

We claim that $\pi\left(\mathrm{Hilb}^{[2,4]}\right) \subset \mathrm{Hilb}^{[2]}$ is isomorphic to the blow-up of $\mathbb{C}^2$ at $(0,0)$. If $I \in \pi\left(\mathrm{Hilb}^{[2,4]}\right)$ then $(0,0)$ is always in the support of I. We may represent I as two point $(0,0)$ and P_I, $P_I \in \mathbb{C}^2$. Then $\pi\left(\mathrm{Hilb}^{[2,4]}\right)$ is isomorphic to the space of (P_I, line l contains $(0,0)$, P_I) which is by definition the blow-up of $\mathbb{C}^2$ at the point $(0,0)$.

Example: $\mathrm{Hilb}^{[3,5]}$

Let $(I,J) \in \mathrm{Hilb}^{[3,5]}$. We recall that $\mu(I) = \dim_{\mathbb{C}} I/\mathfrak{m}I$ is the minimum number of generators of I, and define

$$\mathrm{Hilb}^{[n]}_{m} := \left\{ I \in \mathrm{Hilb}^{[n]} \,\middle|\, \mu(I) = m \right\},\ Br^{[n]}_{m} := \left\{ I \in Br^{[n]} \,\middle|\, \mu(I) = m \right\}.$$

Note that, for $r = 2$, we have the image of the projection $\mathrm{Hilb}^{[n,n+2]} \to \mathrm{Hilb}^{[n]}$ is $\mathrm{Hilb}^{[n]}_{s\geq 1} = \mathrm{Hilb}^{[n]}_{m\geq 2}$. Then we have a decomposition of $\pi : \mathrm{Hilb}^{[3,5]}$ by the inverse image of the projection $\pi : \mathrm{Hilb}^{[3,5]} \to \mathrm{Hilb}^{[3]}_{s\geq 1}$:

$$\mathrm{Hilb}^{[3,5]} = \pi^{-1}\left(\mathrm{Hilb}^{[3]}_{2}\right) \bigcup \pi^{-1}\left(\mathrm{Hilb}^{[3]}_{3}\right).$$

We know that the Briançon variety $Br^{[3]}$ is a disjoint union of a smooth stratum $Br^{[3]}_2$ and a singular point $Br^{[3]}_3 = \left\{\mathfrak{m}^2\right\}$ and $\mathrm{Hilb}^{[3]}_3$ is in fact $Br^{[3]}_3$. First, consider the stratum $\mathrm{Hilb}^{[3]}_2$, the fiber of the projection π over this stratum is parametrized by $Gr_2\left(\mathbb{C}^2\right) \simeq \{pt\}$. Therefore, we have $\pi^{-1}\left(\mathrm{Hilb}^{[3]}_2\right) \subset \mathrm{Hilb}^{[3,5]}$ is isomorphic to the disjoint union of smooth strata:

$$\pi^{-1}\left(\mathrm{Hilb}^{[3]}_{2}\right) \simeq \left(Br^{[1]} \times \left(\mathbb{C}^2 \backslash \{(0,0)\}\right)^{[2]}\right) \cup \left(Br^{[2]} \times \mathbb{C}^2 \backslash \{(0,0)\}\right) \cup Br^{[3]}_{2}.$$

Next, over the singular point $\mathfrak{m}^2 = Br_3^{[3]}$, the fibers of π is parametrized by the Grassmannian $Gr_2(\mathbb{C}^3)$. So $\pi^{-1}\left(\mathrm{Hilb}_3^{[3]}\right)$ is isomorphic to $\mathbb{P}^2$.

To summarize, $\mathrm{Hilb}^{[3,5]}$ is ,motivicly , a union of the smooth strata $\mathrm{Hilb}_3^{[3]}$ and a $\mathbb{P}^2$, the blowup at the singular point $\mathfrak{m}^2$.

2.1.2 Properties of the refined Hilbert scheme

We continue to investigate properties of the refined Hilbert scheme. Most importantly, we have the following theorem:

Theorem 2.1.8 ([33]). *The refined Hilbert scheme* $\mathrm{Hilb}^{[n,n+r]}$ *is smooth, irreducible and of dimension* $2n - r(r-1)$.

Proof. We claim that the dimension of tangent space $T_{(I,J)}\mathrm{Hilb}^{[n,n+r]}$ equals to $2n - r(r-1)$ for every $(I, J) \in \mathrm{Hilb}^{[n,n+r]}$.

Observe that, for each $(I, J) \in \mathrm{Hilb}^{[n,n+r]}$, there are universal sequences:

$$0 \to I \to R \to A \to 0, \tag{2.5}$$
$$0 \to J \to R \to B \to 0, \tag{2.6}$$
$$0 \to J \to I \to C \to 0 \Leftrightarrow 0 \to C \to B \to A \to 0. \tag{2.7}$$

We apply by the functors $\mathrm{Hom}_R(C, -)$ and $\mathrm{Hom}_R(-, C)$ to two exact sequences of (2.7) and consider the following double complex:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(C,C) & \xrightarrow{f_1} & \mathrm{Hom}_R(C,B) & \xrightarrow{f_2} & \mathrm{Hom}_R(C,A) & \longrightarrow & \mathrm{Ext}^1_R(C,C) \\
 & & \downarrow d_1 & & \downarrow d_2 & & \downarrow d_3 & & \\
0 & \longrightarrow & \mathrm{Hom}_R(I,C) & \xrightarrow{g_1} & \mathrm{Hom}_R(I,B) & \xrightarrow{g_2} & \mathrm{Hom}_R(I,A) & \to \cdots & \\
 & & \downarrow e_1 & & \downarrow e_2 & & \downarrow e_3 & & \\
0 & \longrightarrow & \mathrm{Hom}_R(J,C) & \xrightarrow{h_1} & \mathrm{Hom}_R(J,B) & \xrightarrow{h_2} & \mathrm{Hom}_R(J,A) & \to \cdots & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{Ext}^1_R(C,C) & & \vdots & & \vdots & &
\end{array} \tag{2.8}$$

By [8], the tangent space of the incidence variety

$$Z_{n_1,n_2} = \left\{(I,J) \in \mathrm{Hilb}^{[n_1]} \times \mathrm{Hilb}^{[n_2]} \,\middle|\, I \supset J\right\}$$

at a point (I, J) is isomorphic to the kernel of the map

$$\begin{aligned}
\phi_I - \phi_J : \mathrm{Hom}_R(I,A) \oplus \mathrm{Hom}_R(J,B) &\to \mathrm{Hom}_R(J,A) \\
(f,g) &\mapsto \phi_I(f) - \phi_J(g),
\end{aligned}$$

where

$$\phi_I : \mathrm{Hom}_R(I, A) \to \mathrm{Hom}_R(J, A), \quad \alpha : I \to A \mapsto \phi_I(\alpha) = \alpha|_J \tag{2.9}$$
$$\phi_J : \mathrm{Hom}_R(J, B) \to \mathrm{Hom}_R(J, A), \quad \alpha : I \to A \mapsto \phi_J(\alpha) = pr \circ \alpha : J \to B \twoheadrightarrow A \tag{2.10}$$

are two maps following from the condition $I \supset J$. In other words, $(f, g) \in T_I \mathrm{Hilb}^{[n]} \times \mathrm{Hilb}^{[n+r]}$ if their images in $\mathrm{Hom}_R(J, A)$ coincide. Observing the diagram (2.8), there are two maps to $\mathrm{Hom}_R(I, A)$ and $\mathrm{Hom}_R(J, B)$ respectively. But only the contribution from $\mathrm{Hom}_R(I, B)$ is compatible with the inclusion $J \supset \mathfrak{m}I$. Since the maps f_1, d_1 and g_1 are injective, and there is an embedding $\mathrm{Hom}_R(C, C) \hookrightarrow \mathrm{Hom}_R(I, B)$. Moreover, $\mathrm{Hom}_R(C, C)$ is contained in the kernel of maps $g_2 : \mathrm{Hom}_R(I, B) \to \mathrm{Hom}_R(I, A)$ and $e_2 : \mathrm{Hom}_R(I, B) \to \mathrm{Hom}_R(J, B)$. We claim that the quotient $\mathrm{Hom}_R(I, B) / \mathrm{Hom}_R(C, C)$ has constant dimension. First, since $C = I/J$ possess a trivial R-module structure, the dimension $\dim_{\mathbb{C}} \mathrm{Hom}_R(C, C)$ is r^2. Next, applying the functor $\mathrm{Hom}_R(-, B)$ to the exact sequence (2.5), there is a long eact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_R(A, B) \to \mathrm{Hom}_R(R, B) \to \mathrm{Hom}_R(I, B) \\ &\to \mathrm{Ext}^1_R(A, B) \to \mathrm{Ext}^1_R(R, B) \to \mathrm{Ext}^1_R(I, B) \\ &\to \mathrm{Ext}^2_R(A, B) \to \mathrm{Ext}^2_R(R, B) \to \mathrm{Ext}^2_R(I, B) \to 0, \end{aligned}$$

where $\mathrm{Ext}^1_R(R, B) = \mathrm{Ext}^2_R(R, B) = 0$. To compute the dimension of $\mathrm{Hom}_R(I, B)$, we first note that $\dim_R(A, B) = n + r - 1$. It is because the image of the generator 1 of A can be any base element of B except for 1. From the duality, $\mathrm{Ext}^2_R(A, B) \simeq \mathrm{Hom}_R(B, A)^\vee$ has dimension n. Since the Euler characteristic $\chi(A, B) = \sum_{i=0}^{2} (-1)^i \dim_{\mathbb{C}} \mathrm{Ext}^i(A, B)$ equals 0, we have $\dim_{\mathbb{C}} \mathrm{Ext}^1(A, B) = 2n + r - 1$. Now from the long exact sequence

$$0 \to \mathrm{Hom}_R(A, B) \to \mathrm{Hom}_R(R, B) \xleftrightarrow{\alpha} \mathrm{Hom}_R(I, B) \xleftrightarrow{\beta} \mathrm{Ext}^1_R(A, B) \to 0, \tag{2.11}$$

we have short exact sequences

$$\begin{aligned} &0 \to \mathrm{Hom}_R(A, B) \to \mathrm{Hom}_R(R, B) \to \mathrm{Im}(\alpha) \to 0 \\ &0 \to \mathrm{Im}(\alpha) \to \mathrm{Hom}_R(I, B) \to \mathrm{Im}(\beta) = \mathrm{Ext}^1_R(A, B) \to 0. \end{aligned}$$

It follows that

$$\begin{aligned} \dim_{\mathbb{C}} \mathrm{Hom}_R(I, B) &= \dim_{\mathbb{C}} \mathrm{Ext}^1_R(A, B) + \dim_{\mathbb{C}} \mathrm{Im}(\alpha) \\ &= \dim_{\mathbb{C}} \mathrm{Ext}^1_R(A, B) + (\dim_{\mathbb{C}} \mathrm{Hom}_R(R, B) - \dim_{\mathbb{C}} \mathrm{Hom}_R(A, B)) \\ &= 2n + r - 1 - (n + r - (n + r - 1)) = 2n + r. \end{aligned}$$

Thus we have $\dim_{\mathbb{C}} \mathrm{Hom}_R(I, B) / \mathrm{Hom}_R(C, C) = 2n + r - r^2 = 2n - r(r - 1)$.

■

2.2 Torus action, Bialynicki-Birula cell decomposition and homology of $\mathrm{Hilb}^{[n,n+r]}$

Consider the torus action on $\mathrm{Hilb}^{[n,n+r]}$ $t \cdot (I,J) = (t \cdot I, t \cdot J)$, where the fixed points are pairs of monomial ideals (I,J) satisfying $I \supset J \supset \mathfrak{m}I$. If we choose a generic one-parameter subgroup

$$\begin{aligned} \mathbb{C}^* &\to T \\ t &\mapsto (t_1, t_2) = (t^{\alpha_1}, t^{\alpha_2}), \end{aligned}$$

then the T-fixed points set is finite and it induces a cell decomposition of $\mathrm{Hilb}^{[n,n+r]}$. Each cell is represented by a T-fixed point, which is described by a Young diagram $\triangle$ with $|\triangle| = n+r$ and r marked removable boxes. Moreover, by [25], the refined Briançon variety is a deformation retract of $\mathrm{Hilb}^{[n,n+r]}$. For generic one-parameter subgroup $\mathbb{C}^* \to T$, we have the $\pm$-attracting cells

$$S_\triangle := \left\{ (I,J) \in \mathrm{Hilb}^{[n,n+r]} \,\middle|\, \lim_{t\to 0} t \cdot (I,J) \in \triangle \right\}, \tag{2.12}$$

$$U_\triangle := \left\{ (I,J) \in \mathrm{Hilb}^{[n,n+r]} \,\middle|\, \lim_{t\to \infty} t \cdot (I,J) \in \triangle \right\}. \tag{2.13}$$

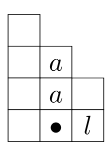
Analogous to lemma 1.2.4, we have

$$\biguplus_{\triangle} S_\triangle = \mathrm{Hilb}^{[n,n+r]} \text{ and } \biguplus_{\triangle} U_\triangle = Br^{[n,n+r]}, \tag{2.14}$$

where the summation runs over all Young diagrams $\triangle$ with $|\triangle| = n+r$ and r marked removable boxes.

The character $\mathrm{ch}_{(I,J)}\mathrm{Hilb}^{[n,n+r]}$ of the T-action at the tangent space at a fixed point (I,J) were calculated by Nakajima and Yoshioka. They also proved a formula for the generating function of the Poincaré polynomials of $\mathrm{Hilb}^{[n,n+r]}$. To state their results, we recall notations $l(\boxed{\bullet})$ and $a(\boxed{\bullet})$:

- the arm-length $a\left(\boxed{\bullet}\right)$ to be the number of the boxes above $\boxed{\bullet}$,
- the leg-length $l\left(\boxed{\bullet}\right)$ to be the number of the boxes on right of $\boxed{\bullet}$.

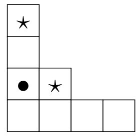


Example:
The arm-length and the leg-length of $\boxed{\bullet}$ are the number of as and ls respectively.

Proposition 2.2.1 ([34])**.** *Let $\triangle_{(I,J)}$ be a fixed point of $\mathrm{Hilb}^{[n,n+r]}$. We say a box b of $\triangle$ is "relevant" if it is neither a marked removable box nor both of the up-most box and right-most box of b are marked removable boxes. We denote by $\triangle' \subset \triangle$ the subdiagram obtained by removing all marked removable boxes $\boxed{\star}$ from $\triangle$ (e.g., Table 2.4).*

We have the character formula of $T_{(I,J)}\mathrm{Hilb}^{[n,n+r]}$:

$$\mathrm{ch}T_{(I,J)}\mathrm{Hilb}^{[n,n+r]} = \sum_{b:\text{relevant}} \left(t_1^{-l_\triangle(b)} t_2^{a_{\triangle'}(b)+1} + t_1^{l_{\triangle'}(b)+1} t_2^{-a_\triangle(b)} \right). \tag{2.15}$$



$\Leftrightarrow (\langle x^4, xy, y^3\rangle, \langle x^4, x^2y, xy^2, y^4\rangle) \in \mathrm{Hilb}^{[6,8]}$

Table 2.4: Empty boxes $\square$ are the relevant boxes. $\triangle' = \triangle \backslash \{\star\}$s.

Proposition 2.2.2 ([34]). *Let r, n be positive integers with $n \geq \binom{r}{2}$. The Poincaré polynomial of* $\mathrm{Hilb}^{[n,n+r]}$ *has the form*

$$P\left(\mathrm{Hilb}^{[n,n+r]}; t\right) = \sum_{\substack{\triangle : |\triangle| = n+r, \\ \text{with } r \text{ marked} \\ \text{removable boxes}}} t^{(2n+3r-r^2)-2w(\triangle)} \tag{2.16}$$

where $w(\triangle)$ is the number of columns in $\triangle$. Moreover, it has the generating function:

$$\sum_{n=\binom{r}{2}}^{\infty} P\left(\mathrm{Hilb}^{[n,n+r]}; t\right) q^n = q^{\binom{r}{2}} \left(\prod_{d=1}^{\infty} \frac{1}{1-t^{d-1}q^d}\right)\left(\prod_{d=1}^{r} \frac{1}{1-t^d q^d}\right). \tag{2.17}$$

Proof. By [30, Cor. 5.10], the dimension of the affine cell equals to the number of negative weights. It is equivalent to count the dimension of sum of weight spaces which satisfy either

(i) the weight of t_2 is negative,

(ii) or the weight of t_2 is 0 and the weight of t_1 is negative.

The second case cannot happen. Therefore, it is number of relevant boxes b with $a(b) > 0$ and is equal to

$$|\triangle| - \binom{r}{2} - w(\triangle) = \frac{1}{2}\left(2n - r^2 + 3r\right) - w(\triangle),$$

and we have

$$P\left(\mathrm{Hilb}^{[n,n+r]}; t\right) = \sum_{\triangle} t^{2\left(n + \frac{-r^2+3r}{2} - w(\triangle)\right)} \tag{2.18}$$

Next, to prove the formula for the generating function, we use the fact that the set of Young diagrams with $n+r$ boxes and r marked removable boxes is in bijection with the set of pairs of Young diagrams $(\triangle_1, \triangle_2)$ such that $\triangle_2$ has at most r columns and $|\triangle_1| + |\triangle_2| = n - \binom{r}{2}$ (cf. [[34], Section 5.5]). Then the equation (2.18) becomes

$$P\left(\mathrm{Hilb}^{[n,n+r]}; t\right) = \sum t^{|\triangle_1| + |\triangle_2| - w(\triangle_1)}, \tag{2.19}$$

where the summation runs over all pairs of Young diagrams $(\triangle_1, \triangle_2)$ such that Y_2 has at most r columns and $|\triangle_1| + |\triangle_2| = n - \binom{r}{2}$. It follows that the generating function is

$$\sum_{n=\binom{r}{2}}^{\infty} P\left(\mathrm{Hilb}^{[n,n+r]}; t\right) q^n = q^{\binom{r}{2}} \left(\prod_{d=1}^{\infty} \frac{1}{1-t^{(d-1)}q^d}\right)\left(\prod_{d=1}^{r} \frac{1}{1-t^d q^d}\right). \tag{2.20}$$

■

Theorem 2.2.3. *The refined Briançon variety $Br^{[n,n+r]}$ is of complex dimension $n - \binom{r}{2}$. Furthermore, the number of top-dimensional components of $Br^{[n,n+r]}$ is equal to the number of Young diagrams with $n+r$ boxes with r columns.*

Proof. Choose a generic one-parameter subgruop $\mathbb{C}^* \to T$ so that we have a cell decomposition of $Br^{[n,n+r]}$. Since $Br^{[n,n+r]}$ is a deformation retract of $\mathrm{Hilb}^{[n,n+r]}$, by equation 2.18 of the Poincaré polynomial of $\mathrm{Hilb}^{[n,n+r]}$, we have that $Br^{[n,n+r]}$ is a disjoint union of affine cells of complex dimension $\left(n + \frac{-r^2+3r-2w(\triangle)}{2}\right)$, where $\triangle \in \left(\mathrm{Hilb}^{[n,n+r]}\right)^T$. We know also each $\triangle$ contains at least r removable boxes $\Rightarrow w(\triangle) \geq r$. We conclude that $\dim_{\mathbb{C}} Br^{[n,n+r]} = n + \frac{-r^2+r}{2}$ and top-dimensional components are indexed by Young diagrams $\triangle$ with exactly r columns and $|\triangle| = n + r$. ■

2.3 Operators on the direct sum of homology groups of $\mathrm{Hilb}^{[n,n+r]}$

In the case of $\mathrm{Hilb}^{[n]}$, the generating function of $P\left(\mathrm{Hilb}^{[n]}; t\right)$ has a form of the infinite product $\sum_n P\left(\mathrm{Hilb}^{[n]}; t\right) q^n = \prod_{d=1}^{\infty} \frac{1}{1-t^{2d-2}q^d}$ (1.17). Grojnowski ([20]) and Nakajima ([29]) show a geometric interpretation as the character of an infinite dimensional Heisenberg algebra action on $\oplus_n H_*\left(\mathrm{Hilb}^{[n]}\right)$ through operators on $\mathbb{H}$.

We observe that in the formula for the generating function of $P\left(\mathrm{Hilb}^{[n,n+r]}; t\right)$:

$$\sum_{n=\binom{r}{2}}^{\infty} P\left(\mathrm{Hilb}^{[n,n+r]}; t\right) q^n = q^{\binom{r}{2}} \left(\prod_{d=1}^{\infty} \frac{1}{1-t^{2d-2}q^d}\right) \left(\prod_{d=1}^{r} \frac{1}{1-t^2q^d}\right),$$

there are two products: an infinite product $\displaystyle\prod_{d=1}^{\infty} \frac{1}{1-t^{2d-2}q^d}$ and a finite product $\prod_{d=1}^{r} \frac{1}{1-t^2q^d}$. This observation suggests that there exists an action of Lie algebra of operators on the infinite direct sum of the homology of the refined Hilbert schemes

$$\widetilde{\mathbb{H}}_r := \bigoplus_{n \geq \binom{r}{2}} H_*\left(\mathrm{Hilb}^{[n,n+r]}\right)$$

with the character (2.15). In particular, there should be infinitely many operators of degree $2(d-1)$ and r operators of degree $2d$ that correspond to $\prod_{d=1}^{\infty} \frac{1}{1-t^{2d-2}q^d}$ and $\prod_{d=1}^{r} \frac{1}{1-t^2q^d}$ respectively.

2.3.1 Operators of Nakajima's type

We give a construction of the operators of degree $(2d-2)$ through the correspondence (cf. 1.2.2). We refer to these operators are of *Nakajima's type* since our construction is analogous to Nakajima's construction for $\mathrm{Hilb}^{[n]}$ (cf. [29, 30]).

Denoted by π_{ij}, π_k the projections from $\mathrm{Hilb}^{[n,n+r]} \times \mathrm{Hilb}^{[n+k,n+k+r]} \times \mathbb{C}^2$ to (i,j)-th and k-th factors respectively. For a positive integer k, the projection π_2 to $\mathrm{Hilb}^{[n+k,n+k+r]}$ is proper, whereas the projection π_1 to $\mathrm{Hilb}^{[n,n+r]}$ is not.

First, let

$$Z(r,k)_n := \left\{((I,J),(K,L),p) \in \mathrm{Hilb}^{[n,n+r]} \times \mathrm{Hilb}^{[n+k,n+k+r]} \times \mathbb{C}^2 \backslash \{(0,0)\} \,\middle|\, I \subseteq K, \mathrm{Supp}(I \backslash K) = \{p\}\right\}. \tag{2.21}$$

We defined an incidence variety $\overline{Z(r,k)_n}$ in $\mathrm{Hilb}^{[n,n+r]} \times \mathrm{Hilb}^{[n+k,n+k+r]} \times \mathbb{C}^2$ by taking the closure of $Z(r,k)_n$.

Proposition 2.3.1. *The variety* $\overline{Z(r,k)}_n$ *has dimension* $2n - r(r-1) + k + 1$.

Proof. Recall that an element of $\overline{Z(r,k)}$ is of the form $((I,J),(K,L),p)$. We decompose $\overline{Z(r,k)}_n$ into strata by the multiplicity $m_{p,I}$ of I at each point p in the support of I:

$$\overline{Z(r,k)}_n = \bigcup_{s=0}^{n} \overline{Z(r,k)}_n(s),$$

where $\overline{Z(r,k)}_n(s) := \{((I,J),(K,L),p) \mid m_{p,I} = s\}$. We claim that $\overline{Z(r,k)}_n(0)$ is in fact the generic part of $\overline{Z(r,k)}_n$. When $s = 0$, $\overline{Z(r,k)}_n(0)$ consists of element $((I,J),(K,L),p)$ such that the point p is not in the support of I and it gives rise to a birational map

$$\begin{aligned} \overline{Z(r,k)}_n(0) &\to \mathrm{Hilb}^{[n,n+r]} \times Br^{[k]}_{\mathbb{C}^2} \\ ((I,J),(K,L),p) &\mapsto ((I,J),B_p)\,. \end{aligned}$$

Then we get the dimension of $\overline{Z(r,k)}_n(0)$

$$\begin{aligned} \dim_{\mathbb{C}} \overline{Z(r,k)}_n(0) &= \dim_{\mathbb{C}} \mathrm{Hilb}^{[n,n+r]} + \dim_{\mathbb{C}} Br^{[k]}_{\mathbb{C}^2} \\ &= 2(n - \binom{r}{2}) + (k+1) = 2n - r(r-1) + k + 1. \end{aligned}$$

Now, for $s > 1$, we have $\dim \overline{Z(r,k)}(s) < 2n - r(r-1) + k + 1$. It is because that if we add/subtract one point, dimension of $\mathrm{Hilb}^{[n,n+r]}$ increases/drops by 2, $Br^{[k]}_{\mathbb{C}^2}$ increases/drops only by 1. We conclude that $\dim_{\mathbb{C}} \overline{Z(r,k)} = 2n - r(r-1) + k + 1$. ∎

It follows that $\overline{Z(r,k)}_n$ defines a class $[\overline{Z(r,k)}_n] \in H_{2n-r(r-1)+k+1}\left(\overline{Z(r,k)}_n\right)$.

If $\alpha \in H_*(\mathbb{C}^2)$, $\beta \in H_*(\mathbb{C}^2)^{BM}$ and $u \in H_i\left(\mathrm{Hilb}^{[n,n+r]}\right)$, then the operators $\beta_k(\cdot)$ of adding k points and the operators $\alpha_k(\cdot)$ of subtracting k points are defined by

$$\alpha_k(u) = \pi_{2*}\left(\left(\pi_1^*(\mathrm{PD}^{-1}u) \cup \pi_3^*(\mathrm{PD}^{-1}\alpha)\right) \cap [\overline{Z(r,k)}_n]\right) \in H_{i+(k-1)}\left(\mathrm{Hilb}^{[n+k,n+k+r]}\right) \tag{2.22}$$

and

$$\beta_k(y) = \pi_{1*}\left(\left(\pi_2^*(\mathrm{PD}^{-1}u) \cup \pi_3^*(\beta)\right) \cap [\overline{Z(r,k)}_n]\right) \in H_*\left(\mathrm{Hilb}^{[n-k,n-k+r]}\right) \tag{2.23}$$

respectively, where α_k has degree $\deg \alpha + k - 1$.

Here, the map $((I,J),(K,L),p) \mapsto \rho(I \setminus K)$ restricted on $\overline{Z(r,k)}_n$ is the relative Hilbert-Chow morphism, which is proper. These operators are well-defined on $\widetilde{\mathbb{H}}_r$:
To ensure that we do get a homology class, we need at least one of pull-back classes $\pi_j^*(\cdot)$ to be compactly supported. $\pi_1^*(\mathrm{PD}^{-1}u)$ is not the pull-back of a proper map whereas $\pi_3^*(\mathrm{PD}^{-1}\alpha)$ is compactly support pull-back by a proper map. $\pi_2^*(\mathrm{PD}^{-1}u)$ is a pull-back through a proper map, so there is no further requirements on $\pi_3^*(\beta)$.

Proposition 2.3.2. *Let $\beta \in H_*(\mathbb{C}^2), \alpha \in \overline{H}_*(\mathbb{C}^2)$. The operators α_k, β_k satisfy the Heisenberg relation*

$$[\alpha_i, \beta_j] = i\delta_{i+j,0}\langle \alpha, \beta\rangle \mathrm{id}_{\widetilde{\mathbb{H}}_r}$$

where the pairing $\langle \alpha, \beta\rangle$ stands for the push-forward a point $f_*(\alpha \cap \beta)$, $f\colon \mathbb{C}^2 \to \{pt\}$.

We note that the only nontrivial choice of such α and β is $\alpha = [pt], \beta = [\mathbb{C}^2]$.

Proof. We proceed a proof that is analogous to the Nakajima's proof for $\mathrm{Hilb}^{[n]}$ by considering separated cases for $i + j = 0$ and $i + j \neq 0$ (cf. [30, Proposition 2.3.2]). The main idea of the proof is that when $i \neq j$, the support of the corresponding correspondence has a dimension strictly less than the expected dimension and the only contribution comes from the case of $i = j$.

By $\alpha_i\beta_j$, we mean the composition which applies first the operator α_i and then the operators β_j and $[\alpha_i, \beta_j] = \alpha_i\beta_j - \beta_j\alpha_i$. We define the following notations:

$$\begin{aligned}
Y_{i,j} &:= \mathrm{Hilb}^{[n,n+r]} \times \mathbb{C}^2 \times \mathrm{Hilb}^{[n+i,n+i+r]} \times \mathbb{C}^2 \times \mathrm{Hilb}^{[n+i+j,n+i+j+r]},\\
c_{ij} &:= [\overline{Z(r,i)}_n][\overline{Z(r,j)}_{n+i}] = \pi_{1245*}\left(\pi_{123}^*[\overline{Z(r,i)}_n]\pi_{345}^*[\overline{Z(r,j)}_{n+i}]\right),\\
C_{ij} &:= \pi_{1245}\left(\pi_{123}^{-1}(\overline{Z(r,i)}_n)\pi_{345}^{-1}(\overline{Z(r,j)}_{n+i})\right),\\
\{p \neq q\} &:= \coprod_{i,j} \{(I,p,q,J) \in \pi_{1245}(Y_{ij}) \mid p \neq q\},\\
\{p = q\} &:= \coprod_{i,j} \{(I,p,q,J) \in \pi_{1245}(Y_{ij}) \mid p = q\},
\end{aligned}$$

where π_- are the projections to $(-)$-th factors.

Note that the map $\mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2 \times \mathbb{C}^2, (p,q) \mapsto (q,p)$ on $\{p \neq q\}$ is bijective. This induces an isomorphism

$$C_{ij} \cap \{p \neq q\} \to C_{ji} \cap \{p \neq q\}$$

by $((I,J),p,q,(I'J')) \mapsto ((I,J),q,p,(I'J'))$. Thus, $\pi_{123}^{-1}\left(\overline{Z(r,i)}_n\right)$ and $\pi_{345}^{-1}\left(\overline{Z(r,j)}_{n+i}\right)$ intersect transversally over $\{(p,q), p \neq q\}$, where the dimension is given by

$$\begin{aligned}
&(2n - r(r-1) + i + 1) + (2(n+i) - r(r-1) + j + 1) - 2(n + i - \binom{r}{2})\\
&= 2n + i + j + 2 - r(r-1).
\end{aligned}$$

On the other hand, the subset $\{(p,q) \mid p = q\}$ of $\mathbb{C}^2 \times \mathbb{C}^2$ is isomorphic to $\mathbb{C}^2$ under the diagonal morphism $\Delta\colon \mathbb{C}^2 \to \mathbb{C}^2 \times \mathbb{C}^2, p \mapsto (p,p)$. This induces a map

$$\begin{gathered}
C_{ij} \cap \{p = q\} \to Z(n, i+j)\\
((I,J),p,q,(K,L)) \mapsto ((I,J), p = q, (K,L)),
\end{gathered}$$

which is compatible with the extended Hilbert-Chow morphism $((I,J),(K,L)) \mapsto \rho(I \setminus K)$ by the fact that $\alpha \cap \beta = \Delta^*(\alpha \times \beta)$. Thus, the component $C_{ij} \cap \{p = q\}$ has dimension $2n - r(r-1) + i + j + 1$, which is less than the expected dimension.

Case of $i+j\neq 0$ and $ij\geq 0$: Suppose that $i>0$ and $j>0$, we first consider the case of $[\alpha_i,\beta_j]$. We observe that the set-theoretical support of the composition $\alpha_i\beta_j$ is given by the $C_{ij}\subset \mathrm{Hilb}^{[n,n+r]}\times\mathbb{C}^2\times\mathbb{C}^2\times\mathrm{Hilb}^{[n+i+j,n+i+j+r]}$, where

$$C_{ij}=\left\{((I,J),p,q,(I'J'))\,\middle|\,\exists(K,L)\in\mathrm{Hilb}^{[n+i,n+i+r]},\rho(K)=\rho(I)+i[p],\rho(I')=\rho(K)+j[q]\right\}.$$

Considering $\mathbb{C}^2\times\mathbb{C}^2=\{p\neq q\}\cup\{p=q\}$, we will discuss the cases $C_{ij}\cap\{p\neq q\}$ and $C_{ij}\cap\{p=q\}$ separately. As we pointed out previously, a generic element of $\overline{Z(r,i)}_n$ is an element (I,J) of $\mathrm{Hilb}^{[n,n+r]}$ where the support of I is disjoint from the point adding. One can identify $C_{ij}\cap\{p\neq q\}$ with the closure of the image of the map

$$\begin{aligned}\mathrm{Hilb}^{[n,n+r]}\times Br^{[i]}_{\mathbb{C}^2}\times Br^{[j]}_{\mathbb{C}^2}&\to\mathrm{Hilb}^{[n,n+r]}\times\mathbb{C}^2\times\mathbb{C}^2\times\mathrm{Hilb}^{[n+i+j,n+i+j+r]}\\ ((I,J),B_p,B_q)&\mapsto((I,J),p,q,(I\cap B_p\cap B_q,J\cap B_p\cap B_q)))\end{aligned}$$

with dimension $2(n-\binom{r}{2})+(i+1)+(j+1)$.

For the case of $C_{ij}\cap\{p=q\}$, under the identification of $\mathbb{C}^2\times\mathbb{C}^2$ with $\mathbb{C}^2$ by the diagonal $\Delta(p,p)=p\in\mathbb{C}^2$, there is a birational map

$$\begin{aligned}C_{ij}\cap\{p=q\}&\to\overline{Z(r,i+j)}_n\\ ((I,J),p,p,(I',J))&\mapsto((I,J),(I',J'),p)\,.\end{aligned}$$

Since $\dim\overline{Z(r,i+j)}_n=2n-r(r-1)+i+j+1$ is less than the expected dimension of c_{ij}, $C_{ij}\cap\{p=q\}$ does contribute to $c_{i,j}$. And similarly, $C_{ji}\cap\{p=q\}$ does not contribution to the intersection for the same dimensional reason.

Now, for the case of $[\alpha_{-i},\beta_{-j}]$, both $C_{-i-j}\cap\{p\neq q\}$ and $C_{-j-i}\cap\{p\neq q\}$ can be identified with the closure of the image of the map

$$\begin{aligned}\mathrm{Hilb}^{[n-i-j,n-i-j+r]}\times Br^{[i]}_{\mathbb{C}^2}\times Br^{[j]}_{\mathbb{C}^2}&\to\mathrm{Hilb}^{[n,n+r]}\times\mathbb{C}^2\times\mathbb{C}^2\times\mathrm{Hilb}^{[n-i-j,n+i+j+r]}\\ ((I,J),B_p,B_q)&\mapsto((I\cap B_p\cap B_q,J\cap B_p\cap B_q),p,q,(I,J))),\end{aligned}$$

where $\mathrm{Hilb}^{[n-i-j,n-i-j+r]}\times Br^{[i]}_{\mathbb{C}^2}\times Br^{[j]}_{\mathbb{C}^2}$ has dimension $2(n-i-j-\binom{r}{2})+(i+1)+(j+1)=2n-i-j+2-r(r-1)$, which is the same as the dimension of c_{-i-j} and c_{-j-i}. Whereas $C_{-i-j}\cap\{p=q\}$ and $C_{-j-i}\cap\{p=q\}$ has strictly smaller dimensions after a similar argument as above.

Thus $c_{-i-j}-c_{-j-i}=0$.

Case of $i+j\neq 0$ and $ij\leq 0$: Without losing generality, we compute for the case $[\alpha_i,\beta_{-j}]$.

$C_{i-j}\cap\{p\neq q\}$ and $C_{-ji}\cap\{p\neq q\}$ can be both identified by the closure of the image of the map

$$\begin{aligned}\mathrm{Hilb}^{[n-j,n-j+r]}\times Br^{[i]}_{\mathbb{C}^2}\times Br^{[j]}_{\mathbb{C}^2}&\to\mathrm{Hilb}^{[n,n+r]}\times\mathbb{C}^2\times\mathbb{C}^2\times\mathrm{Hilb}^{[n+i-j,n+i-j+r]}\\ ((I,J),B_p,B_q)&\mapsto((I\cap B_q,J\cap B_q),p,q,I\cap B_p,J\cap B_p)))\end{aligned}$$

where $(I,J),B_p,B_q$ have disjoint supports. Moreover, $\mathrm{Hilb}^{[n-j,n-j+r]}\times Br^{[i]}_{\mathbb{C}^2}\times Br^{[j]}_{\mathbb{C}^2}$ has dimension $2(n-j-\binom{r}{2})+(i+1)+(j+1)=2n-j+i+2-r(r-1)$.

$C_{i-j} \cap \{p = q\}$ can be parametrized by

$$\begin{aligned} \mathrm{Hilb}^{[n-j,n-j+r]} \times Br^{[i]}_{\mathbb{C}^2} \times Br^{[j]}_{\mathbb{C}^2} &\to \mathrm{Hilb}^{[n,n+r]} \times \Delta^{-1}(\mathbb{C}^2 \times \mathbb{C}^2) \times \mathrm{Hilb}^{[n+i-j,n+i-j+r]} \\ \big((I,J), B_p, B'_p\big) &\mapsto ((I,J), p, (I \cap B_p, J \cap B_p)) \end{aligned}$$

Case of $i + j = 0$: Let

$$\begin{aligned} Y^+ &:= \mathrm{Hilb}^{[n,n+r]} \times \mathbb{C}^2 \times \mathrm{Hilb}^{[n+i,n+i+r]} \times \mathbb{C}^2 \times \mathrm{Hilb}^{[n,n+r]} \\ Y^- &:= \mathrm{Hilb}^{[n,n+r]} \times \mathbb{C}^2 \times \mathrm{Hilb}^{[n-i,n-i+r]} \times \mathbb{C}^2 \times \mathrm{Hilb}^{[n,n+r]}, \end{aligned}$$

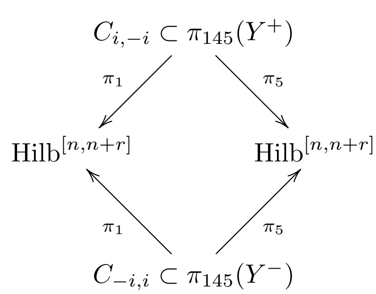
and projections $\pi_{(-)}$ to $(-)$-factors. The compositions $\alpha_i\alpha_{-i}$ and $\alpha_{-i}\alpha_i$ are induced by the classes

$$\begin{aligned} c_{i,-i} &:= \pi_{1245*}\left(\pi^*_{123}([\overline{Z(r,i)}_n]) \cap \pi^*_{345}([\tau\overline{Z(r,i)}_n])\right) \\ c_{-i,i} &:= \pi_{1245*}\left(\pi^*_{123}([\tau\overline{Z(r,i)}_{n-i}]) \cap \pi^*_{345}([\overline{Z(r,i)}_{n-i}])\right) \end{aligned}$$

and are supported on the sets

$$\begin{aligned} C_{i,-i} &:= \pi_{1245}\left(\pi^{-1}_{123}(\overline{Z(r,i)}_n) \cap \pi^{-1}_{345}(\tau\overline{Z(r,i)}_n)\right) \\ &= \left\{((I,J),p,q,(I',J')) \,\middle|\, \exists (K,L) \in \mathrm{Hilb}^{[n+i,n+i+r]} : I \supset K \subset I', \mathrm{Supp}(I/K) = p, \mathrm{Supp}(I'/K) = \{q\}\right\} \\ C_{-i,i} &:= \pi_{1245}\left(\pi^{-1}_{123}(\tau\overline{Z(r,i)}_{n-i}) \cap \pi^{-1}_{345}(\overline{Z(r,i)}_{n-i})\right) \\ &= \left\{((I,J),p,q,(I',J')) \,\middle|\, \exists (\bar{K},\bar{L}) \in \mathrm{Hilb}^{[n-i,n-i+r]} : I \subset \bar{K} \supset I', \mathrm{Supp}(\bar{K}/I) = \{p\}, \mathrm{Supp}(\bar{K}/I') = \{q\}\right\} \end{aligned}$$

with expected dimension $2\,(2n - r(r-1) + i) - (2(n+i) - r(r-1)) = 2n - r(r-1)$.



$C_{i,-i}$ has 2 irreducible components of dimension $2n - r(r-1)$:

- The component $C_{i,-i} \cap \{p \neq q\} = \overline{\phi\left(\mathrm{Hilb}^{[n-i,n-i+r]} \times Br^{[i]}_{\mathbb{C}^2} \times Br^{[i]}_{\mathbb{C}^2}\right)}$ where

  $$\begin{aligned} \phi\colon \mathrm{Hilb}^{[n-i,n-i+r]} \times Br^{[i]}_{\mathbb{C}^2} \times Br^{[i]}_{\mathbb{C}^2} &\to C_{i,-i} \cap \{p \neq q\} \\ ((I,J), B_p, B_q) &\mapsto ((I \cup B_p, J \cup B_p), \rho(B_p), \rho(B_q), (I \cup B_q, J \cup B_q)) \end{aligned}$$

  for the ideals $((I,J), B_p, B_q) \in \mathrm{Hilb}^{[n-i,n-i+r]} \times Br^{[i]}_{\mathbb{C}^2} \times Br^{[i]}_{\mathbb{C}^2}$ having disjoint supports.

- The component $C_{i,-i} \cap \{p = q\}$

  $= \left\{((I,J),p,p,(I',J')) \,\middle|\, \exists(\bar{K},\bar{L}) \in \mathrm{Hilb}^{[n-i,n-i+r]} : I \subset \bar{K} \supset I', \mathrm{Supp}(\bar{K}/I) = \{p\} = \mathrm{Supp}(\bar{K}/I')\right\},$

  which is $\Delta\left(\mathrm{Hilb}^{[n,n+r]}\right)$ the diagonal of $\mathrm{Hilb}^{[n,n+r]}$.

We proceed the same decomposition for $C_{-i,i}$

$$C_{-i,i} = (C_{-i,i} \cap \{p \neq q\}) \cup (C_{-i,i} \cap \{p = q\}).$$

$C_{-i,i} \cap \{p \neq q\}$ can be parametrized by the same map

$$\begin{aligned} \phi\colon \mathrm{Hilb}^{[n-i,n-i+r]} \times Br^{[i]}_{\mathbb{C}^2} \times Br^{[i]}_{\mathbb{C}^2} &\to C_{i,-i} \cap \{p \neq q\} \\ ((I,J),B_p,B_q) &\mapsto ((I \cup B_p, J \cup B_p), \rho(B_p), \rho(B_q), (I \cup B_q, J \cup B_q)). \end{aligned}$$

As for the piece $C_{-i,i} \cap \{p = q\}$, note that $((I,J),q,p,(I',J')) \in C_{-i,i} \cap \{p = q\}$ implies $(I,J) = (I',J') \in \mathrm{Hilb}^{[n,n+r]}$ $\left(s \geq i + \binom{r}{2}\right)$ which is non-generic. Thus, it does not contribute to this class.

Since $C_{i,-i}$ and $C_{-i,i}$ intersects transversally over $\{p \neq q\}$, we have

$$C_{i,-i} - C_{-i,i} = C_{i,-i} \cap \{p = q\} = c\left[\Delta\left(\mathrm{Hilb}^{[n,n+r]}\right)\right]$$

for some constant $N \in \mathbb{C}$. In order to determine the constant N, we consider locally near a generic point of $\left(\pi_{123}^{-1}(\overline{Z(r,i)}_n) \cap \pi_{345}^{-1}(\tau\overline{Z(r,i)}_n)\right)\Big|_{\{p=q\}} \subset \mathrm{Hilb}^{[n,n+r]} \times \mathbb{C}^2 \times \mathrm{Hilb}^{[n+i,n+i+r]}$, which are elements $((I,J),p,(K,L))$ such that $\mathrm{Supp}(I) \cap \{p\} = \emptyset$. Near a generic point $((I,J),p,(K,L))$, the product $\mathrm{Hilb}^{[n,n+r]} \times \mathbb{C}^2 \times \mathrm{Hilb}^{[n+i,n+i+r]} \times \mathbb{C}^2 \times \mathrm{Hilb}^{[n,n+r]}$ can be replaced byy

$$\mathrm{Hilb}^{[n,n+r]} \times \mathbb{C}^2 \times (\mathrm{Hilb}^{[n,n+r]} \times Br^{[i]}_{\mathbb{C}^2}) \times \mathbb{C}^2 \times \mathrm{Hilb}^{[n,n+r]}$$
$$\left(\subset \mathrm{Hilb}^{[n,n+r]} \times \mathbb{C}^2 \times (\mathrm{Hilb}^{[n,n+r]} \times \mathrm{Hilb}^{[i]}) \times \mathbb{C}^2 \times \mathrm{Hilb}^{[n,n+r]}\right)$$

Then the degree N of the diagonal $[C_{i,-i} \cap \{p = q\}]$ is independent of n, r chosen as there is a rational map

$$\mathrm{Hilb}^{[n,n+r]} \times \mathbb{C}^2 \times (\mathrm{Hilb}^{[i]}) \times \mathbb{C}^2 \to$$

$$\begin{array}{ccc} & \mathrm{Hilb}^{[n,n+r]} \times \mathbb{C}^2 \times (\mathrm{Hilb}^{[n,n+r]} \times \mathrm{Hilb}^{[i]}) \times \mathbb{C}^2 \times \mathrm{Hilb}^{[n,n+r]} & \\ \swarrow^{(\pi_1,\pi_4)} & & \searrow^{(\pi_4,\pi_5)} \\ \mathrm{Hilb}^{[n,n+r]} \times \mathbb{C}^2 & & \mathbb{C}^2 \times \mathrm{Hilb}^{[n,n+r]} \end{array}$$

such that the original projections π_1 and π_5 are identity maps $id\colon \mathrm{Hilb}^{[n,n+r]} \to \mathrm{Hilb}^{[n,n+r]}$. We have $\deg \pi_{1245} = \deg \pi_{45}$.

The inclusion

$$(\rho, id)\colon Br_p^{[i]} \to \mathbb{C}^2 \times Br_{\mathbb{C}^2}^{[i]}, B_p \mapsto (\rho(B_p), B_p)$$

gives a local expression of $\overline{Z(r,i)}_n$ of the form $\mathrm{Hilb}^{[n,n+r]} \times \mathbb{C}^2 \times Br_{\mathbb{C}^2}^{[i]}$ in $\mathrm{Hilb}^{[n,n+r]} \times \mathbb{C}^2 \times \mathrm{Hilb}^{[i]}$ by $((I,J), \rho(B_p), B_p) \mapsto ((I,J), p, B_p)$. Furthermore, $\left(\pi_{123}^*([\overline{Z(r,i)}_n]) \cap \pi_{345}^*([\tau \overline{Z(r,i)}_n])\right)\Big|_{\{p=q\}}$ is supported on the intersection

$$\pi_{123}^{-1}\left(\mathrm{Hilb}^{[n,n+r]} \times \mathbb{C}^2 \times Br_{\mathbb{C}^2}^{[i]}\right) \bigcap \pi_{345}^{-1}\left(Br_{\mathbb{C}^2}^{[i]} \times \mathbb{C}^2 \times \mathrm{Hilb}^{[n,n+r]}\right) \subset \left(\mathrm{Hilb}^{[n,n+r]} \times \mathbb{C}^2 \times Br_{\mathbb{C}^2}^{[i]} \times \mathbb{C}^2 \times \mathrm{Hilb}^{[n,n+r]}\right)$$

We obtain a simpler picture after letting $n = \binom{r}{2}$, where $\mathrm{Hilb}^{[n,n+r]}$ is a just a point.

$$\begin{array}{ccc} & \mathrm{Hilb}^{[n,n+r]} \times \mathbb{C}^2 \times \mathrm{Hilb}^{[i]} \times \mathbb{C}^2 \times \mathrm{Hilb}^{[n,n+r]} & \\ \tilde{\pi}_{12} \swarrow & & \searrow \tilde{\pi}_{45} \\ \mathrm{Hilb}^{[n,n+r]} \times \mathbb{C}^2 & & \mathbb{C}^2 \times \mathrm{Hilb}^{[n,n+r]} \end{array} \quad \to \quad \begin{array}{ccc} & \{pt\} \times \mathbb{C}^2 \times \mathrm{Hilb}^{[i]} \times \mathbb{C}^2 \times \{pt\} & \\ \tilde{\pi}_{1} \swarrow & & \searrow \tilde{p}_{3} \\ \{pt\} \times \mathbb{C}^2 & & X \times \{pt\} \end{array}$$

The diagonal in this case is given by

$$N\left[\Delta(\mathbb{C}^2)\right] = \pi_{13*}\left(\left[\mathbb{C}^2 \times (\rho, id)^*(Br_{\mathbb{C}^2}^{[i]})\right] \cap \left[(\rho, id)^*(Br_{\mathbb{C}^2}^{[i]}) \times \mathbb{C}^2\right]\right),$$

for some constant $N \in \mathbb{C}$. After intersecting with $\pi_3^*([q])$, the class of a point $q \in \mathbb{C}^*$, we have

$$\begin{aligned} N\left[\{q\}\right] &= \pi_{1*}\left(\left[\mathbb{C}^2 \times (q, Br_p^{[i]})\right] \bigcap \left[(\rho, id)^*(Br_{\mathbb{C}^2}^{[i]}) \times \{q\}\right]\right) \\ &= \pi_{1*}\left(\left[\mathbb{C}^2 \times Br_p^{[i]}\right] \bigcap \left[(\rho, id)^*(Br_{\mathbb{C}^2}^{[i]})\right]\right) \end{aligned}$$

From this, we see that N equals to intersection number of the classes $\left[Br_p^{[i]}\right]$ and $\left[Br_{\mathbb{C}^2}^{[i]}\right]$ inside $\mathrm{Hilb}^{[n]}$. These numbers N were calculated by various persons ([15, 30]) which is equal to i. We conclude that $c_{i,-i} - c_{-i,i} = i\ \Delta\left(\mathrm{Hilb}^{[n,n+r]}\right)$ and we obtain the final result

$$\alpha_i \alpha_{-i} - \alpha_{-i}, \alpha_i = [\alpha_i, \alpha_{-i}] = i\ \mathrm{id}_{\tilde{\mathbb{H}}}.$$

■

2.3.2 A conjecture on the new type operators

In this subsection, we formulate a conjecture of the correspondence for the new type operators homological degree $2d$ associated to the finite product $\prod_{d=1}^{r} \frac{1}{1-t^{2d}q^d}$ in the formula (2.17) for the generating function of $P\left(\mathrm{Hilb}^{[n,n+r]}; t\right)$.

Notation. We note $I \supset_{\mathfrak{m}} J$, if the ideals I, J satisfy $I \supset J \supset \mathfrak{m}I$.

Definition 2.3.3. Let r, j, n be positive integers and $j \leq r$. We define an incidence variety $Q^r_{n,j}$ of $\mathrm{Hilb}^{[n,n+r]} \times \mathrm{Hilb}^{[n+j,n+j+r]}$:

$$Q^r_{n,j} := \left\{ ((I,J),(K,L)) \in \mathrm{Hilb}^{[n,n+r]} \times \mathrm{Hilb}^{[n+j,n+j+r]} \,\middle|\, I \supset_{\mathfrak{m}} K \supseteq_{\mathfrak{m}} J \supset_{\mathfrak{m}} L \right\}.$$

In particular, when $j = r$, we have

$$Q^r_{n,r} = \left\{ ((I,J),(K,L)) \in \mathrm{Hilb}^{[n,n+r]} \times \mathrm{Hilb}^{[n+r,n+2r]} \,\middle|\, J = K \right\},$$

which is isomorphic to the space parametrized flags of ideals $\{I \supset_{\mathfrak{m}} J \supset_{\mathfrak{m}} L\}$ in $\mathrm{Hilb}^{[n]} \times \mathrm{Hilb}^{[n+r]} \times \mathrm{Hilb}^{[n+2r]}$.

Let $w \in H_*(\{(0,0)\}) \simeq H_*(\{pt\})$ and $0 < j \leq r$. We define the operator $w^r_j \colon H_*(\mathrm{Hilb}^{[n,n+r]}) \to H_*(\mathrm{Hilb}^{[n+j,n+j+r]})$ to be the convolution products associated to the class $[Q^r_{n,j}]$,

$$w^r_j(u) = \pi_{2*}\left((\pi_1^*(\mathrm{PD}^{-1}u) \cup \pi_3^*(\mathrm{PD}^{-1}w)) \cap [Q^r_{n,j}]\right) \in H_*\left(\mathrm{Hilb}^{[n+j,n+j+r]}\right), \tag{2.24}$$

and define w^r_{-j} by reading the diagram backwards. Alternatively, since $\mathrm{Hilb}^{[n,n+r]}$ is smooth, the pairing

$$\langle u, v\rangle = \mathrm{PD}^{-1}u \cup \mathrm{PD}^{-1}v \cap [\mathrm{Hilb}^{n,n+r}],\ u,v \in H_*\left(\mathrm{Hilb}^{[n,n+r]}\right)$$

is non-degenerate. Then one can define $w^r_{-j} \colon H_*(\mathrm{Hilb}^{[n,n+r]}) \to H_*(\mathrm{Hilb}^{[n-j,n-j+r]})$ as the adjoint operators of w^r_j. The operator w^r_j has degree $2\dim_{\mathbb{C}}(\mathrm{Hilb}^{[n+j,n+j+r]}) - 2\dim_{\mathbb{C}}(Q^r_{n,j})$.

For case of $r = 1$, W.-P. Li and Z. Qin constructed a similar new operator for the nested Hilbert scheme $\left\{(I,J) \in \mathrm{Hilb}^{[n]} \times \mathrm{Hilb}^{[n+1]} \,\middle|\, I \supset J\right\}$ in [37]. They actually identified the inverse of the new type operators and the algebra structure the operators.

We establish a conjecture:

Conjecture 2. *Let $N_0 = \binom{r}{2}$ and $v_0 \in \mathrm{Hilb}^{[N_0,N_0+r]}$. Then for each element $v \in \widetilde{\mathbb{H}}$, there exists operators $v_1, \cdots, v_d \in \{\alpha_i, i \in \mathbb{N}\} \cup \left\{w^r_j, 1 \leq j \leq r\right\}$ such that*

$$v = v_1 v_2 \cdots v_d \cdot v_0.$$

Moreover, the space $\widetilde{\mathbb{H}}$ is an infinite-dimensional irreducible representation of this algebra of operators.

Considering the strata of $\mathrm{Hilb}^{[n,n+r]}$:

$$\mathrm{Hilb}^{[n,n+r]}_s := \{(I,J) \,|\, \sigma(I) = s\}.$$

Then $\mathrm{Hilb}^{[n,n+r]}_{\binom{r}{2}}$ is the generic part of $\mathrm{Hilb}^{[n,n+r]}$ since $\dim \mathrm{Hilb}^{[n,n+r]}_s \leq \dim \mathrm{Hilb}^{[n,n+r]}_{s+1}, \forall s \geq \binom{r}{2}$. The more an element $(K,L) \in \mathrm{Hilb}^{[n+j,n+j+r]}$ has points supported at $(0,0)$, the less the dimension of the strata it belongs to. An element $((I,J)(K,L)) \in Q^r_{n,j}$ are related by $J \subset K$ with differences concentrated at $(0,0)$. Thus, if $(I,J) \in \mathrm{Hilb}^{[n,n+r]}$ has minimal value $\sigma(I)$, then those $(K,L) \in \mathrm{Hilb}^{[n+j,n+j+r]}$ will belong to the strata with minimal value $\sigma(K)$ satisfying the condition $J \subset K$.

From formula (1.18), the incidence variety corresponding to the operators of new type should have expected dimension $2\dim_{\mathbb{C}} \mathrm{Hilb}^{[n,n+r]} + j = 2n + j - r(r-1)$. We show that this property holds for $Q^r_{n,j}$.

Proposition 2.3.4. *The incidence variety $Q^r_{n,j}$ has complex dimension $2n+j-r(r-1)$.*

Proof. First, we consider the decomposition of $Q^r_{n,j}$ with respect to $\sigma(I), I \in \mathrm{Hilb}^{[n]}$:

$$Q^r_{n,j}(s) := \left\{((I,J),(J,L)) \in Q^r_{n,j} \,\middle|\, \sigma(I) = s\right\}.$$

Then for each strata $Q^r_{n,j}(s)$, there is a birational map

$$\begin{aligned} \mathrm{Hilb}^{[n-s]} \times Br^{[s,s+r]} \times Br^{[s+j,s+j+r]} &\longrightarrow Q^r_{n,j}(s) \\ (I,(B_1,B_2),(B_3,B_4)) &\longmapsto ((I\cap B_1, I\cap B_2),(I\cap B_3, I\cap B_4)), \end{aligned} \tag{2.25}$$

where $(B_1,B_2),(B_3,B_4)$ satisfy the condition $B_1 \supseteq_{\mathfrak{m}} B_2 \supseteq_{\mathfrak{m}} B_3 \supseteq_{\mathfrak{m}} B_4$. This gives an upper bound the dimension of the strata $Q^r_{n,j}(s)$:

$$\begin{aligned} \dim_{\mathbb{C}} Q^r_{n,j}(s) &\leq \dim_{\mathbb{C}} \left(\mathrm{Hilb}^{[n-s]} \times Br^{[s,s+r]} \times Br^{[s+j,s+j+r]}\right) \\ &= 2(n-s) + (s - \frac{r(r-1)}{2}) + (s+j-\frac{r(r-1)}{2}) \\ &= 2n - r(r-1) + j, \end{aligned}$$

which turns out to be independent of the number s. Therefore, we have an upper bound for $\dim_{\mathbb{C}} Q^r_{n,j}$:

$$\dim_{\mathbb{C}} Q^r_{n,j} \leq 2n - r(r-1) + j.$$

Next, we claim that $Q^r_{n,j}$ has a component that has dimension $2n-r(r-1)+j$. We consider the strata $Q^r_{n,j}(s)$ for $s=\binom{r}{2}$. In this case, $Br^{[s,s+r]} = \{(\mathfrak{m}^{r-1},\mathfrak{m}^r)\}$ is just a point. We want to compute the dimension of the subset

$$S := \left\{(B,\tilde{B}) \in Br^{[s+j,s+j+r]} \,\middle|\, \mathfrak{m}^{r-1} \supset B \supset \mathfrak{m}^r \supset \tilde{B}\right\}.$$

Recall that $\mathrm{Hilb}^{[n,n+r]}$ as well as $Br^{[n,n+r]}$ admit affine cell decompositions induced by a generic $\mathbb{C}^*$-action, where each cell of $\mathrm{Hilb}^{[n,n+r]}$ is represented by a Young diagram $\triangle$ with $n+r$ boxes with r marked removable boxes. In particular, by Corollary 2.2.2, the dimension of a cell of $Br^{[n,n+r]}$ is $n-\binom{r}{2}+r-w(\triangle)$, where $w(\triangle)$ is the number of columns of $\triangle$. We note that the cell of $Br^{[s+j,s+r+j]}$ associated to the fixed point (K,L) that corresponds to the partition $(j+r,r-1,r,2,\ldots,1)$ has dimension $s+j-\binom{r}{2}+r-w(\triangle)=j$ if $s=\binom{r}{2}$. Since the fixed point (I,J) satisfies $\mathfrak{m}^{r-1}\supset K\supset \mathfrak{m}^r \supset L$ and the inclusion relations is invariant under the torus action, we conclude that the subset S has a component of dimension j. Thus, the strata $Q^r_{n,j}(\binom{r}{2})$ has a component of dimension

$$\dim_{\mathbb{C}} \mathrm{Hilb}^{[n-s]} + \dim_{\mathbb{C}} Br^{[s,s+r]} + j = 2n - r(r-1) + j.$$

We conclude that $Q^r_{n,j}$ has dimension $2n-r(r-1)+j$. ■

Chapter 3

Matrix description of the refined Hilbert scheme and example of local coordinates

In this chapter, we give a matrix description of the refined Hilbert scheme. Using this description, we explain an idea of analyzing the local structure of $\mathrm{Hilb}^{[n,n+r]}$ through the examples of $\mathrm{Hilb}^{[2,4]}$ and $\mathrm{Hilb}^{[3,5]}$. This inspire us to study of a special divisor $\mathrm{Hilb}^{[n,n+2]}_{s\geq 2}$ of $\mathrm{Hilb}^{[n,n+2]}$; the subvariety that contains all element (I,J) such that the multiplicity of I at $(0,0)$ is at least two.

3.1 Matrix description of $\mathrm{Hilb}^{[n,n+r]}$

In Chapter 1, we saw that an ideal $I\in\mathrm{Hilb}^{[n]}$ may be represented as a $\mathrm{GL}_n(\mathbb{C})$-orbit of (X,Y,i,j) (Here, j is in fact 0, but we will keep it for the moment). Apply this to the refined Hilbert scheme $\mathrm{Hilb}^{[n,n+r]}\subset\mathrm{Hilb}^{[n]}\times\mathrm{Hilb}^{[n+r]}$. Then an element $(I,J)\in\mathrm{Hilb}^{[n,n+r]}$ corresponds to a pair of matrices (X_I,Y_I,i_I,j_I) and (X_J,Y_J,i_J,j_J) satisfy two conditions coming from the assumption $I\supset J\supset\mathfrak{m}I$:

(i) The action of (X_I,Y_I) and (X_J,Y_J) preserve a fixed flag:

$$I/J\simeq\mathbb{C}^r\twoheadrightarrow R/J\simeq\mathbb{C}^{n+r}\twoheadrightarrow R/I\simeq\mathbb{C}^n \tag{3.1}$$

(ii) (X_I,Y_I) and (X_J,Y_J) act trivially on their quotient $\mathbb{C}^r\simeq I/J$.

The matrices X_I and Y_I are determined by $X:=X_J$ and $Y:=Y_J$ which lie in the subspace $U_{n,n+r}$ of (n+r)×(n+r) matrices $\mathrm{Mat}_{n+r}(\mathbb{C})$ that act nilpotently on the flag $R/J\twoheadrightarrow R/I$ and act trivially on the quotient $\mathbb{C}^r\simeq I/J$. The vector $i:=i_J$ determines the vector $i_I\in\mathbb{C}^n$ and $j:=j_I\in(\mathbb{C}^n)^*$ determines $j_J\in(\mathbb{C}^{n+r})^*$. Since $R/J\supset R/I$, the operators of multiplying x and y on R/J act on R/I can be seen as a restriction of R/J to R/I. Moreover since R acts on $I/J\simeq\mathbb{C}^r$ trivially, we can choose a type of basis such that the first r columns of X and $Y\in U_{n,n+r}\subset\mathrm{Mat}_{n+r}(\mathbb{C})$ are identically 0. Similarly, j has the form $(\underbrace{0,\dots,0}_{r},j_{r+1},\dots,j_{n+r})$ determining by $j_I=(j_{r+1},\dots,j_{n+r})\in(\mathbb{C}^n)^*$.

We define the space of quadruples (X, Y, i, j)

$$\widetilde{\mathcal{M}} = \left\{ (X,Y,i,j) \;\middle|\; \begin{array}{ll} (1) & \text{The entries of } r \text{ first columns of } X, Y \text{ and } j \text{ are } 0. \\ (2) & [X,Y]+ij=0 \\ (3) & \text{Stability condition: there exists no subspace } S \subsetneq V = \mathbb{C}^{n+r} \\ & \text{such that } X(S)\subset S,\ Y(S)\subset S \text{ and } \mathrm{Im} i \subset S. \end{array} \right\} \tag{3.2}$$

That is (X, Y, i, j) have the form

$$(X,Y,i,j) = \left(\overset{r \text{ columns} \quad n \text{ columns}}{\begin{pmatrix} 0\cdots 0 & *\cdots * \\ \vdots \quad \vdots & \vdots \quad \vdots \\ 0\cdots 0 & *\cdots * \end{pmatrix}}, i = \begin{pmatrix} i_1 \\ i_2 \\ \vdots \\ i_{n+r} \end{pmatrix}, j = \begin{pmatrix} \underbrace{0\ldots 0}_{r} & j_{r+1}\ldots j_{r+n} \end{pmatrix} \right).$$

Such type of basis is invariant under the action of the parabolic subgroup

$$\mathrm{P}_{r,n} := \left\{ \left(\begin{array}{c|c} \mathbf{P_1} & \mathbf{P_2} \\ \hline \mathbf{0} & \mathbf{P_4} \end{array} \right) \in \mathrm{GL}_{n+r}(\mathbb{C}) \;\middle|\; P_1 \in \mathrm{GL}_r(\mathbb{C}), P_3 \in \mathrm{GL}_n(\mathbb{C}), P_2 \in \mathrm{Mat}_{r\times n}(\mathbb{C}) \right\}.$$

Each element $g = \left(\begin{array}{c|c} \mathbf{P_1} & \mathbf{P_2} \\ \hline \mathbf{0} & \mathbf{P_4} \end{array} \right) \in \mathrm{P}_{r,n}$ has the inverse of the form $g^{-1} = \left(\begin{array}{c|c} \mathbf{P_1^{-1}} & -\mathbf{P_1^{-1}P_2P_4} \\ \hline 0 & \mathbf{P_4^{-1}} \end{array} \right)$. Hence the bottom-right $n \times n$ submatrix acts by $\mathrm{GL}_n(\mathbb{C})$. We will show that the refined Hilbert scheme $\mathrm{Hilb}^{[n,n+r]}$ can be identified with the quotient $\widetilde{\mathcal{M}}/\mathrm{P}_{r,n}$, where the group $\mathrm{P}_{r,n}$ acts on $\widetilde{\mathcal{M}}$ by

$$g \cdot (X, Y, i, j) \mapsto (gXg^{-1}, gYg^{-1}, gi, jg^{-1}).$$

Here, we show a proof that is analogous to the proof [34, Section 4]. We first show that this quotient space has expected dimension.

Proposition 3.1.1. *The space* $\widetilde{\mathcal{M}} \big/ \mathrm{P}_{r,n}$ *has complex dimension* $2n - r(r-1)$.

Proof. Recall that in 1.13, there is an isomorphism

$$\mathrm{Hilb}^{[n]} \simeq \left\{ (X,Y,i,j) \;\middle|\; \begin{array}{ll} (1) & [X,Y]+ij=0 \\ (2) & \text{(Stability) There exists no subspace } S \subsetneq \mathbb{C}^n \\ & \text{such that } X(S)\subset S,\ Y(S)\subset S \text{ and } \mathrm{Im} i\subset S. \end{array} \right\} \Big/ \mathrm{GL}_n(\mathbb{C}).$$

We consider the following map,

$$\begin{aligned} \theta : \mathrm{End}(\mathbb{C}^{n+r}) \times \mathrm{End}(\mathbb{C}^{n+r}) \times \mathbb{C}^{n+r} \times \left(\mathbb{C}^{n+r}\right)^* &\to \mathrm{End}(\mathbb{C}^{n+r}) \\ (X, Y, i, j) &\longmapsto [X, Y] + ij. \end{aligned}$$

Let $U_{n,n+r} \subset \mathrm{Mat}_{(n+r)}(\mathbb{C})$ be the $n(n+r)$-dimensional subspace of matrices that first r columns are 0 with the $\mathrm{P}_{r,n} \subset \mathrm{GL}_{n+r}$ action that preserving the form of matrix and

$$\tilde{V} = \{(\underbrace{0\ldots,0}_{r}, z_{r+1}, \ldots, z_{r+n}) \in \left(C^{n+r}\right)^*\} \simeq (\mathbb{C}^n)^*.$$

If we let $\tilde{\theta}$ be the restriction of θ to the subspace

$$\tilde{\theta} : U_{n,n+r} \times U_{n,n+r} \times \mathbb{C}^{n+r} \times \tilde{V} \to U_{n,n+r} \tag{3.3}$$
$$(X, Y, i, j) \mapsto [X, Y] + ij. \tag{3.4}$$

then $\widetilde{\mathcal{M}}$ is an open subset of $\tilde{\theta}^{-1}(0)$.

The differential of $\tilde{\theta}$ at a point (X, Y, i, j) is given by

$$d\theta : \underbrace{(\delta X, \delta Y, \delta i, \delta j)}_{\in U_{n,n+r} \times U_{n,n+r} \times \mathbb{C}^{n+r} \times \tilde{V}} \longmapsto [\delta X, Y] + [X, \delta Y] + \delta i\ j + i\ \delta j$$

The dual space of $\mathrm{Mat}_{(n+r)}(\mathbb{C})$ can be identified with itself by the inner product $\langle A, B\rangle := \mathrm{tr}(AB)$. Let $U^T_{n,n+r} \subset \mathrm{Mat}_{(n+r)}(\mathbb{C})$ be the subspace of matrices such that first r rows are 0s.

$$\left(\begin{array}{c|c} \mathbf{0} & A_{r\times n} \\ \hline \mathbf{0} & B_{n\times n} \end{array}\right) \in U_{n,n+r}, \quad \left(\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline C_{n\times r} & D_{n\times n} \end{array}\right) \in U^T_{n,n+r}$$

Table 3.1: Examples of elements of $U_{n,n+r}$ and $U^T_{n,n+r}$.

Consider the inner product

$$U^T_{n,n+r} \times U_{n,n+r} \longrightarrow \mathbb{C}$$
$$(A, B) \mapsto \mathrm{tr}(AB).$$

To show that it is non-degenerate (hence a complete paring), let (in the form of block matrices) $A = \left(\begin{array}{c|c} 0_{r\times r} & 0_{r\times n} \\ \hline A_{n\times r} & A_{n\times n} \end{array}\right) \in U^T_{n,n+r}$ and $B = \left(\begin{array}{c|c} 0_{r\times r} & B_{r\times n} \\ \hline 0_{n\times r} & B_{n\times n} \end{array}\right) \in U_{n,n+r}$. It follows that the trace of the product

$$\mathrm{tr}(AB) = \mathrm{tr}(BA) = \mathrm{tr}\left(\begin{array}{c|c} B_{r\times n}A_{n\times r} & B_{r\times n}A_{n\times n} \\ \hline B_{n\times n}A_{n\times r} & B_{n\times n}A_{n\times n} \end{array}\right) = \mathrm{tr}\left(B_{r\times n}A_{n\times r}\right) + \mathrm{tr}(B_{n\times n}A_{n\times n}).$$

Suppose that $\mathrm{tr}(AB) = \mathrm{tr}\left(B_{r\times n}A_{n\times r}\right) + \mathrm{tr}(B_{n\times n}A_{n\times n}) = 0$ for all $B \in U_{n,n+r}$. Then $A_{n\times n} = 0$ since the trace in non-degenerate on space of square matrices. The block matrix $A_{n\times r}$ also equals 0 by the fact that tr is positive-definite and consider the case of $B_{r\times n}$ is the conjugate transpose.

The differential map $\tilde{\theta}$ is surjective is equivalent to the following statement: Let $A \in U^T_{n,n+r}$. If $\mathrm{tr}\left(Ad\theta(\delta X, \delta Y, \delta i, \delta j)\right) = 0$ for all $(\delta X, \delta Y, \delta i, \delta j)$, then $A = 0$. Since the trace map is linear and invariant under cyclic permutations, we have

$$\begin{aligned}
\mathrm{tr}\left(Ad\theta\right) &= \mathrm{tr}\left(A[\delta X, Y]\right) + \mathrm{tr}\left(A[X, \delta Y]\right) + \mathrm{tr}\left(A\delta i\ j\right) + \mathrm{tr}\left(Ai\ \delta j\right) \\
&= \mathrm{tr}\left(A\delta XY - AY\delta X\right) + \mathrm{tr}\left(AX\delta Y - A\delta YX\right) + \mathrm{tr}\left(A\delta i\ j\right) + \mathrm{tr}\left(Ai\delta j\right) \\
&= \mathrm{tr}\left(YA\delta X - AY\delta X\right) + \mathrm{tr}\left(AX\delta Y - XA\delta Y\right) + \mathrm{tr}\left(jA\delta i\right) + \mathrm{tr}\left(Ai\delta j\right) \\
&= \mathrm{tr}\left([Y, A]\delta X\right) + \mathrm{tr}\left([A, X]\delta Y\right) + \mathrm{tr}\left(jA\delta i\right) + \mathrm{tr}\left(Ai\delta j\right).
\end{aligned}$$

Now, the trace map is non-degenerated and $\mathrm{tr}\left(Ad\theta(\delta X, \delta Y, \delta i, \delta j)\right) = 0$ for all $(\delta X, \delta Y, \delta i, \delta j)$, then $[Y, A]$, $[A, X]$, jA and Ai must vanish. If we consider the subspace spanned by A, it is X, Y invariant from the fact that X and Y commute with A. But i is a cyclic vector by the stability condition, it forces $A = 0$.

Thus $\widetilde{\mathcal{M}}$ is an open subset of $\ker\tilde{\theta}$ of dimension $n(n+r)+2n+r$.

Now, the group $\mathrm{P}_{r,n}$ acts freely on $\widetilde{\mathcal{M}}$, since it is a subgroup of GL_{n+r} which acts freely on $\mathcal{M} \simeq \mathrm{Hilb}^{n+r}$.

Finally, we have the dimension of the quotient space

$$\dim_{\mathbb{C}} \widetilde{\mathcal{M}}\Big/ \mathrm{P}_{r,n} = n(n+r)+2n+r-\big(n(n+r)+r^2\big) = 2n - r(r-1).$$

■

Now, following from Proposition 1.1.14, if a quadruplet (X,Y,i,j) satisfies the relation $[X,Y]+ij=0$ and the stability condition in the formula (3.2) , then we have $j=0$. The condition $[X,Y]+ij=0$ reduces to $[X,Y]=0$ for every $(X,Y,i,j)\in\widetilde{\mathcal{M}}$. Thus, we can rewrite $\widetilde{\mathcal{M}}$:

$$\widetilde{\mathcal{M}} = \left\{ (X,Y,i) \;\middle|\; \begin{array}{ll} (1) & \text{The entries of } r \text{ first columns of } X \text{ and } Y \text{ are } 0. \\ (2) & [X,Y]=0 \\ (3) & \text{Stability condition: there is no } X,\, Y \text{ invariant proper subspace} \\ & \text{in } \mathbb{C}^{n+r} \text{ that contains image of } i. \end{array} \right\} \tag{3.5}$$

Theorem 3.1.2. *The refined Hilbert scheme* $\mathrm{Hilb}^{[n,n+r]}$ *is isomorphic to the quotient space* $\widetilde{\mathcal{M}}\Big/\mathrm{P}_{r,n}$.

Proof. Let $(X,Y,i)\in\widetilde{\mathcal{M}}/\mathrm{P}_{r,n}$. The $\mathrm{P}_{r,n} = \begin{pmatrix} \mathbf{P_{r\times r}} & \mathbf{P_{r\times n}} \\ 0 & \mathbf{P_{n\times n}} \end{pmatrix}$ as a block matrix acts on $n\times n$-submatrices

$$X_n := (X_{ij})_{r+1\le i,j\le n+r}\,,\; Y_n := (Y_{ij})_{r+1\le i,j\le n+r}$$

only by the right-bottom part $\mathbf{P_{n\times n}} \simeq \mathrm{GL}_{\mathbf{n}}(\mathbb{C})$. Let $i_n = (i_k)_{r+1\le k\le n+r}$. Then there are maps

$$\begin{aligned} \phi : R &\to \mathbb{C}^{n+r} \\ f(x,y) &\mapsto f(X,Y)i \\ \psi : R &\to \mathbb{C}^{n} \\ f(x,y) &\mapsto f(X_n,Y_n)i_n. \end{aligned}$$

Both images of maps ϕ and ψ are invariant under the multiplication of X and Y, X_n and Y_n respectively. By the stability condition, ϕ, ψ are surjective. We define then $I=\ker\psi$ and $J=\ker\phi$. This pair of ideals satisfies the properties $I\supset J$ since ψ is a restriction of ϕ. Moreover $I/J\simeq\mathbb{C}^r$ with a trivial R-module structure since the first r columns of X and Y are zeros.

Conversely, let $(I,J)\in\mathrm{Hilb}^{[n,n+r]}$. We have $V^{n+r} := R/J \supset R/I$ by $I\supset J$ and $\mathrm{GL}_{n+r}(\mathbb{C})$ acts on it as change of basis. Let X, Y be matrices of multiplication by x and y on V^{n+r} and $i=1\in R/J$. It follows that X and Y commute. Since 1 multiplied by monomials $x^a y^b$ span R, the stability condition holds. Moreover, the matrices X ,Y preserve $V^{n+r}\twoheadrightarrow V^n$ and act. trivially on $\mathbb{C}^r\simeq I/J$. We may choose a basis such that $\mathbb{C}^r$ is spanned by first r base vectors and X, Y has first r columns filled out with 0:

$$X = \left(\begin{array}{c|c} 0_{r,r} & X_{r,n} \\ \hline 0_{n,r} & X_{n,n} \end{array}\right),\ Y = \left(\begin{array}{c|c} 0_{r,r} & Y_{r,n} \\ \hline 0_{n,r} & Y_{n,n} \end{array}\right),\ i = \begin{pmatrix} i_1 \\ i_2 \\ \vdots \\ i_{n+r} \end{pmatrix}.$$

Let $X_n := X_{n,n}, Y :=_{n,n}$ be the submatrices of order n of (X, Y) act on R/I and $\tilde{i} = (i_{r+1}, \cdots, i_{r+n})^T$. Then $[X_n, Y_n]$ since $[X, Y]$ and X_n, Y_n act on the image of the restriction $\tilde{i}$ spans R/I, otherwise it would contradict to the fact that $i \in \mathbb{C}^{n+r}$ is cyclic. ■

We note that the connection between the Hilbert scheme of points and the variety parameterizing the pairs of square matrices (X, Y), $XY - YX = 0$ has been studied by M. Bulois and L. Evain in [7].

3.1.1 Matrix description: Torus action

The 2-dimensional torus $T \simeq (\mathbb{C}^*)^2$ acts on the refined Hilbert scheme $\mathrm{Hilb}^{[n,n+r]}$, which is isomorphic to $\widetilde{\mathcal{M}}/\mathrm{P}_{r,n}$. We describe this action in term of matrix description.

A $\mathrm{P}_{r,n}$-orbit of $\widetilde{\mathcal{M}}$ that represented by (X, Y, i) in $\mathrm{Hilb}^{[n,n+r]}$ is a T-fixed point if there exists a homomorphism $\phi_t : T \to \mathbb{C}^*$, $\quad (t_1, t_2) \mapsto (g_{t_1}(t), g_{t_2}(t)$ such that

$$\begin{pmatrix} g_{t_1}(t) \\ g_{t_2}(t) \end{pmatrix} \cdot (X\ Y) = \begin{pmatrix} g_{t_1}(t)Xg_{t_1}^{-1}(t) & g_{t_1}(t)Yg_{t_1}^{-1}(t) \\ g_{t_2}(t)Xg_{t_2}^{-1}(t) & g_{t_2}(t)Yg_{t_2}^{-1}(t) \end{pmatrix} = \begin{pmatrix} t_1^{-1}X & Y \\ X & t_2^{-1}Y \end{pmatrix} \tag{3.6}$$

We may assume that $g_{t_1}(t)$ and $g_{t_1}(t)$ are following maximal torus of $\mathrm{GL}_{n+r}(\mathbb{C})$:

$$g_{t_1}(t) = T_\alpha := \begin{pmatrix} t_1^{-\alpha_1} & & 0 \\ & \ddots & \\ 0 & & t_1^{-\alpha_{n+r}} \end{pmatrix},\ g_{t_2}(t) = T_\beta := \begin{pmatrix} t_2^{-\beta_1} & & 0 \\ & \ddots & \\ 0 & & t_2^{-\beta_{n+r}} \end{pmatrix}, \alpha_k, \beta_l \in \mathbb{Z},$$

they act on a $n + r$ square matrix M with weight $-(*_i - *_j)$, where $* = \alpha, \beta$, at ij-the entry of M

$$T_*\left(M_{ij}\right) = \left(t^{-(*_i - *_j)}M_{ij}\right)_{1 \geq i,j \geq n+2}$$

If (I, J) is a standard Young tableau of $n + 2$ boxes with 2 columns, then the matrices (X, Y, i) of this fixed has the properties that.

If X_{ij} and Y_{ij} are entries of 1's for some i, j, then

$$\begin{pmatrix} T_\alpha X T_\alpha^{-1} & T_\alpha Y T_\alpha^{-1} \\ T_\beta X T_\beta^{-1} & T_\beta Y T_\beta^{-1} \end{pmatrix} = \begin{pmatrix} t_1^{-1}X & Y \\ X & t_2^{-1}Y \end{pmatrix} \Rightarrow \begin{cases} (w_{t_1}X_{ij}, w_{t_2}X_{ij}) = (\alpha_j - \alpha_i + 1,\ \beta_j - \beta_i) \\ (w_{t_1}Y_{ij}, w_{t_2}Y_{ij}) = (\alpha_j - \alpha_i,\ \beta_j - \beta_i + 1) \end{cases}$$

where $(w_{t_1}X, w_{t_2}X)$ (and $(w_{t_1}Y, w_{t_2}Y)$ respectively) are multiplicative (t_1, t_2)-weights vectors of the isotropy representation for the entries X_{ij} (and Y_{ij} respectively).

3.2 Matrix description: A case of $\mathrm{Hilb}^{[n,n+2]}$ and Young diagrams

Theorem 3.1.2 says that element of the refined Hilbert scheme is isomorphic to an orbit $\{g \cdot (X, Y, i) \mid g \in \mathrm{P}_{r,n}\}$, $(X, Y, i) \in \widetilde{\mathcal{M}}$. This suggests a way to build an explicit local coordinates by choosing representatives of orbit. we study examples of $\mathrm{Hilb}^{[n,n+2]}$ for $n = 2, 3$. We analyze these space near a fixed point using the matrix description (X, Y, i) of $\mathrm{Hilb}^{[n,n+2]}$, namely by giving local coordinates at fixed point. The model of these local coordinates were calculated by computer. The principle of the calculation works as the following: To each fixed points, we choose a filling of numbers of the corresponding Young diagram. Using the action of $\mathrm{P}_{r,n}$ $g \cdot (X, Y, i) =$ and the commutation relation $[X, Y] = 0$, we fix the value (0 or 1) of certain entries of matrices (X, Y, i) while the codimension stays invariant. We call this procedure a "*normalization*". After proceeding a normalization, these matrix entries are functions of $2n - 2$ complex numbers $z_1, \cdots, z_{n-1}, v_1, \cdots, v_{n-1} \in \mathbb{C}$. Moreover, the torus action lift to the entries of (X, Y) therefore each entry has a weight of T-action. We should check that the T-weights of our local coordinates coincide with the weights of the isotropy representation on tangent space.

We recall from (3.2) in the setting of $r = 2$; an element $(X, Y, i) \in \mathrm{Hilb}^{[n,n+2]}$ consists of

$$X, Y \sim \overset{n \text{ col.}}{\begin{pmatrix} 0 & 0 & * \cdots * \\ \vdots & \vdots & \vdots \quad\ \vdots \\ 0 & 0 & * \cdots * \end{pmatrix}}, i = \begin{pmatrix} i_1 \\ i_2 \\ \vdots \\ i_{n+r} \end{pmatrix} \text{ acts by the subgroup } \mathrm{P}_{2,n} = \left(\begin{array}{cc|ccc} * & * & \overbrace{* & \cdots & *}^{n} \\ * & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & * & \cdots & * \end{array}\right).$$

We show examples of local coordinates system in a small neighborhood of each fixed point using the matrix description of $\mathrm{Hilb}^{[2,4]}$ and $\mathrm{Hilb}^{[3,5]}$. We will apply these local coordinates to analyze the spectrum of ideals (I, J) in $\mathrm{Hilb}^{[2,4]}$ and $\mathrm{Hilb}^{[3,5]}$. The model of these local coordinates are calculated by computer using the principle we explain earlier.

Denoted by $T \simeq \left\{(t_1, t_2) \in (\mathbb{C}^*)^2\right\}$ the two-dimensional torus. By a (t_1, t_2)-*weights* or a T-*weight*, we mean the multiplicative weights (α, β) of the action of t_1 and t_2, respectively.

3.2.1 Example of matrix description: $\mathrm{Hilb}^{[2,4]}$

In the Example 2.1.1, we saw that $\mathrm{Hilb}^{[2,4]}$ is isomorphic to $\mathbb{P}^1$. It has two T-fixed points:

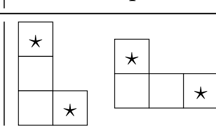
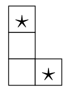
	$\dim_C$	$P(-;t)$	$E(-;t)$	T-fixed points
$\mathrm{Hilb}^{[2,4]}$	2	$1+t$	$t+t^2$	

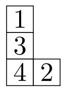
Table 3.2: Poincaré polynomial , E-polynomial and T-fixed points of $\mathrm{Hilb}^{[2,4]}$.

From motivic point of view, $\mathrm{Hilb}^{[2,4]}$ is union of strata $\mathrm{Hilb}^{[2,4]}_{s=1}$ and $Br^{[2,4]}$. Moreover, since $\mathrm{Hilb}^{[2,4]}_{s=1} \simeq \mathbb{C}^{2*} \times Br^{[1,3]}$, an element $(I, J) \in \mathrm{Hilb}^{[2,4]}_{s=1}$ has the form

$$I = \langle x, y\rangle \cap \langle x - a, y - b\rangle = \langle y^2 - by, bx - ay, xy - ay, x^2 - ax\rangle$$
$$J = \langle y^2, xy, x^2\rangle \cap \langle x - a, y - b\rangle.$$

Case of the fixed point $\left(\langle y^2, x\rangle, \langle y^3, xy, x^2\rangle\right)$

We start with the fixed point $\left(\langle y^2, x\rangle, \langle y^3, xy, x^2\rangle\right) \Leftrightarrow$ . Since $\dim_{\mathbb{C}} \mathrm{Hilb}^{[2,4]} = 2$, there are two (t_1, t_2)-weights of the tangent space $\boxed{(0, 1), (1, -1)}$.

We choose the numbering 

and obtain matrices (X, Y) according to this numbering. The torus acts on (X, Y, i) and hence each entry has a (t_1, t_2)-weights. The following matrices w_X, w_y have ij-th entry correspond to the (t_1, t_2)-weights of X_{ij} or Y_{ij}:

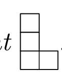
$$w_X = \begin{pmatrix} \begin{pmatrix}1\ 0\end{pmatrix} & \begin{pmatrix}2\ -2\end{pmatrix} & \begin{pmatrix}1\ -1\end{pmatrix} & \begin{pmatrix}1\ -2\end{pmatrix} \\ \begin{pmatrix}0\ 2\end{pmatrix} & \begin{pmatrix}1\ 0\end{pmatrix} & \begin{pmatrix}0\ 1\end{pmatrix} & \begin{pmatrix}0\ 0\end{pmatrix} \\ \begin{pmatrix}1\ 1\end{pmatrix} & \begin{pmatrix}2\ -1\end{pmatrix} & \begin{pmatrix}1\ 0\end{pmatrix} & \begin{pmatrix}1\ -1\end{pmatrix} \\ \begin{pmatrix}1\ 2\end{pmatrix} & \begin{pmatrix}2\ 0\end{pmatrix} & \begin{pmatrix}1\ 1\end{pmatrix} & \begin{pmatrix}1\ 0\end{pmatrix} \end{pmatrix}, w_Y = \begin{pmatrix} \begin{pmatrix}0\ 1\end{pmatrix} & \begin{pmatrix}1\ -1\end{pmatrix} & \begin{pmatrix}0\ 0\end{pmatrix} & \begin{pmatrix}0\ -1\end{pmatrix} \\ \begin{pmatrix}-1\ 3\end{pmatrix} & \begin{pmatrix}0\ 1\end{pmatrix} & \begin{pmatrix}-1\ 2\end{pmatrix} & \begin{pmatrix}-1\ 1\end{pmatrix} \\ \begin{pmatrix}0\ 2\end{pmatrix} & \begin{pmatrix}1\ 0\end{pmatrix} & \begin{pmatrix}0\ 1\end{pmatrix} & \begin{pmatrix}0\ 0\end{pmatrix} \\ \begin{pmatrix}0\ 3\end{pmatrix} & \begin{pmatrix}1\ 1\end{pmatrix} & \begin{pmatrix}0\ 2\end{pmatrix} & \begin{pmatrix}0\ 1\end{pmatrix} \end{pmatrix}$$

We give now the local coordinates near this fixed point

Proposition 3.2.1. *The map*

$$\mathrm{Hilb}^{[2,4]} \to \mathbb{C}^2$$
$$(X, Y, i) \mapsto (z_1, v_1)$$

where $X = \begin{pmatrix} 0 & 0 & z_1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & z_1 v_1 & z_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & v_1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $z_1, v_1 \in \mathbb{C}$ *have* (t_1, t_2)*-weights* $(1, -1)$ *and* $(0, 1)$,

is a local coordinate system of $\mathrm{Hilb}^{[2,4]}$ *near the fixed point* .

Proof. We construct an invertible map from X, Y to the local coordinates $C^{i,j}_{h,k}$ in 1.1.9. Since z_1, v_1 are local coordinate, they should have the same T-action weights. From the character formula (2.17) of the tangent space , they are $(1, -1), (0, 1)$ which corresponds to the vectors

$$xy \mapsto y^2 = C^{1,1}_{0,2} \text{ and}$$
$$y^2 \mapsto y = C^{0,2}_{0,1}.$$

We calculate the product

$$Y^2 i = \begin{pmatrix} 1 \\ 0 \\ v_1 \\ 0 \end{pmatrix}, XYi = \begin{pmatrix} z_1 \\ 0 \\ v_1 z_1 \\ 0 \end{pmatrix}$$

in the ordered basis $\mathbb{C}y^2 \oplus \mathbb{C}x \oplus \mathbb{C}y \oplus 1$. Then by the construction of $C^{i,j}_{h,k}$, we have $v_1 = C^{0,1}_{0,1}$ and $z_1 = C^{1,1}_{0,2}$. It show that $(z_1, v_1) \mapsto (X, Y, i)$ is indeed a local coordinate. ■

The underlying supporting points of $(I, J) \in \text{Hilb}^{[n,n+2]}$ correspond to the eigenvalues of matrices (X, Y, i) associated to (I, J). Also, the condition for a matrix to have 0 as an eigenvalue if and only if it has determinant zero. Since $\in \text{Hilb}^{[n,n+2]} = \text{Hilb}^{[n,n+2]}_{s\geq 1}$ so $(0, 0)$ is always contained in $\text{Supp}(I)$ from the definition and thus we have the multiplicity $\sigma(J) \geq 3$. We can see this phenomenon from our explicit local chart.

Then the condition for $(I, J) \in \text{Hilb}^{[n,n+2]}_{s\geq 2}$ to have another pair of eigenvalues $(0, 0)$ in our local chart (z_1, v_1) is equivalent to require the 1×1 submatrices M_X, M_Y in the associated matrices $(X, Y) \in \text{Mat}^2_{4\times 4}$

$$X = \begin{pmatrix} 0 & 0 & z_1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & z_1 v_1 & z_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & v_1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

both have determinant zero (3.7).

	$M_X = \begin{pmatrix} z_1 v_1 \end{pmatrix}$	$M_Y = \begin{pmatrix} v_1 \end{pmatrix}$
det(= tr)	$C_1(z_1, v_1) = v_1 z_1 = z_1 D_1$	$D_1(z_1, v_1) := v_1$

(3.7)

We knew that $\text{Hilb}^{[n,n+2]}_{s\geq 2}$ is of codimension one, so it is not surprising that the condition of $\det M_X = 0 = \det M_Y$ reduces to a single equation: $\det M_Y = v_1$:

$\sigma(I) = s$	equation	wts of the euqation
$s \geq 2$	$D_1 = v_1 = 0$	$(0, 1)$

Then the section defining $\text{Hilb}^{[2,4]}_{s\geq 2}$ is v_1 in this local coordinate.

Case of the fixed point $\big(\langle x^2, y\rangle, \langle x^3, xy, y^2\rangle\big)$

We consider another fixed point $\begin{smallmatrix} \star & & \\ \square & \square & \star \end{smallmatrix} = \big(\langle x^2, y\rangle, \langle x^3, xy, y^2\rangle\big)$, the $\mathbb{C}^*$-weights of the tangent space at this fixed point is given by

(t_1, t_2)-weight	$(-1, 1), (1, 0)$

.

We fix a numbering $\young(2,431)$. The induced $(\mathbb{C}^*)^2$-action on entries has weights:

$$w_X = \begin{pmatrix} (1\ 0) & (-1\ 1) & (0\ 0) & (-1\ 0) \\ (3\ -1) & (1\ 0) & (2\ -1) & (1\ -1) \\ (2\ 0) & (0\ 1) & (1\ 0) & (0\ 0) \\ (3\ 0) & (1\ 1) & (2\ 0) & (1\ 0) \end{pmatrix}, \quad w_Y = \begin{pmatrix} (0\ 1) & (-2\ 2) & (-1\ 1) & (-2\ 1) \\ (2\ 0) & (0\ 1) & (1\ 0) & (0\ 0) \\ (1\ 1) & (-1\ 2) & (0\ 1) & (-1\ 1) \\ (2\ 1) & (0\ 2) & (1\ 1) & (0\ 1) \end{pmatrix}$$

In a neighborhood of $\young(2,431)$, we have a local coordinate

$$X = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & z_1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & v_1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & v_1 z_1 & v_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

which have two-dimensional $\mathbb{C}^*$ weights

	z_1	v_1
(t_1, t_2)-weights	$(1, 0)$	$(-1, 1)$

.

	$M_X = \begin{pmatrix} z_1 \end{pmatrix}$	$M_Y = \begin{pmatrix} v_1 z_1 \end{pmatrix}$
det = tr	$C_1 = z_1$	$D_1 = v_1 z_1 = v_1 D_1$

The condition of $\det M_X = 0 = \det M_Y$ reduces to a single equation:

$\sigma(I) = s$	equation	$\mathbb{C}^*$-wts of the euqation
$s \geq 2$	$C_1 = z_1 = 0$	$(1, 0)$

Change of coordinates

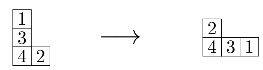
To complete this model of local coordinates for $\mathrm{Hilb}^{[2,4]}$, we need the transition maps of these local coordinates.

Let $$\left(X_{\yng(1,1,2)} = \begin{pmatrix} 0 & 0 & z_1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & v_1 z_1 & z_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, Y_{\yng(1,1,2)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & v_1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) \text{ and } \left(X_{\yng(1,3)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, Y_{\yng(1,3)} = \begin{pmatrix} 0 & 0 & b_1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a_1 b_1 & b_1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right)$$

be elements in local coordinates of $\young(1,3,42)$ and $\young(2,431)$. Let $g = \begin{pmatrix} \frac{1}{z_1^2} & 0 & 0 & 0 \\ -v_1 & -z_1 & 1 & 0 \\ 0 & 0 & \frac{1}{z_1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ be an element of $\mathrm{GL}_5(\mathbb{C})$. Then the action of g takes $\left(X_{\square}, Y_{\square}\right)$ to $\left(X_{\square}, Y_{\square}\right)$: $\left(gX_{\square}\, g^{-1}, gY_{\square}\, g^{-1}\right) =$

$$\left(\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & v_1 z_1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \frac{1}{z_1} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & v_1 & \frac{1}{z_1} \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) = \left(X_{\square}, Y_{\square}\right).$$

Thus the transition map is given by:



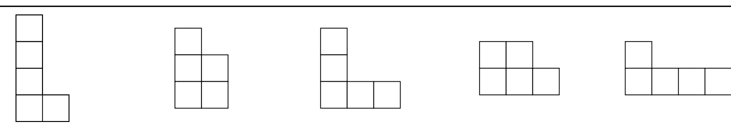
$$(z_1, v_1) \mapsto (b_1^{-1}, a_1 b_1).$$

3.2.2 Example of matrix description: $\mathrm{Hilb}^{[3,5]}$

Our next example is $\mathrm{Hilb}^{[3,5]}$. It is smooth, of $\dim_{\mathbb{C}} = 4$ with five torus fixed points (Table 3.3). It

	$\dim_C$	$P(X;t)$	$E(X;t)$
$\mathrm{Hilb}^{[3,5]}$	4	$1+2t^2+2t^4$	$2t^4+2t^6+t^8$

Fixed points of of $\mathrm{Hilb}^{[3,5]}$



$\langle y^4, xy, x^2\rangle$	$\langle y^3, xy^2, x^2\rangle$	$\langle y^3, xy, x^3\rangle$	$\langle y^2, x^2y, x^3\rangle$	$\langle y^2, xy, x^4\rangle$

Table 3.3: List of fixed points of $\mathrm{Hilb}^{[3,5]}$.

is reducible since $\mathrm{Hilb}^{[3,5]}$ has two top dimensional cells.

Analysis of $Br^{[3,5]}$

We would like to understand the geometry and the topology of $Br^{[3,5]}$ through the projection $Br^{[3,5]} \to Br^{[3]}$.

For this reason, first, we need to analysis the variety $Br^{[3]}$. The Briançon variety $Br^{[3]}$ is singular, irreducible of dimension 3. It consists of a smooth stratum consisting of elements requiring minimum two generators $Br^{[3]}_{m=2}$ (which is also the curvilinear component $Br^{[3]}_{curve}$), together with a singular point $\langle y^2, xy, x^2\rangle = \mathfrak{m}^2$ as a one-point compactification of $Br^{[3]}$.

First, we show that $Br^{[3]}_{m=2}$ is a $\mathbb{C}^1$ bundle of degree 3 over $\mathbb{P}^1$.

We describe the open covers: Let U_x (U_y respectively) be the subset of ideals such that the corresponding initial ideal does not contain x (y respectively). The $\mathbb{P}^1 = \{[a:b]\}$ has canonical open covers $V_a \cup V_b$ where $V_a = \{[a:b] \mid a \neq 0\}$ and $V_b = \{[a:b] \mid b \neq 0\}$ with local charts

$$\begin{aligned} \phi_a : V_a \times \mathbb{C} &\to U_x \\ (a;\tau) &\mapsto \langle y + ax + \tau x^2, x^3 \rangle, \\ \phi_b : V_b \times \mathbb{C} &\to U_y \\ (b;\tau') &\mapsto \langle x + by + \tau' y^2, y^3 \rangle. \end{aligned}$$

A calculation of the transition function $\phi_b^{-1} \circ \phi_a : V_a \times \mathbb{C} \to V_b \times \mathbb{C}$, $(a;\tau) \mapsto (b;\tau')$ yields

$$\begin{aligned} & y + a\left(-by - \tau' y^2\right) + \tau\left(-by - \tau' y^2\right)^2 \left(\in \langle y^3 \rangle_{span}\right) \\ & = (1-ab)y + (\tau b^2 - a\tau')y^2 \mod y^3 \end{aligned}$$

where we get the relations: $\begin{cases} ab = 1 \\ a\tau' = \tau b^2 \end{cases} \Leftrightarrow \begin{cases} b = a^{-1} \\ \tau' = \tau a^{-3} \end{cases}$ and the isomorphism $(a;\tau) \mapsto \left(a^{-1}, \tau a^{-3}\right) = (b;\tau')$. Therefore, the strata $Br^{[3]}_{m=2}$ is a $\mathbb{C}^1$ bundle of degree 3 over $\mathbb{P}^1$.

Second, we observe that the fiber of the projection $\pi_3 : \mathrm{Hilb}^{[3,5]} \to \mathrm{Hilb}^{[3]}$ over $Br^{[3]}_{m=2}$ is a point. To see this, we give the explicit description of the open covers U_x and $U_y \subset Br^{[3]}_{m=2}$:

$$\begin{aligned} U_x &= \{\langle y + ax + \tau x^2, x^3 \rangle\}, \\ U_y &= \{\langle x + by + \tau' y^2, y^3 \rangle\}. \end{aligned}$$

The fact of the dimensions $\dim_{\mathbb{C}} R/\mathfrak{m}I = 5$ and $\dim_{\mathbb{C}} I/\mathfrak{m}I = 2$ implies that $\pi_3^{-1}(I)$ consists a unique element $(I, J = \mathfrak{m}I)$. And the open covers $U_x, U_y \subset \mathrm{Hilb}^{[3]}$ have preimages of π_3

$$\begin{aligned} \pi_3^{-1}(U_x) &= \{\left(I_{[a_1;a_2]}, J_{[a_1;a_2]}\right)\} = \{\left(\langle y + a_1 x + a_2 x^2, x^3 \rangle, \mathfrak{m}\langle y + a_1 x + a_2 x^2, x^3 \rangle\right)\}, \\ \pi_3^{-1}(U_y) &= \{\left(I_{[b_1;b_2]}, J_{[b_1;b_2]}\right)\} = \{\left(\langle x + b_1 y + b_2 y^2, y^3 \rangle, \mathfrak{m}\langle x + b_1 y + b_2 y^2, y^3 \rangle\right)\}. \end{aligned}$$

Next, over the singular point $\mathfrak{m}^2$ of $Br^{[3]}$, the ideals $J \in Br^{[5]}$ with $(\mathfrak{m}^2, J) \in \mathrm{Hilb}^{[3,5]}$ are parametrized by the Grassmannian $Gr(2,3) \simeq \mathbb{P}^2$: since $\dim_{\mathbb{C}} I/\mathfrak{m}I = 3$ and $I/J \leq I/\mathfrak{m}I$ is 2-dimensional subspace. We compute the induced (t_1,t_2)-weights of $\mathbb{P}^2$ in the basis $y^2R \oplus xyR \oplus x^2R \simeq \mathbb{C}^3$. Let $w_1 = wt(y^2) = (0,2)$, $w_2 = wt(xy) = (1,1)$ and $w_3 = wt(x^2) = (2,0)$. The T-weights at each fixed point of $\mathbb{P}^2$ are given as follows: So far, we have seen that, set-theoretically, the Briançon variety

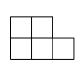
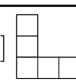
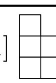
Fixed points of $\{\left(\mathfrak{m}^2, J\right) \mid J \in Br^{[5]}\} \simeq \mathbb{P}^2$	$[1:0:0]$ 	$[0:1:0]$ 	$[0:0:1]$ 
(t_1,t_2)-weight	$w_1 - w_2 = (-1,1)$ $w_1 - w_3 = (-2,2)$	$w_2 - w_1 = (1,-1)$ $w_2 - w_3 = (-1,1)$	$w_3 - w_2 = (1,-1)$ $w_3 - w_1 = (2,-2)$

Table 3.4: T-weights of

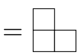
$Br^{[3]}$ is a disjoint union of a $\mathbb{C}^1$-bundle over $\mathbb{P}^1$ together with a singular point. It follows that the refined Briançon variety $Br^{[3,5]}$ is the union of a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ and a $\mathbb{P}^2$ attach to the point of infinity: $\mathfrak{m}^2$ in $Br^{[3]}$. Remember that $Br^{[3,5]}$ has five fixed points, where two of them come from $\mathbb{P}^1$ and the rest three of them come from $\mathbb{P}^2$. We would like to understand what happens near each fixed point, in particular, to know if there are smooth or not. Let (V_a, ϕ_a) and (V_b, ϕ_b)

$$\phi_a : V_a \times \mathbb{C} \to U_x, \phi_a\,(a;\tau) = \langle y + ax + \tau x^2, x^3\rangle,$$
$$\phi_b : V_b \times \mathbb{C} \to U_y, \phi_b\,(b;\tau') = \langle x + by + \tau' y^2, y^3\rangle.$$

be the usual open cover of $\mathbb{P}^1$

Fixing $[a:1] \in \mathbb{P}^1$, we study the behavior of $(I_{a,\tau}, J_{a,\tau}) \in Br^{[3,5]}$ when the ideal $I_{a,\tau} = \langle y + ax + \tau x^2, x^3\rangle$ flows to the point $\mathfrak{m}^2$ along the fiber. For this purpose, we compute the Groebner basis of $I_{a,\tau}$ and $J_{a,\tau}$ with respect to the lexicographic monomial ordering $x > y$,

$$\begin{aligned} I_{a,\tau} =& \langle y^3, a^3x + a^2y + \tau y^2, -a^2x - ay + \tau xy, axy + y^2, xy^2, ax + \tau x^2 + y, ax^2 + xy, x^2y, x^3\rangle \\ J_{a,\tau} =& \langle y^4, a^3xy + a^2y^2 + \tau y^3, -a^2xy - ay^2 + \tau xy^2, axy^2 + y^3, xy^3, a^2x^2 + 2axy + y^2, axy \\ &+ \tau x^2y + y^2, ax^2y + xy^2, x^2y^2, ax^2 + \tau x^3 + xy, ax^3 + x^2y, x^3y, x^4\rangle \end{aligned}$$
.

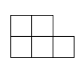
We observe that the Groebner basis of $I_{a,\tau}$ contains elements $\begin{cases} a^3x + a^2y + \tau y^2 \\ -a^2x - ay + \tau xy \\ ax + \tau x^2 + y \end{cases}$ and it tends to the limit $\begin{cases} y^2 \\ xy \\ x^2 \end{cases}$ as $\tau \to \infty$. So the limit $\lim\limits_{\tau\to\infty} I_{a,\tau}$ over the fiber of $[a:1] \in \mathbb{P}^1$ is $\mathfrak{m}^2 =$ . Now we observe what happen to $J_{a,\tau}$ upper in $Br^{[3,5]}$ as $\tau \to \infty$: We look at the following elements of the Groebner basis of $J_{a,\tau}$ that involved τ and let $\tau \to \infty$,

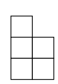
$$\begin{cases} a^3xy + a^2y^2 + \tau y^3 \\ -a^2xy - ay^2 + \tau xy^2 \\ axy + \tau x^2y + y^2 \\ ax^2 + \tau x^3 + xy \end{cases} \xrightarrow{\tau\to\infty} \begin{cases} y^3 \\ xy^2 \\ x^2y \\ x^3 \end{cases}.$$

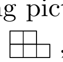
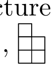
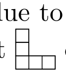
Calculate the Groebner basis of $J_{a,\tau=\infty}$, we get

$$\lim_{\tau\to\infty} J_{a,\tau} = \langle y^3, xy^2, a^2x^2 + 2axy + y^2, x^2y, x^3\rangle.$$

Moreover, $J_{a,\infty}$ goes to two different fixed points of $Gr_2(3) \simeq \mathbb{P}^2$ as $a \to 0$ or ∞:

$$\lim_{a\to 0} J_{a,\infty} = \langle y^2, x^2y, x^3\rangle \Leftrightarrow$$ 

$$\lim_{a\to\infty} J_{a,\infty} = \langle y^3, xy^2, x^2\rangle \Leftrightarrow$$ .

We have the following picture of $\mathrm{Hilb}^{[3,5]}$: The fibers of $\mathbb{P}^1$-bundle glue to a copy of $\mathbb{P}^1$ inside $\mathbb{P}^2$ with two fixed points  ,  and are disjoint to the third fixed point  of $\mathbb{P}^2$.

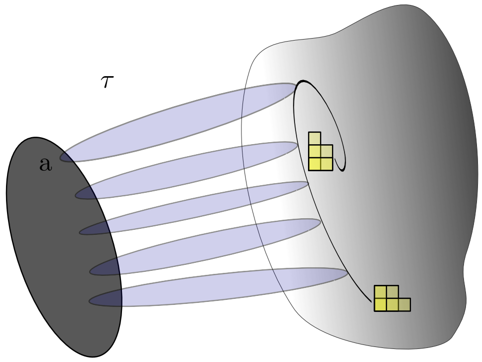
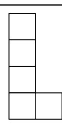
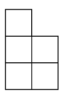
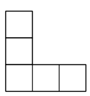
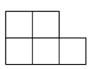
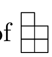
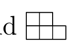
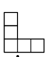
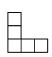
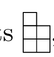
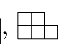


Figure 3.1: $\mathrm{Hilb}^{[3,5]}$.

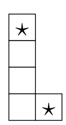
We recall the equations that define the Briaçon variety $Br^{[3,5]}$ in our local coordinates of $\mathrm{Hilb}^{[3,5]}$ at fixed points:

Fixed points	euqation for $s = 3$	Jacobian matrix
	$D_2 = v_2$ $D_1 = v_1$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$
	$C_1 = 2z_1v_1 + z_2v_2$ $D_1 = 2v_2 - z_2v_1$	$\begin{pmatrix} 2z_1 & z_2 & 2v_1 & v_2 \\ -z_2 & 2 & 0 & -v_1 \end{pmatrix}$
	$C_1 = 2z_1v_1 + v_2$ $D_1 = 2z_2v_2 + v_1$	$\begin{pmatrix} 2z_1 & 1 & 2v_1 & 0 \\ 1 & 2z_2 & 0 & 2v_2 \end{pmatrix}$
	$C_1 = 2z_1 - v_1z_2$ $D_1 = v_1z_1 + 2v_2z_2$	$\begin{pmatrix} -z_2 & 0 & 2 & -v_1 \\ z_1 & 2z_2 & v_1 & 2v_2 \end{pmatrix}$

From the above table, we observe that the Jacobian matrices of  and  are both of rank 1 and  has rank 2. We conclude that the fixed point  is smooth and other two points ,  are singular.

Local charts of $\mathrm{Hilb}^{[3,5]}$ near a fixed point

Case of the fixed point $\big(\langle y^3, x\rangle, \langle y^4, xy, x^2\rangle\big)$

We begin with the fixed point $\big(\langle y^3, x\rangle, \langle y^4, xy, x^2\rangle\big) =$  . It has weights of T-action at the tangent space

(t_1, t_2)-weight	$(0,1),(1,-1)$	$(0,2),(1,-2)$

To find a local coordinate near this fixed point, we choose the numbering: $\begin{array}{|c|c}\hline 1 & \\ \hline 3 & \\ \hline 4 & \\ \hline 5 & \multicolumn{1}{|c|}{2} \\ \hline \end{array}$ and obtain:

$$X = \begin{pmatrix} 0 & 0 & z_1 & z_2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & z_1v_1 + z_2v_2 & z_1 & z_2 \\ 0 & 0 & z_1v_2 & z_2v_2 & z_1 - z_2v_1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & v_1 & 1 & 0 \\ 0 & 0 & v_2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

with (t_1, t_2)-weights matrices at associated to X and Y:

$$w_X = \begin{pmatrix} (1\ 0) & (2\ -3) & (1\ -1) & (1\ -2) & (1\ -3) \\ (0\ 3) & (1\ 0) & (0\ 2) & (0\ 1) & (0\ 0) \\ (1\ 1) & (2\ -2) & (1\ 0) & (1\ -1) & (1\ -2) \\ (1\ 2) & (2\ -1) & (1\ 1) & (1\ 0) & (1\ -1) \\ (1\ 3) & (2\ 0) & (1\ 2) & (1\ 1) & (1\ 0) \end{pmatrix}, \; w_Y = \begin{pmatrix} (0\ 1) & (1\ -2) & (0\ 0) & (0\ -1) & (0\ -2) \\ (-1\ 4) & (0\ 1) & (-1\ 3) & (-1\ 2) & (-1\ 1) \\ (0\ 2) & (1\ -1) & (0\ 1) & (0\ 0) & (0\ -1) \\ (0\ 3) & (1\ 0) & (0\ 2) & (0\ 1) & (0\ 0) \\ (0\ 4) & (1\ 1) & (0\ 3) & (0\ 2) & (0\ 1) \end{pmatrix}$$

From the above weight matrices, the coordinates z_1, z_2, v_1, v_2 have weights of T-action

	z_1	z_2	v_1	v_2
(t_1, t_2)-weight	$(1, -1)$	$(1, -2)$	$(0, 1)$	$(0, 2)$

The condition for $(I, J) \in \mathrm{Hilb}^{[n,n+2]}_{s\geq 2}$ to have another pair of eigenvalues $(0, 0)$ in our local chart (z_1, v_1) is equivalent to require the 2×2 submatrices M_X, M_Y in the associated matrices $(X, Y) \in \mathrm{Mat}^2_{5\times 5}$

To investigate the condition that an element $(I, J) \in \mathrm{Hilb}^{[3,5]}$ lie in strata $\mathrm{Hilb}^{[3,5]}_{s\geq i}$ for $i = 2, 3$, we calculate the determinant and trace of M_X and M_Y :

	$M_X = \begin{pmatrix} z_1v_1 + z_2v_2 & z_1 \\ z_1v_2 & z_2v_2 \end{pmatrix}$	$M_Y = \begin{pmatrix} v_1 & 1 \\ v_2 & 0 \end{pmatrix}$
det	$C_2 := v_2(-z_1^2 + z_1z_2v_1 + z_2^2v_2)$ $= D_2(-z_1^2 + z_1z_2v_1 + z_2^2v_2)$	$D_2 := v_2$
tr	$C_1 := z_1v_1 + 2z_2v_2$	$D_1 := v_1$

Similar to the case of $\mathrm{Hilb}^{[2,4]}$, (X, Y) both have extra eigenvalue 0 if determinant of M_X and M_Y vanish. Furthermore, (X, Y) have only eigenvalue $(0, 0)$ if and only if both determinant and trace of M_X, M_Y vanish:

$\sigma(I) = s$	equations	(t_1, t_2)-weight
$s \geq 2$	$D_2 = 0$	$(0, 2)$
$s = 3$	det and trace of $M_X, M_Y = 0$ $\Rightarrow D_2 = 0$ and $D_1 = 0$	$(0, 2), (0, 1)$

Table 3.5: Equations for $\mathrm{Hilb}^{[2,4]}_{s\geq 2}$ and $Br^{[2,4]}$.

Case of the fixed point $\left(\mathfrak{m}^2, \langle y^3, xy^2, x^2\rangle\right)$

The fixed point $\begin{array}{|c|c|}\hline \star & \\ \hline & \star \\ \hline & \\ \hline \end{array} = \left(\mathfrak{m}^2, \langle y^3, xy^2, x^2\rangle\right)$ has weights of $T = (\mathbb{C}^*)^2$ action at the tangent space

(t_1,t_2)-weight	$(0,1),(1,-1)$	$(-1,2),(2,-2)$

We fix a numbering $\begin{array}{|c|c|}\hline 1 & \\ \hline 3 & 2 \\ \hline 5 & 4 \\ \hline \end{array}$ and the torus acts in entries of (X,Y) associated to the numbering with weights:

$$w_X = \begin{pmatrix} (1\ 0) & (2\ -1) & (1\ -1) & (2\ -2) & (1\ -2) \\ (0\ 1) & (1\ 0) & (0\ 0) & (1\ -1) & (0\ -1) \\ (1\ 1) & (2\ 0) & (1\ 0) & (2\ -1) & (1\ -1) \\ (0\ 2) & (1\ 1) & (0\ 1) & (1\ 0) & (0\ 0) \\ (1\ 2) & (2\ 1) & (1\ 1) & (2\ 0) & (1\ 0) \end{pmatrix}, \quad w_Y = \begin{pmatrix} (0\ 1) & (1\ 0) & (0\ 0) & (1\ -1) & (0\ -1) \\ (-1\ 2) & (0\ 1) & (-1\ 1) & (0\ 0) & (-1\ 0) \\ (0\ 2) & (1\ 1) & (0\ 1) & (1\ 0) & (0\ 0) \\ (-1\ 3) & (0\ 2) & (-1\ 2) & (0\ 1) & (-1\ 1) \\ (0\ 3) & (1\ 2) & (0\ 2) & (1\ 1) & (0\ 1) \end{pmatrix}.$$

And the local coordinate is given by

$$X = \begin{pmatrix} 0 & 0 & 0 & z_1 & 0 \\ 0 & 0 & 1 & z_2 & 0 \\ 0 & 0 & z_1v_1 & z_1v_2 & 0 \\ 0 & 0 & v_2 & z_1v_1 + z_2v_2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & v_2 - z_2v_1 & z_1v_1 & 1 \\ 0 & 0 & v_1 & v_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with T-weights

	z_1	z_2	v_1	v_2
(t_1,t_2)-weight	$(2,-2)$	$(1,-1)$	$(-1,2)$	$(0,1)$

To investigate the condition that an element $(I,J) \in \mathrm{Hilb}^{[3,5]}$ lie in strata $\mathrm{Hilb}^{[3,5]}_{s\geq i}$ for $i = 2,3$, we calculate the determinant and trace of M_X and M_Y :

	$M_X = \begin{pmatrix} z_1v_1 & z_1v_2 \\ v_2 & z_1v_1 + z_2v_2 \end{pmatrix}$	$M_Y = \begin{pmatrix} v_2 - z_2v_1 & z_1v_1 \\ v_1 & v_2 \end{pmatrix}$
det	$C_2 = z_1\left(z_1v_1^2 + z_2v_2v_1 - v_2^2\right)$ $= -z_1D_2 = \frac{z_1}{2}\left(v_1C_1 - v_2D_1\right)$	$D_2 = -z_1v_1^2 - z_2v_2v_1 + v_2^2$ $= \frac{1}{2}\left(-v_1C_1 + v_2D_1\right)$
tr	$C_1 = 2z_1v_1 + z_2v_2,$	$D_1 = 2v_2 - z_2v_1$

(X,Y) both have extra eigenvalue 0 if determinant of M_X and M_Y vanish. Furthermore, (X,Y) have only eigenvalue $(0,0)$ if and only if both determinant and trace of M_X, M_Y vanish. We observe that $\mathrm{tr}M_X = 0$ and $\mathrm{tr}M_Y = 0$ implies $\det M_X = \det M_Y = 0$, thus the previous condition for $s = 3$ reduce to two equations $\mathrm{tr}M_X = 0$ and $\mathrm{tr}M_Y = 0$:

$\sigma(I) = s$	equations	(t_1, t_2)-weight
$s \geq 2$	$D_2 = z_1 v_1^2 + z_2 v_2 v_1 - v_2^2 = 0$ $= \frac{1}{2}(v_1 C_1 - v_2 D_1)$	$(0, 2)$
$s = 3$	det and trace of $M_X, M_Y = 0$ $\Rightarrow C_1 = 0$ and $D_1 = 0$	$(0, 1), (1, 0)$

Another observation according this table is that $\det M_X = 0$ has two components: $D_2 = \frac{1}{2}(v_1 C_1 - v_2 D_1) = 0$ defines a hypersurface and the plane $z_1 = 0$.

Case of the fixed point $\left(\mathfrak{m}^2, \langle y^3, xy, x^3 \rangle\right)$

The fixed point $\begin{array}{|c|cc} \hline \star & & \\ \cline{1-1} & & \\ \hline & & \multicolumn{1}{|c|}{\star} \\ \hline \end{array} = \left(\mathfrak{m}^2, \langle y^3, xy, x^3 \rangle\right)$ has weights of torus action at the tangent space

(t_1, t_2)-weight	$(0, 1), (1, -1)$ $\quad (-1, 1), (1, 0)$

.

Fixing a numbering: $\begin{array}{|c|c|c|} \hline 1 \\ \cline{1-1} 4 \\ \hline 5 & 3 & 2 \\ \hline \end{array}$, the torus acts in entries of (X, Y) associated to the numbering with weights:

$$w_X = \begin{pmatrix} (1\ 0) & (3\ {-2}) & (2\ {-2}) & (1\ {-1}) & (1\ {-2}) \\ (-1\ 2) & (1\ 0) & (0\ 0) & (-1\ 1) & (-1\ 0) \\ (0\ 2) & (2\ 0) & (1\ 0) & (0\ 1) & (0\ 0) \\ (1\ 1) & (3\ {-1}) & (2\ {-1}) & (1\ 0) & (1\ {-1}) \\ (1\ 2) & (3\ 0) & (2\ 0) & (1\ 1) & (1\ 0) \end{pmatrix}, \quad w_Y = \begin{pmatrix} (0\ 1) & (2\ {-1}) & (1\ {-1}) & (0\ 0) & (0\ {-1}) \\ (-2\ 3) & (0\ 1) & (-1\ 1) & (-2\ 2) & (-2\ 1) \\ (-1\ 3) & (1\ 1) & (0\ 1) & (-1\ 2) & (-1\ 1) \\ (0\ 2) & (2\ 0) & (1\ 0) & (0\ 1) & (0\ 0) \\ (0\ 3) & (2\ 1) & (1\ 1) & (0\ 2) & (0\ 1) \end{pmatrix}$$

We have local coordinates:

$$X = \begin{pmatrix} 0 & 0 & 0 & z_1 & 0 \\ 0 & 0 & 1 & z_2 & 0 \\ 0 & 0 & z_1 v_1 + v_2 & z_2 v_2 & 1 \\ 0 & 0 & -z_1 v_2 & z_1 v_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & z_1 & 1 & 0 \\ 0 & 0 & z_2 & 0 & 0 \\ 0 & 0 & z_2 v_2 & -z_2 v_1 & 0 \\ 0 & 0 & z_1 v_1 & v_1 + z_2 v_2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

with weights of the torus action

	z_1	z_2	v_1	v_2
(t_1, t_2)-weight	$(1, -1)$	$(-1, 1)$	$(0, 1)$	$(1, 0)$

To investigate the condition that an element $(I, J) \in \mathrm{Hilb}^{[3,5]}$ lie in strata $\mathrm{Hilb}^{[3,5]}_{s \geq i}$ for $i = 2, 3$, we calculate the determinant and trace of M_X and M_Y:

	$M_X = \begin{pmatrix} z_1v_1+v_2 & z_2v_2 \\ -z_1v_2 & z_1v_1 \end{pmatrix}$	$M_Y = \begin{pmatrix} z_2v_2 & -z_2v_1 \\ z_1v_1 & v_1+z_2v_2 \end{pmatrix}$
det	$C_2 = z_1\left(z_1v_1^2+z_2v_2^2+v_2v_1\right)$ $= z_1E = \frac{z_1}{2}\left(v_1C_1+v_2D_1\right)$	$D_2 = z_2\left(z_1v_1^2+z_2v_2^2+v_2v_1\right)$ $= z_2E = \frac{z_2}{2}\left(v_1C_1+v_2D_1\right)$
trace	$C_1 = (2z_1v_1+v_2)$	$D_1 = (2z_2v_2+v_1)$

$$E := z_1v_1^2+z_2v_2^2+v_2v_1 = \frac{1}{2}\left(v_1C_1+v_2D_1\right)$$

Table 3.6: Analysis of submatrices M_X, M_Y

Conditions for (X,Y) contain at least two points at $(0,0)$:

$\sigma(I)=s$	equations	(t_1,t_2)-weight
$s\geq 2$	$E = z_1v_1^2+z_2v_2^2+v_2v_1 = 0$ $\left(=\frac{1}{2}\left(v_1C_1+v_2D_1\right)\right)$	$(1,1)$
$s=3$	det and trace of $M_X, M_Y = 0$ $\Rightarrow C_1 = 0$ and $D_1 = 0$	$(1,0),(0,1)$

Case of the fixed point $\left(\mathfrak{m}^2, \langle y^2, x^2y, x^3\rangle\right)$

$\begin{array}{|c|c|c} \hline & \star & \\ \hline & & \multicolumn{1}{|c|}{\star} \\ \hline \end{array}$ $\left(\mathfrak{m}^2, \langle y^2, x^2y, x^3\rangle\right)$:

(t_1,t_2)-weight	$(-1,1),(1,0)$ $\quad(-2,2),(2,-1)$

$$\begin{array}{|c|c|c} \hline 4 & 1 & \\ \hline 5 & 3 & \multicolumn{1}{|c|}{2} \\ \hline \end{array} : X = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & z_1 & z_2 & 0 \\ 0 & 0 & v_2z_2 & z_1-v_1z_2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & v_1 & 1 & 0 \\ 0 & 0 & v_2 & 0 & 0 \\ 0 & 0 & v_1z_1+v_2z_2 & z_1 & 1 \\ 0 & 0 & v_2z_1 & v_2z_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Two-dimensional weights:

$$w_X = \begin{pmatrix} (1\ 0) & (2\ {-1}) & (1\ {-1}) & (0\ 0) & (0\ {-1}) \\ (0\ 1) & (1\ 0) & (0\ 0) & ({-1}\ 1) & ({-1}\ 0) \\ (1\ 1) & (2\ 0) & (1\ 0) & (0\ 1) & (0\ 0) \\ (2\ 0) & (3\ {-1}) & (2\ {-1}) & (1\ 0) & (1\ {-1}) \\ (2\ 1) & (3\ 0) & (2\ 0) & (1\ 1) & (1\ 0) \end{pmatrix},\ w_Y = \begin{pmatrix} (0\ 1) & (1\ 0) & (0\ 0) & ({-1}\ 1) & ({-1}\ 0) \\ ({-1}\ 2) & (0\ 1) & ({-1}\ 1) & ({-2}\ 2) & ({-2}\ 1) \\ (0\ 2) & (1\ 1) & (0\ 1) & ({-1}\ 2) & ({-1}\ 1) \\ (1\ 1) & (2\ 0) & (1\ 0) & (0\ 1) & (0\ 0) \\ (1\ 2) & (2\ 1) & (1\ 1) & (0\ 2) & (0\ 1) \end{pmatrix}$$

Weights of coordinates z_1, z_2, v_1, v_2:

	z_1	z_2	v_1	v_2
(t_1,t_2)-weight	$(1,0)$	$(2,-1)$	$(-1,1)$	$(-2,2)$

	$M_X = \begin{pmatrix} z_1 & z_2 \\ v_2 z_2 & z_1 - v_1 z_2 \end{pmatrix}$	$M_Y = \begin{pmatrix} v_1 z_1 + v_2 z_2 & z_1 \\ v_2 z_1 & v_2 z_2 \end{pmatrix}$
det	$C_2 = -v_1 z_2 z_1 - v_2 z_2^2 + z_1^2$	$D_2 = v_2 \left(v_1 z_2 z_1 + v_2 z_2^2 - z_1^2 \right)$
trace	$C_1 = 2z_1 - v_1 z_2$	$D_1 = v_1 z_1 + 2v_2 z_2$

Case of the fixed point $\left(\langle y, x^3\rangle, \langle y^2, xy, x^4\rangle\right)$

$$\left(\langle y, x^3\rangle, \langle y^2, xy, x^4\rangle\right) = \begin{array}{|c|c|c|c|}\hline \star & & & \\ \hline & & & \star \\ \hline\end{array} :$$

(t_1,t_2)-weight	$(-2,1),(2,0)$	$(-1,1),(1,0)$

$$\begin{array}{|c|cccc}\hline 1 \\ \hline 5 & 4 & 3 & 2 \\ \hline\end{array} : X = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & z_1 & 1 & 0 \\ 0 & 0 & z_2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & v_1 & v_2 & 0 \\ 0 & 0 & v_1 z_1 + v_2 z_2 & v_1 & v_2 \\ 0 & 0 & v_1 z_2 & v_2 z_2 & v_1 - v_2 z_1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Weights of coordinates z_1, z_2, v_1, v_2:

	z_1	z_2	v_1	v_2
(t_1,t_2)-weight	$(1,0)$	$(2,0)$	$(-1,1)$	$(-2,1)$

	$M_X = \begin{pmatrix} z_1 & 1 \\ z_2 & 0 \end{pmatrix}$	$M_Y = \begin{pmatrix} v_1 z_1 + v_2 z_2 & v_1 \\ v_1 z_2 & v_2 z_2 \end{pmatrix}$
det	$C_2 := z_2$	$D_2 := -z_2 \left(-v_2 v_1 z_1 - v_2^2 z_2 + v_1^2 \right)$
trace	$C_1 := z_1$	$D_1 := z_1 v_1 + 2 z_2 v_2$

Conditions for (X, Y) contain at least two $(0,0)$:

$\sigma(I) = s$	equations	(t_1,t_2)-weight
$s \geq 2$	$C_2 = 0$	$(2,0)$
$s = 3$	det and trace of $M_X, M_Y = 0$ $\Rightarrow C_2 = 0$ and $C_1 = 0$	$(2,0),(1,0)$

Change of Coordinates

To complete the description of the chart of Hilb$^{[3,5]}$ at each fixed point, we need the transition function of coordinates. They can be found using computer software.

Here, we show an example of this calculation of the transition map $\begin{array}{|c|c}\hline 1 \\ \hline 3 \\ \hline 4 \\ \hline 5 & 2 \\ \hline\end{array} \longrightarrow \begin{array}{|c|cc}\hline 1 \\ \hline 3 \\ \hline 5 & 4 & 2 \\ \hline\end{array}$:

Take $g = \begin{pmatrix} \frac{z_1^2+v_2z_2^2}{z_2(v_1z_2-2z_1)} & -z_2 & 1 & 0 & 0 \\ \frac{1}{2z_1z_2-v_1z_2^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{z_2} & 0 & 0 \\ 0 & 0 & v_1-\frac{z_1}{z_2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{PGL}_{2,3}$.

We have a change of basis,

$$\left(gX_{\yng(1,1,1,2)}g^{-1}, gY_{\yng(1,1,1,2)}g^{-1}\right) = \left(\begin{pmatrix} 0 & 0 & 0 & \frac{-z_1^2+v_1z_2z_1+v_2z_2^2}{v_1z_2-2z_1} & 0 \\ 0 & 0 & 1 & \frac{1}{2z_1-v_1z_2} & 0 \\ 0 & 0 & \frac{z_1^2}{z_2}+v_2z_2 & \frac{z_1}{z_2} & 1 \\ 0 & 0 & z_1\left(-\frac{z_1^2}{z_2}+v_1z_1+v_2z_2\right) & z_1\left(v_1-\frac{z_1}{z_2}\right)+v_2z_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \frac{-z_1^2+v_1z_2z_1+v_2z_2^2}{v_1z_2-2z_1} & 1 & 0 \\ 0 & 0 & \frac{1}{2z_1-v_1z_2} & 0 & 0 \\ 0 & 0 & \frac{z_1}{z_2} & \frac{1}{z_2} & 0 \\ 0 & 0 & -\frac{z_1^2}{z_2}+v_1z_1+v_2z_2 & v_1-\frac{z_1}{z_2} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right).$$

Comparing it to the local coordinates near the fixed point

$$\left(X_{\yng(1,3)}, Y_{\yng(1,3)}\right) : \left(\begin{pmatrix} 0 & 0 & 0 & a_1 & 0 \\ 0 & 0 & 1 & a_2 & 0 \\ 0 & 0 & a_1b_1+b_2 & a_2b_2 & 1 \\ 0 & 0 & -a_1b_2 & a_1b_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & a_1 & 1 & 0 \\ 0 & 0 & a_2 & 0 & 0 \\ 0 & 0 & a_2b_2 & -a_2b_1 & 0 \\ 0 & 0 & a_1b_1 & b_1+a_2b_2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right),$$

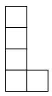
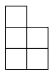
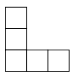
we see that $\left(gX_{\yng(1,1,1,2)}g^{-1}, gY_{\yng(1,1,1,2)}g^{-1}\right)$ is indeed an element of the local coordinates $\left(X_{\yng(1,3)}, Y_{\yng(1,3)}\right)$.

$\mathrm{Hilb}^{[3,5]}$: Change of coordinates f local charts.	
$\yng(1,1,1,2) \to \yng(1,2,2)$	$(z_1,z_2,v_1,v_2) \mapsto (a_1,a_2,b_1,b_2) = \left(v_1z_2z_1+v_2z_2^2-z_1^2, 2z_1-v_1z_2, z_2^{-1}, z_1z_2^{-1}\right)$
$\yng(1,1,1,2) \to \yng(1,3)$	$(z_1,z_2,v_1,v_2) \mapsto (a_1,a_2,b_1,b_2) = \left(\frac{v_1z_2z_1+v_2z_2^2-z_1^2}{v_1z_2-2z_1}, \frac{1}{2z_1-v_1z_2}, v_1-\frac{2z_1}{z_2}, z_1\left(\frac{2z_1}{z_2}-v_1\right)\right)$
$\yng(1,1,1,2) \to \yng(1,4)$	$(z_1,z_2,v_1,v_2) \mapsto (a_1,a_2,b_1,b_2) = \left(v_1z_1+2v_2z_2, v_2\left(z_1^2-v_1z_2z_1-v_2z_2^2\right), z_1^{-1}, \frac{z_2}{z_1\left(-z_1^2+v_1z_2z_1+v_2z_2^2\right)}\right)$
$\yng(1,1,3) \to \yng(1,2,2)$	$(z_1,z_2,v_1,v_2) \mapsto (a_1,a_2,b_1,b_2) = \left(-z_1z_2^{-1}, z_2^{-1}, -v_1z_2, v_2z_2\right)$
$\yng(2,3) \to \yng(1,2,2)$	$(z_1,z_2,v_1,v_2) \mapsto (a_1,a_2,b_1,b_2) = \left(z_1v_1, z_1v_2, \frac{-z_2}{z_1}, \frac{1}{z_1}\right)$

Table 3.7: $\mathrm{Hilb}^{[3,5]}$: Transition maps of local coordinates.

Analysis of divisor $\mathrm{Hilb}^{[3,5]}_{s\geq 2}$ and the corresponding line bundle

We note that the divisor $\mathrm{Hilb}^{[3,5]}_{s\geq 2}$ is the support of the image of the new type correspondences $Q_1^2 \subseteq \mathrm{Hilb}^{[2,4]} \times \mathrm{Hilb}^{[3,5]}$ under the projection $\pi_2 : \mathrm{Hilb}^{[2,4]} \times \mathrm{Hilb}^{[3,5]} \to \mathrm{Hilb}^{[2,4]}$. With our previous calculation, we can actually get more information about this divisor and the corresponding line bundle. First, we have the equations defines the divisor $\mathrm{Hilb}^{[3,5]}_{s\geq 2}$ in each local coordinates:

fixed points	euqation for $s \geq 2$	(t_1,t_2)-weights of v_i, u_i			
		z_1	z_2	v_1	v_2
	$D_2 = v_2 = 0$	$(1,-1)$	$(1,-2)$	$(0,1)$	$(0,2)$
	$D_2 = z_1 v_1^2 + z_2 v_2 v_1 - v_2^2 = 0$ $= \frac{1}{2}\left(v_1 C_1 - v_2 D_1\right)$	$(2,-2)$	$(1,-1)$	$(-1,2)$	$(0,1)$
	$E = z_1 v_1^2 + z_2 v_2^2 + v_2 v_1$ $= \frac{1}{2}\left(v_1 C_1 + v_2 D_1\right)$	$(1,-1)$	$(-1,1)$	$(0,1)$	$(1,0)$

Together with the transition map of local coordinates in Table **??**, we can calculate the gluing map of this the line bundle:

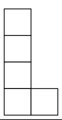
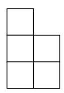
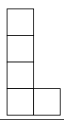
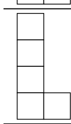
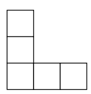
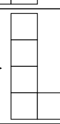
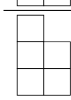
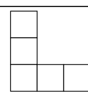
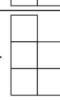
equation of $s \geq 2$ in U_i	equation of $s \geq 2$ of U_j in U_i	transition function
 $: -v_2$	 $\longrightarrow$  $: -v_2$	1
 $: -v_2$	 $\longrightarrow$  $: -v_2\left(2z_1 - v_1 z_2\right)$	$(2z_1 - v_1 z_2)$
 $: -v_1^2 z_1 - v_2 v_1 z_2 + v_2^2$	 $\longrightarrow$  $: z_2\left(-v_1^2 z_1 - v_2 v_1 z_2 + v_2^2\right)$	z_2

Table 3.8: $\mathrm{Hilb}^{[3,5]}$: Gluing map of line bundle $\phi_{ij} : U_i \cap U_j \to \mathbb{C}^*$.

Another interesting information we can get from our examples is the weights of the torus action. We consider determinant line bundles B and F in (3.10) for the case of $\mathrm{Hilb}^{[3,5]}$. Here, we have a list of the (t_1,t_2)-weights of torus action on the fiber $\det B$, $\det F$ and the divisor $\mathrm{Hilb}^{[3,5]}_{s\geq 2}$ at each fixed point: One guess of describing globally the line bundle $L_{s\geq 2}$ associate to $\mathrm{Hilb}^{[3,5]}_{s\geq 2}$ is to express it into product of the existing line bundles $\det B$ and $\det F$. We assume that the weights of two trivial line bundle $\simeq \mathrm{Hilb}^{[3,5]} \times \mathbb{C}$ from the action of $t_1 \in \mathbb{C}^*$ and $t_2 \in \mathbb{C}^*$ are $(a_1, 0)$ and $(0, a_2)$. Denoted by $\mathrm{wt}\,(*)_\lambda$ the weight of torus action of $*$ in the local coordinates of the fixed point λ. Then then following equation should hold at each fixed point λ of $\mathrm{Hilb}^{[3,5]}$:

$$\alpha \cdot \mathrm{wt}(\det B)_\lambda + \beta \cdot \mathrm{wt}(\det F)_\lambda + (a_1, 0) + (0, a_2) = \mathrm{wt}(L_{s\geq 2})_\lambda \tag{3.8}$$

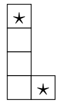
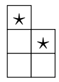
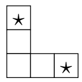
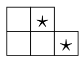
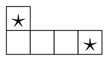
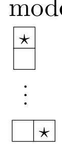
Fixed point	$\det B$	$\det(I/J)$	$\mathrm{Hilb}^{[3,5]}_{s\geq 2}$
	$(0,3)$	$(1,3)$	$(0,2)$
	$(1,1)$	$(1,3)$	$(0,2)$
	$(1,1)$	$(2,3)$	$(1,1)$
	$(1,1)$	$(3,1)$	$(2,0)$
	$(3,0)$	$(3,1)$	$(2,0)$

Table 3.9: (t_1,t_2)-weight of $\det B$, $\det F$ and $L_{s\geq 2}$.

for some fixed integers α, β, a_1 and a_2.
Solving a system of five equations in four unknown α, β, a_1, a_2,we get solution for weight of $L_{s\geq 2}$: $\alpha = 0, \beta = 1, a_1 = -1 a_2 = -1$ It turns our surprisingly that the line bundle associated to the divisor $\mathrm{Hilb}^{[3,5]}_{s\geq 2}$ depends only on the bundle F, that is the quotient (I/J).

3.2.3 Local coordinates of $\mathrm{Hilb}^{[n,n+2]}$ in matrix description; example and conjecture

Motivating from observations of our previous examples, we have a model of local coordinates for the fixed point of $\mathrm{Hilb}^{[n,n+2]}$ that has corresponding young diagram  $\sim \lambda = (n,1)$.

Proposition. *If a fixed point $(I,J) \in \mathrm{Hilb}^{[n,n+2]}$ has the corresponding Young diagram $\triangle_{(I,J)} \sim (n,1)$, then the map*

$$\mathbb{C}^{2n-2} \longrightarrow \widetilde{\mathcal{M}} \to \mathrm{Hilb}^{[n,n+2]}$$

$$(z_1,\dots,z_{n-1},w_1,\dots,w_{n-1}) \mapsto \left(\begin{pmatrix} 0 & 0 & z_1 & \dots & z_{n-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & * & \dots & * & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & w_1 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & w_{n-1} & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right) \tag{3.9}$$

where $* \in \mathbb{C}[z_1,\dots,z_{n-1},w_1,\dots w_{n-1}]$ is a local chart of the top dimensional cell associate to $\triangle_{(I,J)}$ in a small neighborhood.

Proof. We first claim that one can use $P_{2,n}$ to eliminate the entries of X, Y and i that are 0 in (3.9). We choose the numbering $\begin{array}{cc} \boxed{1} & \\ \boxed{3} & \\ \boxed{4} & \\ \vdots & \\ \boxed{} & \boxed{2} \end{array}$ for the fixed point $\triangle_{(I,J)}$. We normalize the vector i to be $(0,\dots,0,1)^T$ using the $n+2$-column of the subgroup group $P_{2,n}$. This implies that X, Y have entries $X_{2,n+2}$, $Y_{1,3}, Y_{k,k+1}$, $k = 3, \dots n+1$ and the diagonal of $P_{2,n}$ equal to 1:

$$P_{2,n} = \begin{pmatrix} g_{11}=1 & g_{12} & g_{13} & \dots & g_{1,n+1} & 0 \\ g_{21} & 1 & & \dots & g_{2,n+1} & 0 \\ 0 & 0 & 1 & \dots & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & & & 1 & 0 \\ 0 & 0 & g_{n+1,3} & & g_{n+2,n+1} & 1 \end{pmatrix}, \ (X,Y,i) = \left(\begin{pmatrix} 0 & 0 & X_{13} & \dots & X_{1,n+1} & X_{1,n+2} \\ 0 & 0 & X_{23} & \dots & X_{2,n+1} & 1 \\ 0 & 0 & X_{33} & \dots & X_{3,n+1} & X_{3,n+1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & Y_{1,4} & \dots & Y_{1,n+2} \\ 0 & 0 & Y_{2,3} & Y_{2,4} & \dots & Y_{2,n+2} \\ 0 & 0 & Y_{3,3} & 1 & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & Y_{n+1,3} & Y_{n+1,4} & \dots & 1 \\ 0 & 0 & Y_{n+2,3} & & & Y_{n+2,n+2} \end{pmatrix}, i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right)$$

Now the entries of X and Y that lie above the 1's can be eliminated by the upper triangular part of $P_{2,n}$:

$P_{2,n}$	(X,Y,i)
$\begin{pmatrix} g_{11}=1 & 0 & 0 & \dots & 0 & 0 \\ g_{21} & 1 & & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & & & 1 & 0 \\ 0 & 0 & g_{n+1,3} & & g_{n+2,n+1} & 1 \end{pmatrix}$	$\left(\begin{pmatrix} 0 & 0 & X_{13} & \dots & X_{1,n+1} & 0 \\ 0 & 0 & X_{23} & \dots & X_{2,n+1} & 1 \\ 0 & 0 & X_{33} & \dots & X_{3,n+1} & X_{3,n+1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 0\dots & 0 \\ 0 & 0 & Y_{2,3} & 0 & \dots & 0 \\ 0 & 0 & Y_{3,3} & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & Y_{n+1,3} & Y_{n+1,4} & \dots & 1 \\ 0 & 0 & Y_{n+2,3} & Y_{n+2,4} & \dots & Y_{n+2,n+2} \end{pmatrix}, i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right)$

We can further put $Y_{2,3}$ and the entries in the power triangular submatrix $\begin{pmatrix} 1 & 0 & \dots & 0 \\ Y_{4,4} & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ Y_{n+1,4} & & \dots & 1 \\ Y_{n+2,4} & & \dots & Y_{n+2,n+2} \end{pmatrix}$ by the rest of the entries in $P_{2,n}$:

$P_{2,n}$	(X,Y,i)
$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & & & 1 & 0 \\ 0 & 0 & 0 & & 0 & 1 \end{pmatrix}$	$\left(\begin{pmatrix} 0 & 0 & X_{1,3} & \dots & X_{1,n+1} & 0 \\ 0 & 0 & X_{2,3} & \dots & X_{2,n+1} & 1 \\ 0 & 0 & X_{3,3} & \dots & X_{3,n+1} & X_{3,n+2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & * & * & * & * \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 0\dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & Y_{3,3} & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & Y_{n+2,3} & 0 & \dots & 0 \end{pmatrix}, i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right)$

Since $[X,Y] = 0$, the last row of X and Y has entries 0. Moreover, the entries in 2nd rows of X are as well 0 except for $X_{2,n+2} = 1$:

$\mathrm{P}_{2,n}$	(X, Y, i)
$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & & & 1 & 0 \\ 0 & 0 & 0 & & 0 & 1 \end{pmatrix}$	$\left(\begin{pmatrix} 0 & 0 & X_{1,3} & \dots & X_{1,n+1} & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & X_{3,3} & \dots & X_{3,n+1} & X_{3,n+2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & X_{n+1,3} & \dots & & X_{n+1,n+2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & Y_{3,3} & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & Y_{n+1,3} & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}, i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right)$

Our next step is to show that the rest of entries $X_{k,l}$ where $3 \leq k \leq n+1$, $3 \leq l \leq n+2$ can be expressed as a function of $X_{1,\alpha}$ and $Y_{\alpha,3}$ for $3 \leq \alpha \leq n+1$. We prove it by inductions. We may assume without losing generality that $3 \leq \alpha \leq n+1$. Considering the commutation relation $[X, Y] = 0$, we observe that $Y_{k,l} = 0$ except for the entries of "1"s and $Y_{3,\alpha}$. Then $[X, Y]_{1,\alpha}$ gives equations $\sum_k X_{1,k} Y_{k,\alpha} = Y_{1,3} X_{3,\alpha} = 1 \cdot X_{3,\alpha} = X_{3,\alpha}$. So each monomial $X_{3,\alpha}$ can be found in an entry of $[X, Y]$. Suppose that for $k \geq 3$, each entry $X_{k,\alpha}$ are functions in $X_{k',\alpha}$ for $k' \leq k$. We want to show that $X_{k+1,\alpha}$ can be expressed by $X_{k',\alpha}, k' \leq k$. This works with the same idea: the monomial $X_{k+1,\alpha}$ appears in $[X, Y]_{k,\alpha}$ from $YX_{k,\alpha} = \sum Y_{k,l} X_{l,\alpha} = Y_{k,k+1} X_{k+1,\alpha} + Y_{k,3} X_{3,\alpha}$ since $Y_{k,l} = 0$ for all $l \neq 3, (k+1)$. Since $XY_{k,\alpha} = \sum X_{k,l} Y_{l,\alpha}$ by assumption is a function in $X_{k',\alpha}$s for $k' \leq k$. Thus the statement is true for $k+1$. We conclude from inductions, $X_{k,l}$ $3 \leq k \leq n+1, 3 \leq l \leq n+2$ can be expressed in terms of 2n-2 variables $z_\alpha = X_{1,2+\alpha}$ $w_\alpha = Y_{2+\alpha,3}$, $\alpha = 1, \ldots, n-1$. ■

For arbitrary h_2, we have a conjecture based on the fact that the set of weights of torus action of local coordinates must coincide with the weights at the tangent space of the fixed point.
We explain an observation of the weights of local coordinate and the weights at the tangent space in the next subsection.

Weights, Young diagrams and top-dimensional cells of $\mathrm{Hilb}^{[n,n+2]}$

In this subsection, we explain an observation of choice of entries in matrix description such that As we have seen in Chapter 2, the 2-dimensional torus T acts on the refined Hilbert scheme. If we choose a generic one-parameter subgroup $\mathbb{C}^* \to T$ of the torus, then the fixed points set coincides with the T-fixed points set and it induced affine cell decompositions of $\mathrm{Hilb}^{[n,n+r]}$. Each cell is indexed by a fixed point of $\mathrm{Hilb}^{[n,n+r]}$. These fixed points can be represented by marked Young diagrams $\triangle$, and we can read off several pieces of information about the cell. For instance, the dimension of the cell is $|\triangle| - \binom{r}{2} - w(\triangle) = \frac{1}{2}\left(2n - r^2 + 3r\right) - w(\triangle)$. In particular, the top-dimensional cells of $\mathrm{Hilb}^{[n,n+r]}$ are in one-to-one correspondence to the Young diagram of $(n+r)$ boxes with a minimum number of columns (i.e., r columns), and r marked removable boxes. Another information that we get from the marked diagram is the weights of the torus action at the tangent space of the fixed points, which presents as arrows pointing from one monomial to another. Here, we are interested in the case of $r = 2$, where the top-dimensional cell corresponds to a marked Young diagram with two columns.
Let $\triangle_\lambda \in \mathrm{Hilb}^{[n,n+2]}$ be a torus fixed point. Assuming that the second column of $\triangle_\lambda$ has h_2 boxes, we choose a numbering of $\triangle_\lambda$ in the way that, counting from the top box, the h_2 boxes of the first column are numbered by $1, 3, \ldots, 1 + 2(h_2 - 1)$ and the h_2 boxes of the second column are numbered by $2, 4, \ldots, 2 + 2(h_2 - 1)$.
We fix a basis for the matrices $X = (X_{ij})_{1 \leq i,j \leq n+2}$, $Y = (Y_{ij})_{1 \leq i,j \leq n+2}$ according to this Young

1	
3	
5	
$\vdots$	
	2
	4
	6

Table 3.10: An example for $h_2 = 3$.

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tableau $\triangle_\lambda$ so that the entries of X and Y satisfy the conditions:

$$\begin{cases} X_{ij} = 1 \text{ if } \triangle_\lambda \text{ contains a subdiagram of the form } \boxed{j}\boxed{i} \\ Y_{kl} = 1 \text{ if } \triangle_\lambda \text{ contains a subdiagram of the form } \begin{smallmatrix}\boxed{k}\\ \boxed{l}\end{smallmatrix}. \end{cases}$$

It follows from the formula (2.15) for the character of the tangent space at a fixed point, if $b_{i,j}$ is a box of $\triangle_\lambda$, then the two tangent vectors associated to $b_{i,j}$ are

$$\left(l_{\triangle'_\lambda(b_{i,j})} + 1, -a_{\triangle_\lambda(b_{i,j})}\right) \text{ and } \left(-l_{\triangle_\lambda(b_{i,j})}, a_{\triangle'(b_{i,j})} + 1\right).$$

We denote the coordinate of a box numbered by k by (k_x, k_y), which correspond to the one-dimensional weight space $\mathbb{C}x^\alpha y^\beta$ of weight (t_1^α, t_2^β). Let (α_k, β_k), $i = 1, \ldots n+2$ be the multiplicative (t_1, t_2) weights. We want to associate vectors inside the Young digram $\triangle_\lambda$ to the weight of torus action at the tangent space of this fixed points. First, let (α_i, β_i), $i = 1, \ldots n+2$ be the multiplicative (t_1, t_2) weights of the torus action. Then the torus acts on the ij-th matrix entry of X and the kl-th matrix entry of Y by

$$\begin{cases} (w_{t_1} X_{ij}, w_{t_2} X_{ij}) = (\alpha_j - \alpha_i + 1,\ \beta_j - \beta_i), \\ (w_{t_1} Y_{ij}, w_{t_2} Y_{ij}) = (\alpha_j - \alpha_i,\ \beta_j - \beta_i + 1). \end{cases}$$

We write the position of of a numbered box $\boxed{i}$ by (i_x, i_y). Then the above statement is equivalent to

$$\begin{cases} (w_{t_1} X_{ij}, w_{t_2} X_{ij}) & = (j_x - i_x + 1, j_y - i_y), \\ (w_{t_1} Y_{ij}, w_{t_2} Y_{ij}) & = (j_x - i, j_y - i_y + 1). \end{cases}$$

for two boxes numbered by i and j.

Second, from the character formula 2.15 each box $b_{i,j}$ of $\triangle_\lambda$ gives rise to a pair of (t_1, t_2)-weights

$$\begin{cases} \left(-l_{\triangle_\lambda(b_{i,j})},\ a_{\triangle'(b_{i,j})} + 1\right), \\ \left(l_{\triangle'(b_{i,j})} + 1,\ -a_{\triangle_\lambda(b_{i,j})}\right) \end{cases}.$$

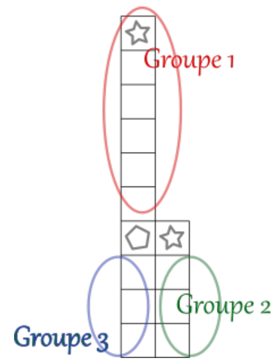
We note that, if an entry $X_{i,j}$ is a local coordinate then near $\triangle_\lambda$, then it must have the same weight. From the above equations, the boxes i, j associated with the entries X_{ij} should satisfy:

$$j_x - i_x = l_{\triangle'_\lambda(b_{i,j})}$$
$$j_y - i_y = -a_{\triangle_\lambda(b_{i,j})}.$$

Similarly we have for Y_{lk}, , the boxes k, l associated the entries Y_{kl} should satisfy:

$$l_x - k_x = -l_{\triangle_\lambda(b_{i,j})}$$
$$l_y - k_y = a_{\triangle'_\lambda(b_{i,j})}.$$

Base on this fact, we describe weight vectors in three different part of Young diagram:

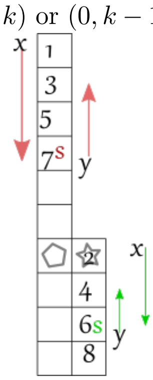


If b_i is a box number by i in **Group 1** or **Group 2** with k boxes above, then the weight vectors associated to this box are

$$\begin{cases} (l_{\triangle'_\lambda(b_{i,j})} + 1, -a_{\triangle_\lambda(b_{i,j})}) = (1, -k) \\ (-l_{\triangle_\lambda(b_{i,j})}, a_{\triangle'_\lambda(b_{i,j})} + 1) = (0, k) \end{cases}.$$

And an entry of X or Y has correct weight if its index corresponds to the boxes inside the Young diagram that are admissible with the vectors $(0, -k)$ or $(0, k-1)$. Among these possible choices, we pick the following entries:

Group 1: X_{1s}, Y_{s3},
Group 2: X_{2s}, Y_{s4}.



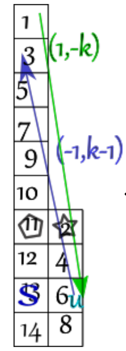
Next, we consider the boxes located in the **Group 3** and assume that the box has k boxes above. In this case, the weight vectors are

$$\begin{cases} (l_{\triangle'_\lambda(b_{i,j})} + 1, -a_{\triangle_\lambda(b_{i,j})}) = (2, -k) \\ (-l_{\triangle_\lambda(b_{i,j})}, a_{\triangle'(b_{i,j})} + 1) = (-1, k). \end{cases}$$

and the entries with corrected weights are the boxes inside the Young diagram that admissible with the vectors

$$\begin{cases} (1, -k) \\ (-1, k-1). \end{cases}$$

Let u be the number of the box on the right of $b_{i,j}$.

We choose, among these possibilities, the entries for Group 3: X_{1u}, Y_{u3}. 

. These entries X_{1u}, Y_{u3} have, by the construction, the desired weights of torus action.
Summing up, we conjecture that the entries we chose form a system of local coordinates in a neighborhood of the fixed point $\triangle_\lambda$.

Conjecture 3. *Let $\triangle_{(I,J)}$ be fixed point of $\mathrm{Hilb}^{[n,n+2]}$ presents a top dimensional cell. We assume that $\triangle_{(I,J)}$ has h boxes at the second column, where $1 \le h \le \frac{n+2}{2} + 1$.*
We choose numbering of in the way that, counting from highest the box, the h_2 boxes of the first column are numbered by $1, 3, \ldots, 1 + 2(h_2 - 1)$ and the h_2 boxes of the second column are numbered by $2, 4, \ldots, 2 + 2(h_2 - 1)$.
Then there exists a local chart $\mathbb{C}^{2(n-1)} \to \mathrm{Hilb}^{[n,n+2]}$ in terms of entries of the matrices (X, Y) near the fixed point.
Then the set of entries of X and Y

$$\begin{cases} X_{1i} & i\text{=1+2*1,}\ldots\text{,1+2(h-1),2h+1, } \ldots\text{, 2h+n+2-3h=n+2-h} \\ X_{2j} & j\text{=2+2,}\ldots\text{, 2+2(h-1)} \\ X_{1k} & k\text{=2+2,}\ldots\text{, 2+2(h-1)} \end{cases}$$

$$\begin{cases} Y_{i3} & i\text{=1+2*1,}\ldots\text{,1+2(h-1),2h+1,}\ldots\text{, 2h+n+2-3h=n+2-h} \\ Y_{j4} & j\text{=2+2,}\ldots\text{, 2+2(h-1)} \\ Y_{k3} & k\text{=2+2,}\ldots\text{, 2+2(h-1)} \end{cases}$$

form a local coordinates from $\mathbb{C}^{2(n-1)} \to \mathrm{Hilb}^{[n,n+2]}$.
Note that the total number of the variables in X (Y, respectively) is

$$\underbrace{(h-1) + (n+2-3h)}_{\#X_{1i}(Y_{i3},\ respectively)} + \underbrace{h-1}_{\#X_{2j}(Y_{j4},\ respectively)} + \underbrace{h-1}_{\#X_{1k}(Y_{k3},\ respectively)}$$

is $(n-1)$.

3.3 A special divisor $\mathrm{Hilb}^{[n,n+2]}_{s\ge 2}$

For $1 \le k \le n$, we define subvarieties of $\mathrm{Hilb}^{[n,n+2]}$ with respect to the number of points I has at $(0,0)$

$$\mathrm{Hilb}^{[n,n+2]}_{(s\ge k)} = \left\{ (I,J) \in \mathrm{Hilb}^{[n,n+2]} \,\middle|\, \sigma(I) := s \ge k \right\}$$

It induces a filtration on $\mathrm{Hilb}^{[n,n+2]}$:

$$\mathrm{Hilb}^{[n,n+2]} = \mathrm{Hilb}^{[n,n+2]}_{(s\geq 1)} \supseteq \mathrm{Hilb}^{[n,n+2]}_{(s\geq 2)} \supseteq \cdots \supseteq \mathrm{Hilb}^{[n,n+2]}_{(s=n)} = Br^{[n,n+2]}.$$

We are particularly interested in the strata when $k = 2$, which is inspired by the analysis in the examples of $\mathrm{Hilb}^{[2,4]}$ and $\mathrm{Hilb}^{[3,5]}$ in the previous section. Where we saw that near each fixed point, $\mathrm{Hilb}^{[3,5]}_{s\geq 2}$ is given by zero locus of a single equation in our local coordinates.
Generally, the subvariety $\mathrm{Hilb}^{[n,n+2]}_{s\geq 2}$ is of codimension one: Since $\mathrm{Hilb}^{[n,n+2]}_{s\geq 2}$ is motivicly isomorphic to the closure of $\mathrm{Hilb}^{[n-2]} \times Br^{[2,4]}$ and $\mathrm{Hilb}^{[n,n+2]}$ is $2n-2$-dimensional, we have

$$\begin{aligned}\dim_{\mathbb{C}} \mathrm{Hilb}^{[n,n+2]}_{s\geq 2} &= \dim_{\mathbb{C}} \mathrm{Hilb}^{[n-2]} + \dim_{\mathbb{C}} B^{[2,4]} \\ &= 2(n-2)+1 \\ &= 2n-3.\end{aligned}$$

In fact, this divisor $\mathrm{Hilb}^{[n,n+2]}_{s\geq 2}$ is the support of the image of the correspondences $Q_1^2 \subseteq \mathrm{Hilb}^{[n,n+2]} \times \mathrm{Hilb}^{[n+1,n+3]}$ of the new type operators under the projection $\mathrm{Hilb}^{[n,n+2]} \times \mathrm{Hilb}^{[n+1,n+3]} \to \mathrm{Hilb}^{[n+1,n+2]}$.
Generally, for any $r \geq 1$, we have:

Proposition 3.3.1. *The support of the image of the projection* $\mathrm{Hilb}^{[n,n+r]} \times \mathrm{Hilb}^{[n+1,n+r+1]} \to \mathrm{Hilb}^{[n+1,n+r+1]}$ *of the correspondence* $Q^r_{1,n}$ *of adding a point at* $(0,0)$ *is equal to the divisor* $\mathrm{Hilb}^{[n+1,n+r+1]}_{s\geq\binom{r}{2}+1} \subset \mathrm{Hilb}^{[n+1,n+r+1]}$.

Proof. The generic component of the refined Hilbert scheme $\mathrm{Hilb}^{[n,n+r]}$ is the strata $\mathrm{Hilb}^{[n,n+r]}_{s\geq\binom{r}{2}}$, where $(I,J) \in \mathrm{Hilb}^{[n,n+r]}$ and I has ar least r points supported at $(0,0)$. Since for any $(I,J) \in \mathrm{Hilb}^{[n,n+r]}$, the minimum number of generators $\mu(I)$ (=$\dim R/I_0$) of I is greater than 1, we can add a point at $(0,0)$ without imposing extra conditions to I. In other words, any elements of $\mathrm{Hilb}^{[n+1,n+r+1]}_{s\geq\binom{r}{2}+1}$ can be realized by adding a $(0,0)$ to an element of $\mathrm{Hilb}^{[n,n+r]}$. ■

We first discuss an example of this line bundle associated to the divisor $\mathrm{Hilb}^{[2,4]}_{s\geq 2}$.

Example 3.3.2. We claim that the line bundle associate to divisor $\mathrm{Hilb}^{[2,4]}_{s\geq 2}$ is the pull-back bundle π^*S, where $\pi : S \to \mathbb{P}^1$ is the tautological lines bundle of $\mathbb{P}^1$:
We have seen that $\mathrm{Hilb}^{[2,4]}$ is isomorphic to the blow-up of $\mathbb{C}^2$ at the origin. Now, we can pull back S though the bundle projection $\pi : S \to \mathbb{P}^1$ to S itself:

$$\begin{array}{ccc} \pi^*S & & \\ \downarrow & & \\ S & \xrightarrow{\pi} & \mathrm{Hilb}^{[2,4]} \end{array} \qquad \text{with fiber} \qquad \begin{array}{c} (\pi^*S)_q = S_q = l_p \\ \downarrow \\ (l,q) \ \in S \end{array}$$

It comes with a canonical section given by $\gamma : S \to \pi^*S$, sending $(l,q) = (l_q, q)$ to the point (on the fiber) q and $\gamma(l,q)$ is zero $\Leftrightarrow q = (0,0) \Leftrightarrow s(I) \geq 2$.

Indeed, from our local coordinates in the previous example, the transition function ψ should satisfy

$$f_{\ydiagram{1,1,2}}(z_1, v_1) = \psi \circ f_{\ydiagram{1,3}}(a_1, b_1) \circ \phi$$

where $f_{\ydiagram{1,1,2}} = v_1$ et $f_{\ydiagram{1,3}} = a_1$ are equations for $\mathrm{Hilb}^{[2,4]}_{s\geq 2}$ in $\mathrm{Hilb}^{[2,4]}$ and $\phi : (z_1, v_1) \mapsto (a_1, b_1) = (v_1 z_1, z_1^{-1})$. We have then

$$v_1 = \psi v_1 z_1 \Leftarrow \psi = \frac{1}{z_1}.$$

We are interested in identifying the corresponding line bundle and the section defining the divisor $\mathrm{Hilb}^{[n,n+2]}_{s\geq 2}$.
We express $(I, J) \in \mathrm{Hilb}^{[n,n+2]}$, $n \geq 1$ by intersection of ideals:

$$\begin{cases} I = I_{p_1} \cap \cdots \cap I_{p_l} \cap I_0, \\ J = J_{p_1} \cap \cdots \cap J_{p_l} \cap J_0 = I_{p_1} \cap \cdots \cap I_{p_l} \cap J_0 \end{cases}$$

for some distinct points $p_1, \cdots, p_l \in X^*$.
Let F be the vector bundle over $\mathrm{Hilb}^{[n,n+2]}$ with fiber

$$F_{(I,J)} = I_0/J_0 = I/J. \tag{3.10}$$

We claim that there is a well-defined map from $F_{(I,J)}$ to the quotient $\mathfrak{m}/\mathfrak{m}^2$. Clearly, the maximal ideal $\mathfrak{m}$ at $(0,0)$ always contains I_0. The rest is to show that J_0 is contained in $\mathfrak{m}^2$:
We observe that if $(I, J) \in \mathrm{Hilb}^{[n,n+2]}$, then $\dim_{\mathbb{C}}(R/J_0) \geq 3$ and the minimum number of generators $\mu(J_0)$ is at least 3 by the condition $I \supseteq_{\mathfrak{m}} J$. Since the maximal ideal $\mathfrak{m}^2$ is the ideal having minimum 3 generators with minimal number of point at $(0,0)$, we conclude that $J_0 \subseteq \mathfrak{m}^2$.
Denoted by M the trivial (but equivariantly non-trivial) vector bundle over $\mathrm{Hilb}^{[n,n+2]}$ with constant fiber

$$M_{(I,J)} = \mathfrak{m} \,/\mathfrak{m}^2 \,. \tag{3.11}$$

We have then a canonical bundle map $f : F \to M$

$$\begin{aligned} F_{(I,J)} &\to M \\ f + J_0 &\mapsto f + \mathfrak{m}^2. \end{aligned}$$

The map $f : F \to M$ induces a map over their determinant

$$\det f : \wedge^{top} F \to \wedge^{top} M.$$

Theorem 3.3.3. *The divisor* $\mathrm{Hilb}^{[n,n+2]}_{s\geq 2} \subset \mathrm{Hilb}^{[n,n+2]}$ *is, set-theoretically, the zero locus given by a section of* $(\wedge^{top} F)^* \otimes \wedge^{top} M$.

Proof. We claim that the bundle map $f : F_{(I,J)} \to M$ at the fiber of an element $(I, J) \in \mathrm{Hilb}^{[n,n+2]}$ is surjective if and only if $\sigma(I) = 1$, that is the ideal I has only one point supported at $(0,0)$.
If $\sigma(I) = 1$, then $(I_0, J_0) = \left(\mathfrak{m}, \mathfrak{m}^2\right)$ and $f : I/J = \mathfrak{m}/\mathfrak{m}^2 \to M$ is the identity map.
If an element $(I, J) \in \mathrm{Hilb}^{[n,n+2]}$ with $\dim_{\mathbb{C}} R/I_0 \geq 2$, then there exists $h \in I_0/J_0$ such that the degree $\deg h \geq 2$. To see this, we consider the cases of different minimum number of generators $\mu(I) =$

2 and $\mu(I) \geq 3$. Let $(I,J) \in \mathrm{Hilb}^{[n,n+2]}$, we first consider the case of $\mu(I) = 2$. This implies that the ideal $I_0 \in Br^{[\sigma(I)]}$ is curvlinear and can be present in the from $\langle y^{\sigma(I)}, x + a_1 y + \cdots + a_{\sigma(I)-1} y^{\sigma(I)-1}\rangle$. By $J = \mathfrak{m}I$, one of two generators the quotient space I/J must have degree ≥ 2 and vanishes under the map $f : F_{(I,J)} \to M = \mathfrak{m}/\mathfrak{m}^2$. Next, assume that $\mu(I) \geq 3$. Since the element $I \in \mathrm{Hilb}^{[n]}$ such that $\mu(I) = 3$ with least $n \in \mathbb{N}$ is the ideal $\langle x^2, xy, y^2\rangle = \mathfrak{m}^2 \in \mathrm{Hilb}^{[3]}$, the generators of $I/\mathfrak{m}I$ are polynomials of degree ≥ 2. This implies that the subspace $I/J \leq I/\mathfrak{m}I$ are also spanned by polynomials of degree higher than 2 and they vanish under the map f. Thus, f is surjective at $(I,J) \in \mathrm{Hilb}^{[n,n+2]}$ if and only if $\sigma(I) = 1$.
By the isomorphism $\mathrm{Hom}\,(F,M) \simeq \Gamma(F^* \otimes M)$, the map $\det f$ is a section of the line bundle $(\wedge^{top} F)^* \otimes \wedge^{top} M$. From above calculation, we see that $\sigma(I) \geq 2 \Leftrightarrow f$ is not an isomorphism $\Leftrightarrow$ the determinant of linear map f vanish. We conclude that the support $\{\det f = 0\}$ is the subvariety $\mathrm{Hilb}_{(s\geq 2)}^{[n,n+2]}$.

■

Another interesting special case of $\mathrm{Hilb}_s^{[n,n+2]}$ is when $s = n$, the refined Briançon variety $\mathrm{Hilb}_{(s=n)}^{[n,n+2]} = Br^{[n,n+2]}$.
The following statement not only for $Br^{[n,n+2]}$ but any $Br^{[n,n+r]}$.
Let $B \to \mathrm{Hilb}^{[n,n+r]}$ be ($\mathrm{Hilb}^{[n]}$) tautological bundle whose fiber at (I,J) is the quotient

$$B_{(I,J)} = R/I.$$

We define bundle maps Σ and $\gamma : B \to \mathcal{O}_{\mathrm{Hilb}^{[n,n+r]}}$ as follows

$$\Sigma : h \mapsto \sum_{p\in \mathrm{Supp}(I)} m_{p,I} h(p) \text{ and}$$
$$\gamma : h \mapsto h(0,0)$$

where Σ send $h \in B$ to the sum of value of f at the supports of I with multiplicities $m_{p,I} = \dim_{\mathbb{C}}(R/I_p)$ and γ is the evaluation at $(0,0)$ which is well-define since for any element (I,J) of $\mathrm{Hilb}^{[n,n+r]}$, the support $\mathrm{Supp}(I)$ contains $(0,0)$.

$$\begin{array}{ccc} B & \xrightarrow{\Sigma,\ \gamma} & \mathcal{O}_{\mathrm{Hilb}^{[n,n+r]}} \\ & \searrow \quad \swarrow & \\ & \mathrm{Hilb}^{[n,n+r]} & \end{array}$$

Note that both Σ and γ are surjective since $1 \in B$. Also both kernel of Σ and γ have $\dim_{\mathbb{C}} = (n-1)$. Let $B' := \ker \Sigma$, a bundle over $\mathrm{Hilb}^{[n,n+r]}$ of rank $(n-1)$.

Theorem 3.3.4. *The refined Briançon variety $Br^{[n,n+r]} = \mathrm{Hilb}_{s=n}^{[n,n+r]}$ is the zero locus of the section γ^* of the bundle $(B')^*$.*

Proof. We show that $B' = \ker\gamma$ if and only if $\rho(I) = (0,0)$. Since the dimension of B' and $\ker\gamma$ are both equal to $n-1$, showing $B' = \ker\gamma$ is equivalent to the statement of $(I,J) \in \mathrm{Hilb}^{[n]}$:

$$\sum_{p\in \mathrm{Supp}(I)} m_{p,I} h(p) = 0 \Leftrightarrow h(0,0) = 0, \forall h \in B'. \tag{3.12}$$

If $(I,J)\in Br^{[n,n+r]}$ then for all $h\in B'$, then

$$\sum_{p\in \mathrm{Supp}(I)} m_{p,I}h(p) = n\cdot h(0,0) = 0 \Leftrightarrow h(0,0)=0.$$

Conversely, let $(I,J)\in \mathrm{Hilb}^{[n,n+r]}$ such that $B'=\ker\gamma$. We claim that $\mathrm{Supp}(I)=\{(0,0)\}$. Suppose it is not true, that $|\mathrm{Supp}(I)|\geq 2$. For any $h\in B'$, we can rewrite $h(0,0)=\sum_{p\in\mathrm{Supp}(I)\backslash(0,0)} c_p h(p) = c$ for some $c_p\in\mathbb{C}$. Then for any $c=h(0,0)\in\mathbb{C}$, one can construct $h\in B'$ such that the sum on the right-hand side equals to c by assigning value to each points in $\mathrm{Supp}(I)\backslash(0,0)$. We have $h(0,0)\neq 0$, a contraction. Therefore, I must be supported at $(0,0)$, and it follows $(I,J)\in Br^{[n,n+r]}$.

■

Remark 3.3.5. Let $\triangle_{top}$ be a fixed point with h_2 boxes on the second column representing a top-dimensional cell. Let $C_{\triangle_{top}}$ be the associated affine cell of $\mathrm{Hilb}^{[n+1,n+3]}$ with the local coordinates $(X_{ij})_{1\leq i,j\leq n+3}, (Y_{ij})_{1\leq i,j\leq n+3}$ in conjecture 3. Then $C_{\triangle_{top}}\cap\pi_2(Q_1^2)\subset\mathrm{Hilb}^{[n+1,n+3]}$ is an algebraic variety given by the zero locus of the determinant equation of the following submatrix:

$$\begin{pmatrix} Y_{33} & \dots & Y_{3(n+2)} \\ \vdots & & \vdots \\ Y_{(n+2)3} & \dots & Y_{(n+2)(n+2)} \end{pmatrix}.$$

In addition, it has a weight of two-dimensional torus action $(0, n-(h_2-1))$.

Chapter 4

The E-polynomials of the refined strata of the Hilbert scheme of points on the plane

In this chapter, we study the geometric and topological invariants of strata associated to the functions μ and σ. We will focus on the *E-polynomial* (or *the virtual Hodge polynomial*, *the Hodge-Deligne polynomial*) whose motivic properties facilitate the calculations. We present a formula for the E-polynomials of the strata of the Hilbert scheme of points on $\mathbb{C}^2$ associated to a fixed number of generators.

4.1 The E-polynomials of the Hilbert scheme of points

Let Z be a complex algebraic variety. P. Deligne established the existence of two filtrations on the j-th cohomology group of Z: the weight filtration W_*

$$0 = W_{-1} \subseteq W_0 \subseteq \cdots \subseteq W_{2j} = H^j(Z)$$

and the Hodge filtration F^*

$$H^j(Z) = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^m \supseteq F^{2j} = 0$$

such that, for each l, the filtration induced by F on the graded piece $\mathrm{gr}_l W := W_l/W_{l-1}$ endows $\mathrm{gr}_l W$ with a pure Hodge structure of weight l. One can define a mixed Hodge structure on the compactly supported cohomology $H_c^*(Z)$ as well (cf. [17]).
We define the compactly supported mixed Hodge polynomial of Z to be the generating function of the compactly supported mixed Hodge numbers $h_c^{p,q;j}(Z) := \dim_{\mathbb{C}} \mathrm{gr}_p F \left(\mathrm{gr}_{p+q} W \left(H_c^j(Z)\right)\right)$:

$$H_c(Z; u, v, s) := \sum_{p,q,j} h_c^{p,q;j}(Z) u^p v^q s^j.$$

For a connected smooth variety Z, the cohomological pairing

$$H^k(Z) \times H_c^{2\dim Z - k}(Z) \to H^{2\dim Z}(Z). \tag{4.1}$$

is a morphism of mixed Hodge structures. Then Poincaré duality implies the equality of mixed Hodge numbers:

$$h_c^{p,q;j}(Z) = h^{\dim Z-p, \dim Z-q; 2\dim Z-j}(Z) \tag{4.2}$$

where $h^{p,q;j} := \dim_{\mathbb{C}} \mathrm{gr}_p F\left(\mathrm{gr}_{p+q} W(H^j(Z))\right)$ are the mixed Hodge numbers of Z. Equivalently, if $H(Z;u,v,s) := \sum_{p,q,j} h^{p,q;j}(Z) u^p v^q s^j$ is the mixed Hodge polynomial of Z, then we have

$$H_c(Z;u,v,s) = u^d v^d s^{2d} H(Z;u^{-1},v^{-1},s^{-1}). \tag{4.3}$$

The specialization $H_c(Z;u,v,s)|_{u=1,v=1}$ is the compactly supported Poincaré polynomial of Z. Moreover, if Z is a connected smooth variety, then by (4.2), we can express the compactly supported Poincaré polynomial in terms of $P(Z;s) := \sum_j \dim_{\mathbb{C}} H^j(Z)s^j$, the Poincaré polynomial of Z as follows:

$$H_c(Z;1,1,s) = P\left(Z;s^{-1}\right) s^{2\dim_{\mathbb{C}} Z}. \tag{4.4}$$

The *E-polynomial* or the *Hodge-Deligne polynomial* of Z is defined to be the compactly supported mixed Hodge polynomial specialized at $s=-1$

$$E(Z;u,v) := \sum_{p,g,j} h_c^{p,q;j}(Z)(-1)^j u^p v^q.$$

Properties of the E-polynomial:

- **Additivity:** If a complex algebraic variety Z is represented as a disjoint union of locally Zariski closed subsets $Z = \cup_{i=1}^n Z_i$, then

$$E(Z;u,v) = E(Z_1;u,v) + \ldots E(Z_n;u,v).$$

- **Factorization on fibrations:** If $f: Z \to Z'$ is a Zariski locally trivial fibration of complex algebraic varieties with fiber F over a closed point, then

$$E(Z) = E(Z') \cdot E(F).$$

  In particular, if Z, Z' are complex algebraic varieties, then

$$E(Z \times Z') = E(Z) \cdot E(Z').$$

- The specialization at $u = v = 1$ gives the topological Euler characteristic

$$E(Z;1,1) = \sum_{p,g,j} (-1)^j h_c^{p,q;j}(Z) = \chi(Z).$$

  When Z is smooth and projective, the specialization

$$E(Z;-u,-v) = \sum_{p,q} \dim H^q(Z,\Omega^p) u^p v^q = \sum_{p,q} h^{p,q} u^p v^q$$

  agrees with the Hodge polynomial of Z, and

$$E(Z;-y,1) = \sum_{p,g,j} (-1)^j h^{p,q;j}(Z) y^p = \chi_y(Z)$$

  is the Hirzebruch χ_y-genus of Z.

We show some examples of how the motivic property facilitates the calculation of the E-polynomials.

Example 4.1.1. The E-polynomial of $\mathbb{C}$ is

$$E(\mathbb{C}; u, v) = \sum_{p,g,j} h_c^{p,q;j}(\mathbb{C})(-1)^j u^p v^q = uv.$$

- Applying the factorization property to $\mathbb{C}^n = \underbrace{\mathbb{C} \times \cdots \mathbb{C}}_{\text{n times}}$, we have

$$E(\mathbb{C}^n; u, v) = (uv)^n.$$

- Using the additivity property and $E(\mathbb{C}^n; u, v) = (uv)^n$, we have

$$E(\mathbb{C} \setminus \{(0,0)\}; u, v) = E(\mathbb{C}; u, v) - E(\{pt\}; u, v) = uv - 1.$$

Notation: Throughout this chapter, t is a variable of degree 2. If the mixed Hodge numbers $h^{p,q;j}(Z)$ of an algebraic variety Z vanish except when $p = q$, then the E-polynomial $E(Z; u, v)$ may be written as a polynomial in $t = uv$. This condition holds for the Hilbert schemes $\mathrm{Hilb}^{[n]}$, $Br^{[n]}$ and the refined Hilbert scheme $\mathrm{Hilb}^{[n,n+r]}$ we will introduce later: their compactly supported mixed Hodge numbers depend only on the mixed Hodge numbers of the base space $\mathbb{C}^2$, which satisfies the above condition. Thus we adopt the simplified notation $E(Z; t)$ in this article for the E-polynomial:

$$E(Z; t) := E(Z; \sqrt{t}, \sqrt{t}) = \sum_{p,q,j} h_c^{p,q;j}(Z) u^p v^q s^j \mid_{u,v=\sqrt{t} \text{ and } s=-1} \tag{4.5}$$

The calculations of the E-polynomials of Hilbert schemes of points on smooth surfaces goes back to the works of J. Cheah and L. Goettche (cf. [8, 10, 19]). A version of their result is the following:

Theorem 4.1.2 ([8, 10])**.** *The generating function of the E-polynomials of a smooth surface S has the form*

$$\sum_{n=0}^{\infty} E\left(S^{[n]}; u, v\right) s^n = \prod_{d=1}^{\infty} \prod_{p,q} \left(\frac{1}{1 - u^{p+d-1} v^{q+d-1} s^d}\right)^{e_{p,q}(S)}, \tag{4.6}$$

where $e_{p,g} := \sum_k (-1)^k h_c^{p,q,k}(Z)$.

Remark 4.1.3. In [8] and [10], the E-polynomial has a different name: the virtual Hodge polynomnial, and $e_{p,g}$ is called the (p, q)-th virtual Hodge number of S.

In particular, for the case of $\mathbb{C}^2$, the formula 4.6 gives

$$\sum_{n=0}^{\infty} E\left(\mathrm{Hilb}^{[n]}; t\right) q^n = \prod_{d=1}^{\infty} \frac{1}{1 - t^{(d+1)} q^d}. \tag{4.7}$$

More generally, According to a theorem of Bialynicki-Birula (cf. [2, 3]), a $\mathbb{C}^*$-action on a smooth projective variety with isolated fixed points induces a decomposition of M into affine spaces $M = \biguplus_{p \in M^T} \mathbb{A}^{\alpha(p)}$, where $\alpha(p)$ is the number of positive weights of T_pM. Thus, in this case, the compactly supported Poincaré polynomial of M is $\sum_{p\in M^T} t^{\alpha(p)}$ and it agrees with the E-polynomial of M. In other words, we have

$$E(M;t) = P\left(M;\sqrt{t}^{-1}\right) t^{\dim_{\mathbb{C}} M} = \sum_{p\in M^T} t^{\alpha(p)}. \tag{4.8}$$

Even though the Hilbert schemes of points on the plane are not projective, this property, nevertheless, holds for $\mathrm{Hilb}^{[n]}$ endowed with the action. Moreover, this $\mathbb{C}^*$ action on $\mathrm{Hilb}^{[n]}$ induces, in parallel, a cell decomposition of the Briançon variety $Br^{[n]} = \biguplus_{p\in\left(\mathrm{Hilb}^{[n]}\right)^T} \mathbb{A}^{2n-\alpha(p)}$, where $\alpha(p)$ is the number of positive weights of $T_p\mathrm{Hilb}^{[n]}$ (cf. [14, 25, 30]). The same arguments go through for $\mathrm{Hilb}^{[n,n+r]}$. This provides that the E-polynomial equals the compactly supported Poincaré polynomial for $\mathrm{Hilb}^{[n]}$, $Br^{[n]}$ and $\mathrm{Hilb}^{[n,n+r]}$.

Example 4.1.4. We recall that the generating function of the Poincaré polynomial of $\mathrm{Hilb}^{[n]}$ has the form

$$\sum_{n=0}^{\infty} P\left(\mathrm{Hilb}^{[n]};\sqrt{t}\right) q^n = \prod_{d=1}^{\infty} \frac{1}{1-t^{d-1}q^d}. \tag{4.9}$$

Applying our previous statement, we have the generating function of the E-polynomials

$$\sum_{n=0}^{\infty} E\left(\mathrm{Hilb}^{[n]};t\right) q^n = \sum_{n=0}^{\infty} t^{2n} P\left(\mathrm{Hilb}^{[n]};\sqrt{t}^{-1}\right) q^n = \prod_{d=1}^{\infty} \frac{1}{1-t^{2d-(d-1)}q^d} = \prod_{d=1}^{\infty} \frac{1}{1-t^{d+1}q^d}, \tag{4.10}$$

which agrees with equation (4.7).

The same arguments go through for $\mathrm{Hilb}^{[n,n+r]}$. Thus E-polynomial and the compactly supported Poincaré polynomial of $\mathrm{Hilb}^{[n,n+r]}$ are also equal.
To find the E-polynomial of $\mathrm{Hilb}^{[n,n+r]}$, we apply the character formula (2.15) at the tangent space of $\mathrm{Hilb}^{[n,n+r]}$ at a T-fixed point.

Proposition 4.1.5. *The generating function of the E-polynomial of* $\mathrm{Hilb}^{[n,n+r]}$ *has the form*

$$\sum_{n=\binom{r}{2}}^{\infty} E\left(\mathrm{Hilb}^{[n,n+r]};t\right) q^n = q^{\binom{r}{2}} \left(\prod_{d=1}^{\infty} \frac{1}{1-t^{(d+1)}q^d}\right)\left(\prod_{d=1}^{r} \frac{1}{1-t^d q^d}\right). \tag{4.11}$$

Proof. Since $\mathrm{Hilb}^{[n,n+r]}$ has the property that the E-polynomial equals the compactly supported Poincaré polynomial, it is equivalent to find the generating function of the compactly supported Poincaré polynomials of $\mathrm{Hilb}^{[n,n+r]}$. We apply an argument as in [[34] Corollary 5.3 and 5.4]. First, we compute the E-polynomial of $\mathrm{Hilb}^{[n,n+r]}$. Unlike their calculation for the Poincaré polynomial, here we count the sum of weights with opposite sign for the compactly supported Poincaré polynomial. This sum corresponds to the total number of boxes $\square$ of $\triangle$ satisfying either

(i) $a_{\triangle}(\square) = 0$ or

(ii) $a_{\triangle'}(\square) + 1 > 0$.

Denote by $|\triangle|$ the number of boxes of $\triangle$ and $w(\triangle)$ the number of columns of $\triangle$. For (i), we have $w(\triangle)-r$ of them, and every relevant box satisfies (ii). Therefore total number is $n-\binom{r+1}{2}+w(\triangle)$ and we have

$$E\left(\mathrm{Hilb}^{[n,n+r]};t\right)=\sum t^{n-\binom{r+1}{2}+w(\triangle)} \tag{4.12}$$

where the summation runs over all Young diagrams with $|\triangle|=n+r$ and r marked removable boxes. Next, to find the generating function of $E\left(\mathrm{Hilb}^{[n,n+r]};t\right)$, we use the fact that the set of Young diagrams with $n+r$ boxes and r marked removable boxes is in bijection with the set of pairs of Young diagrams $(\triangle_1,\triangle_2)$ such that $\triangle_2$ has at most r columns and $|\triangle_1|+|\triangle_2|=n-\binom{r}{2}$ (cf. [[34], Section 5.5]). Then the equation (4.12) becomes

$$E\left(\mathrm{Hilb}^{[n,n+r]};t\right)=\sum t^{|\triangle_1|+|\triangle_2|+w(\triangle_1)}, \tag{4.13}$$

where the summation runs over all pairs of Young diagrams $(\triangle_1,\triangle_2)$ such that $\triangle_2$ has at most r columns and $|\triangle_1|+|\triangle_2|=n-\binom{r}{2}$. It follows that the generating function is

$$\sum_{n=\binom{r}{2}}^{\infty} E\left(\mathrm{Hilb}^{[n,n+r]};t\right)q^n=q^{\binom{r}{2}}\left(\prod_{d=1}^{\infty}\frac{1}{1-t^{(d+1)}q^d}\right)\left(\prod_{d=1}^{r}\frac{1}{1-t^dq^d}\right). \tag{4.14}$$

■

4.2 The E-polynomials of the refined strata $\mathrm{Hilb}_m^{[n]}$ and $Br_m^{[n]}$

We present a refinement of these formulas by calculating the E-polynomials of the strata

$$Br_m^{[n]}:=\left\{I\in Br^{[n]}\,\middle|\,\mu(I)=m\right\}\text{ and }\mathrm{Hilb}_m^{[n]}:=\left\{I\in \mathrm{Hilb}^{[n]}\,\middle|\,\mu(I)=m\right\}$$

associated to the invariant $\mu(I)=\dim_{\mathbb{C}} I/\mathfrak{m}I$. The results are the following.

Theorem 4.2.1. *The generating function of the E-polynomials of the strata $Br_m^{[n]}$ of the Briançon variety is given by*

$$\sum_{n=0}^{\infty}E\left(Br_m^{[n]};t\right)q^n=\prod_{i=1}^{m-1}\frac{1}{1-t^{i+1}}\sum_{a=1}^{m}(-1)^{a+1}t^{\binom{a}{2}+m-1}\begin{bmatrix}m\\a\end{bmatrix}_t\left(\prod_{k=0}^{\infty}\frac{1-q^kt^{k-a}}{1-q^kt^{k-1}}\right). \tag{4.15}$$

where $\begin{bmatrix}n\\k\end{bmatrix}_q=\begin{cases}\prod_{i=0}^{k-1}\frac{1-q^{n-i}}{1-q^{i+1}} & \text{if } 1\le k\le n\\ 1 & \text{if } k=0\\ 0 & \text{if } k>n.\end{cases}$ *stands for the q-binomial coefficients.*

Remark 4.2.2. (I) Note that the summand indexed by $a=1$, in fact, equals the constant term of the rest of the summation. Thus the formula stays true for $q\ge 1$ if we let the summation from $a=2$ to m.

(II) We observe that the formulas (4.15) are rather elegant and preserve some of the structure related to modular forms and partition functions, and thus suggest a possible link with geometric representation theory.

Theorem 4.2.3. *The generating function of the E-polynomial of the strata* $\mathrm{Hilb}_m^{[n]}$ *is*

$$\sum_{n=0}^{\infty} E\left(\mathrm{Hilb}_m^{[n]};t\right)q^n = \prod_{i=1}^{m-1}\frac{1}{1-t^{i+1}}\sum_{a=1}^{m}(-1)^a t^{\binom{a}{2}+m}\begin{bmatrix} m \\ a \end{bmatrix}_t \left(\prod_{k=0}^{\infty}\frac{1-q^k t^{k-a}}{1-q^k t^{k+1}}\right). \tag{4.16}$$

Example 4.2.4. We list some examples of the generating function from Theorem 4.2.3:

- **Case of** $m=2$:

$$\begin{aligned}\sum_{n=0}^{\infty} E\left(Br_2^{[n]};t\right)q^n &= \frac{t}{1-t} + \frac{t^2}{t^2-1}\prod_{k=0}^{\infty}\frac{1-t^{k-2}q^k}{1-t^{k-1}q^k}\\ &= \frac{t}{1-t}\left(1-\prod_{k=1}^{\infty}\frac{1-t^{k-2}q^k}{1-t^{k-1}q^k}\right).\end{aligned}$$

  We note that $\prod_{k=1}^{\infty}\frac{1-t^{k-2}q^k}{1-t^{k-1}q^k}=1$ when $t=1$. This implies that this infinite product is divisible by $\frac{1}{1-t}$, which is not clear by just looking at the original formula.

- **Case of** $m=3$:

$$\begin{aligned}\sum_{n=0}^{\infty} E\left(Br_3^{[n]};t\right)q^n &= \frac{t^2}{(1-t)(1-t^2)} - \frac{t^3}{(1-t)(1-t^2)}\prod_{k=0}^{\infty}\frac{1-t^{k-2}q^k}{1-t^{k-1}q^k}\\ &\quad + \frac{t^5}{(1-t^2)(1-t^3)}\prod_{k=0}^{\infty}\frac{1-t^{k-3}q^k}{1-t^{k-1}q^k}\\ &= \frac{t^2}{(1-t^2)(1-t)} - \frac{t^2}{(1-t)^2}\prod_{k=1}^{\infty}\frac{1-t^{k-2}q^k}{1-t^{k-1}q^k} + \frac{t^3}{(1-t^2)(1-t)}\prod_{k=1}^{\infty}\frac{1-t^{k-3}q^k}{1-t^{k-1}q^k}.\end{aligned}$$

- **Case of** $m=4$:

$$\begin{aligned}\sum_{n=0}^{\infty} E\left(Br_4^{[n]};t\right)q^n =& \frac{t^3}{(1-t)(1-t^2)(1-t^3)} - \frac{t^4}{(1-t)(1-t^2)^2}\left(\prod_{d=0}^{\infty}\frac{1-t^{d-2}q^d}{1-t^{d-1}q^d}\right)\\ &+ \frac{t^6}{(1-t)(1-t^2)(1-t^3)}\left(\prod_{d=0}^{\infty}\frac{1-t^{d-3}q^d}{1-t^{d-1}q^d}\right)\\ &- \frac{t^9}{(1-t^2)(1-t^3)(1-t^4)}\left(\prod_{d=0}^{\infty}\frac{1-t^{d-4}q^d}{1-t^{d-1}q^d}\right).\end{aligned}$$

We observe that the formulas are rather elegant and preserve some structures related to modular forms and partition functions, and thus suggest a possible link with geometric representation theory. The key idea of the proof suggested to us by A. Oblomkov is that the fibers of the projection $\pi:\mathrm{Hilb}^{[n,n+r]}\to\mathrm{Hilb}^{[n]}$ over $\mathrm{Hilb}_m^{[n]}=\{I\,|\,\mu(I)=m\}$ is a Grassmannian:

$$\begin{array}{ccc} \mathrm{Gr}_r\left(\mathbb{C}^m\right) \to \pi^{-1}\left(\mathrm{Hilb}_m^{[n]}\right) \subseteq \mathrm{Hilb}^{[n,n+r]} \\ \big\downarrow \pi \\ \mathrm{Hilb}_m^{[n]} \subseteq \mathrm{Hilb}^{[n]} \end{array} \tag{4.17}$$

This idea appears in [36] in the context of a conjecture relating the HOMFLY polynomial of the link of a plane curve singularity C to the E-polynomials of the Hilbert schemes of points supported on C. We will proceed the proof using this key observation together with the additivity and multiplicativity for fibrations of the E-polynomial.

4.2.1 A formula for generating function of the E-polynomials of $Br_m^{[n]}$

Let

$$\mathrm{Hilb}_m^{[n]} = \left\{ I \in \mathrm{Hilb}^{[n]} \,\middle|\, \mu(I) = m \right\} \text{ and } \mathrm{Hilb}_m^{[n]}(s) = \left\{ I \in \mathrm{Hilb}^{[n]} \,\middle|\, \mu(I) = m, \sigma(I) = s \right\}$$

the refined Hilbert scheme strata associated to the function μ and σ. In particular, when $s = n$, we have

$$\mathrm{Hilb}_m^{[n]}(n) = Br_m^{[n]} = \left\{ I \in Br^{[n]} \,\middle|\, \mu(I) = m \right\}. \tag{4.18}$$

We first consider the refined Hilbert scheme strata

$$\mathrm{Hilb}_m^{[n]} := \left\{ I \in \mathrm{Hilb}^{[n]} \,\middle|\, \mu(I) = m \right\},$$

and the induced decomposition of $\mathrm{Hilb}^{[n]}$ with respect to μ

$$\mathrm{Hilb}^{[n]} = \biguplus_m \mathrm{Hilb}_m^{[n]}.$$

If $J \in \mathrm{Hilb}^{[n,n+r]}$ satisfies the condition $I \supset J \supseteq \mathfrak{m}I$, then J is fully determined by its image in $I/\mathfrak{m}I$. Thus for a fixed $I \in \mathrm{Hilb}^{[n]}$ with $\mu(I) = m$, the set of $J \in H^{[n+r]}$ such that $(I, J) \in \mathrm{Hilb}^{[n,n+r]}$ is parametrized by the Grassmannian of r-dimensional subspaces of $I/\mathfrak{m}I \simeq \mathbb{C}^m$. Over each stratum $\mathrm{Hilb}_m^{[n]}$ of $\mathrm{Hilb}^{[n]}$, the projection map $\mathrm{Hilb}^{[n,n+r]} \to \mathrm{Hilb}^{[n]}$ has fibers $Gr(r, \mathbb{C}^m)$ at each ideal $I \in \mathrm{Hilb}_m^{[n]}$.
Since Grassmannians $Gr(r, \mathbb{C}^m)$ are projective and smooth, their E-polynomials and Poincaré polynomials are equal:

$$E\left(Gr(r, \mathbb{C}^m); t\right) = P\left(Gr(r, \mathbb{C}^m); \sqrt{t}\right) = \prod_{i=1}^{r} \frac{1 - t^{(m-i+1)}}{1 - t^i} = \begin{bmatrix} m \\ r \end{bmatrix}_t.$$

This Grassmannian bundle structure together with the motivic property of the E-polynomial imply the following equality

$$E\left(\mathrm{Hilb}^{[n,n+r]}; t\right) = \sum_{m=1}^{\mu_n^{\max}} E\left(\mathrm{Hilb}_m^{[n]}; t\right) E\left(Gr_r(\mathbb{C}^m); t\right) = \sum_{m}^{\mu_n^{\max}} E\left(\mathrm{Hilb}_m^{[n]}; t\right) \begin{bmatrix} m \\ r \end{bmatrix}_t, \tag{4.19}$$

where $\mu_n^{\max} := \max\left\{ \mu(I) \,\middle|\, I \in \mathrm{Hilb}^{[n]} \right\} = \max\left\{ \mu(I) \,\middle|\, I \in \left(\mathrm{Hilb}^{[n]}\right)^T \right\}$.

Furthermore, presenting an ideal $I \in \mathrm{Hilb}^{[n]}$ as an intersection of ideals $I_0 \cap I'$, where I_0 is the part supported at $(0,0)$ and I' s the part supported at $Y_0 := \mathbb{C}^2 \setminus \{(0,0)\}$, we obtain the decomposition

$$\mathrm{Hilb}_m^{[n]} \simeq \biguplus_{s=0}^{n} \left(Br_m^{[s]} \times Y_0^{[n-s]} \right). \tag{4.20}$$

Using motivic properties of the E-polynomial, we can conclude

$$E\left(\mathrm{Hilb}_m^{[n]};t\right)=\sum_{s=0}^{n}E\left(Y_0^{[n-s]};t\right)\cdot E\left(Br_m^{[s]};t\right). \tag{4.21}$$

Our goal is to find the E-polynomials of $H_m^{[k]}$ and $Br_m^{[k]}$ using equations in (4.19) and (4.21). We will consider all $H_m^{[k]}$ and $Br_m^{[k]}$, $m,k\in\mathbb{N}$, at the same time and we define following infinite matrices: $\mathcal{X}:=\left(E\left(H_i^{[j]};t\right)\right)_{i\geq1,j\geq0}$, $\mathcal{B}:=\left(E\left(Br_i^{[j]};t\right)\right)_{i\geq1,j\geq0}$, $\mathcal{R}:=\left(E\left(H^{[j,j+i]};t\right)\right)_{i\geq1,j\geq0}$,

$$\mathcal{G}:=\left(E\left(Gr_i(\mathbb{C}^j);t\right)\right)_{i,j\geq1}=\left(\begin{bmatrix}j\\i\end{bmatrix}_t\right)_{i,j\geq1}\text{ and }\mathcal{A}:=\left(E\left(Y_0^{[j-i]};t\right)\right)_{i,j\geq1}.$$

Proposition 4.2.5. *The matrices $\mathcal{X},\mathcal{B},\mathcal{R},\mathcal{G}$ and $\mathcal{A}$ satisfy*

$$\mathcal{G}\mathcal{X}=\mathcal{R}\text{ and }\mathcal{B}\mathcal{A}=\mathcal{X}.$$

Proof. A direct calculation of matrix products gives

$$\begin{aligned}\mathcal{G}\mathcal{X}&=\left(\textstyle\sum_{k=1}^{\infty}\mathcal{G}_{ik}\mathcal{X}_{kj}\right)_{i\geq1,j\geq0}\\&=\left(\textstyle\sum_{k=1}^{\infty}E\left(Gr_i(\mathbb{C}^k);t\right)E\left(H_k^{[j]};t\right)\right)_{i\geq1,j\geq0}\\(\text{by equation (4.19) })&=\left(E\left(H^{[j,j+i]};t\right)\right)_{i\geq1,j\geq0}=\mathcal{R}.\end{aligned}$$

Similarly, we have the product $\mathcal{B}\mathcal{A}$

$$\begin{aligned}\mathcal{B}\mathcal{A}&=\left(\textstyle\sum_{k=0}^{\infty}\mathcal{B}_{ik}\mathcal{A}_{kj}\right)_{i\geq1,j\geq1}\\&=\left(\textstyle\sum_{k=0}^{\infty}E\left(Br_i^{[k]};t\right)E\left(Y_0^{[j-k]};t\right)\right)_{i\geq1,j\geq1}\\(\text{by equation (4.21) })&=\left(E\left(H_i^{[j]};t\right)\right)_{i\geq1,j\geq0}=\mathcal{X}.\end{aligned}$$

■

By definition, $\mathcal{G}$ and $\mathcal{A}$ are upper triangular matrices with 1s on the diagonal, so they are invertible with upper triangular inverses. Then the matrices $\mathcal{X}$ and $\mathcal{B}$ may be expressed as products of matrices

$$\begin{cases}\mathcal{X}&=\mathcal{G}^{-1}\mathcal{R}\\\mathcal{B}&=\mathcal{X}\mathcal{A}^{-1}=\mathcal{G}^{-1}\mathcal{R}\mathcal{A}^{-1}\end{cases} \tag{4.22}$$

Thus to compute the E-polynomials $E\left(\mathrm{Hilb}_m^{[n]};t\right)=\mathcal{X}_{m,n}$ and $E\left(Br_m^{[n]};t\right)=\mathcal{B}_{m,n}$, it is sufficient to find the inverse matrices of $\mathcal{G}$ and $\mathcal{A}$.

Proposition 4.2.6. *The matrix $\mathcal{G}=\left(\begin{bmatrix}j\\i\end{bmatrix}_t\right)_{i,j\geq1}$ has the inverse*

$$\mathcal{G}^{-1}=\left((-1)^{j-i}t^{\binom{j-i}{2}}\begin{bmatrix}j\\i\end{bmatrix}_t\right)_{i,j\geq1}.$$

Proof. Let $\mathcal{G}^{-1}$ denote the inverse of $\mathcal{G}$. Denoted by δ_{ij} the Kronecker delta function, the ij-th entry of the matrix product $\mathcal{G}^{-1}\mathcal{G}$ that is by definition

$$(\mathcal{G}^{-1}\mathcal{G})_{ij} = \sum_{k=1}^{\infty} \mathcal{G}_{ik}^{-1}\mathcal{G}_{kj} = \sum_{k=1}^{j} \mathcal{G}_{ik}^{-1} \begin{bmatrix} j \\ k \end{bmatrix}_t = \delta_{ij}. \tag{4.23}$$

We apply the following orthogonality relation for the q-binomial coefficients (cf. [11] p.118-p.119): For every $0 \le i \le j$ one has

$$\delta_{ij} = \sum_{k=i}^{j} (-1)^{k-i} q^{\binom{k-i}{2}} \begin{bmatrix} j \\ k \end{bmatrix}_q \begin{bmatrix} k \\ i \end{bmatrix}_q \overset{\left(\left[\begin{smallmatrix} k \\ i \end{smallmatrix}\right]_q = 0 \text{ if } k < i\right)}{=\!=\!=\!=} \sum_{k=1}^{j} (-1)^{k-i} q^{\binom{k-i}{2}} \begin{bmatrix} j \\ k \end{bmatrix}_q \begin{bmatrix} k \\ i \end{bmatrix}_q,$$

and we obtain

$$\sum_{k=1}^{j} \mathcal{G}_{ik}^{-1} \begin{bmatrix} j \\ k \end{bmatrix}_t = \delta_{ij} = \sum_{k=1}^{j} (-1)^{k-i} t^{\binom{k-i}{2}} \begin{bmatrix} j \\ k \end{bmatrix}_t \begin{bmatrix} k \\ i \end{bmatrix}_t. \tag{4.24}$$

By comparing the coefficients of $\left[\begin{smallmatrix} j \\ k \end{smallmatrix}\right]_t$ in (4.24), we have

$$\mathcal{G}_{ik}^{-1} = (-1)^{k-i} t^{\binom{k-i}{2}} \begin{bmatrix} k \\ i \end{bmatrix}_t.$$

■

Our next step is to calculate the inverse of $\mathcal{A}$. To this end, we first need the generating function of E-polynomials of the Hilbert scheme of points on the punctured plane Y_0.

Proposition 4.2.7 ([8, 10, 19]). *The E-polynomial $E\left(Y_0^{[n]}; t\right)$ of the Hilbert scheme of points on the punctured complex plane Y_0 has the generating function*

$$\sum_{n=0}^{\infty} E\left(Y_0^{[n]}; t\right) q^n = \prod_{d=1}^{\infty} \frac{1 - t^{d-1}q^d}{1 - t^{d+1}q^d}.$$

Here, we give a direct proof using the knowledge of $E\left(\mathrm{Hilb}^{[n,n+r]}; t\right)$ and $E(Br^{[n]}; t)$.

Proof. First, we observe that from the decomposition in (4.20): $\mathrm{Hilb}^{[n]} \simeq \biguplus_{s=0}^{n} Y_0^{[s]} \times Br^{[n-s]}$, we have a bijective morphism

$$\biguplus_{s=0}^{n} Y_0^{[s]} \times Br^{[n-s]} \to \mathrm{Hilb}^{[n]}$$

by sending a pair of subschemes in $Y_0^{[s]} \times Br^{[n-s]}$ to the union of the two.
We recall that the formula (4.7) for the generating function of the E-polynomials of $\mathrm{Hilb}^{[n]}$ has the form

$$\sum_{n=0}^{\infty} E\left(\mathrm{Hilb}^{[n]}; t\right) q^n = \prod_{d=1}^{\infty} \frac{1}{1 - t^{d+1}q^d}. \tag{4.25}$$

Applying the motivicity of the E-polynomial to this decomposition 4.20, we obtain the equality

$$\sum_{n=0}^{\infty} E\left(\mathrm{Hilb}^{[n]};t\right) q^n = \sum_{n=0}^{\infty}\left(\sum_{s=0}^{n} E\left(Y_0^{[s]};t\right) E\left(Br^{[n-s]};t\right)\right) q^n. \tag{4.26}$$

After the change of variable $k = n - s$, the right-hand side of the equation ((4.26)) has the form of a doubly infinite summation

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\sum_{s=0}^{n} E\left(Y_0^{[s]};t\right) E\left(Br^{[n-s]};t\right)\right) q^n &\overset{(k=n-s)}{=\!=\!=} \sum_{k=0}^{\infty}\left(\sum_{s=0}^{\infty} E\left(Y_0^{[s]};t\right) E\left(Br^{[k]};t\right)\right) q^{s+k} \\ &= \sum_{k=0}^{\infty}\sum_{s=0}^{\infty} E\left(Y_0^{[s]};t\right) q^s E\left(Br^{[k]};t\right) q^k \\ &\overset{(s,\ k \text{ are independent})}{=\!=\!=\!=\!=\!=} \left(\sum_{k=0}^{\infty} E\left(Br^{[k]};t\right) q^k\right)\left(\sum_{s=0}^{\infty} E\left(Y_0^{[s]};t\right) q^s\right).\end{aligned} \tag{4.27}$$

As we pointed out in the beginning of the section, the Briançon variety $Br^{[k]}$ admits a cell decomposition as well and thus $E\left(Br^{[k]};t\right) = P\left(Br^{[k]};\sqrt{t}^{-1}\right) t^{\dim_{\mathbb{C}} Br^{[k]}}$. We replace pieces $E\left(Br^{[k]};t\right)$ by $P\left(Br^{[k]};\sqrt{t}^{-1}\right) t^{\dim_{\mathbb{C}} Br^{[k]}}$ in equation (4.27):

$$\left(\sum_{k=0}^{\infty} P\left(Br^{[k]};\sqrt{t}^{-1}\right) t^{\dim_{\mathbb{C}} Br^{[k]}} q^k\right)\left(\sum_{s=0}^{\infty} E\left(Y_0^{[s]};t\right) q^s\right),$$

which is equal to

$$\left(\sum_{k=0}^{\infty} P\left(H^{[k]};\sqrt{t}\right) q^k\right)\left(\sum_{s=0}^{\infty} E\left(Y_0^{[s]};t\right) q^s\right), \tag{4.28}$$

since $\mathrm{Hilb}^{[n]}$ and $Br^{[n]}$ are homotopic. Recall that $P\left(\mathrm{Hilb}^{[n]};\sqrt{t}\right)$ has the generating function

$$\sum_{n=0}^{\infty} P\left(\mathrm{Hilb}^{[n]};\sqrt{t}\right) q^n = \prod_{d=1}^{\infty}\frac{1}{1-t^{d-1}q^d}.$$

Thus we have

$$\sum_{n=0}^{\infty} E\left(\mathrm{Hilb}^{[n]};t\right) q^n = \prod_{d=1}^{\infty}\frac{1}{1-t^{d+1}q^d} = \left(\prod_{d=1}^{\infty}\frac{1}{1-t^{d-1}q^d}\right)\left(\sum_{s=0}^{\infty} E\left(Y_0^{[s]};t\right) q^s\right).$$

Therefore

$$\sum_{n=0}^{\infty} E\left(Y_0^{[n]};t\right) q^n = \prod_{d=1}^{\infty}\frac{1-t^{d-1}q^d}{1-t^{d+1}q^d}.$$

■

Example 4.2.8. We list some E-polynomials $E\left(Y_0^{[n]};t\right)$ for $0 \le n \le 8$:

n	$E\left(Y_0^{[n]};t\right)$
0	1
1	t^2-1
2	$t^4+t^3-t^2-t$
3	$t^6+t^5-2t^3-t^2+t$
4	$t^8+t^7+t^6-t^5-3t^4+t^2$
5	$t^{10}+t^9+t^8-3t^6-3t^5+t^4+2t^3$
6	$t^{12}+t^{11}+t^{10}+t^9-t^8-4t^7-3t^6+3t^5+2t^4-t^3$
7	$t^{14}+t^{13}+t^{12}+t^{11}-3t^9-6t^8-t^7+5t^6+2t^5-t^4$
8	$t^{16}+t^{15}+t^{14}+t^{13}+t^{12}-t^{11}-5t^{10}-6t^9+t^8+7t^7+t^6-2t^5$

Before stating the result about the matrix $\mathcal{A}^{-1}$, we define the *dual E-polynomial* $\breve{E}$ of a complex variety Z

$$\breve{E}(Z;t) := H(Z;\sqrt{t},\sqrt{t},-1). \tag{4.29}$$

When Z is a connected and smooth, we can compute $\breve{E}(Z;t)$ from $E(Z;t)$ by Poincaré duality (4.1):

$$\breve{E}(Z;t) = E\left(Z;t^{-1}\right)t^{\dim_{\mathbb{C}} Z}.$$

Corollary 4.2.9. *The dual E-polynomial of $Y_0^{[n]}$ has generating function*

$$\sum_{n=0}^{\infty} \breve{E}\left(Y_0^{[n]};t\right)q^n = \prod_{d=1}^{\infty} \frac{1-t^{d+1}q^d}{1-t^{d-1}q^d}.$$

Proof. Since the Hilbert scheme $Y_0^{[n]}$ is a complex $4n$-dimensional smooth variety, the Poincaré duality is compatible with the mixed Hodge structure on cohomology, and we have

$$\breve{E}\left(Y_0^{[n]};t\right) = t^{\dim_{\mathbb{C}} Y_0^{[n]}} E\left(Y_0^{[n]};t^{-1}\right) = t^{2n} E\left(Y_0^{[n]};t^{-1}\right).$$

Then from the definition of the generating function of $\breve{E}$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \breve{E}\left(Y_0^{[n]};t\right)q^n &= \sum_{n=0}^{\infty} E\left(Y_0^{[n]};t^{-1}\right)t^{2n}q^n \overset{(\tilde{q}=t^2q)}{=} \prod_{d=1}^{\infty} \frac{1-t^{1-d}\tilde{q}^d}{1-t^{-d-1}\tilde{q}^d} \\ &= \prod_{d=1}^{\infty} \frac{1-t^{1-d}(t^2q)^d}{1-t^{-d-1}(t^2q)^d} = \prod_{d=1}^{\infty} \frac{1-t^{d+1}q^d}{1-t^{d-1}q^d}.\end{aligned}$$

■

We are now ready to calculate $\mathcal{A}^{-1}$.

Proposition 4.2.10. *The matrix $\mathcal{A}$ has the inverse*

$$\mathcal{A}^{-1} = \left(\mathcal{A}^{-1}_{ij}\right) = \left(\breve{E}\left(Y_0^{[j-i]}\right)\right)$$

$$\mathcal{A}^{-1} = \begin{pmatrix} 1 & \breve{E}\left(Y_0^{[1]}\right) & \breve{E}\left(Y_0^{[2]}\right) & \breve{E}\left(Y_0^{[3]}\right) & & \breve{E}\left(Y_0^{[k]}\right) & \\ 0 & 1 & \breve{E}\left(Y_0^{[1]}\right) & \breve{E}\left(Y_0^{[2]}\right) & \dots & \breve{E}\left(Y_0^{[k-1]}\right) & \dots \\ \vdots & 0 & 1 & \breve{E}\left(Y_0^{[1]}\right) & & \vdots & \\ & & 0 & 1 & & \breve{E}\left(Y_0^{[k-j]}\right) & \\ & & \vdots & 0 & & \vdots & \\ & & & & & 1 & \end{pmatrix}, \tag{4.30}$$

where $\breve{E}\left(Y_0^{[j-i]}\right) = t^{2(j-i)}E\left(Y_0^{[j-i]};t^{-1}\right)$ *is the dual E-polynomial of* $Y_0^{[j-i]}$*.*

Proof. Let $\mathcal{C}$ be the infinite matrix in (4.30). The ij-th entry of the matrix $\mathcal{AC}$ is given by the sum

$$\begin{aligned}\left(\mathcal{A}\mathcal{A}^{-1}\right)_{ij} &= \sum_{k=i}^{j} E\left(Y_0^{[k-i]};t\right)\breve{E}\left(Y_0^{[j-k]};t\right) \\ &= \sum_{l=0}^{j-i} E\left(Y_0^{[l]};t\right)\breve{E}\left(Y_0^{[j-i-l]};t\right).\end{aligned}$$

Note that Proposition 4.2.7 and Corollary 4.2.9 yield the product of the generating functions

$$\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} E\left(Y_0^{[i]};t\right)\breve{E}\left(Y_0^{[n-i]};t\right)\right)q^n = \left(\sum_{n=0}^{\infty} E\left(Y_0^{[n]};t\right)q^n\right)\left(\sum_{n=0}^{\infty}\breve{E}\left(Y_0^{[n]};t\right)q^n\right) = 1.$$

Therefore, $(\mathcal{AC})_{ij} = \delta_{ij}$ and it implies $\mathcal{C} = \mathcal{A}^{-1}$. ■

We are now ready to prove Theorem 4.2.1.

Proof of Theorem 4.2.1. The E-polynomial $E\left(Br_m^{[n]};t\right)$ is equal to the (m,n)-th entry of the matrix product $\mathcal{B} = \mathcal{X}\mathcal{A}^{-1} = \mathcal{G}^{-1}\mathcal{R}\mathcal{A}^{-1}$ which is given by the sum

$$\sum_k\sum_j \mathcal{G}^{-1}_{mk}\mathcal{R}_{kj}\mathcal{A}^{-1}_{jn} = \sum_{k=m}^{\mu_n^{\max}}\sum_{j=\binom{k}{2}}^{n}(-1)^{k-m}t^{\binom{k-m}{2}}\begin{bmatrix}k\\m\end{bmatrix}_t E\left(H^{[j,j+k]};t\right)\breve{E}\left(Y_0^{[n-j]};t\right).$$

Substituting this into the generating function, we obtain

$$\sum_{n=0}^{\infty}E\left(Br_m^{[n]};t\right)q^n = \sum_{n=0}^{\infty}\left(\sum_{k=m}^{\mu_n^{\max}}\sum_{j=\binom{k}{2}}^{n}(-1)^{k-m}t^{\binom{k-m}{2}}\begin{bmatrix}k\\m\end{bmatrix}_t E\left(\mathrm{Hilb}^{[j,j+k]};t\right)\breve{E}\left(Y_0^{[n-j]};t\right)\right)q^n.$$

Here we can let the indexes k and j run from 0 to ∞, and the infinite sum does not change, since the space $\mathrm{Hilb}^{[j,j+k]} = \emptyset$ if $j \le \binom{k}{2}$ by Proposition 2.1.5 and $\begin{bmatrix} k \\ m \end{bmatrix}_t = 0$ if $k \le m$. Recall that the generating functions of $E\left(\mathrm{Hilb}^{[n,n+r]};t\right)$ and $\breve{E}\left(Y_0^{[n]};t\right)$ are given by $q^{\binom{r}{2}}\left(\prod_{d=1}^{\infty}\frac{1}{1-t^{(d+1)}q^d}\right)\left(\prod_{d=1}^{r}\frac{1}{1-t^dq^d}\right)$ and $\prod_{d=1}^{\infty}\frac{1-t^{d+1}q^d}{1-t^{d-1}q^d}$ respectively. Then the generating function

$$\sum_{n=0}^{\infty} E\left(Br_m^{[n]};t\right)q^n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{k-m}t^{\binom{k-m}{2}}\begin{bmatrix} k \\ m \end{bmatrix}_t E\left(\mathrm{Hilb}^{[j,j+k]};t\right)\breve{E}\left(Y_0^{[n-j]};t\right)\right)q^n$$

is equal to the product

$$\begin{aligned}\left(\prod_{d=1}^{\infty}\frac{1}{1-t^{d+1}q^d}\right)&\left(\prod_{d=1}^{\infty}\frac{1-t^{d+1}q^d}{1-t^{d-1}q^d}\right)\left(\sum_{k=0}^{\infty}q^{\binom{k}{2}}(-1)^{k-m}t^{\binom{k-m}{2}}\begin{bmatrix} k \\ m \end{bmatrix}_t\prod_{d=1}^{k}\frac{1}{1-t^dq^d}\right)\\ &= \left(\prod_{d=1}^{\infty}\frac{1}{1-t^{d-1}q^d}\right)\sum_{k=0}^{\infty}(tq)^{\binom{k}{2}}(-1)^{k-m}t^{-km+\binom{m+1}{2}}\begin{bmatrix} k \\ m \end{bmatrix}_t\frac{1}{(tq)_k}, \qquad (4.31)\end{aligned}$$

where $(tq)_k := \prod_{d=1}^{k}\left(1-t^dq^d\right)$. To continue the proof, we need the following lemma.

Lemma 4.2.11. *We have*

$$\sum_{i=0}^{m}(-1)^{m+i}t^{km-\binom{m}{2}+\binom{i}{2}-ik}\begin{bmatrix} m \\ i \end{bmatrix}_t = \prod_{i=0}^{m-1}\left(1-t^{k-i}\right). \qquad (4.32)$$

Proof of Lemma 4.2.11. Recall the Gauss's binomial formula (cf. [27, p.29]):

$$\prod_{k=0}^{n-1}(1+aq^k) = \sum_{k=0}^{n}q^{\binom{k}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_q a^k.$$

Applying the Gauss's binomial formula with $a=-t^{-k}, q=t$, we obtain

$$\begin{aligned}\prod_{i=0}^{m-1}\left(1-t^{k-i}\right) &= (-1)^m t^{mk-\binom{m}{2}}\prod_{i=0}^{m-1}\left(1+(-t^{-k})t^i\right)\\ &= (-1)^m t^{mk-\binom{m}{2}}\sum_{i=0}^{m}t^{\binom{i}{2}}\begin{bmatrix} m \\ i \end{bmatrix}_t(-t^{-k})^i\\ &= \sum_{i=0}^{m}(-1)^{m+i}t^{mk-\binom{m}{2}+\binom{i}{2}-ik}\begin{bmatrix} m \\ i \end{bmatrix}_t.\end{aligned}$$

∎

We continue the calculation of the generating function of $E\left(Br_m^{[n]};t\right)$. We write $\begin{bmatrix} k \\ m \end{bmatrix}_t = \prod_{i=0}^{m-1}\frac{1-t^{k-i}}{1-t^{i+1}}$ and apply Lemma 4.2.11 to the product $\prod_{i=0}^{m-1}\left(1-t^{k-i}\right)$. We substitute it into the generating function

$$\begin{aligned}
\sum_{n=0}^{\infty} E\left(Br_m^{[n]};t\right) q^n &= \left(\prod_{d=1}^{\infty} \frac{1}{1-t^{d-1}q^d}\right) \sum_{k=0}^{\infty} (tq)^{\binom{k}{2}} (-1)^{k-m} t^{-km+\binom{m+1}{2}} \begin{bmatrix} k \\ m \end{bmatrix}_t \frac{1}{(tq)_k} \\
&= \left(\prod_{d=1}^{\infty} \frac{1}{1-t^{d-1}q^d}\right) (-1)^m \sum_{k=0}^{\infty} \frac{(-1)^k (tq)^{\binom{k}{2}}}{(tq)_k} t^{-km+\binom{m+1}{2}} \\
&\times \left(\frac{\sum_{a=0}^{m} (-1)^{m+a} t^{km-\binom{m}{2}+\binom{a}{2}-ak} \begin{bmatrix} m \\ a \end{bmatrix}_t}{\prod_{i=0}^{m-1} \left(1-t^{i+1}\right)} \right) \\
&= \left(\prod_{d=1}^{\infty} \frac{1}{1-t^{d-1}q^d}\right) \prod_{i=0}^{m-1} \frac{1}{1-t^{i+1}} \sum_{a=0}^{m} (-1)^a t^{m+\binom{a}{2}} \begin{bmatrix} m \\ a \end{bmatrix}_t \sum_{k=0}^{\infty} \frac{(-1)^k (tq)^{\binom{k}{2}} t^{-ak}}{(tq)_k}.
\end{aligned}$$

Recall the Euler identity (cf. [11]):

$$(z)_{\infty} = \sum_{n=0}^{\infty} \frac{(-1)^n z^n q^{\binom{n}{2}}}{(q)_n} = \prod_{n=0}^{\infty} (1-zq^n). \tag{4.33}$$

We apply the identity (4.33) to the infinite sum $\sum_{k=0}^{\infty} \frac{(-1)^k (tq)^{\binom{k}{2}} t^{-ak}}{(tq)_k}$ with change of variables $\begin{cases} q \mapsto tq, \\ z \mapsto t^{-a} \end{cases}$.

Finally, we arrive at the result:

$$\begin{aligned}
\sum_{n=0}^{\infty} E\left(Br_m^{[n]};t\right) q^n &= \left(\prod_{d=1}^{\infty} \frac{1}{1-t^{d-1}q^d}\right) \prod_{i=0}^{m-1} \frac{1}{1-t^{i+1}} \sum_{a=0}^{m} (-1)^a t^{m+\binom{a}{2}} \begin{bmatrix} m \\ a \end{bmatrix}_t \prod_{k=0}^{\infty} (1-t^{-a}(tq)^k) \\
&= \prod_{i=0}^{m-1} \frac{1}{1-t^{i+1}} \sum_{a=0}^{m} (-1)^a t^{m+\binom{a}{2}} \begin{bmatrix} m \\ a \end{bmatrix}_t \left(1-t^{0-1}\right) \left(\prod_{d=0}^{\infty} \frac{1-t^{d-a}q^d}{1-t^{d-1}q^d}\right) \\
&= \prod_{i=1}^{m-1} \frac{1}{1-t^{i+1}} \sum_{a=0}^{m} (-1)^{a+1} t^{m-1+\binom{a}{2}} \begin{bmatrix} m \\ a \end{bmatrix}_t \left(\prod_{d=0}^{\infty} \frac{1-t^{d-a}q^d}{1-t^{d-1}q^d}\right).
\end{aligned}$$

Note that the infinite product $\left(\prod_{d=0}^{\infty} \frac{1-t^{d-a}q^d}{1-t^{d-1}q^d}\right) = 0$ when $a=0$, and we have

$$\sum_{n=0}^{\infty} E\left(Br_m^{[n]};t\right) q^n = \prod_{i=1}^{m-1} \frac{1}{1-t^{i+1}} \sum_{a=1}^{m} (-1)^{a+1} t^{m-1+\binom{a}{2}} \begin{bmatrix} m \\ a \end{bmatrix}_t \left(\prod_{d=0}^{\infty} \frac{1-t^{d-a}q^d}{1-t^{d-1}q^d}\right).$$

■

4.2.2 A formula for the generating function of the E-polynomials of $\mathrm{Hilb}_m^{[n]}$

Using Theorem 4.2.1, we calculate the E-polynomial of the refined strata $\mathrm{Hilb}_m^{[n]}$ as well..

Proposition 4.2.12. *The E-polynomial of the refined stratum* $\mathrm{Hilb}_m^{[n]}$ *has the generating function*

$$\sum_{n=0}^{\infty} E\left(\mathrm{Hilb}_m^{[n]};t\right) q^n = \prod_{i=1}^{m-1} \frac{1}{1-t^{i+1}} \sum_{a=1}^{m} (-1)^a t^{\binom{a}{2}+m} \begin{bmatrix} m \\ a \end{bmatrix}_t \left(\prod_{k=0}^{\infty} \frac{1-q^k t^{k-a}}{1-q^k t^{k+1}} \right). \tag{4.34}$$

Remark 4.2.13. One can also obtain this formula directly from the entries of matrix products $\mathcal{X} = \mathcal{G}^{-1}\Gamma$ in equations (4.22) with a similar argument as in the proof of Theorem 4.2.1.

Proof. From relation (4.21), the E-polynomial of $\mathrm{Hilb}_m^{[n]}$ is a sum

$$E\left(\mathrm{Hilb}_m^{[n]};t\right) = \sum_{s=0}^{n} E\left((\mathbb{C}^2\backslash\{(0,0)\})^{[n-s]};t\right) E\left(Br_m^{[s]};t\right).$$

Then the generating function of the E-polynomial $E\left(\mathrm{Hilb}_m^{[n]};t\right)$

$$\begin{aligned}
\sum_{n=0}^{\infty} E\left(\mathrm{Hilb}_m^{[n]};t\right) q^n &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} E\left((\mathbb{C}^2\backslash\{(0,0)\})^{[n-j]};t\right) E\left(Br_m^{[j]};t\right) \right) q^n \\
&= \left(\sum_{n=0}^{\infty} E\left(\mathbb{C}^2\backslash\{(0,0)\}^{[n]};t\right) q^n \right) \left(\sum_{n=0}^{\infty} E\left(Br_m^{[n]};t\right) q^n \right).
\end{aligned}$$

Applying the formula of the generating functions in (4.2.7) and (4.2.1), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} E\left(\mathrm{Hilb}_m^{[n]};t\right) q^n &= \left(\prod_{d=1}^{\infty} \frac{1-t^{d-1}q^d}{1-t^{d+1}q^d} \right) \left(\prod_{d=1}^{m-1} \frac{1}{1-t^{i+1}} \sum_{a=1}^{m} (-1)^{a+1} t^{m-1+\binom{a}{2}} \begin{bmatrix} m \\ a \end{bmatrix}_t \left(\prod_{d=0}^{\infty} \frac{1-t^{d-a}q^d}{1-t^{d-1}q^d} \right) \right) \\
&= \prod_{d=1}^{m-1} \frac{1}{1-t^{i+1}} \sum_{a=1}^{m} (-1)^{a+1} t^{m+\binom{a}{2}} \begin{bmatrix} m \\ a \end{bmatrix}_t \left(\frac{t^{-1}(1-t)}{1-t^{-1}} \right) \left(\prod_{d=0}^{\infty} \frac{1-t^{d-a}q^d}{1-t^{d+1}q^d} \right) \\
&= \prod_{d=1}^{m-1} \frac{1}{1-t^{i+1}} \sum_{a=1}^{m} (-1)^{a} t^{m+\binom{a}{2}} \begin{bmatrix} m \\ a \end{bmatrix}_t \left(\prod_{d=0}^{\infty} \frac{1-t^{d-a}q^d}{1-t^{d+1}q^d} \right).
\end{aligned}$$

∎

4.3 Euler characteristics of the refined strata.

The specializations of $E\left(Br_m^{[n]};t\right)$ at $t=1$ is the topological Euler characteristic of $Br_m^{[n]}$

$$\chi(Br_m^{[n]}) = E\left(Br_m^{[n]};1\right),$$

which equals to $\chi\left(\mathrm{Hilb}_m^{[n]}\right)$. It is, however, difficult to obtain an explicit formula for the generating function of $\chi\left(Br_m^{[n]}\right)$ from formula (4.15) in Theorem 4.2.1. If we consider the specialization of the equation (4.31) in the proof of Theorem 4.2.1

$$\sum_{n=0}^{\infty} E\left(Br_m^{[n]};t\right) q^n = \left(\prod_{d=1}^{\infty} \frac{1}{1-t^{d-1}q^d}\right) \sum_{k=0}^{\infty} (tq)^{\binom{k}{2}} (-1)^{k-m} t^{-km+\binom{m+1}{2}} \begin{bmatrix} k \\ m \end{bmatrix}_t \frac{1}{(tq)_k}$$

at $t=1$, we obtain a formula for the generating function of $\chi(Br_m^{[n]})$:

$$\sum_{n=0}^{\infty} \chi\left(Br_m^{[n]}\right) q^n = \left(\prod_{d=1}^{\infty} \frac{1}{1-q^d}\right) \sum_{k=0}^{\infty} \frac{(-1)^{k-m} q^{\binom{k}{2}}}{(q)_k} \binom{k}{m}. \tag{4.35}$$

n	$m=2$	$m=3$	$m=4$	$m=5$
0	0	0	0	0
1	1	0	0	0
2	2	0	0	0
3	2	1	0	0
4	3	2	0	0
5	2	5	0	0
6	4	6	1	0
7	2	11	2	0

Table 4.1: Examples of $\chi\left(Br_m^{[n]}\right)$.

Tables

Table 4.2: Examples of E-polynomials $E\left(Br_m^{[n]};t\right)$.

n	$E\left(B_1^{[n]};t\right)$	$E\left(Br_2^{[n]};t\right)$	$E\left(Br_3^{[n]};t\right)$	$E\left(Br_4^{[n]};t\right)$	$E\left(Br_5^{[n]};t\right)$
0	1	0	0	0	0
1	0	1	0	0	0
2	0	$t+1$	0	0	0
3	0	t^2+t	1	0	0
4	0	t^3+2t^2	$t+1$	0	0
5	0	t^4+2t^3-t	$2t^2+2t+1$	0	0
6	0	$t^5+3t^4+t^3-t^2$	$2t^3+3t^2+t$	1	0
7	0	$t^6+3t^5+t^4-2t^3-t^2$	$3t^4+5t^3+3t^2$	$t+1$	0
8	0	$t^7+4t^6+2t^5-2t^4-t^3$	$3t^5+7t^4+4t^3-t$	$2t^2+2t+1$	0
9	0	$t^8+4t^7+3t^6-3t^5-2t^4$	$4t^6+9t^5+7t^4-2t^2-t$	$3t^3+4t^2+2t+1$	0
10	0	$t^9+5t^8+4t^7-3t^6-3t^5$	$4t^7+12t^6+10t^5+t^4-3t^3-2t^2$	$4t^4+6t^3+4t^2+t$	1
11	0	$t^{10}+5t^9+5t^8-4t^7-5t^6$	$5t^8+15t^7+15t^6+2t^5-5t^4-4t^3-t^2$	$5t^5+10t^4+7t^3+3t^2$	$t+1$
12	0	$t^{11}+6t^{10}+7t^9-3t^8-6t^7+t^5$	$5t^9+18t^8+19t^7+4t^6-8t^5-7t^4-2t^3$	$7t^6+14t^5+12t^4+5t^3-t$	$2t^2+2t+1$
13	0	$t^{12}+6t^{11}+8t^{10}-4t^9-9t^8-t^7+t^6$	$6t^{10}+22t^9+27t^8+7t^7-10t^6-11t^5-4t^4$	$8t^7+20t^6+18t^5+9t^4-2t^2-t$	$3t^3+4t^2+2t+1$
14	0	$t^{13}+7t^{12}+10t^{11}-3t^{10}-11t^9-2t^8+2t^7$	$6t^{11}+26t^{10}+34t^9+12t^8-13t^7-16t^6-6t^5+t^3$	$10t^8+26t^7+27t^6+13t^5-5t^3-3t^2-t$	$5t^4+7t^3+5t^2+2t+1$

Table 4.3: Examples of the E-polynomials $\mathrm{Hilb}_m^{[n]}$.

	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$
n=0	1	0	0	0	0
1	t^2-1	1	0	0	0
2	$t^4+t^3-t^2-t$	t^2+t	0	0	0
3	$t^6+t^5-2t^3-t^2+t$	$t^4+2t^3+t^2-t-1$	1	0	0
4	$t^8+t^7+t^6-t^5-3t^4+t^2$	$t^6+2t^5+3t^4-2t^2-t$	t^2+t	0	0
5	$t^{10}+t^9+t^8-3t^6-3t^5+t^4+2t^3$	$t^8+2t^7+4t^6+3t^5-2t^4-4t^3-2t^2$	$t^4+2t^3+2t^2$	0	0
6	$t^{12}+t^{11}+t^{10}+t^9-t^8-4t^7-3t^6+3t^5+2t^4-t^3$	$t^{10}+2t^9+4t^8+5t^7+2t^6-5t^5-6t^4-t^3+t^2+t$	$t^6+2t^5+4t^4+2t^3-t^2-t-1$	1	0
7	$t^{14}+t^{13}+t^{12}+t^{11}-3t^9-6t^8-t^7+5t^6+2t^5-t^4$	$t^{12}+2t^{11}+4t^{10}+6t^9+6t^8-t^7-10t^6-8t^5-t^4+2t^3+t^2$	$t^8+2t^7+5t^6+6t^5+2t^4-2t^3-2t^2-t$	t^2+t	0
8	$t^{16}+t^{15}+t^{14}+t^{13}+t^{12}-t^{11}-5t^{10}-6t^9+t^8+7t^7+t^6-2t^5$	$t^{14}+2t^{13}+4t^{12}+6t^{11}+8t^{10}+5t^9-6t^8-15t^7-9t^6+2t^5+4t^4+2t^3$	$t^{10}+2t^9+5t^8+8t^7+8t^6-5t^4-4t^3-2t^2$	$t^4+2t^3+2t^2$	0
9	$t^{18}+t^{17}+t^{16}+t^{15}+t^{14}-3t^{12}-7t^{11}-6t^{10}+5t^9+9t^8-3t^6$	$t^{16}+2t^{15}+4t^{14}+6t^{13}+9t^{12}+9t^{11}+2t^{10}-14t^9-22t^8-9t^7+5t^6+7t^5+3t^4$	$t^{12}+2t^{11}+5t^{10}+9t^9+13t^8+9t^7-3t^6-9t^5-7t^4-3t^3$	$t^6+2t^5+4t^4+3t^3$	0
10	$t^{20}+t^{19}+t^{18}+t^{17}+t^{16}+t^{15}-t^{14}-5t^{13}-9t^{12}-4t^{11}+9t^{10}+10t^9-3t^8-4t^7+t^6$	$t^{18}+2t^{17}+4t^{16}+6t^{15}+9t^{14}+11t^{13}+9t^{12}-4t^{11}-24t^{10}-27t^9-6t^8+12t^7+10t^6+3t^5-t^4-t^3$	$t^{14}+2t^{13}+5t^{12}+9t^{11}+15t^{10}+17t^9+8t^8-10t^7-16t^6-10t^5-3t^4+2t^3+t^2+t$	$t^8+2t^7+5t^6+7t^5+4t^4-t^3-t^2-t-1$	1
11	$t^{22}+t^{21}+t^{20}+t^{19}+t^{18}+t^{17}-3t^{15}-8t^{14}-10t^{13}-t^{12}+15t^{11}+10t^{10}-6t^9-4t^8+t^7$	$t^{20}+2t^{19}+4t^{18}+6t^{17}+9t^{16}+12t^{15}+13t^{14}+6t^{13}-14t^{12}-37t^{11}-33t^{10}+19t^8+14t^7+3t^6-2t^5-t^4$	$t^{16}+2t^{15}+5t^{14}+9t^{13}+16t^{12}+22t^{11}+22t^{10}+4t^9-20t^8-24t^7-14t^6-2t^5+3t^4+2t^3+t^2$	$t^{10}+2t^9+5t^8+9t^7+11t^6+4t^5-2t^4-2t^3-2t^2-t$	t^2+t
12	$t^{24}+t^{23}+t^{22}+t^{21}+t^{20}+t^{19}+t^{18}-t^{17}-5t^{16}-10t^{15}-10t^{14}+5t^{13}+19t^{12}+8t^{11}-10t^{10}-5t^9+2t^8$	$t^{22}+2t^{21}+4t^{20}+6t^{19}+9t^{18}+12t^{17}+15t^{16}+13t^{15}-28t^{13}-51t^{12}-33t^{11}+11t^{10}+31t^9+18t^8+2t^7-4t^6-2t^5$	$t^{18}+2t^{17}+5t^{16}+9t^{15}+16t^{14}+24t^{13}+31t^{12}+23t^{11}-6t^{10}-36t^9-36t^8-17t^7+7t^5+4t^4+2t^3$	$t^{12}+2t^{11}+5t^{10}+10t^9+16t^8+15t^7+4t^6-5t^5-5t^4-4t^3-2t^2$	$t^4+2t^3+2t^2$
13	$t^{26}+t^{25}+t^{24}+t^{23}+t^{22}+t^{21}+t^{20}-3t^{18}-8t^{17}-13t^{16}-8t^{15}+11t^{14}+25t^{13}+5t^{12}-15t^{11}-4t^{10}+3t^9$	$t^{24}+2t^{23}+4t^{22}+6t^{21}+9t^{20}+12t^{19}+16t^{18}+17t^{17}+11t^{16}-11t^{15}-47t^{14}-66t^{13}-32t^{12}+27t^{11}+43t^{10}+21t^9-t^8-7t^7-3t^6$	$t^{20}+2t^{19}+5t^{18}+9t^{17}+16t^{16}+25t^{15}+36t^{14}+39t^{13}+22t^{12}-22t^{11}-57t^{10}-48t^9-19t^8+6t^7+12t^6+7t^5+3t^4$	$t^{14}+2t^{13}+5t^{12}+10t^{11}+18t^{10}+24t^9+20t^8+t^7-10t^6-9t^5-7t^4-3t^3$	$t^6+2t^5+4t^4+3t^3$
14	$t^{28}+t^{27}+t^{26}+t^{25}+t^{24}+t^{23}+t^{22}+t^{21}-t^{20}-5t^{19}-11t^{18}-14t^{17}-5t^{16}+20t^{15}+28t^{14}-2t^{13}-20t^{12}-3t^{11}+5t^{10}$	$t^{26}+2t^{25}+4t^{24}+6t^{23}+9t^{22}+12t^{21}+16t^{20}+19t^{19}+18t^{18}+5t^{17}-27t^{16}-69t^{15}-79t^{14}-20t^{13}+50t^{12}+59t^{11}+22t^{10}-7t^9-12t^8-5t^7$	$t^{22}+2t^{21}+5t^{20}+9t^{19}+16t^{18}+25t^{17}+38t^{16}+48t^{15}+46t^{14}+12t^{13}-49t^{12}-85t^{11}-63t^{10}-16t^9+16t^8+22t^7+12t^6+5t^5$	$t^{16}+2t^{15}+5t^{14}+10t^{13}+19t^{12}+29t^{11}+36t^{10}+23t^9-5t^8-19t^7-17t^6-12t^5-5t^4$	$t^8+2t^7+5t^6+7t^5+5t^4$

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