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Wilson surface theory

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«Wilson Surface Theory»

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Résumé

Le présent travail décrit la théorie des surfaces de Wilson. Nous commençons par la définition d'une observable de ligne de Wilson dans les théories de jauge, à partir de laquelle nous construisons d'abord une observable de surface de Wilson, puis une théorie de surface de Wilson à 2 dimensions indépendante pouvant interagir topologiquement avec d'autres théories de jauge.

Cette thèse contient les résultats suivants:

- Une intégrale de chemin à 1 dimension sur des orbites coadjointes, correspondant à une ligne de Wilson, est remplacée par un modèle sigma topologique à 2 dimensions. Nous montrons que ce modèle sigma est une extension équivariante de la forme symplectique de Kirillov sur l'orbite coadjointe. Cela nous permet de définir des observables de surface de Wilson sur des surfaces bidimensionnelles arbitraires (y compris fermées).

- Nous donnons une nouvelle description par une intégrale de chemin pour les lignes et les surfaces de Wilson en termes de modèle sigma de Poisson.

- La théorie de surface de Wilson est formulée en tant que théorie undépendente - une théorie topologique bidimensionnelle des champs quantiques avec un espace de Hilbert à 1 dimension. Cette théorie a le Lagrangien d'une théorie BF avec une contrainte sur le champ B. Nous avons calculé les fonctions de partition sur toutes les surfaces à 2 dimensions et pour tous les types topologiques de fibrés de jauge. Sur une surface fermée, la théorie de surface de Wilson définit un invariant topologique du fibré G-principal.

- La théorie de surface de Wilson peut interagir avec une théorie de jauge via la topologie des fibrés principaux. Nous décrivons ces interactions avec les théories BF et de Yang-Mills à 2 dimensions et en déduisons des formules pour les fonctions de partition modifiées par les interactions.

- Nous calculons explicitement les fonctions de partition de la théorie bidimensionnelle de Yang-Mills en interaction avec une surface de Wilson pour les cas

$G = U(1)$, $G = SO(3)$, $G = SU(N)/\mathbb{Z}_m$, $G = Spin(4l)/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ et obtenons une formule générale pour tout groupe de Lie connexe compact.

Abstract

The present work describes the theory of Wilson surfaces. It starts with a definition of a Wilson line observable in gauge theories, from which we construct first a Wilson surface observable, and then an independent 2-dimensional Wilson surface theory which can interact topologically with background gauge theories.

This thesis contains the following results:

- A 1-dimensional path integral over coadjoint orbits, corresponding to a Wilson line observable, is replaced by a 2-dimensional topological sigma model. We show that this sigma model is an equivariant extension of the Kirillov symplectic form on the coadjoint orbit. This allows us to define Wilson surface observables on arbitrary (including closed) 2-dimensional surfaces.

- A new path integral description for Wilson lines and surfaces in terms of Poisson sigma model is given.

- Wilson surface theory is formulated as a separate 2-dimensional topological quantum field theory with a 1-dimensional Hilbert space. It has a Lagrangian of a BF theory with a constraint on the B-field. We computed partition functions on all 2-dimensional surfaces and for all topological types of gauge group bundles. On a closed surface, the Wilson surface theory defines a topological invariant of the principal G-bundle.

- The Wilson surface theory can interact with some background gauge theory through the topology of principal bundles. We describe these interactions with 2-dimensional BF and Yang-Mills theories and derive formulas for the partition functions modified by the interactions.

- We compute explicitly the partition functions of the 2-dimensional Yang-Mills theory interacting with a Wilson surface for the cases $G = U(1)$, $G = SO(3)$,$G = SU(N)/\mathbb{Z}_m$, $G = Spin(4l)/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and obtain a general formula for any compact connected Lie group.

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- to God[1] for playing dice, so much in my favour.

[1]Einstein convention God:=Universe is implied.

Contents

Chapter 1

Background material

1.1 Gauge theories in physics and mathematics

Gauge theories lie at the heart of our current understanding of the fundamantal laws of nature.

The key feature of these theories is their invariance under some local transformations forming a gauge group G.

The electromagnetism is described by an abelian $G = U(1)$ gauge theory. Its non-abelian generalisation, pure Yang-Mills theory, describes the gluons' interactions. And the Standard Model of elementary particles is a non-abelian gauge theory with a gauge group $G = U(1) \times SU(2) \times SU(3)$, it provides a unified model for electromagnetic, weak and strong interactions and is famous by its predictive power. Even gravity in general relativity can in some sense be treated as a gauge theory if for the gauge group we take the group of diffeomorphisms of the spacetime manifold.

In this section we are going to introduce gauge theories in a physical language, and then provide a mathematical description of the concept.

1.1.1 Yang-Mills theory

For the purposes of this thesis the most important example of a gauge theory is going to be the Yang-Mills theory which we discuss in more detail [10, 14].

Let G be a compact Lie group and $\mathfrak{g}$ be its Lie algebra. We denote by T^a the basis of the algebra $\mathfrak{g}$. These are the generators of the group G. They satisfy the commutation relations:

$$[T^a, T^b] = f_c^{ab} T^c, \tag{1.1}$$

where f_c^{ab} are called the structure constants of the group G, and sommation is implied over the repeated indices.

Recall that an element $g \in G$ near the identity can always be represented as $g = e^{i\theta_a T^a}$, where θ_a are the coordinates parametrizing the manifold.

The vector fields A_μ^a, or gauge potential, take values in the Lie algebra of the group G. The number of the fields corresponds to the number of generators of G. Thus in case of electromagnetism with $G = U(1)$, there is only one gauge field giving rise to a photon after quantization. For QCD the gauge group appears to be $G = SU(3)$ with $3^2 - 1 = 8$ generators, and 8 gauge fields describe 8 gluons. It is convenient to consider A_μ in matrix representation of the algebra $\mathfrak{g}$. Then the potential can be expanded in terms of the generators T^a:

$$A_\mu(x) = A_\mu^a(x) T_a. \tag{1.2}$$

To construct the Lagrangian the covariant derivative is introduced:

$$D_\mu = (\partial_\mu + A_\mu). \tag{1.3}$$

The covariant derivative transforms as: $D_\mu \mapsto g D_\mu g^{-1}$. This condition defines the gauge transformation for the potential $A_\mu \mapsto A_\mu^g$:

$$\partial_\mu + A_\mu^g = g(\partial_\mu + A_\mu) g^{-1},$$

$$\begin{aligned} A_\mu^g &= -g\partial_\mu g^{-1} + g A_\mu g^{-1} + \partial_\mu = -\partial_\mu + g g^{-1}(\partial_\mu g) g^{-1} + g A_\mu g^{-1} + \partial_\mu, \\ A_\mu^g &= g A_\mu g^{-1} + \partial_\mu g g^{-1}. \end{aligned} \tag{1.4}$$

The field strength tensor $F_{\mu\nu}$ is defined as follows:

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{1.5}$$

Being a function of A_μ^a the field strength tensor $F_{\mu\nu}$ takes values in the Lie algebra $\mathfrak{g}$ and can be expanded in the generators T^a basis:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{bc}^a A_\mu^b A_\nu^c. \tag{1.6}$$

Note that in electromagnetism the structure constants are zero (as well as the commutator $[A_\mu, A_\nu]$) and the field strength tensor is linear in A_μ. This is not the case for a non-abelian theory. Also, in a non-abelian theory the field stregth $F_{\mu\nu}$ is not gauge invariant. By construction it transforms in the same way as the covariant derivative: $F_{\mu\nu} \to g F_{\mu\nu} g^{-1}$.

The gauge invariant kinetic term will constitute the free Lagrangian. It is obtained by taking the trace on the Lie algebra of the squared field strength:

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4g^2}F^a_{\mu\nu}F^{\mu\nu}_a, \tag{1.7}$$

where g is the Yang-Mills theory coupling and we used the normalization for T^a such that $Tr(T^aT^b) = \frac{1}{2}\delta^{ab}$.

The mentioned non-linearity of $F_{\mu\nu}$ results in the fact that cubic and quartic terms in A_μ are present in the Lagrangian. They describe self-interactions of the non-abelian gauge field. Thus already the free YM Lagrangian describes a non-trivial interacting field theory.

The action functional for the Yang-Mills theory:

$$S_{YM} = -\frac{1}{4g^2}\int d^4x F^a_{\mu\nu}F^{\mu\nu}_a. \tag{1.8}$$

Then the equations of motion obtained from this action are:

$$\partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = 0. \tag{1.9}$$

Or

$$D_\mu F^{\mu\nu} = 0. \tag{1.10}$$

We can define a dual tensor $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\delta}F_{\lambda\delta}$, to the field strength tensor $F_{\mu\nu}$ and construct the expression $D_\mu\tilde{F}^{\mu\nu} = \partial_\mu\tilde{F}^{\mu\nu} + [A_\mu, \tilde{F}^{\mu\nu}]$ which vanishes due to the antisymmetric properties of $F_{\lambda\delta}$ and $\epsilon^{\mu\nu\lambda\delta}$ tensors. This provides another field equation also known as Bianchi identity:

$$D_\mu\tilde{F}^{\mu\nu} = 0. \tag{1.11}$$

1.1.2 Yang-Mills theory geometrically

Gauge theories are geometric in character. Their natural mathematical description is the theory of principal bundles, connections on them and the curvatures of these connections [2, 11, 13].

Let G be a compact Lie group, $P \to N$ be a principal G-bundle over the space-time manifold N. Then for the tangent bundle TP to the total space there is a well defined subbundle, the vertical bundle $VP \hookrightarrow TP$. In more detail, to every element $\xi \in \mathfrak{g}$ we associate the fundamental vector field on P: $\xi^P \in \mathfrak{X}(P) \subset TP$. This vector field will be tangent to the fiber G and thus $\xi^P \in VP$. The map $\mathfrak{g} \to V_uP$, $u \in P$, is an isomorphism of vector spaces.

Definition 1.1.1. The connection on the principal G-bundle is the choice of a horizontal subbundle $HP \hookrightarrow TP$, which is an invariant complement to VP in TP, such that

$$\begin{aligned} &(i) \quad TP = VP \oplus HP \\ &(ii) \quad H_{gu}P = g_*(H_uP), \;\; u \in P, \;\; g \in G, \\ &\qquad\;\; g_* \text{ is a map induced by the action } G \times P \to P. \end{aligned}$$

It is convenient to choose the horizontal subbundle as the kernel of some map. More precisely, a connection can be defined as a bundle projection map $\mathcal{A} : TP \to VP$, then $HP = \ker(\mathcal{A})$. For practical computations this is achieved by introducing on P a connection $\mathfrak{g}$-valued 1-form $\mathcal{A} \in \Omega^1(P, \mathfrak{g})$

Definition 1.1.2. A connection $\mathfrak{g}$-valued 1-form $\mathcal{A} \in \Omega^1(P, \mathfrak{g})$ is a projection of T_uP to the vertical subspace $V_uP \simeq \mathfrak{g}$, satisfying the following conditions:

$$\begin{aligned} &(i) \quad \imath_{\xi^P}\mathcal{A} = \xi \;\; \forall \xi \in \mathfrak{g}, \\ &(ii) \quad \mathcal{A} \text{ is G} - \text{equivariant}, \text{ i. e. } g^*\mathcal{A} = g\mathcal{A}g^{-1}, \;\; g \in G. \end{aligned}$$

Then the horizontal subspace is given by:

$$H_uP \equiv \{X \in T_uP \,|\, \imath_X \mathcal{A} = 0\}. \tag{1.12}$$

The definitions 1.1.1 and 1.1.2 are equivalent.

A connection 1-form can be defined locally on the base manifold N. Let $\{U_i\}$ be an open cover of N and $\sigma_i : U_i \to \pi^{-1}(U_i)$ be a local section for each U_i. Then we can define a $\mathfrak{g}$-valued 1-form A_i on each U_i:

$$A_i = \sigma_i^* \mathcal{A} \in \Omega^1(U_i, \mathfrak{g}). \tag{1.13}$$

Note that if the principal G-bundle is trivial, there exists a global section $\sigma : N \to P$ and the connection 1-form on the base manifold is given globally by $A = \sigma^*\mathcal{A} \in \Omega^1(N, \mathfrak{g})$.

For a non-trivial bundle on the intersections $U_i \cap U_j$ the local forms agree in the following way:

$$A_j = t_{ij} A_i t_{ij}^{-1} - dt_{ij} t_{ij}^{-1}, \tag{1.14}$$

where $t_{ij} : U_i \cap U_j \to G$ are the transition functions.

Hence, if we know a local section σ_i and a $\mathfrak{g}$-valued 1-form A_i on each U_i we can define a global connection 1-form $\mathcal{A} \in \Omega^1(P, \mathfrak{g})$ on the bundle, such that $A_i = \sigma_i^*\mathcal{A}$, by the formula

$$\mathcal{A}|_{\pi^{-1}(U_i)} = g_i \pi^* A_i g_i^{-1} - dg_i g_i^{-1}, \tag{1.15}$$

where d is the exterior differentiation on P and g_i is the local trivialization defined by $\varphi_i(u) = (p, g_i)$ for $u = g_i \sigma_i(p)$ and $\varphi_i : P \to U_i \times G$.

In this formula we recognize the gauge transformation for the Yang-Mills potential (1.4). And the local 1-forms $A_i = \sigma_i^* \mathcal{A}$ on the base manifold are identified with the gauge potential.

The covariant derivative operator is induced by the connection on the principal G-bundle in the following way. Recall that a *horizontal* n-form with values in the Lie algebra $\omega \in \Omega^n(P, \mathfrak{g})$ is a form satisfying the condition $\imath_{\xi_P}\omega = 0$ for $\xi \in \mathfrak{g}$ and the fundamental vector field $\xi_P \in VP$.

Then the connection gives rise to an equivariant projection operator $P^h_{\mathcal{A}}$: $\Omega^n(P, \mathfrak{g}) \to \Omega^n_{hor}(P, \mathfrak{g})$. And the covariant derivative is the following composition:

$$d_{\mathcal{A}} = P^h_{\mathcal{A}} \circ d : \Omega^n(P, \mathfrak{g}) \to \Omega^{n+1}_{hor}(P, \mathfrak{g}). \tag{1.16}$$

Definition 1.1.3. Let $\mathcal{A} \in \Omega^1(P, \mathfrak{g})$ be a connection 1-form on the principal G-bundle $P \to N$. The curvature of the connection is the $\mathfrak{g}$-valued 2-form $\mathcal{F}_{\mathcal{A}} \in \Omega^2(P, \mathfrak{g})$ given by the covariant derivative of the connection:

$$\mathcal{F}_{\mathcal{A}} = d_{\mathcal{A}}\mathcal{A}.$$

This 2-form is horizontal by definition (the covariant derivative maps the forms into the horizontal subspace). Taking the curvature 2-form on two vector fields $X, Y \in TP$ it is easy to show that $\mathcal{F}_{\mathcal{A}}$ is equivariant:

$$\begin{aligned} g^*\mathcal{F}_{\mathcal{A}}(X,Y) &= \mathcal{F}_{\mathcal{A}}(g_*X, g_*Y) = d\mathcal{A}(g_*X^H, g_*Y^H) = dg^*\mathcal{A}(X^H, Y^H) = \\ &= d(g\mathcal{A}(X^H, Y^H)g^{-1}) = g(d\mathcal{A}(X^H, Y^H))g^{-1}) \\ &= g\mathcal{F}_{\mathcal{A}}(X,Y)g^{-1}, \end{aligned} \tag{1.17}$$

where we used the fact that the map g_* preserves the horizontal subspaces and commutes with the differential, i.e. $(g_*X)^H = g_*X^H$ and $dg^* = g^*d$.

For practical computations it is useful to introduces the explicit expression for the covariant derivative operator action on the connection

$$d_{\mathcal{A}}\mathcal{A} = d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] \tag{1.18}$$

and thus define the $\mathcal{F}_{\mathcal{A}}$ explicitly in terms of $\mathcal{A}$ and its ordinary differential:

$$\mathcal{F}_{\mathcal{A}} = d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}]. \tag{1.19}$$

The form $d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] \in \Omega^2(P, \mathfrak{g})$ is horizontal. This can be seen by contracting it with some horizontal vector field ξ_P:

$$\imath_{\xi_P}(d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}]) = \mathcal{L}_{\xi_P}\mathcal{A} - [\mathcal{A}, \imath_{\xi_P}\mathcal{A}] = [\mathcal{A}, \xi] - [\mathcal{A}, \xi] = 0.$$

On the other hand, its Lie derivative satisfies the condition $\mathcal{L}_{\xi_P}\mathcal{F}_\mathcal{A} = [\mathcal{F}_\mathcal{A}, \xi]$:

$$\begin{aligned}
\mathcal{L}_{\xi_P}(d\mathcal{A} + \tfrac{1}{2}[\mathcal{A},\mathcal{A}]) &= (d\imath_{\xi_P} + \imath_{\xi_P}d)(d\mathcal{A} + \tfrac{1}{2}[\mathcal{A},\mathcal{A}]) \\
&= \tfrac{1}{2}\imath_{\xi_P}d[\mathcal{A},\mathcal{A}] + d\imath_{\xi_P}d\mathcal{A} + \tfrac{1}{2}d\imath_{\xi_P}[\mathcal{A},\mathcal{A}] \\
&= [\imath_{\xi_P}d\mathcal{A},\mathcal{A}] + [d\mathcal{A},\imath_{\xi_P}\mathcal{A}] + d\mathcal{L}_{\xi_P}\mathcal{A} + [d\imath_{\xi_P}\mathcal{A},\mathcal{A}] + [\imath_{\xi_P}\mathcal{A}, d\mathcal{A}] \\
&= [\mathcal{L}_{\xi_P}\mathcal{A},\mathcal{A}] + [d\mathcal{A},\xi] + d\mathcal{L}_{\xi_P}\mathcal{A} + [\xi, d\mathcal{A}] \\
&= [[\mathcal{A},\xi],\mathcal{A}] + d[\mathcal{A},\xi] = [\tfrac{1}{2}[\mathcal{A},\mathcal{A}],\xi] + [d\mathcal{A},\xi] \\
&= [d\mathcal{A} + \tfrac{1}{2}[\mathcal{A},\mathcal{A}],\xi].
\end{aligned}$$

One possible interpretation is that curvature measures the quantity by which the Lie bracket of horizontal vector fields fails to be horizontal.

Proposition 1.1.4. *If $X, Y \in HP$ are horizontal vector fields on P, then $\mathcal{A}([X,Y]) = -\mathcal{F}_\mathcal{A}(X,Y)$.*

Proof. Recall that $\mathcal{A}$ vanishes on horizontal vectors, then

$$\mathcal{F}_\mathcal{A}(X,Y) = d\mathcal{A}(X^H, Y^H) = d\mathcal{A}(X,Y) = X\mathcal{A}(Y) - Y\mathcal{A}(X) - \mathcal{A}([X,Y]) = -\mathcal{A}([X,Y]).$$

Another possible interpretation is that curvature measures for how much the covariant differential fails to be a true differential.[1]

Proposition 1.1.5. *For $\omega \in \Omega^n(P,\mathfrak{g})$, $(d_\mathcal{A})^2\omega = [\mathcal{F}_\mathcal{A}, \omega]$.*

Proof. See [11]

The curvature is covariantly constant. This important property is called **the Bianchi identity**:

$$d_\mathcal{A}\mathcal{F}_\mathcal{A} = 0. \tag{1.20}$$

The identity is easily verified:

$$d_\mathcal{A}\mathcal{F}_\mathcal{A} = d\mathcal{F}_\mathcal{A} + [\mathcal{A},\mathcal{F}_\mathcal{A}] = \frac{1}{2}d[\mathcal{A},\mathcal{A}] + [\mathcal{A}, d\mathcal{A}] + \frac{1}{2}[[\mathcal{A},\mathcal{A}],\mathcal{A}] = 0,$$

as the first two terms cancel each other and the last one vanishes by the Jacobi identity for $\mathfrak{g}$.

The curvature 2-form can be defined locally on the base manifold N. Let $\{U_i\}$ be an open cover of N and $\sigma_i : U_i \to \pi^{-1}(U_i)$ be a local section for each U_i. Then we can define a $\mathfrak{g}$-valued 2-form F_i on each U_i:

$$F_i = \sigma_i^*\mathcal{F}_\mathcal{A} \in \Omega^2(U_i, \mathfrak{g}). \tag{1.21}$$

[1] The exterior differential satisfies the property $d^2 = 0$.

Note again that if the principal G-bundle is trivial, there exists a global section $\sigma : N \to P$ and the curvature 2-form on the base manifold is given globally by $F = \sigma^*\mathcal{F}_{\mathcal{A}} \in \Omega^2(N, \mathfrak{g})$.

In terms of local potential we find:

$$F = \sigma^*\mathcal{F}_{\mathcal{A}} = \sigma^*(d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}]) = d\sigma^*\mathcal{A} + \frac{1}{2}[\sigma^*\mathcal{A}, \sigma^*\mathcal{A}] = dA + \frac{1}{2}[A, A].$$

This form is identified with the Yang-Mills field strength.

Note that on the overlapping $U_i \cap U_j$ the local curvature forms should satisfy the compatibility condition:

$$F_i = t_{ij}F_j t_{ij}^{-1},$$

where the t_{ij} are the transition functions. This condition corresponds to the gauge transformation for the Yang-Mills field strength.

Let G be a compact connected Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra, and Tr an invariant scalar product on $\mathfrak{g}$. Consider a 4-manifold N and let P be a principal G-bundle $P \to N$. A connection $\mathcal{A}$ is given by a $\mathfrak{g}$-valued 1-form on P. The curvature of the connection $\mathcal{A}$ is a $\mathfrak{g}$-valued 2-form on P given by the formula $\mathcal{F}_{\mathcal{A}} = d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}]$.

Assuming that $P \to N$ is a trivial bundle, we can choose a section $\sigma : N \to P$ and define the gauge field $A = \sigma^*\mathcal{A}$ on N. Changing a section is equivalent to a gauge transformation of the gauge field A:

$$A \mapsto A^g = gAg^{-1} - dgg^{-1} \tag{1.22}$$

for $g : N \to G$.

Then the curvature 2-form on the manifold N is defined by $F_A = \sigma^*\mathcal{F}_{\mathcal{A}} = dA + \frac{1}{2}[A, A]$. It gauge transforms as $F_A \mapsto F_A^g = gF_Ag^{-1}$.

Now we can formulate the Yang-Mills action:

$$S_{YM}(A) = -\frac{1}{2g^2}\int_N \mathrm{Tr}F_A \wedge *F_A, \tag{1.23}$$

where g^2 is the gauge coupling constant and $*F_A$ is the Hodge dual of the curvature. Note that the quantity $\mathrm{Tr}F_A \wedge *F_A$ is invariant under the gauge transformation (1.22).

The variation of the action with respect to the gauge potential A yields the equation of motion:

$$d_A * F_A = d * F_A + [A, *F_A] = 0. \tag{1.24}$$

Another field equation is provided by the Bianchi identity (1.20). It is automatically satisfied being just a geometric consequence of the definition of the curvature:

$$d_AF_A = dF_A + [A, F_A] = 0. \tag{1.25}$$

1.2 Equivariant cohomology

1.2.1 Borel model

The equivariant cohomology [4, 5, 7, 12] appears when one is interested in the properties of manifolds (or topological spaces) with a group action $G \times M \to M$.

Let G be a compact Lie group. When it acts freely on a manifold M the quotient space of this action M/G is also a manifold. Free G action means that for $g \in G$ and $a \in M$ $ga = a$ only if $g = id_G$. Then the cohomology of such a quotient manifold $H^*(M/G)$ is a nice object (and it will coinside with the equivariant cohomology of M).

However, if G-action on M is not free the space M/G is usually singular. But still useful information can be obtained by constructing the *homotopy quotient* (also called *Borel construction*) and its cohomology will provide an efficient replacement for the cohomology of the space M/G.

The idea of the Borel construction is to replace M with some space $\tilde{M}$ on which the group G acts freely and which is homotopically equivalent to the original manifold M. One can always find a contractible space EG on which G acts freely. Then the product $M \times EG = \tilde{M} \simeq M$ is homotopically equivalent to M and G acts on it freely.

The contractible space EG in its turn is an important example of a principal G-bundle: $EG \to BG = EG/G$ called the classifying bundle for G.

$M \times EG \to (M \times EG)/G$ is a principle G-bundle and its base manifold $(M \times EG)/G$ is precisely the *homotopy quotient*.

Definition 1.2.1. The equivariant cohomology on M is the ordinary cohomology on $(M \times EG)/G$: $H^*_G(M) = H^*((M \times EG)/G)$.

1.2.2 The Weil model of equivariant cohomology

The topological definition (1.2.1) of equivariant cohomology can be reformulated in algebraic terms. In more detail, the Weil algebra is introduced:

$$W_G := S\mathfrak{g}^* \otimes \wedge\mathfrak{g}^*, \tag{1.26}$$

which is constructed by taking the product of the symmetric and exterior algebras of the dual space to $\mathfrak{g}$. It is graded by assigning degree 2 to generators of $S\mathfrak{g}^*$ and degree 1 to generators of $\wedge\mathfrak{g}^*$,

$$W_G^l = \oplus_{j+2k=l} S^k\mathfrak{g}^* \otimes \wedge^j\mathfrak{g}^*. \tag{1.27}$$

The elements $a, f \in W_G \otimes \mathfrak{g}$ are constructed as follows: a is a element in $\wedge^1 \mathfrak{g}^* \otimes \mathfrak{g}$ defined by the canonical pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. Similarly, $f \in S^1 \mathfrak{g}^* \otimes \mathfrak{g}$.

There is an action of a Lie superalgebra $\mathcal{G}$ on the Weil algebra, which is defined as follows. Let M be a smooth manifold, G be a compact connected Lie group acting on M and $\mathfrak{g}$ be the Lie algebra of G. To every $\xi \in \mathfrak{g}$ we associate the fundamental vector field $\xi_M \in \mathfrak{X}(M)$.

The three operator: de Rham differential d, contractions $\imath_\xi^M$ and Lie derivatives L_ξ^M act on the space of differential forms $\Omega^*(M)$. They satisfy the relations

$$[\imath_\xi^M, \imath_\eta^M] = 0,\ [L_\xi^M, \imath_\eta^M] = \imath_{[\xi,\eta]}^M,\ [L_\xi^M, L_\eta^M] = L_{[\xi,\eta]}^M,$$

$$[d, \imath_\xi^M] = L_\xi^M,\ [d, L_\xi^M] = 0,\ [d,d] = 0,$$

where $[-,-]$ is a supercommutator.

These operators form a Lie superalgebra $\mathcal{G}$. One of the possible representations of $\mathcal{G}$ are spaces of differential forms $\Omega^*(M)$. Another representation is precisely the Weil algebra W_G. $\mathcal{G}$ acts on the elements $a, f \in W_G \otimes \mathfrak{g}$:

$$da = f - \frac{1}{2}[a,a], \quad df = [f,a]. \tag{1.28}$$

$$\imath_\xi^W a = \xi, \quad \imath_\xi^W f = 0. \tag{1.29}$$

$$L_\xi^W f = [f,\xi],\ \ L_\xi^W a = [a,\xi]. \tag{1.30}$$

One can think of a as a universal connection on a principal G-bundle. Then, the first formula in (3.17) gives the standard definition of the curvature and the second one is the Bianchi identity.

As representations of $\mathcal{G}$ are carried by both $\Omega^*(M)$ and W_G the diagonal action on the tensor product can be defined. Thus d, $\imath_\xi$, L_ξ act on $\Omega^*(M) \otimes W_G$ as follows:

$$\begin{aligned} L_\xi &= L_\xi^M \otimes 1 + 1 \otimes L_\xi^W, \\ \imath_\xi &= \imath_\xi^M \otimes 1 + 1 \otimes \imath_\xi^W, \\ d &= d \otimes 1 + 1 \otimes d. \end{aligned} \tag{1.31}$$

The Weil algebra W_G provides the algebraic analogue of the space of differential forms on the classifying bundle EG. The cohomology of the Weil algebra is trivial, which is analogous to the space EG being contractible. And replacing the manifold M by the space $M \times EG$ with a free G-action finds its algebraic analog in replacing the space of differential forms $\Omega^*(M, \mathfrak{g})$ by the tensor product $\Omega^*(M) \otimes W_G$.

The space $\Omega^*_G(M)$ of equivariant forms on M is then defined as the basic part of $\Omega^*(M) \otimes W_G$:

$$\Omega^*_G(M) := \{\alpha \in \Omega^*(M) \otimes W_G | L_\xi \alpha = 0, \imath_\xi \alpha = 0\}.$$

Theorem 1.2.2. *The equivariant cohomology on M is given by*

$$H^*_G(M) = H^*(\Omega^*_G(M), d \otimes 1 + 1 \otimes d). \tag{1.32}$$

Proof. See [7]

The theorem 1.2.2 defines the Weil model of equivariant cohomology.

1.3 Geometry of coadjoint orbits

1.3.1 Symplectic and Poisson manifolds

Definition 1.3.1. A *symplectic manifold* is a pair (M, ω) where M^{2n} is an even-dimensional smooth manifold endowed with a *symplectic structure* $\omega \in \Omega^2(M)$, a closed non-degenerate differential 2-form:

$$d\omega = 0 \text{ and } \forall p \in M,\ \forall X \neq 0\ \exists Y \text{ s.t. } \omega(X, Y) \neq 0,\ X, Y \in T_pM$$

In local coordinates the symplectic form can be represented as $\omega = \omega_{ij}(x)dx^i dx^j$ with $(\omega_{ij}(x))$ antisymmetric non-degenerate matrix. Thus a symplectic manifold is always even-dimensional, such matrices don't exist in odd dimensions.

Non-degeneracy implies the existance of the isomorphism between the tangent and cotangent bundles $\omega : TM \to T^*M$ with the inverse $\pi = \omega^{-1}$. The bijection between vector fields and 1-forms on M is given by:

$$\imath_X \omega = \theta, \tag{1.33}$$

where $X \in \mathcal{X}(M)$ is a vector field and $\theta \in \Omega^1(M)$ is a 1-form on M. In local coordinates it can be represented as $\langle \omega_{ij}dx^i dx^j, X^i \frac{\partial}{\partial x^i} \rangle = \omega_{ij}X^i dx^j = \theta_j dx^j$. In case when the 1-one form is closed and given by $\theta = dH$ for a differentiable function $H \in C^\infty(M)$, the vector field $X_H \in \mathcal{X}(M)$ is called Hamiltonian.

Then to each differentiable function $H \in C^\infty(M)$ we can assign a vector field in the following way:

$$\omega(X_H, Y) = dH(Y),\ H \in C^\infty(M),\ X \in \mathcal{X}_{Ham}(M),\ Y \in \mathcal{X}(M) \tag{1.34}$$

This leads to defining a binary operation called *Poisson bracket* on the vector space of differentiable functions on M.

Definition 1.3.2. The *Poisson bracket* $\{\cdot,\cdot\} : C^\infty(M)\times C^\infty(M)\to C^\infty(M)$ on a symplectic manifold (M,ω) is a bilinear map on the space of differentiable functions $C^\infty(M)$ given by

$$\{f,g\} = \omega(X_f, X_g) \tag{1.35}$$

for $f,g\in C^\infty(M)$. The bracket satisfies the following properties:

- $\{f,g\} = \omega(X_f,X_g) = -\omega(X_g,X_f) = -\{g,f\}$ (antisymmetry),
- $\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0$ (Jacobi identity),
- $\{fg,h\} = f\{g,h\}+g\{f,h\}$ (Leibniz rule).

The first two conditions mean that the Poisson bracket defines a Lie bracket on the space $C^\infty(M)$. The formula for the bracket can be written just in terms of functions and Hamiltonian vector fields on M:

$$\{f,g\} = \omega(X_f,X_g) = df(X_g) = \imath_{X_g} df = \mathcal{L}_{X_g} f = -\mathcal{L}_{X_f} g.$$

A manifold together with the Poisson bracket is a Poisson manifold.

Recall that the inverse of the symplectic 2-form ω is a bi-vector defining a map $\pi : T^*M\to TM$. In local coordinates it can be expressed as $\pi = \pi^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}$.

Theorem 1.3.3. *The bivector $\pi=\omega^{-1}$ is Poisson and it defines the bracket $\{f,g\} = \pi(df\wedge dg)$.*

Thus every symplectic manifold (M,ω) is also a Poisson manifold. However, the converse is not always true: for a given Poisson manifold (M,π) the map $\pi : T^*M\to TM$ is not invertible in general and it is not possible to define $\pi^{-1}=\omega$. Neverthless (M,π) can be represented as a disjoint uninon of smooth even-dimensional submanifolds which are called *symplectic leaves*. On each symplectic leaf Σ the map $\pi : T^*\Sigma\to T\Sigma$ is a bijection. Consequently, on every leaf there exists a differential 2-form $\omega=\pi^{-1}\in\Omega^2(\Sigma)$ such that $\omega(X_f,X_g)=\{f,g\}$. The form $\omega =\in \Omega^2(\Sigma)$ is closed and non-degenerate, hence it defines a symplectic structure on (Σ,ω).

1.3.2 Coadjoint orbits as symplectic manifolds

Let G be a compact Lie group. We use the following notation pretending that G is a matrix group (by Ado Theorem, this is actually true). The adjoint representation $g\mapsto Ad(g)$ defines the the action of G on its Lie algebra $\mathfrak{g}$ which is given by matrix conjugation: $Ad(g)=g\xi g^{-1}$ for $g\in G$ and $\xi\in\mathfrak{g}$.

The dual to the adjoint representation is the *coadjoint* representation: $Ad^*(g) := Ad(g^{-1})^*$. This is the action of the group G on the dual of its Lie algebra $\mathfrak{g}^*$. By definition

$$\langle Ad^*(g)\lambda,\xi\rangle = \langle\lambda, Ad(g^{-1})\xi\rangle, \tag{1.36}$$

where $g \in G$, $\xi \in \mathfrak{g}$, $\lambda \in \mathfrak{g}^*$ and $\langle \cdot, \cdot \rangle$ is the canonical pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. Recall that for a semisimple Lie group the algebra $\mathfrak{g}$ and its dual are isomorphic: $\mathfrak{g} \simeq \mathfrak{g}^*$, hence the coadgoint action can be identified with the adjoint and denoted by

$$Ad(g^{-1})^*\lambda = g^{-1}\lambda g \tag{1.37}$$

for $\lambda \in \mathfrak{g}^*$ and $g \in G$.

Definition 1.3.4. A *coadjoint orbit* (or an orbit of the G action on $\mathfrak{g}^*$) is the set of elements of $\mathfrak{g}^*$:

$$\mathcal{O}_\lambda = \{g^{-1}\lambda g | \lambda \in \mathfrak{g}^*, g \in G\}. \tag{1.38}$$

Recall that the Lie algebra generators satisfy the condition $[T^a, T^b] = f_c^{ab} T^c$. The generators can be associated with the coordinate functions on $\mathfrak{g}^*$.

Theorem 1.3.5. ***(Kirillov-Kostant-Souriau)****[9] Let $f \in C^\infty(\mathfrak{g}^*)$, $X, Y \in \mathfrak{g}$, $\xi \in \mathfrak{g}^*$, such that $f_X(\xi) =< \xi, X >$. Then there exists the unique Poisson bivector π, such that the Poisson bracket is $\{f_X, f_Y\} = f_{[X,Y]}$.*

This bivector field can be written in terms of coordinates on $\mathfrak{g}^*$ as:

$$\pi = f_{ab}^c T_c \frac{\partial}{\partial T_a} \frac{\partial}{\partial T_b}. \tag{1.39}$$

Then the Poisson bracket on $\mathfrak{g}^*$ can be expressed as:

$$\{f, g\} = f_{ab}^c T_c \frac{\partial f}{\partial T_a} \frac{\partial g}{\partial T_b}, \tag{1.40}$$

where $f, g \in C^\infty(\mathfrak{g}^*)$. The space of linear functions on $\mathfrak{g}^*$ is closed under this operation as the bivector π has linear coefficients and thus is the Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ is a Lie subalgebra in $C^\infty(\mathfrak{g}^*)$.

This shows that $\mathfrak{g}^*$, the dual of the Lie algebra $\mathfrak{g}$, is a Poisson manifold.

Theorem 1.3.6. ***(Kirillov-Kostant-Souriau)*** *[8] Coadjoint orbits $\mathcal{O}_\lambda \subset \mathfrak{g}^*$ are symplectic leaves of $\mathfrak{g}^*$.*

By the theorem 1.3.6 the coadjoint orbits are symplectic manifolds $(\mathcal{O}_\lambda, \hat{\varpi}_\lambda)$. The symplectic structure $\hat{\varpi}_\lambda$ on $\mathcal{O}_\lambda$ is defined as follows. Let us take the Mauer-Cartan form$\theta = g^{-1}dg \in \Omega^1(G, \mathfrak{g})$, which is a $\mathfrak{g}$-valued 1-form on the group G. Consider a principal H_λ-bundle $G \to \mathcal{O}_\lambda \simeq G/H_\lambda$, where H_λ is the stabilizer of λ. For a given point $\lambda \in \mathcal{O}_\lambda$ we define a 2-form on G taking values in $\mathbb{R}$:

$$\varpi_\lambda = -\operatorname{Tr} \lambda d\theta = -\operatorname{Tr} \lambda (g^{-1}dg)^2 \in \Omega^2(G). \tag{1.41}$$

This 2-form is basic and by definition closed. Recall that $\Omega^*(G)_{basic} \simeq \Omega^*(\mathcal{O}_\lambda)$. Then the form ϖ_λ descends on $\mathcal{O}_\lambda$ to the unique form $\hat{\varpi}_\lambda \in \Omega^2(\mathcal{O}_\lambda)$, i.e. for $\varpi_\lambda \in \Omega^2_{basic}(G)$ there exists the unique $\hat{\varpi}_\lambda \in \Omega^2(\mathcal{O}_\lambda)$ such that $\pi^*\hat{\varpi}_\lambda = \varpi_\lambda$. This 2-form $\hat{\varpi}_\lambda$, known also as Kirillov form [8], is G-invariant, closed, non-degenerate and thus defines the symplectic structure on $\mathcal{O}_\lambda$.

1.3.3 Quantization of coadjoint orbits

One of the reasons why coadjoint orbits arise in gauge theories is that upon quantization they produce gauge invariant quantities defined on curves - Wilson line observables. Let G be a compact connected Lie group, $\mathfrak{g}$ its Lie algebra, $P \to N$ be a principal G-bundle over a manifold N, $\Gamma \subset N$ a closed curve embedded in N, and $A \in \Omega^1(P, \mathfrak{g})$ be a $\mathfrak{g}$-valued one-form on P.

Then a Wilson line (Wilson loop) is given by the holonomy of the gauge field A along Γ, and a finite dimensional irreducible representation R of G:

$$W_\Gamma^R(A) = \mathrm{Tr}_R\, P \exp\left(\int_\Gamma A^R\right). \tag{1.42}$$

The general idea of the quantization is to associate a Hilbert space to a classical system, whereas classical observables become linear operators on the Hilbert space. A coadjoint orbit, as a symplectic manifold, thus represents the phase space of a classical system.

Canonical quantization promotes canonical coordinates $\{T^a\}$ (they are coordinate functions on $\mathfrak{g}^*$) to operators $\hat{T}^a$. The Poisson bracket $\{T^a, T^b\} = f_c^{ab} T^c$ becomes the commutator $[\hat{T}^a, \hat{T}^b] = -i f_c^{ab} \hat{T}^c$.

The classical Hamiltonian is a function on the classical phase space: $H = T^a A_a(t)$, where $A_a(t)$ is a background field. The corresponding Hamiltonian operator acting on the Hilbert space is $\hat{H} = \hat{T}^a A_a(t)$.

Then the partition function of the quantized system takes the form

$$Z_{\mathcal{O}_\lambda} = \mathrm{Tr}_{\mathcal{H}}\, P \exp\left(-i \int_\Gamma \langle \hat{T}, A \rangle\right) = \mathrm{Tr}_{\mathcal{H}}\, P \exp\left(-i \int_\Gamma A^a(t) \hat{T}_a dt\right). \tag{1.43}$$

This is exactly a Wilson line, if we identify the representation R with the Hilbert space $\mathcal{H}$ of the system. To show how this identification works, we will go briefly through the basic steps of geometric quantization for the coadjoint orbits.

Geometrically, a coadjoint orbit $\mathcal{O}_\lambda$ is quantized as a Kähler manifold [3, 8]. Indeed, one can show that there exists a G-invariant complex structure $\mathcal{J}_\lambda$ on $\mathcal{O}_\lambda$, compatible with ϖ_λ, i.e. $\varpi_\lambda(\cdot, \cdot \mathcal{J}_\lambda)$ is a Kähler metric on $\mathcal{O}_\lambda$, such that ϖ_λ is a Kähler form.

In what follows, we assume that $\lambda \in \mathfrak{h}^*$ is regular, then $\mathfrak{h}_\lambda = \mathfrak{h}$ and $H_\lambda = H$ is a maximal torus subgroup. Next, one defines a line bundle over $\mathcal{O}_\lambda \simeq G/H$. Such bundles are classified by the elements of the second cohomology of the orbit $H^2(G/H, \mathbb{Z})$. At the same time, the existence of a principal H-bundle over the same base space $G \to G/H$ implies

$$H^2(G/H, \mathbb{Z}) = H^1(H, \mathbb{Z}) = Hom(H, U(1)) \simeq \Lambda^*_G, \tag{1.44}$$

where $\Lambda^*_G \subset \mathfrak{h}^*$ is the lattice of integral weights of G, hence they classify the line bundles over $\mathcal{O}_\lambda$.

Let $\mathcal{L}(\mu) = G \times_H \mathbb{C}$ be a line bundle over G/H determined by the weight μ. To become a prequantum bundle, $\mathcal{L}(\mu)$ must satisfy Bohr-Sommerfeld condition, i.e. the symplectic form ϖ_λ must be obtained from the curvature of the prequantum line bundle as $F_{\nabla_\mu} = i\varpi_\lambda$. This is indeed the case when $\lambda = \mu$ is quantized a weight of G. So we define $\mathcal{L}(\lambda)$ to be prequantum line bundle over the orbit $\mathcal{O}_\lambda$. The procedure is finished by identifying the Hilbert space of the system as the space of the holomorphic sections of the line bundle $\mathcal{O}_\lambda$:

$$\mathcal{H} = \Gamma_{hol}(\mathcal{O}_\lambda, \mathcal{L}(\lambda)) \simeq R_\lambda, \tag{1.45}$$

where the isomorphism is given by the Borel-Weil theorem.

Theorem 1.3.7. ***(Borel-Weil)*** *The space $\Gamma_{hol}(\mathcal{O}_\lambda, \mathcal{L}(\lambda))$ is non-zero exactly when $\lambda \in \Lambda^*_{G,+}$, where $\Lambda^*_{G,+}$ is the lattice of dominant weights, and in this case $\Gamma_{hol}(\mathcal{O}_\lambda, \mathcal{L}(\lambda))$ is an irreducible representation R_λ of G with highest weight λ.*

1.3.4 Path integral quantization of a coadjoint orbit

By definition a path integral reproduces time evolution of a quantum system described by Hamiltonian formalism. Hence if instead of canonical quantization we apply a path integral formalism to quantizing the coadjoin orbit we obtain a different presentation for the same object, for a Wilson line observable.

First let us see how the path integral quantization of a symplectic phase space works. Consider a physical system whose phase space is given by a symplectic manifold (M, ω, H) with the symplectic structure $\omega \in \Omega^2(M)$ and the Hamiltonian $H \in C^\infty(M)$. Let x_i be coordinates on M. The Poisson bracket on M will govern the time evolution of the system:

$$\dot{x}_i = \{H, x_i\}. \tag{1.46}$$

The action functional on the phase space (M, ω, H) is given by

$$S = \int (\alpha - H) dt, \tag{1.47}$$

where α is the symplectic potential solving the equation $\omega = d\alpha$ at least locally. Recall an example from classical mechanics with $\omega = dp^i \wedge dq^i$, $\alpha = p_i dq^i$. The action functional for the system would be

$$S = \int (p_i \dot{q}_i - H(p, q, t))\, dt.$$

And after path integral quantization the partition function of the system becomes

$$\int \mathcal{D}x e^{iS}. \tag{1.48}$$

A path integral quantization of coadjoint orbits described by Alekseev-Faddeev-Shatashvili [1] works as follows. Let $b : \Gamma \to \mathfrak{g}^*$ be an auxiliary field defined on the curve Γ and taking values in O_λ. In addition, we introduce a field $g : \Gamma \to G$ which takes values in G, with the property $b(s) = g(s)\lambda g(s)^{-1}$. The symplectic structure ϖ_O on the orbit is given by the formula (1.41), the symplectic potential α_O is $\alpha_O = \langle \lambda, g^{-1}dg \rangle$, and the Hamiltonian $H = -\langle \lambda, g^{-1}Ag \rangle$. Further on we denote the canonical pairing $\langle \cdot, \cdot \rangle$ by $\mathrm{Tr}(\cdot\ \cdot)$. Then the partition function is

$$Z_{\mathcal{O}_\lambda} = \int \mathcal{D}g\, e^{iS_\lambda(A,g)}, \tag{1.49}$$

where

$$S_\lambda(A, g) = \int_\Gamma \mathrm{Tr}\, \lambda(g^{-1}dg + g^{-1}Ag) = \int_\Gamma \mathrm{Tr}\, b(dgg^{-1} + A) \tag{1.50}$$

is the action functional 1.47.

This construction becomes the basepoint for the Wilson surface theory studied further on in the main part of the thesis.

Chapter 2

Description of results

Diakonov-Petrov formula for a Wilson line

This thesis studies several path integral presentations for a Wilson line, a non-local observable playing a very important role in gauge theories. The path ordered exponential *Pexp*, denoting the holonomy of the gauge field in the formula (1.42) for a Wilson line, is usually not easy to work with in practical computations. That is why another descriptions for (1.42) are interesting.

One path integral presentation, involving quantization of coadjoint orbits and discussed in detail in section 1.3.4, was invented by Alekseev, Faddev, Shatashvili.

Another presentation for a Wilson line was developped by Diakonov and Petrov [6]. The idea is to consider a surface Σ for which the curve Γ is a boundary. Then, applying the Stokes theorem to Alekseev-Faddeev-Shatashvili Lagrangian in (1.47), we can provide a new beautiful formula for the Wilson line. This time the action functional is defined on a 2-dimensional submanifold:

$$S_\lambda(A,g) = \int_\Gamma \operatorname{Tr} b(dgg^{-1} + A) = \int_\Sigma d\operatorname{Tr} b(dgg^{-1} + A) = \int_\Sigma \operatorname{Tr} b(F_A - (d_A g g^{-1})^2), \tag{2.1}$$

where d_A is the covariant derivative with respect to the gauge field A, and F_A is its curvature.

The Lagrangian of this formula is defined on the coadjoint orbit $\mathcal{O}_\lambda$, as well as Alekseev-Faddeev-Shatashvili Lagrangian of (1.47). Coadjoint orbit are symplectic manifolds and are endowed with the canonical Kirillov symplectic form. We show that the Lagrangian in (2.1) is an equivariant extension of the Kirillov form. This interpretation lets us define the action functional (2.1) on any orientable surface, even a closed one, hence define a Wilson surface observable.

Poisson sigma model for a Wilson surface observable

The construction of the Poisson sigma model of a Wilson surface is based on the action functional (2.1).

We introduce an auxiliary $\mathfrak{g}$-valued field $a \in \Omega^1_{hor}(P, \mathfrak{g})^G$. The new action is:

$$\begin{aligned} S_\lambda(a, b, A) &= \int_\Sigma \mathrm{Tr}(b(F_A - (dgg^{-1} + A)^2 + (dgg^{-1} + A + a)^2) \\ &= \int_\Sigma \mathrm{Tr}(b(d(A + a) + (A + a)^2)), \end{aligned} \tag{2.2}$$

Obviously, if the auxiliary field a is integrated out, this action reduces to the Diakonov-Petrov action. The resulting expression coinsides with the Poisson sigma model for coadjoin orbits, but there is a new detail: now a connection $A + a$ is defined on the same principal bundle, combining the background field A and the auxiliary field a.

Wilson surface theory

The study of Wilson surfaces started as a study of surface observables in some other gauge theory. The definition (2.2) allowed us to construct Wilson surface theory as a separate 2-dimensional model, without any theory in the background:

$$S_\lambda(a, b, A) = \int_\Sigma \mathrm{Tr}(bF_{A+a}). \tag{2.3}$$

As a Lagrangian field theory, it has an action of BF-theory with a constraint on the B-field to belong to a certain coadjoint orbit. This coadjoint orbit is the parameter of the theory. As a quantum theory, it has a 1-dimensional Hilbert space associated to a circle. We computed partition functions on all 2-dimensional surfaces and for all topological types of gauge group bundles.

In more detail, principal G-bundles $P \to \Sigma$ over a closed surface Σ are classified by the elements $\gamma \in \pi_1(G)$ in the fundamental group G. The gauge group can be viewed as $G = \tilde{G}/\Gamma$, where $\tilde{G}$ is the universal cover of G and $\Gamma \subset Z(\tilde{G})$ is a subgroup of the center of $\tilde{G}$. For each element $\gamma \in \pi_1(G)$ in the fundamental group there exists a corresponding element $C_\gamma \in \Gamma \subset Z(\tilde{G})$ in the center of the covering group, since $\Gamma \cong \pi_1(G)$. Then the partition function for a particular equivalence class of principal bundles $P \to \Sigma$, defined by $\gamma \in \pi_1(G)$, is given by:

$$Z^\Sigma_{WS}(C_\gamma, \lambda) = \frac{\chi_\lambda(C_\gamma)}{d_\lambda}, \tag{2.4}$$

where $\chi_\lambda(C_\gamma)$ is a value of the character χ_λ on the element C_γ and d_λ is the dimension of the representation.

Topological interactions and exact results for 2-dimensional gauge theories

In this thesis we study topological interactions of the Wilson surface theory with a background gauge theory, in particular 2-dimensional BF and Yang-Mills theories. The two gauge connections A and $A+a$ are defined on the same principal bundle, this is how the two theories interact through topology. The presence of the Wilson surface modifies the partition function of the background theory multiplying by a phase (2.4) the individual contributions for each class of bundles:

$$Z^{interact} = \sum_{\gamma \in \pi_1(G)} Z^{backgr}(C_\gamma) \cdot e^{i\varphi_\gamma}. \tag{2.5}$$

The Yang-Mills theory in 2 dimensions is exactly solvable, so it comes in very convenient to test our model and compute explicitly the partition functions in the presence of a Wilson surface for concrete gauge groups. In the present work we study in detail the examples of the gauge groups $G = U(1)$, $G = SO(3)$,$G = SU(N)/\mathbb{Z}_m$ (m devides N), $G = Spin(4l)/\mathbb{Z}_2 \oplus \mathbb{Z}_2$. And we obtain a formula of the partition function for any compact connected Lie group $G = \tilde{G}/\Gamma$.

Thus, without a Wilson surface, the 2-dimensional Yang-Mills partition function is:

$$Z_{YM}(\tau) = \sum_{R_\mu(G=\tilde{G}/\Gamma)} d_{R_\mu}^{2-2g} e^{-\tau C_2(R_\mu)}, \tag{2.6}$$

where τ is a parameter absorbing the Yang-Mills coupling and the area of the surface, μ is the highest weight of a representation R_μ, $C_2(R_\mu)$ is its quadratic Casimir, and the sum goes over such representations of the covering group $\tilde{G}$ that descend to the representations of $G = \tilde{G}/\Gamma$.

The presence of a Wilson surface changes the partition function:

$$Z_{YM}^{\lambda}(\tau) = \sum_{R_{\mu+\lambda}(G=\tilde{G}/\Gamma)} dim_{R_\mu}^{2-2g} e^{-\tau C_2(R_\mu)}, \tag{2.7}$$

where the sum now is over such representations $R_\mu(\tilde{G})$, that the representations of $\tilde{G}$ with the highest weight $\mu + \lambda$ would correspond to the representations of $G = \tilde{G}/\Gamma$.

The ideas, discribed briefly in this chapter, are developped thoroughly in chapters 3 and 4, corresponding to the following publications:

A. Alekseev, O. Chekeres, P. Mnev, *Wilson surface observables from equivariant cohomology*, J. High Energ. Phys. (2015) 2015: 93, [arXiv:1507.06343].

O. Chekeres, *Quantum Wilson surfaces and topological interactions*, J. High Energ. Phys. (2019) 2019: 30, [arXiv:1805.10992].

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Chapter 3

Wilson surface observables from equivariant cohomology

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Dedicated to Ludwig Faddeev on the occasion of his 3^4*th anniversary*

Abstract: Wilson lines in gauge theories admit several path integral descriptions. The first one (due to Alekseev-Faddeev-Shatashvili) uses path integrals over coadjoint orbits. The second one (due to Diakonov-Petrov) replaces a 1-dimensional path integral with a 2-dimensional topological σ-model. We show that this σ-model is defined by the equivariant extension of the Kirillov symplectic form on the coadjoint orbit. This allows to define the corresponding observable on arbitrary 2-dimensional surfaces, including closed surfaces. We give a new path integral presentation of Wilson lines in terms of Poisson σ-models, and we test this presentation in the framework of the 2-dimensional Yang-Mills theory. On a closed surface, our Wilson surface observable turns out to be nontrivial for G non-simply connected (and trivial for G simply connected), in particular we study in detail the cases $G = U(1)$ and $G = SO(3)$.

Keyword: Wilson line, Wilson surface, equivariant cohomology, 2d Yang-Mills, gauge theories

1 Introduction

Wilson line observables play an important role in gauge theories. Such an observable is defined by a closed curve Γ and a representation of the gauge group R. It is described by the formula

$$W_\Gamma^R = \mathrm{Tr}_R\, P\, \exp\left(\int_\Gamma A\right),$$

where A is a gauge field.

Wilson lines admit several interesting path integral presentations. The first one is due to Alekseev-Faddeev-Shatashvili [1], and it involves the auxiliary field $b = g\lambda g^{-1}$, where g is a group element defined on the curve Γ and λ is the label of the coadjoint orbit which corresponds to the representation R. The formula reads

$$W_\Gamma^R(A) = \int \mathcal{D}g\, e^{iS_\lambda(A,g)},$$

where the auxiliary action $S_\lambda(A,g)$ is given by the expression

$$S_\lambda(A,g) = \int_\Gamma \mathrm{Tr}\,\lambda(g^{-1}dg + g^{-1}Ag) = \int_\Gamma \mathrm{Tr}\, b(dgg^{-1} + A).$$

The second presentation is due to Diakonov-Petrov, and a path integral of an auxiliary 2-dimensional field theory is defined on the surface Σ bounding the curve Γ,

$$W_\Gamma^R(A) = \int \mathcal{D}g\, e^{i\int_\Sigma \mathrm{DP}_\lambda(A,g)}.$$

In [2], this presentation was used in the discussion of the area law for Wilson lines in a gauge theory with confinement.

In this article, we study the relation between the two path integral presentations. Our first result is a beautiful formula for the Diakonov-Petrov Lagrangian

$$\mathrm{DP}_\lambda(A,g) = \mathrm{Tr}\, b(F_A - (d_A g g^{-1})^2), \tag{3.1}$$

where F_A is the field strength and $d_A g = dg + Ag$. With this definition, the Diakonov-Petrov action can be defined on closed surfaces and on surfaces with multiple boundary components.

Our second result is the interpretation of the formula (3.1) in terms of equivariant cohomology. Coadjoint orbits carry the canonical Kirillov symplectic form. We show that the expression $\mathrm{DP}_\lambda(A,g)$ is the equivariant extension of this symplectic form corresponding to the action of the gauge group on the orbit.

Next, we use the Diakonov-Petrov formula to give a new path integral presentation of Wilson lines in terms of the Poisson σ-model. In our case, the Poisson σ-model reduces to a 2-dimensional BF-theory

$$S = \int \operatorname{Tr} b(d(A+\alpha) + (A+\alpha)^2),$$

where the field b takes values in the coadjoint orbit, A is the external gauge field and α is the auxiliary gauge field of the Poisson σ-model. In case when A is a connection on a non trivial bundle, the expression $A+\alpha$ defines another connection on the same bundle.

We test our theory of Wilson surfaces in the case of the 2-dimensional Yang-Mills (2YM) theory. In this case, the YM theory is exactly solvable, and one can see the effects of adding a Wilson surface observable in the partition function. We look in detail at the cases of $G = U(1)$, $G = SU(2)$ and $G = SO(3)$. For $G = SU(2)$, a simply connected group, a Wilson surface of any spin doesn't change the partition function for a closed surface. That is, the corresponding observable turns out to be trivial. For $G = U(1)$, a Wilson surface observable carries a parameter $\lambda \in \mathbb{R}$. For $\lambda \in \mathbb{Z}$ the observable is trivial, and it is nontrivial for $\lambda \notin \mathbb{Z}$. In the case of $G = SO(3)$, the Wilson surface observable is a $\mathbb{Z}_2$-valued topological invariant of the $SO(3)$-bundle, and and we observe an interesting relation

$$e^{i\int DP_{1/2}(A,g)} = \sqrt{e^{i\int DP_1(A,g)}}$$

between surface observables of integer and half-integer spin. Observables of integer spin are trivial (as in the case of $G = SU(2)$). But the observables of half-integer spin prove to be nontrivial, and one can see how they change the 2YM partition function on a closed surface.

Recently, $\mathbb{Z}_2$-valued topological invariants appeared in the theory of topological insulators [3, 4, 5, 6, 7]. A possible relation of our results with these invariants is to be explored.

Gauge invariant quantities assigned to surfaces rather than curves were extensively studied in literature [8, 9, 10, 11]. Our approach differs from most other approaches in that we have no higher gauge fields in the game. Our starting point is the standard gauge field which is a 1-form with values in a Lie algebra. The surface observable is obtained by using a non-abelian Stokes formula from a Wilson line observable. It is surprising that this construction allows an invariant formulation suitable for surfaces with many boundary components and for closed surfaces. It is also surprising that even for closed surfaces one may construct nontrivial observables (as in the case of $G = U(1)$ or $G = SO(3)$).

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2 A simple example: the case of $G = U(1)$

We start with a simple example of the first Chern class of a principal circle bundle over an orientable surface Σ. Let $P \to \Sigma$ be a principal $U(1)$-bundle, and let $\tilde{\mathfrak{a}} \in \Omega^1(P)$ be a connection on P. Then, $F_{\tilde{\mathfrak{a}}} = d\tilde{\mathfrak{a}}$ is the curvature of P and the first Chern form. It is basic and descends to a 2-form on Σ. If Σ is closed, then

$$c_1(P) = \frac{1}{2\pi} \int_\Sigma F_{\tilde{\mathfrak{a}}} \tag{3.2}$$

is an integer called the first Chern number of P. One can view the defining equation for the curvature, $F_{\tilde{\mathfrak{a}}} = d\tilde{\mathfrak{a}}$ as the definition of the 1-dimensional Chern-Simons form [12, 13],

$$F_{\tilde{\mathfrak{a}}} = d\,\mathrm{CS}_1(\tilde{\mathfrak{a}}),$$

where $\mathrm{CS}_1(\tilde{\mathfrak{a}}) = \tilde{\mathfrak{a}}$.

Assume that the surface Σ is connected, orientable and has a nontrivial boundary $\Gamma = \partial\Sigma \neq \emptyset$. Then, the circle bundle P is necessarily trivial. Let's choose a global section $\sigma : \Sigma \to P$ and define the gauge field $a = \sigma^*\tilde{\mathfrak{a}}$ so that $F_a = \sigma^* F_{\tilde{\mathfrak{a}}}$. Then, one can define a quantity $S(a)$ associated to Σ via

$$S(a) = \int_\Sigma F_a.$$

We can think of this expression as of the simplest surface observable associated to the surface Σ. Using the Stokes formula, we obtain

$$S(a) = \int_\Sigma F_a = \int_\Sigma da = \int_\Gamma a. \tag{3.3}$$

To distinguish between various connections appearing in the paper, in 1-dimensional CS-theory we use the Gothic letter $\tilde{\mathfrak{a}}$ for the connection 1-form on the principal bundle and the latin letter a for the 1-form representing this connection on the manifold.

Let $\phi : \Sigma \to U(1)$ and consider the gauge transformation $a^\phi = a + d\phi$. Then, the curvature F_a is gauge invariant, and so is the expression for the surface observable $\int_\Sigma F_a$.

However, in the expression $\int_\Gamma a$ the gauge invariance is lost! Indeed,

$$\int_\Gamma a^\phi = \int_\Gamma (a + d\phi) = \int_\Gamma a + (\phi(2\pi) - \phi(0))\,. \tag{3.4}$$

In general, $\phi(2\pi) - \phi(0) = 2\pi n$ with $n \in \mathbb{Z}$. The fact that the left hand side of (3.3) is gauge invariant and the right hand side is not may appear as a contradiction. But there is a solution to this puzzle: for gauge transformations defined on the surface Σ (that is, $\phi : \Sigma \to U(1)$) we have $\phi(2\pi) = \phi(0)$, and the extra term $\phi(2\pi) - \phi(0)$ always vanishes.

If we use the expression $S(a) = \int_\Gamma a$ and admit arbitrary gauge transformations defined on Γ, it is the exponential $\exp(iS(a))$ which becomes gauge invariant, while $S(a)$ is not gauge invariant in general.

3 Two path integral presentations of Wilson lines

In this Section, we shall discuss an example of the bulk-boundary correspondence based on the path integral description of Wilson lines in gauge theory.

3.1 The Alekseev-Faddev-Shatashvili (AFS) formula

Let G be a compact connected Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra, and Tr an invariant scalar product on $\mathfrak{g}$. Consider a manifold N, and let P be a principal G-bundle $P \to N$. A connection $\mathcal{A}$ is given by a $\mathfrak{g}$-valued 1-form on P.

Let $\Gamma \subset N$ be a closed curve. The restriction of P to a closed curve Γ is always trivial (since the group G is connected). Hence, we can use a connection 1-form $A \in \Omega^1(\Gamma, \mathfrak{g})$ instead of $\mathcal{A}$.

A Wilson line [14, 15, 16] is an observable defined by the holonomy of the gauge field A along a closed curve Γ, embedded in N, and a finite dimensional representation R of G. It is given by the formula

$$W^R_\Gamma(A) = \mathrm{Tr}_R\, P \exp\left(\int_\Gamma A^R\right), \tag{3.5}$$

For general non-abelian case we use the capital Gothic letter $\mathcal{A}$ to denote the connection 1-form on the principal bundle and the capital letter A for the 1-form representing this connection on the manifold.

where A^R is a matrix-valued 1-form obtained by taking A in the representation R,

$$A^R = A_i^a \, t_a^R \, dx^i. \tag{3.6}$$

The path ordered exponential $P\exp$ stands for the holonomy of the gauge field,

$$P\exp\left(\int_\Gamma A^R\right) = 1 + \int_\Gamma A^R + \frac{1}{2!}\int_{s_1>s_2} A^R(s_1)\wedge A^R(s_2) + \dots \tag{3.7}$$

where s_1, s_2 are parameters on the curve Γ.

Irreducible finite dimensional representations of G are in one-to-one correspondence with integral coadjoint orbits $O \subset \mathfrak{g}^*$ [17]. In more detail, an irreducible representation is uniquely determined by its highest weight $\lambda \in \mathfrak{h}^*$, where $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$ [18, 19]. Then, one can associate to λ the orbit of the coadjoint action in the space $\mathfrak{g}^*$. By abuse of notation, we write it as a matrix conjugation

$$\mathrm{Ad}_g^*(\lambda) = g\lambda g^{-1}, \tag{3.8}$$

and denote the coadjoint orbit by

$$O_\lambda = \{g\lambda g^{-1}; g \in G\}. \tag{3.9}$$

A path integral presentation of Wilson lines described by Alekseev-Faddeev-Shatashvili (see also [23, 20, 21, 24, 22]) works as follows. Let $b : \Gamma \to \mathfrak{g}^*$ be an auxiliary field defined on the curve Γ and taking values in O_λ. In addition, we introduce a field $g : \Gamma \to G$ which takes values in G, with the property $b(s) = g(s)\lambda g(s)^{-1}$. Then,

$$W_\Gamma^R = \int \mathcal{D}g \, e^{iS_\lambda(A,g)}, \tag{3.10}$$

where

$$S_\lambda(A,g) = \int_\Gamma \mathrm{Tr}\,\lambda(g^{-1}dg + g^{-1}Ag) = \int_\Gamma \mathrm{Tr}\, b(dgg^{-1} + A). \tag{3.11}$$

We can introduce the differential form

$$\mathrm{AFS}_\lambda(A,g) = \mathrm{Tr}\,\lambda(g^{-1}dg + g^{-1}Ag) = \mathrm{Tr}\, b(dgg^{-1} + A), \tag{3.12}$$

such that

$$S_\lambda(A,g) = \int_\Gamma \mathrm{AFS}_\lambda(A,g).$$

This action is invariant with respect to the following gauge transformations with parameter $h : \Gamma \to G$:

$$g \mapsto hg, \;\; A \mapsto A^h = hAh^{-1} - dhh^{-1}, \;\; b \mapsto hbh^{-1}.$$

Summation is implied over the repeated indices.

Indeed, it is easy to check that

$$\begin{aligned} AFS_\lambda(A^h, hg) &= \mathrm{Tr}\,\lambda((hg)^{-1}d(hg) + (hg)^{-1}(hAh^{-1} - dhh^{-1})hg) \\ &= \mathrm{Tr}\,\lambda(g^{-1}dg + g^{-1}Ag) \\ &= AFS_\lambda(A, g). \end{aligned}$$

Hence, $S_\lambda(A^h, hg) = S_\lambda(A, g)$.

It is interesting to consider another class of gauge transformations

$$g \mapsto gt^{-1},\ t \in H_\lambda,$$

where H_λ is the subgroup of G preserving λ under the coadjoint action:

$$H_\lambda = \{h \in G;\ h\lambda h^{-1} = \lambda\}.$$

For $\lambda \in \mathfrak{h}^*$ generic, H_λ is the Cartan subgroup of G. However, $S_\lambda(A, g)$ is not invariant under these transformations. Instead, it acquires an additional term:

$$S_\lambda(gt^{-1}, A) = S_\lambda(g, A) - \int_\Gamma \mathrm{Tr}\lambda(dtt^{-1}) = S_\lambda(g, A) - 2\pi \mathrm{Tr}(\lambda\vec{n}), \tag{3.13}$$

where $\vec{n} = \int_\Gamma dtt^{-1}$. The components of $\vec{n}$ are the winding numbers of the map $t : \Gamma \to H_\lambda$. The exponential $e^{ikS_\lambda(g,A)}$ is gauge invariant if

$$k\,\mathrm{Tr}\,(\lambda\vec{n}) \in \mathbb{Z}$$

for vectors $\vec{n} \in \mathbb{Z}^r$. If λ is an integral weight, then $Tr(\lambda\vec{n})$ is always an integer. This requires the coefficients k to be quantized.

3.2 The Diakonov-Petrov (DP) formula

The second presentation of the Wilson line is due to Diakonov and Petrov who came up with the following formula:

$$\mathrm{DP} = d\,\mathrm{AFS}_\lambda(A, g). \tag{3.14}$$

Expanding the form $d\,\mathrm{AFS}_\lambda(A, g)$ we obtain the expression for the Diakonov-Petrov Lagrangian in terms of a matrix-valued 1-form A and a matrix-valued function g:

$$
\begin{aligned}
d\,\mathrm{AFS}_\lambda(A,g) &= d\ \mathrm{Tr}\,\lambda(g^{-1}dg+g^{-1}Ag)\\
&= \mathrm{Tr}\,\lambda(dg^{-1}dg+dg^{-1}Ag+g^{-1}dAg-g^{-1}Adg)\\
&= \mathrm{Tr}\,\lambda(-g^{-1}dgg^{-1}dg-g^{-1}dgg^{-1}Ag+g^{-1}dAg-g^{-1}Agg^{-1}dg)\\
&= \mathrm{Tr}\,\lambda(g^{-1}F_Ag-\tfrac{1}{2}g^{-1}[A,A]g-\tfrac{1}{2}[g^{-1}dg,g^{-1}dg]-[g^{-1}dg,g^{-1}Ag])\\
&= \mathrm{Tr}\,b(F_A-\tfrac{1}{2}[dgg^{-1}+A,dgg^{-1}+A])\\
&= \mathrm{Tr}\,b\,(F_A-(d_Agg^{-1})^2)\\
&= \mathrm{DP}_\lambda(A,g),
\end{aligned}
$$

where we used $dA=F_A-\frac{1}{2}[A,A]$, $dg^{-1}=-g^{-1}dgg^{-1}$ and $d_Ag=dg+Ag$.

Now consider a surface Σ bounded by the curve Γ. One can use the Stokes formula to give a new expression for the action $S_\lambda(A,g)$,

$$
S_\lambda(A,g)=\int_\Gamma \mathrm{AFS}_\lambda(A,g)=\int_\Sigma \mathrm{DP}_\lambda(A,g)=\int_\Sigma \mathrm{Tr}\,b\left(F_A-(d_Agg^{-1})^2\right).
$$

The right hand side is manifestly gauge invariant whereas the expression $\int_\Gamma \mathrm{AFS}_\lambda(A,g)$ is not. The explanation is similar to the one that we encounter in the case of $G=U(1)$: for gauge transformations $g:\Sigma\to G$, both expressions for the action are gauge invariant; which is not necessarily the case for the integral of $\mathrm{AFS}_\lambda(A,g)$ if one considers a gauge transformation $g:\Gamma\to G$.

The path integral is taken over all configurations of the field g including all boundary values.

4 Equivariant cohomology approach

The main result of this Section is the equivariant cohomology [25, 27, 26, 28] interpretation of the Diakonov-Petrov formula.

4.1 The Weil model of equivariant cohomology

Let M be a smooth manifold, G be a compact connected Lie group acting on M and $\mathfrak{g}$ be the Lie algebra of G. To every $\xi\in\mathfrak{g}$ we associate the fundamental vector field $\xi_M\in\mathfrak{X}(M)$.

The de Rham differential d, contractions $\imath_\xi^M$ and Lie derivatives L_ξ^M act on the space of differential forms $\Omega^*(M)$, and satisfy the relations

$$
[\imath_\xi^M,\imath_\eta^M]=0,\quad [L_\xi^M,\imath_\eta^M]=\imath_{[\xi,\eta]}^M,\quad [L_\xi^M,L_\eta^M]=L_{[\xi,\eta]}^M,\quad [d,\imath_\xi^M]=L_\xi^M,\quad [d,L_\xi^M]=0,\quad [d,d]=0,
$$

where $[-,-]$ stands for a supercommutator.

Together, they form a Lie superalgebra $\mathcal{G}$. One natural class of representations of $\mathcal{G}$ are spaces of differential forms $\Omega^*(M)$. Another representation is the Weil algebra:

$$W_G := S\mathfrak{g}^* \otimes \wedge\mathfrak{g}^*, \tag{3.15}$$

which is constructed by taking the product of the symmetric and exterior algebras of the dual space to $\mathfrak{g}$. The Weil algebra is graded by assigning degree 2 to generators of $S\mathfrak{g}^*$ and degree 1 to generators of $\wedge\mathfrak{g}^*$,

$$W_G^l = \oplus_{j+2k=l} S^k\mathfrak{g}^* \otimes \wedge^j\mathfrak{g}^*. \tag{3.16}$$

It is convenient to introduce elements $a, f \in W_G \otimes \mathfrak{g}$ constructed as follows: a is a element in $\wedge^1\mathfrak{g}^* \otimes \mathfrak{g}$ defined by the canonical pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. Similarly, $f \in S^1\mathfrak{g}^* \otimes \mathfrak{g}$.

The superalgebra $\mathcal{G}$ acts on W_G as follows:

$$da = f - \frac{1}{2}[a,a], \qquad df = [f,a]. \tag{3.17}$$

$$\imath_\xi^W a = \xi, \qquad \imath_\xi^W f = 0. \tag{3.18}$$

$$L_\xi^W f = [f,\xi], \qquad L_\xi^W a = [a,\xi]. \tag{3.19}$$

One can think of a as a universal connection on a principal G-bundle. Then, the first formula in (3.17) gives the standard definition of the curvature and the second one is the Bianchi identity.

As representations of $\mathcal{G}$ are carried by both $\Omega^*(M)$ and W_G the diagonal action on the tensor product can be defined. Thus d, $\imath_\xi$, L_ξ act on $\Omega^*(M) \otimes W_G$ as follows:

$$\begin{aligned} L_\xi &= L_\xi^M \otimes 1 + 1 \otimes L_\xi^W, \\ \imath_\xi &= \imath_\xi^M \otimes 1 + 1 \otimes \imath_\xi^W, \\ d &= d \otimes 1 + 1 \otimes d. \end{aligned} \tag{3.20}$$

The space $\Omega_G^*(M)$ of equivariant forms on M is then defined as the basic part of $\Omega^*(M) \otimes W_G$:

$$\Omega_G^*(M) := \{\alpha \in \Omega^*(M) \otimes W_G | L_\xi\alpha = 0, \imath_\xi\alpha = 0\}.$$

And the Weil model of equivariant cohomology on M is defined as:

$$H_G^*(M) = H^*(\Omega_G^*(M), d \otimes 1 + 1 \otimes d). \tag{3.21}$$

4.2 The form $\mathrm{DP}_\lambda(A,g)$ as an equivariant cocycle

Now we consider in more detail the Diakonov-Petrov action, where the form

$$\mathrm{DP}_\lambda(A,g) = \operatorname{Tr}\lambda(g^{-1}F_A g - \frac{1}{2}g^{-1}[A,A]g - \frac{1}{2}[g^{-1}dg, g^{-1}dg] - [g^{-1}dg, g^{-1}Ag]) \quad (3.22)$$

is of particular interest. Here A is the gauge field on N and F_A is its curvature. We can now construct an equivariant differential form on the orbit O_λ by replacing A with a and F_A with f. The resulting element has the form

$$\mathrm{DP}_\lambda(a,g) = \operatorname{Tr}\lambda(g^{-1}fg - \frac{1}{2}g^{-1}[a,a]g - \frac{1}{2}[g^{-1}dg, g^{-1}dg] - [g^{-1}dg, g^{-1}ag]) \quad (3.23)$$

It is a well-known fact that the coadjoint orbits $O_\lambda \subset \mathfrak{g}^*$ (discussed in the previous Section) are symplectic manifolds (O_λ, ϖ_O). Here the symplectic form ϖ_O on O_λ can be identified as one of the terms in $\mathrm{DP}_\lambda(a,g)$:

$$\varpi_O = -\operatorname{Tr}\lambda(g^{-1}dg)^2 = -\operatorname{Tr} b(dgg^{-1})^2. \quad (3.24)$$

This is a closed and non-degerate 2-form also known as the Kirillov form.

Our first claim is that $\mathrm{DP}_\lambda(a,g)$ is equivariantly closed. Indeed,

$$\mathrm{DP}_\lambda(a,g) = d\,\mathrm{AFS}_\lambda(a,g) = d\operatorname{Tr}\lambda(g^{-1}dg + g^{-1}ag).$$

Hence, $d\,\mathrm{DP}_\lambda(a,g) = 0$.

Futhermore, applying the combined contraction gives:

$$\imath_\xi \mathrm{DP}_\lambda(a,g) = \operatorname{Tr} b(-\frac{1}{2}[-\xi gg^{-1}+\xi, dgg^{-1}+a] + \frac{1}{2}[dgg^{-1}+a, -\xi gg^{-1}+\xi]) = 0, \quad (3.25)$$

where we have used that $\imath_\xi(dg) = -\xi g$.

Closedness and horizontality of $\mathrm{DP}_\lambda(a,g)$ imply vanishing of its Lie derivative:

$$L_\xi \mathrm{DP}_\lambda(a,g) = (\imath_\xi d + d\imath_\xi)\mathrm{DP}_\lambda(a,g) = 0. \quad (3.26)$$

The two conditions $L_\xi \mathrm{DP}_\lambda(a,g) = 0$, $\imath_\xi \mathrm{DP}_\lambda(a,g) = 0$ being satisfied, $\mathrm{DP}_\lambda(a,g)$ is an equivariant differential form on the coadjoint orbit O_λ. Since it is equivariantly closed, we can view it as an equivariant extension of the Kirillov symplectic form.

5 Poisson σ-model formula for a Wilson line

In this Section, we introduce another description for a Wilson line observable in terms of a 2-dimensional path integral.

Recall that for a Poisson manifold (M, π) with Poisson structure

$$\pi = \frac{1}{2}\,\pi^{ij}\,\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j},$$

the corresponding Poisson σ-model [29, 31, 30] is defined by the action

$$S^\pi(X, \alpha) = \int_\Sigma \left(\alpha_i dX^i + \frac{1}{2}\pi^{ij}(X)\alpha_i \wedge \alpha_j\right).$$

Here $X : \Sigma \to M$ is a map of the surface Σ to the target space M, $X^i = x^i \circ X$ are its components, and α_i are 1-forms on Σ representing gauge fields of the Poisson σ-model. In case when π^{ij} is invertible, one can integrate out the fields α to obtain an integral of the symplectic form $\omega_{ij} = (\pi^{-1})^{ji}$:

$$S_\omega = \frac{1}{2}\int_\Sigma \omega_{ij} dX^i \wedge dX^j. \tag{3.27}$$

In particular, when Σ is a surface with boundary the path integral is taken over all X, including boundary values, and over the auxiliary fields α which vanish on the boundary:

$$\int_{\alpha|_{\partial\Sigma}=0} DX D\alpha\, e^{iS^\pi(X,\alpha)} = \int DX e^{iS_\omega(X)}.$$

Now let the target space be $\mathcal{O}_\lambda$, a coadjoint orbit of G passing through the point λ. (For this part of the discussion we avoid unnecessary complexity and work with a trivial G-bundle over the world-sheet Σ.) The form ω is given by (3.24), and the action (3.27) reads

$$S_{\varpi_O} = -\int_\Sigma \mathrm{Tr}\,\lambda(g^{-1}dg)^2 = -\int_\Sigma \mathrm{Tr}\, b(dgg^{-1})^2.$$

To construct the Poisson σ-model on the coadjoint orbit, we have to add the auxiliary gauge fields $\alpha \in \Omega^1(\Sigma, \mathfrak{g})$. The action S^π is given by

$$S^\pi(b, \alpha) = \int_\Sigma \mathrm{Tr}\, b(d\alpha + \alpha^2). \tag{3.28}$$

And again, for Σ a surface with boundary we impose the condition that the auxiliary field α vanishes on the boundary. In fact, in the expression (3.28) one can identify

If further on we want to perform gluing on the surface observable, instead of fixed $\alpha|_{\partial\Sigma} = 0$ we then allow a family of boundary conditions $\alpha|_{\partial\Sigma} = \alpha_0$ and integrate over α_0 for gluing.

the 2-dimensional BF-theory [32] with an extra condition that b belongs to a fixed coadjoint orbit.

To make the relationship between S_{ϖ_O} and S^π more transparent, we rewrite the latter as follows:

$$\int_\Sigma \mathrm{Tr}\, b(d\alpha + \alpha^2) = \int_\Sigma \mathrm{Tr}\, b\left((dgg^{-1} + \alpha)^2 - (dgg^{-1})^2\right), \tag{3.29}$$

where we have used integration by parts and the fact that $\mathrm{Tr} b[dgg^{-1}, \alpha] = \mathrm{Tr}[b, dgg^{-1}]\alpha = -\mathrm{Tr}(db)\alpha$. It is now clear that integrating out α yields the expression for S_{ϖ_O}.

Recall that the Diakonov-Petrov form (3.22) is an equivariant extention of the Kirillov form and thus the Diakonov-Petrov action

$$S_{DP} = \int_\Sigma \mathrm{Tr} b(F_A - (d_A gg^{-1})^2)$$

can be viewed as a version of the action S_{ϖ_O} interacting with the external gauge field A. Introducing an auxiliary gauge field α and performing the transformations similar to (3.29) we define the Poisson σ-model version of this action by

$$\begin{aligned} S_\sigma(b, A, \alpha) &= \textstyle\int_\Sigma \mathrm{Tr}\, b\,(F_A + (d_A gg^{-1} + \alpha)^2 - (d_A gg^{-1})^2) \\ &= \textstyle\int_\Sigma \mathrm{Tr}\, b\,(F_A + d_A\alpha + \alpha^2) \\ &= \textstyle\int_\Sigma \mathrm{Tr}\, b\,(d(A+\alpha) + (A+\alpha)^2)\,. \end{aligned} \tag{3.30}$$

From the first line, it is obvious that integrating out α yields the Diakonov-Petrov action. Surprizingly, the final expression coincides with the Poisson σ-model for the coadjoint orbit (3.28), but with the new gauge field $A + \alpha$ combining the background field A and the gauge field of the Poisson σ-model α.

Up to now all the argumentation above works in the case when A is a gauge field on a trivial G-bundle over Σ. If this is not the case, a further analysis is required. In fact, for a nontrivial geometric setup all the formulas still hold. However, one should be more precise about where the fields take values. Let $P \to \Sigma$ be a (possibly nontrivial) principal G-bundle and $\mathcal{A} \in \Omega^1(P, \mathfrak{g})$ be a connection on P. The curvature $F_\mathcal{A} = d\mathcal{A} + \mathcal{A}^2$ belongs to $\Omega^2_{\mathrm{hor}}(P, \mathfrak{g})^G$, the space of horizontal G-invariant 2-forms with values in $\mathfrak{g}$. Since the expression

$$\int_\Sigma \mathrm{Tr}\, bF_\mathcal{A}$$

makes part of the action $S_\sigma(b, \mathcal{A}, \alpha)$, the field b must take values in $\Omega^0_{\mathrm{hor}}(P, \mathfrak{g})^G$, that is in G-invariant functions on P with values in $\mathfrak{g}$. The expression $\mathrm{Tr}\, bF_\mathcal{A}$ is then a horizontal invariant 2-form on P, and it descends to Σ. In a similar fashion $\alpha \in \Omega^1_{\rm hor}(P, \mathfrak{g})^G$, and interestingly the combination $\mathcal{A} + \alpha$ defines a new connection on P.

6 The interpretation of surface observables in terms of topology of principal bundles

In this Section, we give an interpretation of the surface observables in terms of the first Chern class of the bundle [33, 34, 35, 36] with the structure group H_λ.

Let N be a manifold (space-time), $P \to N$ be a principle G-bundle over N, $\Sigma \subset N$ be a submanifold of N, and $P|_\Sigma \to \Sigma$ be the restriction of the principal bundle P to Σ. Assume that over Σ the structure group G of P reduces to a subgroup $H \subset G$. That is, Σ carries a principal H-bundle $Q \to \Sigma$, and there is an H-equivariant inclusion $i : Q \to P|_\Sigma$.

The bundle $Q \to \Sigma$ is a pull-back of the universal bundle $EH \to BH$ under the map $\sigma : \Sigma \to BH$. It induces a map in cohomology $\sigma^* : H^*(BH) \to H^*(\Sigma)$. Since Σ is 2-dimensional, we are particularly interested in the cohomology group $H^2(BH) \cong {\rm char}(\mathfrak{h}) \subset \mathfrak{h}^*$. Here ${\rm char}(\mathfrak{h})$ is the set of characters of the Lie algebra $\mathfrak{h}$,

$$ {\rm char}(\mathfrak{h}) = \{\lambda \in \mathfrak{h}^*;\ \langle \lambda, [x, y]\rangle = 0 \text{ for all } x, y \in \mathfrak{h}\}. \tag{3.31} $$

For a closed surface Σ, we will show that the surface observable of the previous sections is given by

$$ \int_\Sigma \sigma^* c_\lambda, $$

where $c_\lambda \in H^2(BH)$ is the image of $\lambda \in {\rm char}(\mathfrak{h})$.

In more detail, the form ${\rm DP}_\lambda(a, g)$ defined on a universal G-bundle can be viewed as an element of $\Omega^2_G(P \times Q, \mathfrak{g})$. Let $\pi : \mathfrak{g} \to \mathfrak{h}$ be an H-equivariant projection from $\mathfrak{g}$ to $\mathfrak{h}$, and let

$$ \mathfrak{a} = \pi(g^{-1}dg + g^{-1}\mathcal{A}g). \tag{3.32} $$

This is an element of $\Omega^1(P \times Q, \mathfrak{h})$. We will show that

$$ {\rm DP}_\lambda(\mathcal{A}, g) = {\rm Tr}\, \lambda(d\mathfrak{a} + \mathfrak{a}^2). \tag{3.33} $$

First, observe that the condition (3.31) implies ${\rm Tr}\, \lambda(\mathfrak{a}^2) = 0$. Indeed, $\mathfrak{a}^2 = \frac{1}{2}[\mathfrak{a}, \mathfrak{a}]$ and $\mathfrak{a}$ takes values in $\mathfrak{h}$. Thus, we are interested in the expression

$$ {\rm Tr}\, \lambda\, d\mathfrak{a} = {\rm Tr}\, \lambda\, (d\mathfrak{a} + \mathfrak{a}^2). $$

Second, recall the structure of the invariant pairing between the elements of $\mathfrak{h}$ and its dual $\mathfrak{h}^*$, $\lambda \in \mathfrak{h}^*$ being an element of the dual to the Cartan subalgebra of $\mathfrak{g}$. One can view $\mathfrak{g}$ as a direct sum of the subalgebra $\mathfrak{h}$ and its invariant complement $\mathfrak{p}$ (that is, $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$):

$$\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{p}.$$

Then the invariant product between two elements is defined in the following way:

$$\mathrm{Tr}\,(\lambda x) = \langle \lambda, x \rangle \;\; for \;\; \lambda \in \mathfrak{h}^*,\; x \in \mathfrak{h},$$

$$\mathrm{Tr}\,(\lambda y) = 0 \;\; for \;\; \lambda \in \mathfrak{h}^*,\; y \in \mathfrak{p}.$$

And the product of λ with an element of $\mathfrak{g}$ under projection to $\mathfrak{h}$ is the same as the product of λ with this element itself:

$$\mathrm{Tr}\,(\lambda\, \pi(x+y)) = \mathrm{Tr}\,(\lambda x) = \mathrm{Tr}\,(\lambda(x+y)) = \langle \lambda, x \rangle.$$

Thus the following direct computation proves our claim:

$$\begin{aligned}
\mathrm{Tr}\,\lambda(d\mathfrak{a} + \mathfrak{a}^2) &= \mathrm{Tr}\,\lambda d\mathfrak{a} \\
&= \mathrm{Tr}\,\lambda\pi(d(g^{-1}dg + g^{-1}\mathcal{A}g)) \\
&= \mathrm{Tr}\,\lambda\pi(g^{-1}F_{\mathcal{A}}g - \tfrac{1}{2}g^{-1}[\mathcal{A},\mathcal{A}]g - \tfrac{1}{2}[g^{-1}dg, g^{-1}dg] - [g^{-1}dg, g^{-1}\mathcal{A}g]) \\
&= \mathrm{Tr}\,\lambda\pi(g^{-1}F_{\mathcal{A}}g - (g^{-1}d_{\mathcal{A}}g)^2) \\
&= \mathrm{Tr}\,\lambda(g^{-1}F_{\mathcal{A}}g - (g^{-1}d_{\mathcal{A}}g)^2) \\
&= \mathrm{DP}_\lambda(\mathcal{A}, g).
\end{aligned}$$

Hence, we conclude

$$\int_\Sigma \mathrm{DP}_\lambda(\mathcal{A}, g) = \int_\Sigma \mathrm{Tr}\,\lambda(d\mathfrak{a} + \mathfrak{a}^2) = \int_\Sigma \sigma^* c_\lambda, \tag{3.34}$$

as required. Thus, our more sophisticated definition for a surface observable in the case of G non-abelian is structurally the same as the one in the case of $G = U(1)$. And the value of the observable can be identified with the first Chern number $c_1(Q)$ of the bundle $Q \subset P$ over Σ.

7 Application to 2-dimensional Yang-Mills theory

In this Section, we study the effects of adding a Wilson surface in the 2D Yang-Mills theory. Since this theory is exactly solvable [37, 38, 39, 40, 41, 46, 44, 45, 43, 42], one obtains explicit formulas for partition function in the presence of a Wilson surface.

7.1 2-dimensional Yang-Mills theory

Recall that the action of the Yang-Mills theory is given by

$$S_{YM}(A) = \frac{1}{4e^2}\int_\Sigma \mathrm{Tr}F_A * F_A, \tag{3.35}$$

where $*$ is the Hodge dual defined by the metric on the surface Σ. In the first order formalism, this action can be rewritten as

$$S(A,B) = \int_\Sigma \mathrm{Tr}(BF_A + \frac{e^2}{2}B^2 d^2\sigma), \tag{3.36}$$

where B is an auxiliary field and $d^2\sigma$ is the area element on Σ.

The canonical quantization of the theory on a cylinder $S^1 \times \mathbb{R}$ gives rise to the Hilbert space with a basis $\chi_R(w)$, where R runs through the set of irreducible representations of G, and

$$w = \mathrm{Pexp}\int_{S^1} A$$

is the holonomy of A around S^1. The partition function of the theory on an orientable surface of genus g with r boundary components reads

$$Z(\tau, w_1, \ldots, w_r) = \sum_R d_R^{2-2g-r} e^{-\tau C_2(R)} \chi_R(w_1)\ldots\chi_R(w_r). \tag{3.37}$$

Here $\tau = \frac{e^2}{2}\sigma$, σ the area of the surface, $w_1, \ldots, w_r$ are holonomies of A around the boundary components, the sum runs over all irreducible representations of G, and $C_2(R)$ is the quadratic Casimir in the representation R. In particular, for a closed surface one obtains

$$Z(\tau) = \sum_R d_R^{2-2g} e^{-\tau C_2(R)}. \tag{3.38}$$

For more details see e. g. [47, 48].

7.2 U(1)

As a warm up example, we consider the case $G = U(1)$. The partition function for 2D-YM with Wilson surface can be obtained through Hamiltonian formalism. The idea is to construct the new Hamiltonian which already contains the Wilson surface. The formula for the partition function would be:

$$Z_\lambda(\tau) = \mathrm{Tr}\, e^{-itH_\lambda}, \tag{3.39}$$

where H_λ is the Hamiltonian of the theory perturbed by a Wilson surface operator:

Let $w = e^{i\int_0^L dx A_1}$ be a holonomy around a curve of a constant time slice. The representations of $U(1)$ are labelled by integers $n \in \mathbb{Z}$. The characters of the representations are $\chi_n(w) = w^n$. Then the eigenvalues of the Hamiltonian are given by:

$$H_\lambda \chi_n(w) = \frac{e^2}{2} L(n-\lambda)^2 \chi_n(w),$$

where L is the length of the cylinder on which the quantization takes place. And the partition function becomes:

$$Z_\lambda(\tau) = \sum_{n\in\mathbb{Z}} e^{-i\tau(n-\lambda)^2}. \tag{3.40}$$

Notice that the label of the representation n gets a shift by $-\lambda$ due to the presence of the Wilson surface.

On the other hand, recall that in the case of $G = U(1)$ the Wilson surface observable $S(A)$ coincides with the first Chern class of the corresponding $U(1)$-bundle. Hence, after adding a factor $\exp(iS(A))$ in the definition of the partition function the calculation of the path integral yields

$$Z_\lambda(\tau) = \int DA\, e^{iS_{YM}+iS_\lambda} = \beta \sum_{m\in\mathbb{Z}} e^{i\pi^2 m^2/\tau + i2\pi\lambda m}. \tag{3.41}$$

where β is a constant and the meaning of the parameter m is the first Chern number of the $U(1)$-bundle over Σ.

A nice observation can be made that the partition function computed from the Hamiltonian formalism (the sum over the representations) is related through the Poisson resummation to the partition function obtained from the functional integral formalism (the sum over the first Chern number of the U(1)-bundles over Σ):

$$\sum_{n\in\mathbb{Z}} e^{-i\tau(n-\lambda)^2} = \sqrt{\frac{\pi}{i\tau}} \sum_{m\in\mathbb{Z}} e^{i\pi^2 m^2/\tau + i2\pi\lambda m}.$$

From this resummation we can deduce the value of the constant $\beta = \sqrt{\frac{\pi}{i\tau}}$.

7.3 G arbitrary

We shall use the Poisson σ-model action representing the Wilson surface observable. In the first order formalism, the total action reads

$$S = \int_\Sigma \mathrm{Tr}\left(BF_A + \frac{e^2}{2} B^2 d^2\sigma + bF_{A+\alpha} \right), \tag{3.42}$$

where $F_{A+\alpha} = d(A+\alpha) + (A+\alpha)^2$ is the field strength of the gauge field $(A+\alpha)$ calculated in equation (4.4). Hence, we obtain two non-interacting BF-theories. The first one has a Hamiltonian $\frac{e^2}{2}\mathrm{Tr}B^2$, and the field B can vary in $\mathfrak{g}^*$. The second one has vanishing Hamiltonian, and the field b takes values in the coadjoint orbit passing through λ. The Hilbert space of such a system has a basis

$$\psi_R(A,\alpha) = \chi_R(w(A))\chi_{R_\lambda}(w(A+\alpha)), \tag{3.43}$$

where R runs through irreducible representations of G, R_λ is the irreducible representation corresponding to the coadjoint orbit passing through λ, and

$$w(A) = \mathrm{Pexp}\int_{S^1} A\,, \qquad w(A+\alpha) = \mathrm{Pexp}\int_{S^1}(A+\alpha).$$

Since only the first BF-theory contributes in the Hamiltonian, its eigenvalue on $\psi_R(A,\alpha)$ is given by $C_2(R)$, as in the pure YM theory.

On a surface of genus g with r boundary components, the partition function reads

$$Z_\lambda(\tau,A,\alpha)) = d_{R_\lambda}^{-r}\sum_R d_R^{2-2g-r} e^{-\tau C_2(R)}\chi_R(w_1(A))\ldots\chi_R(w_r(A))\chi_{R_\lambda}(w_1(A{+}\alpha))\ldots\chi_{R_\lambda}(w_r(A{+}\alpha)) \tag{3.44}$$

In particular, for a closed surface we obtain exactly the same expression as for the pure YM theory:

$$Z_\lambda(\tau) = \sum_R d_R^{2-2g} e^{-\tau C_2(R)} = Z(\tau).$$

The Wilson surface observable which has no effect on the partition function is hardly of any interest. However, while considering the cases when R_λ is a projective representation of the gauge group G, we obtain nontrivial results. The difference is transparent in the $U(1)$ example discussed in the previous subsection. If in formulas (3.41) and (3.40) λ is an integral weight of the $U(1)$ representations, the shift of the Chern number in the exponential is not visible. If we allow λ to be a weight of a representation of $\mathbb{R}$ (that is, a projective representation of $U(1)$) the partition function changes. The same principle is applies to a more interesting example of the gauge group $SO(3)$ and its universal cover $SU(2)$ which we discuss in more detail in the next section.

7.4 SU(2) and SO(3) in Hamiltonian formalism

We now apply the general formalism of the previous section to the case of $G = SU(2)$ and $G = SO(3)$. We keep the notation $Z(\tau)$ for the partition function of the free 2YM theory and $Z_\lambda(\tau)$ for the theory with a Wilson surface of spin λ.

For **G=SU(2)**, irreducible representations are labeled by integer and half-integer spins j. The dimension of such a representation is $2j+1$, and the quadratic Casimir element is $\left(j+\frac{1}{2}\right)^2$. The corresponding formula for the partition function on the surface of genus g with r boundary components reads

$$\begin{aligned} Z_\lambda(\tau, A, \alpha) \quad = \quad & (2\lambda+1)^{-r} \sum_{j\in\mathbb{Z}^{\geq 0}/2} (2j+1)^{2-2g-r} e^{-\tau(j+1/2)^2} \times \\ & \times \chi_j(w_1(A))...\chi_j(w_r(A))\chi_\lambda(w_1(A+\alpha))...\chi_\lambda(w_r(A+\alpha)). \end{aligned} \tag{3.45}$$

Here the sum is over non negative integer and half-integer values of j, and the characters of irreducible representations are defined by formula

$$\chi_j \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} = \frac{c^{j+1/2} - c^{-(j+1/2)}}{c^{1/2} - c^{-1/2}}. \tag{3.46}$$

For the closed surface, we get the following result

$$Z_{\lambda, SU(2)}(\tau) = \sum_{j\in\mathbb{Z}^{\geq 0}/2} (2j+1)^{2-2g} e^{-\tau(j+1/2)^2} \tag{3.47}$$

which coincides with the partition function $Z(\tau)$ without Wilson surface.

In the case of **G=SO(3)**, the partition function for a closed surface may be calculated as a sum of contributions of the trivial and nontrivial bundles over Σ [49]:

$$Z_{\mathrm{SO(3)}}(\tau) = Z^{\mathrm{triv}}(\tau) + Z^{\mathrm{nontriv}}(\tau). \tag{3.48}$$

Here the contribution of the trivial bundle

$$Z^{\mathrm{triv}}(\tau) = \frac{1}{2} Z_{SU(2)}(\tau).$$

To obtain the contribution of the nontrivial representation, we consider the partition function of the surface with a disc removed, in such a way that one boundary component appears:

$$Z(\tau, w) = \sum_{j\in\mathbb{Z}^{\geq 0}/2} (2j+1)^{1-2g} e^{-\tau(j+1/2)^2} \chi_j(w), \tag{3.49}$$

and then put $w=-e$ (the nontrivial central element of $SU(2)$) to get

$$\begin{aligned} Z^{\mathrm{nontriv}}(\tau) \quad & = \quad \tfrac{1}{2} Z(\tau, -e) \\ & = \quad \tfrac{1}{2} \sum_{j\in\mathbb{Z}^{\geq 0}/2} (2j+1)^{1-2g} e^{-\tau(j+1/2)^2} \chi_j(-e) \\ & = \quad \tfrac{1}{2} \sum_{j\in\mathbb{Z}^{\geq 0}/2} (-1)^{2j} (2j+1)^{2-2g} e^{-\tau(j+1/2)^2}. \end{aligned} \tag{3.50}$$

The resulting partition function for $SO(3)$ yields

$$Z_{\mathrm{SO}(3)}(\tau) = \sum_{j \in \mathbb{Z}^{\geq 0}} (2j+1)^{2-2g} e^{-\tau(j+1/2)^2},$$

where the sum is now over non negative integer spins, as expected.

Now we apply the same procedure to a partition function with Wilson surface of spin λ. For the trivial part we obtain

$$Z_{\lambda}^{\mathrm{triv}}(\tau) = \frac{1}{2} \sum_{j \in \mathbb{Z}^{\geq 0}/2} (2j+1)^{2-2g} e^{-\tau(j+1/2)^2}, \tag{3.51}$$

as before. And the nontrivial contribution yields:

$$\begin{aligned} Z_{\lambda}^{\mathrm{nontriv}}(\tau) &= \tfrac{1}{2} Z_{\lambda}(\tau, -e, -e) \\ &= \tfrac{1}{2(2\lambda+1)} \textstyle\sum_{j \in \mathbb{Z}^{\geq 0}/2} (2j+1)^{1-2g} e^{-\tau(j+1/2)^2} \chi_j(-e) \chi_\lambda(-e) \\ &= \tfrac{(-1)^{2\lambda}}{2} \textstyle\sum_{j \in \mathbb{Z}^{\geq 0}/2} (-1)^{2j} (2j+1)^{2-2g} e^{-\tau(j+1/2)^2}. \end{aligned} \tag{3.52}$$

If the spin of the Wilson surface observable λ is an integer, summing up the two contributions reproduces the same answer as for the theory without Wilson surface. However, if λ is half-integer, we have $(-1)^{2\lambda} = -1$ and partition function is changed by the presence of the observable:

$$Z_{\lambda,\mathrm{SO}(3)}(\tau) = \sum_{j \in \frac{1}{2} + \mathbb{Z}^{\geq 0}} (2j+1)^{2-2g} e^{-\tau(j+1/2)^2}, \tag{3.53}$$

where the sum is over j which now take only half-integer values!

Thus the formula for the partition function in 2YM with a Wilson surface observable for the gauge group $G = SO(3)$ is given by

$$\begin{aligned} Z_{\lambda,\mathrm{SO}(3)}(\tau) &= \textstyle\sum_{j \in \mathbb{Z}^{\geq 0}} (2j+1)^{2-2g} e^{-\tau(j+1/2)^2}, \qquad \lambda \in \mathbb{Z}^{\geq 0}, \\ Z_{\lambda,\mathrm{SO}(3)}(\tau) &= \textstyle\sum_{j \in \frac{1}{2} + \mathbb{Z}^{\geq 0}} (2j+1)^{2-2g} e^{-\tau(j+1/2)^2}, \qquad \lambda \in \tfrac{1}{2} + \mathbb{Z}^{\geq 0}. \end{aligned} \tag{3.54}$$

Note, that we observe a shift of the representation labels j, similar to the $U(1)$ case formula (3.40).

7.5 Topological approach to SU(2) and SO(3) partition functions

In this Section, we would like to confirm by path integral computations the results obtained previously for $G = SO(3)$. Recall that for pure $SO(3)$ Yang-Mills theory on a closed surface we have

$$Z(\tau) = \int \mathcal{D}A\mathcal{D}g\, e^{iS_{YM}} = Z^{\text{triv}}(\tau) + Z^{\text{nontriv}}(\tau),$$

where on the right hand side we split the contributions of trivial and nontrivial $SO(3)$-bundles over the surface.

For the theory with Wilson surface observable, the partition function is

$$Z_{\lambda,SO(3)}(\tau) = \int \mathcal{D}A\mathcal{D}g\, e^{iS_{YM}+i\int DP_\lambda(A,g)}. \tag{3.55}$$

It turns out that the Wilson surface factor $e^{i\int DP_\lambda(A,g)}$ only depends on the topological type of the bundle and not on the particular gauge field choices on this bundle.

Recall the topological description of the observable given in Section 6 and assume that the field g defines a circle subbundle $Q \subset P$. Then according to the formula (3.34), the term of the action corresponding to the Wilson surface is

$$S_\lambda = \int DP_\lambda(A,g) = \int \operatorname{Tr} \lambda d\mathfrak{a} = 2\pi\lambda c_1(Q). \tag{3.56}$$

Here $\mathfrak{a} \in \Omega^1(Q, \mathrm{Lie}(S^1))$ is a connection 1-form on the subbundle Q taking values in the Lie algebra of S^1, $c_1(Q) = \frac{1}{2\pi}\int_\Sigma d\mathfrak{a}$ is the first Chern number of the S^1-subbundle over Σ and λ is the coefficient of the Wilson surface with the meaning of spin.

Note that for 2YM the $SO(3)$-bundle and the S^1-subbundle are over the same base-space Σ. Hence the $SO(3)$-bundle $P \to \Sigma$ is completely determined by its subbundle $Q \to \Sigma$. In more detail, since $S^1 \subset SO(3)$ we are allowed to use the transition functions $\phi \in S^1$ of Q as transition functions of P. The $SO(3)$-bundle is then given by $P = SO(3) \times_{S^1} Q$.

The transition function of Q is $\phi : S^1 \to S^1$, and the transition function of P is $\hat{\phi} : S^1 \to S^1 \to SO(3)$. It is obtained from ϕ by composition with the embedding of the maximal torus $S^1 \to SO(3)$. The equivalence class of the bundle P is determined by the homotopy class $[\hat{\phi}] \in \pi^1(SO(3)) = \mathbb{Z}_2 = \{+1, -1\}$. At the same time the equivalence class of the subbundle Q is determined by $[\phi] \in \pi^1(S^1) = \mathbb{Z}$ which corresponds to an integer winding number and hence to the first Chern number

This is a particular instance of the construction of relative bundles (see [50]) used to describe topological defects in [51].

$c_1(Q)$. The relation between the winding number and $c_1(Q)$ is transparent from the formula (3.4). The induced map between the bundles equivalence classes $\mathbb{Z} \to \mathbb{Z}_2$ maps even Chern numbers to the trivial element of $\mathbb{Z}_2$ and odd Chern numbers to the nontrivial one [52]. Then an $SO(3)$-bundle defined by an S^1-subbundle with an even Chern class is trivial, and an $SO(3)$-bundle defined by a subbundle with an odd Chern class is necessarily nontrivial.

Another interpretation of the map $\mathbb{Z} \to \mathbb{Z}_2$ between the bundles equivalence classes can be given. For Σ connected and orientable, an $SO(3)$-bundle $P \to \Sigma$ is completely determined, up to a bundle isomorphism, by its 2nd Stiefel-Whitney class $w_2 \in H^2(\Sigma, \mathbb{Z}_2) \simeq \mathbb{Z}_2$, while a $U(1)$-bundle Q is determined by its 1st Chern class $c_1 \in H^2(\Sigma, \mathbb{Z}) \simeq \mathbb{Z}$. For $P = Q \times_{U(1)} SO(3)$, these classes are related by $w_2(P) = c_1(Q) \mod 2$.

The Wilson surface factor in the functional integral then gives:

$$\begin{aligned} e^{i2\pi\lambda c_1(Q)} &= +1, && \text{P is trivial,} \\ e^{i2\pi\lambda c_1(Q)} &= +1, && \text{P is nontrivial, } \lambda \in \mathbb{Z}^{\geq 0}, \\ e^{i2\pi\lambda c_1(Q)} &= -1, && \text{P is nontrivial, } \lambda \in \tfrac{1}{2} + \mathbb{Z}^{\geq 0}. \end{aligned} \tag{3.57}$$

Note that the exponentials for surface observables for $\lambda = 1/2$ and for $\lambda = 1$ are related as follows:

$$e^{i\int DP_1(A,g)} = \left(e^{i\int DP_{1/2}(A,g)}\right)^2$$

or

$$e^{i\int DP_{1/2}(A,g)} = \sqrt{e^{i\int DP_1(A,g)}}.$$

That is, the surface observable for half-integer spin is a nontrivial square root of the observable for integer spin.

For λ half-integer the nontrivial part of the partition function acquires a factor -1 and becomes

$$Z_\lambda^{\text{nontriv}}(\tau) = -\frac{1}{2} \sum_{j \in \mathbb{Z}^{\geq 0}/2} (-1)^{2j} (2j+1)^{1-2g} e^{-\tau(j+1/2)^2}.$$

This leads to the following formula:

$$\begin{aligned} Z_{\lambda,SO(3)}(\tau) &= Z^{\text{triv}}(\tau) + Z^{\text{nontriv}}(\tau), && \lambda \in \mathbb{Z}^{\geq 0} \\ Z_{\lambda,SO(3)}(\tau) &= Z^{\text{triv}}(\tau) - Z^{\text{nontriv}}(\tau), && \lambda \in \tfrac{1}{2} + \mathbb{Z}^{\geq 0}. \end{aligned} \tag{3.58}$$

Summing up the trivial and nontrivial contributions reproduces exactly the result of the equation (3.54). For the $SU(2)$ gauge group the bundle P is necessarily trivial and the Wilson surface does not affect the 2YM partition function.

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Chapter 4

Quantum Wilson surfaces and topological interactions

Olga Chekeres

Abstract: We introduce the description of a Wilson surface as a 2-dimensional topological quantum field theory with a 1-dimensional Hilbert space. On a closed surface, the Wilson surface theory defines a topological invariant of the principal G-bundle $P \to \Sigma$. Interestingly, it can interact topologically with 2-dimensional Yang-Mills and BF theories modifying their partition functions. This gives a new interpretation of the results obtained in [1]. We compute explicitly the partition function of the 2-dimensional Yang-Mills theory interacting with a Wilson surface for the cases $G = SU(N)/\mathbb{Z}_m$, $G = Spin(4l)/(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ and obtain a general formula for any compact connected Lie group.

1 Introduction

The discussion of surface observables in gauge theories has been ongoing for quite some time. Wilson surfaces, domain walls, surface defects etc appear in many domains of physics and mathematics, from gauge theories to condensed matter. They

Keyword: Wilson surface, topological interactions, 2d Yang-Mills, gauge theories

have been studied extensively in literature [2, 3, 4, 5, 6, 7, 8]. In most cases a 1-dimensional observable, namely a Wilson line [9, 10, 11, 12, 13, 14, 15, 16, 17], is generalized to 2 dimensions by introducing higher gauge fields defined on surfaces. Our approah is different, it is based as well on a definition of a Wilson line, but doesn't involve introducing higher gauge fields.

The first result of this article is a formulation of the Wilson surface as a new 2-dimensional topological quantum field theory. We define its action, the Hilbert space and compute its partition function.

The basis for our construction is a 1-form standard gauge field taking values in a Lie algebra. A Wilson surface is defined by an orientable surface Σ and an irreducible representation R_λ of the gauge group G. In [1] we obtained its description as a 2-dimensional topological σ-model:

$$S_\lambda(a,b,A) = \int_\Sigma \mathrm{Tr}(b(d(A+a)+(A+a)^2)) = \int_\Sigma \mathrm{Tr}(bF_{A+a}), \tag{4.1}$$

where $\lambda \in \Lambda^*$ is the highest weight of the representation R_λ, b is a scalar field taking values in $\mathfrak{g}^*$ and constrained to be a conjugate of $\lambda \in \Lambda^*$, A is a background gauge field, and a is an auxiliary gauge field. Interpreting $A+a$ as a new gauge field allows us to interpret a Wilson surface as an independent 2-dimensional BF-theory with a constraint on the B-field.

In this article we start with the action functional (4.1) from [1] and canonically quantize it to obtain the partition function formula for the Wilson surface theory. To describe the formula we first recall some topological facts. Principal G-bundles $P \to \Sigma$ over a closed surface Σ are classified by the elements $\gamma \in \pi_1(G)$ in the fundamental group of the gauge group G [18, 19]. The gauge group can be represented as $G = \tilde{G}/\Gamma$, where $\tilde{G}$ is the universal cover of G and $\Gamma \subset Z(\tilde{G})$ is a proper subgroup of the center of $\tilde{G}$. Since $\Gamma \cong \pi_1(G)$, for every element $\gamma \in \pi_1(G)$ in the fundamental group there exists a corresponding element $C_\gamma \in \Gamma \subset Z(\tilde{G})$ in the center of the covering group. Then the partition function for a particular equivalence class of principal bundles $P \to \Sigma$, defined by $\gamma \in \pi_1(G)$, is given by:

$$Z^\Sigma_{WS}(C_\gamma,\lambda) = \frac{\chi_\lambda(C_\gamma)}{d_\lambda} = e^{i\varphi_\gamma} \in U(1), \tag{4.2}$$

where $\chi_\lambda(C_\gamma)$ is a value of the character χ_λ on the element C_γ and d_λ is the dimension of the representation. This is a 2-dimentional topological quantum field theory with a 1-dimensional Hilbert space.

Our second result is the description of topological interactions of Wilson surfaces with 2-dimensional topological gauge theories, e.g. with BF or Yang-Mills theories. Their partition functions on a surface Σ are obtained by summation over all the classes of principal G-bundles defined over the given surface [20, 21, 22]. When we insert a Wilson surface into 2-dimensional BF or Yang-Mills theory, it interacts topologically with the background gauge theory, as the gauge connections A and $A+a$ are defined on the same principal G-bundle $P \to \Sigma$. The presence of a Wilson surface modifies the partition function of the background theory multiplying by a phase (4.2) the individual contributions for each class of principal bundles:

$$Z^{interact} = \sum_{\gamma \in \pi_1(G)} Z^{backgr}(C_\gamma) \cdot e^{i\varphi_\gamma}. \tag{4.3}$$

In subsection 3.1 we describe explicitly the action and the partition function of 2-dimensional BF-theory interacting with the Wilson surface theory.

The formulas of the partition function for 2-dimentional Yang-Mills theory with an insertion of the Wilson surface observable were first obtained in [1] for the cases of $G = U(1), SU(2), SO(3)$. In subsection 3.2 of the current article we write the general formula for 2-dimensional Yang-Mills theory with a Wilson surface and give it a new interpretation of the topological interaction.

Since Yang-Mills theory in 2 dimensions is exactly solvable, we can compute explicitly the partition functions in the presence of a Wilson surface for concrete gauge groups. The examples of $G = U(1), SU(2), SO(3)$ were computed in [1]. In this article we compute the partition functions of 2-dimensional Yang-Mills theory with a Wilson surface for a larger class of gauge groups. In particular, we study in detail the examples of the gauge groups $G = SU(N)/\mathbb{Z}_m$ (m divides N) and $G = Spin(4N)/(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$.

In case of $G = SU(N)/\mathbb{Z}_m$ the fundamental group is $\pi_1(G) \cong \mathbb{Z}_m$ and the Wilson surface phase $e^{i\varphi_k}$ is defined by the angle:

$$\varphi_k = \frac{2\pi k}{m} \cdot [\lambda],$$

where $k = 0, 1, ..., m-1$ labels the elements of $\pi_1(G)$, and $[\lambda] \in \mathbb{Z}_m$ is an integer mod m denoting the equivalence class of the highest weight λ characterising the Wilson surface. For $G = Spin(4N)/(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ the fundamental group is $\pi_1(G) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and the Wilson surface is defined by the angle:

$$\varphi_{k_1,k_2} = \pi(k_1[\lambda_1] + k_2[\lambda_2]),$$

where a pair (k_1, k_2) labels the elements of $\pi_1(G)$, with $k_1, k_2 \in \{0, 1\}$, and $[\lambda_1], [\lambda_2] \in \mathbb{Z}_2$ are integers modulo 2 given by two linear combinations of the components of same highest weight λ characterizing the Wilson surface.

Eventually, we obtain the formula of the partition function for 2D-YM with a Wilson surface for any compact connected Lie group $G = \tilde{G}/\Gamma$. In this case the Wilson surface phase is defined by the angle:

$$\varphi_{k_1,...,k_i} = \sum_i k_i < \lambda, c_i >= 2\pi \sum_i \frac{k_i}{m_i}[\lambda_i].$$

Here we account for the most general case, when the fundamental group of G is given by a product of i cyclic groups: $\pi_1(G) = \mathbb{Z}_{m_1} \times ...... \times \mathbb{Z}_{m_i}$. Then m_i is the number of elements in $\mathbb{Z}_{m_i}$, the index $k_i = 0, ..., m_i - 1$ labels the elements in the i-th factor, $k_i c_i \in \mathfrak{h} \subset Lie(\tilde{G})$ is an element of the Cartan subalgebra such that $e^{i\sum_i k_i c_i} = C_{k_1,...,k_i} \in \Gamma \cong \pi_1(G)$ is a central element of the covering group $\tilde{G}$, $\lambda \in \mathfrak{h}^*$ is the highest weight of the irreducible representation of G characterizing the Wilson surface, $<,>$ is the invariant scalar product defined on $Lie(\tilde{G})$ and $[\lambda_i] \in \mathbb{Z}_{m_i}$ are integers modulo m_i given by i linear combinations of the components of same highest weight λ.

For a closed surface the Wilson surface is nontrivial for G non-simply connected, and it is not visible ($e^{i\varphi} = 1$) for G simply connected. Also the value of λ plays a role: for λ being the highest weight of a representation of the gauge group G itself the Wilson surface is trivial, and it is nontrivial if λ labels a representation of the universal cover $\tilde{G}$ which does not descend to G. On a closed surface, the partition function of the Wilson surface is a topological invariant of the principal G-bundle.

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2 Wilson surface theory

2.1 Wilson surface observables

Recall the construction of Wilson surface observables from [1]. Let G be the gauge group, $\mathfrak{g}$ its Lie algebra, $(x, y) \to \mathrm{Tr}(xy)$ an invariant scalar product on $\mathfrak{g}$ and P a principal G-bundle over a surface Σ. We denote by $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra and by $\Lambda^* \subset \mathfrak{h}^*$ the weight lattice.

A Wilson surface observable is described by an auxiliary 2-dimensional gauge theory on the surface Σ. The fields in this theory are a $\mathfrak{g}^*$-valued scalar field b and a $\mathfrak{g}$-valued 1-form a. The action depends on the following data: the background gauge field A and the weight $\lambda \in \Lambda^*$. For a trivial G-bundle it is given by

$$\begin{array}{rcl} S_\lambda(a,b,A) & = & \int_\Sigma \mathrm{Tr}(b(F_A - (dgg^{-1}+A)^2 + (dgg^{-1}+A+a)^2) \\ & = & \int_\Sigma \mathrm{Tr}(b(d(A+a)+(A+a)^2)), \end{array} \tag{4.4}$$

where we identified $\mathfrak{g}^*$ with $\mathfrak{g}$ using the scalar product and integrated by parts using the equality $\mathrm{Tr} b[dgg^{-1}, a] = \mathrm{Tr}[b, dgg^{-1}]a = -\mathrm{Tr}(db)a$. The field $b = g\lambda g^{-1}$ belongs to the same conjugacy class as the fixed element λ, the combination $A + a$ is a new gauge field. The fields obey the following gauge transformation rules: by $h : \Sigma \to G$ they transform as

$$g \mapsto hg\,,\ A \mapsto hAh^{-1} - dhh^{-1}\,, b \mapsto hbh^{-1}\,,\ a \mapsto hah^{-1}.$$

Note that integrating out a in (4.4) yields the Diakonov-Petrov action [23, 1] for a Wilson surface:

$$S_{DP} = \int_\Sigma \mathrm{Tr}(b(F_A - (dgg^{-1}+A)^2). \tag{4.5}$$

The construction (4.4) also works on nontrivial bundles. $A \in \Omega^1(P, \mathfrak{g})$ is the connection on P, its curvature $F_A \in \Omega^2_{hor}(P, \mathfrak{g})^G$ is a horizontal 2-form taking values in $\mathfrak{g}$. The auxiliary gauge field a is such that $a \in \Omega^1_{hor}(P, \mathfrak{g})^G$ and the sum $A+a$ defines a new connection on P with a curvature $F_{A+a} = d(A+a)+(A+a)^2$, $F_{A+a} \in \Omega^2_{hor}(P, \mathfrak{g})^G$. The field b takes values in $\Omega^0_{hor}(P, \mathfrak{g}^*)^G$. The combination $\mathrm{Tr}(bF_{A+a})$ is then a basic 2-form which decends to Σ. One can show this in the following way. The 2-form bF_{A+a} is G-equivariant, i.e. under the gauge transformations it transforms as $b^h F_{A^h+a^h} = hbF_{A+a}h^{-1}$, yeilding $\mathrm{Tr}(b^h F_{A^h+a^h}) = \mathrm{Tr}(bF_{A+a})$. Acting by the contraction we obtain:

$$\imath_{\xi^\sharp}\mathrm{Tr}(bF_{A+a}) = \imath_{\xi^\sharp}\mathrm{Tr}\left(b(F_A+a^2+dgg^{-1}a+adgg^{-1}+Aa+aA)\right) = \mathrm{Tr}\left(b(-\xi a+a\xi+\xi a-a\xi)\right) = 0,$$

where $\xi \in \mathfrak{g}$ induces the fundamental vector field $\xi^\sharp \in \mathfrak{X}(\Sigma)$, and we have used that $\imath_{\xi^\sharp}(dg) = -\xi g$, $\imath_{\xi^\sharp} A = \xi$ (by definition of connection), $\imath_{\xi^\sharp} F_A = 0$, $\imath_{\xi^\sharp} a = 0$ (F_A and a are horizontal). This computation proves that G-invariant form $\mathrm{Tr}(bF_{A+a})$ is horizontal, and hence basic.

The meaning of $\lambda \in \Lambda^*_+$ is as follows. The integral weights of the representations of G form the weight lattice $\Lambda^* \subset \mathfrak{h}^*$. Dominant integral weights $\Lambda^*_+ \subset \Lambda^*$ are in one to one correspondence with irreducible representations of G [24, 25]. The element $\lambda \in \Lambda^*_+$ is the highest weight of some representation of G, it is a parameter characterising the Wilson surface. For example, for $G = SU(2)$ or $G = SO(3)$ we talk about a Wilson surface of spin λ. Note that in case when G is not simply connected but is a quotient $G = \tilde{G}/\Gamma$, where $\tilde{G}$ is its universal cover and $\Gamma \subset Z(\tilde{G})$ is a subgroup of the center of $\tilde{G}$, the weight lattices are related as $\Lambda^*_G \subset \Lambda^*_{\tilde{G}}$. A representation of $\tilde{G}$ can be considered as a projective representation of G, and λ is allowed to take values in $\Lambda^*_{\tilde{G}}$, and, as we will see later, these are exactly the values which describe the presence of nontrivial Wilson surfaces.

2.2 Quantum Wilson surfaces

The action for a Wilson surface (4.4) was first constructed in [1] to represent a surface observable within some other gauge theory. In the current paper we define it to be a separate 2-dimensional quantum field theory. Its action is a 2-dimensional BF theory [26] on Σ, with respect to the new connection $A + a$:

$$S_\lambda(a, b, A) = \int_\Sigma \mathrm{Tr}(bF_{A+a}). \tag{4.6}$$

There is just one subtlety which distinguishes the Wilson surface theory from an ordinary BF-theory: its b-field, as described in the previous subsection, is restricted to take values in the coadjoint orbit of λ, $\mathcal{O}_\lambda \subset \mathfrak{g}^*$. This affects, as we will see later, the definition of the Hilbert space of the Wilson surface and the structure of its partition function. Nevertheless, as a starting point for our formulation we recall canonical quantization of an ordinary BF-theory on a surface and the derivation of its partition function first done by Migdal in [27] and then discussed in more detail by Witten in [20, 21].

For simplicity let first Σ be a cylinder C, the G-bundle P will be necessarily trivial. We chose space and time coordinates (x, t) in a way that the boundary of C is given by two closed curves γ_1 and γ_2, situated on equal time slices, and x is a periodic coordinate of period L. We associate to γ_i a gauge invariant wave function $\psi(A)$ which is a function of holonomy U_i of A around γ_i:

$$\psi[U_i] = \psi[Pe^{\int_{\gamma_i} A}].$$

The Hilbert space $\mathcal{H}_\gamma$ of such a theory is given by G-invariant L^2 functions on G. $\mathcal{H}_\gamma$ admits a natural basis in terms of characters of representations, and any wave function $\psi(A) \in \mathcal{H}_\gamma$ has an expansion in characters $\chi_R(U)$, where R is an irreducible

More precisely, in [27] Migdal derives the formulas for 2-dimensional Yang-Mills theory. Then in [20, 21] Witten describes this derivation in more detail and considers the case of the topological 2-dimensional Yang-Mills (action 2.14 in [20]) taking the vanishing coupling constant limit, which is equivalent to 2-dimensional BF-theory.

representation. The boundaries are oriented, the wave functions on the incoming and outgoing boundary components are denoted by $\overline{\chi_R}(U_i)$ and $\chi_R(U_j)$ respectively.

In BF-theory the Hamiltonian vanishes (as expected for a topological field theory), so the partition function on a cylinder (equation 4.12 in [21]) reduces to:

$$Z^C_{BF}(U_1, U_2) = \sum_R \overline{\chi_R}(U_1)\chi_R(U_2). \tag{4.7}$$

For a generic surface with genus g and r boundary components the BF partition function (equation 2.68 in [20]) reads:

$$Z^\Sigma_{BF}(U_1, ..., U_r) = \sum_R d_R^{2-2g-r}\chi_R(U_1)...\chi_R(U_r), \tag{4.8}$$

where d_R is the dimension of the irreducible representation R and all the boundaries are chosen to be outgoing.

To obtain the formula for a closed surface we proceed as follows. The partition function will necessarily depend on the equivalence class of G-bundle over Σ. Recall the classification of principal G-bundles over Σ by the elements of the fundamental group of G: $\pi_1(G) \cong \Gamma \subset Z(\tilde{G})$. Consider a surface with just one puncture, i.e. one boundary component. Gluing this puncture to an infinitesimal disc yeilds a closed surface, and this operation is described by identifying $U = C_i$, where $C_i \in \Gamma$ is a central element of $\tilde{G}$.

The contribution to the partition function of each class $[P]$ of a principal G-bundle over the surface is given by (equation 4.26 in [21]):

$$Z^\Sigma_{BF}(C_i) = \sum_R d_R^{1-2g}\chi_R(C_i). \tag{4.9}$$

The total partition function for BF-theory on a closed surface Σ is then a sum over equivalence classes of principal G-bundles over Σ (equation 4.27 in [21]):

$$Z^\Sigma_{BF} = \frac{1}{\#\Gamma}\sum_{C_i\in\Gamma}\sum_R d_R^{1-2g}\chi_R(C_i), \tag{4.10}$$

where $\#\Gamma$ is the cardinality of Γ.

Note that the sum over R in (4.10) converges only for surfaces Σ with genus $g > 1$.

Using the tools from [20, 21] described above we can now define our Wilson surface theory. It is a very simple 2-dimensional gauge theory and in what follows we are going to explain how to construct its partition function. In contrast with BF-theory, the $\mathfrak{g}^*$-valued field $b = g\lambda g^{-1}$ is now the conjugation of the same fixed element

$\lambda \in \Lambda^*_{\tilde{G}}$. The Hilbert space becomes one-dimensional choosing just one irreducible representation R_λ, and the partition function is just a phase. The normalization of the states is such that $|| < \chi_\lambda(U)|\chi_\lambda(U) > ||^2 = 1$.

Any orientable surface can be obtained by gluing together some elementary components, namely discs and pairs of pants. So it is sufficient to define the states corresponding to these elementary components and then the definition of the Wilson surface for any orientable surface will follow. A natural way to describe the state corresponding to a disc would be to assign a character of holonomy U of the connection $A+a$ to the boundary of the disc and a character of the identity element e of the holonomy group to the second "no-boundary" of the disc. This formulation was chosen by Witten in [20, 21], and it would give the following:

$$Z^{disc}_o(U,\lambda) = \chi_\lambda(U)\overline{\chi_\lambda}(e), \tag{4.11}$$

where the orientation is chosen in a way that U is a holonomy of the connection $A+a$ around an outgoing boundary, and e is a holonomy around an incoming boundary. In contrast with BF-theory this formulation is not good for our Wilson surface with a 1-dimensional Hilbert space for the following reason: the value of the character of the identity $\overline{\chi_\lambda}(e)$ yields the dimension of the representation, so the state $Z^{disc}_o(U,\lambda)$ will not satisfy the normalization condition $|| < \chi_\lambda(U)|\chi_\lambda(U) > ||^2 = 1$. This motivates us to introduce a normalization factor $\frac{1}{d_\lambda}$ into the definition of the disc state:

$$Z^{disc}_{WS}(U,\lambda) = \chi_\lambda(U)\frac{\overline{\chi_\lambda}(e)}{d_\lambda} = \chi_\lambda(U), \tag{4.12}$$

where d_λ is the dimension of the irreducible representation corresponding to the highest weight λ.

The Wilson surface on a pair of pants is naturally defined by the formula:

$$Z^{p-o-p}_{WS}(U_1,U_2,U_3,\lambda) = \overline{\chi_\lambda}(U_1)\chi_\lambda(U_2)\chi_\lambda(U_3), \tag{4.13}$$

where U_1, U_2, U_3 are holonomies of $A+a$ around one incoming and 2 outgoing boundary components.

Then any other orientable surface can be obtained by gluing those components together, i.e. assigning the same holonomy element U_i to the incoming and outgoing boundaries which we want to glue, and integrating over it using the property of the characters:

$$\int dU_i\overline{\chi_\lambda}(U_i)\chi_\lambda(U_i) = 1.$$

The partition function for a surface of arbitrary genus with r boundary components is thus a product of r states living on the boundaries (here chosen to be outgoing):

$$Z^{\Sigma}_{WS}(U_1,....,U_r,\lambda) = \chi_\lambda(U_1)......\chi_\lambda(U_r).$$

To get the expression for a closed surface we start with a surface with one boundary component $Z^{\Sigma}_{WS}(U_1, \lambda) = \chi_\lambda(U_1)$ and glue it to a disc $Z^{disc}_{WS}(U_1, \lambda) = \overline{\chi_\lambda(U_1)}\frac{\chi_\lambda(e)}{d_\lambda}$, the factor $\chi_\lambda(U_1)$ disappears after the integration. But here there is a subtlety: unlike over an open surface, a principal G-bundle over a closed surface in general can be nontrivial, so, keeping the normalization factor $\frac{1}{d_\lambda}$, we replace the identity element of holonomy e by a central element C_i corresponding to a particular class $[P]$ of the bundle $P \to \Sigma$. This yields the formula of the Wilson surface theory for a closed surface for a class $[P]$ of principal bundles $P \to \Sigma$:

$$Z^{\Sigma}_{WS}(C_i, \lambda) = \frac{\chi_\lambda(C_i)}{d_\lambda}. \tag{4.14}$$

For a nontrivial central element C_i this is an element of $U(1)$: $Z^{\Sigma}_{WS}(C_i, \lambda) \in U(1)$. In case when P is a trivial bundle, the Wilson surface is always trivial: $Z^{\Sigma}_{WS}(P_{triv}, \lambda) = 1$.

3 Topological interactions with 2-dimensional gauges theories

In general case our observable could be understood as a surface defect embedded into a higher dimensional space-time. But in this paper we want to test it in the context of 2-dimensional gauge theories, which can be solved exactly. In this case the Wilson surface is a “global” observable, defined on the entire 2-dimensional space-time Σ.

3.1 BF-theory with a Wilson surface

The action functional for BF-theory with a Wilson surface is:

$$S^{\lambda}_{BF}(A, a, B, b) = \int_\Sigma \mathrm{Tr}(BF_A) + \int_\Sigma \mathrm{Tr}(bF_{A+a}), \tag{4.15}$$

where $B \in \Omega^0(P, \mathfrak{g}^*)^G$, $A \in \Omega^1(P, \mathfrak{g})$ is the background gauge field, $F_A \in \Omega^2_{hor}(P, \mathfrak{g})^G$ its curvature.

These two BF-theories are not completely independent, they interact topologically: the connections A and $A + a$ are defined on the same principal G-bundle, so the characters $\chi_R(U(A))$ and $\chi_\lambda(U(A + a))$ are taken on the same central element.

Then the partition function for a closed surface with a Wilson surface of weight λ is obtained by taking a product of partition functions (4.9), (4.14) defined in the previous section for each class $[P]$ and then summing over all the equivalence classes:

$$Z^{\lambda}_{BF} = \frac{1}{\#\Gamma} \sum_{C_i \in \Gamma} \frac{\chi_\lambda(C_i)}{d_\lambda} \sum_R d_R^{1-2g} \chi_R(C_i). \tag{4.16}$$

3.2 2D-YM theory with a Wilson surface

In this subsection we derive of the partition function for 2D-YM with a Wilson surface observable. In [1] the partition function formulas were obtained for the cases of $U(1)$, $SU(2)$ and $SO(3)$ gauge groups. Now we quantize the whole system of 2-dimensional Yang-Mills theory with a Wilson surface inserted. Also we are giving to the presence of the Wilson surface a new interpretation: it is viewed as a field theory interacting with the background Yang-Mills theory through topology of the principal bundle.

Recall 2D-YM in the first order formalism. The action functional for the theory with a Wilson surface [1] is:

$$S^{\lambda}_{YM}(A, a, B, b) = \int_\Sigma \mathrm{Tr}(BF_A + \frac{e^2}{2} B^2 d^2\sigma) + \int_\Sigma \mathrm{Tr}(bF_{A+a}), \tag{4.17}$$

where B is an auxiliary field taking values in $\mathfrak{g}^*$ and $d^2\sigma$ is the area element on Σ. Again, we see that the action splits into two theories interacting topologically through the connections A and $A + a$ defined on the same principal bundle. The partition function for each class $[P]$ will be a product of the partition function for 2D-YM and the partition function for the Wilson surface.

Canonical quantization of this interacting theory goes as follows. The Hamiltonian of the first theory is $H = \frac{e^2}{2}\mathrm{Tr}B^2$. The Hamiltonian of the second theory vanishes. The basis for the Hilbert space, as it was described in [1], is given by gauge invariant functions $\psi_R(A, a) = \chi_R(U(A))\chi_\lambda(U(A + a))$, where R runs through the irreps of G, λ chooses one irrep of G, and $U(A)$, $U(A + a)$ are holonomies of the connections A and $A + a$ respectively. The eigenvalues of the Hamiltonian on $\psi(A, a)$ are given by quadratic Casimir $C_2(R)$ of the representation R, just like for the 2D-YM without Wilson surface, as only the Hamiltonian of 2D-YM contributes to the total theory.

Then the time evolution operator takes value $e^{-\tau C_2(R)}$ on the functions $\psi_R(A, a)$, where $\tau = \frac{e^2}{2}\sigma$ absorbs the YM coupling constant e^2 and the area of the surface σ.

First, we obtain the partition function for each class $[P]$ which is a product of a Wilson surface phase (4.14) and the 2D-YM partition function $Z_{YM}(\tau) = \sum_R d_R^{1-2g} e^{-\tau C_2(R)} \chi_R(C_i)$. Then the total formula of the partition function for a closed surface with a Wilson surface of weight λ is given by the sum over all the equivalence classes $[P]$:

$$Z^{\lambda}_{YM}(\tau) = \frac{1}{\#\Gamma}\sum_{C_i\in\Gamma}\sum_{R} d_R^{1-2g} e^{-\tau C_2(R)}\chi_R(C_i)\frac{\chi_\lambda(C_i)}{d_\lambda}. \tag{4.18}$$

4 Exact results for 2D-YM theory interacting with a Wilson surface

Yang-Mills theory in 2 dimensions is exactly solvable [27, 28, 29, 30, 31, 32, 33, 34, 35, 36], this allows us to obtain explicit formulas for the partition function in the presence of a Wilson surface. In [1] we computed the partition functions for 2D Yang-Mills with a Wilson surface for the gauge groups $U(1)$, $SU(2)$ and $SO(3)$. Now we are going to generalize this result to G being any compact connected Lie group.

4.1 SU(N) and the groups covered by SU(N)

To visualize the result of topological interactions with a Wilson surface, we first perform a detailed computation for the case of $\tilde{G} = SU(N)$. The center of $SU(N)$ is given by: $Z(SU(N)) = \{e^{\frac{2\pi i k}{N}} Id_N \,|\, k = 0, ..., N-1\} = \mathbb{Z}_N$. And the subgroups of the centre are $\Gamma = \mathbb{Z}_m$ where m divides N. We consider $G = SU(N)/\mathbb{Z}_m$.

The rank of $SU(N)$ is equal to $N-1$, i.e. the basis of Cartan subalgebra $\mathfrak{h}$ has $N-1$ elements. In the defining presentation the basis of $\mathfrak{h}$ is given by: $h_n = \mathrm{diag}(0......., 1, -1, ...0)$ with matrix elements $h_{nn} = 1$, $h_{n+1,n+1} = -1$. Then any element $h \in \mathfrak{h}$ can be represented in terms of the basis as $h = \sum_{n=1}^{N-1} a_n h_n$, with $a_n \in \mathbb{R}$ linear coefficients. Exponentiating elements $h \in \mathfrak{h}$ we obtain the maximal torus of $SU(N)$: $H = e^{ih} = e^{i\sum a_n h_n} = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta_1}, \mathrm{e}^{\mathrm{i}\theta_2}, ..., \mathrm{e}^{\mathrm{i}\theta_{\mathrm{N}-1}}, \mathrm{e}^{-\mathrm{i}\sum_{\mathrm{i}=1}^{\mathrm{N}-1}\theta_\mathrm{i}}) \in \mathrm{T}$.

The center $Z(SU(N))$, and hence its proper subgroup $\Gamma \subset Z(SU(N)) \subset T$, is a subgroup of the maximal torus: $\Gamma \ni C_k = e^{i\theta_k} Id_N \in T$, with $\theta_k = 2\pi k/m$.

Consider the elements of the Cartan subalgebra $c_k = \mathrm{diag}(\theta_\mathrm{k}, \theta_\mathrm{k}, ..., -(\mathrm{N}-1)\theta_\mathrm{k})_{\mathrm{N}\times\mathrm{N}} \in \mathfrak{h}$, such that $C_k = e^{ic_k} \in Z(SU(N))$. In terms of the basis of $\mathfrak{h}$ they are given as follows:

$c_k = \mathrm{diag}(\theta_\mathrm{k}, \theta_\mathrm{k}, ..., -(\mathrm{N}-1)\theta_\mathrm{k}) = \theta_\mathrm{k}\cdot\mathrm{diag}(1, 1, ..., 1, -(\mathrm{N}-1)) = \theta_\mathrm{k}\cdot\sum_{\mathrm{n}=1}^{\mathrm{N}-1}\mathrm{n}\mathrm{h}_\mathrm{n}$.
Then the central elements of $SU(N)$ are given by $C_k = e^{i\theta_k\sum_{n=1}^{N-1} nh_n} \in Z(SU(N))$.

The irreducible representations of $SU(N)$ are labeled by highest weights with $N-1$ independent elements: $\mu = (\mu_1, ..., \mu_{N-1})$.

The central elements in the representation R_μ of highest weight μ are obtained as: $R_\mu(e^{i\theta_k\sum_{n=1}^{N-1} nh_n}) = e^{i\theta_k\sum_{n=1}^{N-1} nR_\mu(h_n)}$.

The natural choice for the basis of R_μ is in terms of the weight vectors. In this basis $R_\mu(h_n)$ are diagonal and yield weights while acting on the basis vectors. A central element C_k is a multiple of identity, therefore $R_\mu(C_k)$ has to be a multiple of identity as well, so it's enough to compute it just on the highest weight vector:

$$R_\mu(C_k) = e^{i\frac{2\pi k}{m}\sum_{n=1}^{N-1} n\mu_n} \cdot Id_{d_{R_\mu}}. \tag{4.19}$$

The linear combination $\sum_{n=1}^{N-1} n\mu_n$ is an integer, but the expression (4.19) depends only on the value of this sum modulo m, as $e^{i\frac{2\pi k}{m}\sum_{n=1}^{N-1} n\mu_n} = e^{i\frac{2\pi k}{m}(\sum_{n=1}^{N-1} n\mu_n + m)}$. This allows us to define the equivalence classes of the highest weight μ:

$$[\mu] \equiv [\sum_{n=1}^{N-1} n\mu_n] \in \mathbb{Z}_m. \tag{4.20}$$

Note that the irreps of $\tilde{G}$ descend to the irreps of G if $\sum_{n=1}^{N-1} n\mu_n = 0 \mod m$. In terms of weight lattices $\Lambda^*_G \subset \Lambda^*_{\tilde{G}} \subset \mathfrak{h}^*$, where $\Lambda^*_{\tilde{G}}$ is the weight lattice for $SU(N)$, Λ^*_G is the weight lattice for $G = SU(N)/\mathbb{Z}_m$.

The characters of the central elements in the representations R_μ are as follows:

$$\chi_{R_\mu}(C_k) = Tr(e^{i\frac{2\pi k}{m}\sum_{n=1}^{N-1} n\mu_n} \cdot Id_{d_{R_\mu}}) = d_{R_\mu} \cdot (e^{i\frac{2\pi k}{m}})^{[\mu]}. \tag{4.21}$$

We keep the notation $Z_{YM}(\tau)$ for the partition function of the free 2D-YM theory and $Z^\lambda_{YM}(\tau)$ for the theory with a Wilson surface of the highest weight λ.

Without Wilson surface the partition function for $SU(N)/\mathbb{Z}_m$ is given by

$$Z_{YM}(\tau) = \frac{1}{m}\sum_{k=0}^{m-1}\sum_{R_\mu(SU(N))} {d_{R_\mu}}^{1-2g} e^{-\tau C_2(R_\mu)} \chi_{R_\mu}(C_k), \tag{4.22}$$

where the sum is over the representations R_μ of $SU(N)$, and k labels central elements in the subgroup Γ.

The computation gives the following result:

$$\begin{aligned} Z_{YM}(\tau) &= \tfrac{1}{m}\textstyle\sum_{k=0}^{m-1}\sum_{R_\mu(SU(N))} d^{2-2g}_{R_\mu} e^{-\tau C_2(R_\mu)} (e^{i\frac{2\pi k}{m}})^{[\mu]} \\ &= \textstyle\sum_{R_\mu(SU(N))} d^{2-2g}_{R_\mu} e^{-\tau C_2(R_\mu)} \tfrac{1}{m}\sum_{k=0}^{m-1} (e^{i\frac{2\pi k}{m}})^{[\mu]}, \end{aligned} \tag{4.23}$$

where the sum over k is equal to m for $[\mu] = 0$ and zero otherwise. The condition corresponds to those representations of $SU(N)$ in which the central elements are all trivial, that is to the representations of $SU(N)/\mathbb{Z}_m$:

$$Z_{YM}(\tau) = \sum_{R_\mu(G=SU(N)/\mathbb{Z}_m)} d^{2-2g}_{R_\mu} e^{-\tau C_2(R_\mu)}. \tag{4.24}$$

Now let us introduce a Wilson surface of weight λ:

$$\begin{aligned} Z^{\lambda}_{YM}(\tau) &= \tfrac{1}{\#\Gamma}\sum_{k=0}^{m-1}\sum_{R_\mu(SU(N))} d_{R_\mu}^{1-2g} e^{-\tau C_2(R_\mu)} \chi_{R_\mu}(C_k)\tfrac{\chi_\lambda(C_k)}{d_\lambda} \\ &= \tfrac{1}{m}\sum_{k=0}^{m-1}\sum_{R_\mu(SU(N))} d_{R_\mu}^{1-2g} e^{-\tau C_2(R_\mu)} \tfrac{d_\lambda\cdot(e^{i\frac{2\pi k}{m}})^{[\lambda]}}{d_\lambda} d_R\cdot(e^{i\frac{2\pi k}{m}})^{[\mu]} \\ &= \tfrac{1}{m}\sum_{k=0}^{m-1}\sum_{R_\mu(SU(N))} d_{R_\mu}^{2-2g} e^{-\tau C_2(R_\mu)} (e^{i\frac{2\pi k}{m}})^{[\mu]+[\lambda]}, \end{aligned} \tag{4.25}$$

where $[\lambda] = [\sum_{n=1}^{N-1} n\lambda_n] \in \mathbb{Z}_m$ are equivalence classes of the Wilson surface weight λ. In more detail, we consider a quotient map $\Lambda^*_{\tilde{G}} \ni \lambda \mapsto (\sum_{n=1}^{N-1} n\lambda_n)_{mod\, m} \in \mathbb{Z}_m$ and the highest weights for Wilson surfaces will belong to equivalence classes $[\lambda] \in \Lambda^*_{\tilde{G}}/\Lambda^*_G \cong \mathbb{Z}_m$. Note that in case when $\lambda \in \Lambda^*_G$, i.e. $\sum_{n=1}^{N-1} n\lambda_n = 0 \mod m$, the Wilson surface is not visible: $Z^\lambda(\tau) = Z(\tau)$.

The sum over k in (4.25) is different from zero only for $[\mu]+[\lambda]=0$, and the partition function formula for 2D-YM with a Wilson surface yields:

$$Z^{\lambda}_{YM}(\tau) = \sum_{\mu\in\Lambda^*_{\tilde{G}},\,[\mu+\lambda]=0} dim_{R_\mu}^{2-2g} e^{-\tau C_2(R_\mu)} = \sum_{\mu\in[-\lambda]} dim_{R_\mu}^{2-2g} e^{-\tau C_2(R_\mu)},$$

where the sum now goes over the representations $R_{\mu+\lambda}$ of $G = SU(N)/\mathbb{Z}_m$ of highest weights $\mu+\lambda$, i.e. $[\mu] = [-\lambda] \in \Lambda^*_{\tilde{G}}/\Lambda^*_G$.

Note that in case when $G = SU(N)$, i.e. the gauge group is simply connected, the presence of the Wilson surface makes no impact on the partition function. Let us look at this situation in more detail. There is just one class of principal $SU(N)$-bundles over a surface Σ – trivial $SU(N)$-bundle. The $SU(N)$ partition function without Wilson surface is given by:

$$Z^{SU(N)}_{YM}(\tau) = \sum_{R_\mu} d_{R_\mu}^{1-2g} e^{-\tau C_2(R_\mu)} \chi_R(e) = \sum_{R_\mu} d_{R_\mu}^{2-2g} e^{-\tau C_2(R_\mu)}, \tag{4.26}$$

where e is the identity.

And adding a Wilson surface of weight λ leaves the partition function unchanged:

$$Z^{\lambda,SU(N)}_{YM}(\tau) = \sum_{R_\mu} d_{R_\mu}^{1-2g} e^{-\tau C_2(R_\mu)} \chi_R(e)\frac{\chi_\lambda(e)}{d_\lambda} = \sum_{R_\mu} d_{R_\mu}^{2-2g} e^{-\tau C_2(R_\mu)}. \tag{4.27}$$

4.2 Generalization for any compact connected Lie group

The result explained in the explicit example of the previous section remains valid for all compact connected Lie groups. All of them (with the exception of exceptional ones) have as a universal cover one of the following groups: $SU(N)$, $Spin(N)$, $Sp(N)$ and can be obtained by taking a quotient by a subgroup Γ of the center.

The case of $\tilde{G} = SU(N)$ has been discussed in the previous section. For $Spin(N)$, $N \geq 3$ the data is as follows:

$$Z(Spin(N)) = \begin{cases} \mathbb{Z}_2 & \text{if } N = 2l+1,\ \Gamma = \mathbb{Z}_2, \\ \mathbb{Z}_4 & \text{if } N = 4l+2,\ \Gamma = \mathbb{Z}_2 \text{ or } \Gamma = \mathbb{Z}_4, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } N = 4l,\quad \Gamma = \mathbb{Z}_2 \quad \text{or} \quad \Gamma = \mathbb{Z}_2 \oplus \mathbb{Z}_2. \end{cases} \tag{4.28}$$

The group $Sp(N)$ has the center $Z(Sp(N)) = \mathbb{Z}_2$.

Among the exceptional groups only E_6 and E_7 are interesting for our purposes, the rest of them (G_2, F_4 and E_8) are simply connected and have a trivial center. The real compact forms of E_6 and E_7 are not simply connected. The universal cover of E_6 has the center $Z(\tilde{E}_6) = \mathbb{Z}_3$, and the universal cover of E_7 has the center $Z(\tilde{E}_7) = \mathbb{Z}_2$.

We consider the gauge group $G = \tilde{G}/\Gamma$. The center of the cover $Z(\tilde{G}) \subset T$ is a subgroup of the maximal torus. T is given by the elements $H = e^{ih} \in T$, where $h \in \mathfrak{h}$ is in the Cartan subalgebra.

The irreducible representations of $\tilde{G}$ are labeled by highest weight with n independent elements, where n is the rank of $\tilde{G}$: $(\mu_1, ..., \mu_n)$.

In most cases the center of $\tilde{G}$, or its proper subgroup Γ, is given by $\mathbb{Z}_m$ for some $m \in \mathbb{Z}$, and the calculation looks similar to the $\tilde{G} = SU(N)$ example. But in general Γ can be represented by a product of i cyclic groups: $\Gamma = \mathbb{Z}_{m_1} \times ...... \times \mathbb{Z}_{m_i}$. We take the elements $\sum_i k_i c_i \in \mathfrak{h}$ in the Cartan subalgebra $\mathfrak{h}$ of $\tilde{G}$ and exponentiate them to get the central elements $C_{k_1 ... k_i} = e^{i \sum_i k_i c_i} \in Z(\tilde{G})$. Here we account for the structure of Γ: the index i refers to the i-th factor in the product and the coefficient k_i labels the elements inside each factor $\mathbb{Z}_{m_i}$.

In terms of the basis of the Cartan subalgebra $h_j \in \mathfrak{h}$ we can express $\sum_i k_i c_i = \sum_i 2\pi \frac{k_i}{m_i} \sum_{j=1}^n (a_i)_j h_j$, where n is the dimension of $\mathfrak{h}$, m_i is the number of the elements in $\mathbb{Z}_{m_i}$ and $(a_i)_j$ are real linear coefficients describing c_i and depending on the choice of a basis h_i.

The representation R_μ of a central element $C_{k_1,...,k_i}$ is given by the formula: $R_\mu(C_{k_1,...,k_i}) = R_\mu(e^{i \sum_i k_i c_i}) = e^{i \sum_i k_i R_\mu(c_i)}$. The characters of the central elements in the representations R_μ are:

$$\chi_{R_\mu}(C_{k_1,...,k_i}) = Tr(e^{i \sum_i k_i R_\mu(c_i)}) = d_{R\mu} \cdot e^{i \sum_i k_i <\mu, c_i>} = d_{R\mu} \cdot e^{i2\pi \sum_i \frac{k_i}{m_i}[\mu_i]}. \tag{4.29}$$

Here we have rewritten the pairing $\sum_i k_i < \mu, c_i >$ in the following way: $<\mu, \sum_i k_i c_i >= i2\pi \sum_i \frac{k_i}{m_i} \sum_{j=1}^n (a_i)_j \mu_j = i2\pi \sum_i \frac{k_i}{m_i}[\mu_i]$, where $(a_i)_j$ are linear coefficients producing different linear combinations of μ_js for each k_i-th element. The i different linear combinations $\sum_{j=1}^n (a_i)_j \mu_j \in \mathbb{Z}$ define i types of equivalence classes of the highest weight μ: $[\sum_{j=1}^n (a_i)_j \mu_j] \equiv [\mu_i] \in \mathbb{Z}_{m_i}$, where $[\mu_i]$ is an integer modulo m_i.

Without Wilson surface the partition function for G is given by:

$$Z_{YM}(\tau) = \sum_i \frac{1}{m_i} \sum_{k_i=0}^{m_i-1} \sum_{R_\mu(G)} d_{R_\mu}^{1-2g} e^{-\tau C_2(R_\mu)} \chi_{R_\mu}(C_{k_1,\ldots,k_i}), \tag{4.30}$$

where the sum is over the representation R_μ of $G = \tilde{G}/\Gamma$ and i coefficients k_i label a central element in the subgroup Γ.

Using (4.29) we compute:

$$\begin{aligned} Z_{YM}(\tau) &= \sum_i \tfrac{1}{m_i} \sum_{k_i=0}^{m_i-1} \sum_{R_\mu(G)} d_{R_\mu}^{2-2g} e^{-\tau C_2(R_\mu)} \cdot e^{ik_i<\mu,c_i>} \\ &= \sum_{R_\mu(G)} d_{R_\mu}^{2-2g} e^{-\tau C_2(R_\mu)} \sum_i \tfrac{1}{m_i} \sum_{k_i=0}^{m_i} e^{i2\pi \frac{k_i}{m_i}[\mu_i]}. \end{aligned} \tag{4.31}$$

Here each sum over k_i in the second line is different from zero and is equal to m_i only if $\mu_i = 0 \bmod m_i$ (i.e. $[\mu_i] = 0$) . This condition corresponds to choosing only those representations of $\tilde{G}$ in which the elements $C_{k_1,\ldots,k_i} \in \Gamma$ are all trivial, i.e. the representations of $G = \tilde{G}/\Gamma$:

$$Z_{YM}(\tau) = \sum_{R_\mu(G=\tilde{G}/\Gamma)} d_{R_\mu}^{2-2g} e^{-\tau C_2(R_\mu)}. \tag{4.32}$$

Now let us introduce a Wilson surface of weight λ. Just like any highest weight, λ will belong to i types of equivalence classes defined by the pairing $k_i < \lambda, c_i >= 2\pi \frac{k_i}{m_i} \sum_{j=1}^n (a_i)_j \lambda_j$, where $(a_i)_j$ are real linear coefficients for the pairing with the k_i-th element and $\sum_{j=1}^n (a_i)_j \lambda_j$ is an integer. Then the weight λ will be characterised by belonging to i types of equivalence classes: $[\lambda_i] = [\sum_{j=1}^n (a_i)_j \lambda_j] \in \mathbb{Z}_{m_i}$. The partition function with a Wilson surface of weight λ is given by:

$$\begin{aligned} Z_{YM}^\lambda(\tau) &= \sum_i \tfrac{1}{m_i} \sum_{k_i=0}^{m_i-1} \sum_{R_\mu(G)} d_{R_\mu}^{1-2g} e^{-\tau C_2(R_\mu)} \chi_{R_\mu}(C_{k_1,\ldots,k_i}) \tfrac{\chi_\lambda(C_{k_1,\ldots,k_i})}{d_\lambda} \\ &= \sum_i \tfrac{1}{m_i} \sum_{k_i=0}^{m_i-1} \sum_{R_\mu(G)} d_{R_\mu}^{2-2g} e^{-\tau C_2(R_\mu)} e^{ik_i<\mu,c_i>} e^{ik_i<\lambda,c_i>} \\ &= \sum_i \tfrac{1}{m_i} \sum_{k_i=0}^{m_i-1} \sum_{R_\mu(G)} d_R^{2-2g} e^{-\tau C_2(R_\mu)} e^{i2\pi \frac{k_i}{m_i}[\mu_i]} e^{i2\pi \frac{k_i}{m_i}[\lambda_i]} \\ &= \sum_{R_\mu(G)} d_R^{2-2g} e^{-\tau C_2(R_\mu)} \sum_i \tfrac{1}{m_i} \sum_{k_i=0}^{m_i-1} e^{i2\pi \frac{k_i}{m_i}[\mu_i+\lambda_i]}. \end{aligned} \tag{4.33}$$

Now each sum over k_i is different from zero and is equal to m_i only if $[\mu_i + \lambda_i] = 0$ for all i. This results in:

$$Z^{\lambda}_{YM}(\tau) = \sum_{R_{\mu+\lambda}(G=\tilde{G}/\Gamma)} dim^{2-2g}_{R_\mu} e^{-\tau C_2(R_\mu)}, \tag{4.34}$$

where the sum is over such representations $R_\mu(\tilde{G})$, that the representations of $\tilde{G}$ with the highest weight $\mu + \lambda$ would correspond to the representations of $G = \tilde{G}/\Gamma$.

4.3 Example of $G = Spin(4l)/(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$

Now let us illustrate the formulas (4.33), (4.34) with an example of a gauge group with $\pi_1(G) \cong \Gamma = \mathbb{Z}_{m_1} \times ...... \times \mathbb{Z}_{m_i}$. The covering group $\tilde{G} = Spin(4l)$ has the center given by a product of two copies of $\mathbb{Z}_2$: $Z(Spin(4l)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. If we factorize by the entire center we get $G = Spin(4l)/\mathbb{Z}_2 \oplus \mathbb{Z}_2$. We start with the central elements of $\tilde{G} = Spin(4l)$: $C_{k_1k_2} = e^{i(k_1c_1+k_2c_2)} \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$, where $k_1c_1 + k_2c_2 = \pi k_1 \sum_{i=1}^n (a_1)_i h_i + \pi k_2 \sum_{i=1}^n (a_2)_i h_i \in \mathfrak{h}$ are in the Cartan subalgebra of $Spin(4l)$, the coefficients $k_j = 0, 1$ label the elements in the j-th copy of $\mathbb{Z}_2$, and $(a_1)_i, (a_2)_i$ are real coefficients describing the elements c_1 and c_2 respectively and depending on the choice of a basis h_i.

The representation R_μ of a central element $C_{k_1k_2}$ is given by the formula: $R_\mu(C_{k_1k_2}) = R_\mu(e^{i(k_1c_1+k_2c_2)}) = e^{i(k_1R_\mu(c_1)+k_2R_\mu(c_2))}$. The characters of the central elements in the representations R_μ are:

$$\chi_{R_\mu}(C_{k_1k_2}) = Tr(e^{i(k_1R_\mu(c_1)+k_2R_\mu(c_2))}) = d_{R\mu} \cdot e^{i(k_1<\mu,c_1>+k_2<\mu,c_2>)} = d_{R\mu} \cdot e^{i\pi(k_1[\mu_1]+k_2[\mu_2])}. \tag{4.35}$$

Here we've computed the pairing $< \mu, k_1c_1 + k_2c_2 >$ explicitly: $i\pi(k_1 \sum_{i=1}^n (a_1)_i \mu_i + k_2 \sum_{i=1}^n (a_2)_i \mu_i) \equiv i\pi(k_1[\mu_1] + k_2[\mu_2])$. We denote by $[\mu_1]$ and $[\mu_2]$ two different linear combinations ($\sum_{i=1}^n (a_1)_i \mu_i \in \mathbb{Z}$ and $\sum_{i=1}^n (a_2)_i \mu_i \in \mathbb{Z}$) of the components of the same highest weight μ modulo 2.

Without Wilson surface the partition function for $G = Spin(4l)/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is given by:

$$\begin{aligned}
Z_{YM}(\tau) &= \tfrac{1}{4} \sum_{k_1=0}^1 \sum_{k_2=0}^1 \sum_{R_\mu(Spin(4l))} d^{1-2g}_{R_\mu} e^{-\tau C_2(R_\mu)} \chi_{R_\mu}(C_{k_1k_2}) \\
&= \tfrac{1}{4} \sum_{k_1=0}^1 \sum_{k_2=0}^1 \sum_{R_\mu(Spin(4l))} d^{2-2g}_{R_\mu} e^{-\tau C_2(R_\mu)} \cdot e^{i(k_1<\mu,c_1>+k_2<\mu,c_2>)} \\
&= \sum_{R_\mu(Spin(4l))} d^{2-2g}_{R_\mu} e^{-\tau C_2(R_\mu)} \cdot \tfrac{1}{4} \sum_{k_1=0}^1 \sum_{k_2=0}^1 e^{i\pi(k_1[\mu_1]+k_2[\mu_2])} \\
&= \sum_{R_\mu(Spin(4l)/(\mathbb{Z}_2\oplus\mathbb{Z}_2))} d^{2-2g}_{R_\mu} e^{-\tau C_2(R_\mu)}.
\end{aligned} \tag{4.36}$$

In more detail, the sum in the first line of (4.36) runs over the representations R_μ of $Spin(4l)$ and in the last line - over the representations R_μ of $Spin(4l)/(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$. The change happens for the following reason. Each sum over k_i in the second line is equal to zero, or to 2 if $\sum_{i=1}^n a_i\mu_i$ is even, i.e. $[\mu_i] = 0$. This condition corresponds to chosing only those representations of $Spin(4l)$ in which the elements $C_{k_1k_2} \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$ are all trivial, i.e. the representations of $Spin(4l)/(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$.

When we introduce a Wilson surface of weight λ, it will involve defining two equivalence classes for λ from the pairing $< \lambda, k_ic_i >$: $[\lambda_1] = [\sum_{j=1}^n (a_1)_j\lambda_j] \in \mathbb{Z}_2$ and $[\lambda_2] = [\sum_{j=1}^n (a_2)_j\lambda_j] \in \mathbb{Z}_2$. The partition function in the presence of a Wilson surface is modified in the following way:

$$\begin{aligned}
Z^\lambda_{YM}(\tau) &= \frac{1}{4}\sum_{k_1=0}^1\sum_{k_2=0}^1\sum_{R_\mu(Spin(4l))} d_{R_\mu}^{1-2g} e^{-\tau C_2(R_\mu)}\chi_{R_\mu}(C_{k_1k_2})\frac{\chi_\lambda(C_{k_1k_2})}{d_\lambda} \\
&= \frac{1}{4}\sum_{k_1=0}^1\sum_{k_2=0}^1\sum_{R_\mu(Spin(4l))} d_{R_\mu}^{2-2g} e^{-\tau C_2(R-\mu)} e^{i<\mu,k_1c_1+k_2c_2>} e^{i<\lambda,k_1c_1+k_2c_2>} \\
&= \frac{1}{4}\sum_{k_1=0}^1\sum_{k_2=0}^1\sum_{R_\mu(Spin(4l))} d_R^{2-2g} e^{-\tau C_2(R)} e^{i(k_1[\mu_1]+k_2[\mu_2])} e^{i(k_1[\lambda_1]+k_2[\lambda_2])} \\
&= \sum_{R_\mu(Spin(4l))} d_R^{2-2g} e^{-\tau C_2(R)} \frac{1}{4}\sum_{k_1=0}^1\sum_{k_2=0}^1 e^{i\pi(k_1[\mu_1+\lambda_1]+k_2[\mu_2+\lambda_2])} \\
&= \sum_{\mu_1\in[-\lambda_1],\,\mu_2\in[-\lambda_2]} dim_{R_\mu}^{2-2g} e^{-\tau C_2(R_\mu)},
\end{aligned} \tag{4.37}$$

Here the sum over each k_i is different from zero only for $\mu_i + \lambda_i$ even. This condition reduces the sum in the last line to the sum over such representations $R_\mu(Spin(4l))$ that the representations of $Spin(4l)$ with the highest weight $\mu + \lambda$ would correspond to the representations of $Spin(4l)/(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$.

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