Structural and Spectral Properties of Schreier Graphs of Spinal Groups

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Desafiar la perspectiva del fracaso a la que estamos condenados

Roberto Iniesta

Résumé

Les groupes agissant sur des arbres enracinés constituent une classe très intéressant de groupes, dont beaucoup montrent des propriétés uniques. Comme exemple principal, le groupe de Grigorchuk est un 2-groupe qui est de type fini mais pas de présentation finie, tous ses quotients non-triviaux sont finis, il est moyennable mais pas élémentairement moyennable et il est le premier exemple de groupe à croissance intermédiaire.

Cette thèse se concentre sur les groupes spinaux, une famille concrète de groupes agissant sur des arbres enracinés. En restreignant le type de générateurs possibles, leur étude devient abordable. Pourtant, les groupes spinaux contiennent toujours des exemples remarquables de groupes agissant sur des arbres enracinés. Il est particulièrement intéressant de constater comme certaines propriétés de ces exemples peuvent ou ne peuvent pas se généraliser pour autres groupes similaires.

Les graphes de Cayley sont des représentations de l'action d'un groupe de type fini sur lui-même par multiplication à gauche. Les graphes de Schreier généralisent cette notion pour des actions quelconques. Étant donné que les groupes agissant sur des arbres enracinés possèdent une action naturelle sur l'arbre, les graphes de Schreier par rapport à cette action deviennent un outil clé pour leur étude.

Dans cette thèse, nous construisons les graphes de Schreier associés à l'action des groupes spinaux sur chacun des niveaux finis de l'arbre, ainsi que sur son bord, par rapport à un système de générateurs naturel. Pour les graphes infinies, nous trouvons leur nombre de buts et classes d'isomorphisme, les deux comme des graphes avec et sans étiquettes. En outre, nous étudions le système dynamique donné par les graphes de Schreier considérés comme un sous-ensemble de l'espace topologique des graphes marqués.

Nous étudions ensuite les graphes de Cayley et de Schreier des groupes spinaux depuis une perspective de théorie spectrale des graphes, en contribuant avec des exemples additionnels à la courte liste des types de spectre connus pour des graphes de Cayley ou de Schreier. En particulier, nous trouvons le spectre des opérateurs d'adjacence sur les graphes de Schreier des groupes spinaux avec deux méthodes différentes, par des approximations finies des graphes et à l'aide de fonctions de renormalisation. Nous concluons que ce spectre est l'union de deux intervalles si le groupe agit sur l'arbre binaire ou l'union d'un ensemble de Cantor avec un ensemble dénombrable de points qui s'accumulent sur lui autrement. Pour le cas binaire, de plus, le spectre de l'opérateur d'adjacence sur le graphe de Cayley coincide avec celui sur le graphe de Schreier.

Par la suite, nous étendons l'analyse spectrale en traitant les mesures spectrales de l'opérateur d'adjacence sur les graphes de Schreier. Pour le cas binaire, nous obtenons explicitement la densité des mesures spectrales sur tous les graphes de Schreier sauf une orbite, et donnons la densité d'une mesure spectrale pour cette orbite. Les deux sont absolument continues par rapport à la mesure de Lebesgue.

Nous poursuivons l'analyse spectrale avec le cas non-binaire, en trouvant les fonctions propres de l'opérateur d'adjacence explicitement. Pour les graphes de Schreier associés à un sous-ensemble explicite du bord de l'arbre à mesure uniforme de Bernoulli égale à un, nous montrons que toutes les mesures spectrales sont discrètes.

Également, nous présentons des exemples de graphes de Schreier pour lesquels les mesures spectrales possèdent une partie singulière non triviale. Nous donnons une décomposition de l'espace de fonctions sur certaines graphes de Schreier comme la somme directe des espaces propres et un sous-espace explicite, et nous montrons que la mesure spectrale de toute fonction dans le deuxième est singulière.

Finalement, nous concluons cette thèse en étudiant des notions de basse complexité sur les systèmes dynamiques formés par les graphes de Schreier. Nous rappelons ces notions pour des sous-décalages linéaires et donnons des généralisations dans le contexte des systèmes dynamiques de Schreier, et nous caractérisons quand elles sont satisfaites pour le cas des groupes spinaux.

Abstract

Groups acting on rooted trees constitute a very interesting class of groups which features countless groups exhibiting rather uncommon properties. As main example, Grigorchuk's group is a finitely generated, not finitely presented 2-group which is just infinite, amenable but not elementary amenable and the first example of a group with intermediate growth.

This thesis focuses on spinal groups, a particular family of groups acting on rooted trees. By restricting the type of generators such groups may have, their study becomes tractable. However spinal groups still contain many of the remarkable examples of groups acting on rooted trees. It is of special interest to see how known properties of such examples generalize or fail to generalize for other similar groups.

Cayley graphs are representations of the action of a finitely generated group on itself by left-multiplication. Schreier graphs generalize this notion by representing any other action. Provided that groups acting on rooted trees are naturally equipped with their action on the tree, the Schreier graphs associated with this action become a very useful tool for their study.

In this thesis, we construct the Schreier graphs associated with the action of spinal groups on each of the finite levels of the tree, as well as its boundary, with respect to a natural spinal generating set. For the infinite graphs, we discuss their number of ends and isomorphism classes, both as labeled and unlabeled graphs. In addition, we study the dynamical system given by the Schreier graphs regarded as a subset of the topological space of marked graphs.

Afterwards, we study Cayley and Schreier graphs of spinal groups from a spectral graph theory perspective, thus contributing with additional examples to the short list of known shapes of spectra of Cayley and Schreier graphs. In particular, we compute the spectrum of the adjacency operator on Schreier graphs of spinal groups via two approaches, namely, by finite approximation of the graphs and using renormalization maps. We find that this spectrum is the union of two intervals if the group acts on the binary tree or the union of a Cantor set with a countable set accumulating on it otherwise. For the binary case, moreover, the spectrum of the adjacency operator on the Cayley graph coincides with that of the Schreier graph.

Next, we extend the spectral analysis by discussing the spectral measures of the adjacency operator on the Schreier graphs. For the binary case, we compute the explicit density of the spectral measures for all Schreier graphs except one orbit, and also give the density of one spectral measure for that orbit. Both of them are absolutely continuous with respect to the Lebesgue measure.

The spectral analysis is continued with the non-binary case, by finding the eigenfunctions of the adjacency operator explicitly. For the Schreier graphs of an explicit subset of points in the boundary of the tree of uniform Bernoulli measure one, we show that all spectral measures are purely discrete, so the spectrum is pure point.

We also exhibit examples of Schreier graphs for which the spectral measures have nontrivial singular continuous part. We provide a decomposition of the space of functions on certain Schreier graphs as the direct sum of the eigenspaces and an explicit subspace, and show that any function in the latter has a purely singular continuous spectral measure.

Finally, we conclude this thesis with the study of several notions of low complexity on the dynamical systems formed by Schreier graphs. We recall such notions for linear subshifts and give their generalizations in the context of Schreier dynamical systems, and characterize when they are satisfied for the case of spinal groups.

Resum

Els grups que actuen sobre arbres amb arrel constitueixen una classe de grups de gran interès, que presenta un nombre considerable de grups amb propietats poc freqüents. Com a exemple principal, el grup de Grigorchuk és un 2-grup finitament generat però no finitament presentat, tots els seus quocients no trivials són finits, és amenable però no elementalment amenable i és el primer exemple de grup amb creixement intermedi.

Aquesta tesi se centra en els grups espinals, una família concreta de grups que actuen sobre arbres amb arrel. Restringint el tipus de generadors possibles, el seu estudi esdevé abordable. No obstant això, els grups espinals encara contenen exemples notables de grups que actuen sobre arbres amb arrel. És particularment interessant constatar com certes propietats d'aquests exemples poden o no poden generalitzar-se per a altres grups similars.

Els grafs de Cayley són representacions de l'acció per multiplicació a l'esquerra d'un grup finitament generat sobre si mateix. Els grafs de Schreier generalitzen aquesta noció representant qualsevol altra acció. Atès que els grups que actuen sobre arbres amb arrel poseeixen una acció natural sobre l'arbre, els grafs de Schreier associats esdevenen una eina molt important per al seu estudi.

En aquesta tesi, construïm els grafs de Schreier associats a l'acció dels grups espinals sobre cadascun dels nivells de l'arbre, així com sobre la seva vora, respecte d'un conjunt generador natural. Per als grafs infinits, parlem sobre el seu nombre d'extrems i les seves classes d'isomorfisme, com a grafs amb i sense etiquetes. A més, estudiem el sistema dinàmic donat pels grafs de Schreier com a subespai de l'espai topològic de grafs marcats.

Després, estudiem els grafs de Cayley i de Schreier dels grups espinals des del punt de vista de la teoria espectral de grafs, contribuent amb exemples addicionals a la curta llista de tipus d'espectres coneguts per a grafs de Cayley i de Schreier. En particular, trobem l'espectre de l'operador d'adjacència sobre els grafs de Schreier dels grups espinals emprant dos mètodes diferents, per aproximació per grafs finits i utilitzant funcions de renormalització. Observem que aquest espectre és la unió de dos intervals si el grup actua sobre l'arbre binari o la unió d'un conjunt de Cantor amb un conjunt numerable que s'hi acumula altrament. Per

al cas binari, a més, l'espectre de l'operador d'adjacència sobre el graf de Cayley coincideix amb aquell sobre el graf de Schreier.

Tot seguit, estenem l'anàlisi espectral parlant de les mesures espectrals de l'operador d'adjacència sobre els grafs de Schreier. Per al cas binari, calculem explícitament la densitat de les mesures espectrals per a tots els grafs de Schreier llevat d'una òrbita, i també donem la densitat d'una mesura espectral per a aquesta òrbita. Totes dues són absolutament contínues respecte de la mesura de Lebesgue.

Continuem l'anàlisi espectral amb el cas no binari, trobant les funcions pròpies de l'operador d'adjacència de manera explícita. Per als grafs de Schreier d'un subconjunt explícit de punts de la vora de l'arbre de mesura uniforme de Bernoulli u, provem que totes les mesures espectrals són purament discretes.

També exhibim exemples de grafs de Schreier per als quals les mesures espectrals tenen una component singularment contínua no trivial. Proporcionem una descomposició de l'espai de funcions sobre aquests grafs de Schreier com a suma directa dels espais propis amb un subespai explícit, i provem que la mesura espectral de tota funció d'aquest subespai és singularment contínua.

Per acabar, concloem aquesta tesi amb l'estudi de diverses nocions de baixa complexitat sobre els sistemes dinàmics formats pels grafs de Schreier. Recordem aquestes nocions per a subdecalatges lineals i en donem generalitzacions en el context de sistemes dinàmics de Schreier, i caracteritzem quan aquestes generalitzacions són satisfetes en el cas dels grups espinals.

Resumen

Los grupos que actúan sobre árboles con raíz constituyen una clase de grupos muy interesante, que presenta una cantidad considerable de grupos con propiedades poco comunes. Como ejemplo principal, el grupo de Grigorchuk es un 2-grupo finintamente generado pero no finitamente presentado, todos sus cocientes no triviales son finitos, es amenable pero no elementalmente amenable y es el primer ejemplo de grupo con crecimiento intermedio.

Esta tesis se centra en los grupos espinales, una familia concreta de grupos que actúan sobre árboles con raíz. Restringiendo el tipo de generadores posibles, su estudio se hace abordable. Sin embargo, los grupos espinales aún contienen ejemplos notables de grupos que actúan sobre árboles con raíz. Es particularmente interesante comprobar cómo ciertas propiedades de dichos ejemplos pueden o no generalizarse para otros grupos similares.

Los grafos de Cayley son representaciones de la acción por multiplicación a la izquierda de un grupo finitamente generado sobre sí mismo. Los grafos de Schreier generalizan esta noción representando cualquier otra acción. Como los grupos que actúan sobre árboles con raíz poseen una acción natural sobre el árbol, los grafos de Schreier asociados constituyen una herramienta muy importante para su estudio.

En esta tesis, construimos los grafos de Schreier asociados a la acción de los grupos espinales sobre cada uno de los niveles del árbol, así como sobre su borde, respecto a un conjunto generador natural. Para los grafos infinitos, tratamos su número de extremos y sus clases de isomorfismo, como grafos con y sin etiquetar. Además, estudiamos el sistema dinámico dado por los grafos de Schreier como subespacio del espacio topológico de grafos marcados.

Posteriormente, estudiamos los grafos de Cayley y de Schreier de los grupos espinales desde el punto de vista de la teoría espectral de grafos, contribuyendo con ejemplos adicionales a la corta lista de tipos de espectros conocidos para grafos de Cayley y de Schreier. En particular, encontramos el espectro del operador de adyacencia sobre los grafos de Schreier de los grupos espinales a través de dos métodos diferentes, por aproximación por grafos finitos y utilizando funciones de renormalización. Observamos que dicho espectro es

la unión de dos intervalos si el grupo actúa sobre el árbol binario o la unión de un conjunto de Cantor con un conjunto numerable que se le acumula si no. Para el caso binario, además, el espectro del operador de adyacencia sobre el grafo de Cayley coincide con aquél sobre el grafo de Schreier.

A continuación, extendemos el análisis espectral tratando las medidas espectrales del operador de adyacencia sobre los grafos de Schreier. Para el caso binario, calculamos explícitamente la densidad de las medidas espectrales para todos los grafos de Schreier salvo una órbita, y también damos la densidad de una medida espectral para esta órbita. Ambas son absolutamente contínuas respecto a la medida de Lebesgue.

Continuamos el análisis espectral con el caso no binario, hallando las funciones propias del operador de adyacencia de manera explícita. Para los grafos de Schreier de un subconjunto explícito de puntos del borde del árbol de medida uniforme de Bernoulli uno, demostramos que todas las medidas espectrales son puramente discretas.

También exhibimos ejemplos de grafos de Schreier para los cuales las medidas espectrales tienen una componente singularmente contínua no trivial. Proporcionamos una descomposición del espacio de funciones sobre estos grafos de Schreier como suma directa de los espacios propios y un subespacio explícito, y demostramos que la medida espectral de toda función de dicho subespacio es singularmente contínua.

Para terminar, concluimos esta tesis con el estudio de varias nociones de baja complejidad sobre los sistemas dinámicos formados por los grafos de Schreier. Recordamos estas nociones para subdecalajes lineales y damos generalizaciones en el contexto de sistemas dinámicos de Schreier, y caracterizamos cuando estas generalizaciones se satisfacen en el caso de los grupos espinales.

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Contents

Chapter 1

Introduction

The main focus of this thesis lies in the intersection between the theory of groups acting on rooted trees and spectral analysis on graphs. These particularly rich group actions give rise to certain families of graphs which exhibit uncommon properties, notably from the point of view of spectral graph theory. Their special structure allows to develop a variety of methods (finite approximation, renormalization, dynamical simulation) that can serve as an approach to spectral problems.

The motivation for the study of groups acting on rooted trees responds to the fact that they characterize residually finite groups. Even though residually finite groups had been studied since earlier, as well as other related aspects like automata, wreath products or branching subgroup structures, groups acting on rooted trees owe their importance to the definition of a group of interval exchange transformations by Grigorchuk in 1980 [31]. This group arose as an answer to the general Burnside problem, which asked whether an infinite, finitely generated group could be periodic. Even though this was not the first example of such a group, it turned out to be an incredibly interesting example. In addition to being a finitely generated, infinite periodic group, it features many other striking properties, the most important of which is arguably its intermediate growth.

The definition of Grigorchuk's group in terms of interval exchange transformations was reformulated in terms of automorphisms of the binary rooted tree. This inspiring example opened a new page in the study of groups acting on rooted trees. Ever since, this subject has been widely studied and has provided numerous examples of new groups with special properties. Moreover, such active research gave rise to some important abstract definitions of group properties, as for instance self-similar or branch groups.

Grigorchuk provided an uncountable family $\mathcal{G}$ of groups of intermediate growth acting on the binary tree in [32], and in [33] he generalized it to the p-ary tree, for any prime p. The Gupta-Sidki p-groups were introduced in [47]. As time passed, the generalizations diversified and became more sophisticated. In this thesis, we focus on one of them, spinal groups.

The family of spinal groups was defined in [7] as a subclass of a larger family of groups introduced in [34], and later studied in [5, 65], with the purpose of generalizing Grigorchuk's family $\mathcal{G}$, as mentioned above. Spinal groups recover and extend this family, providing an uncountable family of groups $\mathcal{S}_{d,m}$ for every $d \geq 2$ and $m \geq 1$, in such a way that $\mathcal{S}_{2,2}$ coincides with $\mathcal{G}$. More generally, each $\mathcal{S}_{d,m}$ is a self-similar family of groups which is a generalization of a self-similar group, in the same way that $\mathcal{G}$ is a self-similar family of groups generalizing Grigorchuk's group. All spinal groups are amenable [49].

Spinal groups constitute a considerably rich source of examples, yet its study remains tractable, which allows to explore how several properties of Grigorchuk's group extend or do not extend to other cases.

For any finitely generated group G, and finite generating set S, we naturally consider its Cayley graph $\mathrm{Cay}(G, S)$. Cayley graphs have long been employed as powerful tools in geometric group theory and also studied in their own right as interesting examples of vertex-transitive graphs. One way of regarding Cayley graphs is as representations of the action of G on itself by left multiplication. Schreier graphs arise as representations of generic actions of finitely generated groups, in particular on topological or measured spaces. Despite losing vertex-transitivity, Schreier graphs are still regular, oriented, edge-labeled graphs, which often exhibit interesting properties. If the action is free, then Schreier graphs on every orbit are isomorphic to the Cayley graph. A new motivation to consider Schreier graphs comes from the recent interest in non-free actions [67].

For G acting on a set $\mathcal{X}$, we obtain a family of Schreier graphs $(\Gamma_x)_{x\in\mathcal{X}}$. Under mild conditions, Schreier graphs are generally locally isomorphic but not necessarily isomorphic. The Schreier graphs $(\Gamma_x)_{x\in\mathcal{X}}$ of an action of a group G on a topological space or a measured space $\mathcal{X}$ gives rise to an interesting dynamical system. The space of marked graphs $\mathcal{G}_{*,S}$ is equipped with the natural topology of local convergence, a basis of which is formed by subsets of graphs with a fixed ball around the marked point. If we consider Schreier graphs as marked graphs, the map $x \mapsto (\Gamma_x, x)$ provides a way of transporting the action of G on $\mathcal{X}$ to the graphs themselves, by shifting the marked point.

There are in the literature numerous examples of the study of Schreier graphs, for actions of groups in families such as branch groups, Thompson groups [61, 62] or automata groups [42, 43, 18, 2, 12, 3, 21]. They have been useful both as interesting families of graphs in their own right (spectral theory [43, 15], expansion [42, 26], growth [42, 10, 11], statistical physics models [19, 20, 56], automata theory [42, 43, 18, 12, 15, 21], dynamical systems [21]) and as tools in the study of the acting group (amenability [61, 62], random walks [64], maximal subgroups [30], and even the construction of simple groups of intermediate growth [58]).

Cayley graphs and Schreier graphs of finitely generated groups constitute an important class of examples in spectral graph theory. Conversely, spectral properties of the Laplace operator play a significant role in the theory of random walks on groups and more generally in geometric group theory. The first natural question that arises is whether the spectrum determines the group in some way, or, as formulated by Alain Valette in [66] paraphrasing Mark Kac, *Can one hear the shape of a group?*.

Spectral computations are notoriously difficult, and the variety of methods limited. Indeed, very few examples of spectra for infinite graphs or infinite families of graphs have been explicitly computed. Not even qualitative results are abundant. The list of known shapes of spectra of Cayley graphs is short: an interval, the union of an interval with one or two isolated points, the union of an interval with countably many points accumulating to a point outside the interval [41], the union of two disjoint intervals (Grigorchuk's group [24]) and the union of two disjoint intervals with one or two isolated points (free products of two finite cyclic groups [16]). For Schreier graphs, the first examples exhibiting a Cantor set or the union of a Cantor set and a countable set of points accumulating on it as spectrum were obtained in [4]. It is still open whether Cantor spectrum can occur on a Cayley graph.

Another interesting spectral property is the spectral measure type, albeit very little is known in this regard. Lamplighter groups remain the only family of infinite groups for which the spectral measure has been shown to be purely discrete (see [44] for the original result on the lamplighter over $\mathbb{Z}$, and [51] for a generalization to lamplighters with arbitrary bases). It is also the case for typical Schreier graphs of the Basilica group [15]. In [60] an example of a graph was shown to have a nontrivial singular continuous part. This graph happens to be the Schreier graph of the Hanoi group on three pegs with respect to the stabilizer of a neighborhood of a singular point for the action of the group on the boundary of the ternary tree. There are so far no examples of Cayley graphs with nontrivial singular continuous part.

Recently, a new interesting connection has emerged, relating the study of certain self-similar groups and families of groups with quasicrystals or subshifts of low complexity [68, 54, 55, 63]. In particular, Schreier graphs of Grigorchuk's group have been related with linear subshifts of aperiodic type. This connection has allowed the computation of spectra of Laplacians corresponding to weighted random walks [38, 39], thus providing examples with purely singular continuous spectra. In particular, Schreier graphs of spinal groups acting on the binary tree can be encoded as simple Toeplitz subshifts. However, this correspondence is lost for spinal groups acting on trees of higher degree, because the Schreier graphs do not have linear structure in that case.

There are plenty of open questions in this area, and the link between self-similar groups and subshifts of low complexity represents fertile ground for new research to develop.

The main objective of this thesis is the study of different aspects of Schreier graphs of spinal groups, in order to prove that they exhibit rare properties for which other known examples are few to none. The features we consider range from purely graph theory to spectral graph theory, with an insight in dynamical systems.

In Chapter 3, we thoroughly describe the Schreier graphs associated with the action of spinal groups on both the tree and its boundary. We first give an inductive method to construct them in Sections 3.1 and 3.2. Then, in Sections 3.3 to 3.6, we discuss some geometric properties including their number of ends and their isomorphism classes, both as labeled and unlabeled graphs. We conclude the chapter in Section 3.7 by studying the dynamical system formed by the space of Schreier graphs as a subset of the space of marked graphs with local convergence.

Chapter 4 is devoted to the computation of several Laplacian spectra on Schreier graphs of spinal groups. We provide two different approaches, in Sections 4.1 and 4.2. The former is by finite approximation, so the spectra on the finite Schreier graphs is used in order to find the spectra on the infinite Schreier graphs. The latter uses renormalization maps to find the spectra on the infinite graphs directly, even on limit graphs in the space of Schreier graphs. However, despite being more elegant than the former, it does not apply for all spinal groups. For spinal groups acting on the binary tree, we find that the spectrum on any Schreier graph is the union of two intervals, while for the rest it is the union of a Cantor set and a countable set of points accumulating on it.

In addition, we prove in Section 4.3 that the spectrum on the Cayley graph, for spinal groups acting on the binary tree, coincides with that of the Schreier graphs (see [23]), and in Section 4.4 we study the dependence of the spectrum on the chosen generating set.

Spectral measures on Schreier graphs of spinal groups are discussed in Chapter 5. First, we explicitly compute the empirical spectral measure for the Schreier graphs of any spinal group, in Section 5.1. The rest of the chapter exhibits the different possibilities of spectral measure types for Schreier graphs of spinal groups, again finding remarkable differences between the binary and the non-binary cases. For spinal groups acting on the binary tree, we provide in Section 5.2 the explicit densities of the spectral measures on the typical Schreier graphs and on one of the isolated Schreier graphs. Both are absolutely continuous with respect to the Lebesgue measure.

In Section 5.3, for each spinal group acting on a non-binary tree, we present a family of Schreier graphs with purely discrete spectral measures. We do so by explicitly computing the eigenfunctions and showing that they form a complete system. This family of Schreier graphs corresponds to a set of points of uniform Bernoulli measure one in the boundary of the tree ∂T_d. The proof of the pure point spectrum and some results about the eigenfunctions are, to the best of our knowledge, new in the literature, even for the Fabrykowski-Gupta group, whose spectrum was computed in [4].

Additionally, in Section 5.4 we exhibit certain Schreier graphs of spinal groups for which the spectral measures have nontrivial singular continuous part. In particular, we decompose the space of functions on the graphs as the direct sum of the eigenspaces plus an explicit subspace, all of whose functions have purely singular continuous spectral measures. The graphs for which this happens arise naturally as limit of Schreier graphs of points in the boundary of the tree in the space of marked graphs with local convergence.

Finally, in Chapter 6 we recover the setting of Schreier dynamical systems from [35] and accommodate in it all Schreier graphs of spinal groups. Such dynamical systems have been studied in [68] for Grigorchuk's group, in [18] for the Basilica group and in [61, 62] for Thompson's groups. We review some classical notions of low complexity for linear subshifts and, in hopes of broadening the bridge between the study of Schreier graphs and the theory of linear subshifts, we extend them to the framework of Schreier dynamical systems and characterize when the dynamical systems defined by spinal groups satisfy them in this more general setup.

Chapter 2

Preliminaries

In this chapter, we gather the definitions and some basic results of the main objects discussed in the thesis.

2.1 Groups acting on rooted trees

Let $d \geq 2$ be an integer and let T_d be the d-ary infinite rooted tree. If we consider the alphabet $X = \{0, \dots, d-1\}$, vertices in T_d are in bijection with the set X^* of finite words in X. The root is represented by the empty word $\emptyset$ and, if v represents a vertex, vi represents its i-th child, for $i \in X$. For $n \geq 0$, the n-th level of T_d is the set of vertices of T_d which are at distance n from the root. It is in bijection with the set X^n of words on X of length n.

The group of all graph automorphisms of T_d is denoted $\mathrm{Aut}(T_d)$. Any automorphism of T_d must fix the root, as it is the only vertex with degree d, and hence must map X^n to itself, for every $n \geq 0$. Any automorphism $g \in \mathrm{Aut}(T_d)$ can be inductively described by the permutation $\tau \in \mathrm{Sym}(X)$ it induces on the vertices of the first level and its projections $g_i \in \mathrm{Aut}(T_d)$, $i = 0, \dots, d-1$, to the d subtrees attached at the root. Symbolically it can be written as

$$g = \tau(g_0, \dots, g_{d-1}),$$

where, for every $v = v_0 \dots v_{n-1} \in X^n$,

$$g(v_0 \dots v_{n-1}) = \tau(v_0) g_{v_0}(v_1 \dots v_{n-1}).$$

We may extend this to any vertex of the tree by defining the *projection* g_v of an element $g \in \mathrm{Aut}(T_d)$ on a vertex $v \in X^*$ as the automorphism satisfying, for every $w \in X^*$,

$$g(vw) = g(v) g_v(w).$$

Some natural subgroups to look at in the context of groups acting on rooted trees are vertex stabilizers. For $G \leq \mathrm{Aut}(T_d)$, the *stabilizer of a vertex* $v \in X^n$ is the subgroup

$$\mathrm{Stab}_G(v) = \{g \in G \mid g(v) = v\}.$$

Furthermore, we consider the *level stabilizers*, which are the subgroups

$$\mathrm{Stab}_G(n) = \bigcap_{v \in X^n} \mathrm{Stab}_G(v),$$

for every $n \geq 1$. Notice that they are of finite index for any $G \in \mathrm{Aut}(T_d)$.

In addition to the tree T_d, we consider its *boundary* ∂T_d, which is the set of infinite, non-backtracking paths from the root. This is a compact space that arises naturally from the structure of T_d. The boundary ∂T_d is in bijection with the set $X^{\mathbb{N}}$ of infinite sequences in the set $X^{\mathbb{N}}$. The action of a group on T_d can be naturally extended to its boundary ∂T_d due to the fact that any automorphism of T_d preserves prefixes. As for vertices of the tree, we also consider the stabilizers of points in the boundary. The stabilizer of a point $\xi \in X^{\mathbb{N}}$ is the subgroup

$$\mathrm{Stab}_G(\xi) = \{g \in G \mid g(\xi) = \xi\}.$$

Due to the self-similar nature of the tree, there is a natural transformation of the boundary which corresponds to identifying any of the subtrees rooted at the first level with the whole tree. This is the *shift* map σ on the boundary of the tree, and corresponds to delete the first digit of any point in the boundary. Namely,

$$\begin{array}{rccc} \sigma : & X^{\mathbb{N}} & \to & X^{\mathbb{N}} \\ & \xi_0\xi_1\xi_2\ldots & \mapsto & \xi_1\xi_2\ldots \end{array}.$$

Finally, let us introduce some broad families of groups acting on rooted trees. The family of spinal groups that we consider throughout this thesis is not fully contained in any of them, but most of its remarkable examples lie precisely in the intersection of these families.

Let $G \leq \mathrm{Aut}(T_d)$ be a group acting on T_d. We say that G is *self-similar* if for every $g \in G$ and every $v \in X^*$, its projections g_v belong to G. Equivalently, if for every $i \in X$, its projections g_i belong to G.

Another class of groups acting on rooted trees we would like to introduce is that of automata groups. In order to do so, we first need to define Mealy automata.

A *finite-state Mealy automaton* is a tuple $\mathcal{A} = (Q, \Sigma, \phi, \psi)$ such that Q is a finite set called the set of states, Σ is a finite alphabet, $\phi : Q \times \Sigma \to Q$ is a map called the transition function and $\psi : Q \times \Sigma \to \Sigma$ is another map called the output function. We call an automaton $\mathcal{A}$ invertible if, for every $q \in Q$, the map $\psi_q : \Sigma \to \Sigma$, defined as $\psi_q(x) = \psi(q, x)$, is a bijection of Σ.

A very convenient way of representing automata is via a directed graph, with set of vertices Q, and the following edges: for every $q \in Q$ and $x \in \Sigma$, there is an edge from q to $\phi(q, x)$, labeled by $x|\psi(q, x)$.

Every state $q \in Q$ of an invertible automaton $\mathcal{A}$ induces an automorphism of the rooted tree T_d, with $d = |\Sigma|$, which we abusively denote q as well, recursively given by $q(\emptyset) = \emptyset$ and $q(xv) = \psi(q, x)\phi(q, x)(v)$, for every $x \in \Sigma$, $v \in \Sigma^*$. We say that the group generated by an invertible automaton $\mathcal{A}$ is the group $G_{\mathcal{A}} \leq \mathrm{Aut}(T_d)$ generated by the automorphisms of T_d induced by all $q \in Q$. Conversely, we say that a group $G \leq \mathrm{Aut}(T_d)$ is an *automaton group* if there exists an invertible automaton $\mathcal{A}$ such that $G = G_{\mathcal{A}}$.

Example 2.1.1. Grigorchuk's group is the subgroup of $\mathrm{Aut}(T_2)$ generated by the following automorphisms:

$$\begin{aligned} a &= \tau(1, 1) \\ b &= (a, c) \\ c &= (a, d) \\ d &= (1, b) \end{aligned},$$

where τ is the nontrivial element of $\mathrm{Sym}(X)$. See Figure 2.1 for an illustration of the generators. Notice the relations $a^2 = b^2 = c^2 = d^2 = bcd = 1$. Grigorchuk's group is a finitely generated infinite 2-group, and it is not finitely presented. It is just infinite, i.e. any nontrivial quotient is finite. It has intermediate growth, and it is amenable but not elementary amenable. Grigorchuk's group is self-similar and an automaton group.

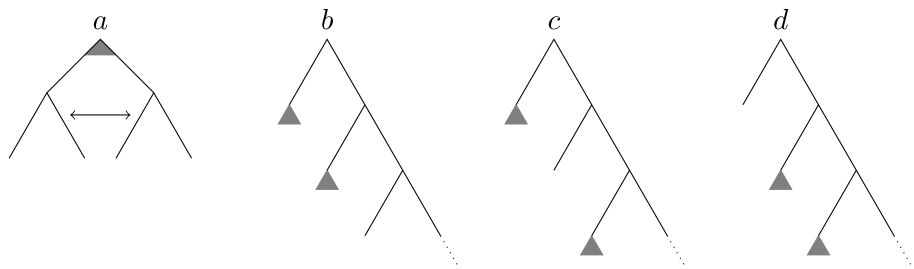


Figure 2.1: The generators of Grigorchuk's group.

Example 2.1.2. Another well-known example of group acting on the binary tree is the Basilica group. Again setting τ to be the nontrivial permutation of $\mathrm{Aut}(X)$, it is generated by the following automorphisms, depicted in Figure 2.2.

$$\begin{aligned} a &= (1, b) \\ b &= \tau(a, 1) \end{aligned}.$$

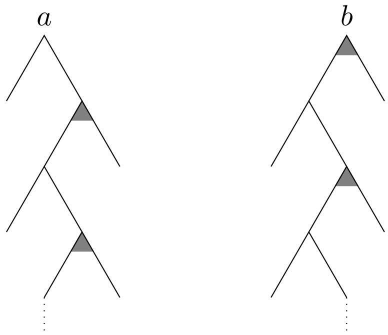


Figure 2.2: The generators of the Basilica group.

The Basilica group was introduced in [45] as a self-similar automaton group, and its Schreier graphs were fully described in [18]. It has exponential growth, because a and b generate a free non-abelian semigroup, and it is amenable but not elementary amenable.

Example 2.1.3. A very interesting example of a group acting on a non-binary rooted tree is the Fabrykowski-Gupta group. First introduced in [28], it remains arguably the simplest nontrivial example of group acting on the ternary tree. It is defined by the following automorphisms:

$$\begin{aligned} a &= \tau(1,1,1) \\ b &= (a,1,b) \end{aligned},$$

with τ being the 3-cycle (012) in $\mathrm{Sym}(X)$, so that a cyclically permutes the subtrees on the first level. The generators are illustrated in Figure 2.3.

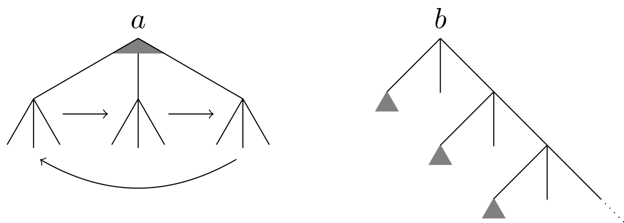


Figure 2.3: The generators of the Fabrykowski-Gupta group.

The Fabrykowski-Gupta group is also a self-similar automaton group, and was shown to have intermediate growth in [6].

Example 2.1.4. The Hanoi towers group (on k pegs, $k \geq 3$) is another example of group acting on the k-ary tree. For $k = 3$, it is the subgroup of $\mathrm{Aut}(T_3)$ generated by

$$\begin{aligned} a &= (01)(1,1,a) \\ b &= (02)(1,b,1) \\ c &= (12)(c,1,1) \end{aligned} \ ,$$

see Figure 2.4 for a depiction of the generators.

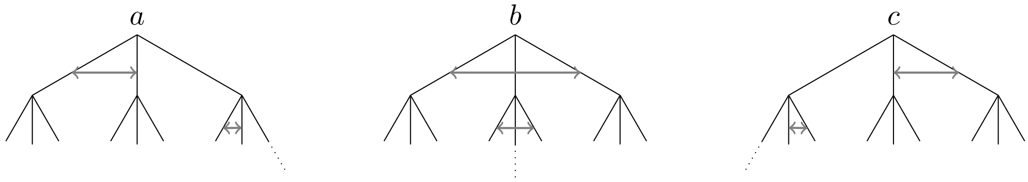


Figure 2.4: The generators of the Hanoi towers group on three pegs.

The Hanoi towers groups appeared in [42] as a model of the homonymous combinatorial problem. Indeed, the Hanoi towers problem (on k pegs) can be reformulated in terms of distances between vertices in the Schreier graphs, which happen to be finite approximations of the Sierpiński gasket. The Hanoi towers group (on k pegs) is a self-similar automaton group for each $k \geq 3$. It is known to be amenable but not elementary amenable for $k = 3$. However, for $k \geq 4$, the question of amenability is still open.

2.2 Spinal groups

Let us now define the family of groups acting on rooted trees which will constitute the focus of the thesis, the family of *spinal groups*. This family was defined in [7] as a subclass of a family of groups defined in [34], and was further studied in [5, 65]. It contains a variety of examples of groups with very interesting properties. Within this family, we find branch groups, groups of intermediate growth, just-infinite groups or infinite finitely-generated torsion groups. The family of spinal groups is indexed by three parameters: the width of the tree $d \geq 2$, an integer $m \geq 1$ and a sequence ω on a finite alphabet $\Omega = \Omega_{d,m}$.

Consider the automorphism $a \in \mathrm{Aut}(T_d)$ defined by

$$a = (01 \dots d-1)(1, \dots, 1).$$

This automorphism cyclically permutes the vertices of the first level, and hence the subtrees rooted at them. Equivalently, a is the automorphism defined by

$$a(iv) = (i+1)v,$$

for any $i \in X$ and $v \in X^*$, where the sum is taken modulo d. We set $A = \langle a \rangle \leq \mathrm{Aut}(T_d)$, and we have $A \cong \mathbb{Z}/d\mathbb{Z}$.

Let now $m \geq 1$ be an integer and $B = (\mathbb{Z}/d\mathbb{Z})^m$, and let $\omega = (\omega_n)_{n\geq 0} \in \mathrm{Epi}(B, A)^{\mathbb{N}}$ be a sequence of epimorphisms from B to A. For every $b \in B$, we define the automorphism $b_\omega \in \mathrm{Aut}(T_d)$ as follows

$$b_\omega = (\omega_0(b), 1, \ldots, 1, b_{\sigma\omega}),$$

where σ is the map which deletes the first epimorphism of ω. These automorphisms are often called *spinal* automorphisms, as they fix the rightmost ray of the tree (the *spine*). They act as a^j on the subtree rooted at $(d-1)^r 0$, if $\omega_r(b) = a^j$. Equivalently, the automorphisms b_ω are defined by

$$b_\omega((d-1)^r iv) = \begin{cases} (d-1)^r 0 \omega_r(b)(v) & \text{if } i = 0 \\ (d-1)^r iv & \text{otherwise} \end{cases},$$

for any $r \geq 0$, $i \in X$ and $v \in X^*$. A picture of the generators can be found in Figure 2.5. Notice that every vertex different from the root can be written as $(d-1)^r iv$, so the definition is complete. Let $B_\omega = \langle b_\omega \mid b \in B \rangle \leq \mathrm{Aut}(T_d)$.

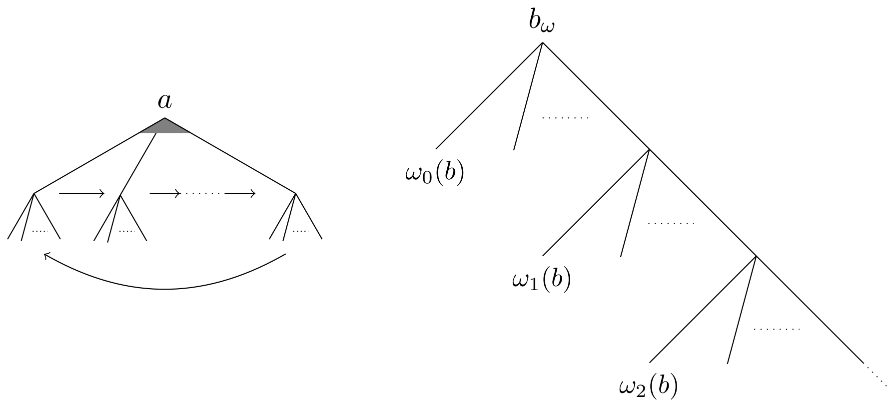


Figure 2.5: Both types of generators of a spinal group. The spinal generating set contains all the powers of a and b_ω for all $b \in B$.

The set $\mathrm{Epi}(B, A)$ of epimorphisms from B to A is always nonempty, as $m \geq 1$. In particular, every b_ω stabilizes the first level of the tree. Let $\Omega = \Omega_{d,m} \subset \mathrm{Epi}(B, A)^{\mathbb{N}}$ be the set of sequences satisfying the condition

$$\forall i \geq 0, \quad \bigcap_{j \geq i} \mathrm{Ker}(\omega_j) = 1. \tag{2.1}$$

Then, for every $\omega \in \Omega$, the action of B_ω on T_d and ∂T_d is faithful.

We say that $G_\omega = \langle A, B_\omega \rangle$ is the *spinal group* defined by d, m and ω. We will abuse notation and write $B = B_\omega$. The action of any spinal group on T_d is transitive at every level. Spinal groups admit a natural generating set, $S = (A \cup B) \setminus \{1\}$, which we call *spinal generating set*. Unless stated otherwise, we consider spinal groups with this generating set.

The definition of spinal groups in [7, 5] slightly differs from the definition given above. It allows for more general groups A and B, but at the same time restricts Ω by assuming an extra condition, namely

$$\forall i \geq 0, \quad \bigcup_{j \geq i} \mathrm{Ker}(\omega_j) = B. \tag{2.2}$$

This condition ensures that the groups G_ω are torsion. In this thesis, we will use the term *spinal groups* for the groups G_ω with $A = \mathbb{Z}/d\mathbb{Z}$, $B = (\mathbb{Z}/d\mathbb{Z})^m$ and $\omega \in \Omega$ as defined above, not assuming this extra condition and thus considering both torsion and non-torsion groups. For fixed $d \geq 2$ and $m \geq 1$, we denote $\mathcal{S}_{d,m}$ the family of spinal groups defined by d, m and any sequence $\omega \in \Omega$.

It was proven in [7] that all torsion spinal groups are of intermediate growth. Also all groups in $\mathcal{S}_{2,2}$ have intermediate growth, as shown in [32]. The same proof can be extended to $\mathcal{S}_{2,m}$, $m \geq 2$. Moreover, in [29] it is proved that almost all groups in $\mathcal{S}_{3,m}$ with respect to the uniform Bernoulli measure on $\Omega_{3,m}$ have intermediate growth too. The problem is open for non-torsion groups in $\mathcal{S}_{d,m}$, with $d \geq 4$ and $m \geq 1$. The main result in [49] implies that all spinal groups are amenable.

Example 2.2.1. Let $d = 2$ and $m = 1$. Let also π be the only epimorphism in $\mathrm{Epi}(B, A)$. In this case, Ω contains only the constant sequence $\pi^{\mathbb{N}}$. The spinal group $G_{\pi^{\mathbb{N}}}$ is generated by two involutions a and b, and is in fact isomorphic to the infinite dihedral group D_∞. This is the only example of spinal group which is not a branch group, and its behavior will often be different to the rest of spinal groups.

Example 2.2.2. Let $d = 2$ and $m = 2$. Then, $B = \{1, b, c, d\}$, with $b^2 = c^2 = d^2 = bcd = 1$. The set $\mathrm{Epi}(B, A)$ has three epimorphisms, let us call them π_b, π_c and π_d, in such a way that $\pi_x(x) = 1$, for $x \in \{b, c, d\}$. The family $\mathcal{S}_{2,2}$ is the uncountable family of groups of intermediate growth constructed by Grigorchuk in [32]. Their spinal generating set is precisely $S = \{a, b, c, d\}$.

In particular, the first group known to have intermediate growth, so-called Grigorchuk's group, corresponds to the spinal group $G_\omega \in \mathcal{S}_{2,2}$ with $\omega = (\pi_d \pi_c \pi_b)^{\mathbb{N}}$. This group is an outstanding example in the context of groups acting on rooted trees. In addition of having intermediate growth, it is a branch group, it is a 2-group, it is not finitely presented, it is just-infinite and is amenable but not elementary amenable. In fact, the very definition of spinal groups arises as a generalization of this group.

Another notable example in $\mathcal{S}_{2,2}$ is the so-called Grigorchuk-Erschler group. This is the spinal group $G_\omega \in \mathcal{S}_{2,2}$ with $\omega = (\pi_d \pi_c)^{\mathbb{N}}$. It also has intermediate growth, but the element ab has infinite order, so it is not torsion.

Example 2.2.3. Let $d = 3$, $m = 1$ and $\omega = \pi^{\mathbb{N}}$, where $\pi \in \operatorname{Epi}(B, A)$ is the epimorphism mapping the generator b of B to a. The spinal group G_ω defined by these parameters is the Fabrykowski-Gupta group. It was shown in [6] that it has intermediate growth. Its spinal generating set is $S = \{a, a^{-1}, b, b^{-1}\}$.

Example 2.2.4. Let p be an odd prime, and let $d = p$ and $m = 2$, so that $A = \{1, a, \dots, a^{p-1}\}$ and $B = \langle b, c\rangle$. For every $i \in X$, define $\pi_i \in \operatorname{Epi}(B, A)$ as $\pi_i(b) = a$, $\pi_i(c) = a^i$, and $\pi \in \operatorname{Epi}(B, A)$ as $\pi(b) = 1$, $\pi(c) = a$. The family defined by spinal groups with $d = p$, $m = 2$ and $\omega \in \Omega_{p,2} \cap \{\pi_0, \dots, \pi_{d-1}, \pi\}^{\mathbb{N}}$ is precisely the one defined in [33], in one of the first attempts to generalize the family $\mathcal{S}_{2,2}$ for trees of higher degree.

Example 2.2.5. Let p be a prime, $m \geq 1$ and f a polynomial in $\mathbb{Z}/p\mathbb{Z}[x]$ of degree m. Following [65], if we set $A = \mathbb{Z}/p\mathbb{Z}$ and $B = (\mathbb{Z}/p\mathbb{Z})^m$ we can define a group $G_{p,f}$ acting on T_p. The choice of f is equivalent to the choice of $\alpha \in \operatorname{Epi}(B, A)$ and $\rho \in \operatorname{Aut}(B)$. This defines a family of groups which we call Šunić groups.

This family of groups is recovered as a family of spinal groups with $d = p$, keeping $m \geq 1$ and setting $\omega = \omega_0\omega_1 \dots$ such that $\omega_n = \alpha \circ \rho^n$, for every $n \geq 0$. Some examples of groups in this family are the infinite dihedral, Grigorchuk's group or Fabrykowski-Gupta group. However, by construction the sequence ω is always periodic, which is not necessarily the case for spinal groups in general.

Proposition 2.2.6. *Let G_ω be a spinal group. Then, G_ω is self-similar if and only if G_ω is a Šunić group. Equivalently, if and only if for every $n \geq 0$ there exists $\rho \in \operatorname{Aut}(B)$ such that $\omega_n = \omega_0 \circ \rho^n$.*

Proof. It is clear from the definition that any Šunić group is self-similar. If G_ω is a self-similar spinal group, then for every $b \in B$ there must exist some $b' \in B$ such that $b' = b|_{d-1}$, so $\omega_{n+1}(b) = \omega_n(b')$ for every $n \geq 0$. If there was $b'' \in B$ for which this was also true, then $b'(b'')^{-1}$ would be in $\operatorname{Ker}(\omega_n)$ for every $n \geq 0$, which would violate condition (2.1) unless $b' = b''$. Therefore b' is unique, and thus we can define an endomorphism $\rho : B \to B$ as $\rho(b) = b'$. If $1 \neq b \in \operatorname{Ker}(\rho)$, then $\omega_n(b) = 1$ for every $n \geq 1$, which again would violate condition (2.1), hence ρ is an automorphism. If we define α to be ω_0, we are in the setting of a Šunić group. □

Example 2.2.7. According to [57], an automorphism group of a rooted tree T is an iterated monodromy group of a post-critically finite backward iteration of polynomials if and only if it is generated by a dendroid set of automorphisms of T.

We will omit the details, but it can be proven that any spinal group G_ω for which at most m different $\operatorname{Ker}(\omega_i)$ occur, for all $i \geq 0$, admits a dendroid generating set. This means that there are many iterated monodromy groups in the spinal family, self-similar or not. A particular case of iterated monodromy groups of a sequence of polynomials is when this sequence is the constant sequence $f^{\mathbb{N}}$, and so we call the iterated monodromy group $IMG(f)$. In our setting, this happens only when $m = 1$. Further details can be found in [57].

2.3 Schreier graphs

We now define the graphs which constitute the main objects of study of the thesis, so-called Schreier graphs. Let G be a group, generated by a finite set S, and let $H \leq G$. The *Schreier graph of G associated with H with respect to S*, denoted $\mathrm{Sch}(G, H, S)$, is the graph whose vertex set is the set of lateral classes G/H with the following edges: for every $s \in S$ and $gH \in G/H$, there is an edge from gH to sgH labeled by s. Notice that Schreier graphs may have loops or multiple edges, and their edges are oriented and labeled by the finite generating set S. We say that a s-edge is an edge labeled by $s \in S$. Similarly, if $T \subset S$, a T-edge is an edge labeled by a generator $t \in T$. Schreier graphs are $|S|$-regular, since every vertex gH has exactly one outgoing edge labeled by each $s \in S$. Cayley graphs are a particular case of Schreier graphs, where H is the trivial subgroup.

We usually consider Schreier graphs of groups $G \leq \mathrm{Aut}(T_d)$, with $H = \mathrm{Stab}_G(x)$, for $x \in X^* \sqcup X^{\mathbb{N}}$. If the action of G is transitive at every level of the tree, as is the case for spinal groups, then the Schreier graph $\mathrm{Sch}(G, \mathrm{Stab}_G(v), S)$ coincides for every $v \in X^n$. Consequently, for any spinal group G_ω, we denote $\Gamma_n = \mathrm{Sch}(G_\omega, \mathrm{Stab}_{G_\omega}(v), S)$, where $v \in X^n$ and S is the spinal generating set. Similarly, for $\xi \in X^{\mathbb{N}}$, we write $\Gamma_\xi = \mathrm{Sch}(G_\omega, \mathrm{Stab}_G(\xi), S)$.

There is a natural bijection between the sets $G/\mathrm{Stab}_G(x)$ and the orbits Gx. Hence, any Schreier graph can be regarded as the so-called orbital Schreier graph, with vertex set Gx instead of $G/\mathrm{Stab}_G(x)$. The edges are then given by the action, so for every $s \in S$ and $y \in Gx$ there is a s-edge from y to $s(y)$. In our context, these two perspectives are equivalent, and we will use the latter when describing Schreier graphs, usually labeling vertices of the graph with the sets X^n or $G_\omega\xi$, the orbits of vertices or boundary points, respectively.

Example 2.3.1. Consider the action of a on the first level of the tree X. The Schreier graph associated with the generating set $\{a\}$ has vertex set X and is a d-cycle, as shown in Figure 2.6.

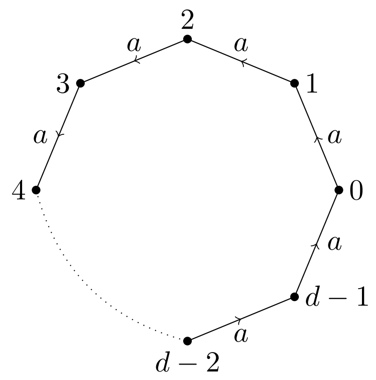


Figure 2.6: Schreier graph of the action of $\langle a \rangle$ on X with generating set $\{a\}$.

2.4 Spectral properties

2.4.1 Adjacency operator

Many of the results in the thesis involve the spectral properties of the Laplacian operator on a graph. Instead, we will consider the adjacency operator, the study of which is equivalent to the study of the Laplacian for regular graphs. Let Γ be a graph, and let us abuse notation by denoting its vertex set also by Γ. Then $\ell^2(\Gamma)$ is the space of square-summable functions on the vertices, which is a Hilbert space. The *adjacency operator of* Γ is the operator $\Delta_\Gamma : \ell^2(\Gamma) \to \ell^2(\Gamma)$, defined by

$$\Delta_\Gamma f(v) = \sum_{w \sim v} f(w),$$

where $\sim$ denotes the adjacency relation between vertices.

This operator is self-adjoint. If Γ is k-regular, we may normalize the adjacency operator, and then $M_\Gamma = \frac{1}{k}\Delta_\Gamma$ is the Markov operator on Γ. Similarly, the operator $1 - M_\Gamma$ is usually called the Laplacian operator on Γ.

In the thesis we study adjacency operators on Schreier or Cayley graphs of spinal groups. For every $n \geq 0$, we shall write Δ_n for the adjacency operator on Γ_n, while, for $\xi \in X^{\mathbb{N}}$, Δ_ξ shall denote the adjacency operator on Γ_ξ. More precisely, they are defined by

$$\Delta_n f(v) = \sum_{s \in S} f(sv), \qquad \Delta_\xi f(\xi') = \sum_{s \in S} f(s\xi').$$

2.4.2 Spectrum

We particularly interest ourselves in the spectra of these operators. If $\mathcal{H}$ is a Hilbert space over $\mathbb{C}$ and $T : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator, the *spectrum of* T is the set

$$\mathrm{sp}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not invertible}\}.$$

The spectrum of T is contained in the ball of radius $\|T\|$. If T is self-adjoint, then by the spectral theorem its spectrum is contained in the real line. Therefore, in that case, we have $\mathrm{sp}(T) \subset [-\|T\|, \|T\|]$.

An *eigenvalue* of T is a solution $\lambda \in \mathbb{C}$ to the equation $Tv = \lambda v$, for $v \in \mathcal{H}$, $v \neq 0$. An *eigenvector* (or *eigenfunction*) of T of eigenvalue $\lambda \in \mathbb{C}$ is an element $v \in \mathcal{H}$ for which the equation $Tv = \lambda v$ holds. Notice that every eigenvalue of T belongs to $\mathrm{sp}(T)$. If $\mathcal{H}$ has finite dimension, then $\mathrm{sp}(T)$ is exactly the set of eigenvalues of T. If $\mathcal{H}$ is infinite-dimensional, $\mathrm{sp}(T)$ may contain other values.

In this thesis, we consider $\mathcal{H} = \ell^2(\Gamma)$, with Γ being a Schreier or Cayley graph of some spinal group. We compute spectra of adjacency operators on such graphs, which are $|S|$-regular. These operators are always self-adjoint, and their operator norm is exactly $|S|$. Their spectra are therefore contained in the interval $[-|S|, |S|]$.

2.4.3 Spectral measures

Spectral measures constitute another important object of study of this thesis. Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ be a bounded, linear self-adjoint operator. For every $v, w \in \mathcal{H}$, there is a unique measure $\mu_{v,w}$ on $\mathrm{sp}(T)$ satisfying, for every $n \geq 0$,

$$\int_{\mathrm{sp}(T)} x^n \, d\mu_{v,w}(x) = \langle T^n v, w \rangle.$$

We shall call this measure the *spectral measure of T associated with v and w*. Notice that $\mu_{v,w}(\mathrm{sp}(T)) = \langle v, w \rangle$, so $\mu_{v,w}$ is a finite measure, but could in general take negative values. For $w = v$, the measure $\mu_{v,v}$ is positive. In that case, we call it the *spectral measure of T associated with v*, and denote it simply $\mu_v = \mu_{v,v}$.

If $\mathcal{H} = \ell^2(\Gamma)$ for some graph Γ, we may consider the Dirac delta function δ_p, for $p \in \Gamma$, which takes value 1 on p and vanishes everywhere else. The spectral measure associated with such a function, denoted $\mu_p = \mu_{\delta_p}$, is sometimes called the *Kesten spectral measure of p*.

Spectral measures are interesting as they provide more detailed information than just the spectrum of an operator as a set. The spectral measures are supported on the spectrum. According to a refinement of Lebesgue's decomposition Theorem, for any $v \in \mathcal{H}$, the spectral measure $\mu = \mu_v$ can be decomposed into three mutually singular measures $\mu = \mu_{pp} + \mu_{ac} + \mu_{sc}$, where μ_{pp} is a pure point or discrete measure, consisting only of atoms with positive measure, μ_{ac} is absolutely continuous with respect to the Lebesgue measure, and μ_{sc} is called the singular continuous part. This allows us to define three subspaces $\mathcal{H}_{pp}$, $\mathcal{H}_{ac}$ and $\mathcal{H}_{sc}$ of $\mathcal{H}$, containing vectors whose spectral measures are absolutely continuous with each of μ_{pp}, μ_{ac} and μ_{sc}, respectively. These subspaces are invariant under T, and in virtue of the spectral theorem, we have $\mathcal{H} = \mathcal{H}_{pp} \oplus \mathcal{H}_{ac} \oplus \mathcal{H}_{sc}$.

In this thesis we study how different Schreier graphs of spinal groups provide different outcomes in this decomposition for the adjacency operator. We exhibit for instance Schreier graphs where the spectral measures are all absolutely continuous with respect to the Lebesgue measure. We also show Schreier graphs with pure point spectrum, or, equivalently, for which there exists a basis of eigenfunctions. Furthermore, we present examples of Schreier graphs with nontrivial singular continuous spectrum.

Chapter 3

Construction of the Schreier graphs

In this chapter we present the construction of Schreier graphs of spinal groups. We shall start by describing the Schreier graphs associated with the action of a spinal group on the n-th level of the tree X^n, and then we will provide some tools to study the action on $X^{\mathbb{N}}$, the boundary of the tree.

We discuss the number of ends of any given Schreier graph, and we characterize when two Schreier graphs are isomorphic as marked graphs, both as edge-labeled and unlabeled graphs. We also compute the size of the isomorphism classes in terms of the uniform Bernoulli measure μ on $X^{\mathbb{N}}$. Moreover, we provide a way of encoding any Schreier graph Γ_ξ as a tree with two equivalent definitions, one purely combinatorial and the other purely geometrical.

Finally, one can transfer an action of a finitely generating group on a topological space to an action on the so-called space of marked Schreier graphs. We do so for the action of any spinal group on $X^{\mathbb{N}}$, and we study the new action with respect to the original one.

3.1 Schreier graphs on X^n

Let $d \geq 2$, $m \geq 1$ and $\omega \in \Omega$, and let G_ω be the spinal group defined by these parameters. Let also S be the spinal generating set for G_ω. Our goal is to describe the graphs Γ_n for every $n \geq 1$. We recall that $\Gamma_n = \mathrm{Sch}(G_\omega, \mathrm{Stab}_{G_\omega}(v), S)$ for every $v \in X^n$, as the action of G_ω on X^n is transitive. Let us start by describing the Schreier graph on the first level.

Lemma 3.1.1. *The Schreier graph Γ_1, associated with the action of G_ω on X and spinal generating set S, is the graph with vertex set X and edges:*

- $\forall i \in X$, $\forall k = 1, \ldots, d-1$, *an a^k-edge from i to $i + k$.*
- $\forall i \in X$, $\forall b \in B$, *a b-loop on i.*

Proof. For every $i \in X$, i is mapped to $i+1$ by a, so it is mapped to $i+k$ by a^k. Moreover, every $b \in B$ stabilizes all vertices in the first level, so $b(i) = i$ for every i. □

We describe the rest of graphs Γ_n recursively. In order to do so, let us define the following operation on graphs.

Definition 3.1.2. Let Γ be a graph and $v \in \Gamma$ be a vertex. We set $\tilde{\Gamma}$ to be a copy of Γ but with all loops on v removed. Let Λ be a finite graph on k vertices, with $k \geq 2$, which we call $p_0, \ldots, p_{k-1}$.

Let $\tilde{\Gamma}_0, \ldots, \tilde{\Gamma}_{k-1}$ be k disjoint copies of $\tilde{\Gamma}$, with $v_0, \ldots, v_{k-1}$ be the vertices corresponding to v in $\tilde{\Gamma}$, respectively. We define the graph $\operatorname{Star}(\Lambda, \Gamma, v)$ as the disjoint union of Λ and all $\tilde{\Gamma}_i$, identifying p_i with v_i for every $i = 0, \ldots, k-1$. Namely,

$$\operatorname{Star}(\Lambda, \Gamma, v) = \left(\Lambda \sqcup \bigsqcup_{i=0}^{k-1} \tilde{\Gamma}_i \right) / \{p_i = v_i \mid i = 0, \ldots, k-1\}.$$

Observe that no edges are identified in the process. We will slightly abuse notation by calling Γ_i the i-th copy of Γ in $\operatorname{Star}(\Lambda, \Gamma, v)$. If the graphs Γ and Λ are edge-labeled, so is $\operatorname{Star}(\Lambda, \Gamma, v)$. The vertex set of $\operatorname{Star}(\Lambda, \Gamma, v)$ is in bijection with $\Gamma \times \{0, \ldots, k-1\}$.

The Star graph operation is similar to the inflation of graphs used in [9] and [12], but taking loops into account. We may now use it in order to recursively describe the graphs Γ_n.

Proposition 3.1.3. *The Schreier graph Γ_n, associated with the action of G_ω on X^n and spinal generating set S, is defined by*

$$\Gamma_1 = \operatorname{Star}(\Theta, \Xi, \emptyset), \qquad \Gamma_{n+1} = \operatorname{Star}(\Lambda_{\omega_{n-1}}, \Gamma_n, (d-1)^{n-1}0) \quad \forall n \geq 1,$$

where Ξ, Θ and Λ_π are the following finite graphs:

- *Ξ is the graph with vertex set $\{\emptyset\}$ which has $d^m - 1$ loops, each labeled with a different element $b \in B \setminus \{1\}$.*
- *Θ is the graph with vertex set X, which has an edge labeled by a^k from i to $i + k$, for every $i \in X$ and for every $k = 1, \ldots, d-1$.*
- *If $\pi \in \operatorname{Epi}(B, A)$, Λ_π is the graph with vertex set X with the following edges: for every $b \in B \setminus \{1\}$, let k be such that $\pi(b) = a^k$. Then, for every $i \in X$, there is an edge from i to $i + k$. Notice that this implies adding loops for all $b \in \operatorname{Ker}(\pi)$.*

Some examples of these graphs can be found in Figure 3.1.

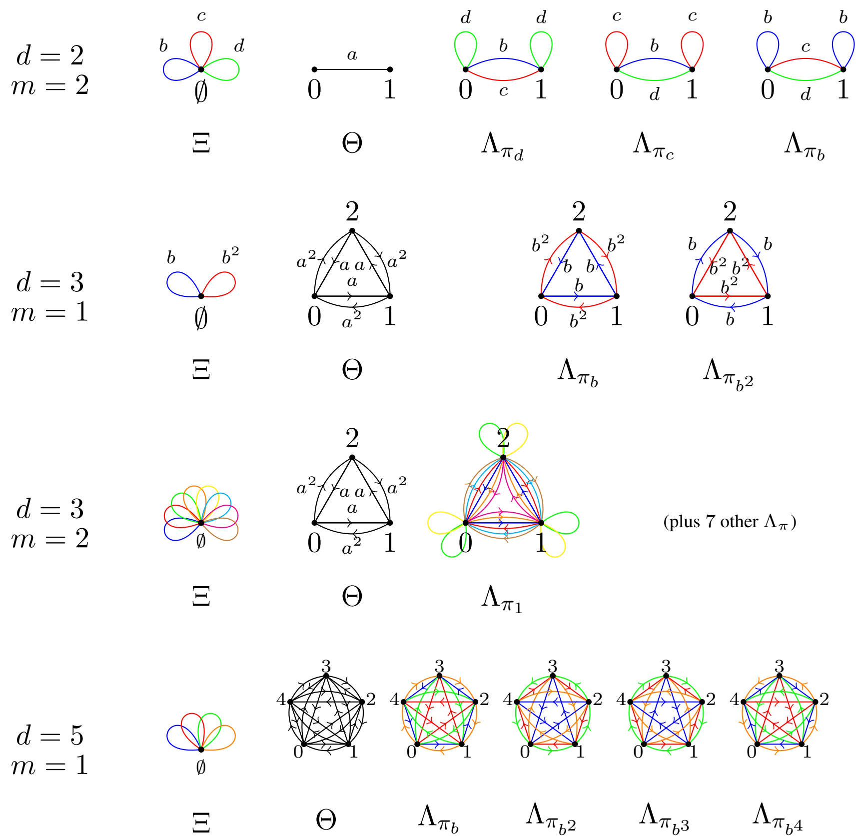


Figure 3.1: Blocks composing the Schreier graphs of spinal groups for some values of $d \geq 2$ and $m \geq 1$.

Proof. The claim about Γ_1 is the object of Lemma 3.1.1. Now assume $n \geq 1$, and let $\Gamma'_{n+1} = \mathrm{Star}(\Lambda_{\omega_{n-1}}, \Gamma_n, (d-1)^{n-1}0)$. By construction, their vertex sets are both X^{n+1}, so let us prove that their edges are the same. Let $v = v_0 \dots v_n \in X^{n+1}$, we will prove that its set of outgoing edges is the same in both Γ_{n+1} and Γ'_{n+1}. Let us call $w = v_0 \dots v_{n-1} \in X^n$, so that $v = wv_n$.

In Γ_{n+1}, there is exactly one outgoing edge for every generator $s \in S$ from v to $s(v)$. So, for every $s \in S$, we must prove that in Γ'_{n+1} there is exactly one outgoing s-edge from v to $s(v)$. Notice that $s(v) = s(wv_n) = s(w)v_n$ unless $w = (d-1)^{n-1}0$ and $s \in B$.

Suppose first that $w \neq (d-1)^{n-1}0$. In that case, in Γ_n, w has an outgoing s-edge towards $s(w)$. Moreover, as $w \neq (d-1)^{n-1}0$, the outgoing s-edge from v cannot go outside its copy of Γ_n. Therefore, v must have an outgoing s-edge to $s(w)v_n$, and in fact $s(w)v_n = s(v)$, again because $w \neq (d-1)^{n-1}0$.

Assume now that $w = (d-1)^{n-1}0$. If $s \in A$, $s(w) \neq w$, so the outgoing s-edge is not a loop, and so it is not removed in the construction of Γ'_{n+1}, hence there is an outgoing s-edge from v to $s(w)v_n$ in Γ'_{n+1}. Because $s \in A$ changes only the first digit of any vertex, we have $s(v) = s(w)v_n$.

Finally, suppose $w = (d-1)^{n-1}0$ and $s \in B$. In this case $s(w) = w$, so the edge in Γ_n is a loop and is indeed removed in the construction of Γ'_{n+1}. But the vertex v is identified with the vertex labeled by v_n in $\Lambda_{\omega_{n-1}}$, which has an outgoing s-edge towards the vertex labeled by $v_n + k$ in $\Lambda_{\omega_{n-1}}$, where $\omega_{n-1}(s) = a^k$. Therefore, in Γ'_{n+1}, the vertex v has an outgoing s-edge towards the vertex $w(v_n + k)$. As it turns out, $s(v) = s((d-1)^{n-1}0v_n) = (d-1)^{n-1}0\omega_{n-1}(s)(v_n) = w(v_n + k)$, which completes the proof. □

Remark 3.1.4. With this characterization, notice that $\operatorname{diam}(\Gamma_n) = 2^n - 1$.

Let us use Proposition 3.1.3 to obtain the Schreier graphs of some notable examples of spinal groups.

Example 3.1.5. The finite Schreier graphs of the infinite dihedral group ($d = 2$, $m = 1$), depicted in Figure 3.2 are segments with labels alternating between a and b.

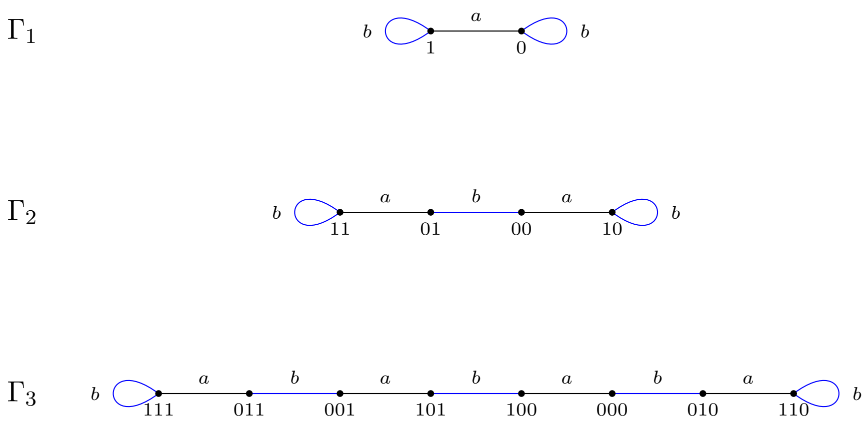


Figure 3.2: Finite Schreier graphs for the infinite dihedral group.

Example 3.1.6. The finite Schreier graphs for Grigorchuk's group were already described in [4], and more generally for any spinal group with $d = 2$ and $m = 2$ in [55]. Some examples of such graphs can be found in Figure 3.3.

Example 3.1.7. The simplest example with $d = 3$ and $m = 1$ is the Fabrykowski-Gupta group, some finite Schreier graphs of which, described in [4], are displayed in Figure 3.4.

Example 3.1.8. In order to give an example with higher d, Figure 3.5 shows some finite Schreier graphs for the spinal group with $d = 5$, $m = 1$ and constant sequence $\omega = \pi^{\mathbb{N}}$, with $\pi(b) = a$.

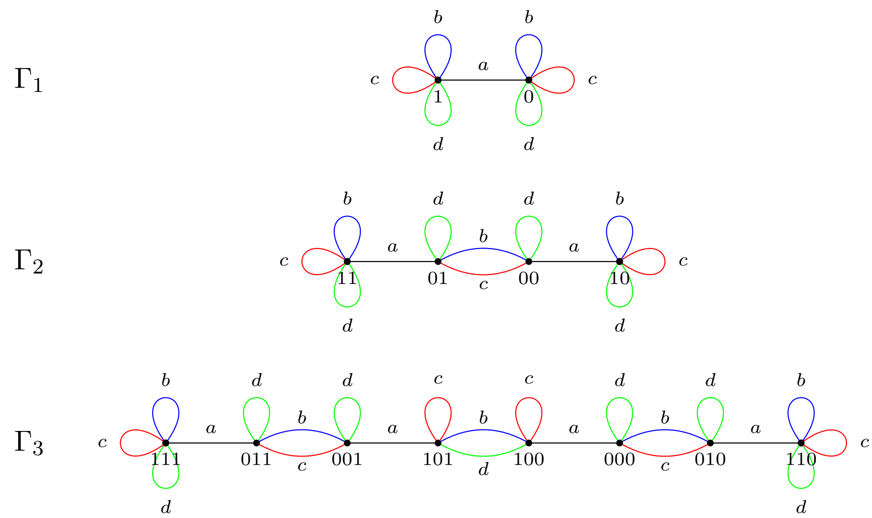


Figure 3.3: Finite Schreier graphs for Grigorchuk's group.

3.2 Schreier graphs on $X^{\mathbb{N}}$

Let $d \geq 2$, $m \geq 1$ and $\omega \in \Omega$, and let G_ω be the spinal group defined by these parameters. Let also S be the spinal generating set for G_ω. We now want to describe the Schreier graphs associated with the action of G_ω on the boundary of the tree $X^{\mathbb{N}}$. More precisely, for every $\xi \in X^{\mathbb{N}}$, we are interested in the graph $\Gamma_\xi = \mathrm{Sch}(G_\omega, \mathrm{Stab}_{G_\omega}(\xi), S)$.

Notice that $X^{\mathbb{N}}$ is endowed with the shift operator $\sigma : X^{\mathbb{N}} \to X^{\mathbb{N}}$, which removes the first letter of a sequence. We say that two points $\xi, \eta \in X^{\mathbb{N}}$ are *cofinal* if they differ at most in finitely many letters, that is, if there exists some $r \geq 0$ such that $\sigma^r(\xi) = \sigma^r(\eta)$. Cofinality is an equivalence relation, and we call the equivalence class of ξ its *cofinality class*, denoted $\mathrm{Cof}(\xi)$.

Proposition 3.2.1. *For every $\xi \in X^{\mathbb{N}}$, $G_\omega \xi = \mathrm{Cof}(\xi)$.*

Proof. Because a only changes the first digit of the sequence and any $b \in B$ either fixes the sequence or changes the first digit after a prefix $(d-1)^n 0$, any generator changes at most one digit of the sequence. Let $\eta \in G_\omega \xi$, so there exists $g \in G$ such that $\eta = g\xi$. Then, g changes at most $|g|_S$ digits in ξ, which implies that η and ξ are cofinal.

Conversely, we can check that starting from ξ and performing transformations corresponding to the generators (changing the first letter and changing the first letter after a specific pattern), we can obtain any point η cofinal with ξ. □

Proposition 3.2.2. *Let $\xi \in X^{\mathbb{N}}$. The Schreier graph Γ_ξ, associated with the action of G_ω on the orbit of ξ, has vertex set* $\mathrm{Cof}(\xi)$ *the following edges:*

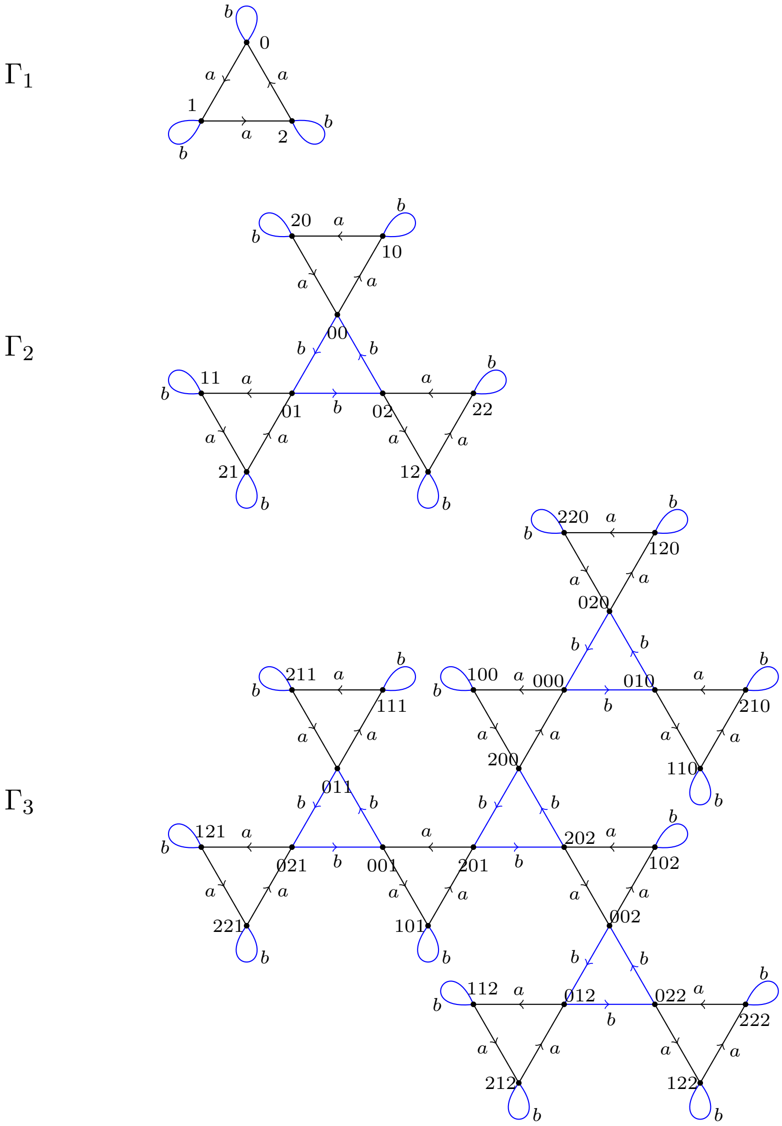


Figure 3.4: Finite Schreier graphs for the Fabrykowski-Gupta group. For clarity, only generators a and b are drawn. Edges labeled by their inverses are the same but reversed.

- $\forall \eta = \eta_0\eta_1 \cdots \in \mathrm{Cof}(\xi)$, $\forall k = 1, \ldots, d-1$, *there is an edge from* η *to* $(\eta_0+k)\eta_1\eta_2\ldots$ *labeled by* a^k.

- $\forall \eta = \eta_0\eta_1 \cdots \in \mathrm{Cof}(\xi)$, $\forall b \in B \setminus \{1\}$,

  - *If the* n*-prefix of* η *is* $(d-1)^{n-1}0$ *for some* $n \geq 1$, *then there is a* b*-edge from* η *to* $(d-1)^{n-1}0(\eta_n+k)\sigma^{n+1}\eta$, *where* k *is such that* $\omega_{n-1}(b) = a^k$.
  - *Otherwise there is a loop at* η *labeled by* b.

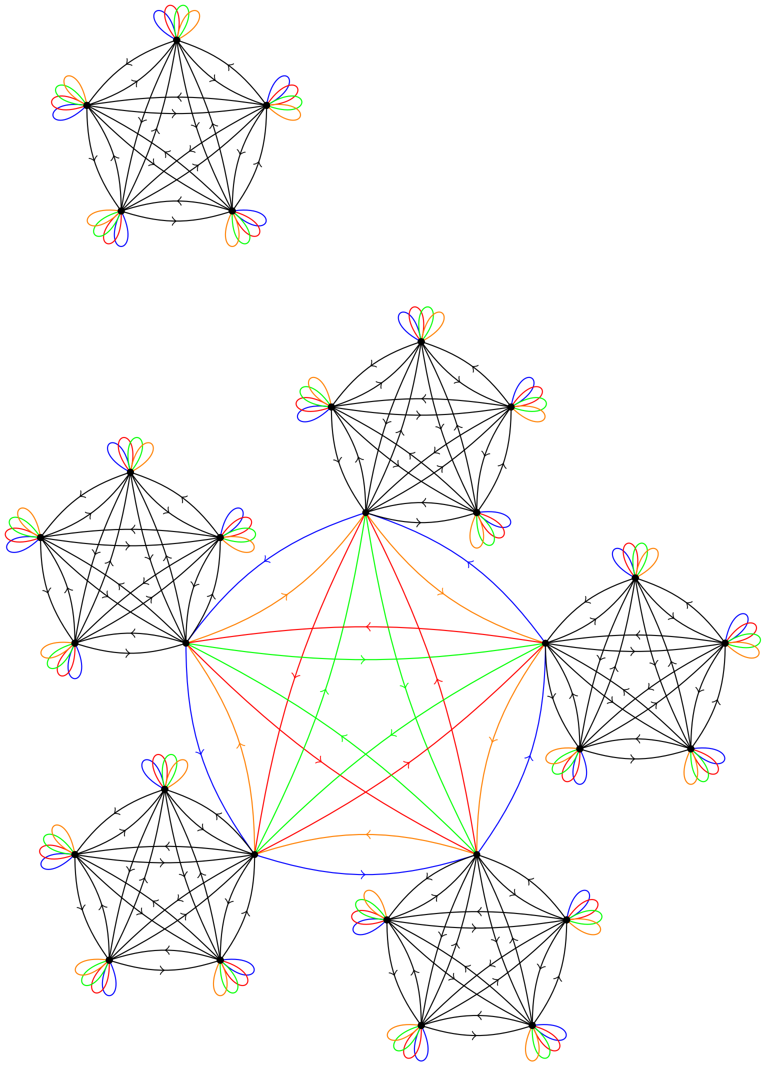


Figure 3.5: Finite Schreier graphs for the spinal group with $d = 5$, $m = 1$ and $\omega = \pi^{\mathbb{N}}$. For clarity, vertex and edge labelings have been omitted. Black edges correspond to powers of a, colored edges correspond to powers of b.

Proof. Proposition 3.2.1 shows that the orbit of ξ is its cofinality class, and the edges follow from the definition of the action of A and B on $X^{\mathbb{N}}$. $\square$

Definition 3.2.3. Let $\xi \in X^{\mathbb{N}}$ and $\eta \in \mathrm{Cof}(\xi)$. We define the following subgraphs of Γ_ξ, for every $n \geq 0$:

$$\Gamma_\eta^n = X^n \sigma^n \eta = \{v\sigma^n\eta \mid v \in X^n\},$$

$$\Lambda_\eta^n = (d-1)^n 0 X \sigma^{n+2}\eta = \{(d-1)^n 0 i \sigma^{n+2}\eta \mid i \in X\}.$$

Notice that Γ_η^n has d^n vertices and contains η, for every $n \geq 0$, Moreover, the subgraph Γ_η^n does not depend on $\eta_0 \ldots \eta_{n-1}$, the prefix of η of length n. The subgraphs $\{\Gamma_\eta^n\}_{\eta \in \mathrm{Cof}(\xi)}$ are either disjoint or coincident, and every vertex of Γ_ξ belongs to exactly one of them. Furthermore, no vertex in Γ_η^n has outgoing edges to $\Gamma_\xi \setminus \Gamma_\eta^n$ except for $(d-1)^{n-1}0\sigma^n\eta$ and possibly $(d-1)^n\sigma^n\eta$, depending on whether it is fixed by B or not.

The subgraph Λ_η^n has exactly d vertices, and it does not depend on $\eta_0 \ldots \eta_{n+1}$, the prefix of η of length $n+2$. If η is fixed by B, then it does not belong to Λ_η^n for any $n \geq 0$. Otherwise, η belongs to exactly one Λ_η^n, for some $n \geq 0$. Observe that Λ_η^n contains only B-edges. In fact, Λ_η^n is isomorphic to Λ_{ω_n} from Proposition 3.1.3 and Figure 3.1.

Proposition 3.2.4. *Let $\xi \in X^{\mathbb{N}}$, $\eta \in \mathrm{Cof}(\xi)$ and $n \geq 1$. Let $\bar{\Gamma}_\eta^n$ be the graph obtained from Γ_η^n after removing all loops on the vertices $(d-1)^{n-1}0\sigma^n\eta$ and $(d-1)^n\sigma^n\eta$. Similarly, let $\bar{\Gamma}_n$ be the graph obtained from Γ_n after removing all loops on the vertices $(d-1)^{n-1}0$ and $(d-1)^n$. Then $\bar{\Gamma}_\eta^n$ and $\bar{\Gamma}_n$ are isomorphic. The position of η in Γ_η^n depends only on its n-prefix $\eta_0 \ldots \eta_{n-1}$.*

Proof. Consider the bijection between the vertex sets given by $\varphi : v\sigma^n\eta \mapsto v$. Let $v \in X^n$ and $s \in S$, and let us prove that $v\sigma^n\eta$ has an outgoing s-edge to $s(v)\sigma^n\eta$ in Γ_η^n iff v has an outgoing s-edge to $s(v)$ in $\bar{\Gamma}_n$.

Suppose first that v is different from $(d-1)^{n-1}0$ and $(d-1)^n$, so no loops are removed on v nor $v\sigma^n\eta$, and therefore v has an outgoing s-edge to $s(v)$. For such v, we have $s(v\sigma^n\eta) = s(v)\sigma^n\eta$, and so $\varphi(s(v\sigma^n\eta)) = \varphi(s(v)\sigma^n\eta) = s(v)$.

Suppose now that v is $(d-1)^{n-1}0$ or $(d-1)^n$. If $s \in A$, again $s(v\sigma^n\eta) = s(v)\sigma^n\eta$, and so $\varphi(s(v\sigma^n\eta)) = \varphi(s(v)\sigma^n\eta) = s(v)$. Because $s(v) \neq v$, the edge is not a loop and hence is not removed in neither of the graphs.

Assume $s \in B$. As $v = (d-1)^{n-1}0$ or $v = (d-1)^n$, $s(v) = v$ and so v has a s-loop in Γ_n, which is removed in $\bar{\Gamma}_n$, so v has no outgoing s-edge in $\bar{\Gamma}_n$.

In Γ_ξ, $v\sigma^n\eta$ has an outgoing s-edge to $s(v)s|_v(\sigma^n\eta) = vs|_v(\sigma^n\eta)$. Two things may happen. If $s|_v(\sigma^n\eta) = \sigma^n\eta$, then $v\sigma^n\eta$ has an s-loop in Γ_η^n, which is later removed in $\bar{\Gamma}_\eta^n$. If $s|_v(\sigma^n\eta) \neq \sigma^n\eta$, then the s-edge goes from $v\sigma^n\eta$ to a vertex in $\Gamma_\xi \setminus \Gamma_\eta^n$, hence the vertex $v\sigma^n\eta$ has no outgoing s-edge in the subgraph Γ_η^n. In any of the two cases, $v\sigma^n\eta$ has no outgoing s-edge in $\bar{\Gamma}_\eta^n$. □

Proposition 3.2.4 is a powerful tool in order to describe the graph Γ_ξ. The subgraphs Γ_η^n are isomorphic to Γ_n up to the loops at two vertices. We will often abuse notation and say that these subgraphs Γ_η^n are copies of Γ_n in Γ_ξ.

For every $n \geq 1$, we can then regard Γ_ξ as the disjoint union of copies of Γ_η^n, joined together through the subgraphs Λ_η^r, $r \geq n-1$.

Remark 3.2.5. Let $\xi \in X^{\mathbb{N}}$, $\eta \in \mathrm{Cof}(\xi)$ and $n \geq 0$. We denote by $\mathcal{B}_v(r)$ the ball centered at a vertex v of radius $r \geq 0$. Combining Propositions 3.2.4 and 3.1.3, we can check the following properties.

$$\bigcup_{v \in \Lambda_\eta^n} \mathcal{B}_v(2^{n+1} - 1) = \Gamma_\eta^{n+2}. \tag{3.1}$$

$$\bigcup_{v \in \Lambda_\eta^n} \mathcal{B}_v(2^k - 1) = X^k (d-1)^{n-k} 0 X \sigma^{n+2} \eta, \quad 0 \leq k \leq n. \tag{3.2}$$

Remark 3.2.6. After we fix $d \geq 2$ and $m \geq 1$, the Schreier graphs Γ_n for spinal groups $G_\omega \in \mathcal{S}_{d,m}$ only differ in the labeling of the B-edges, for every $n \geq 0$. Indeed, by Proposition 3.1.3 the construction depends on ω only in the graphs Λ_π, for $\pi \in \mathrm{Epi}(B, A)$, which implies that these graphs are all isomorphic as unlabeled graphs.

As we will see later in Section 3.5, for every $d \geq 2$ and $m \geq 1$, once $\xi \in X^{\mathbb{N}}$ is fixed all graphs Γ_ξ are isomorphic as unlabeled graphs for every spinal group $G_\omega \in \mathcal{S}_{d,m}$. This allows to extend the results about the Schreier graphs of one particular example $G_\omega \in \mathcal{S}_{d,m}$ to all groups in $\mathcal{S}_{d,m}$, as long as these results do not depend on the edge labeling.

Similarly, if we only fix $d \geq 2$, the unlabeled Schreier graphs of spinal groups in $\mathcal{S}_{d,m}$ are obtained from those for spinal groups in $\mathcal{S}_{d,1}$, but with additional B-loops and multiple B-edges. Again this is true because the same holds for the graphs Λ_π, for $\pi \in \mathrm{Epi}(B, A)$, which are the building blocks of the Schreier graphs.

Example 3.2.7. There are two infinite Schreier graphs up to isomorphism for the infinite dihedral group ($d = 2$, $m = 1$), shown in Figure 3.6. One corresponds to the orbit of the point $1^{\mathbb{N}}$, while the other is the Schreier graph of any other point ξ in $X^{\mathbb{N}}$.

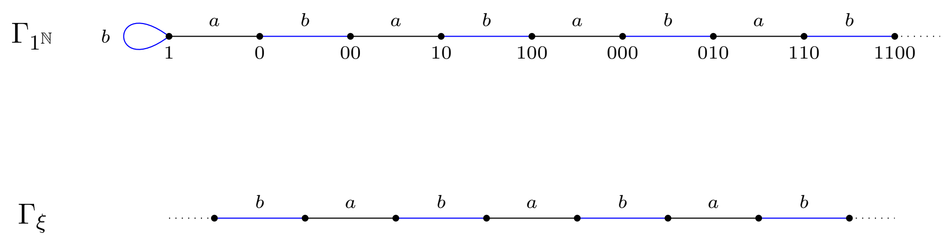


Figure 3.6: Infinite Schreier graphs for the infinite dihedral group. Vertex labels in $\Gamma_{1^{\mathbb{N}}}$ are to be concatenated with $1^{\mathbb{N}}$.

Example 3.2.8. We exhibit two examples of infinite Schreier graphs for Grigorchuk's group ($d = 2$, $m = 2$). The graphs $\Gamma_{1^{\mathbb{N}}}$ and $\Gamma_{0^{\mathbb{N}}}$ are a one-ended line and a two-ended line, respectively. They are depicted in Figure 3.7.

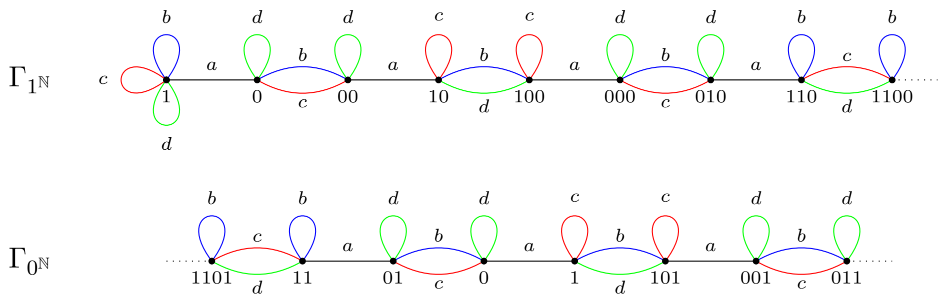


Figure 3.7: Infinite Schreier graphs for Grigorchuk's group. Vertex labels in $\Gamma_{1^{\mathbb{N}}}$ or $\Gamma_{0^{\mathbb{N}}}$ are to be concatenated with $1^{\mathbb{N}}$ or $0^{\mathbb{N}}$, respectively.

Example 3.2.9. Figure 3.8 illustrates two infinite Schreier graphs for the Fabrykowski-Gupta group ($d = 3$, $m = 1$). As we will see later in this chapter (Theorem 3.3.1), the graph $\Gamma_{2^{\mathbb{N}}}$ is one-ended and $\Gamma_{0^{\mathbb{N}}}$ is two-ended.

3.3 Number of ends

When studying infinite graphs, an important invariant is the number of ends. In [18], the authors count the number of ends of the Schreier graphs associated with the Basilica group. The method described in [12] allows to compute the number of ends of Schreier graphs of self-similar automata groups. We will use it for spinal groups to see that their Schreier graphs are either one or two-ended.

We say that a graph Γ has k ends, or is k-ended, if, for every finite subgraph $F \subset \Gamma$, the subgraph $\Gamma \setminus F$ has not more than k infinite connected components, and this k is minimal. The number of ends is a property of unlabeled, unmarked graphs, and loops or multiple edges do not play any role. Therefore, the number of ends of a Schreier graph Γ_ξ of a spinal group depends only on d and $\xi \in X^{\mathbb{N}}$, but not on m or the sequence $\omega \in \Omega_{d,m}$ (see Remark 3.2.6). We can therefore count the number of ends for Schreier graphs of the spinal groups G_d defined, for each $d \geq 2$, by $m = 1$ and $\omega = \pi^{\mathbb{N}}$, where π maps the generator of b of B to a. The groups G_d happen to be automata groups (see Figure 3.9), so we can apply the results on the ends of Schreier graphs of automata groups from [12].

Theorem 3.3.1. *Let G_ω be a spinal group and $\xi \in X^{\mathbb{N}}$. We partition $X^{\mathbb{N}} = E_1 \sqcup E_2$ with*

$$E_2 = X^*\{0, d-1\}^{\mathbb{N}} \setminus \mathrm{Cof}((d-1)^{\mathbb{N}}), \quad E_1 = X^{\mathbb{N}} \setminus E_2.$$

Then Γ_ξ is 2-ended if and only if $\xi \in E_2$ and Γ_ξ is 1-ended if and only if $\xi \in E_1$.

Proof. As mentioned above, it suffices to show the claim for the groups G_d, $d \geq 2$. G_2 is the infinite dihedral group, for which $E_2 = X^{\mathbb{N}} \setminus \mathrm{Cof}(1^{\mathbb{N}})$ and $E_1 = \mathrm{Cof}(1^{\mathbb{N}})$. Any $\xi \in E_2$

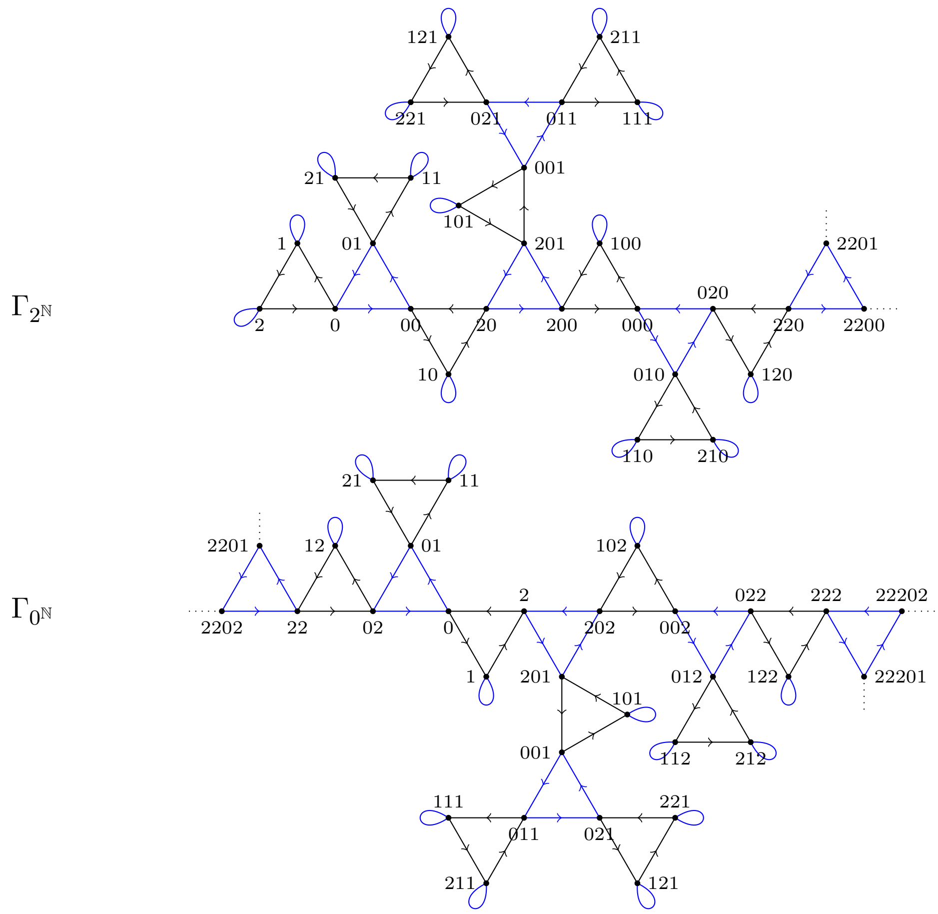


Figure 3.8: Infinite Schreier graphs for the Fabrykowski-Gupta group. For clarity, only generators a (in black) and b (in blue) are drawn. Edges labeled by their inverses are the same but reversed. Vertex labels in $\Gamma_{2^{\mathbb{N}}}$ or $\Gamma_{0^{\mathbb{N}}}$ are to be concatenated with $2^{\mathbb{N}}$ or $0^{\mathbb{N}}$, respectively.

has trivial stabilizer, so all the corresponding Schreier graphs are isomorphic to the Cayley graph, a two-ended line alternating a and b-edges. The stabilizer of $1^{\mathbb{N}}$ contains B, so $\Gamma_{1^{\mathbb{N}}}$ is a one-ended line. Section 5.2 in [12] shows it for the Fabrykowski-Gupta group G_3. The same proof applies for $d > 3$, by replacing 2 by $d-1$ and 1 by $\{1, \dots, d-2\}$. □

The set $X^{\mathbb{N}}$ is naturally equipped with the uniform Bernoulli measure μ. Let us compute the measure of the sets E_1 and E_2 for $d \geq 2$.

Lemma 3.3.2. *If $E \subset X^{\mathbb{N}}$ such that $\mu(E) = 0$, then $\mu(X^*E) = 0$.*

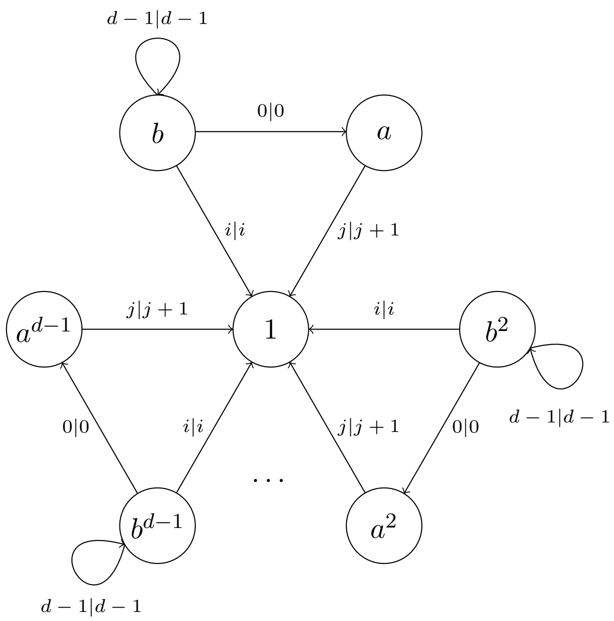


Figure 3.9: Finite-state automaton generating G_d. The letters i and j stand for any $i \in X \setminus \{0, d-1\}$ and $j \in X$, respectively.

Proof. We can decompose $X^*E = \bigsqcup_{n\geq 0} \bigsqcup_{w\in X^n} wE$, so

$$\mu(X^*E) = \sum_{n\geq 0} \sum_{w\in X^n} \mu(wE) = \sum_{n\geq 0} \sum_{w\in X^n} \frac{\mu(E)}{d^n} = \sum_{n\geq 0} \mu(E) = 0.$$

□

Theorem 3.3.3. *Let G_ω be a spinal group and E_1 and E_2 as in Theorem 3.3.1.*

- *If $d = 2$, then $\mu(E_1) = 0$ and $\mu(E_2) = 1$.*
- *If $d \geq 3$, then $\mu(E_1) = 1$ and $\mu(E_2) = 0$.*

Proof. For $d = 2$, we have $E_1 = \mathrm{Cof}(1^{\mathbb{N}})$ and $E_2 = X^{\mathbb{N}} \setminus \mathrm{Cof}(1^{\mathbb{N}})$. By Lemma 3.3.2, $\mu(E_1) = \mu(X^*1^{\mathbb{N}}) = \mu(1^{\mathbb{N}}) = 0$.

For $d \geq 3$, we have $\mu(E_2) \leq \mu(X^*\{0, d-1\}^{\mathbb{N}})$. Decomposing after the first digit, we obtain $\mu(\{0, d-1\}^{\mathbb{N}}) = \frac{2}{d}\mu(\{0, d-1\}^{\mathbb{N}})$, so $\mu(\{0, d-1\}^{\mathbb{N}}) = 0$. Again using Lemma 3.3.2, we obtain $\mu(E_2) = 0$. □

Theorem 3.3.3 shows a difference between binary spinal groups and the rest. For spinal groups with $d = 2$, all Schreier graphs are two-ended except for one orbit, while for $d \geq 3$ the set of points of the boundary whose Schreier graph is one-ended has measure one, even though there are uncountably many orbits with two-ended graphs.

3.4 Isomorphism classes of labeled Schreier graphs

Let us call the isomorphism class of (Γ_ξ, ξ) as a marked labeled (unlabeled) graph its labeled (unlabeled) isomorphism class. The measure of an isomorphism class is the uniform Bernoulli measure of the set of boundary points $\xi \in X^{\mathbb{N}}$ for which (Γ_ξ, ξ) is in the isomorphism class.

If the Schreier graphs (Γ_ξ, ξ) are considered as marked, labeled graphs, the answer to the isomorphism problem is immediate.

Proposition 3.4.1. *Let G_ω be a spinal group with $\omega \in \Omega_{d,m}$, excluding the case $d = 2$, $m = 1$. For $\xi, \eta \in X^{\mathbb{N}}$, the graphs (Γ_ξ, ξ) and (Γ_η, η) are isomorphic as marked labeled graphs if and only if $\xi = \eta$.*

Proof. For any branch group, Proposition 2.2 in [35] implies that the stabilizers of different points of $X^{\mathbb{N}}$ are different. Since the graph Γ_ξ is the Schreier graph of a spinal group G_ω with respect to the stabilizer of the point $\xi \in X^{\mathbb{N}}$, two graphs (Γ_ξ, ξ) and (Γ_η, η) cannot be isomorphic as marked labeled graphs unless $\xi = \eta$. □

Corollary 3.4.2. *There are uncountably many labeled isomorphism classes of marked Schreier graphs (Γ_ξ, ξ), each of measure zero.*

Notice that there is only one exception in Proposition 3.4.1. The only spinal group with $d = 2, m = 1$ is the infinite dihedral group, which is the only spinal group which is not a branch group. All points $\xi \in X^{\mathbb{N}} \setminus \mathrm{Cof}(1^{\mathbb{N}})$ give rise to Schreier graphs (Γ_ξ, ξ) isomorphic to the Cayley graph. Therefore, all these graphs are isomorphic as marked edge-labeled graphs. The remaining graphs (Γ_ξ, ξ) for $\xi \in \mathrm{Cof}(1^{\mathbb{N}})$ are pairwise non-isomorphic, as they are different markings of the same graph $\Gamma_{1^{\mathbb{N}}}$, which is a one-ended line. Hence, for the infinite dihedral there is one labeled isomorphism class of measure one, and countably many of zero measure.

3.5 Isomorphism classes of unlabeled Schreier graphs

Let us now consider the isomorphism problem for Schreier graphs as marked unlabeled graphs. Unlike for the labeled case, the unlabeled isomorphism classes are nontrivial.

3.5.1 Binary case

For the case of the infinite dihedral ($d = 2$, $m = 2$), the unlabeled isomorphism classes coincide with the labeled ones. Let us start with the study of the rest of spinal groups with $d = 2$.

Proposition 3.5.1. *Let G_ω be a spinal group with $\omega \in \Omega_{d,m}$, with $d = 2$, $m \geq 2$. For $\xi, \eta \in X^{\mathbb{N}}$, $\xi \neq \eta$, the graphs (Γ_ξ, ξ) and (Γ_η, η) are isomorphic as marked unlabeled graphs if and only if $\xi, \eta \notin \mathrm{Cof}(1^{\mathbb{N}})$.*

Proof. For $\xi \notin \mathrm{Cof}(1^{\mathbb{N}})$, Γ_ξ is a two-ended line. Each vertex has $2^{m-1} - 1$ loops, one edge to one neighbor and 2^{m-1} edges to the other neighbor. The graph does not depend on ξ, so they are all isomorphic. If $\xi, \eta \in \mathrm{Cof}(1^{\mathbb{N}})$, the distance to $1^{\mathbb{N}}$ is different for each of them, so there are no nontrivial isomorphisms. □

Corollary 3.5.2. *For $d = 2$, there is one unlabeled isomorphism class of marked Schreier graphs (Γ_ξ, ξ) of full measure, and countably many unlabeled isomorphism classes of measure zero.*

3.5.2 Non-binary case

Let us now describe the possible isomorphisms of marked unlabeled Schreier graphs for spinal groups with $d \geq 3$. Let G_ω be a spinal group with $\omega \in \Omega_{d,m}$, with $d \geq 3$, $m \geq 1$. We start by giving some necessary conditions.

Lemma 3.5.3. *Let $\xi, \eta \in X^{\mathbb{N}}$, and let $\varphi : (\Gamma_\xi, \xi) \to (\Gamma_\eta, \eta)$ be an isomorphism of marked unlabeled graphs. For every $\xi' \in \mathrm{Cof}(\xi)$ and $n \geq 0$,*

$$\varphi(\Lambda^n_{\xi'}) = \Lambda^n_{\eta'} \quad \text{and} \quad \varphi(\Gamma^n_{\xi'}) = \Gamma^n_{\eta'},$$

where $\eta' = \varphi(\xi')$.

Proof. We will abuse notation and denote subgraphs and their vertex sets in the same way. Notice that, even though we consider isomorphisms between unlabeled graphs, vertices that are fixed by B must be mapped to vertices that are fixed by B, otherwise they have a different number of loops. The second statement is trivial for $n = 0$, as ξ' is mapped to $\eta' = \varphi(\xi')$. Let us show the second statement for $n = 1$.

We know that $\Gamma^1_{\xi'} = X\sigma(\xi')$. Since $1\sigma(\xi')$ is fixed by B, then $\mathcal{B}_{1\sigma(\xi')}(1) = \Gamma^1_{\xi'}$. Hence

$$\varphi(\Gamma^1_{\xi'}) = \varphi(\mathcal{B}_{1\sigma(\xi')}(1)) = \mathcal{B}_{\varphi(1\sigma(\xi'))}(1) \supset \Gamma^1_{\varphi(1\sigma(\xi'))} = \Gamma^1_{\eta'}.$$

The last equality comes from the fact that $\varphi(1\sigma(\xi'))$ is also fixed by B, and so can only be joined by an A-edge to η'. The other inclusion then follows from the cardinality of the sets.

Let us now prove the first statement for all $n \geq 0$. Let $\xi'' \in \Lambda^n_{\xi'}$, therefore $\xi'' = (d-1)^n 0 i \sigma^{n+2}(\xi')$, and denote $\eta'' = \varphi(\xi'')$. Since ξ'' is not fixed by B, its image η'' is also not fixed by B, so $\eta'' \in \Lambda^k_{\eta'}$ for some $k \geq 0$. We decompose $\mathcal{B}_{\xi''}(1) = \Gamma^1_{\xi''} \sqcup \left(\Lambda^n_{\xi'} \setminus \{\xi''\}\right)$ and similarly $\mathcal{B}_{\eta''}(1) = \Gamma^1_{\eta''} \sqcup \left(\Lambda^k_{\eta'} \setminus \{\eta''\}\right)$. The isomorphism φ maps one ball onto the other, but by the previous case, it maps $\Gamma^1_{\xi''}$ to $\Gamma^1_{\eta''}$. Hence, it must map $\Lambda^n_{\xi'}$ to $\Lambda^k_{\eta'}$. We can suppose without loss of generality that $k \leq n$. To show $k = n$, let us suppose $k+1 \leq n$ for a contradiction. Using Remark 3.2.5, notice that

$$\left| \bigcup_{v \in \Lambda^n_{\xi'}} \mathcal{B}_v(2^{k+1}) \right| = \left| \bigcup_{v \in \Lambda^n_{\xi'}} \mathcal{B}_v(2^{k+1}-1) \right| + d(d-1) =$$

$$= \left| X^{k+1}(d-1)^{n-k-1} 0 X \sigma^{n+2}(\xi') \right| + d(d-1) = d^{k+2} + d(d-1),$$

while

$$\left| \bigcup_{v \in \Lambda^k_{\eta'}} \mathcal{B}_v(2^{k+1}) \right| = \left| \bigcup_{v \in \Lambda^k_{\eta'}} \mathcal{B}_v(2^{k+1}-1) \right| + p(d-1) =$$

$$= \left| \Gamma^{k+2}_{\eta'} \right| + p(d-1) = d^{k+2} + p(d-1),$$

with $p = 1$ or $p = 2$ depending on whether $\sigma(\eta')$ is fixed by B or not, respectively. In any case, one ball must be mapped onto the other, so $p = d \geq 3$, which is a contradiction. Therefore $\Lambda^n_{\xi'}$ must be mapped to $\Lambda^n_{\eta'}$.

Finally, let us prove the second claim for $n \geq 2$. Again by Remark 3.2.5 we have

$$\varphi(\Gamma^n_{\xi'}) = \varphi\left(\bigcup_{v \in \Lambda^{n-2}_{\xi'}} \mathcal{B}_v(2^{n-1}-1) \right) = \bigcup_{v \in \Lambda^{n-2}_{\xi'}} \varphi\left(\mathcal{B}_v(2^{n-1}-1)\right) =$$

$$= \bigcup_{v \in \Lambda^{n-2}_{\xi'}} \mathcal{B}_{\varphi(v)}(2^{n-1}-1) = \bigcup_{v \in \Lambda^{n-2}_{\eta'}} \mathcal{B}_v(2^{n-1}-1) = \Gamma^n_{\eta'}.$$

□

The important consequence of this Lemma is that, even if we consider unlabeled graphs, isomorphisms are not allowed to map A-edges to B-edges or vice versa. Furthermore, for every $n \geq 0$, they are only allowed to map copies of Γ_n to copies of Γ_n, which is quite restrictive.

Lemma 3.5.4. *Let $\xi, \eta \in X^{\mathbb{N}}$, and let $\varphi : (\Gamma_\xi, \xi) \to (\Gamma_\eta, \eta)$ be an isomorphism of marked unlabeled graphs. For every $n \geq 0$, $\xi_n = 0$ if and only if $\eta_n = 0$.*

Proof. The case $n = 0$ is a consequence of the first statement in Lemma 3.5.3. Let $n \geq 1$ such that $\xi_n = 0$.

The second statement in Lemma 3.5.3 implies that $\Gamma^n_\xi = X^n 0 \sigma^{n+1}(\xi)$ is mapped to $\Gamma^n_\eta = X^n \eta_n \sigma^{n+1}(\eta)$. If $\eta_n = d-1$, then the latter has a vertex in $\Lambda^k_{\eta'}$, for some $\eta' \in \mathrm{Cof}(\eta)$ and $k \geq n+1$, while the former does not have any vertex in $\Lambda^k_{\xi'}$ for any $\xi' \in \mathrm{Cof}(\xi)$ and

$k \geq n+1$. By the first statement in Lemma 3.5.3, this is a contradiction, hence $\eta_n \neq d-1$. Suppose also $\eta_n \neq 0$ for a contradiction. In that case, the vertex $(d-1)^n \eta_n \sigma^{n+1}(\eta)$ is fixed by B, while $(d-1)^n 0 \sigma^{n+1}(\xi)$ is not. Now by Remark 3.2.5 we have

$$\left| \bigcup_{v \in \Lambda_\xi^{n-2}} \mathcal{B}_v(2^{n-1}) \right| = \left| \bigcup_{v \in \Lambda_\xi^{n-2}} \mathcal{B}_v(2^{n-1}-1) \right| + 2(d-1) =$$

$$= \left|\Gamma_\xi^n\right| + 2(d-1) = d^n + 2(d-1),$$

and

$$\left| \bigcup_{v \in \Lambda_\eta^{n-2}} \mathcal{B}_v(2^{n-1}) \right| = \left| \bigcup_{v \in \Lambda_\eta^{n-2}} \mathcal{B}_v(2^{n-1}-1) \right| + d - 1 =$$

$$= \left|\Gamma_\eta^n\right| + d - 1 = d^n + d - 1,$$

which is also a contradiction. Hence $\eta_n = 0$. □

Lemma 3.5.4 states that for two points in $X^{\mathbb{N}}$ to have isomorphic marked unlabeled Schreier graphs they must have zeros in the same positions. This excludes any isomorphism between graphs of points with finitely and infinitely many zeros. Moreover, if we write $\xi \in X^{\mathbb{N}}$ as $\xi = w_0 0 w_1 0 \dots$, with $w_k \in (X \setminus \{0\})^*$, then its image must have the form $\varphi(\xi) = \tilde{w}_0 0 \tilde{w}_1 0 \dots$, with $|w_k| = |\tilde{w}_k|$ for every $k \geq 0$.

Let $n \geq 0$ and $0 \leq m \leq n-1$. The sets

$$Y_m = (X \setminus \{0\})^{n-m-1} (X \setminus \{0, d-1\})(d-1)^m, \quad Y_n = \{(d-1)^n\}$$

define a partition of the set $(X \setminus \{0\})^n = \sqcup_{m=0}^n Y_m$.

Definition 3.5.5. We call $\xi, \eta \in X^{\mathbb{N}}$ *compatible* (and we denote this by $\xi \sim \eta$) if they are of the form $\xi = w_0 0 w_1 0 \dots$, $\eta = \tilde{w}_0 0 \tilde{w}_1 0 \dots$, with $w_k, \tilde{w}_k \in (X \setminus \{0\})^* \sqcup (X \setminus \{0\})^{\mathbb{N}}$, $|w_k| = |\tilde{w}_k|$, and $w_k \in Y_{m_k}$ if and only if $\tilde{w}_k \in Y_{m_k}$, for every $k \geq 0$ such that $|w_k| < \infty$. Notice that compatibility is an equivalence relation on $X^{\mathbb{N}}$.

Proposition 3.5.6. *Let $\xi, \eta \in X^{\mathbb{N}}$, and let $\varphi : (\Gamma_\xi, \xi) \to (\Gamma_\eta, \eta)$ be an isomorphism of marked unlabeled graphs. Then, $\xi \sim \eta$.*

Proof. By Lemma 3.5.4, we can assume that $\xi = w_0 0 w_1 0 \dots$ and $\eta = \tilde{w}_0 0 \tilde{w}_1 0 \dots$ with $w_k, \tilde{w}_k \in (X \setminus \{0\})^* \sqcup (X \setminus \{0\})^{\mathbb{N}}$ and $|w_k| = |\tilde{w}_k|$ for every $k \geq 0$ such that $|w_k| < \infty$. Let us assume, for a contradiction, that there exists some $k \geq 0$ such that, if $n = |w_k|$, $w_k \in Y_m$ and $\tilde{w}_k \in Y_{m'}$, with $m < m'$. Let also $N_r = \sum_{s=0}^{r-1}(|w_s|+1)$, so that $\sigma^{N_k}(\xi) = w_k 0 w_{k+1} 0 \dots$ and $\sigma^{N_k}(\eta) = \tilde{w}_k 0 \tilde{w}_{k+1} 0 \dots$.

Assume first $m < m' = n$, then $w_k = wi(d-1)^m$ and $\tilde{w}_k = (d-1)^n$, for some $i \in X \setminus \{0, d-1\}$ and $w \in (X \setminus \{0\})^{n-m-1}$. We exclude the case $k = 0$ because then B would fix ξ but not η, which is absurd. Hence we assume $k \geq 1$ and so $N_k > 0$. We define the following subgraphs of Γ_ξ and Γ_η, respectively:

$$\Gamma_\xi^{N_k} = X^{N_k} wi(d-1)^m 0 \sigma^{N_{k+1}}(\xi),$$

$$\Gamma_\eta^{N_k} = X^{N_k} (d-1)^n 0 \sigma^{N_{k+1}}(\eta).$$

These subgraphs are copies of Γ_{N_k} which respectively contain ξ and η. By Lemma 3.5.3, we have $\varphi(\Gamma_\xi^{N_k}) = \Gamma_\eta^{N_k}$. The latter contains the vertex $(d-1)^{N_k+n} 0 \sigma^{N_{k+1}}(\eta)$, which belongs to $\Lambda_{\eta'}^{N_k+n}$ for some $\eta' \in \mathrm{Cof}(\eta)$. However, since $i \neq 0, d-1$, the former does not contain any vertex which belongs to $\Lambda_{\xi'}^M$ for any $\xi' \in \mathrm{Cof}(\xi)$ and $M \geq N_k$. Again by Lemma 3.5.3, this is a contradiction.

Assume now $m < m' < n$. Then $m < n-1$ or equivalently $n - m - 1 > 0$. In this case, we consider the following subgraphs of Γ_ξ and Γ_η, respectively:

$$\Gamma_\xi^{N_k+n-m-1} = X^{N_k+n-m-1} i(d-1)^m 0 \sigma^{N_{k+1}}(\xi),$$

$$\Gamma_\eta^{N_k+n-m-1} = X^{N_k+n-m-1} (d-1)^{m+1} 0 \sigma^{N_{k+1}}(\eta).$$

These subgraphs are copies of $\Gamma_{N_k+n-m-1}$ which respectively contain ξ and η. Again Lemma 3.5.3 implies $\varphi(\Gamma_\xi^{N_k+n-m-1}) = \Gamma_\eta^{N_k+n-m-1}$. Similarly to the previous case, the latter contains the vertex $(d-1)^{N_k+n} 0 \sigma^{N_{k+1}}(\eta)$, which belongs to $\Lambda_{\eta'}^{N_k+n}$ for some $\eta' \in \mathrm{Cof}(\eta)$. In the former, there is not any vertex belonging to $\Lambda_{\xi'}^M$, for any $\xi' \in \mathrm{Cof}(\xi)$ and $M \geq N_k + n - m - 1$. As $n - m - 1 > 0$, once more by Lemma 3.5.3 this is a contradiction. □

We have shown that compatibility is a necessary condition to find an isomorphism of marked unlabeled graphs. Our next goal will be to prove the converse. Namely, that for any two compatible points there exists an isomorphism mapping their graphs to each other. To this purpose, let use introduce the following notation.

Definition 3.5.7. Let $\xi, \xi' \in X^{\mathbb{N}}$. We define $R = R_{\xi,\xi'} = \min\{s \mid \sigma^s(\xi) = \sigma^s(\xi')\}$. Notice that $R < \infty$ if and only if $\xi' \in \mathrm{Cof}(\xi)$. Also note that $R = 0$ if and only if $\xi' = \xi$ and that otherwise, by minimality, $\xi_{R-1} \neq \xi'_{R-1}$.

Definition 3.5.8. Let $\xi, \eta \in X^{\mathbb{N}}$. We define $\tau_n = \tau_{n,\xi,\eta} = (\xi_n\ \eta_n) \in \mathrm{Sym}(X)$. Let us also define the map $\varphi = \varphi_{\xi,\eta} : (\Gamma_\xi, \xi) \to (\Gamma_\eta, \eta)$ as

$$\varphi(\xi') = \begin{cases} \eta & \text{if } \xi' = \xi \\ \xi'_0 \dots \xi'_{R-2} \tau_{R-1}(\xi'_{R-1}) \sigma^R(\eta) & \text{if } \xi' \neq \xi, \quad \text{with } R = R_{\xi,\xi'} \end{cases}.$$

Remark 3.5.9. Suppose $\xi \sim \xi'$, and let $\eta' = \varphi(\xi')$.

1. $R_{\eta,\eta'} = R_{\xi,\xi'}$. It is clear that $R_{\eta,\eta'} \leq R_{\xi,\xi'}$, and if it was strictly smaller, then, setting $n = R_{\xi,\xi'} - 1$, we would have $\eta_n = \tau_n(\xi'_n)$, and so $\xi_n = \xi'_n$, which is a contradiction with the minimality of $R_{\xi,\xi'}$.

2. $\psi = \varphi_{\eta,\xi}$ is the inverse of $\varphi = \varphi_{\xi,\eta}$. Indeed, $\psi(\varphi(\xi)) = \xi$, and, for $\xi' \neq \xi$,

$$\psi(\varphi(\xi')) = \psi\left(\xi'_0 \dots \xi'_{R-2}\tau_{R-1}(\xi'_{R-1})\sigma^R(\eta)\right) =$$

$$= \xi'_0 \dots \xi'_{R-2}\tau^2_{R-1}(\xi'_{R-1})\sigma^R(\xi) = \xi'_0 \dots \xi'_{R-2}\xi'_{R-1}\sigma^R(\xi) = \xi'.$$

Hence, φ is a bijection between the vertex sets of Γ_ξ and Γ_η.

Proposition 3.5.10. *Let $\xi, \eta \in X^{\mathbb{N}}$ such that $\xi \sim \eta$. Then $\varphi_{\xi,\eta} : (\Gamma_\xi, \xi) \to (\Gamma_\eta, \eta)$ is an isomorphism of marked unlabeled graphs.*

Proof. Let $\varphi = \varphi_{\xi,\eta}$. We already proved in Remark 3.5.9 that φ is a bijection between the sets of vertices, so we only have to prove that edges are preserved. Every A or B-edge joins vertices in the same $\Gamma^1_{\xi'}$ or $\Lambda^n_{\xi'}$, respectively, for some $\xi' \in \mathrm{Cof}(\xi)$ and $n \geq 0$. Let then $\xi' \in \mathrm{Cof}(\xi)$, $n \geq 0$, set $\eta' = \varphi(\xi')$ and let us prove $\varphi(\Gamma^1_{\xi'}) = \Gamma^1_{\eta'}$ and $\varphi(\Lambda^n_{\xi'}) = \Lambda^n_{\eta'}$.

For the first claim, let us consider two cases. If $\xi \in \Gamma^1_{\xi'}$, then we have

$$\varphi(\Gamma^1_{\xi'}) = \varphi(\Gamma^1_\xi) = \varphi\left(\{\xi\} \sqcup \{j\sigma(\xi) \mid j \neq \xi_0\}\right) =$$

$$= \{\eta\} \sqcup \{\tau_0(j)\sigma(\eta) \mid j \neq \xi_0\} = \{\eta\} \sqcup \{j\sigma(\eta) \mid j \neq \eta_0\} = \Gamma^1_\eta.$$

Since $R_{\xi,\xi'} \leq 1$, by Remark 3.5.9 $R_{\eta,\eta'} \leq 1$, so $\Gamma^1_\eta = \Gamma^1_{\eta'}$.

Assume now $\xi \notin \Gamma^1_{\xi'}$, so $R = R_{\xi,\xi'} \geq 2$. In that case, $R_{\xi,v} = R$ for all $v \in \Gamma^1_{\xi'}$. Then

$$\varphi(\Gamma^1_{\xi'}) = \varphi(\{j\xi'_1 \dots \xi'_{R-1}\sigma^R(\xi) \mid j \in X\}) =$$

$$= \{j\xi'_1 \dots \xi'_{R-2}\tau_{R-1}(\xi'_{R-1})\sigma^R(\eta) \mid j \in X\} = \Gamma^1_{\eta'}.$$

To prove the second claim, set $R = \min_{v \in \Lambda^n_{\xi'}} R_{\xi,v}$. In this case, R cannot be $n+2$, since v_{n+1} takes all values in X when v runs through $\Lambda^n_{\xi'}$. We will consider four different cases, depending on the value of R.

Suppose first $R = 0$. This means that $\xi \in \Lambda^n_{\xi'}$. Moreover, $R_{\xi,v} = n+2$ for all $v \in \Lambda^n_{\xi'}$ except for $v = \xi$. Therefore,

$$\varphi(\Lambda^n_{\xi'}) = \varphi(\{\xi\} \sqcup \{(d-1)^n 0 j \sigma^{n+2}(\xi) \mid j \neq \xi_{n+1}\}) =$$

$$= \{\eta\} \sqcup \{(d-1)^n 0 \tau_{n+1}(j)\sigma^{n+2}(\eta) \mid j \neq \xi_{n+1}\} =$$

$$= \{\eta\} \sqcup \{(d-1)^n 0 j \sigma^{n+2}(\eta) \mid j \neq \eta_{n+1}\} =$$

$$= \{(d-1)^n 0 j \sigma^{n+2}(\eta) \mid j \in X\} = \Lambda^n_\eta.$$

And $\Lambda^n_\eta = \Lambda^n_{\eta'}$, because $R_{\eta,\eta'} = R_{\xi,\xi'} \leq n+2$, so $\sigma^{n+2}(\eta) = \sigma^{n+2}(\eta')$.

Suppose now $1 \leq R \leq n$. For the vertex $v = (d-1)^n 0 \xi_{n+1} \sigma^{n+2}(\xi') \in \Lambda^n_{\xi'}$ we must have $R_{\xi,v} = R \leq n$. This implies $\xi = \xi_0 \ldots \xi_{R-1}(d-1)^{n-R} 0 \xi_{n+1} \sigma^{n+2}(\xi)$ and $\sigma^{n+2}(\xi) = \sigma^{n+2}(\xi')$. Additionally, since $\xi \sim \eta$, we can write $\eta = \eta_0 \ldots \eta_{R-1}(d-1)^{n-R} 0 \eta_{n+1} \sigma^{n+2}(\eta)$. For every $w \in \Lambda^n_{\xi'}$, $w \neq v$, we have $R_{\xi,w} = n+2$. Then,

$$\varphi(\Lambda^n_{\xi'}) = \varphi\left(\{v\} \sqcup \{(d-1)^n 0 j \sigma^{n+2}(\xi) \mid j \neq \xi_{n+1}\}\right) =$$

$$= \{(d-1)^{R-1} \tau_{R-1}(d-1)(d-1)^{n-R} 0 \eta_{n+1} \sigma^{n+2}(\eta)\} \sqcup$$

$$\sqcup \{(d-1)^n 0 \tau_{n+1}(j) \sigma^{n+2}(\eta) \mid j \neq \xi_{n+1}\} =$$

$$= \{(d-1)^n 0 \eta_{n+1} \sigma^{n+2}(\eta)\} \sqcup \{(d-1)^n 0 j \sigma^{n+2}(\xi) \mid j \neq \eta_{n+1}\} = \Lambda^n_\eta.$$

where we used $\tau_{R-1}(d-1) = d-1$ because $\xi \sim \eta$. In addition, again $R_{\eta,\eta'} = R_{\xi,\xi'} \leq n+2$, so $\Lambda^n_\eta = \Lambda^n_{\eta'}$.

The third case is $R = n+1$. The vertex $v = (d-1)^n 0 \xi_{n+1} \sigma^{n+2}(\xi') \in \Lambda^n_{\xi'}$ satisfies $R_{\xi,v} = R = n+1$. This implies $\sigma^{n+2}(\xi) = \sigma^{n+2}(\xi')$. For any other $w \in \Lambda^n_{\xi'}$, we must have $R_{\xi,w} = n+2$. Therefore,

$$\varphi(\Lambda^n_{\xi'}) = \varphi(\{v\} \sqcup \{(d-1)^n 0 j \sigma^{n+2}(\xi) \mid j \neq \xi_{n+1}\}) =$$

$$= \{(d-1)^n \tau_n(0) \eta_{n+1} \sigma^{n+2}(\eta)\} \sqcup \{(d-1)^n 0 \tau_{n+1}(j) \sigma^{n+2}(\eta) \mid j \neq \xi_{n+1}\} =$$

$$= \{(d-1)^n 0 \eta_{n+1} \sigma^{n+2}(\eta)\} \sqcup \{(d-1)^n 0 j \sigma^{n+2}(\eta) \mid j \neq \eta_{n+1}\} = \Lambda^n_\eta,$$

where we used $\tau_n(0) = 0$ because of Lemma 3.5.4 and provided that $\xi \sim \eta$. Once more, $R_{\eta,\eta'} = R_{\xi,\xi'} \leq n+2$, so $\Lambda^n_\eta = \Lambda^n_{\eta'}$.

Finally, assume $R \geq n+3$. We can write $\Lambda^n_{\xi'} = \{(d-1)^n 0 j \sigma^{n+2}(\xi') \mid j \in X\} = \{(d-1)^n 0 j \xi'_{n+2} \ldots \xi'_{R-1} \sigma^R(\xi') \mid j \in X\}$. In this case, $R_{\xi,v} = R$ for every $v \in \Lambda^n_{\xi'}$, and also $R_{\xi,\xi'} = R$, which implies $\sigma^R(\xi) = \sigma^R(\xi')$. Then,

$$\varphi(\Lambda^n_{\xi'}) = \varphi(\{(d-1)^n 0 j \xi'_{n+2} \ldots \xi'_{R-1} \sigma^R(\xi) \mid j \in X\}) =$$

$$= \{(d-1)^n 0 j \xi'_{n+2} \ldots \xi'_{R-2} \tau_{R-1}(\xi'_{R-1}) \sigma^R(\eta) \mid j \in X\} =$$

$$= \Lambda^n_{\xi'_0 \ldots \xi'_{R-2} \tau_{R-1}(\xi'_{R-1}) \sigma^R(\eta)},$$

and notice that $\eta' = \xi'_0 \ldots \xi'_{R-2} \tau_{R-1}(\xi'_{R-1}) \sigma^R(\eta)$, so

$$\Lambda^n_{\xi'_0 \ldots \xi'_{R-2} \tau_{R-1}(\xi'_{R-1}) \sigma^R(\eta)} = \Lambda^n_{\eta'}.$$

□

Theorem 3.5.11. *Let G_ω be a spinal group with $\omega \in \Omega_{d,m}$, with $d \geq 3$, $m \geq 1$. For $\xi, \eta \in X^{\mathbb{N}}$, the graphs (Γ_ξ, ξ) and (Γ_η, η) are isomorphic as marked unlabeled graphs if and only if $\xi \sim \eta$.*

Proof. One direction of the equivalence is the object of Proposition 3.5.6, while the converse is that of Proposition 3.5.10. □

If we consider Schreier graphs as unmarked graphs, we can adapt Theorem 3.5.11 easily.

Corollary 3.5.12. *Let G_ω be a spinal group with $\omega \in \Omega_{d,m}$, with $d \geq 3$, $m \geq 1$. For $\xi, \eta \in X^{\mathbb{N}}$, the graphs Γ_ξ and Γ_η are isomorphic as unmarked unlabeled graphs if and only if there exists $\eta' \in \mathrm{Cof}(\eta)$ such that $\xi \sim \eta'$.*

Proof. Let $\varphi : \Gamma_\xi \to \Gamma_\eta$ be an isomorphism of unmarked graphs. Then $\eta' = \varphi(\xi) \in \mathrm{Cof}(\eta)$, and so $\tilde{\varphi} : (\Gamma_\xi, \xi) \to (\Gamma_\eta, \eta')$ is an isomorphism of marked graphs. By Theorem 3.5.11, $\xi \sim \eta'$.

Conversely, if there exists $\eta' \in \mathrm{Cof}(\eta)$ such that $\xi \sim \eta'$, again by Theorem 3.5.11 (Γ_ξ, ξ) and (Γ_η, η') are isomorphic as marked graphs, and so in particular Γ_ξ and Γ_η are isomorphic as unmarked graphs. □

Finally, let us compute the measure of each isomorphism class.

Theorem 3.5.13. *For $d \geq 3$, there are uncountably many unlabeled isomorphism classes of marked Schreier graphs (Γ_ξ, ξ), each of measure zero.*

Proof. Let $\xi \in X^{\mathbb{N}}$, and let $C = \mathrm{Comp}(\xi) \subset X^{\mathbb{N}}$ be its compatibility class. We will show $\mu(C) = 0$.

Suppose first that ξ has finitely many zeros. In that case, there exists $N \geq 0$, $w \in X^N$ and $\xi' \in (X \setminus \{0\})^{\mathbb{N}}$ such that $\xi = w\xi'$. Moreover, $C \subset X^N (X \setminus \{0\})^{\mathbb{N}}$, so

$$\mu(C) \leq \mu(X^N (X \setminus \{0\})^{\mathbb{N}}) = \mu((X \setminus \{0\})^{\mathbb{N}}) = 0.$$

Assume now that ξ has infinitely many zeros, so there is an infinite sequence of words $(w_k)_k$, $w_k \in (X \setminus \{0\})^*$ such that $\xi = w_0 0 w_1 0 \ldots$. Let $n_k = |w_k|$ and $C_k = \mathrm{Comp}(w_k 0 w_{k+1} 0 \ldots)$. By Lemma 3.5.4, any compatible point must have the same zeros at the same positions, so we have

$$\mu(C) \leq \mu(X^{n_0} 0 C_1) = \frac{1}{d}\mu(C_1).$$

If we iterate this inequality, we have that, for any $k \geq 0$,

$$\mu(C) \leq \frac{1}{d^k}\mu(C_k).$$

In particular, for any $\varepsilon > 0$, by choosing k so that $\frac{1}{d^k} \leq \varepsilon$, $\mu(C) \leq \frac{1}{d^k}\mu(C_k) \leq \varepsilon$. Hence, $\mu(C) = 0$. □

Corollary 3.5.14. *For $d \geq 3$, there are uncountably many unlabeled isomorphism classes of unmarked Schreier graphs Γ_ξ, each of measure zero.*

Proof. Let $\xi \in X^{\mathbb{N}}$, and let $\mathrm{Comp}(\xi) \subset X^{\mathbb{N}}$ be its compatibility class. Let us define $D = \cup_{g \in G_\omega} \mathrm{Comp}(g\xi)$. In light of Corollary 3.5.12, Γ_ξ is isomorphic to Γ_η if and only if $\eta \in D$. Theorem 3.5.13 implies that $\mu(\mathrm{Comp}(\xi)) = 0$. Since the action preserves the measure μ, for every $g \in G_\omega$, we have $\mu(\mathrm{Comp}(g\xi)) = 0$. Therefore, we conclude that $\mu(D) = 0$. □

3.6 Encoding of the Schreier graphs

In order to better understand the infinite Schreier graphs Γ_ξ of spinal groups, we will encode their structure in sequences of integers characterizing its marked unlabeled isomorphism class. Each sequence may be used to build a tree describing the shape of Γ_ξ. This purely combinatorial object serves as a way to visualize the infinite Schreier graphs. For spinal groups on the binary tree, by Proposition 3.5.1 we know that all their unlabeled Schreier graphs are isomorphic except for one orbit. Therefore, we assume $d \geq 3$ for all the section unless stated otherwise.

Definition 3.6.1. Let $\xi \in X^{\mathbb{N}}$. We define its associated infinite sequence $\{c_n\}_{n\geq 1}$ by $c_1 = 0$ and, for every $n \geq 0$,

$$c_{n+2} = \begin{cases} 0 & \text{if } \xi_0 \dots \xi_n = (d-1)^n 0 \\ k+1 & \text{if } \xi_{k+1} \dots \xi_n = (d-1)^{n-k-1} 0 \text{ and } \xi_k \neq d-1 \\ n+1 & \text{if } \xi_n \neq 0 \end{cases} .$$

The value of c_{n+2} depends only on $\xi_0 \dots \xi_n$, the prefix of length $n+1$ of ξ. Intuitively, c_{n+2} is the minimal length of the prefix we have to change to $\xi_0 \dots \xi_n$ to obtain $(d-1)^n 0$. Notice therefore that $c_n < n$ for every $n \geq 1$.

Proposition 3.6.2. *Let $\xi, \xi' \in X^{\mathbb{N}}$, and let $\{c_n\}_{n\geq 1}$ and $\{c'_n\}_{n\geq 1}$ be, respectively, their associated sequences. Then $\xi \sim \xi'$ if and only if $c_n = c'_n$ for every $n \geq 1$.*

Proof. First, suppose that ξ and ξ' are compatible. This means that $\xi_n = 0$ if and only if $\xi'_n = 0$, and hence $c_{n+2} = c'_{n+2} = n+1$ for every $n \geq 0$ such that $\xi_n, \xi'_n \neq 0$. Now let $n \geq 0$ such that $\xi_n, \xi'_n = 0$. We can assume for a contradiction that $c_{n+2} > c'_{n+2}$, in which case we have, for some $i \in X$, $i \neq 0, d-1$,

	$\dots$	k	$k+1$	$\dots$	$n-1$	n	$\dots$
ξ	$\dots$	i	$d-1$	$\dots$	$d-1$	0	$\dots$
ξ'	$\dots$	$d-1$	$d-1$	$\dots$	$d-1$	0	$\dots$

We decompose ξ as $w_0 0 w_1 0 \dots$ and ξ' as $\tilde{w}_0 0 \tilde{w}_1 0 \dots$, with $|w_k| = |\tilde{w}_k|$ for every k, and let r be such that ξ_k belongs to w_k and, respectively, ξ'_k belongs to $\tilde{w}_k$. We see from the table above that w_r and $\tilde{w}_r$ cannot belong to the same Y_m, which is a contradiction with the fact that ξ and ξ' are compatible.

Conversely, let us suppose that $c_{n+2} = c'_{n+2}$, for every $n \geq 0$. Now let $n \geq 0$ such that $\xi_n = 0$, so $c_{n+2} < n+1$, and thus $c'_{n+2} < n+1$. This implies that $\xi'_n = 0$, so ξ and ξ' do have zeros at the same positions. To see that they are indeed compatible, assume that they are not, and thus that there exists some n such that $\xi_n = \xi'_n = 0$ for which there is some $k < n$ satisfying $\xi_{k+1} \dots \xi_n = \xi'_{k+1} \dots \xi'_n = (d-1)^{n-k-1}0$ and $\xi_k \neq \xi'_k = d-1$, without loss of generality. In that case, we are actually in the same situation as in the table above, which implies that $c_{n+2} > c'_{n+2}$, which contradicts the fact that the sequences are equal. Hence ξ and ξ' must be compatible. □

Definition 3.6.3. Let $\xi \in X^{\mathbb{N}}$, with associated sequence $\{c_n\}_{n\geq 1}$. We define the tree of ξ, denoted $T(\xi)$, as the graph with vertex set $\mathbb{N} = \{0, 1, 2, \dots\}$ and undirected edges between n and c_n for every $n \geq 1$. The following properties are easy to check:

- As $c_n < n$ for every $n \geq 1$, the graph $T(\xi)$ is indeed a tree.
- Any vertex $n \geq 1$ has exactly 1 neighbor k such that $k < n$.
- The degree of any vertex $n \geq 1$ is between 1 and 3. The degree of the vertex 0 is between 1 and 2.
- $T(\xi)$ and $\{c_n\}_{n\geq 1}$ contain precisely the same amount of information. We can construct $T(\xi)$ knowing $\{c_n\}_{n\geq 1}$ and vice versa.

Remark 3.6.4. We can regard $T(\xi)$ as a tree with an induced ordering in the set of vertices. Whenever we say that two such trees are equal, we assume that this ordering in the vertices is preserved.

Proposition 3.6.5. *Let $\xi, \xi' \in X^{\mathbb{N}}$, and let $T(\xi)$ and $T(\xi')$ be, respectively, their associated trees. Then $\xi \sim \xi'$ if and only if $T(\xi) = T(\xi')$.*

Proof. By Proposition 3.6.2, ξ and ξ' are compatible if and only if their associated sequences are the same, which happens if and only if $T(\xi) = T(\xi')$. □

The notion of the associated tree of a point $\xi \in X^{\mathbb{N}}$ can be used to visualize a schema of the graph Γ_ξ. In order to do that, we will introduce an alternative, purely geometric definition for $T(\xi)$, which will make clear how the tree describes the shape of the graph.

Definition 3.6.6. Let $\xi \in X^{\mathbb{N}}$. Consider the sequence $\{\Gamma_\xi^n\}_{n\geq 0}$ of finite subgraphs of Γ_ξ. Recall that $\Gamma_\xi^n = X^n\sigma^n\xi$ for every $n \geq 0$, see Definition 3.2.3. As Γ_ξ^{n+1} contains Γ_ξ^n for every $n \geq 0$, we have an ascending sequence of finite subgraphs, such that $\bigcup_{n\geq 0} \Gamma_\xi^n = \Gamma_\xi$. In some sense, these subgraphs play the role of balls centered at ξ, although they are much more convenient to describe Γ_ξ.

In addition, let us define the notion of center of the subgraphs Γ_ξ^n. For $n \geq 2$, we define the *center* of Γ_ξ^n as its subgraph Λ_ξ^{n-2}. Recall that $\Lambda_\xi^{n-2} = (d-1)^{n-2}0X\sigma^n\xi$, see Definition 3.2.3. For convenience, we set the centers $\Lambda_\xi^{-2} = \Gamma_\xi^0 = \{\xi\}$ and $\Lambda_\xi^{-1} = \Gamma_\xi^1 = X\sigma\xi$.

Notice that, for every $n \geq 0$, Γ_ξ^n contains all centers Λ_ξ^k for $k = -2, \dots, n-2$, and does not contain any center Λ_ξ^k, for $k \geq n-1$. Moreover, Γ_ξ^n intersects one or two other centers. It intersects Λ_ξ^{n-1} at the vertex $(d-1)^{n-1}0\sigma^n\xi$ if $n \geq 1$. It intersects another center Λ_ξ^k, for $k \geq n$, if and only if the vertex $(d-1)^n\sigma^n\xi$ is not fixed by B.

Definition 3.6.7. Let $\xi \in X^{\mathbb{N}}$. We define the graph $T'(\xi)$ as the graph with vertex set $\mathbb{N} = \{0, 1, 2, \dots\}$ and undirected edges between k and n for each $k > n \geq 0$ satisfying one of these two conditions:

- Λ_ξ^{n-2} and Λ_ξ^{k-2} have non-empty intersection, and this intersection is not contained in any other Λ_ξ^l, for $l \leq n-3$.
- Λ_ξ^{n-2} and Λ_ξ^{k-2} do not intersect, but there exists a path in Γ_ξ between them satisfying:
  - The endpoint in Λ_ξ^{n-2} does not belong to Λ_ξ^l, for any $l \leq n-3$.
  - The endpoint in Λ_ξ^{k-2} does not belong to Λ_ξ^l, for any $l \leq k-3$.
  - The inner vertices of the path do not belong to Λ_ξ^l for any $l \geq -2$.

Notice that as $\Lambda_\xi^{-2} = \{\xi\} \subset X\sigma\xi = \Lambda_\xi^{-1}$, there is always an edge from 1 to 0.

These conditions illustrate how the centers of Γ_ξ are connected. This is sufficient in order to know how the subgraphs Γ_ξ^n are connected, and hence to recover the shape of Γ_ξ itself. The following notion of blocking centers simplifies the evaluation of these conditions.

Definition 3.6.8. Let $r, s \geq 1$. We say that the centers $\Lambda_\xi^{n_1-2}, \dots, \Lambda_\xi^{n_r-2}$ *block* the centers $\Lambda_\xi^{k_1-2}, \dots, \Lambda_\xi^{k_s-2}$ if any path from any of $\Lambda_\xi^{k_1-2}, \dots, \Lambda_\xi^{k_s-2}$ to any other center Λ_ξ^{l-2}, with $l \neq n_1, \dots, n_r$ and $l \neq k_1, \dots, k_r$, contains an inner vertex in one of $\Lambda_\xi^{n_1-2}, \dots, \Lambda_\xi^{n_r-2}$.

In that case, the vertices $k_1, \dots, k_s$ of $T'(\xi)$ can only have edges among them or to $n_1, \dots, n_r$, but they cannot have edges to any other l.

Proposition 3.6.9. *For every $\xi \in X^{\mathbb{N}}$, $T(\xi) = T'(\xi)$. Equivalently, $T(\xi) \cong T'(\xi)$ and they have the same vertex ordering.*

Proof. It follows from the definition that both graphs have an edge between 1 and 0. Let us show that, for every $n \geq 2$, the set of edges between $n+2$ and $\{0, \dots, n+1\}$ is the same for $T(\xi)$ and $T'(\xi)$. Notice that in the first, there is always exactly one such edge, which goes from $n+2$ to c_{n+2}.

Let $n \geq 0$, and define $v = (d-1)^n 0 \sigma^{n+1}(\xi)$, the only vertex in the intersection of Γ_ξ^{n+1} and Λ_ξ^n. Let also p be the smallest index such that $v \in \Lambda_\xi^{p-2}$. If p was $2, \dots, n+1$, then $v_{p-2} = 0$, which contradicts the definition of v. So the only possibilities are $p = 0, 1, n+2$.

If $p = 0$, then $v = \xi$, so $c_{n+2} = 0$. On the other hand, because every path from Λ_ξ^n to any of Λ_ξ^l with $l \leq n-1$ goes through v, the only edge between $n+2$ and $\{0, \dots, n+1\}$ joins $n+2$ and 0. If $p = 1$, $v = (d-1)\sigma(\xi)$, so $c_{n+2} = 1$. Similarly, in $T'(\xi)$, we have the edge $n+2$ to 1 for the same reason as before. Now assume $p = n+2$ for the rest of the proof. Λ_{n+2} is the only center containing v. We have three possibilities, according to the value of ξ_n.

If $\xi_n \neq 0, d-1$, then $c_{n+2} = n+1$, and Γ_ξ^n is only connected to $\Gamma_\xi \setminus \Gamma_\xi^n$ through the vertex $u = (d-1)^{n-1} 0 \sigma^n(\xi)$. Because this vertex u belongs to Λ_ξ^{n-1}, this means that Λ_ξ^{n-1} blocks the centers Λ_ξ^l for every $l \leq n-2$. Therefore, the only edge in $T'(\xi)$ between $n+2$ and $\{0, \dots, n+1\}$ is from $n+2$ to $n+1$.

Suppose that $\xi_n = d-1$, so again $c_{n+2} = n+1$. Let u be the vertex $(d-1)^{n-1} 0 \sigma^n(\xi)$ and w be the vertex $(d-1)^n \sigma^n(\xi)$. Notice that $u \in \Lambda_\xi^{n-1} \cap \Gamma_\xi^n$ and $w \in \Gamma_\xi^n$, and Γ_ξ^n is only connected to $\Gamma_\xi \setminus \Gamma_\xi^n$ through u and possibly w. If w is fixed by B, then we are in the same situation as above: Λ_ξ^{n-1} blocks Λ_ξ^l for every $l \leq n-2$. Otherwise, $w \in \Lambda_\xi^r$, for some $r \geq n+1$, and then Λ_ξ^{n-1} and Λ_ξ^r block the centers Λ_ξ^l for every $l \leq n-2$. In any case, there is only one edge in $T'(\xi)$ between $n+2$ and $\{0, \dots, n+1\}$, joining $n+2$ and $n+1$.

Assume now for the rest of the proof $\xi_n = 0$, and let k be the smallest such that $\xi_{k+1} \dots \xi_n = (d-1)^{n-k-1} 0$. As $v \neq \xi$, necessarily $k \geq 1$, and by minimality $\xi_k \neq d-1$. In this case, we have $c_{n+2} = k+1$.

Let $u = (d-1)^k 0 \sigma^{k+1}(\xi) \in \Lambda_\xi^k$, and observe that Γ_ξ^{k+1} is connected to $\Gamma_\xi^{n+1} \setminus \Gamma_\xi^{k+1}$ only through the vertex u. The former contains all centers Λ_ξ^l for every $l \leq k-1$, while the latter contains all centers Λ_ξ^l with $k \leq l \leq n-1$. Any path from Λ_ξ^n to Λ_ξ^l with $k \leq l \leq n-1$ must then contain the vertex $u \in \Lambda_\xi^k$, which prevents any edge in $T'(\xi)$ from $n+2$ to $\{k+2, \dots, n+1\}$.

Now Γ_ξ^k is connected to $\Gamma_\xi \setminus \Gamma_\xi^k$ only through the vertices $u' = (d-1)^{k-1} 0 \sigma^k(\xi) \in \Lambda_\xi^{k-1}$ and possibly $w = (d-1)^k \sigma^k(\xi)$. If w is fixed by B, then Λ_ξ^{k-1} blocks all centers Λ_ξ^l for $l \leq k-2$. Otherwise, we have $\xi_k = 0$, and in fact $u = w$. Then Λ_ξ^{k-1} and Λ_ξ^k block all centers Λ_ξ^l for $l \leq k-2$. In any case, this prevents any edge in $T'(\xi)$ from $n+2$ to $\{0, \dots, k\}$. Hence, the only possible edge from $n+2$ to $\{0, \dots, n+1\}$ is to $k+1$.

Finally, any path within $X^k(d-1)\sigma^{k+1}(\xi)$ joining v and $(d-1)^{k-1} 0 (d-1) \sigma^{k+1}(\xi) \in$

Λ_ξ^{k-1} connects the centers Λ_ξ^n and Λ_ξ^{k-1} without intersecting any other center. Hence, there is indeed an edge in $T'(\xi)$ between $n+2$ and $k+1$. In any of the cases we showed that $T(\xi)$ and $T'(\xi)$ have the same edges between $n+2$ and $\{0,\dots,n+1\}$ for every $n \geq 0$. This suffices to conclude that $T(\xi)$ and $T'(\xi)$ are the same tree. □

Remark 3.6.10. Recall that, by Theorem 3.5.11, (Γ_ξ, ξ) and (Γ_η, η) are isomorphic as marked unlabeled graphs if and only if ξ and η are compatible. We also proved in Proposition 3.6.2 that ξ and η are compatible if and only if their associated sequences $\{c_n\}_{n\geq 1}$ coincide, or, equivalently, by Proposition 3.6.5, if and only if $T(\xi)$ and $T(\eta)$ are the same tree (with the induced ordering). Now Proposition 3.6.9 implies that we have two ways of building this tree. The first way is purely combinatorial, only using the sequence $\{c_n\}_{n\geq 1}$, which is obtained straightforward from the digits of ξ. The second is purely geometrical, as it is based in paths between the centers inside Γ_ξ.

Remark 3.6.11. These two additional characterizations of compatibility, namely the associated sequence and tree, are useful in both directions. On the one hand, we can take any $\xi \in X^{\mathbb{N}}$, find its associated sequence $\{c_n\}_{n\geq 1}$ and from it build its tree $T(\xi)$. We may now use the tree to recover the Schreier graph Γ_ξ as follows:

1. Let Γ be an unlabeled copy of the graph Γ_1 without loops at the vertices $u_1 = 0$ and $u_2 = d-1$. Let also z be the vertex labeled by ξ_0. Call the vertices u and v *open* and the rest *closed*.

2. For every $n \geq 0$, do the following:

   (i) Let Λ_n be an unlabeled copy of Λ_{ω_n} from Proposition 3.1.3. Call all its vertices open. Let $v \in \Lambda_n$.

   (ii) If $c_{n+2} = 0$, identify v with z in Γ and call the identified vertex *closed*.
   If $c_{n+2} = 1$ and u_1 is open, identify v with u_1 in Γ and call the identified vertex closed. Otherwise identify v with u_2 in Γ and call the identified vertex closed.
   If $c_{n+2} = k+2$ for $k \geq 0$, let $w \in \Lambda_k$ be an open vertex. Let Γ'_{k+1} be a copy of Γ_{k+1} without loops on the vertices $v' = (d-1)^{k+1}$ and $w' = (d-1)^k 0$. Call all its vertices closed. In Γ, identify v with v' and w with w', and call both identified vertices closed.

3. Finally, for every $n \geq 0$ and every open vertex $w \in \Lambda_n$, let Γ'_{n+1} be a copy of Γ_{n+1} without loops on the vertex $w' = (d-1)^k 0$. Call all its vertices closed. In Γ, identify w with w', and call the identified vertex closed.

4. The resulting marked graph (Γ, z) is isomorphic to the Schreier graph (Γ_ξ, ξ), with Λ_n corresponding to the center Λ_ξ^n for every $n \geq 0$.

On the other hand, we can work backwards if we start from a marked graph (Γ, z) and we want to know if there exists some $\xi \in X^{\mathbb{N}}$ such that $(\Gamma, z) \cong (\Gamma_\xi, \xi)$. In this case, we can build the tree from Γ using the second definition, and extract the sequence $\{c_n\}_{n\geq 1}$ from it. Finally, we only have to use its definition to unravel a set of conditions on the digits of ξ, which will determine its compatibility class.

We may also wonder what happens if instead of prescribing a graph we prescribe a tree T (or, equivalently, a sequence $\{c_n\}_{n\geq 1}$) and we try to find $\xi \in X^{\mathbb{N}}$ such that $T = T(\xi)$. As it turns out, not every tree or sequence can be realized as the associated of point $\xi \in X^{\mathbb{N}}$. Let us characterize the sequences $\{c_n\}_{n\geq 1}$ arising in such a way.

Proposition 3.6.12. *Let $\{d_n\}_{n\geq 1} \subset \mathbb{N}$. $\{d_n\}_{n\geq 1}$ is the associated sequence of some $\xi \in X^{\mathbb{N}}$ if and only if*

- $d_n \leq n - 1, \quad \forall n \geq 1.$
- $d_{n+2} \neq n + 1 \Longrightarrow d_{r+2} \geq n + 1, \quad \forall r > n.$

Proof. Every sequence defined as in Definition 3.6.1 satisfies these conditions. For the converse, let $\{d_n\}_{n\geq 1}$ satisfy both conditions. Let $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. We define a point $\xi \in X^{\mathbb{N}}$ in the following way:

1. Set $I = \{n \in \mathbb{N}^* \mid d_{n+2} < n + 1\}$ and $J = \{n \in \mathbb{N}^* \mid d_{n+2} = n + 1\}$. By the first condition, $I \sqcup J = \mathbb{N}^*$.
2. If $d_{n+2} = 0$ for some $n \in I$, we define $\xi_0 \dots \xi_n = (d-1)^n 0$. By the second condition, there can only be one such n, so this is well defined.
3. For each k such that $0 \leq k \leq n - 1$, if $d_{n+2} = k + 1$ for some $n \in I$, then we define $\xi_{k+1} \dots \xi_n = (d - 1)^{n-k-1} 0$. This is well defined because if $n' \in I$, with $n' < n$, then by the second condition $d_{n+2} \geq n' + 1$, so they define disjoint sets of digits of ξ.
4. After this procedure, we set any undefined digit of ξ to 1.

Let $\{c_n\}_{n\geq 1}$ be the associated sequence of this ξ. Let us verify that, indeed, $d_n = c_n$ for every $n \geq 1$. First, we have $c_1 = d_1 = 1$, and $c_{n+2} = n + 1$ for every $n \geq 0$ such that $\xi_n \neq 0$. By construction, $\xi_n \neq 0$ if and only if $n \in J$, equivalently if $d_{n+2} = n + 1$.

Now let $n \geq 0$ such that $c_{n+2} = 0$. Then, $\xi_0 \dots \xi_n = (d - 1)^n 0$. Again by construction of ξ, $d_{n+2} = 0$ too.

Finally, if $c_{n+2} = k + 1$ with $0 \leq k \leq n - 1$, then $\xi_{k+1} \dots \xi_n = (d - 1)^{n-k-1} 0$, and $\xi_k \neq d - 1$. But then once again d_{n+2} must also be $k + 1$ by construction of ξ. ∎

In addition, we can use the tree description to illustrate the computation of the growth of the graphs Γ_ξ, previously shown in [9].

Proposition 3.6.13. *Let G_ω be a spinal group with $\omega \in \Omega_{d,m}$ with $d \geq 2$ and $m \geq 1$, and let $\xi \in X^{\mathbb{N}}$. The growth of the Schreier graph Γ_ξ is polynomial of degree $\log_2(d)$.*

Proof. Without loss of generality (see Remark 3.2.6), let us prove the statement for spinal groups G_d, defined by $m = 1$ and $\omega = \pi^{\mathbb{N}}$, with π mapping the generator of B to the generator of A. Moreover, in [10] it is proved that the growth rate of the graphs Γ_ξ is the same for every $\xi \in X^{\mathbb{N}}$, so let us restrict to the point $\xi = (d-1)^{\mathbb{N}}$.

Consider the tree $T(\xi)$, which is a one-ended line with vertex labels in $\mathbb{N}$ and edges $(n, n+1)$ for every $n \geq 0$. Identifying each vertex n with the center Λ_ξ^{n-2}, and each edge $(n, n+1)$ with the finite subgraph connecting Λ_ξ^{n-2} and Λ_ξ^{n-1} in Γ_ξ, we observe that, for any $n \geq 1$, the subtree $\{0, \ldots, n\}$ represents the finite subgraph Γ_ξ^{n-1}. This subgraph is, up to some loops, isomorphic to the finite Schreier graph Γ_{n-1}, and therefore has diameter $2^{n-1}-1$ and contains d^{n-1} vertices. Moreover, the ball of radius $2^{n-1}-1$ around ξ coincides with Γ_ξ^{n-1}.

Let $r \geq 0$ and define k such that $2^{k-1} \leq r \leq 2^k - 1$. The subgraph represented by $\{0, \ldots, k\}$, of diameter $2^{k-1} - 1$, is then contained in the ball of radius r around ξ, and this is contained in the subgraph represented by $\{0, \ldots, k+1\}$, of diameter $2^k - 1$. The size of the ball is then bounded between d^{k-1} and d^k. Since $k - 1 = \lceil \log_2(r) \rceil$, it is bounded between $d^{\lceil \log_2(r) \rceil}$ and $d^{\lceil \log_2(r) \rceil + 1}$, and is thus equivalent to $r^{\log_2(d)}$, as

$$d^{\log_2(r)} = d^{\log_2(d) \log_d(r)} = r^{\log_2(d)}.$$

□

Let us now illustrate the computation of the sequences $\{c_n\}_{n\geq 1}$ and the trees $T(\xi)$ with some examples.

Consider the Fabrykowski-Gupta group ($d = 3$, $m = 1$). We will compute the associated sequences $\{c_n\}_{n\geq 1}$ using Definition 3.6.1 and display the associated trees for the points $2^{\mathbb{N}}$, $0^{\mathbb{N}}$, $(110)^{\mathbb{N}}$ and $(210)^{\mathbb{N}}$ in $X^{\mathbb{N}}$. The first, third and fourth yield one-ended Schreier graphs (see Theorem 3.3.1), and the second a two-ended Schreier graph. For comparison, the last two are compatible points, so by Theorem 3.5.11 their unlabeled Schreier graphs are isomorphic.

In the illustrating Figures 3.10, 3.11, 3.12 and 3.13, vertex labels are to be concatenated with the appropriate shift of ξ. Subgraphs denoted Γ_n are copies within Γ_ξ of the finite Schreier graphs Γ_n, in the sense of Proposition 3.2.4. The associated tree $T(\xi)$ is overlapped to show its relation with Γ_ξ.

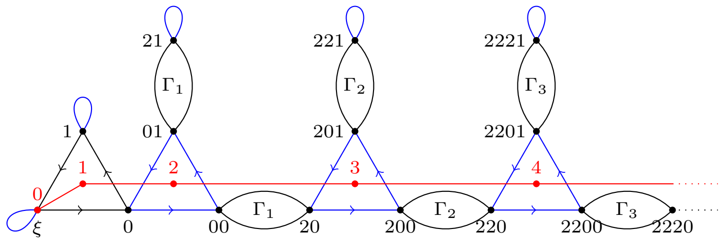


Figure 3.10: Schema of the Schreier graph Γ_ξ for the Fabrykowski-Gupta group with $\xi = 2^{\mathbb{N}}$. The tree $T(\xi)$ is overlapped in red.

Example 3.6.14. Let $\xi = 2^{\mathbb{N}}$. Its sequence $\{c_n\}_{n\geq 1}$ satisfies $c_{n+2} = n + 1$ for every $n \geq 1$, so it is $(0, 1, 2, 3, 4, 5, 6, \dots)$. The associated tree $T(\xi)$ is then a one-ended line labeled by the natural numbers in order. A schema of Γ_ξ and the tree $T(\xi)$ are displayed in Figure 3.10.

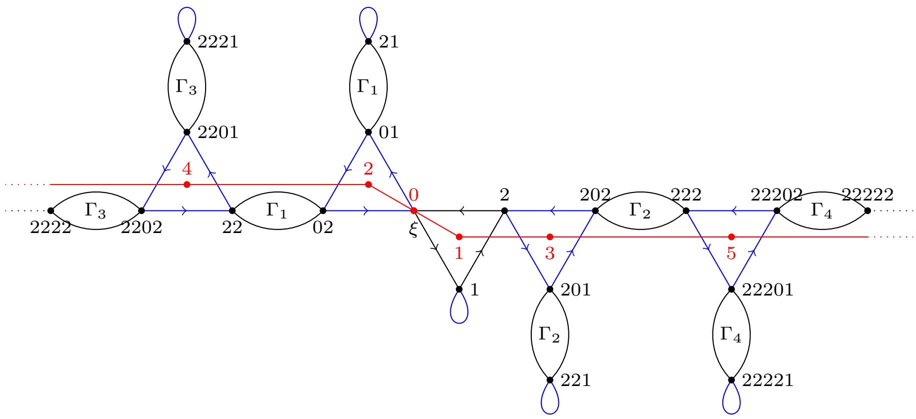


Figure 3.11: Schema of the Schreier graph Γ_ξ for the Fabrykowski-Gupta group with $\xi = 0^{\mathbb{N}}$. The tree $T(\xi)$ is overlapped in red.

Example 3.6.15. Let $\xi = 0^{\mathbb{N}}$. Its sequence $\{c_n\}_{n\geq 1}$ satisfies $c_{n+2} = n$ for every $n \geq 0$, so it is $(0, 0, 1, 2, 3, 4, 5, \dots)$. The associated tree $T(\xi)$ is then a two-ended line labeled by increasing even numbers on one side of 0 and increasing odd numbers on the other. A schema of Γ_ξ and the tree $T(\xi)$ are displayed in Figure 3.11.

Example 3.6.16. Let $\xi = (210)^{\mathbb{N}}$. Its sequence $\{c_n\}_{n\geq 1}$ is given by $c_1 = 0$ and, for $n \geq 0$,

$$c_{n+2} = \begin{cases} n & \text{if } n \equiv 2 \mod 3 \\ n+1 & \text{if } n \not\equiv 2 \mod 3 \end{cases},$$

so it is the sequence $(0, 1, 2, 2, 4, 5, 5, \dots)$. The associated tree $T(\xi)$ is the one-ended tree displayed in Figure 3.12 overlapped on a schema of the Schreier graph Γ_ξ.

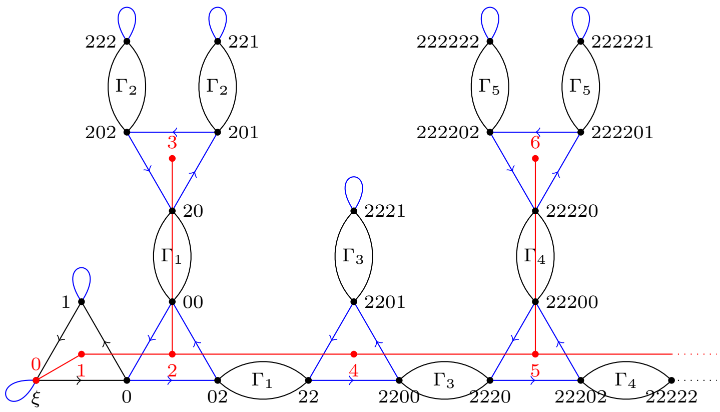


Figure 3.12: Schema of the Schreier graph Γ_ξ for the Fabrykowski-Gupta group with $\xi = (210)^{\mathbb{N}}$. The tree $T(\xi)$ is overlapped in red.

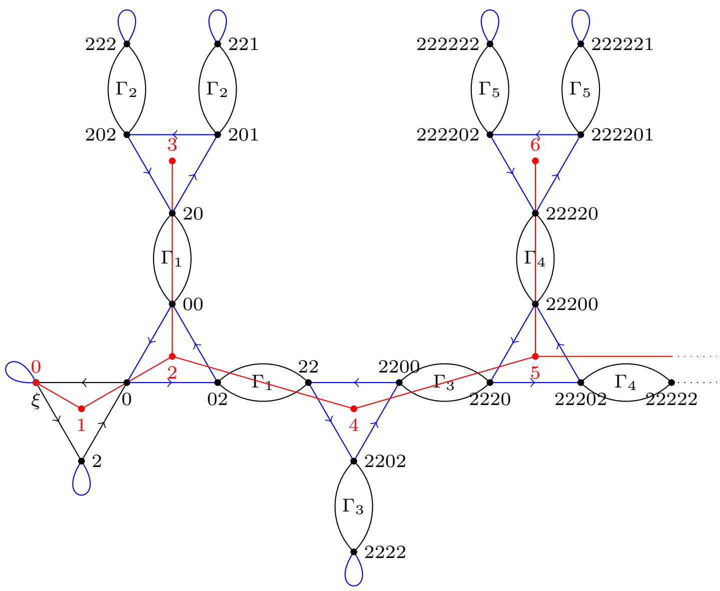


Figure 3.13: Schema of the Schreier graph Γ_ξ for the Fabrykowski-Gupta group with $\xi = (110)^{\mathbb{N}}$. The tree $T(\xi)$ is overlapped in red.

Example 3.6.17. Let $\xi = (110)^{\mathbb{N}}$. As it is compatible with $(210)^{\mathbb{N}}$, their associated sequences coincide, hence $\{c_n\}_{n\geq 1}$ is given by $c_1 = 0$ and, for $n \geq 0$,

$$c_{n+2} = \begin{cases} n & \text{if } n \equiv 2 \mod 3 \\ n+1 & \text{if } n \not\equiv 2 \mod 3 \end{cases},$$

so it is the sequence $(0, 1, 2, 2, 4, 5, 5, \dots)$. A schema of the Schreier graph Γ_ξ is shown in Figure 3.13, together with the tree $T(\xi)$. By comparing Figures 3.12 and 3.13, it becomes evident that the unlabeled Schreier graphs are isomorphic and that both associated trees coincide.

3.7 Space of Schreier graphs

For the action of a group G with a finite generating set S on a topological space $\mathcal{X}$ by homeomorphisms, it is particularly interesting to consider Schreier graphs as marked graphs, by adding a marked vertex. We then regard Schreier graphs as elements of $\mathcal{G}_{*,S}$, the space of marked graphs with edge labels in S, which is equipped with the topology of local convergence (or Gromov-Hausdorff topology). A basis for this topology is formed by the so-called cylinder sets, each of which contains all graphs with isomorphic r-balls around the marked vertex, for $r \geq 0$. Namely, by the sets

$$C_{(\Gamma,x)}(r) = \{(\Gamma', x') \in \mathcal{G}_{*,S} \mid \mathcal{B}_{(\Gamma,x)}(r) \cong \mathcal{B}_{(\Gamma',x')}(r)\},$$

for given $(\Gamma, x) \in \mathcal{G}_{*,S}$ and $r \geq 0$. Note that we write (Γ, x) for a graph Γ marked at the vertex x. We can define a map Sch from $\mathcal{X}$ to the space of marked, directed, labeled graphs $\mathcal{G}_{*,S}$ as

$$\begin{array}{rccl} \mathrm{Sch}: & \mathcal{X} & \to & \mathcal{G}_{*,S} \\ & x & \mapsto & (\Gamma_x, x) \end{array},$$

so we obtain a family Schreier graphs $(\Gamma_x, x)_{x\in\mathcal{X}}$.

If G is a finitely generated group of automorphisms of a rooted spherically homogeneous tree T that acts transitively of all levels of the tree then, for each boundary point $\xi \in X^{\mathbb{N}}$, the sequence of finite Schreier graphs $(\Gamma_n, \xi_0 \dots \xi_{n-1})$ converges to (Γ_ξ, ξ). Let now G_ω be a spinal group with spinal generating set S, with $\omega \in \Omega_{d,m}$ for $d \geq 2$, $m \geq 1$.

Proposition 3.7.1. *Let $\xi \in X^{\mathbb{N}}$. The map* Sch *is continuous at the point ξ if and only if $\xi \notin \mathrm{Cof}((d-1)^{\mathbb{N}})$.*

Proof. Let ξ be a point in the boundary, and let $\mathrm{Sch}(\xi)$ be its image. A neighborhood of $\mathrm{Sch}(\xi)$ is a set of marked graphs such that, for some $r \geq 0$, their balls of radius r around the marked vertex are isomorphic to $\mathcal{B} = \mathcal{B}_\xi(r)$. Fix $r \geq 1$ and let U be the corresponding neighborhood of $\mathrm{Sch}(\xi)$.

Let now $R \geq 0$ be such that $\mathcal{B} \subset \Gamma_\xi^R$ and such that $\mathcal{B}$ does not contain the vertices $(d-1)^{R-1}0\sigma^R\xi$ and $(d-1)^R\sigma^R\xi$. There is always an R satisfying the first condition and such that $\mathcal{B}$ does not contain the first vertex, but to ensure that it does not contain the second, we need $\xi \notin \mathrm{Cof}((d-1)^{\mathbb{N}})$.

Now consider the neighborhood $\xi_0 \dots \xi_{R-1}X^{\mathbb{N}}$ of ξ, and let η be a point of this neighborhood. We need to show that $\mathrm{Sch}(\eta) \in U$, so that $\mathcal{B}' = \mathcal{B}_\eta(r)$ is isomorphic to $\mathcal{B}$. Using

Proposition 3.2.4, both $\mathcal{B}$ and $\mathcal{B}'$ are isomorphic to the ball of radius r around $\xi_0 \dots \xi_{R-1}$ in Γ_R. Hence, Sch is continuous everywhere outside $\mathrm{Cof}((d-1)^{\mathbb{N}})$.

For points $\xi = w(d-1)^{\mathbb{N}} \in \mathrm{Cof}((d-1)^{\mathbb{N}})$, consider the sequence $(w(d-1)^n 0^{\mathbb{N}})_n$, which converges to ξ. Any ball centered at $w(d-1)^n 0^{\mathbb{N}}$ big enough will contain $(d-1)^{|w|+n}0^{\mathbb{N}}$, which is not fixed by B, and hence outgoing edges which are not loops for some $b \in B$, on that vertex, while $(d-1)^{\mathbb{N}}$ is fixed by B, and therefore all its outgoing B-edges are loops. The balls will not be isomorphic, and so Sch is not continuous on $\mathrm{Cof}((d-1)^{\mathbb{N}})$. □

If $\mathcal{X}_0 \subset \mathcal{X}$ is the set of continuity points of the map Sch, we define the *space of Schreier graphs* as $\mathcal{G}_{G,\mathcal{X}} = \overline{\mathrm{Sch}(\mathcal{X}_0)}$, the closure of the image of the continuity points of the map Sch. The group G acts on the space of Schreier graphs by shifting the marked vertex. In order to describe the space of Schreier graphs for spinal groups, in Theorem 3.7.3, we first need the following Lemma.

Lemma 3.7.2. *Let $\xi, \eta \in X^{\mathbb{N}}$. If $B_\xi(r)$ and $B_\eta(r)$ are isomorphic, then ξ and η share a prefix of length $\lfloor \log_2(r) \rfloor$.*

Proof. Let $k = \lfloor \log_2(r) \rfloor$, so we have $r \geq 2^k$. By Proposition 3.2.4 we know that the subgraphs Γ_ξ^k and Γ_η^k are copies of Γ_k. Because the diameter of Γ_k is $2^k - 1$ (see Remark 3.1.4), they must be fully contained in $B_\xi(r)$ and $B_\eta(r)$, respectively. The isomorphism between the balls must then restrict to an isomorphism between Γ_ξ^k and Γ_η^k, which maps ξ to η. Both are mapped to the same vertex of Γ_k, hence their prefixes of length k must coincide. □

Theorem 3.7.3. *Let G_ω be a spinal group, with $\omega \in \Omega_{d,m}$, for any $d \geq 2, m \geq 1$ except $d = 2, m = 1$. Then*

1. *The map* $\mathrm{Sch} : X^{\mathbb{N}} \to \mathcal{G}_{*,S}$ *is injective. Its continuity points are* $X^{\mathbb{N}} \setminus \mathrm{Cof}((d-1)^{\mathbb{N}})$.
2. *The set of isolated points of* $\overline{\mathrm{Sch}(X^{\mathbb{N}})}$ *is* $\mathrm{Sch}(\mathrm{Cof}(1^{\mathbb{N}}))$*, if* $d = 2$.
3. *The set* $\overline{\mathrm{Sch}(X^{\mathbb{N}})}$ *does not have isolated points, if* $d \geq 3$.
4. *The space of Schreier graphs* $\mathcal{G}_{G_\omega, X^{\mathbb{N}}}$ *contains a countable set which consists of finitely many d-ended graphs with arbitrary marked vertex. These graphs are* $\Gamma_\pi = \mathrm{Star}(\Lambda_\pi, \Gamma_{(d-1)^{\mathbb{N}}}, (d-1)^{\mathbb{N}})$*, for every* $\pi \in \mathrm{Epi}(B, A)$ *repeating infinitely often in* ω.
5. *The space of Schreier graphs* $\mathcal{G}_{G_\omega, X^{\mathbb{N}}}$ *is the disjoint union of either* $\mathrm{Sch}(X^{\mathbb{N}} \setminus \mathrm{Cof}(1^{\mathbb{N}}))$ *if* $d = 2$ *or* $\mathrm{Sch}(X^{\mathbb{N}})$ *if* $d \geq 3$ *with this countable set.*

Proof. The injectivity of Sch follows from the fact that spinal groups except $d = 2, m = 1$ are branch groups [5], and the stabilizers of boundary points are all different for the action of a branch group on the boundary of the tree (see Proposition 2.2. in [35]). In Proposition 3.7.1 we find the continuity points of Sch.

Let $(\xi^{(n)})_n$ be a sequence of continuity points of Sch in $X^{\mathbb{N}}$ such that $(\Gamma_{\xi^{(n)}}, \xi^{(n)})$ converges to some marked graph (Γ, ξ). For any $r \geq 0$, there exists some $N \geq 0$ such that, for every $n \geq N$, the balls $\mathcal{B}_{\xi^{(n)}}(r)$ are all isomorphic. By Lemma 3.7.2, for every $n \geq N$, all $\xi^{(n)}$ share the same prefix of length $\lfloor \log_2(r) \rfloor$. Hence, $(\xi^{(n)})_n$ converges to a point $\xi^{(\infty)} \in X^{\mathbb{N}}$.

If $\xi^{(\infty)}$ is a continuity point, then by continuity $(\Gamma, \xi) = (\Gamma_{\xi^{(\infty)}}, \xi^{(\infty)})$, so assume the opposite, which means that $\xi^{(\infty)} \in \mathrm{Cof}((d-1)^{\mathbb{N}})$. By continuity of the action, we can assume without loss of generality that $\xi^{(\infty)} = (d-1)^{\mathbb{N}}$. Now two things can happen: either $\xi^{(n)}$ is fixed by B for all large enough n or not. If that is the case, then $(\Gamma, \xi) = (\Gamma_{(d-1)^{\mathbb{N}}}, (d-1)^{\mathbb{N}})$, so graphs corresponding to points in $\mathrm{Cof}((d-1)^{\mathbb{N}})$ are not isolated. Notice that this cannot happen if $d = 2$, as no continuity point is fixed by B, and so they cannot approximate $(d-1)^{\mathbb{N}}$.

Suppose now that $\xi^{(n)}$ is not fixed by B for all large enough n. This means that $\xi^{(n)}$ has a prefix $(d-1)^{k_n}0$, for all n large enough. Because the balls $\mathcal{B}_{\xi^{(n)}}(1)$ must all be isomorphic, this means that, for every $b \in B$, if the image of $(d-1)^{k_n}0i\sigma^{k_n+2}\xi^{(n)}$ by b is $(d-1)^{k_n}0j\sigma^{k_n+2}\xi^{(n)}$, then the image of $(d-1)^{k_{n'}}0i\sigma^{k_{n'}+2}\xi^{(n')}$ by b must be $(d-1)^{k_{n'}}0j\sigma^{k_{n'}+2}\xi^{(n')}$, for every $n' \geq n$. Equivalently, ω_{k_n} is the same epimorphism π for every n large enough. This can only happen and does happen for epimorphisms π repeating infinitely often in ω. Therefore, the graph (Γ, ξ) coincides with (Γ_π, η), with η being any vertex of Λ_π. $\square$

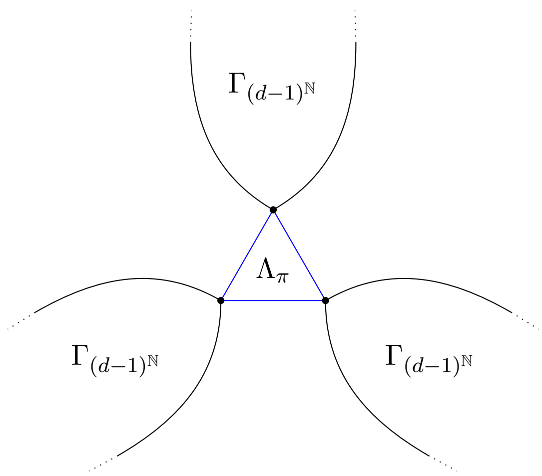


Figure 3.14: Graph Γ_π for $d = 3$.

Remark 3.7.4. Notice that the d-ended graphs Γ_π arising in Theorem 3.7.3 have the product $\mathrm{Cof}((d-1)^{\mathbb{N}}) \times X$ as vertex set and the following edges:

- For every $\eta \in \mathrm{Cof}((d-1)^{\mathbb{N}}) \setminus \{(d-1)^{\mathbb{N}}\}$, $i \in X$ and $s \in S$, there is an s-edge from (η, i) to $(s\eta, i)$.
- For every $i \in X$ and $k = 1, \ldots, d-1$, there is an a^k-edge from $((d-1)^{\mathbb{N}}, i)$ to $(a^k(d-1)^{\mathbb{N}}, i)$.
- For every $i \in X$ and $b \in B \setminus \{1\}$, there is a b-edge from $((d-1)^{\mathbb{N}}, i)$ to $((d-1)^{\mathbb{N}}, \pi(b)i)$.

If we consider the map $(\xi, i) \mapsto \xi$, it becomes clear that Γ_π is a d-covering of $\Gamma_{(d-1)^{\mathbb{N}}}$, for any $\pi \in \mathrm{Epi}(B, A)$. While the graphs Γ_ξ are Schreier graphs of subgroups $\mathrm{Stab}_G(\xi)$, for $m = 1$ the graphs Γ_π are Schreier graphs of $\mathrm{Stab}_G(N(\xi))$, the pointwise stabilizer of a neighborhood $N(\xi)$ of a point $\xi \in \mathrm{Cof}((d-1)^{\mathbb{N}})$. More generally, for $m \geq 1$, the Schreier graphs associated with $\mathrm{Stab}_G(N(\xi))$ are d^{m-1}-coverings of the graphs Γ_π.

Chapter 4

Spectra on Schreier and Cayley graphs

This chapter is devoted to the computation of the spectra of different adjacency operators on Schreier and Cayley graphs of spinal groups with respect to the spinal generating set. We present two ways of computing the spectra of the adjacency operators Δ_ξ on the Schreier graphs Γ_ξ, with $\xi \in X^{\mathbb{N}}$. In the former, we compute first the spectra of the adjacency operators Δ_n on the finite Schreier graphs Γ_n, for $n \geq 0$, and then use a finite approximation argument to find the spectra on Γ_ξ. This approach involves the actual computation of the characteristic polynomials for the adjacency matrices of the graphs, via the Schur complement. The latter is more elegant, and allows to compute the spectra of Δ_ξ without the finite approximations. We use two renormalization maps in the space of graphs, under which the set of Schreier graphs is fixed, and establish relations between the operators Δ_ξ and $\Delta_{\sigma\xi}$. Nevertheless, this approach is not as general as the former, as it is restricted to spinal groups with $m = 1$.

Afterwards, we prove that the spectra on the Cayley graphs coincide with those on the infinite Schreier graphs for spinal groups with $d = 2$, hence providing another negative answer to the question "Can one hear the shape of a group?", within the context of spinal groups.

Finally, we explore the dependence of these spectra with respect to the generating set, and provide some examples to illustrate the different phenomena that may occur.

Let G_ω be a spinal group with parameters $d \geq 2$, $m \geq 1$ and $\omega \in \Omega_{d,m}$. Unless stated otherwise, we consider the spinal generating set $S = (A \cup B) \setminus \{1\}$.

4.1 Spectra on Schreier graphs via finite approximation

In this section, we shall first compute the spectra $\mathrm{sp}(\Delta_n)$ in Theorem 4.1.7, for every $n \geq 0$, using the strategy in [4], namely by finding a recurrence relation between $\mathrm{sp}(\Delta_n)$ and

$\mathrm{sp}(\Delta_{n-1})$ and solving it to explicitly obtain the spectrum of Δ_n, with $n \geq 0$. This will allow us later to find $\mathrm{sp}(\Delta_\xi)$ for every $\xi \in X^{\mathbb{N}}$ in Theorem 4.1.10.

4.1.1 Spectra on finite Schreier graphs

Since the Schreier graphs Γ_n are finite, let us start by computing the matrix of Δ_n. The set of vertices of Γ_n is X^n, let us order them lexicographically. The matrix of Δ_n is a matrix of size $d^n \times d^n$. We will usually write it as a $d \times d$ block matrix where each block is a matrix of size $d^{n-1} \times d^{n-1}$. A block denoted by a scalar is the corresponding multiple of the identity matrix $I_{d^{n-1}}$.

Lemma 4.1.1. *For every $n \geq 0$, the matrix of Δ_n is $A_n + B_n$, where*

$$A_0 = d - 1, \qquad B_0 = d^m - 1,$$

and, if $n \geq 1$,

$$A_n = \begin{pmatrix} 0 & 1 & \dots & 1 & 1 \\ 1 & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 0 & 1 \\ 1 & 1 & \dots & 1 & 0 \end{pmatrix} \in M_{d^n}(\mathbb{R}),$$

$$B_n = \begin{pmatrix} d^{m-1}A_{n-1} + d^{m-1} - 1 & & & & \\ & d^{m-1} - 1 & & & \\ & & \ddots & & \\ & & & d^{m-1} - 1 & \\ & & & & B_{n-1} \end{pmatrix} \in M_{d^n}(\mathbb{R}).$$

Proof. In order to write the adjacency matrix of Γ_n, let us first write the adjacency matrices associated with each of the generators in S we consider. For a generator $s \in S$, we denote its associated adjacency matrix for Γ_n by s_n, and if $u, v \in X^n$, its coefficient (u, v) is 1 if $s(u) = v$ and 0 otherwise.

Recall that a is the generator of A, and it permutes the subtrees of the first level cyclically. This means that, for every $i \in X$ and $v \in X^{n-1}$, $a(iv) = (i+1)v$. We can therefore write the adjacency matrix of this generator as

$$a_0 = 1, \qquad a_n = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix}, \quad \forall n \geq 1.$$

Now, for any $b \in B$ and $k \geq 0$, if we write the matrix

$$b_{0,k} = 1 \qquad b_{n,k} = \begin{pmatrix} \omega_k(b)_{n-1} & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & b_{n-1,k+1} \end{pmatrix}, \quad \forall k \geq 0, \quad \forall n \geq 1,$$

then the adjacency matrix for Γ_n associated with b is $b_n = b_{n,0}$.

Recall that these matrices have size $d^n \times d^n$, so every block is itself a matrix of size $d^{n-1} \times d^{n-1}$, and scalars denote the corresponding multiple of the identity matrix $I_{d^{n-1}}$.

Now notice that, for every $n \geq 0$,

$$\sum_{i=1}^{d-1} a_n^i = A_n, \qquad \sum_{b \in B \setminus \{1\}} b_n = B_n.$$

The former is clear, and, for the latter, we have, for every $k \geq 0$,

$$\sum_{b \in B \setminus \{1\}} b_{n,k} = \begin{pmatrix} \sum\limits_{b \in B \setminus \{1\}} \omega_k(b)_{n-1} & & & & \\ & d^m - 1 & & & \\ & & \ddots & & \\ & & & d^m - 1 & \\ & & & & \sum\limits_{b \in B \setminus \{1\}} b_{n-1,k+1} \end{pmatrix} =$$

$$= \begin{pmatrix} d^{m-1} A_{n-1} + d^{m-1} - 1 & & & & \\ & d^m - 1 & & & \\ & & \ddots & & \\ & & & d^m - 1 & \\ & & & & \sum\limits_{b \in B \setminus \{1\}} b_{n-1,k+1} \end{pmatrix}.$$

The sum in the first block does not depend on k, since ω_k is an epimorphism and all elements of A have exactly d^{m-1} preimages in B. Hence we can inductively conclude that $\sum\limits_{b \in B \setminus \{1\}} b_n = B_n$. Finally, the matrix of Δ_n is then the sum of s_n for every $s \in S$:

$$\sum_{s \in S} s_n = \sum_{i=1}^{d-1} a_n^i + \sum_{b \in B \setminus \{1\}} b_n = A_n + B_n.$$

□

If we now try to find the characteristic polynomial of Δ_n, we will not find any explicit relation with that of Δ_{n-1}. Instead, we consider the matrix

$$Q_n(p,q) := B_n + pA_n - q.$$

These additional parameters will allow us to find a relation between the determinant of $Q_n(p,q)$ and $Q_{n-1}(p',q')$, for some different p' and q'. According to Lemma 4.1.1, by setting $p=1, q=0$, we recover the matrix of Δ_n, so more specifically we want to find $\mathrm{sp}(\Delta_n) = \{q \mid |Q_n(1,q)| = 0\}$.

As mentioned above, the strategy consists of two steps. First, we will provide the relation between the determinants of $Q_n(p,q)$ and $Q_{n-1}(p',q')$ (Proposition 4.1.4). Second, we will solve this recurrence to find a factorization of the determinant of $Q_n(p,q)$ (Proposition 4.1.6).

Our computations involve matrices of the form $rA_n + s$, with $r, s \in \mathbb{R}$, so let us start with the following result, which will be useful later on.

Lemma 4.1.2. *Let $r, s, r', s' \in \mathbb{R}$. Then,*

1. $A_n^2 = (d-2)A_n + d - 1$.
2. $|rA_n + s| = \left[(s-r)^{d-1}(s+(d-1)r)\right]^{d^{n-1}}$.
3. $(rA_n+s)^{-1} = \frac{1}{(r-s)(s+(d-1)r)}(rA_n - (d-2)r - s)$.
4. $(rA_n+s)(r'A_n+s') = [(d-2)rr' + rs' + r's]A_n + (d-1)rr' + ss'$.

Proof. For (1), if we square A_n then we will get a sum of $d-1$ ones for elements in the diagonal and $d-2$ ones for the rest, which shows the claim.

For (2), we have

$$|rA_n+s| = \begin{vmatrix} s & r & \dots & r & r \\ r & s & \dots & r & r \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r & r & \dots & s & r \\ r & r & \dots & r & s \end{vmatrix} = \begin{vmatrix} s-r & 0 & \dots & 0 & r-s \\ 0 & s-r & \dots & 0 & r-s \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & s-r & r-s \\ r & r & \dots & r & s \end{vmatrix} =$$

$$= (s-r)^{(d-1)d^{n-1}} \left| s - \frac{(d-1)r(r-s)}{s-r} \right| = \left[(s-r)^{d-1}(s+(d-1)r)\right]^{d^{n-1}}.$$

For (3), we can verify, using (1), that

$$(rA_n+s)\left(rA_n - (d-2)r - s\right) = r^2A_n^2 - (d-2)r^2A_n - (d-2)rs - s^2 =$$

$$= r^2\left((d-2)A_n + d - 1\right) - (d-2)r^2A_n - (d-2)rs - s^2 = (r-s)(s+(d-1)r).$$

Claim (4) can be checked directly, again using (1). □

Proposition 4.1.3. *For $n = 0$ and $n = 1$, we have*

$$|Q_0(p,q)| = \alpha + p \quad \textit{and} \quad |Q_1(p,q)| = (\alpha + p)\beta^{d-1},$$

where

$$\alpha = \alpha(p,q) := d^m - 1 - q + (d-2)p$$

and

$$\beta = \beta(p,q) := d^m - 1 - q - p.$$

Proof. By direct computation,

$$|Q_0(p,q)| = B_0 + pA_0 - q = d^m - 1 - q + (d-1)p = \alpha + p,$$

$$|Q_1(p,q)| = |B_1 + pA_1 - q| = |pA_1 + d^m - 1 - q| =$$

$$= (d^m - 1 - q + (d-1)p)(d^m - 1 - q - p)^{d-1} = (\alpha + p)\beta^{d-1}.$$

□

We are now ready to compute the determinant of $Q_n(p,q)$ for $n \geq 2$.

Proposition 4.1.4. *For $n \geq 2$, we have*

$$|Q_n(p,q)| = (\alpha\beta^{d^2-3d+1}\gamma^{d-1})^{d^{n-2}} \left|Q_{n-1}(p',q')\right|,$$

with α and β as in Proposition 4.1.3,

$$p' := \frac{d^{m-1}\beta}{\alpha\gamma}p^2 \quad \textit{and} \quad q' := q + \frac{(d-1)\delta}{\alpha\gamma}p^2,$$

where

$$\gamma = \gamma(p,q) := q^2 - ((d-3)p + d^m - 2)q - ((d-2)p^2 + (d-3)p + d^m - 1),$$

$$\delta = \delta(p,q) := q^2 - ((d-3)p + d^m + d^{m-1} - 2)q - ((d-2)p^2 + (d^{m-1} + d - 3)p - d^{2m-1} + d^m + d^{m-1} - 1).$$

Proof. We start computing the determinant of $|Q_n(p,q)|$ directly, performing elementary transformations of rows and columns in determinants.

$$|Q_n(p,q)| = \begin{vmatrix} d^{m-1}A_{n-1} + d^{m-1} - 1 - q & p & \dots & p & p \\ p & d^m - 1 - q & \dots & p & p \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p & p & \dots & d^m - 1 - q & p \\ p & p & \dots & p & B_{n-1} - q \end{vmatrix} =$$

$$= \left|\begin{array}{ccc|cc} \beta+p & \dots & p & p & p \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ p & \dots & \beta+p & p & p \\ \hline p & \dots & p & d^{m-1}A_{n-1}+d^{m-1}-1-q & p \\ p & \dots & p & p & B_{n-1}-q \end{array}\right| =$$

$$= \left|\begin{array}{ccc|cc} \beta & & & p+q+1-d^{m-1}-d^{m-1}A_{n-1} & 0 \\ & \ddots & & \vdots & \vdots \\ & & \beta & p+q+1-d^{m-1}-d^{m-1}A_{n-1} & 0 \\ \hline p & \dots & p & d^{m-1}A_{n-1}+d^{m-1}-1-q & p \\ p & \dots & p & p & B_{n-1}-q \end{array}\right| =$$

$$= \left|\begin{array}{ccc|cc} \beta & & & p+q+1-d^{m-1}-d^{m-1}A_{n-1} & 0 \\ & \ddots & & \vdots & \vdots \\ & & \beta & p+q+1-d^{m-1}-d^{m-1}A_{n-1} & 0 \\ \hline 0 & \dots & 0 & d^{m-1}A_{n-1}+d^{m-1}-1-q-\frac{(d-2)p(p+q+1-d^{m-1}-d^{m-1}A_{n-1})}{\beta} & p \\ 0 & \dots & 0 & p-\frac{(d-2)p(p+q+1-d^{m-1}-d^{m-1}A_{n-1})}{\beta} & B_{n-1}-q \end{array}\right| =$$

$$= (\beta^{d-2})^{d^{n-1}} \left|\begin{array}{cc} d^{m-1}A_{n-1}+d^{m-1}-1-q-\frac{(d-2)p(p+q+1-d^{m-1}-d^{m-1}A_{n-1})}{\beta} & p \\ p\left(1-\frac{(d-2)(p+q+1-d^{m-1}-d^{m-1}A_{n-1})}{\beta}\right) & B_{n-1}-q \end{array}\right| =$$

$$= (\beta^{d-3})^{d^{n-1}} \left|\begin{array}{cc} \beta(d^{m-1}A_{n-1}+d^{m-1}-1-q)-(d-2)p(p+q+1-d^{m-1}-d^{m-1}A_{n-1}) & p^2 \\ \beta-(d-2)(p+q+1-d^{m-1}-d^{m-1}A_{n-1}) & B_{n-1}-q \end{array}\right| =$$

$$= (\beta^{d-3})^{d^{n-1}} \left|\begin{array}{cc} d^{m-1}(\alpha-p)(A_{n-1}+1)+\gamma & p^2 \\ (d-2)d^{m-1}(A_{n-1}-(d-1))+(d-1)\beta & B_{n-1}-q \end{array}\right| .$$

We set for convenience

$$C_n := d^{m-1}(\alpha-p)(A_n+1)+\gamma,$$

$$D_n := (d-2)d^{m-1}(A_n-(d-1))+(d-1)\beta.$$

We continue the computation of the determinant by taking the first Schur complement. Namely, whenever a matrix P is invertible, we have equality $\left|\begin{array}{cc} P & Q \\ R & S \end{array}\right| = |P||S-RP^{-1}Q|$:

$$|Q_n(p,q)| = (\beta^{d-3})^{d^{n-1}} \left|\begin{array}{cc} C_{n-1} & p^2 \\ D_{n-1} & B_{n-1}-q \end{array}\right| =$$

$$= (\beta^{d-3})^{d^{n-1}} |C_{n-1}| \left|B_{n-1}-q-p^2D_{n-1}C_{n-1}^{-1}\right| .$$

Let us now compute these two determinants. First, using Lemma 4.1.2 with $r = d^{m-1}(\alpha - p)$ and $s = \gamma + d^{m-1}(\alpha - p)$, we obtain

$$|C_n| = \Big[(s-r)^{d-1}(s+(d-1)r)\Big]^{d^{n-1}} = (\alpha\beta\gamma^{d-1})^{d^{n-1}},$$

as well as

$$C_n^{-1} = \frac{-1}{\alpha\beta\gamma}\left[d^{m-1}(\alpha-p)(A_n-(d-1))-\gamma\right].$$

Similarly, again by Lemma 4.1.2 but now with

$$\begin{aligned} r &= (d-2)d^{m-1}, \quad & s &= (d-1)\left(\beta-(d-2)d^{m-1}\right), \\ r' &= d^{m-1}(\alpha-p), \quad & s' &= -(\gamma+(d-1)d^{m-1}(\alpha-p)), \end{aligned}$$

we find

$$D_nC_n^{-1} = \frac{-1}{\alpha\gamma}\left[d^{m-1}\beta A_n-(d-1)\delta\right].$$

Indeed,

$$\begin{gathered} (d-2)rr' + rs' + r's = \\ = (d-2)rr' - r(\gamma+(d-1)r') + r'(d-1)(\beta-r) = \\ = -drr' - r\gamma + (d-1)\beta r' = \\ = d^{m-1}\left[(d-1)\beta(\alpha-p)-(d-2)(\gamma+d^m(\alpha-p))\right] = \\ = d^{m-1}\beta[\alpha-(d-1)p] = d^{m-1}\beta^2, \end{gathered}$$

and

$$\begin{gathered} (d-1)rr' + ss' = \\ = (d-1)rr' - (d-1)(\beta-r)(\gamma+(d-1)r') = \\ = (d-1)(drr' + r\gamma - \beta(\gamma+(d-1)r')) = \\ = (d-1)(r(\gamma+dr') - \beta(\gamma+(d-1)r')) = \\ = (d-1)(r\alpha\beta - \beta(\gamma+(d-1)r')) = \\ = -(d-1)\beta(\gamma+(d-1)d^{m-1}(\alpha-p)-(d-2)d^{m-1}\alpha) = \\ = -(d-1)\beta(\gamma+d^{m-1}(\alpha+(d-1)p)) = \\ = -(d-1)\beta(\gamma+d^{m-1}\beta) = -(d-1)\beta\delta. \end{gathered}$$

Therefore,

$$\begin{gathered} B_{n-1} - q - p^2D_{n-1}C_{n-1}^{-1} = \\ B_{n-1} - q + \frac{p^2}{\alpha\gamma}\left[d^{m-1}\beta A_{n-1}-(d-1)\delta\right] = \end{gathered}$$

$$B_{n-1} + \frac{d^{m-1}\beta}{\alpha\gamma}p^2 A_{n-1} - \left(q + \frac{(d-1)\delta}{\alpha\gamma}p^2\right) =$$
$$= B_{n-1} + p' A_{n-1} - q' =$$
$$= Q_{n-1}(p', q').$$

Finally, we conclude the computation of the determinant of $Q_n(p,q)$:

$$|Q_n(p,q)| = (\beta^{d-3})^{d^{n-1}} |C_{n-1}| \left|B_{n-1} - q - p^2 D_{n-1} C_{n-1}^{-1}\right| =$$
$$= (\beta^{d-3})^{d^{n-1}} (\alpha\beta\gamma^{d-1})^{d^{n-2}} \left|Q_{n-1}(p', q')\right| =$$
$$= (\alpha\beta^{d^2-3d+1}\gamma^{d-1})^{d^{n-2}} \left|Q_{n-1}(p', q')\right|.$$

□

This concludes the first part of the strategy, finding a recurrence relation between the determinants of $Q_n(p,q)$ and $Q_{n-1}(p',q')$. For the next part, we need to unfold this recurrence relation to get a factorization of $|Q_n(p,q)|$. Proposition 4.1.3 provides it for $n = 0, 1$. Let us inductively compute it for $n \geq 2$.

Proposition 4.1.5. *For $n = 2$, we have*

$$|Q_2(p,q)| = (\alpha + p)\beta^{(d-2)d+1} H_0^{d-1},$$

where

$$H_x := H_x(p,q) = q^2 - ((d-2)p + d^m - 2)q - ((d-1)p^2 + (d^{m-1}x + d - 2)p + d^m - 1).$$

Proof. Let $\alpha' := \alpha(p', q')$ and $\beta' := \beta(p', q')$ following the definition in Proposition 4.1.3. Then, by that Proposition and Proposition 4.1.4,

$$|Q_2(p,q)| = \alpha\beta^{d^2-3d+1}\gamma^{d-1} \left|Q_1(p', q')\right| = \alpha\beta^{d^2-3d+1}\gamma^{d-1}(\alpha' + p')\beta'^{d-1}.$$

We can verify the following relations

$$\alpha' + p' = \frac{\beta}{\alpha}(\alpha + p), \quad \beta' = \frac{\beta}{\gamma} H_0.$$

Therefore,

$$|Q_2(p,q)| = \alpha\beta^{d^2-3d+1}\gamma^{d-1}\frac{\beta}{\alpha}(\alpha + p)\left(\frac{\beta}{\gamma}H_0\right)^{d-1} =$$
$$= (\alpha + p)\beta^{(d-2)d+1} H_0^{d-1}.$$

□

The motivation for the definition of the polynomials H_x from Proposition 4.1.5 will become apparent in Proposition 4.1.6. They form a family of polynomials in p and q indexed by the point $x \in \mathbb{R}$. For different values of $x \in \mathbb{R}$, the equation $H_x = 0$ defines different hyperbolas in p and q.

Proposition 4.1.6. *For any $n \geq 2$, we have the factorization*

$$|Q_n(p,q)| = (\alpha + p)\beta^{(d-2)d^{n-1}+1} \prod_{k=0}^{n-2} \prod_{x \in F^{-k}(0)} H_x^{(d-2)d^{n-k-2}+1},$$

with α and β as in Proposition 4.1.3, H_x as in Proposition 4.1.5 and F being the map

$$F(x) = x^2 - d(d-1).$$

Proof. The case $n = 2$ is shown in Proposition 4.1.5. We use again the recurrence in Proposition 4.1.4 to show the result for $n \geq 3$ inductively. Let $H'_x := H_x(p', q')$. We can verify

$$H'_x = \frac{\beta}{\alpha\gamma} \prod_{y \in F^{-1}(x)} H_y.$$

Using this relation and the fact that, for any $k \geq 0$, $|F^{-k}(0)| = 2^k$, we have

$$|Q_n(p,q)| = \left(\alpha\beta^{d^2-3d+1}\gamma^{d-1}\right)^{d^{n-2}} \left|Q_{n-1}(p',q')\right| =$$

$$= \left(\alpha\beta^{d^2-3d+1}\gamma^{d-1}\right)^{d^{n-2}} (\alpha' + p')\beta'^{(d-2)d^{n-2}+1} \prod_{k=0}^{n-3} \prod_{x \in F^{-k}(0)} H_x'^{(d-2)d^{n-k-3}+1} =$$

$$= \left(\alpha\beta^{d^2-3d+1}\gamma^{d-1}\right)^{d^{n-2}} \frac{\beta}{\alpha}(\alpha+p) \left(\frac{\beta}{\gamma}H_0\right)^{(d-2)d^{n-2}+1} \prod_{k=0}^{n-3} \prod_{x \in F^{-k}(0)} \left(\frac{\beta}{\alpha\gamma} \prod_{y \in F^{-1}(x)} H_y\right)^{(d-2)d^{n-k-3}+1} =$$

$$= (\alpha\gamma)^{d^{n-2}-1}\beta^{(d^2-2d-1)d^{n-2}+2}(\alpha+p)H_0^{(d-2)d^{n-2}+1} \prod_{k=0}^{n-3} \left(\frac{\beta}{\alpha\gamma}\right)^{2^k((d-2)d^{n-k-3}+1)} \prod_{x \in F^{-(k+1)}(0)} H_x^{(d-2)d^{n-k-3}+1} =$$

$$= (\alpha\gamma)^{d^{n-2}-1}\beta^{(d^2-2d-1)d^{n-2}+2}(\alpha+p)H_0^{(d-2)d^{n-2}+1} \left(\frac{\beta}{\alpha\gamma}\right)^{d^{n-2}-1} \prod_{k=1}^{n-2} \prod_{x \in F^{-k}(0)} H_x^{(d-2)d^{n-k-2}+1} =$$

$$= (\alpha + p)\beta^{(d^2-2d)d^{n-2}+1} \prod_{k=0}^{n-2} \prod_{x \in F^{-k}(0)} H_x^{(d-2)d^{n-k-2}+1} =$$

$$= (\alpha + p)\beta^{(d-2)d^{n-1}+1} \prod_{k=0}^{n-2} \prod_{x \in F^{-k}(0)} H_x^{(d-2)d^{n-k-2}+1}.$$

□

The relation between the determinants of $Q_n(p,q)$ and $Q_{n-1}(p',q')$ is given by the substitution $p \mapsto p'$, $q \mapsto q'$. For Q_2, one of the factors of the determinant is the polynomial we called H_0. To compute the determinant of Q_3, we have to develop H_0'. It is in this analysis that the polynomials H_x and the map F arise. They are the link between H_x' and H_y that allows us to unfold the recurrence.

From the factorization in Proposition 4.1.6 we can extract $\mathrm{sp}(\Delta_n)$, as we mentioned above, by setting $p = 1$. Recall that $|S| = d^m + d - 2$.

Theorem 4.1.7. *Let G_ω be a spinal group with $d \geq 2$, $m \geq 1$ and $\omega \in \Omega_{d,m}$, and let Δ_n be the adjacency operator for the spinal generating set S on the Schreier graph Γ_n, for $n \geq 0$. We have*

$$\mathrm{sp}(\Delta_0) = \{|S|\},$$

$$\mathrm{sp}(\Delta_1) = \{|S|, |S| - d\},$$

and, for $n \geq 2$,

$$\mathrm{sp}(\Delta_n) = \{|S|, |S| - d\} \cup \psi^{-1}\left(\bigcup_{k=0}^{n-2} F^{-k}(0)\right),$$

where $F(x) = x^2 - d(d-1)$ and $\psi(x) = \frac{1}{d^{m-1}}(x^2 - (|S|-2)x - (|S|+d-2))$.

Proof. We already established that $\mathrm{sp}(\Delta_n) = \{q \mid |Q_n(1,q)| = 0\}$. By Propositions 4.1.3 and 4.1.6, the determinant vanishes in the following cases:

- $\alpha + 1 = 0$. Then $|S| \in \mathrm{sp}(\Delta_n)$ with multiplicity 1, for every $n \geq 0$.
- $\beta = 0$. Then $d^m - 2 = |S| - d \in \mathrm{sp}(\Delta_n)$ with multiplicity $(d-2)d^{n-1} + 1$, if $n \geq 1$.
- $H_x = 0$, for some $x \in F^{-k}(0)$ with $0 \leq k \leq n-2$. This implies that

  $$\frac{|S|-2}{2} \pm \sqrt{\left(\frac{|S|-2}{2}\right)^2 + |S| + d - 2 + d^{m-1}x} \in \mathrm{sp}(\Delta_n)$$

  each with multiplicity $(d-2)d^{n-k-2}+1$. These two eigenvalues are the two preimages of x by the map ψ defined above.

□

Remark 4.1.8. Note that, for spinal groups with $m = 1$, we have the equality $\psi(x) = F(x - (d-2))$. In that case, we can rewrite

$$\mathrm{sp}(\Delta_n) = \{|S|\} \cup \bigcup_{k=0}^{n-1} G^{-k}(d-2), \quad \forall n \geq 1,$$

with $G(x) = \psi(x) + d - 2 = F(x - (d-2)) + d - 2$.

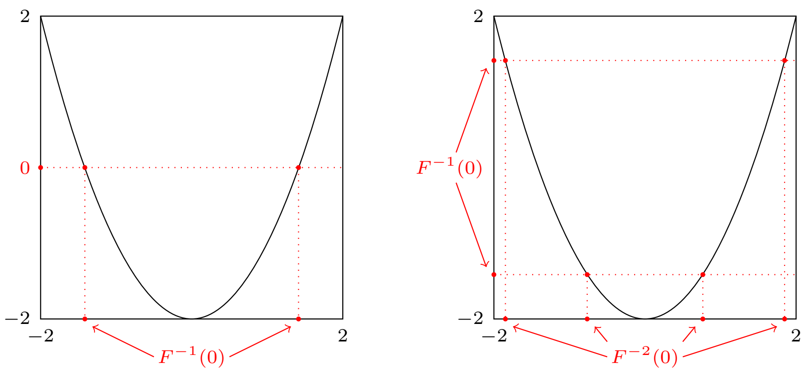


Figure 4.1: Construction of the preimages of 0 by F for $d = 2$. The set of all preimages is dense in the interval $[-2, 2]$.

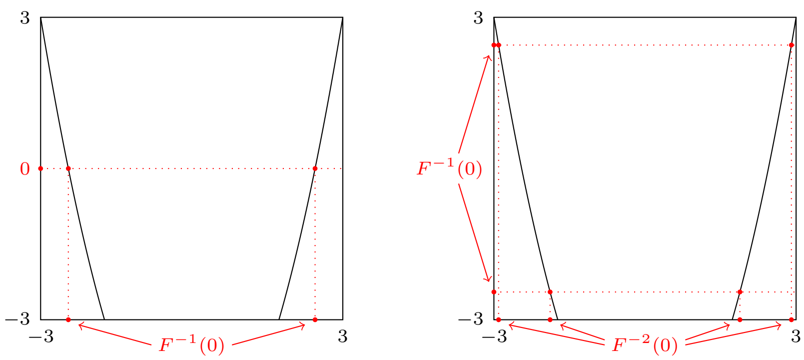


Figure 4.2: Construction of the preimages of 0 by F for $d = 3$. The set of all preimages accumulates on a Cantor set.

Remark 4.1.9. The map ψ is symmetric about its minimal point $\frac{|S|-2}{2}$, and satisfies

$$\psi^{-1}(d) = \{|S|, -2\}\,, \qquad \psi^{-1}(-d) = \left\{ \frac{|S|-2}{2} \pm \sqrt{\left(\frac{|S|-2}{2}\right)^2 + 2(d-2)} \right\},$$

$$\psi^{-1}(-d(d-1)) = \{|S| - d, d-2\}\,.$$

The preimages of 0 by the map F are contained in $[-d, d]$, as shown in Figures 4.1 and 4.2 for $d = 2, 3$, and they accumulate on its Julia set, which is the entire interval $[-2, 2]$ if $d = 2$ or a Cantor set if $d > 2$.

4.1.2 Spectra on infinite Schreier graphs

Let us now find the spectra $\mathrm{sp}(\Delta_\xi)$ on the Schreier graphs Γ_ξ, for $\xi \in X^{\mathbb{N}}$. Having computed $\mathrm{sp}(\Delta_n)$, for any $n \geq 0$, in Theorem 4.1.7, finding $\mathrm{sp}(\Delta_\xi)$ becomes immediate, as shown next in Theorem 4.1.10.

Theorem 4.1.10. *Let G_ω be a spinal group with $d \geq 2$, $m \geq 1$ and $\omega \in \Omega_{d,m}$, and let Δ_ξ be the adjacency operator for the spinal generating set S on the Schreier graph Γ_ξ. Then, for every $\xi \in X^{\mathbb{N}}$, we have*

$$\mathrm{sp}(\Delta_\xi) = \{|S| - d\} \cup \psi^{-1}\left(\overline{\bigcup_{n\geq 0} F^{-n}(0)}\right), \tag{4.1}$$

where $F(x) = x^2 - d(d-1)$ and $\psi(x) = \frac{1}{d^{m-1}}(x^2 - (|S|-2)x - (|S|+d-2))$. In particular, $\mathrm{sp}(\Delta_\xi)$ does not depend on ξ.

For $d = 2$, we have $\mathrm{sp}(\Delta_\xi) = [-2, 0] \cup [2^m - 2, 2^m]$.

For $d > 2$, we can decompose

$$\mathrm{sp}(\Delta_\xi) = \Lambda \cup K,$$

where Λ is a Cantor set of zero Lebesgue measure and $K = \{|S| - d\} \cup \psi^{-1}\left(\bigcup_{n\geq 0} F^{-n}(0)\right)$ is a countable set of isolated points accumulating on Λ.

Proof. For every $\xi \in X^{\mathbb{N}}$, we have the equality

$$\mathrm{sp}(\Delta_\xi) = \overline{\bigcup_{n\geq 0} \mathrm{sp}(\Delta_n)}.$$

Indeed, the inclusion $\subset$ follows from Theorem 3.4.9 in [22] by an argument of weak containment of representations. The other inclusion holds if Γ_ξ is amenable. Since the graphs Γ_ξ have polynomial growth (see Proposition 3.6.13 and [9]), they are amenable. The description of $\mathrm{sp}(\Delta_\xi)$ follows from this equality and Theorem 4.1.7. The value $|S|$ is obtained as the limit of the sequence $(\psi_1^{-1} \circ F_1^{-n}(0))_n$, where ψ_1^{-1} and F_1^{-1} denote the positive branch of the inverse of ψ and F, respectively, so it is not necessary to add it explicitly.

For $d = 2$, the map F is $F(x) = x^2 - 2$, whose Julia set is the interval $[-2, 2]$, and ψ becomes $\psi(x) = \frac{1}{2^{m-1}}(x^2 - (2^m - 2)x - 2^m)$. For any $y \in [-2, 2]$, we find its preimages x by ψ:

$$y = \psi(x) \iff x^2 - (2^m - 2)x - (2^m + 2^{m-1}y) = 0,$$

$$x = 2^{m-1} - 1 \pm \sqrt{(2^{m-1} - 1)^2 + 2^m + 2^{m-1}y}.$$

Hence, if $y \in [-2,2]$, then

$$x \in 2^{m-1} - 1 \pm \sqrt{(2^{m-1}-1)^2 + 2^m + 2^{m-1}[-2,2]} =$$
$$= 2^{m-1} - 1 \pm \sqrt{4^{m-1} + 1 + [-2^m, 2^m]} =$$
$$= 2^{m-1} - 1 \pm \sqrt{[(2^{m-1}-1)^2, (2^{m-1}+1)^2]} =$$
$$= 2^{m-1} - 1 \pm [2^{m-1}-1, 2^{m-1}+1] =$$
$$= [-2,0] \cup [2^m - 2, 2^m].$$

If $d \geq 3$, then the Julia set of F is a Cantor set of zero Lebesgue measure [53], and its two preimages by ψ are still Cantor sets lying on the different sides of the minimal point of ψ, $\frac{|S|-2}{2}$. This completes the proof. Intuitively, we can regard K as two infinite trees (one for each branch of the inverse of ψ), and Λ as their boundary. □

Remark 4.1.11. Recall that for spinal groups with $m = 1$ we had the equality $\psi(x) = F(x - (d-2))$. In this case, we may now rewrite

$$\mathrm{sp}(\Delta_\xi) = \overline{\bigcup_{n \geq 0} G^{-n}(d-2)}, \tag{4.2}$$

with $G(x) = \psi(x) + d - 2 = F(x - (d-2)) + d - 2$. In this case $\mathrm{sp}(\Delta_\xi) = \Lambda \cup K$, with Λ being the Julia set of G and $K = \bigcup_{n \geq 0} G^{-n}(d-2)$ accumulating on Λ. Notice that Λ is an interval if and only if $d = 2$, and otherwise it is a Cantor set [53].

Even though we discuss spectral measures in detail in Chapter 5, it is worth already noting a direct consequence of Theorem 4.1.10 for spinal groups with $d \geq 3$.

Corollary 4.1.12. *Let G_ω be a spinal group with $d \geq 3$, $m \geq 1$ and $\omega \in \Omega_{d,m}$, and let Δ_ξ be the adjacency operator for the spinal generating set S on the Schreier graph Γ_ξ. Then, for every $\xi \in X^{\mathbb{N}}$, any spectral measure of Δ_ξ has trivial absolutely continuous part.*

Proof. Any spectral measure of Δ_ξ has support contained in $\mathrm{sp}(\Delta_\xi)$, which has Lebesgue measure zero for $d \geq 3$. Its absolutely continuous part must therefore be trivial. □

4.2 Spectra on Schreier graphs via renormalization

The goal of this section is to provide another approach to compute the spectra of the adjacency operators Δ_ξ on the Schreier graphs Γ_ξ of spinal groups with $m = 1$ using a renormalization approach, in Theorem 4.2.8. Even though for Δ_ξ it is a particular case of Theorem 4.1.10, the strategy is more elegant, and the result is valid not only for the graphs Γ_ξ, $\xi \in X^{\mathbb{N}}$, but also for the d-ended graphs Γ_π from Theorem 3.7.3, so for all graphs in the space of Schreier graphs $\mathcal{G}_{G_\omega, X^{\mathbb{N}}}$. This method was developed by Quint in [60] for a graph related with the Pascal graph. In that case, the renormalization maps are an example of a more general construction studied in [48] called the para-line graph. Nevertheless, such construction is not suitable for our case, so we use a different approach to define the renormalization maps.

Let G_ω be a spinal group with $d \geq 3$, $m = 1$ and $\omega \in \Omega_{d,1}$. Let $\xi \in X^{\mathbb{N}}$ and Γ_ξ be its Schreier graph with respect to the spinal generating set S, and let us write $\ell^2_\xi = \ell^2(\Gamma_\xi)$. Recall that the shift map $\sigma : X^{\mathbb{N}} \to X^{\mathbb{N}}$ removes the first digit of any point in $X^{\mathbb{N}}$. This shift induces an operator $\Pi^* : \ell^2_{\sigma\xi} \to \ell^2_\xi$, defined as $\Pi^* f(\eta) = f(\sigma\eta)$. Conversely, we consider as well the operator $\Pi : \ell^2_\xi \to \ell^2_{\sigma\xi}$, defined as $\Pi f(\eta) = \sum_{i \in X} f(i\eta)$. As it turns out, Π and Π^* are adjoint operators, and we have the relation $\Pi\Pi^* = d$.

Geometrically, we can think of Γ_ξ as an inflated version of $\Gamma_{\sigma\xi}$, where every vertex has been replaced by a graph on d vertices. If $f \in \ell^2_{\sigma\xi}$ assigns a certain value to a vertex in $\Gamma_{\sigma\xi}$, then $\Pi^* f$ assigns the same value to all d vertices replacing it. Conversely, if $f \in \ell^2_\xi$, then Πf assigns to a vertex the sum of the values of f at the d corresponding vertices in Γ_ξ. See Figure 4.3 for a description of Π^*.

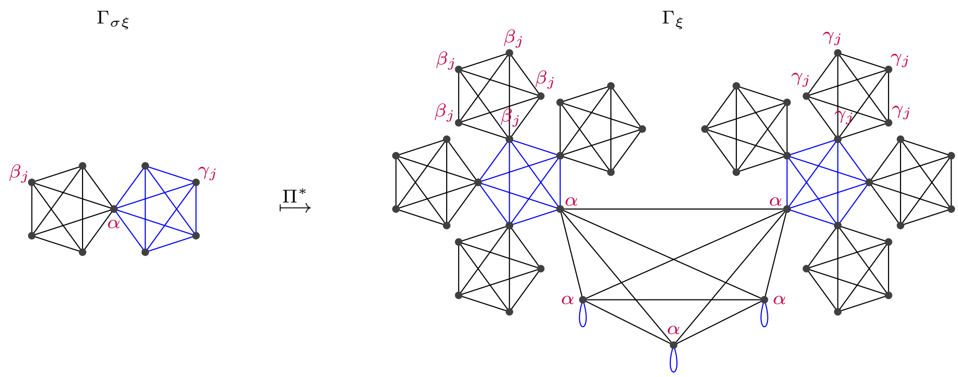


Figure 4.3: The operator Π^* copies the value of the function at each vertex p in $\Gamma_{\sigma\xi}$ to the corresponding d vertices ip, $i \in X$, in Γ_ξ.

In a similar way, we may define the renormalization maps Π^* and Π for Γ_π as well. Let G_ω be a spinal group with $d \geq 3$, $m = 1$ and $\omega \in \Omega_{d,1}$. Let $\pi \in \mathrm{Epi}(B, A)$ occurring infinitely often in ω. The d-ended Schreier graphs Γ_π from Theorem 3.7.3 correspond to Schreier graphs of neighborhoods of the point $(d-1)^{\mathbb{N}} \in X^{\mathbb{N}}$ (see Section 3.7).

Recall that the graphs Γ_π have vertex set $\mathrm{Cof}((d-1)^{\mathbb{N}}) \times X$, and can be decomposed as d copies of $\Gamma_{(d-1)^{\mathbb{N}}}$ joined by a copy of Λ_π. Let $\ell^2_\pi = \ell^2(\Gamma_\pi)$ and Δ_π be the adjacency operator on Γ_π. We can extend the renormalization maps Π^* and Π to the graphs Γ_π in a natural way as $\Pi^*, \Pi : \ell^2_\pi \to \ell^2_\pi$ as $\Pi^* f(\eta, i) = f(\sigma\eta, i)$ and $\Pi f(\eta, i) = \sum_{j \in X} f(j\eta, i)$.

These operators Π^* and Π are the renormalization maps that we will use in order to relate Δ_ξ with $\Delta_{\sigma\xi}$ and Δ_π with itself in order to find their spectra.

Lemma 4.2.1. *Let G be the quadratic polynomial $G(x) = x^2 - 2(d-2)x - 2(d-1)$. We have*

$$G(\Delta_\xi)\Pi^* = \Pi^*\Delta_{\sigma\xi}, \qquad \Pi G(\Delta_\xi) = \Delta_{\sigma\xi}\Pi,$$
$$G(\Delta_\pi)\Pi^* = \Pi^*\Delta_\pi, \qquad \Pi G(\Delta_\pi) = \Delta_\pi\Pi.$$

Proof. We only provide the proof for Δ_ξ, as the proof for Δ_π is analogous. Let $f \in \ell^2_{\sigma\xi}$, and let $p \in \Gamma_{\sigma\xi}$. Let q_j and r_j be its A and B-neighbors, respectively, for $j = 1 \dots d-1$. Let us write α, β_j and γ_j to denote the values of f at p, q_j and r_j, respectively, and let us set $\beta = \sum_{j=1}^{d-1} \beta_j$ and $\gamma = \sum_{j=1}^{d-1} \gamma_j$. We have, for any $i \in X$,

$$\Pi^* f(ip) = f(p) = \alpha,$$

$$\Delta_\xi \Pi^* f(ip) = \begin{cases} \beta + (d-1)\alpha, & i = 0 \\ 2(d-1)\alpha, & i \neq 0, d-1 \\ \gamma + (d-1)\alpha, & i = d-1 \end{cases}$$

and

$$\Delta_\xi^2 \Pi^* f(ip) = \begin{cases} (2d-3)\beta + (d-1)\alpha + \gamma + (d-1)\alpha + 2(d-2)(d-1)\alpha, & i = 0 \\ 2(d-1)^2\alpha + \beta + \gamma + 2(d-1)\alpha + 2(d-1)(d-3)\alpha, & i \neq 0, d-1 \\ (2d-3)\gamma + (d-1)\alpha + \beta + (d-1)\alpha + 2(d-2)(d-1)\alpha, & i = d-1 \end{cases} =$$

$$= \begin{cases} 2(d-1)^2\alpha + (2d-3)\beta + \gamma, & i = 0 \\ 2(d-1)(2d-3)\alpha + \beta + \gamma, & i \neq 0, d-1 \\ 2(d-1)^2\alpha + (2d-3)\gamma + \beta, & i = d-1 \end{cases}.$$

On the other hand, we have that $\Pi^*\Delta_{\sigma\xi} f(ip) = \Delta_{\sigma\xi} f(p) = \beta + \gamma$. Hence,

$$G(\Delta_\xi)\Pi^* f(ip) = (\Delta_\xi^2 - 2(d-2)\Delta_\xi - 2(d-1))\Pi^* f(ip) =$$

$$= \begin{cases} 2(d-1)^2\alpha + (2d-3)\beta + \gamma - 2(d-2)\beta - 2(d-2)(d-1)\alpha - 2(d-1)\alpha, & i = 0 \\ 2(d-1)(2d-3)\alpha + \beta + \gamma - 4(d-2)(d-1)\alpha - 2(d-1)\alpha, & i \neq 0, d-1 \\ 2(d-1)^2\alpha + (2d-3)\gamma + \beta - 2(d-2)\gamma - 2(d-2)(d-1)\alpha - 2(d-1)\alpha, & i = d-1 \end{cases} =$$

$$= \beta + \gamma = \Pi^*\Delta_{\sigma\xi} f(ip).$$

This shows the first relation while the second one is its dual, as Δ_ξ, $\Delta_{\sigma\xi}$ are self-adjoint operators. □

Remark 4.2.2. The map G arising in Lemma 4.2.1 is the same as the map G defined in Remark 4.1.11 relating the maps ψ and F from Theorem 4.1.10 for $m = 1$. For spinal groups with $m \geq 2$, there does not exist any quadratic map playing the role of G in Lemma 4.2.1, which limits this approach to spinal groups with $m = 1$.

We will now use the equalities in Lemma 4.2.1 to find a relation between the spectrum of $\Delta_{\sigma\xi}$ (Δ_π) with that of $\Delta_\xi|_H$ ($\Delta_\pi|_H$), the restriction of Δ_ξ (Δ_π) to a subspace H of ℓ^2_ξ (ℓ^2_π). In order to do that, we will need a general result from functional analysis. We only include its statement (Lemma 4.2.3), for its proof, see [60].

Lemma 4.2.3. *Let $\mathcal{H}$ be a Hilbert space and T a self-adjoint bounded operator of $\mathcal{H}$. Let $q \in \mathbb{R}[x]$ be a quadratic polynomial. Let $K \subset \mathcal{H}$ be a closed subspace such that $q(T)K \subset K$ and K and TK generate $\mathcal{H}$. Then $q(\mathrm{sp}(T)) = \mathrm{sp}(q(T)|_K)$. Moreover, if $T^{-1}K \cap K = 0$, then $\mathrm{sp}(T) = q^{-1}(\mathrm{sp}(q(T)|_K))$.*

Let $H = \langle \Pi^*\ell^2_{\sigma\xi}, \Delta_\xi\Pi^*\ell^2_{\sigma\xi}\rangle$ (alternatively $H = \langle \Pi^*\ell^2_\pi, \Delta_\pi\Pi^*\ell^2_\pi\rangle$) be the subspace of ℓ^2_ξ (ℓ^2_π) generated by the images of the operators Π^* and $\Delta_\xi\Pi^*$ ($\Delta_\pi\Pi^*$). Now we are ready to establish the relation between $\mathrm{sp}(\Delta_\xi|_H)$ and $\mathrm{sp}(\Delta_{\sigma\xi})$ ($\mathrm{sp}(\Delta_\pi|_H)$ and $\mathrm{sp}(\Delta_\pi)$).

Proposition 4.2.4. *For every $\xi \in X^{\mathbb{N}}$, H is invariant under Δ_ξ (Δ_π). Moreover,*

$$\mathrm{sp}(\Delta_\xi|_H) = G^{-1}(\mathrm{sp}(\Delta_{\sigma\xi})),$$

$$\mathrm{sp}(\Delta_\pi|_H) = G^{-1}(\mathrm{sp}(\Delta_\pi)).$$

Proof. We provide again the proof only for Δ_ξ, as for Δ_π it is analogous. For any $f \in \ell^2_{\sigma\xi}$, $\Delta_\xi\Pi^* f \in H$ by definition and, by Lemma 4.2.1,

$$\Delta_\xi^2\Pi^* f = \Pi^*\Delta_{\sigma\xi} f + 2(d-2)\Delta_\xi\Pi^* f + 2(d-1)\Pi^* f \in H.$$

Therefore H is invariant under Δ_ξ.

Let K be the image of Π^*. $\frac{1}{\sqrt{d}}\Pi^*$ is an isometry from $\ell^2_{\sigma\xi}$ to K, as $\Pi\Pi^* = d$, which implies that $\mathrm{sp}(G(\Delta_\xi)|_K) = \mathrm{sp}(\Delta_{\sigma\xi})$ by Lemma 4.2.1. Now set $L = \{f \in \ell^2_{\sigma\xi} \mid \Delta_\xi\Pi^* f \in K\}$ and let $f \in L$. Let $p \in \Gamma_{\sigma\xi}$, and let q_j, r_j be its A and B-neighbors, respectively, for $j = 1, \ldots, d-1$. Let $\alpha, \beta_j, \gamma_j$ denote the values of f at p, q_j and r_j, respectively, and set $\beta = \sum_{j=1}^{d-1}\beta_j$ and $\gamma = \sum_{j=1}^{d-1}\gamma_j$. Recall from the proof of Lemma 4.2.1 that, for every $i \in X$, we have

$$\Delta_\xi\Pi^* f(ip) = \begin{cases} \beta + (d-1)\alpha, & i = 0 \\ 2(d-1)\alpha, & i \neq 0, d-1 \\ \gamma + (d-1)\alpha, & i = d-1 \end{cases}.$$

Since $\Delta_\xi\Pi^* f \in K$, it must be constant over the vertices of Γ_ξ of the form ip, $i \in X$. This means that $\beta = \gamma = (d-1)\alpha$. Similarly, for every $j = 1, \ldots, d-1$, we deduce $(d-1)\beta_j = \beta - \beta_j + \alpha$ and $(d-1)\gamma_j = \gamma - \gamma_j + \alpha$. Equivalently, that $\beta_j = \gamma_j = \alpha$ for every $j = 1, \ldots, d-1$. Thus f is constant, and hence $f = 0$, so $L = 0$.

Now let $g \in \Delta_\xi^{-1}K \cap K$, so both $g, \Delta_\xi g \in K$. There exists $h \in \ell^2_{\sigma\xi}$ such that $g = \Pi^* h$, and so $\Delta_\xi \Pi^* h = \Delta_\xi g \in K$. Consequently, $h \in L$, but then $h = 0$ and therefore $g = 0$, so $\Delta_\xi^{-1}K \cap K = 0$. We may now use Lemma 4.2.3 to obtain

$$\mathrm{sp}(\Delta_\xi|_H) = G^{-1}(\mathrm{sp}(G(\Delta_\xi)|_K)) = G^{-1}(\mathrm{sp}(\Delta_{\sigma\xi})).$$

□

Proposition 4.2.4 establishes a relation between the part of the spectrum of Δ_ξ (Δ_π) which corresponds to the subspace H in ℓ^2_ξ (ℓ^2_π) to the whole spectrum on $\ell^2_{\sigma\xi}$ (ℓ^2_π), via the quadratic map G. We now want to find the part of the spectrum corresponding to the orthogonal complement of H, denoted $H^\perp$. Once we know both parts, we will be able to find an explicit description of the spectrum of Δ_ξ (Δ_π).

Lemma 4.2.5. $\mathrm{sp}(\Delta_\xi|_{H^\perp}) = \mathrm{sp}(\Delta_\pi|_{H^\perp}) = \{d-2, -2\}$.

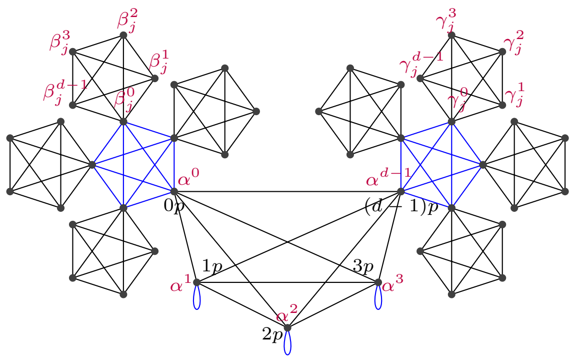


Figure 4.4: Values of the function $f \in H^\perp$ at the vertices ip, iq_j and ir_j of Γ_ξ, for $i \in X$ and $j = 1, \dots, d-1$.

Proof. Once again the proof for Δ_π is analogous so we do not include it. Let $f \in H^\perp$. Let $p \in \Gamma_{\sigma\xi}$, and let q_j, r_j be its A and B-neighbors, respectively, for $j = 1, \dots, d-1$. For each $i \in X$, set $\alpha^i = f(ip)$, $\beta_j^i = f(iq_j)$, $\gamma_j^i = f(ir_j)$, as in Figure 4.4. Because $f \in H^\perp$, we have

$$0 = \langle f, \Pi^* \delta_p \rangle = \langle \Pi f, \delta_p \rangle = \Pi f(p) = \sum_{i \in X} f(ip) = \sum_{i \in X} \alpha^i.$$

Likewise, we obtain, for every $j = 1, \dots, d-1$,

$$\sum_{i \in X} \beta_j^i = \sum_{i \in X} \gamma_j^i = 0.$$

As in the proof of Lemma 4.2.1, we will find $\Delta_\xi f(ip)$ and $\Delta_\xi^2 f(ip)$. In order to simplify notation, we will write $\beta = \sum_{j=1}^{d-1} \beta_j^0$ and $\gamma = \sum_{j=1}^{d-1} \gamma_j^{d-1}$. Using the equalities above, we have

$$\Delta_\xi f(ip) = \begin{cases} \beta - \alpha^0, & i = 0 \\ (d-2)\alpha^i, & i \neq 0, d-1 \\ \gamma - \alpha^{d-1}, & i = d-1 \end{cases}.$$

Again because $f \in H^\perp$, we obtain

$$0 = \langle f, \Delta_\xi \Pi^* \delta_p \rangle = \langle \Pi \Delta_\xi f, \delta_p \rangle = \Pi \Delta_\xi f(p) = \sum_{i \in X} \Delta_\xi f(ip) =$$

$$= \beta - \alpha^0 + (d-2) \sum_{i \in X, i \neq 0, d-1} \alpha^i + \gamma - \alpha^{d-1} =$$

$$= \beta + \gamma - (d-1)(\alpha^0 + \alpha^{d-1}).$$

Furthermore, again using the equalities we have established,

$$\Delta_\xi^2 f(ip) = \begin{cases} \sum_{j=1}^{d-1}(\alpha^0 + \beta - 2\beta_j^0) + (d-2)\sum_{k \in X, k \neq 0, d-1} \alpha^k + \gamma - \alpha^{d-1}, & i = 0 \\ (d-1)(d-2)\alpha^i + (d-2)\sum_{k \neq 0, i, d-1} \alpha^k + \beta - \alpha^0 + \gamma - \alpha^{d-1}, & i \neq 0, d-1 \\ \sum_{j=1}^{d-1}(\alpha^{d-1} + \gamma - 2\gamma_j^0) + (d-2)\sum_{k \in X, k \neq 0, d-1} \alpha^k + \beta - \alpha^0, & i = d-1 \end{cases} =$$

$$= \begin{cases} (d-1)\alpha^0 + (d-3)\beta - (d-2)(\alpha^0 + \alpha^{d-1}) + \gamma - \alpha^{d-1}, & i = 0 \\ (d-1)(d-2)\alpha^i + (d-2)\sum_{k \neq 0, i, d-1} \alpha^k + (d-2)(\alpha^0 + \alpha^{d-1}), & i \neq 0, d-1 \\ (d-1)\alpha^{d-1} + (d-3)\gamma - (d-2)(\alpha^0 + \alpha^{d-1}) + \beta - \alpha^0, & i = d-1 \end{cases} =$$

$$= \begin{cases} d\alpha^0 + (d-4)\beta, & i = 0 \\ (d-2)^2\alpha^i, & i \neq 0, d-1 \\ d\alpha^{d-1} + (d-4)\gamma, & i = d-1 \end{cases}.$$

This implies that in $H^\perp$ we have the equality $\Delta_\xi^2 - (d-4)\Delta_\xi - 2(d-2) = 0$. Indeed,

$$(\Delta_\xi^2 - (d-4)\Delta_\xi) f(ip) = \begin{cases} d\alpha^0 + (d-4)\alpha^0, & i = 0 \\ 2(d-2)\alpha^i, & i \neq 0, d-1 \\ d\alpha^{d-1} + (d-4)\alpha^{d-1}, & i = d-1 \end{cases} = 2(d-2)f(ip).$$

Equivalently, in $H^\perp$ we have $(\Delta_\xi - (d-2))(\Delta_\xi + 2) = 0$. This shows that $\mathrm{sp}(\Delta_\xi|_{H^\perp}) = \{d-2, -2\}$. □

For $x \in \mathrm{sp}(\Delta_\xi)$ ($\mathrm{sp}(\Delta_\pi)$), let us denote by E_x the eigenspace of Δ_ξ (Δ_π) associated with x in ℓ^2_ξ (ℓ^2_π). We will now find the eigenspaces associated with $d-2$ and -2. To that end, we need the following result.

Lemma 4.2.6. $2(d-1)$ *and* $-(d-1)^2 - 1$ *are not eigenvalues of* Δ_ξ *(*Δ_π*).*

Proof. The analogous proof for Δ_π is omitted. Let $f \in \ell^2_\xi$ be an eigenfunction of Δ of eigenvalue $2(d-1)$. Since $f \in \ell^2_\xi$, f must be bounded and reach its maximum value on some vertices. Let Σ be the set of vertices of Γ_ξ attaining that maximum. If $p \in \Sigma$ and q_j are its neighbors, $j = 1, \ldots, 2(d-1)$ because Γ_ξ is $2(d-1)$-regular, then $2(d-1)f(p) = \Delta f(p) = \sum_{j=1}^{2(d-1)} f(q_j)$. But $f(q_j) \leq f(p)$, and therefore, $f(q_j) = f(p)$ for all $j = 1, \ldots, 2(d-1)$, so $q_j \in \Sigma$ for every $j = 1, \ldots, 2(d-1)$. Since Γ_ξ is connected, f must be constant, and since $f \in \ell^2_\xi$, f must be zero.

For the second claim, we know that $\|\Delta\| = |S| = 2(d-1)$, and since $(d-1)^2 + 1 > 2(d-1)$ for every $d > 2$, $-(d-1)^2 - 1$ cannot be in $\mathrm{sp}(\Delta)$. □

Recall from Definition 3.2.3 that Γ^1_η is the subgraph $X\sigma(\eta)$ of Γ_ξ, for $\eta \in \mathrm{Cof}(\xi)$. We will call any such subgraph an A-piece. Similarly, Λ^n_η is the subgraph $(d-1)^n 0 X \sigma^{n+2}(\eta)$ of Γ_ξ, for $n \geq 0$ and $\eta \in \mathrm{Cof}(\xi)$. We will call such subgraph an n-piece, and, more generally, we will call any n-piece a B-piece. Finally, for convenience, we will sometimes call A-pieces (-1)-pieces. In Figures 4.3 and 4.4, B-pieces are drawn in blue, while A-pieces are drawn in black.

Notice that every vertex belongs to exactly one A-piece and every vertex not fixed by B belongs to exactly one B-piece. Moreover, every A-piece has d vertices, joined together only by A-edges. More precisely, for every $k = 1, \ldots, d-1$, any vertex $i\eta$ from an A-piece in Γ_ξ has an a^k-edge to $(i+k)\eta$ in the same A-piece. Similarly, any B-piece has also d vertices, joined together by only B-edges. For every $b \in B \setminus \{1\}$, any vertex $(d-1)^n 0 i\eta$ from a B-piece in Γ_ξ has a b-edge to $(d-1)^n 0(i+k)\eta$ in the same B-piece, if $\omega_n(b) = a^k$.

For the graphs Γ_π, we may extend these notions in a natural way. A-pieces (equivalently (-1)-pieces) are subgraphs of the form $(\Gamma^1_\eta, i) = (X\sigma\eta, i)$, for $\eta \in \mathrm{Cof}((d-1)^{\mathbb{N}})$ and $i \in X$. n-pieces are subgraphs of the form $(\Lambda^n_\eta, i) = ((d-1)^n 0 X \sigma^{n+2}(\eta), i)$, for $n \geq 0$, $\eta \in \mathrm{Cof}(\xi)$ and $i \in X$. We define the subgraph $((d-1)^{\mathbb{N}}, X)$ to be the only ∞-piece. Again, we call any n-piece a B-piece, for $n \geq 0$ or $n = \infty$.

Intuitively, A and B-pieces in Γ_π are exactly those within the d copies of $\Gamma_{(d-1)^{\mathbb{N}}}$, with the exception of the new ∞-piece which joins the d copies together.

Lemma 4.2.7. *The eigenspace E_{-2} is trivial, while E_{d-2} is the subspace of $H^\perp$ given by*

$$E_{d-2} = \{f \in \ell^2_\xi \mid \Pi f = 0, \quad f \text{ constant on } B\text{-pieces}\}.$$

Proof. Once more, we provide only the proof for Δ_ξ, as that for Δ_π is analogous. Let us first show that both E_{d-2} and E_{-2} are contained in $H^\perp$. Let $f \in E_{d-2}$. Then, by Lemma 4.2.1,

$$\Delta_{\sigma\xi} \Pi f = \Pi G(\Delta_\xi) f = G(d-2)\Pi f = (-(d-1)^2 - 1)\Pi f.$$

But Lemma 4.2.6 then implies $\Pi f = 0$. In addition, $\Pi\Delta_\xi f = (d-2)\Pi f = 0$. Now, for every $g \in \ell^2_{\sigma\xi}$, we have

$$\langle f, \Pi^* g\rangle = \langle \Pi f, g\rangle = 0,$$

and

$$\langle f, \Delta_\xi \Pi^* g\rangle = \langle \Pi\Delta_\xi f, g\rangle = 0.$$

Consequently, $f \in H^\perp$.

Let now $f \in E_{-2}$. Again by Lemma 4.2.1, we have

$$\Delta_{\sigma\xi}\Pi f = \Pi G(\Delta_\xi) f = G(-2)\Pi f = 2(d-1)\Pi f.$$

But because of Lemma 4.2.6, $\Pi f = 0$. Similarly, $\Pi\Delta_\xi f = -2\Pi f = 0$, and so analogously as before, $f \in H^\perp$. This shows that H does not contain any eigenfunction of eigenvalue $d-2$ nor -2.

Let us show now that $E_{d-2} = \{f \in \ell^2_\xi \mid \Pi f = 0, \quad f \text{ constant on } B\text{-pieces}\}$. First, if $f \in \ell^2_\xi$ is constant on B-pieces and satisfies $\Pi f = 0$, then, for any $ip \in \Gamma_\xi$,

$$\Delta_\xi f(ip) = \Pi f(p) - f(ip) + (d-1) f(ip) = (d-2) f(ip).$$

Conversely, if $f \in E_{d-2}$, we already know that $\Pi f = 0$. Let $ip_j \in \Gamma_\xi$ be the vertices of a B-piece, with $j \in X$, and let $\alpha_j = f(ip_j)$, for all $j \in X$. Then,

$$\Delta_\xi f(ip_j) = \Pi f(p_j) - \alpha_j + \sum_{k\neq j} \alpha_k = -\alpha_j + \sum_{k\neq j}\alpha_k.$$

Since $\Delta_\xi f = (d-2)f$, we obtain $(d-1)\alpha_j - \sum_{k\neq j}\alpha_k = 0$ for every $j \in X$. This corresponds to the following system of equations:

$$\begin{pmatrix} d-1 & -1 & \dots & -1 & -1 \\ -1 & d-1 & \dots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \dots & d-1 & -1 \\ -1 & -1 & \dots & -1 & d-1 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{d-2} \\ \alpha_{d-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix},$$

all of whose solutions are constant vectors. Hence, f is constant on B-pieces.

Finally, let us show $E_{-2} = 0$. Let $f \in E_{-2}$. Again, we know that $\Pi f = 0$. For every $ip \in \Gamma_\xi$ fixed by B, ip has $d-1$ loops labeled by the generators in $B \setminus \{1\}$. Therefore,

$$\Delta_\xi f(ip) = \Pi f(p) - f(ip) + (d-1) f(ip) = (d-2) f(ip).$$

Since $\Delta_\xi f = -2f$, we obtain $df(ip) = 0$, so f must vanish at every vertex fixed by B. In addition, if ip_j are the vertices of a B-piece, for $j \in X$, then, for every $j \in X$,

$$\Delta_\xi f(ip_j) = \Pi f(p_j) - f(ip_j) + \sum_{k\neq j} f(ip_k) = -f(ip_j) + \sum_{k\neq j} f(ip_k).$$

But $\Delta_\xi f = -2f$, so $\sum_{j\in X} f(ip_j) = 0$, or equivalently that f has zero sum on all B-pieces.

For the remaining part of the proof, we will consider the subgraphs $\Gamma_\eta^n = X^n\sigma^n(\eta)$ of Γ_ξ, for $n \geq 1$, $\eta \in \mathrm{Cof}(\xi)$ (see Definition 3.2.3). Notice that such subgraphs can only be connected to $\Gamma_\xi \setminus \Gamma_\eta^n$ by two vertices: $(d-1)^{n-1}0\sigma^n(\eta)$ and $(d-1)^n\sigma^n(\eta)$. We shall call them the extremes of Γ_η^n. We claim that for every subgraph Γ_η^n of Γ_ξ, f is antisymmetric at its extremes, i.e., that $f((d-1)^{n-1}0\sigma^n(\eta)) = -f((d-1)^n\sigma^n(\eta))$. Indeed, we proceed by induction on n.

If $n = 1$, the subgraphs Γ_η^1 for $\eta \in \mathrm{Cof}(\xi)$ are precisely A-pieces. Since $\Pi f = 0$, the sum of the values of f at its d vertices is 0. However, f vanishes on vertices whose first digit is different from $0, d-1$, as they are fixed by B, and is hence antisymmetric on the remaining two vertices, which are the extremes of Γ_η^1.

Assume the claim to be true for $n \geq 1$, and consider Γ_η^{n+1}, for $\eta \in \mathrm{Cof}(\xi)$. We decompose Γ_η^{n+1} as d subgraphs $\Gamma_{\zeta^{(i)}}^n$, $i \in X$, where $\zeta_k^{(i)} = \eta_k$ for all $k \in \mathbb{N} \setminus \{n\}$ and $\zeta_n^{(i)} = i$. Each of these subgraphs has extremes $p_i = (d-1)^n i\sigma^{n+1}(\eta)$ and $q_i = (d-1)^{n-1}0i\sigma^{n+1}(\eta)$, $i \in X$, and the q_i form a B-piece. Our goal is to show that $f(p_0) = -f(p_{d-1})$.

Notice that the vertices p_i, $i \neq 0, d-1$, are fixed by B, and so $f(p_i) = 0$, $i \neq 0, d-1$. By induction hypothesis we have $f(p_i) = -f(q_i)$ for every $i \in X$, so in particular $f(q_i) = 0$, for $i \neq 0, d-1$. Finally, the q_i form a B-piece, so $\sum_{i\in X} f(q_i) = 0$, which implies $f(q_0) = -f(q_{d-1})$. Now we conclude

$$f(p_{d-1}) = -f(q_{d-1}) = f(q_0) = -f(p_0).$$

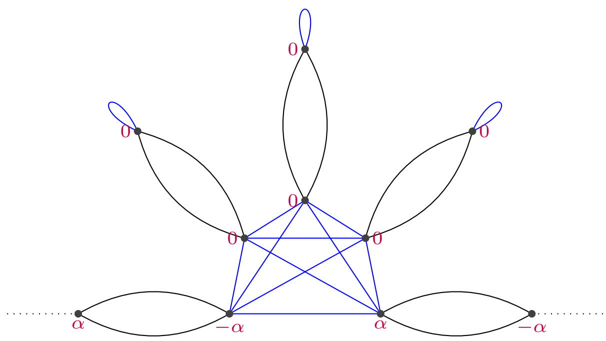


Figure 4.5: Any (-2)-eigenfunction f is antisymmetric on the extremes of the subgraph Γ_ξ^n, for all $n \geq 1$ and $\eta \in \mathrm{Cof}(\xi)$. Therefore it must be zero.

Hence if $f(p) = \alpha \neq 0$ for some $p \in \Gamma_\xi$, we can find an infinite path on Γ_ξ on which f has alternating values $\alpha, -\alpha$, which is a contradiction with the fact that f is square summable. Then $f = 0$ and so $E_{-2} = 0$. □

Notice that for $d = 2$ the eigenspace E_{d-2} is also trivial. We may now conclude with the explicit computation of the spectrum of the adjacency operator Δ_ξ (Δ_π) on the Schreier graphs Γ_ξ (Γ_π).

Theorem 4.2.8. *Let G_ω be a spinal group with $d \geq 3$, $m = 1$ and $\omega \in \Omega_{d,1}$. Let Δ be the adjacency operator on any Schreier graph Γ in the space of graphs $\mathcal{G}_{G_\omega, X^{\mathbb{N}}}$, i.e., either Γ_ξ for $\xi \in X^{\mathbb{N}}$ or Γ_π for $\pi \in \mathrm{Epi}(B, A)$ repeating infinitely often in ω. Then,*

$$\mathrm{sp}(\Delta) = \Lambda \cup \bigcup_{n \geq 0} G^{-n}(d-2), \tag{4.3}$$

where $G(x) = x^2 - 2(d-2)x - 2(d-1)$ and Λ is its Julia set, which is a Cantor set of zero Lebesgue measure. In particular, $\mathrm{sp}(\Delta)$ does not depend on ξ nor π.

Proof. We only provide the proof for Δ_ξ, since for Δ_π it is analogous. By induction on $n \geq 0$, and in parallel for all the graphs Γ_ξ with $\xi \in X^{\mathbb{N}}$, let us show that $G^{-n}(d-2) \subset \mathrm{sp}(\Delta_\xi)$. The case $n = 0$ is a consequence of Lemma 4.2.5. Suppose that $\bigcup_{k=0}^{n} G^{-k}(d-2) \subset \mathrm{sp}(\Delta_{\sigma\xi})$ for some $n \geq 1$. Then, by Proposition 4.2.4, $\bigcup_{k=0}^{n+1} G^{-k}(d-2) \subset \mathrm{sp}(\Delta_\xi)$. Hence $\bigcup_{n \geq 0} G^{-n}(d-2) \subset \mathrm{sp}(\Delta_\xi)$. Since $\mathrm{sp}(\Delta_\xi)$ is closed, and Λ is the adherence of this set, $\Lambda \subset \mathrm{sp}(\Delta_\xi)$ too.

Now let $x \in \mathrm{sp}(\Delta_\xi)$ such that $x \notin \bigcup_{n \geq 0} G^{-n}(d-2)$. In that case, Proposition 4.2.4 and Lemma 4.2.5 imply that $G^n(x) \in \mathrm{sp}(\Delta_{\sigma^n(\xi)})$ for every $n \geq 0$. Therefore the sequence $G^n(x)$ is bounded, which means that $x \in \Lambda$. □

Remark 4.2.9. The map G is the same map from Remark 4.1.11, so it satisfies $G(x) = \psi(x) + d - 2 = F(x - (d-2)) + d - 2$, where ψ and F are the maps from Theorem 4.1.10 with $m = 1$.

As with Corollary 4.1.12, the Cantor spectrum in Theorem 4.2.8 allows us to conclude an immediate consequence about spectral measures.

Corollary 4.2.10. *Let G_ω be a spinal group with $d \geq 3$, $m = 1$ and $\omega \in \Omega_{d,1}$. Let Δ be the adjacency operator on any Schreier graph Γ in the space of graphs $\overline{\mathrm{Sch}(X^{\mathbb{N}})}$, i.e., either Γ_ξ for $\xi \in X^{\mathbb{N}}$ or Γ_π for $\pi \in \mathrm{Epi}(B, A)$ repeating infinitely often in ω. Then any spectral measure of Δ has trivial absolutely continuous part.*

Proof. The support of any spectral measure of Δ is contained in $\mathrm{sp}(\Delta)$. Since the Lebesgue measure of $\mathrm{sp}(\Delta)$ is zero, its absolutely continuous part must be trivial. □

4.3 Spectra on Cayley graphs

We are now interested in finding the spectrum of the adjacency operator on Cayley graphs of spinal groups. Note that for the rest of the chapter we will consider the Markov operator, i.e., the adjacency operator normalized to have operator norm ≤ 1. In particular, the spectra of the adjacency operator and the Markov operator on a regular graph differ only by a factor equal to the degree of the graph. If G is a group generated by a finite set S, we denote the Markov operator on its Cayley graph $\mathrm{Cay}(G, S)$ with respect to S by M_G. For any $H \leq G$, we denote the Markov operator on the Schreier graph $\mathrm{Sch}(G, H, S)$ by M_H.

For a countable group G, the left-regular representation λ_G of G is the unitary representation on the Hilbert space $\ell^2(G)$ given by

$$\lambda_G(g)f(x) = f(g^{-1}x).$$

Similarly, if $H \leq G$, the quasi-regular representation $\lambda_{G/H}$ of G on $\ell^2(G/H)$ is given by

$$\lambda_{G/H}(g)f(xH) = f(g^{-1}xH).$$

Both representations can be extended to the group algebra $\mathbb{C}[G]$ by bounded operators, setting, for every $t = \sum_{g\in G} c_g g \in \mathbb{C}[G]$, $\lambda_G(t) = \sum_{g\in G} c_g \lambda_G(g)$ and $\lambda_{G/H}(t) = \sum_{g\in G} c_g \lambda_{G/H}(g)$. If we set $u = \frac{1}{|S|}\sum_{s\in S} s \in \mathbb{C}[G]$, the representations $\lambda_G(u)$ and $\lambda_{G/H}(u)$ are the same operators as M_G and M_H.

There is a notion of weak containment of unitary representations of a group. We will use the characterization shown in [22], which says that a unitary representation ρ of G is *weakly contained* in another unitary representation η of G (denoted $\rho \prec \eta$) if and only if $\mathrm{sp}(\rho(t)) \subset \mathrm{sp}(\eta(t))$ for every $t \in \mathbb{C}[G]$. Amenability and weak containment of unitary representations are strongly related by Hulanicki's Theorem, which we now recall.

Theorem 4.3.1 (Hulanicki's Theorem)**.** *Let G be a locally compact group, and let λ_G be its left-regular representation. G is amenable if and only if λ_G weakly contains any unitary representation of G.*

Let G_ω be a spinal group with $d = 2$, $m \geq 1$ and $\omega \in \Omega_{2,m}$ and spinal generating set S. We write $M_\xi = M_{\mathrm{Stab}_{G_\omega}(\xi)} = \frac{1}{|S|}\Delta_\xi$. Since all spinal groups with $d = 2$ are amenable [49], Hulanicki's Theorem implies that $\mathrm{sp}(M_{G_\omega}) \supset \mathrm{sp}(M_\xi)$, for every $\xi \in X^{\mathbb{N}}$. Recall that, as a consequence of Theorem 4.1.10, $\mathrm{sp}(M_\xi)$ does not depend on ξ. We now want to prove the other containment again using Hulanicki's Theorem. This implies that there are uncountably many isospectral non quasi-isometric spinal groups, as was shown in [24]. In fact, the proof we provide is a variation of their proof for the smaller family $\mathcal{S}_{2,2}$ of spinal groups, which uses a version of Hulanicki's Theorem for graphs. For us it is enough to use the classical version of Hulanicki's Theorem stated above.

Theorem 4.3.2. *Let G_ω be a spinal group with $d = 2$, $m \geq 1$ and $\omega \in \Omega_{2,m}$, and let M_ξ be the Markov operator with respect to the spinal generating set S on the Schreier graph Γ_ξ. Then, for every $\xi \in X^{\mathbb{N}}$, we have*

$$\mathrm{sp}(M_{G_\omega}) = \mathrm{sp}(M_\xi) = \left[-\frac{1}{2^{m-1}}, 0\right] \cup \left[1 - \frac{1}{2^{m-1}}, 1\right]. \tag{4.4}$$

Proof. Theorem 4.1.10 implies the second equality, for any $\xi \in X^{\mathbb{N}}$. Since G_ω is amenable, we know by Hulanicki's Theorem that $\lambda_{G_\omega/H} \prec \lambda_{G_\omega}$, where $H = \mathrm{Stab}_{G_\omega}(\xi)$, for $\xi \in X^{\mathbb{N}}$. Considering $u = \frac{1}{|S|}\sum_{s\in S} s \in \mathbb{C}[G_\omega]$, this implies that $\mathrm{sp}(\lambda_{G_\omega/H}(u)) \subset \mathrm{sp}(\lambda_{G_\omega}(u))$, or, equivalently, $\mathrm{sp}(M_\xi) \subset \mathrm{sp}(M_{G_\omega})$, for any $\xi \in X^{\mathbb{N}}$. To prove the other inclusion, consider the element $t \in \mathbb{C}[G_\omega]$ defined as follows:

$$t = \frac{1}{2^{m-1}} \sum_{b\in B} b - 1.$$

Observe that $t^2 = 1$. Indeed,

$$\begin{aligned} t^2 &= \left(\frac{1}{2^{m-1}} \sum_{b\in B} b - 1\right)^2 = \\ &= \frac{1}{4^{m-1}} \sum_{b\in B}\sum_{b'\in B} bb' + 1 - \frac{1}{2^{m-2}} \sum_{b\in B} b = \\ &= \frac{1}{4^{m-1}} \sum_{b\in B}\sum_{c\in B} c + 1 - \frac{1}{2^{m-2}} \sum_{b\in B} b = \\ &= \frac{1}{2^{m-2}} \sum_{c\in B} c + 1 - \frac{1}{2^{m-2}} \sum_{b\in B} b = 1. \end{aligned}$$

It follows that the subgroup $D = \langle a, t\rangle$ of the group algebra $\mathbb{C}[G_\omega]$ is a dihedral group (in fact infinite), as $a^2 = t^2 = 1$.

Let $\rho = \lambda_{G_\omega}|_D$ be the restriction of the regular representation to $D \subset \mathbb{C}[G_\omega]$. Since both a and t are involutions, $\rho(a)$ and $\rho(t)$ are unitary operators, and hence ρ is a unitary representation. By Hulanicki's Theorem, provided that D is amenable, we have that $\rho \prec \lambda_D$, where λ_D is the regular representation of D on $\ell^2(D)$. This implies that $\mathrm{sp}(\rho(w)) \subset \mathrm{sp}(\lambda_D(w))$ for every $w \in \mathbb{C}[G_\omega]$. Notice that $u = \frac{a}{2^m} + \frac{t}{2} + \frac{2^{m-1}-1}{2^m} \in \mathbb{C}[D]$, so we have $\mathrm{sp}(M_{G_\omega}) = \mathrm{sp}(\lambda_{G_\omega}(u)) = \mathrm{sp}(\rho(u)) \subset \mathrm{sp}(\lambda_D(u))$.

We only have to compute the latter, which is not hard to do as it corresponds to the spectrum of the Markov operator associated with a random walk on $\mathbb{Z}$ with 2-periodic probabilities. In particular, the probability of staying at a vertex is $\frac{2^{m-1}-1}{2^m}$, and the probabilities of moving to a neighbor are 2-periodic of values $\frac{1}{2}$ and $\frac{1}{2^m}$.

To find the spectrum of a 2-periodic graph we can use the elements of Floquet-Bloch theory (see for instance [8]). As the graph is 2-periodic, a fundamental domain parametrized by $k \in [-\pi, \pi]$ is given by any two consecutive vertices. Using the 2-periodicity of the graph we build a 2×2 matrix for each k with the transition probabilities, find its eigenvalues and we take the closure of their union for all k. The computations are shown below.

$$0 = \begin{vmatrix} \frac{2^{m-1}-1}{2^m} - x & \frac{1}{2^m} + \frac{1}{2}e^{-ik} \\ \frac{1}{2^m} + \frac{1}{2}e^{ik} & \frac{2^{m-1}-1}{2^m} - x \end{vmatrix} = \left(\frac{2^{m-1}-1}{2^m} - x\right)^2 - \left(\frac{1}{4^m} + \frac{1}{4} + \frac{1}{2^m}\cos(k)\right).$$

$$x = \frac{2^{m-1}-1}{2^m} \pm \frac{1}{2^m}\sqrt{4^{m-1} + 1 + 2^m\cos(k)}.$$

$$\mathrm{sp}(\lambda_D(u)) = \bigcup_{k \in [-\pi,\pi]} \mathrm{sp}\left(\lambda_D(u)_k\right) =$$

$$= \frac{2^{m-1}-1}{2^m} \pm \frac{1}{2^m}[2^{m-1}-1, 2^{m-1}+1] = \left[-\frac{1}{2^{m-1}}, 0\right] \cup \left[1 - \frac{1}{2^{m-1}}, 1\right].$$

□

Remark 4.3.3. As $m \to \infty$, $\mathrm{sp}(M_{G_\omega})$ shrink from two intervals to two points.

We can therefore conclude in Corollary 4.3.4 that, for spinal groups, as for other classes of groups [23, 24], the spectrum on the Cayley graph does not determine the group.

Corollary 4.3.4 (See [24]). *There are uncountably many pairwise non quasi-isometric isospectral spinal groups.*

Proof. Theorem 4.3.2 shows that $\mathrm{sp}(M_{G_\omega})$ for spinal groups with $d = 2$ depends only on m. For instance, for $m = 2$, we obtain the family of groups $\mathcal{S}_{2,2}$ defined by Grigorchuk in [32]. This family contains uncountably many groups with different growth function, which is a quasi-isometric invariant. Hence, there are uncountably many isospectral spinal groups which are pairwise non quasi-isometric. □

4.4 Dependence on the generating set

All the results discussed so far in this chapter concerned spinal groups with the spinal generating set $S = (A \cup B) \setminus \{1\}$. One might also wonder what are the spectra like if we consider different generating sets, for instance minimal ones. Some progress has been achieved, for example in [38] the authors characterize whether the spectra of the weighted Markov operator on the Schreier graphs of Grigorchuk's group are either a union of intervals or a Cantor set, depending on the weights on its usual generators.

For spinal groups acting on the binary tree ($d = 2$), the infinite Schreier graphs Γ_ξ have linear shape. The Schreier graphs of a minimal generating set can then be obtained by erasing double edges in the Schreier graph Γ_ξ corresponding to the spinal generators S. This can be translated into considering a Markov operator on Γ_ξ with non-uniform distribution of probabilities on S. The spectra of such anisotropic Markov operators were studied in [39]. Their analysis implies the next result. As in Section 4.3, M_ξ denotes the Markov operator on the Schreier graph Γ_ξ with respect to the spinal generating set S, for $\xi \in X^{\mathbb{N}}$, and M_{G_ω} denotes the Markov operator on the Cayley graph also with respect to the spinal generating set S. For other generating sets T, we will explicitly write Γ_ξ^T, M_ξ^T and $M_{G_\omega}^T$.

Proposition 4.4.1. *Let G_ω be a spinal group with $d = 2$, $m \geq 2$ and $\omega \in \Omega_{d,m}$, and let M_ξ^T be the Markov operator on the graph Γ_ξ^T, with generating set $T \subset S$. For $\pi \in \mathrm{Epi}(B, A)$, we define $q_\pi = |T \cap B \setminus \mathrm{Ker}(\pi)|$. Two cases may occur:*

- *If the numbers q_π are all equal over $\pi \in \mathrm{Epi}(B, A)$ appearing infinitely often in ω, then $\mathrm{sp}(M_\xi^T)$ is a union of intervals.*
- *Otherwise, $\mathrm{sp}(M_\xi^T)$ is a Cantor set of zero Lebesgue measure.*

Proof. First notice that $a \in T$, or else T would generate a finite group. We observe that we can relabel the vertices in Γ_ξ^T by $\mathbb{Z}$ in such a way that the number of edges between them is the following:

- There is one a-edge between any vertex $v \in 2\mathbb{Z}$ and $v + 1$.
- There are $|T \cap B \setminus \mathrm{Ker}(\omega_0)| = q_{\omega_0}$ edges, between any vertex $v \in 4\mathbb{Z} + 1$ and $v + 1$, and $|T \cap \mathrm{Ker}(\omega_0)|$ loops on each of v, $v + 1$.
- There are $|T \cap B \setminus \mathrm{Ker}(\omega_1)| = q_{\omega_1}$ edges between any vertex $v \in 8\mathbb{Z} + 3$ and $v + 1$, and $|T \cap \mathrm{Ker}(\omega_1)|$ loops on each of v, $v + 1$.

  ...

- In general, for every $i \geq 0$, there are $|T \cap B \setminus \mathrm{Ker}(\omega_i)|$ edges between any vertex $v \in 2^{i+2}\mathbb{Z} + 2^{i+1} - 1$ and $v + 1$, and $|T \cap \mathrm{Ker}(\omega_i)|$ loops on each of v, $v + 1$.

Hence, the simple random walk on Γ_ξ^T is given by a weighted random walk on $\mathbb{Z}$, defined by the following probabilities:

- Probability of $\frac{1}{|T|}$ of transitioning between any vertex $v \in 2\mathbb{Z}$ and $v + 1$.
- For every $i \geq 0$, probability of $\frac{q_{\omega_i}}{|T|}$ of transitioning between any vertex $v \in 2^{i+2}\mathbb{Z} + 2^{i+1} - 1$ and $v + 1$.
- For every $i \geq 0$, probability of $1 - \frac{q_{\omega_i}}{|T|} - \frac{1}{|T|}$ of staying at any vertex $v \in 2^{i+2}\mathbb{Z} + 2^{i+1} - 1$ or $v + 1$.

These probabilities follow a periodic pattern if and only if the numbers q_π are all equal for every $\pi \in \mathrm{Epi}(B, A)$ occurring infinitely often in ω. If this is not the case, we may use Corollary 7.2 in [39] to obtain that $\mathrm{sp}(M_\xi^T)$ is a Cantor set of Lebesgue measure zero.

Suppose now that q_π are all equal for every $\pi \in \mathrm{Epi}(B, A)$ occurring infinitely often in ω, so the probabilities are periodic on $\mathbb{Z}$, let l be that period. We may compute $\mathrm{sp}(M_\xi^T)$ using Floquet-Bloch theory. To do so, we first compute the spectrum of a fundamental domain, parametrized by $k \in [-\pi, \pi]$, with some boundary conditions. This gives a set of eigenvalues $\{x_1(k), \ldots, x_l(k)\}$, which are the roots of a polynomial of degree l. These polynomials only depend on $\cos(k)$. Now $\mathrm{sp}(M_\xi^T)$ is just the union of these sets of eigenvalues for all $k \in [-\pi, \pi]$. Since the roots will vary continuously as $\cos(k) \in [-1, 1]$, this union will be a union of at least one and at most l intervals. □

This result provides, for any generating set $T \subset S$, a characterization of the type of spectrum on the Schreier graphs in terms of the numbers q_π. We already know from Theorem 4.1.10 that the second option in Proposition 4.4.1 is realized when $T = S$, the spinal generating set. The following result states that, except one degenerate example corresponding to $G_\omega = D_\infty$ ($d = 2$, $m = 1$), every spinal group on the binary tree has a generating set which gives a Cantor spectrum.

Corollary 4.4.2. *For every spinal group G_ω with $d = 2$, $m \geq 2$ and $\omega \in \Omega_{d,m}$ there exists a minimal generating set $T \subset S$ for which $\mathrm{sp}(M_\xi^T)$ is a Cantor set of Lebesgue measure zero.*

Proof. Let $\pi, \pi' \in \mathrm{Epi}(B, A)$ be two different epimorphisms occurring infinitely often in ω. Recall that B is a vector space over $\mathbb{Z}/2\mathbb{Z}$, and let $K = \mathrm{Ker}(\pi)$ and $K' = \mathrm{Ker}(\pi')$. We know that $[B : K] = [B : K'] = 2$, and $[K : K \cap K'] = 2$ because π' surjects K onto A with kernel $K \cap K'$, since $K \neq K'$. Hence, we have $[B : K \cap K'] = 4$. In particular, we can choose $m - 2$ elements $x_1, \ldots, x_{m-2} \in K \cap K'$ which generate $K \cap K'$. Moreover, we can choose elements $y \in K \setminus K'$ and $y' \in K' \setminus K$ to complete the generating set to one of K or K', respectively, and such that $\{x_1, \ldots, x_{m-2}, y, y'\}$ generate B.

If we now define $T = \{a, x_1, x_2, \ldots, x_{m-2}, y, yy'\} \subset S$, it is clear that it is a minimal generating set for G_ω, since $|T| = m+1$. Moreover, we have $q_\pi = |T \cap B \setminus K| = |\{yy'\}| = 1$ and $q_{\pi'} = |T \cap B \setminus K'| = |\{y, yy'\}| = 2$. By Proposition 4.4.1, $\mathrm{sp}(M_\xi^T)$ is a Cantor set of Lebesgue measure zero. □

We can also find a condition on the generating set $T \subset S$ under which the spectrum on the Schreier graphs is one interval, for certain spinal groups G_ω.

Proposition 4.4.3. *Let G_ω be a spinal group with $d = 2$, $m \geq 2$ and $\omega \in \Omega_{2,m}$, with generating set $T \subset S$. If $q_{\omega_i} = 1$ for every $i \geq 0$, then $\mathrm{sp}(M_\xi^T)$ is the interval $[1 - \frac{4}{|T|}, 1]$.*

Proof. In the proof of Proposition 4.4.1, we established that the simple random walk on Γ_ξ^T is given by a weighted random walk on $\mathbb{Z}$. Since $q_{\omega_i} = 1$ for every $i \geq 0$, the probabilities are reduced to:

- Probability $\frac{1}{|T|}$ of transitioning between any vertex $v \in \mathbb{Z}$ and $v+1$.
- Probability $1 - \frac{2}{|T|}$ of staying at any vertex $v \in \mathbb{Z}$.

The fact that the probabilities are periodic allows us to use Floquet-Bloch theory in order to find $\mathrm{sp}(M_\xi^T)$, and since the period is 1, the computation is rather simple. The only eigenvalue of a fundamental domain, parametrized by $k \in [-\pi, \pi]$, is the solution of the equation

$$0 = 1 - \frac{2}{|T|} + \frac{1}{|T|}e^{ik} + \frac{1}{|T|}e^{-ik} - x = 1 - \frac{2}{|T|} + \frac{2}{|T|}\cos(k) - x \Longrightarrow x = 1 - \frac{2}{|T|} + \frac{2}{|T|}\cos(k).$$

Finally,

$$\mathrm{sp}(M_\xi^T) = \bigcup_{k\in[-\pi,\pi]} \mathrm{sp}(M_\xi^T(k)) = \bigcup_{k\in[-\pi,\pi]} \left\{1 - \frac{2}{|T|} + \frac{2}{|T|}\cos(k)\right\} = \left[1 - \frac{4}{|T|}, 1\right].$$

□

Recall that self-similar groups within the family of spinal groups were studied by Šunić [65]. For every d and m, there are finitely many of them, and they can be specified in terms of an epimorphism $\alpha \in \mathrm{Epi}(B, A)$ and an automorphism $\rho \in \mathrm{Aut}(B)$. The groups in Šunić's family are then the spinal groups defined by the periodic sequence $\omega = (\omega_n)_n$ given by $\omega_n = \alpha \circ \rho^n$. Moreover, it was shown that any of these groups admits a natural minimal Šunić generating set $T = \{a, b_1, \ldots, b_m\}$, contained in the spinal generating set S, such that

$$a = (1,1)\sigma \quad b_1 = (1, b_2) \quad b_2 = (1, b_3) \quad \ldots \quad b_{m-1} = (1, b_m) \quad b_m = (a, b'),$$

for some $b' \in B$. Notice that, for $i = 1, \ldots, m-1$, $\alpha(b_i) = 1$ and $\rho(b_i) = b_{i+1}$, while $\alpha(b_m) = a$ and $\rho(b_m) = b'$. The choice of this $b' \in B$ in such a way that ρ is an automorphism determines the group. It was also shown in [65] that a Šunić group is infinite torsion if and only if all ρ-orbits intersect $\mathrm{Ker}(\alpha)$.

Example 4.4.4. Grigorchuk's group is the group G in Šunić's family with $d = 2$, $m = 2$, $A = \{1, a\}$, $B = \{1, b_1, b_2, b_1b_2\}$ and $\rho(b_2) = b_1b_2$. With the standard notation b, c, d for the generators, we have $b_1 = d$, $b_2 = b$ and $b_1b_2 = c$. The only nontrivial ρ-orbit is $b_1 \mapsto b_2 \mapsto b_1b_2 \mapsto b_1$, which intersects $\mathrm{Ker}(\alpha)$ at b_1, hence the group is infinite torsion. The minimal Šunić generating set is $T = \{a, b_1, b_2\}$ and the spinal generating set is $S = \{a, b_1, b_2, b_1b_2\}$, with

$$a = \tau(1,1) \quad b_1 = (1, b_2) \quad b_2 = (a, b_1b_2) \quad b_1b_2 = (a, b_1),$$

where τ is the nontrivial element of $\mathrm{Sym}(X)$. By Theorem 4.3.2, we have

$$\mathrm{sp}(M_\xi) = \mathrm{sp}(M_G) = \left[-\frac{1}{2}, 0\right] \cup \left[\frac{1}{2}, 1\right].$$

We may consider any of the minimal generating sets $T_{b_1} = \{a, b_2, b_1 b_2\}$, $T_{b_2} = \{a, b_1, b_1 b_2\}$ or $T = \{a, b_1, b_2\}$. In that case, thanks to Proposition 4.4.1, any of $\mathrm{sp}(M_\xi^{T_{b_1}})$, $\mathrm{sp}(M_\xi^{T_{b_2}})$ and $\mathrm{sp}(M_\xi^T)$ is a Cantor set, for any $\xi \in X^{\mathbb{N}}$. It would be interesting to know, for any of these minimal generating sets, what is the spectrum on the Cayley graph. So far, we only know it must contain this Cantor set.

Example 4.4.5. One natural choice in the construction of Šunić groups above is to take $d = 2$ and ρ such that $b' = b_1$. This gives one group for each $m \geq 2$, which we call G_m. We have

$$a = \tau(1,1) \quad b_1 = (1, b_2) \quad b_2 = (1, b_3) \quad \ldots \quad b_{m-1} = (1, b_m) \quad b_m = (a, b_1),$$

again for τ being the nontrivial element of $\mathrm{Sym}(X)$. The element $ab_1 \ldots b_m$ is of infinite order. We consider two generating sets: the spinal generating set $S = (A \cup B) \setminus \{1\}$, of size 2^m, and the Šunić minimal generating set $T = \{a, b_1, \ldots, b_m\}$, of size $m+1$. On the one hand, Theorem 4.3.2 yields that, for any $\xi \in X^{\mathbb{N}}$,

$$\mathrm{sp}(M_\xi) = \mathrm{sp}(M_{G_m}) = \left[-\frac{1}{2^{m-1}}, 0\right] \cup \left[1 - \frac{1}{2^{m-1}}, 1\right].$$

On the other hand,

$$\mathrm{sp}(M_\xi^T) = \left[\frac{m-3}{m+1}, 1\right].$$

Indeed, for any two-ended Γ_ξ^T, the simple random walk translates into the weighted random walk on $\mathbb{Z}$ with probability $\frac{1}{m+1}$ of moving to a neighbor and probability $\frac{m-1}{m+1}$ of staying on any vertex. By taking a one-vertex fundamental domain parametrized by $k \in [-\pi, \pi]$ and using Floquet-Bloch theory, we have:

$$0 = \frac{m-1}{m+1} + \frac{1}{m+1}(e^{ik} + e^{-ik}) - x \Longrightarrow x = \frac{1}{m+1}(m - 1 + 2\cos(k))$$

$$\mathrm{sp}\left(M_\xi^T\right) = \bigcup_{k \in [-\pi,\pi]} \mathrm{sp}\left(M_\xi^T(k)\right) = \bigcup_{k \in [-\pi,\pi]} \left\{\frac{1}{m+1}(m - 1 + 2\cos(k))\right\} = \left[\frac{m-3}{m+1}, 1\right].$$

Proposition 4.4.6. *Let G be a Šunić group with $d = 2$ and $m \geq 2$, with minimal Šunić generating set T. Then the spectrum on the Schreier graph with respect to T is $\left[\frac{m-3}{m+1}, 1\right]$ if $G = G_m$ or a Cantor set of zero Lebesgue measure otherwise.*

Proof. We found above the spectrum on the Schreier graphs for the groups G_m. Suppose now that $\mathrm{sp}(M_\xi^T)$ is a union of intervals. By Proposition 4.4.1, we have that the numbers q_π are all equal over $\pi \in \mathrm{Epi}(B, A)$ occurring infinitely often in ω. By definition of the minimal Šunić generating set T, we know that $q_{\omega_0} = m - 1$, so, as ω is periodic, $q_{\omega_n} = m - 1$ for every $n \geq 0$.

Now, for any $k = 0, \dots, m-1$, we know that $\omega_k(b_{m-k}) = \omega_0 \rho^k(b_{m-k}) = \omega_0(b_m) = a$, so $b_{m-k} \notin \mathrm{Ker}(\omega_k)$. As $q_{\omega_k} = m-1$, the only possibility is that, for every $j = 0, \dots, m-1$, $b_j \in \mathrm{Ker}(\omega_k)$ if and only if $j \neq k$. In particular, this implies that $b_m = (a, b_1)$, so that we are in fact in the case of the group G_m. □

We also have

Lemma 4.4.7. *The Cayley graph of G_m with generating set T is bipartite, for any $m \geq 2$.*

Proof. We only have to show that all relations in the group G_m have even length. Let w be a freely reduced word on T, and let $|w|$ represent its length and $|w|_t$ the number of times the generator $t \in T$ occurs in w.

Suppose that w represents the identity element of G_m. In that case, $|w|_a$ must be even, or otherwise its action on the first level would be nontrivial. This allows us to write the word w as a product of b_i and b_i^a. Let w_0 and w_1 be the two projections of the word w into the first level, before reduction. Let us look at the decomposition of a generator $t \in T$. If $t = a$, then it decomposes as 1 on both subtrees. If $t = b_i$, then it decomposes as b_{i+1} on the right and as 1 on the left, or as a if $i = m$. Notice that the decomposition of b_i^a is that of b_i exchanging the two projections.

It is clear that both w_0 and w_1 represent the identity, too. Hence, $|w_0|_a$ and $|w_1|_a$ must both be even as well. But $|w_0|_a + |w_1|_a = |w|_{b_m}$, so $|w|_{b_m}$ must also be even.

By iterating this argument we can conclude that $|w|$ must be even. In general, let w_u be the projection of w onto the vertex u in X^k, the k-th level of the tree, for $1 \geq k \geq m$. For any $u \in X^k$, w_u must represent the identity, and hence $|w_u|_a$ must be even. But tracing back the a's occurring in w_u we obtain

$$\sum_{u \in L_k} |w_u|_a = \sum_{u \in L_{k-1}} |w_u|_{b_m} = \cdots = \sum_{u \in L_1} |w_u|_{b_{m-k+2}} = |w|_{b_{m-k+1}}.$$

This shows that $|w|_t$ is even, for every $t \in T$, which implies that $|w|$ is indeed even and hence that the Cayley graph with the generating set T is bipartite. □

The spectrum of a bipartite graph is symmetric about 0. At the same time, for amenable groups, the spectrum on any Schreier graph is contained in the spectrum on the Cayley graph. Hence we have,

$$\mathrm{sp}(M_{G_m}^T) \supset -\mathrm{sp}(M_\xi^T) \cup \mathrm{sp}(M_\xi^T) = \left[-1, \frac{3-m}{m+1}\right] \cup \left[\frac{m-3}{m+1}, 1\right].$$

Two cases are of special interest: $m = 2$ and $m = 3$. In these cases, the union of the two intervals above is the whole interval $[-1, 1]$ and we can thus conclude that the spectrum of the Cayley graphs of G_2 and G_3 with respect to the minimal Šunić generating set is the whole interval $[-1, 1]$. For $m \geq 4$, the union of intervals is actually disjoint, so we can only conclude that the spectrum of the Cayley graph contains two intervals $[-1, -\beta]$ and $[\beta, 1]$, with $\beta > 0$.

Note that the group G_2 was studied in [27] and is therefore sometimes called the Grigorchuk-Erschler group. It is the only self-similar group in the Grigorchuk family (spinal groups with $d = 2$ and $m = 2$) besides Grigorchuk's group. The group G_3 is known as Grigorchuk's overgroup [4] because it contains Grigorchuk's group as a subgroup. Indeed, the automorphisms b_2b_3, b_1b_3, b_1b_2 are the generators b, c, d of Grigorchuk's group.

Corollary 4.4.8. *For the Grigorchuk-Erschler group G_2 and Grigorchuk's overgroup G_3 the spectrum of the Cayley graph is a union of two disjoint intervals with respect to the spinal generating set and the interval $[-1, 1]$ with respect to the minimal Šunić generating set.*

The groups G_m with respect to the minimal Šunić generating set T satisfy the first condition in Proposition 4.4.1. In particular, for every $n \geq 0$, the numbers q_{ω_n} are all equal to $m - 1$, hence the spectrum on the Schreier graphs is a union of intervals, as we computed above. Nevertheless, they are the only Šunić groups to satisfy this condition for T, as shown in Proposition 4.4.6.

Example 4.4.9. Another non-torsion example in Šunić's family of self-similar groups is the group G given by $d = 2$, $m = 3$ and ρ such that $\rho(b_3) = b_1b_2b_3$. We have

$$a = (1, 1)\sigma \quad b_1 = (1, b_2) \quad b_2 = (1, b_3) \quad b_3 = (a, b_1b_2b_3).$$

Indeed, the element b_1b_3 decomposes as (a, b_1b_3), so it constitutes a ρ-orbit which does not intersect $\mathrm{Ker}(\alpha)$. Hence, the element ab_1b_3 is of infinite order. This group contains the subgroup $\langle a, b_2b_3, b_1b_2, b_1b_3\rangle$, which is isomorphic to the Grigorchuk-Erschler group. For the spinal generating set S, we have

$$\mathrm{sp}(M_\xi) = \mathrm{sp}(M_G) = \left[-\frac{1}{4}, 0\right] \cup \left[\frac{3}{4}, 1\right].$$

Its minimal Šunić generating set is $\{a, b_1, b_2, b_3\}$. As G is not G_m, by Proposition 4.4.6 we can conclude that $\mathrm{sp}(M_\xi^T)$ is a Cantor set of zero Lebesgue measure. More precisely, the sequence ω is 4-periodic and we have $q_{\omega_0} = q_{\omega_3} = 2$ while $q_{\omega_1} = q_{\omega_2} = 1$.

Chapter 5

Spectral measures

This chapter extends the study of the spectral properties of adjacency operators on Schreier graphs of spinal groups by computing their spectral measures. The spectrum of the adjacency operator Δ_ξ on the Schreier graph Γ_ξ associated with a point $\xi \in X^{\mathbb{N}}$ is a subset of $\mathbb{R}$. Spectral measures are measures with support in the spectrum, and provide more information than just the spectrum as a set. For instance, a function in $\ell^2_\xi = \ell^2(\Gamma_\xi)$ projects trivially onto a given eigenspace of eigenvalue x if and only if its associated spectral measure vanishes for the set $\{x\}$.

The chapter is divided in four sections. We first compute the so-called empirical spectral measure, a measure which only depends on the finite Schreier graphs Γ_n and represents an averaging of all spectral measures. Then we exhibit three different types of spectral measures that may occur for infinite Schreier graphs of spinal groups. For spinal groups with $d = 2$, we show that all spectral measures on the graphs Γ_ξ are absolutely continuous with respect to the Lebesgue measure and give their density explicitly. For spinal groups with $d \geq 3$, we construct a complete basis of eigenfunctions for $\ell^2_\xi = \ell^2(\Gamma_\xi)$ for all ξ in a subset of $X^{\mathbb{N}}$ of uniform Bernoulli measure one, which implies that all spectral measures on the graphs Γ_ξ of these groups are discrete. Finally, we consider the Schreier graphs Γ_π that occur as accumulation points of graphs $\{(\Gamma_\xi, \xi)\}_\xi$ in the space of Schreier graphs $\mathcal{G}_{G_\omega, X^{\mathbb{N}}}$ (see Theorem 3.7.3). We decompose the space $\ell^2_\pi = \ell^2(\Gamma_\pi)$ as the direct sum of two nontrivial subspaces, whose functions have associated spectral measures which are singular continuous and discrete, respectively, thus providing new examples of Schreier graphs with nontrivial singular continuous component in the spectral measures.

5.1 Empirical spectral measure

In the proof of Theorem 4.1.7 we actually found the explicit multiplicities of the eigenvalues of Δ_n. These multiplicities allow us to compute the so-called empirical spectral measure (or density of states) of the graphs $\{\Gamma_n\}_n$.

Definition 5.1.1. Let $\{Y_n\}_n$ be a sequence of finite graphs, and, for every $n \geq 0$ let Δ_n be the adjacency operator on Y_n and ν_n be the counting measure on $\mathrm{sp}(\Delta_n)$. Namely,

$$\nu_n = \frac{1}{|Y_n|} \sum_{x \in \mathrm{sp}(\Delta_n)} \delta_x,$$

where δ_x takes value 1 at x and vanishes anywhere else, and the eigenvalues are counted with multiplicity. Following [46], if the measures ν_n weakly converge to a measure ν, we call ν the *empirical spectral measure* or *density of states* of $\{Y_n\}_n$.

Theorem 5.1.2. *Let G_ω be a spinal group with $d \geq 2$, $m \geq 1$ and $\omega \in \Omega_{d,m}$. Let ν be the empirical spectral measure of $\{\Gamma_n\}_n$. If $d = 2$, ν is absolutely continuous with respect to the Lebesgue measure. Its density is given by the function*

$$g(x) = \frac{|x - 2^{m-1} + 1|}{\pi \sqrt{x(x+2)(2^m - x)(x - 2^m + 2)}}. \tag{5.1}$$

If $d \geq 3$, then ν is discrete. More precisely,

$$\nu = \frac{d-2}{d} \delta_{|S|-d} + \sum_{n \geq 0} \frac{d-2}{d^{n+2}} \sum_{x \in \psi^{-1}(F^{-n}(0))} \delta_x.$$

Proof. Let ν_n be the counting measure on the spectrum of Δ_n, i.e.

$$\nu_n = \frac{1}{d^n} \sum_{x \in \mathrm{sp}(\Delta_n)} \delta_x.$$

From the multiplicities computed in the proof of Theorem 4.1.7, we have that $\nu_0 = \delta_{|S|}$, $\nu_2 = \frac{1}{d}(\delta_{|S|} + (d-1)\delta_{|S|-d})$ and, for $n \geq 2$,

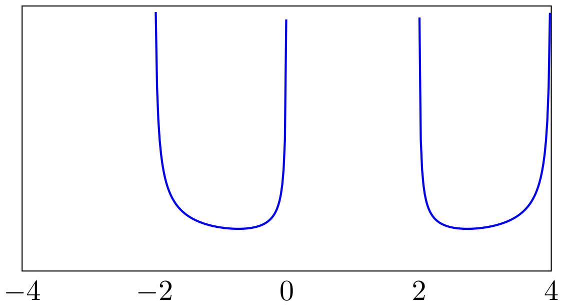
$$\nu_n = \frac{1}{d^n} \left(\delta_{|S|} + \left((d-2)d^{n-1} + 1\right) \delta_{|S|-d} + \sum_{k=0}^{n-2} \left((d-2)d^{n-k-2} + 1\right) \sum_{x \in \psi^{-1}(F^{-k}(0))} \delta_x \right).$$

For $d \geq 3$, we observe, taking the limit as $n \to \infty$, the measure

$$\nu = \frac{d-2}{d} \delta_{|S|-d} + \sum_{n \geq 0} \frac{d-2}{d^{n+2}} \sum_{x \in \psi^{-1}(F^{-n}(0))} \delta_x.$$

as in the statement.

For $d = 2$, all the multiplicities of the eigenvalues in the finite graphs are 1, or equivalently, every eigenvalue of Δ_n has the same mass, $\frac{1}{d^n}$. When taking the limit, the measure of each atom tends to zero and the set of eigenvalues becomes dense in either one ($m = 1$) or two ($m \geq 2$) intervals. Therefore, any set of positive empirical spectral measure is the union



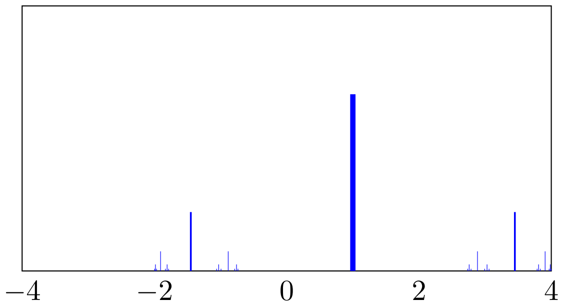


Figure 5.1: Density functions of the empirical spectral measure ν for Grigorchuk's group ($d = 2, m = 2$, above) and the Fabrykowski-Gupta group ($d = 3, m = 1$, below).

of cones of the set of preimages of $F(x) = x^2 - d(d-1)$ plus their closure, which has positive Lebesgue measure. Hence, ν is absolutely continuous with respect to the Lebesgue measure. We can find its precise density if we notice the following, for $d = 2$ and $n \geq 1$:

$$\mathrm{sp}(\Delta_n) = \left\{ \frac{2^m - 2}{2} + \epsilon\sqrt{4^{m-1} + 1 + 2^m \cos\theta} \;\middle|\; \epsilon \in \{\pm 1\}, \theta \in \frac{2\pi\mathbb{Z}}{2^n} \right\} \setminus \{0, -2\}.$$

Indeed, from the proof of Theorem 4.1.7 we recover the two branches of the inverse of ψ:

$$\psi_\epsilon^{-1}(x) = \frac{2^m - 2}{2} + \epsilon\sqrt{4^{m-1} + 1 + 2^{m-1}x}.$$

Any $x \in F^{-k}(0)$ can be written as $x = \pm\sqrt{2 + y}$, with $y \in F^{-(k-1)}(0)$. We can hence complete the proof of the equality above by induction, using the trigonometric identity $2\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{2 + 2\cos(\theta)}$. This allows us to find an injective, measure-preserving map $\chi : [0, \pi] \times \{0, 1\} \to \mathbb{R}$, defined by

$$\chi(\theta, \epsilon) = \frac{2^m - 2}{2} + \epsilon\sqrt{4^{m-1} + 1 + 2^m \cos\theta},$$

with the spectrum uniformly distributed with respect to the Lebesgue measure λ on $[0, \pi] \times \{0, 1\}$. The empirical spectral measure of any subset $E \subset \mathbb{R}$ is then given by $\nu(E) =$

$\lambda(\chi^{-1}(E))$. The density $g(x)$ of ν is thus given by

$$g(x) = \frac{1}{2\pi} \frac{d}{dx} \chi^{-1}(x),$$

which coincides with the expression in the statement. □

5.2 Absolutely continuous spectral measures

This section is devoted to the computation of the spectral measures of the adjacency operator Δ_ξ on Schreier graphs Γ_ξ of spinal groups with $d = 2$, for $\xi \in X^{\mathbb{N}}$. Our goal is to prove that the Kesten spectral measure μ_ξ coincides with the empirical spectral measure ν, for every $\xi \in X^{\mathbb{N}} \setminus \mathrm{Cof}(1^{\mathbb{N}})$, whose density is given in Theorem 5.1.2 and displayed in Figure 5.1. Additionally, we compute the density of the Kesten spectral measure $\mu_{1^{\mathbb{N}}}$.

Theorem 5.2.1. *Let G_ω be a spinal group with $d = 2$, $m \geq 1$ and $\omega \in \Omega_{2,m}$. Let Δ_ξ be the adjacency operator on the Schreier graph Γ_ξ, for $\xi \in X^{\mathbb{N}}$. The Kesten spectral measure μ_ξ is absolutely continuous with respect to the Lebesgue measure. If $\xi \in X^{\mathbb{N}} \setminus \mathrm{Cof}(1^{\mathbb{N}})$, then μ_ξ coincides with the empirical spectral measure ν (see Theorem 5.1.2).*

Proof. First recall that for any $\xi \in X^{\mathbb{N}} \setminus \mathrm{Cof}(1^{\mathbb{N}})$ the graph Γ_ξ is a two-ended line, and does not depend on ξ up to the edge labels (see Proposition 3.5.1). More precisely, every vertex has $2^{m-1} - 1$ loops, $2^m - 2^{m-1}$ edges to one neighbor and one edge to the other neighbor. The simple random walk on such graphs is described by the Markov chain on $\mathbb{Z}$ with probability $\frac{1}{2} - \frac{1}{2^m}$ of staying at any vertex, and alternating probabilities $\frac{1}{2}$ and $\frac{1}{2^m}$ on the other edges. This implies that the Kesten spectral measures μ_ξ do not depend on this point ξ, except for ξ in the orbit of $1^{\mathbb{N}}$.

The empirical spectral measure ν is the integral of the Kesten measures μ_ξ over all $X^{\mathbb{N}}$ (see Theorem 10.8 in [35]), but we just showed that they are all equal in a subset of $X^{\mathbb{N}}$ of measure one. Hence, we necessarily have $\mu_\xi = \nu$ for every ξ in that subset. □

Let us now compute the explicit density of the spectral measure $\mu_{1^{\mathbb{N}}}$ to illustrate the difference with the typical case.

Theorem 5.2.2. *Let G_ω be a spinal group with $d = 2$, $m \geq 1$ and $\omega \in \Omega_{2,m}$. Let Δ_ξ be the adjacency operator on the Schreier graph Γ_ξ, for $\xi \in X^{\mathbb{N}}$. The Kesten spectral measure $\mu_{1^{\mathbb{N}}}$ has density*

$$h(x) = \frac{|x(x+2)|}{2^m \pi \sqrt{x(x+2)(2^m - x)(x - 2^m + 2)}}. \tag{5.2}$$

Proof. We compute the density $h(x)$ of $\mu_{1^{\mathbb{N}}}$ with an approach similar to that in [36]. It uses the fact that the Stieltjes transform of the density of a spectral measure of the Markov operator on a graph coincides with its moment-generating function.

We know that the Schreier graph $\Gamma_{1^{\mathbb{N}}}$ is one-ended, with the following adjacency matrix:

$$T = \begin{pmatrix} 2^m - 1 & 1 & & & \\ 1 & 2^{m-1} - 1 & 2^{m-1} & & \\ & 2^{m-1} & 2^{m-1} - 1 & 1 & \\ & & 1 & 2^{m-1} - 1 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}.$$

Let $\Gamma^0_{1^{\mathbb{N}}}$ be the graph $\Gamma_{1^{\mathbb{N}}}$ after removing $2^{m-1} - 1$ loops at every vertex. Its adjacency matrix is then

$$T^0 = \begin{pmatrix} 2^{m-1} & 1 & & & \\ 1 & 0 & 2^{m-1} & & \\ & 2^{m-1} & 0 & 1 & \\ & & 1 & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix},$$

and it represents the matrix of the operator $\Delta_{1^{\mathbb{N}}} - (2^{m-1} - 1)$. By Theorem 4.1.10, its spectrum is $I \cup J$, with $I = [-2^{m-1} - 1, -2^{m-1} + 1]$ and $J = [2^{m-1} - 1, 2^{m-1} + 1]$.

Notice that the simple random walk on $\Gamma^0_{1^{\mathbb{N}}}$ is given by the Markov chain on $\mathbb{Z}$ with 2-periodic probabilities $\frac{1}{2^{m-1}+1}$ and $\frac{2^{m-1}}{2^{m-1}+1}$. Let $\varphi(t) = \sum_{n \geq 0} P^n_{1^{\mathbb{N}}, 1^{\mathbb{N}}} t^n$ be the Green function of the random walk, with the coefficients being the probabilities of return to $1^{\mathbb{N}}$ after n steps. We can define the moment-generating function of T^0

$$M(z) = \frac{1}{z} \sum_{n \geq 0} \frac{m_n}{z^n}$$

where m_n is top left coefficient of the matrix $(T^0)^n$. Our goal is to prove that this moment-generating function $M(z)$ is the Stieltjes transform of the following density

$$\tilde{h}(x) = \frac{1}{2^m \pi} \sqrt{\frac{(x + 2^{m-1} - 1)(x + 2^{m-1} + 1)}{(x - 2^{m-1} + 1)(2^{m-1} + 1 - x)}}.$$

This will show that the density of the spectral measure for the operator $\Delta_{1^{\mathbb{N}}} - (2^{m-1} - 1)$ is $\tilde{h}$, which will conclude the proof, provided that $h(x) = \tilde{h}(x - (2^{m-1} - 1))$, and so h will be the density of the spectral measure $\mu_{1^{\mathbb{N}}}$ for the operator $\Delta_{1^{\mathbb{N}}}$.

There is a correspondence between the moment-generating functions of Jacobi matrices and continued fractions (see [1], for instance). In the case of T^0, we have

$$M(z) = \cfrac{1}{z - 2^{m-1} - \cfrac{1}{z - \cfrac{4^{m-1}}{z - \cfrac{1}{z - \ddots}}}}.$$

If we write

$$\rho = \cfrac{1}{z - \cfrac{4^{m-1}}{z - \cfrac{1}{z - \ddots}}},$$

then the relation $\rho = \cfrac{1}{z - \cfrac{4^{m-1}}{z - \rho}}$ is satisfied. Therefore ρ is a root of the quadratic equation

$$z\rho^2 + (4^{m-1} - 1 - z^2)\rho + z = 0.$$

With the appropriate choice of the branch of the square root, we find

$$\rho = \frac{z^2 + 1 - 4^{m-1} - \sqrt{(z - 2^{m-1} - 1)(z - 2^{m-1} + 1)(z + 2^{m-1} - 1)(z + 2^{m-1} + 1)}}{2z},$$

and so

$$M(z) = \frac{1}{z - 2^{m-1} - \rho} =$$

$$= \frac{2z}{z^2 - 2^m z + 4^{m-1} - 1 + \sqrt{(z - 2^{m-1} - 1)(z - 2^{m-1} + 1)(z + 2^{m-1} - 1)(z + 2^{m-1} + 1)}} =$$

$$= -\frac{1}{2^m}\left(1 - \sqrt{\frac{(z + 2^{m-1} - 1)(z + 2^{m-1} + 1)}{(z - 2^{m-1} + 1)(z - 2^{m-1} + 1)}}\right).$$

With the expression of the moment-generating function $M(z)$, we just need to find the Stieltjes transform for the density $\tilde{h}(x)$.

The Stieltjes transform of a measure μ on $\mathbb{R}$ is defined as

$$S_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(x)}{z - x}.$$

So the Stieltjes transform of the measure with density $\tilde{h}(x)$ is

$$S(z) = \frac{1}{2^m \pi} \int_{\mathbb{R}} \frac{1}{z - x} \sqrt{\frac{(x + 2^{m-1} - 1)(x + 2^{m-1} + 1)}{(x - 2^{m-1} + 1)(2^{m-1} + 1 - x)}} dx =$$

$$= \frac{1}{2^m\pi}\int_{I\cup J}\frac{1}{z-x}\sqrt{\frac{(x+2^{m-1}-1)(x+2^{m-1}+1)}{(x-2^{m-1}+1)(2^{m-1}+1-x)}}dx.$$

In order to solve this integral, let $z\in\mathbb{C}\setminus(I\cup J)$, and let γ_I and γ_J be closed curves in $\mathbb{C}$ around I and J respectively so that z lies in their exterior. Let $\gamma=\gamma_I\cup\gamma_J$. Consider now

$$k(w)=\frac{1}{z-w}\sqrt{\frac{(w+2^{m-1}-1)(w+2^{m-1}+1)}{(w-2^{m-1}+1)(w-2^{m-1}-1)}},$$

and let us first solve the integral $I(z)=\int_\gamma k(w)dw$.

Let C_ε be the circle of center z and radius ε. Choose ε small enough so that it does not intersect γ. Finally, let C_R be a circle big enough so that both C_ε and γ lie entirely on the inside. We have $I(z)=\int_{C_R}k(w)dw-\int_{C_\varepsilon}k(w)dw$. We will find an expression for $I(z)$ by computing the residues of k at $w=\infty$ and $w=z$. On the one hand,

$$\int_{C_R}k(w)dw=-\int_{C_{1/R}}k(1/u)\frac{du}{u^2}=$$

$$=-\int_{C_{1/R}}\frac{1}{z-1/u}\sqrt{\frac{(1/u+2^{m-1}-1)(1/u+2^{m-1}+1)}{(1/u-2^{m-1}+1)(1/u-2^{m-1}-1)}}\frac{du}{u^2}=$$

$$=-\int_{C_{1/R}}\frac{1}{u(uz-1)}\sqrt{\frac{(1+(2^{m-1}-1)u)(1+(2^{m-1}+1)u)}{(1-(2^{m-1}-1)u)(1-(2^{m-1}+1)u)}}du=$$

$$=\lim_{u\to 0}\frac{-2\pi i}{uz-1}\sqrt{\frac{(1+(2^{m-1}-1)u)(1+(2^{m-1}+1)u)}{(1-(2^{m-1}-1)u)(1-(2^{m-1}+1)u)}}=2\pi i.$$

On the other hand,

$$\int_{C_\varepsilon}k(w)dw=\lim_{w\to z}2\pi i\sqrt{\frac{(w+2^{m-1}-1)(w+2^{m-1}+1)}{(w-2^{m-1}+1)(w-2^{m-1}-1)}}=$$

$$=2\pi i\sqrt{\frac{(z+2^{m-1}-1)(z+2^{m-1}+1)}{(z-2^{m-1}+1)(z-2^{m-1}-1)}}.$$

Therefore,

$$I(z)=2\pi i\left(1-\sqrt{\frac{(z+2^{m-1}-1)(z+2^{m-1}+1)}{(z-2^{m-1}+1)(z-2^{m-1}-1)}}\right)=-2^{m+1}\pi i M(z).$$

Taking the limit as ε tends to zero, we have

$$\lim_{\varepsilon\to 0}I(z)=2\int_{I\cup J}\frac{1}{z-x}\sqrt{\frac{(x+2^{m-1}-1)(x+2^{m-1}+1)}{(x-2^{m-1}+1)(x-2^{m-1}-1)}}dx=$$

$$=-2i\int_{I\cup J}\frac{1}{z-x}\sqrt{\frac{(x+2^{m-1}-1)(x+2^{m-1}+1)}{(x-2^{m-1}+1)(2^{m-1}+1-x)}}dx=-2^{m+1}\pi i S(z).$$

Finally, putting together both equalities, we have $S(z)=M(z)$. □

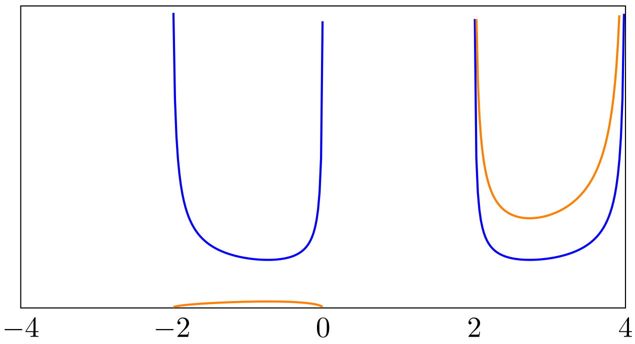


Figure 5.2: Densities of the Kesten spectral measures μ_ξ for Grigorchuk's group ($d = 2$ and $m = 2$). In blue, the symmetric density corresponds to any point $\xi \in X^{\mathbb{N}} \setminus \mathrm{Cof}(1^{\mathbb{N}})$, whose Schreier graphs are two-ended lines. In orange, the asymmetric density corresponds to the point $1^{\mathbb{N}}$, whose Schreier graph is a one-ended line.

Recall that there are uncountably many spinal groups with $d = 2$ and a given m, which are isospectral in virtue of Theorem 4.1.10. Moreover, for those groups, Theorem 5.2.1 concludes that, for a subset of boundary points of measure one, the Kesten spectral measures μ_ξ on the Schreier graphs Γ_ξ coincide. It would be very interesting to determine the Kesten spectral measures of the adjacency operator on the Cayley graphs of these groups.

5.3 Discrete spectral measures

In this section, we consider the spectral measures of the adjacency operator Δ_ξ on the Schreier graphs Γ_ξ of spinal groups with $d \geq 3$. Our goal is to prove that all of them are discrete, for every ξ in an explicit subset of $X^{\mathbb{N}}$ of uniform Bernoulli measure one. We prove this by explicitly constructing a complete basis of eigenfunctions of ℓ^2_ξ, all of which have finite support. To do so, we first find the eigenfunctions of the adjacency operators Δ_n on the finite Schreier graphs Γ_n, for $n \geq 0$, in Proposition 5.3.3 and Corollary 5.3.4. Next, we translate them to the infinite graphs Γ_ξ, in Theorem 5.3.5. Finally, we conclude by showing in Theorem 5.3.9 that these eigenfunctions form a complete system for all ξ in an explicit subset of $X^{\mathbb{N}}$ of uniform Bernoulli measure one. This implies in Corollary 5.3.10 that the spectrum of Δ_ξ is pure point in that case.

5.3.1 Eigenfunctions of Δ_n

Let us start by computing the eigenfunctions of Δ_n, for $n \geq 0$. Since $\mathrm{sp}(\Delta_n) \subset \mathrm{sp}(\Delta_{n+1})$ for every $n \geq 0$, for every eigenvalue x of Δ_n there exists $N \geq 1$ such that $x \in \mathrm{sp}(\Delta_N) \setminus \mathrm{sp}(\Delta_{N-1})$. We will first construct a basis of the x-eigenspace of Δ_N, and use it to construct a basis of the x-eigenspace of Δ_n, for every $n \geq N$. Let us write $\ell^2_n = \ell^2(\Gamma_n)$ and $\ell^2_\xi = \ell^2(\Gamma_\xi)$.

We define a notion of antisymmetry on the graphs Γ_n, which we will exploit to find the eigenfunctions. Let $\tau_i = (i, i+1) \in \mathrm{Sym}(X)$, for $i \in \{0, \ldots, d-2\}$, and let $\Phi^i_n : \Gamma_n \to \Gamma_n$ be the automorphisms of Γ_n defined by

$$\Phi^i_n(v_0 \ldots v_{n-1}) = v_0 \ldots v_{n-2}\tau_i(v_{n-1}),$$

for every $v = v_0 \ldots v_{n-1} \in X^n$. Recall that the graph Γ_n can be decomposed as d copies of Γ_{n-1} each of which is connected to the others only through one vertex. Intuitively, Φ^i_n exchanges the i-th and $(i+1)$-th copies of Γ_{n-1} in this decomposition. We will say that $f \in \ell^2_n$ is *antisymmetric* with respect to Φ^i_n if $f = -f \circ \Phi^i_n$. In particular, this implies that f is supported only on the i-th and $(i+1)$-th copies of Γ_{n-1} in Γ_n.

Proposition 5.3.1. *Let $N \geq 1$ and $x \in \mathrm{sp}(\Delta_N) \setminus \mathrm{sp}(\Delta_{N-1})$. There is a basis $\mathcal{F}_{x,N} = \{f_0, \ldots, f_{d-2}\}$ of the x-eigenspace of Δ_N, such that f_i is antisymmetric with respect to Φ^i_n, for every $i \in \{0, \ldots, d-2\}$. In particular, each f_i is supported in $X^{N-1}i \sqcup X^{N-1}(i+1)$.*

Proof. We know that the multiplicity of x in $\mathrm{sp}(M_N)$ is exactly $d-1$, as we computed in the proof of Theorem 4.1.10. Due to the symmetry of Γ_N, given a x-eigenfuction $f \in \ell^2_n$, we know that $f_i := f - f \circ \Phi^i_n$ is antisymmetric with respect to Φ^i_n and is still an x-eigenfunction, for any $i \in \{0, \ldots, d-2\}$. Furthermore, the fact that these functions are linearly independent becomes clear upon examination of their supports. □

Now, using the notation from Proposition 5.3.1, we partition the basis $\mathcal{F}_{x,N}$ into the following three parts:

$$\mathcal{F}^A_{x,N} := \{f_{d-2}\}, \qquad \mathcal{F}^B_{x,N} := \{f_0\}, \qquad \mathcal{F}^C_{x,N} := \mathcal{F}_{x,N} \setminus \{f_0, f_{d-2}\}.$$

We would like to translate these eigenfunctions from Γ_N to Γ_n, for any $n \geq N$. Recall that the graph Γ_{n+1} consists of d copies of Γ_n joined together by a central piece. We will take advantage of this decomposition. Let $n \geq 1$ and $i \in X$. We define the following linear operators (see Figure 5.4):

$$\rho^i_n : \ell^2_n \to \ell^2_{n+1}, \qquad \rho^i_n f(vj) = f(v)\delta_{i,j},$$

where $\delta_{i,j}$ is 1 if $i = j$ or 0 otherwise. We set $\rho_n = \sum_{i \in X} \rho^i_n$.

Now, in order to get the eigenfunctions of Δ_{n+1} from those of Δ_n, we apply these transition functions ρ^i_n in the following way, for any $n \geq N$,

$$\mathcal{F}^A_{x,n+1} := \rho^{d-1}_n(\mathcal{F}^A_{x,n}), \qquad \mathcal{F}^B_{x,n+1} := \rho^0_n(\mathcal{F}^A_{x,n}),$$

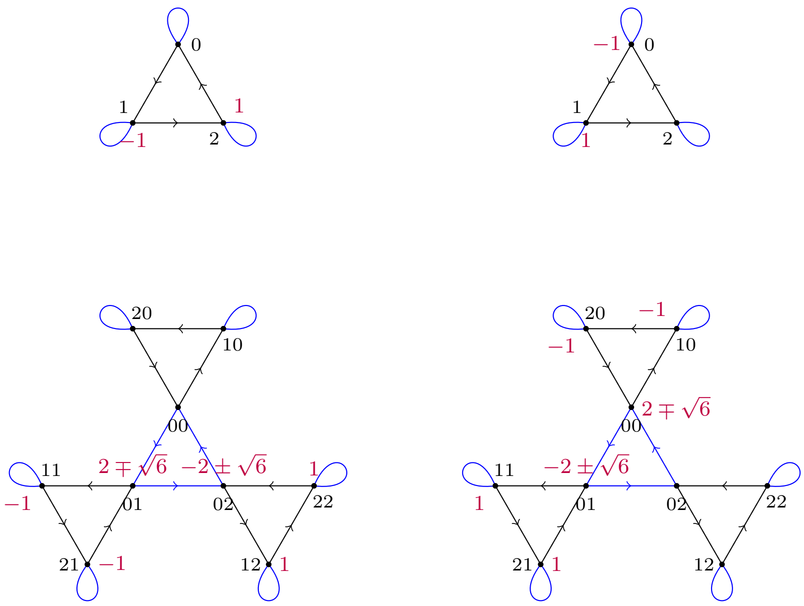


Figure 5.3: Eigenfunctions of eigenvalues $x = 1$ and $x = 1 \pm \sqrt{6}$ on the graphs Γ_1 and Γ_2 for the Fabrykowski-Gupta group ($d = 3$, $m = 1$).

$$\mathcal{F}^C_{x,n+1} := \bigsqcup_{i \neq 0, d-1} \rho_n^i(\mathcal{F}^A_{x,n}) \sqcup \rho_n(\mathcal{F}^B_{x,n}) \sqcup \bigsqcup_{i \in X} \rho_n^i(\mathcal{F}^C_{x,n}).$$

Finally, we set $\mathcal{F}_{x,n} := \mathcal{F}^A_{x,n} \sqcup \mathcal{F}^B_{x,n} \sqcup \mathcal{F}^C_{x,n}$.

Remark 5.3.2. One can look at the supports of the functions in $\mathcal{F}^A_{x,n}$, $\mathcal{F}^B_{x,n}$ and $\mathcal{F}^C_{x,n}$ to verify that these three sets are disjoint for every $n \geq N$. Moreover, the following statements can be inductively proven, for every $n \geq N$:

$$|\mathcal{F}^A_{x,n}| = |\mathcal{F}^B_{x,n}| = 1,$$

$$|\mathcal{F}^C_{x,n}| = (d-2)d^{n-N} - 1,$$

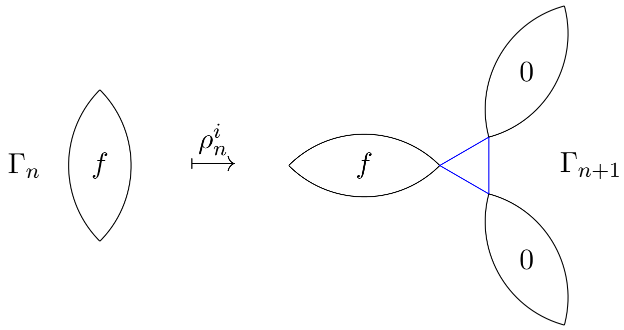
$$|\mathcal{F}_{x,n}| = (d-2)d^{n-N} + 1,$$

Furthermore, notice that, by construction, the following statements hold, for every $n \geq N$:

$$\forall f \in \mathcal{F}_{x,n} \setminus \mathcal{F}^A_{x,n}, \quad f((d-1)^n) = 0,$$

$$\forall f \in \mathcal{F}_{x,n} \setminus \mathcal{F}^B_{x,n}, \quad f((d-1)^{n-1}0) = 0.$$

Proposition 5.3.3. *Let $N \geq 1$ and $x \in \mathrm{sp}(\Delta_N) \setminus \mathrm{sp}(\Delta_{N-1})$. Then $\mathcal{F}_{x,n}$ is a basis of the x-eigenspace of Δ_n, for every $n \geq N$.*



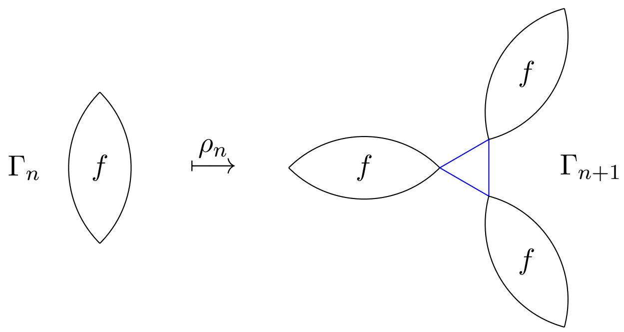


Figure 5.4: Transition operators ρ_n^i and ρ_n. The former copies f on the i-th copy of Γ_n in Γ_{n+1} and vanishes elsewhere; the latter copies f on all the copies of Γ_n.

Proof. Let us proceed by induction on n, with the base case $n = N$ covered in Proposition 5.3.1.

Let $f \in \mathcal{F}_{x,n}$ be an x-eigenfunction of Δ_n. and let $v \in X^n$, $j \in X$ and $s \in S$. On the one hand we have

$$\rho_n^i \Delta_n f(vj) = \Delta_n f(v)\delta_{i,j} = \sum_{s \in S} f(s(v))\delta_{i,j}.$$

On the other hand, if $v \neq (d-1)^{n-1}0$, we have $s(vj) = s(v)j$. In that case,

$$\Delta_{n+1}\rho_n^i f(vj) = \sum_{s \in S} \rho_n^i f(s(vj)) = \sum_{s \in S} \rho_n^i f(s(v)j) = \sum_{s \in S} f(s(v))\delta_{i,j}.$$

So we have $\Delta_{n+1}\rho_n^i f(vj) = \rho_n^i \Delta_n f(vj) = x\rho_n^i f(vj)$ if $v \neq (d-1)^{n-1}0$. For $v = (d-1)^{n-1}0$, we need to further decompose the sums. Since v is fixed by all $b \in B$,

$$\rho_n^i \Delta_n f(vj) = \Delta_n f(v)\delta_{i,j} =$$

$$= \sum_{k=1}^{d-1} f(a^k(v))\delta_{i,j} + \sum_{1 \neq b \in B} f(b(v))\delta_{i,j} =$$

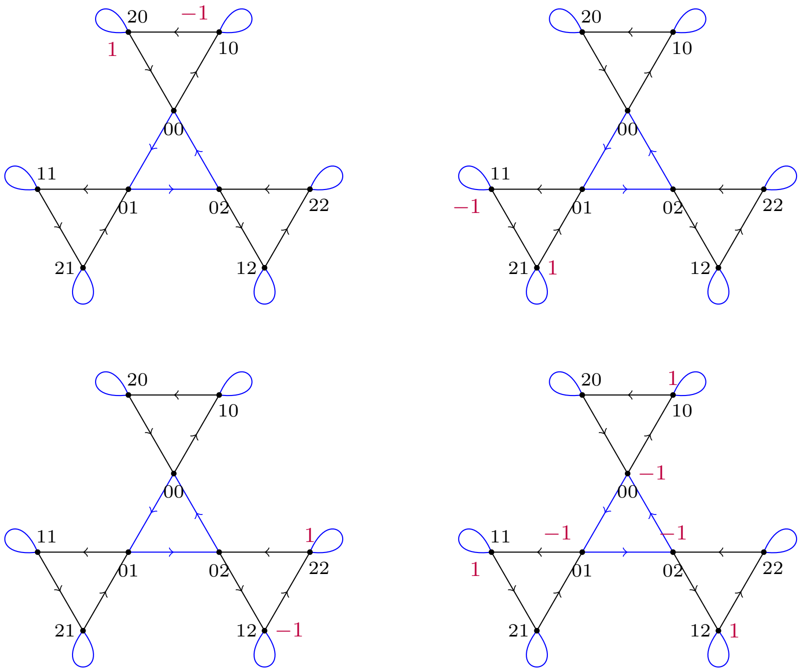


Figure 5.5: Eigenfunctions of eigenvalue $x = 1$ on the graph Γ_2 transferred from Γ_1 via the operators ρ_1^i and ρ_1 for the Fabrykowski-Gupta group ($d = 3$, $m = 1$).

$$= \sum_{k=1}^{d-1} f(a^k(v))\delta_{i,j} + \sum_{1\neq b\in B} f(v)\delta_{i,j},$$

and

$$\Delta_{n+1}\rho_n^i f(vj) =$$
$$= \sum_{k=1}^{d-1} \rho_n^i f(a^k(vj)) + \sum_{1\neq b\in B} \rho_n^i f(b(vj)) =$$
$$= \sum_{k=1}^{d-1} \rho_n^i f(a^k(v)j) + \sum_{1\neq b\in B} \rho_n^i f(v\omega_{n-1}(b)(j)) =$$
$$= \sum_{k=1}^{d-1} f(a^k(v))\delta_{i,j} + \sum_{1\neq b\in B} f(v)\delta_{i,\omega_{n-1}(b)(j)}.$$

By substracting both expressions, we get

$$(\Delta_{n+1} - x)\rho_n^i f(vj) = \sum_{1\neq b\in B} f(v)(\delta_{i,\omega_{n-1}(b)(j)} - \delta_{i,j}).$$

We observe now, by Remark 5.3.2, that if $f \in \mathcal{F}_{x,n} \setminus \mathcal{F}_{x,n}^B$, by construction, we have $f(v) = 0$, and so $\rho_n^i f$ is an x-eigenfunction of Δ_{n+1}. Else, if $f \in \mathcal{F}_{x,n}^B$, we add the equations for all $i \in X$:

$$(\Delta_{n+1} - x)\rho_n f(vj) = \sum_{1 \neq b \in B} f(v) \sum_{i \in X} (\delta_{i,\omega_{n-1}(b)(j)} - \delta_{i,j}) = 0.$$

In this case, $\rho_n f$ is an eigenfunction of Δ_{n+1}.

To conclude, we can inductively verify that the functions in $\mathcal{F}_{x,n+1}$ are linearly independent by looking at the supports of the images of the functions from $\mathcal{F}_{x,n}$ by ρ_n^i and ρ_n. Finally, we already know that $|\mathcal{F}_{x,n+1}| = (d-2)d^{n+1-N} + 1$, which equals the multiplicity of x for Δ_{n+1}, and so the dimension of the x-eigenspace of Δ_{n+1}. □

In Proposition 5.3.3, we have constructed a basis of the x-eigenspace of Δ_n, for every $n \geq N$. We can obtain a basis of ℓ_n^2 of eigenfunctions of Δ_n if we take the union of the bases of each of the x-eigenspaces for every $x \in \mathrm{sp}(\Delta_n)$. For convenience, let us set $\mathcal{F}_{1,n}$ to be the singleton containing the constant function equal to one in ℓ_n^2, for $n \geq 0$.

Corollary 5.3.4. *The set*

$$\bigsqcup_{x \in \mathrm{sp}(\Delta_n)} \mathcal{F}_{x,n}$$

is a basis of ℓ_n^2 that consists of eigenfunctions of Δ_n.

5.3.2 Eigenfunctions of Δ_ξ

We are now ready to describe the eigenfunctions of the adjacency operator Δ_ξ on the Schreier graph Γ_ξ, with $\xi \in X^{\mathbb{N}}$. We do so by transferring the eigenfunctions on the finite graphs Γ_n to Γ_ξ via the transfer operators defined next. For $n \geq 0$, we set

$$\tilde{\rho}_n : \ell_n^2 \to \ell_\xi^2, \qquad \tilde{\rho}_n f(\eta) = f(\eta_0 \ldots \eta_{n-1})\delta_{\sigma^n(\xi),\sigma^n(\eta)}.$$

where again $\delta_{\zeta,\zeta'} = 1$ if $\zeta = \zeta'$ or vanishes otherwise. Intuitively, $\tilde{\rho}_n = \cdots \circ \rho_{n+1}^{\xi_{n+1}} \circ \rho_n^{\xi_n}$. We define the following set:

$$\mathcal{F}_x := \bigcup_{n \geq N} \tilde{\rho}_n(\mathcal{F}_{x,n}^C).$$

If there exists $r \geq 0$ such that $\sigma^r(\xi) = (d-1)^{\mathbb{N}}$, let it be minimal and set $R = \max\{r, N\}$. In that case, we also include the function $\tilde{\rho}_R(\mathcal{F}_{x,R}^A)$ in the definition of $\mathcal{F}_x$.

Theorem 5.3.5. *Let $N \geq 1$ and $x \in \mathrm{sp}(\Delta_N) \setminus \mathrm{sp}(\Delta_{N-1})$. Then every $f \in \mathcal{F}_x$ is a x-eigenfunction of Δ_ξ, for every $\xi \in X^{\mathbb{N}}$.*

Proof. Let $n \geq N$ and $f \in \mathcal{F}_{x,n}^C$. We will show that $\tilde{\rho}_n f$ is a x-eigenfunction of M_ξ. Let $\eta \in \mathrm{Cof}(\xi)$ and denote by v its prefix of length n, so that $\eta = v\sigma^n(\eta)$. Assume first that v is not $(d-1)^n$ nor $(d-1)^{n-1}0$. In that case, for any $s \in S$, $s(\eta) = s(v\sigma^n(\eta)) = s(v)\sigma^n(\eta)$. On the one hand,

$$\Delta_\xi \tilde{\rho}_n f(\eta) = \sum_{s\in S} \tilde{\rho}_n f(s(\eta)) = \sum_{s\in S} \tilde{\rho}_n f(s(v)\sigma^n(\eta)) =$$

$$= \sum_{s\in S} f(s(v))\delta_{\sigma^n(\xi),\sigma^n(\eta)}.$$

On the other hand,

$$\tilde{\rho}_n \Delta_n f(\eta) = \Delta_n f(v)\delta_{\sigma^n(\xi),\sigma^n(\eta)} = \sum_{s\in S} f(s(v))\delta_{\sigma^n(\xi),\sigma^n(\eta)}.$$

We observe that both expressions are equal. Therefore, as f is an x-eigenfunction of Δ_n,

$$\Delta_\xi \tilde{\rho}_n f(\eta) = \tilde{\rho}_n \Delta_n f(\eta) = x\tilde{\rho}_n f(\eta).$$

Assume now $v = (d-1)^n, (d-1)^{n-1}0$. In that case, as $f \in \mathcal{F}^C_{x,n}$, we have $f(v) = 0$ (see Remark 5.3.2). Then,

$$\Delta_\xi \tilde{\rho}_n f(\eta) = \sum_{s\in S} \tilde{\rho}_n f(s(\eta)) = \sum_{s\in S} \tilde{\rho}_n f(s(v\sigma^n(\eta))) =$$

$$= \sum_{s\in S} \tilde{\rho}_n f(s(v)s_v(\sigma^n(\eta))) = \sum_{s\in S} f(s(v))\delta_{\sigma^n(\xi),s_v(\sigma^n(\eta))}.$$

In addition,

$$\tilde{\rho}_n \Delta_n f(\eta) = \Delta_n f(v)\delta_{\sigma^n(\xi),\sigma^n(\eta)} = \sum_{s\in S} f(s(v))\delta_{\sigma^n(\xi),\sigma^n(\eta)}.$$

We have two cases: either $s \in A$, which means that s_v is trivial, or $s \in B$, so $s(v) = v$ and then $f(s(v)) = f(v) = 0$. In any case, the two expressions above coincide. Consequently, $\Delta_\xi \tilde{\rho}_n f(\eta) = \tilde{\rho}_n \Delta_n f(\eta) = x\tilde{\rho}_n f(\eta)$ as well for $v = (d-1)^n, (d-1)^{n-1}0$, which shows that f is a x-eigenfunction of Δ_ξ.

The case ξ cofinal with $(d-1)^{\mathbb{N}}$ and $f \in \mathcal{F}^A_{x,R}$ is proven in a very similar way. The only difference is that now $f(v)$ is not necessarily zero for $v = (d-1)^R$. However, for any $s \in B$, $\delta_{\sigma^n(\xi),s_v(\sigma^n(\eta))} = \delta_{s_v^{-1}(\sigma^n(\xi)),\sigma^n(\eta)} = \delta_{\sigma^n(\xi),\sigma^n(\eta)}$, since $\sigma^n(\xi) = (d-1)^{\mathbb{N}}$ is fixed by s_v^{-1}. Therefore, both expressions are equal and the statement remains true for this case too. □

Remark 5.3.6. Notice that if $x \in \mathrm{sp}(\Delta_N) \setminus \mathrm{sp}(\Delta_{N-1})$, the size of the support of any $f \in \mathcal{F}_x$ is either $2d^{N-1}$ or $2d^N$.

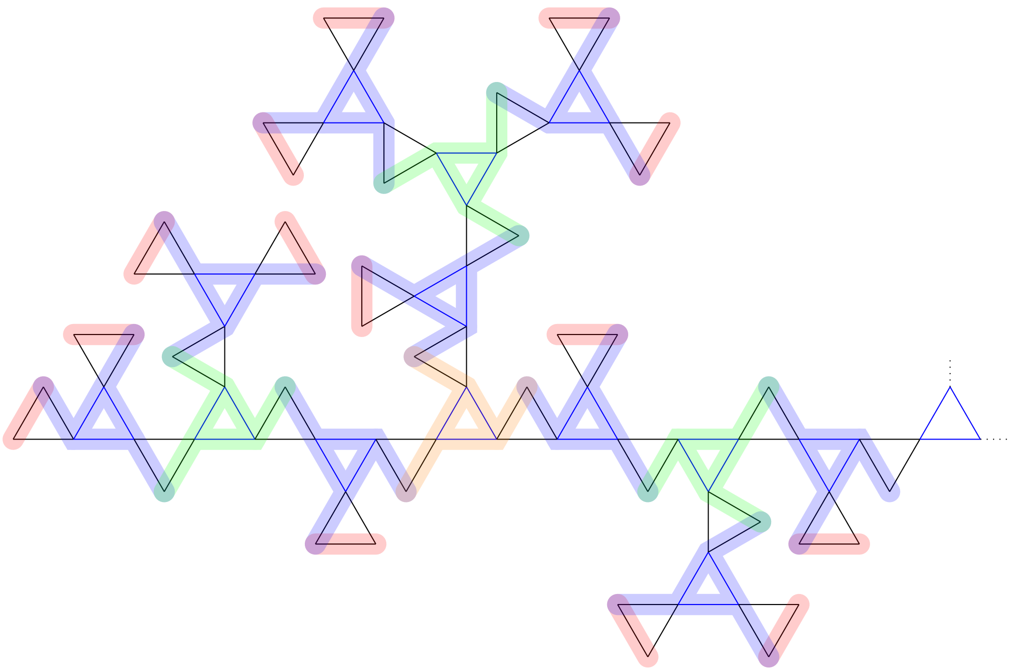


Figure 5.6: Supports of eigenfunctions of Δ_ξ of eigenvalue 1 for the Fabrykowski-Gupta group ($d = 3$, $m = 1$).

Finally, we introduce the set

$$\mathcal{F} := \bigcup_{N\geq 1} \bigcup_{\substack{x\in \mathrm{sp}(\Delta_N) \\ x\notin \mathrm{sp}(\Delta_{N-1})}} \mathcal{F}_x.$$

We want to prove that $\mathcal{F}$ is a complete set of eigenfunctions of Δ_ξ, i.e. that $\langle \mathcal{F} \rangle = \ell^2_\xi$. To that end, we will use spaces of antisymmetric functions, as in [15]. First, let us define the space of antisymmetric functions on Γ_n as

$$\ell^2_{n,a} = \langle f \in \ell^2_n \mid \exists i \in \{1, \ldots, d-2\}, \quad f = -f \circ \Phi^i_n \rangle.$$

Notice that we exclude $i = 0$ in this definition. The reason is the fact that antisymmetric functions with respect to Φ^0_n do not vanish on the vertex $(d-1)^{n-1}0$, which is the one through which the copies of Γ_n in Γ_{n+1} are connected. Eigenfunctions of Δ_n not vanishing on that vertex do not transfer directly to eigenfunctions of Δ_{n+1} via the operators ρ^i_n, as manifested in Proposition 5.3.3 and the construction of $\mathcal{F}_{x,n}$.

Lemma 5.3.7 below gives a basis for the finite-dimensional antisymmetric subspaces $\ell^2_{n,a}$. We will later define $\ell^2_{\xi,a}$ as an extension of the antisymmetric subspaces $\ell^2_{n,a}$ to the infinite graph Γ_ξ, and prove in Lemma 5.3.8 that $\ell^2_{\xi,a}$ is actually contained in $\langle \mathcal{F} \rangle$. Finally, we will use this result to show in Theorem 5.3.9 that $\mathcal{F}$ is a complete set of eigenfunctions of Δ_ξ.

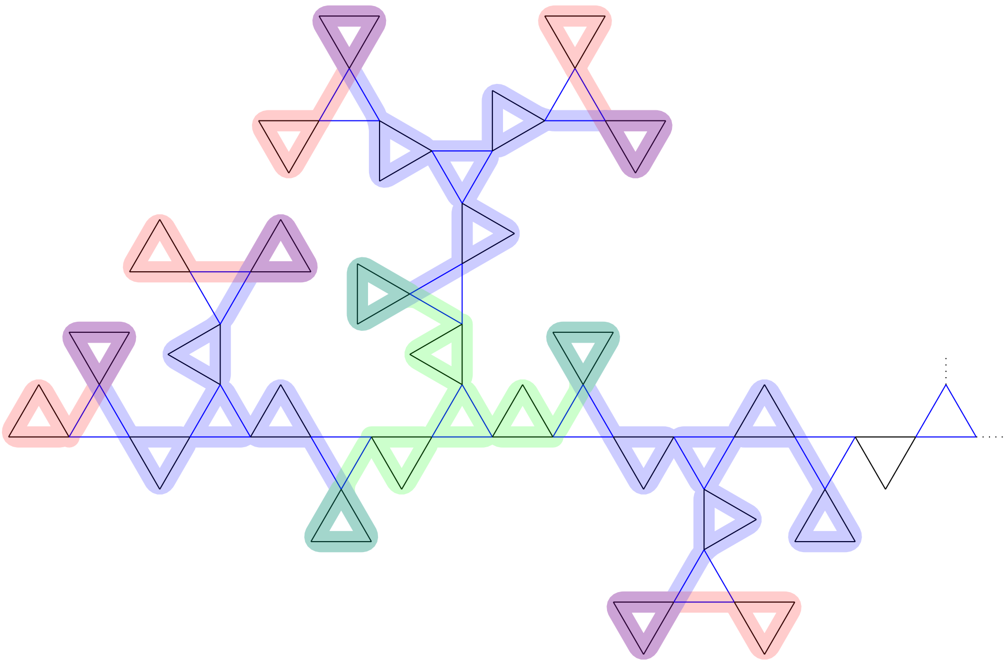


Figure 5.7: Supports of eigenfunctions of Δ_ξ of eigenvalue $1 \pm \sqrt{6}$ for the Fabrykowski-Gupta group ($d = 3$, $m = 1$).

Lemma 5.3.7. *For every $n \geq 1$, $\ell^2_{n,a}$ admits the basis*

$$\bigsqcup_{N=1}^{n-1} \bigsqcup_{\substack{x \in \mathrm{sp}(\Delta_N) \\ x \notin \mathrm{sp}(\Delta_{N-1})}} \bigsqcup_{i=1}^{d-2} (\rho^{i+1}_{n-1} - \rho^i_{n-1})(\mathcal{F}^A_{x,n-1} \sqcup \mathcal{F}^C_{x,n-1}) \quad \sqcup \quad \bigsqcup_{\substack{x \in \mathrm{sp}(\Delta_n) \\ x \notin \mathrm{sp}(\Delta_{n-1})}} \mathcal{F}^A_{x,n} \sqcup \mathcal{F}^C_{x,n}.$$

Proof. First notice that all these functions belong to $\ell^2_{n,a}$ by construction. Indeed, let $1 \leq N \leq n-1$ and $x \in \mathrm{sp}(\Delta_N) \setminus \mathrm{sp}(\Delta_{N-1})$. Let also $f \in \mathcal{F}^A_{x,n-1} \sqcup \mathcal{F}^C_{x,n-1}$ and $1 \leq i \leq d-2$. On the one hand we have

$$\left((\rho^{i+1}_{n-1} - \rho^i_{n-1}) f \circ \Phi^i_n\right)(v_0 \dots v_{n-1}) = (\rho^{i+1}_{n-1} - \rho^i_{n-1}) f(v_0 \dots v_{n-2} \tau_i(v_{n-1})) =$$

$$= f(v_0 \dots v_{n-2})(\delta_{i+1,\tau_i(v_{n-1})} - \delta_{i,\tau_i(v_{n-1})}).$$

On the other hand,

$$(\rho^{i+1}_{n-1} - \rho^i_{n-1}) f(v_0 \dots v_{n-1}) = f(v_0 \dots v_{n-2})(\delta_{i+1,v_{n-1}} - \delta_{i,v_{n-1}}).$$

If $v_{n-1} \neq i, i+1$, then $\tau_i(v_{n-1}) = v_{n-1}$ and so both expressions vanish. Otherwise, since τ_i exchanges i and $i+1$, the first expression equals the second with opposite sign.

If we now take $x \in \mathrm{sp}(\Delta_n) \setminus \mathrm{sp}(\Delta_{n-1})$ and $f \in \mathcal{F}^A_{x,n} \sqcup \mathcal{F}^C_{x,n}$, the fact that f is antisymmetric follows from Proposition 5.3.1.

Finally, we will check that the dimension of both subspaces is the same. We know by construction that $\dim(\ell^2_{n,a}) = (d-2)d^{n-1}$ and that the functions in the statement are linearly independent, again inductively and using the fact that we know their supports. The number of functions is

$$\sum_{N=1}^{n-1} \sum_{\substack{x\in\mathrm{sp}(\Delta_N)\\ x\notin\mathrm{sp}(\Delta_{N-1})}} \sum_{i=1}^{d-2} |\mathcal{F}^A_{x,n-1} \sqcup \mathcal{F}^C_{x,n-1}| \quad + \quad \sum_{\substack{x\in\mathrm{sp}(\Delta_n)\\ x\notin\mathrm{sp}(\Delta_{n-1})}} |\mathcal{F}^A_{x,n} \sqcup \mathcal{F}^C_{x,n}| =$$

$$= \sum_{N=1}^{n-1} \sum_{\substack{x\in\mathrm{sp}(\Delta_N)\\ x\notin\mathrm{sp}(\Delta_{N-1})}} \sum_{i=1}^{d-2} (d-2)d^{n-1-N} \quad + \quad \sum_{\substack{x\in\mathrm{sp}(\Delta_n)\\ x\notin\mathrm{sp}(\Delta_{n-1})}} (d-2) =$$

$$= (d-2)^2 \sum_{N=1}^{n-1} d^{n-1-N} |\operatorname{sp}(\Delta_N) \setminus \operatorname{sp}(\Delta_{N-1})| \; + \; (d-2)|\operatorname{sp}(\Delta_n) \setminus \operatorname{sp}(\Delta_{n-1})| =$$

$$= (d-2)^2 \sum_{N=1}^{n-1} d^{n-1-N} 2^{N-1} + (d-2)2^{n-1} =$$

$$= (d-2)(d^{n-1} - 2^{n-1}) + (d-2)2^{n-1} =$$

$$= (d-2)d^{n-1} = \dim(\ell^2_{n,a}).$$

□

Let $P_n : \ell^2_\xi \to \ell^2_\xi$ be the orthogonal projector to the subspace of functions with support in $\tilde{\Gamma}_n := \Gamma^n_\xi = X^n\sigma^n(\xi)$, so that $P_n f(\eta) = f(\eta)\delta_{\sigma^n(\xi),\sigma^n(\eta)}$. Let also $P'_n : \ell^2_\xi \to \ell^2_n$ be the operator defined by $P'_n f(v) = f(v\sigma^n(\xi))$, so that $P'_n \circ \tilde{\rho}_n$ is the identity in ℓ^2_n. We define the space of antisymmetric functions on Γ_ξ as

$$\ell^2_{\xi,a} = \langle f \in \ell^2_\xi \mid \exists n \in I_\xi, \quad \operatorname{supp}(f) \subset \tilde{\Gamma}_n, \quad P'_n f \in \ell^2_{n,a} \rangle,$$

where $I_\xi = \{n \in \mathbb{N} \mid \forall r \geq 0,\ (d-1)^r 0$ is not a prefix of $\sigma^n(\xi)\}$ is the set of indices n such that $\sigma^n(\xi)$ is fixed by B. Equivalently, it is the set of indices for which $\tilde{\Gamma}_n$ is connected to the rest of Γ_ξ by just one vertex.

Lemma 5.3.8. *The space of antisymmetric functions $\ell^2_{\xi,a}$ is contained in $\langle \mathcal{F} \rangle$.*

Proof. Let $f \in \ell^2_{\xi,a}$. Then there exists some $n \in I_\xi$ such that $\operatorname{supp}(f) \subset \tilde{\Gamma}_n$ and $P'_n f \in \ell^2_{n,a}$. In particular, either there exists some $r \geq 0$ such that $\xi_n \dots \xi_{n+r} = (d-1)^r j$, with $j \neq 0, d-1$, or $\sigma^n(\xi) = (d-1)^{\mathbb{N}}$, in which case we set $r = \infty$. Let T_n be the basis of $\ell^2_{n,a}$ from Lemma 5.3.7. We claim that for every $h \in T_n$, $\tilde{\rho}_n h \in \langle \mathcal{F} \rangle$.

Indeed, let $1 \leq N \leq n-1$, $x \in \operatorname{sp}(\Delta_N) \setminus \operatorname{sp}(\Delta_{N-1})$ and $i \in \{1, \dots, d-1\}$. Given a function $g \in \mathcal{F}^A_{x,n-1} \sqcup \mathcal{F}^C_{x,n-1}$, since $i \neq 0$, we have $\rho^i_{n-1} g \in \mathcal{F}^A_{x,n} \sqcup \mathcal{F}^C_{x,n}$. If $r = \infty$, then

$\tilde{\rho}_n \rho_{n-1}^i g \in \mathcal{F}$ directly. Otherwise, then as $\xi_n \ldots \xi_{n+r} = (d-1)^r j$ and $j \neq 0, d-1$, we have

$$\rho_{n+r}^{\xi_{n+r}} \cdots \rho_n^{\xi_n} \rho_{n-1}^i g = \rho_{n+r}^j \rho_{n+r-1}^{d-1} \cdots \rho_n^{d-1} \rho_{n-1}^i g \in \mathcal{F}_{x,n+r+1}^C.$$

Therefore,

$$\tilde{\rho}_n \rho_{n-1}^i g = \tilde{\rho}_{n+r+1} \rho_{n+r}^{\xi_{n+r}} \cdots \rho_n^{\xi_n} \rho_{n-1}^i g \in \mathcal{F}.$$

Hence, for any generator $h \in T_n$ of the form $h = (\rho_{n-1}^{i+1} - \rho_{n-1}^i)g$, we showed $\tilde{\rho}_n h \in \langle \mathcal{F} \rangle$.

Similarly, let $x \in \mathrm{sp}(\Delta_n) \setminus \mathrm{sp}(\Delta_{n-1})$, and $g \in \mathcal{F}_{x,n}^A \sqcup \mathcal{F}_{x,n}^C$. If $r = \infty$, then again directly $\tilde{\rho}_n g \in \mathcal{F}$. Otherwise, since $\xi_n \ldots \xi_{n+r} = (d-1)^r j$ and $j \neq 0, d-1$, we have

$$\rho_{n+r}^{\xi_{n+r}} \cdots \rho_n^{\xi_n} g = \rho_{n+r}^j \rho_{n+r-1}^{d-1} \cdots \rho_n^{d-1} g \in \mathcal{F}_{x,n+r+1}^C.$$

This implies

$$\tilde{\rho}_n g = \tilde{\rho}_{n+r+1} \rho_{n+r}^{\xi_{n+r}} \cdots \rho_n^{\xi_n} g \in \mathcal{F}.$$

Finally, for every generator h of T_n, of the form $h = g$, $\tilde{\rho}_n h$ must also be in $\langle \mathcal{F} \rangle$.

To conclude, since $P'_n f \in \ell^2_{n,a}$, let us decompose $P'_n f = \sum_i c_i h_i$, with $c_i \in \mathbb{R}$ and $h_i \in T$. The support of f is contained in $\tilde{\Gamma}_n$, so $f = \tilde{\rho}_n P'_n f = \sum_i c_i \tilde{\rho}_n h_i \in \langle \mathcal{F} \rangle$, and hence $\ell^2_{\xi,a}$ is contained in $\langle \mathcal{F} \rangle$. □

Our next step is to show that $\ell^2_{\xi,a}$ is dense in ℓ^2_ξ. However, we are only able to do this under some extra conditions on $\xi \in X^{\mathbb{N}}$, which fortunately define a subset W of uniform Bernoulli measure one in $X^{\mathbb{N}}$. Observe that the antisymmetric subspace $\ell^2_{\xi,a}$ is of infinite dimension if and only if the set $I_\xi = \{n \in \mathbb{N} \mid \forall r \geq 0,\ (d-1)^r 0 \text{ is not a prefix of } \sigma^n(\xi)\}$ is infinite. Equivalently, if and only if Γ_ξ is one-ended (see Theorems 3.3.1 and 3.3.3).

To prove that $\mathcal{F}$ is a complete system of eigenfunctions, we need in fact a slightly stronger condition than Γ_ξ being one-ended. We will need not only that I_ξ is infinite, but also that it contains consecutive pairs k and $k+1$ infinitely often. Let us consider the subset $W \subset X^{\mathbb{N}}$ defined as $W = \{\xi \in X^{\mathbb{N}} \mid k, k+1 \in I_\xi \text{ for infinitely many } k\}$. Note that this set only depends on d, and does not depend on m nor on $\omega \in \Omega_{d,m}$.

Theorem 5.3.9. *Let G_ω be a spinal group with $d \geq 3$, $m \geq 1$ and $\omega \in \Omega_{d,m}$, and let Δ_ξ be the adjacency operator on the Schreier graph Γ_ξ, for $\xi \in X^{\mathbb{N}}$. If $\xi \in W$, then $\mathcal{F}$ is a complete system of finitely supported eigenfunctions of Δ_ξ. The set W has uniform Bernoulli measure 1 in $X^{\mathbb{N}}$.*

Proof. Let μ be the uniform Bernoulli measure on $X^{\mathbb{N}}$. We show first that the set W has measure one. Indeed, the set W can be rewritten as

$$W = \{\xi \in X^{\mathbb{N}} \mid \forall l \geq 0,\ \exists k \geq l,\ k, k+1 \in I_\xi\},$$

and so its complement is X^*Z, with

$$Z = \{\xi \in X^{\mathbb{N}} \mid \forall k \geq 0,\ k \notin I_\xi \text{ or } k+1 \notin I_\xi\}.$$

Notice that if, for some $k \in \mathbb{N}$, $\xi_k \neq 0, d-1$, then $k \in I_\xi$. Equivalently, for any $k \notin I_\xi$, then necessarily $\xi_k = 0, d-1$. Therefore, for any point $\xi \in Z$, either $0 \notin I_\xi$ or $1 \notin I_\xi$, which implies that at least one of ξ_0, ξ_1 is 0 or $d-1$. Hence,

$$\mu(Z) \leq \left(1 - \left(\frac{d-2}{d}\right)^2\right)\mu(Z) \Longrightarrow \mu(Z) = 0.$$

Finally, $\mu(W) = 1 - \mu(X^{\mathbb{N}} \setminus W) = 1 - \mu(X^*Z) = 1 - \mu(Z) = 1$.

We will now prove that the set $\mathcal{F}$ is complete for every $\xi \in W \subset X^{\mathbb{N}}$. Let $f \in \langle \mathcal{F} \rangle^\perp$, and let us show that $f = 0$. In particular, by Lemma 5.3.8, $f \perp \ell^2_{\xi,a}$. Now let n be the smallest such that $\|P_n f\|_{\ell^2} \geq \frac{4}{5}\|f\|_{\ell^2}$ and both $n, n+1 \in I_\xi$, which exists as $\xi \in W$. Our goal is to define an antisymmetric function approximating $P_n f$. Because $P_n f$ concentrates the major part of the norm of f, and $f \perp \ell^2_{\xi,a}$, this function will allow us to derive an inequality concluding that f must be zero.

As $n, n+1 \in I_\xi$, both ξ_n and ξ_{n+1} are not 0. Let $i \in \{1, \ldots, d-2\}$ such that $\xi_n \in \{i, i+1\}$. Define $g := \rho_n^{\xi_n} P'_n f - \rho_n^{\xi_n} P'_n f \circ \Phi^i_{n+1} \in \ell^2_{n+1,a} \subset \ell^2_{n+1}$ and also $h := \tilde{\rho}_{n+1} g \in \ell^2_{\xi,a} \subset \ell^2_\xi$. Then,

$$\begin{aligned} 0 = \langle f, h\rangle_{\ell^2_\xi} &= \langle P'_{n+1} f, g\rangle_{\ell^2_{n+1}} = \\ &= \langle P'_{n+1} f, \rho_n^{\xi_n} P'_n f\rangle_{\ell^2_{n+1}} - \langle P'_{n+1} f, \rho_n^{\xi_n} P'_n f \circ \Phi^i_{n+1}\rangle_{\ell^2_{n+1}} = \\ &= \left\|\rho_n^{\xi_n} P'_n f\right\|^2_{\ell^2_{n+1}} - \langle P'_{n+1} f \circ \Phi^i_{n+1}, \rho_n^{\xi_n} P'_n f\rangle_{\ell^2_{n+1}} = \\ &= \|P_n f\|^2_{\ell^2_\xi} - \langle Q_n f, P_n f\rangle_{\ell^2_\xi}, \end{aligned}$$

with $Q_n : \ell^2_\xi \to \ell^2_\xi$ defined as

$$Q_n f(\eta) = f(\eta_0 \ldots \eta_{n-1} \tau_i(\eta_n) \sigma^{n+1}(\eta)) \delta_{\sigma^n(\eta), \sigma^n(\xi)}.$$

Notice that the function $Q_n f$ is supported in $\tilde{\Gamma}_n$ and its values are those of f on the subgraph $X^n \tau_i(\xi_n) \sigma^{n+1}(\xi) = \Gamma^n_{\xi_0 \ldots \xi_{n-1} \tau_i(\xi_n) \sigma^{n+1}(\xi)}$. Therefore, its norm is not greater than the norm of $f - P_n f$, supported in $\Gamma_\xi \setminus \tilde{\Gamma}_n$, so $\|Q_n f\|_{\ell^2_\xi} \leq \|f - P_n f\|_{\ell^2_\xi}$. Now we have, using the Cauchy-Schwarz inequality,

$$\begin{aligned} 0 &= \|P_n f\|^2_{\ell^2_\xi} - \langle Q_n f, P_n f\rangle_{\ell^2_\xi} \geq \\ &\geq \frac{4^2}{5^2}\|f\|^2_{\ell^2_\xi} - \|Q_n f\|_{\ell^2_\xi} \|P_n f\|_{\ell^2_\xi} \geq \end{aligned}$$

$$\geq \frac{4^2}{5^2}\|f\|^2_{\ell^2_\xi} - \|f - P_n f\|_{\ell^2_\xi}\|f\|_{\ell^2_\xi} \geq$$
$$\geq \frac{4^2}{5^2}\|f\|^2_{\ell^2_\xi} - \sqrt{1-\frac{4^2}{5^2}}\|f\|^2_{\ell^2_\xi} =$$
$$\geq \left(\frac{16}{25} - \frac{3}{5}\right)\|f\|^2_{\ell^2_\xi} = \frac{1}{25}\|f\|^2_{\ell^2_\xi}.$$

Hence $f = 0$. □

We conclude the section by translating Theorem 5.3.9 in terms of the spectral measures of Δ_ξ.

Corollary 5.3.10. *Let G_ω be a spinal group with $d \geq 3$, $m \geq 1$ and $\omega \in \Omega_{d,m}$, and let Δ_ξ be the adjacency operator on the Schreier graph Γ_ξ, for $\xi \in X^{\mathbb{N}}$. If $\xi \in W$, then all spectral measures of Δ_ξ are discrete and supported on the set of eigenvalues $\{|S| - d\} \cup \psi^{-1}(\bigcup_{n\geq 0} F^{-n}(0))$ (see Theorem 4.1.10), i.e. the spectrum of Δ_ξ is pure point. The set W has uniform Bernoulli measure 1 in $X^{\mathbb{N}}$.*

5.4 Singular continuous spectral measures

In order to illustrate the different possibilities for spectral measures of adjacency operators on Schreier graphs of spinal groups, the goal of this section is to exhibit some Schreier graphs for which the spectral measures of the adjacency operator associated with certain explicit functions have nontrivial singular continuous part. In particular, we will decompose the space ℓ^2_π as the direct sum of the eigenspaces of the adjacency operator and an explicit subspace Φ, whose functions have singular continuous spectral measures. To prove that, we recover the renormalization maps Π^* and Π as well as the quadratic map $G(x) = x^2 - 2(d-2)x - 2(d-1)$ from Section 4.2 and use a version of the Ruelle-Perron-Frobenius Theorem (Theorem 5.4.10) as main tool. For this section, let G_ω be a spinal group with $d \geq 3$ and $m = 1$, and $\omega \in \Omega_{d,1}$.

Recall that the graphs Γ_π have vertex set $\mathrm{Cof}((d-1)^{\mathbb{N}}) \times X$, and can be decomposed as d copies of $\Gamma_{(d-1)^{\mathbb{N}}}$ joined by a copy of Λ_π. We set $\ell^2_\pi = \ell^2(\Gamma_\pi)$ and let Δ_π be the adjacency operator on Γ_π. Recall that the renormalization maps $\Pi^*, \Pi : \ell^2_\pi \to \ell^2_\pi$ as $\Pi^* f(\eta, i) = f(\sigma\eta, i)$ on the graphs Γ_π are defined as $\Pi f(\eta, i) = \sum_{j\in X} f(j\eta, i)$.

Notice the nontrivial symmetries that arise in the graph Γ_π. For any $\tau \in \mathrm{Sym}(X)$, the map $(\eta, i) \mapsto (\eta, \tau(i))$ is a graph automorphism of Γ_π. This corresponds to exchanging the copies of $\Gamma_{(d-1)^{\mathbb{N}}}$ according to τ.

5.4.1 Eigenfunctions of Δ_π

In order to describe the spectral measures, we will first give in Proposition 5.4.3 an isomorphism between the eigenspaces of the adjacency operators on the Schreier graphs. As in

Section 4.2, the statements are valid for both Δ_ξ and Δ_π. However, here we shall only state them for the latter, since it is for which the singular continuous spectral measures appear.

Lemma 5.4.1. *We have*

$$\Pi\Delta_\pi\Pi^* = 2(d-1)^2 + \Delta_\pi$$

Proof. Let $f \in \ell^2_\pi$, and let $p \in \Gamma_\pi$. Let q_j and r_j be its A and B-neighbors, respectively, for $j = 1 \dots d-1$. Let us write α, β_j and γ_j to denote the values of f at p, q_j and r_j, respectively, and let us set $\beta = \sum_{j=1}^{d-1} \beta_j$ and $\gamma = \sum_{j=1}^{d-1} \gamma_j$. Using computations similar to those in the proof of Lemma 4.2.1, we have

$$\Pi\Delta_\pi\Pi^* f(p) = \beta + \gamma + 2(d-1)\alpha + 2(d-2)(d-1)\alpha =$$

$$= \beta + \gamma + 2(d-1)^2\alpha = (2(d-1)^2 + \Delta_\pi)f(p).$$

□

Corollary 5.4.2. *Let $f, g \in \ell^2_\pi$. Let $\alpha = \frac{2(d-1)^2}{d}$ and $\beta = \frac{1}{d}$. Then*

$$\langle \Delta_\pi\Pi^* f, \Pi^* g\rangle = \alpha\langle \Pi^* f, \Pi^* g\rangle + \beta\langle G(\Delta_\pi)\Pi^* f, \Pi^* g\rangle.$$

Proof. By Lemmas 5.4.1 and 4.2.1, and the fact that $\Pi\Pi^* = d$,

$$\langle \Delta_\pi\Pi^* f, \Pi^* g\rangle = \langle \Pi\Delta_\pi\Pi^* f, g\rangle = 2(d-1)^2\langle f, g\rangle + \langle \Delta_\pi f, g\rangle =$$

$$= \frac{2(d-1)^2}{d}\langle \Pi\Pi^* f, g\rangle + \frac{1}{d}\langle \Pi\Pi^*\Delta_\pi f, g\rangle =$$

$$= \frac{2(d-1)^2}{d}\langle \Pi^* f, \Pi^* g\rangle + \frac{1}{d}\langle G(\Delta_\pi)\Pi^* f, \Pi^* g\rangle.$$

□

Let H be the subspace generated by the image of Π^* and $\Delta_\pi\Pi^*$, and recall that it is invariant under Δ_π by Proposition 4.2.4. We denote by E_x the eigenspace of Δ_π of eigenvalue $x \in \mathrm{sp}(\Delta_\pi)$. For any $x \in \mathbb{R}$, we define the operator $R_x : \ell^2_\pi \to H \subset \ell^2_\pi$ as

$$R_x f = (x - 2(d-2))\Pi^* f + \Delta_\pi\Pi^* f.$$

Proposition 5.4.3. *Let $x \in \mathrm{sp}(\Delta_\pi)$. Then x is an eigenvalue of $\Delta_\pi|_H$ if and only if $G(x)$ is an eigenvalue of Δ_π. Moreover, R_x induces an isomorphism between the eigenspaces of Δ_π $E_{G(x)}$ and E_x.*

Proof. Let $f \in H$ such that $\Delta_\pi f = xf$. We must have either $\Pi f \neq 0$ or $\Pi\Delta_\pi f \neq 0$, or otherwise, for every $g \in \ell^2_\pi$,

$$0 = \langle \Pi f, g\rangle = \langle f, \Pi^* g\rangle, \qquad 0 = \langle \Pi\Delta_\pi f, g\rangle = \langle f, \Delta_\pi \Pi^* g\rangle,$$

and so $f \in H^\perp$, which then implies $f = 0$. Moreover, as $\Pi\Delta_\pi f = x\Pi f$, we conclude that $\Pi f \neq 0$. By Lemma 4.2.1, $\Delta_\pi \Pi f = \Pi G(\Delta_\pi) f = G(x)\Pi f$, which means that Πf is an eigenfunction of Δ_π of eigenvalue $G(x)$.

Conversely, let $g \in \ell^2_\pi$ such that $\Delta_\pi g = G(x) g$. We have, by Lemma 4.2.1,

$$\Delta_\pi R_x g = (x - 2(d-2))\Delta_\pi \Pi^* g + \Delta_\pi^2 \Pi^* g =$$
$$= (x - 2(d-2))\Delta_\pi \Pi^* g + \Pi^* \Delta_\pi g + 2(d-2)\Delta_\pi \Pi^* g + 2(d-1)\Pi^* g =$$
$$= x\Delta_\pi \Pi^* g + G(x)\Pi^* g + 2(d-1)\Pi^* g =$$
$$= x\Delta_\pi \Pi^* g + (x^2 - 2(d-2)x - 2(d-1))\Pi^* g + 2(d-1)\Pi^* g =$$
$$= x\Delta_\pi \Pi^* g + x(x - 2(d-2))\Pi^* g =$$
$$= x((x - 2(d-2))\Pi^* g + \Delta_\pi \Pi^* g) = xR_x g.$$

Hence $R_x g$ is an eigenfunction of $\Delta_\pi|_H$ of eigenvalue x.

Let us now prove that the restriction of R_x to $E_{G(x)}$ is an isomorphism between $E_{G(x)}$ and E_x. Let $g \in E_{G(x)}$. Notice that, by Lemma 5.4.1,

$$\Pi R_x g = (x - 2(d-2))\Pi\Pi^* g + \Pi\Delta_\pi \Pi^* g =$$
$$= d(x - 2(d-2))g + 2(d-1)^2 g + \Delta_\pi g =$$
$$= d(x - 2(d-2))g + 2(d-1)^2 g + G(x) g =$$
$$= (x^2 + (4-d)x - 2(d-2))g =$$
$$= (x+2)(x - (d-2))g.$$

By Lemma 4.2.7, we know that $x \neq -2, d-2$, so $R_x|_{E_{G(x)}}$ is indeed an injective operator. Finally, let us show that $R_x|_{E_{G(x)}}$ is surjective. Let $f \in E_x \subset H$, and assume that f is orthogonal to $R_x(E_{G(x)})$. In that case, for every $g \in E_{G(x)}$,

$$0 = \langle f, R_x g\rangle = (x - 2(d-2))\langle f, \Pi^* g\rangle + \langle f, \Delta_\pi \Pi^* g\rangle =$$
$$= (x - 2(d-2))\langle \Pi f, g\rangle + \langle \Pi\Delta_\pi f, g\rangle =$$
$$= (x - 2(d-2))\langle \Pi f, g\rangle + x\langle \Pi f, g\rangle =$$
$$= 2(x - (d-2))\langle \Pi f, g\rangle.$$

Hence, $\Pi f = 0$. Since $f \in H$, by the argument at the beginning of the proof we must have $f = 0$. □

Recall from Section 4.2 and Definition 3.2.3 that A-pieces (equivalently (-1)-pieces) in Γ_π are subgraphs of the form $(\Gamma^1_\eta, i) = (X\sigma\eta, i)$, for $\eta \in \mathrm{Cof}((d-1)^{\mathbb{N}})$ and $i \in X$. Similarly, n-pieces are subgraphs of the form $(\Lambda^n_\eta, i) = ((d-1)^n 0 X \sigma^{n+2}(\eta), i)$, for $n \geq 0$, $\eta \in \mathrm{Cof}((d-1)^{\mathbb{N}})$ and $i \in X$, and the only ∞-piece is the subgraph $((d-1)^{\mathbb{N}}, X)$. Finally, we called any n-piece a B-piece, for $n \geq 0$ or $n = \infty$.

Proposition 5.4.4. *Let $x \in G^{-n}(d-2)$, with $n \geq 0$, and $f \in E_x$. Then f has zero sum on $(n-1)$-pieces and is constant on N-pieces, for every $N \geq n$.*

Proof. We proceed by induction on n, in parallel for all the graphs Γ_ξ, $\xi \in X^{\mathbb{N}}$. The case $n = 0$ is a consequence of Lemma 4.2.7.

Let $f \in E_x$. As $n \geq 1$, $x \neq d-2$, so by Lemmas 4.2.5 and 4.2.7 we have $f \in H$. By Proposition 5.4.3, there exists $g \in E_{G(x)}$ such that $f = R_x g$. By induction hypothesis, g has zero sum on $(n-2)$-pieces, and is constant for N-pieces for all $N \geq n-1$. Let us show that f has zero sum on $(n-1)$-pieces, and is constant for N-pieces for all $N \geq n$.

Let $k \geq 0$ and let p_i be the vertices of a k-piece in Γ_π, for $i \in X$. Abusing notation by writing $\sigma(\eta, i) = (\sigma\eta, i)$, the vertices $q_i = \sigma p_i$ form a $(k-1)$-piece in Γ_π, for $i \in X$. Let δ_p be the function with value 1 at the vertex p and zero everywhere else. We have

$$\Pi^* g(p_i) = \langle \Pi^* g, \delta_{p_i} \rangle = \langle g, \Pi\delta_{p_i} \rangle = \langle g, \delta_{q_i} \rangle = g(q_i),$$

$$\Delta_\pi \Pi^* g(p_i) = \langle \Delta_\pi \Pi^* g, \delta_{p_i} \rangle = \langle g, \Pi\Delta_\pi \delta_{p_i} \rangle = \langle g, (d-1)\delta_{q_i} + \sum_{j \neq i} \delta_{q_j} \rangle =$$
$$= (d-2)\langle g, \delta_{q_i} \rangle + \sum_{j \in X} \langle g, \delta_{q_j} \rangle = (d-2) g(q_i) + \sum_{j \in X} g(q_j)$$

and so

$$f(p_i) = R_x g(p_i) = (x - 2(d-2))\Pi^* g(p_i) + \Delta_\pi \Pi^* g(p_i) =$$
$$= (x - 2(d-2)) g(q_i) + (d-2) g(q_i) + \sum_{j \in X} g(q_j) =$$
$$= (x - (d-2)) g(q_i) + \sum_{j \in X} g(q_j).$$

If $k = n-1$, then the vertices q_j form a $(n-2)$-piece, so $\sum_{j \in X} g(q_j) = 0$ by hypothesis. In that case,

$$\sum_{i \in X} f(p_i) = (x - (d-2)) \sum_{i \in X} g(q_i) = 0.$$

So f has zero sum on n-pieces.

If $k \geq n$, then the vertices q_j form a $k-1$-piece, with $k-1 \geq n-1$, and so g is constant on all q_j, $j \in X$. Let us write $\gamma = g(q_j)$. Then

$$f(p_i) = (x - (d-2))\gamma + d\gamma = (x+2)\gamma$$

is constant over the vertices p_i. Therefore, f is constant on all N-pieces, with $N \geq n$. □

Notice that Proposition 5.4.4 implies that all eigenfunctions are constant on the unique ∞-piece of Γ_π.

Corollary 5.4.5. *Let $x \in \bigcup_{n\geq 0} G^{-n}(d-2)$. Then the eigenspace E_x has infinite dimension and is generated by finitely-supported functions.*

Proof. The result is true for $x = d-2$ by Lemma 4.2.7, and it extends to any other $x \in \bigcup_{n\geq 0} G^{-n}(d-2)$ via Proposition 5.4.3. $\square$

5.4.2 Spectral measures of Δ_π

We are now ready to study the spectral measures of the adjacency operator Δ_π on the Schreier graph Γ_π. We will find an explicit subspace of ℓ^2_π for which the spectral measures are singular continuous. More precisely, we will decompose ℓ^2_π as the direct sum of the eigenspaces of Δ_π plus a subspace Φ, whose functions have purely singular continuous spectral measures. We follow a strategy based on [60].

Let us start by relating the spectral measure of a function $f \in \ell^2_\pi$ with that of $\Pi^* f \in \ell^2_\pi$. To do so, we will use as main tool the transfer operator $L_{q,\kappa}$. An exposition of results concerning operators of this type can be found in [59].

Definition 5.4.6. Let $q \in \mathbb{R}[x]$ be the quadratic polynomial $q(x) = (x-u)^2 + t$, with $u, t \in \mathbb{R}$. Let $\kappa : \mathbb{R} \setminus \{u\} \to \mathbb{R}$ be a measurable function. We define the transfer operator $L_{q,\kappa}$ as

$$L_{q,\kappa}\alpha(y) = \sum_{x \in q^{-1}(y)} \kappa(x)\alpha(x),$$

for every measurable function $\alpha : \mathbb{R} \setminus \{u\} \to \mathbb{R}$ and $y \in (t, \infty)$.

Let μ be a positive measure on (t,∞). If, for μ-almost every $y \in (t,\infty)$, κ is positive on both preimages of y by q, then, after performing a change of variables, there is a measure ν on $\mathbb{R} \setminus \{u\}$ such that

$$\int_{\mathbb{R}\setminus\{u\}} \alpha \, d\nu = \int_{(t,\infty)} L_{q,\kappa}\alpha \, d\mu.$$

for every positive measurable function $\alpha : \mathbb{R} \setminus \{u\} \to \mathbb{R}$. We shall denote this measure ν by $L^*_{q,\kappa}\mu$.

In order to find the relation between the spectral measures of Δ_π associated with $f \in \ell^2_\pi$ and $\Pi^* f \in \ell^2_\pi$, we need the following general result about these transfer operators, the proof of which can be found in [59].

Lemma 5.4.7. *Let $\mathcal{H}$ be a Hilbert space and T a self-adjoint bounded operator of $\mathcal{H}$. Let $K \subset \mathcal{H}$ be a closed subspace such that $q(T)K \subset K$ and K and TK generate $\mathcal{H}$. Suppose that there exist $\alpha, \beta \in \mathbb{R}$ such that, for every $v, w \in \mathcal{H}$, $\langle Tv, w\rangle = \alpha\langle v, w\rangle + \beta\langle q(T)v, w\rangle$.*

Let $v \in K$ and let μ and μ' be the spectral measures of T and $q(T)$ associated with v, respectively. If $\mu'(\{t\}) = 0$, then $\mu(\{u\}) = 0$ and $\mu = L^*_{q,\theta}\mu'$, with $\theta : \mathbb{R} \setminus \{u\} \to \mathbb{R}$ given by

$$\theta(x) = \frac{1}{2}\left(1 + \frac{\alpha - u + \beta q(x)}{x - u}\right).$$

For every $x \in \operatorname{sp}(T)$, $x \neq u$, we have $\theta(x) \geq 0$.

We may now apply Lemma 5.4.7 to our situation.

Proposition 5.4.8. *Let $f \in \ell^2_\pi$. Let μ_f and $\mu_{\Pi^* f}$ be the spectral measures of Δ_π associated with f and $\Pi^* f$, respectively. Then $\mu_{\Pi^* f}(\{d-2\}) = 0$ and $\mu_{\Pi^* f} = L^*_{G,\theta}\mu_f$, with $\theta(x) = \frac{x+2}{2}$.*

Proof. Let $\mu'_{\Pi^* f}$ be the spectral measure of $G(\Delta_\pi)$ associated with $\Pi^* f$. By Lemma 4.2.1, notice that $\mu_f = \frac{\mu'_{\Pi^* f}}{d}$. Indeed, for any $n \geq 0$,

$$\langle G(\Delta_\pi)^n \Pi^* f, \Pi^* f\rangle = \langle \Pi^* \Delta_\pi^n f, \Pi^* f\rangle = d\langle \Delta_\pi^n f, f\rangle.$$

We find ourselves in the precise situation of Lemma 5.4.7, with $\mathcal{H} = H$, $T = \Delta_\pi$, $q = G$, so that $u = d-2$ and $t = -(d-1)^2 - 1$, and $K = \Pi^*\ell^2_\pi$. By Corollary 5.4.2, $\alpha = \frac{2(d-1)^2}{d}$ and $\beta = \frac{1}{d}$. In addition, we know that $\mu'_{\Pi^* f}(\{-(d-1)^2 - 1\}) = 0$ after Lemma 4.2.6. Since $\mu_{\Pi^* f}$ is the spectral measure of Δ_π associated with $\Pi^* f$, Lemma 5.4.7 implies that $\mu_{\Pi^* f}(\{d-2\}) = 0$ and $\mu_{\Pi^* f} = L^*_{G,\theta'}\mu'_{\Pi^* f}$, with

$$\theta'(x) = \frac{1}{2}\left(1 + \frac{\alpha - u + \beta q(x)}{x - u}\right) = \frac{x+2}{2d}.$$

Finally, because $\mu_f = \frac{\mu'_{\Pi^* f}}{d}$, if we set $\theta(x) = \frac{x+2}{2}$, we have

$$\mu_{\Pi^* f} = L^*_{G,\theta'}\mu'_{\Pi^* f} = L^*_{G,\theta'}(d\,\mu_f) = L^*_{G,\theta}\mu_f.$$

□

We now need to use a version of the Ruelle-Perron-Frobenius Theorem, which is stated in terms of full shifts on binary alphabets. In our case, we shall prove that the dynamical system given by the action of G on its Julia set Λ is conjugate to the shift on $\{0,1\}^{\mathbb{N}}$.

Lemma 5.4.9. *The dynamical system (Λ, G) is conjugate to the full shift on $\{0,1\}^{\mathbb{N}}$.*

Proof. By Remark 4.1.11, we know that $\Lambda \subset [-2, 2(d-1)]$. In fact, $\Lambda \subset G^{-1}(\Lambda) = I_0 \cup I_1$, where $I_0 = [-2, d-2-\sqrt{d(d-2)}] \cup I_1 = [d-2+\sqrt{d(d-2)}, 2(d-1)]$.

We may define a map $\Lambda \to \{0,1\}^{\mathbb{N}}$ as follows. For every $x \in \Lambda$, we assign the sequence $(\varepsilon_n)_{n\geq 0}$ such that $G^n(x) \in I_{\varepsilon_n}$, for every $n \geq 0$. This map is well-defined since $\Lambda \subset I_0 \cup I_1$. Let us show that it is a bijection.

Let $x, y \in \Lambda$ have the same associated sequence $(\varepsilon_n)_{n\geq 0}$. Notice that

$$|G(x) - G(y)| = |x^2 - 2(d-2)x - y^2 + 2(d-2)y| =$$

$$= |x^2 - y^2 - 2(d-2)(x-y)| = |x - (d-2) + y - (d-2)||x-y|.$$

However, x and y both lie on the same side of $d-2$, so

$$|x - (d-2) + y - (d-2)| = |x - (d-2)| + |y - (d-2)| \geq 2\sqrt{d(d-2)} \geq 3.$$

Hence $|G(x)-G(y)| \geq 3|x-y|$. If $x \neq y$, then for some $n \geq 0$ the distance $|G^n(x)-G^n(y)|$ would be greater than the length of the intervals I_0, I_1 and so $G^n(x)$ and $G^n(y)$ would not lie in the same one, which contradicts the fact that x and y have the same associated sequence. Therefore the map is injective. To prove that it is surjective, let us define the intervals

$$I_{\varepsilon_0\ldots\varepsilon_n} = \bigcap_{k=0}^{n} G^{-k}(I_{\varepsilon_k}) = I_{\varepsilon_0} \cap G^{-1}(I_{\varepsilon_1\ldots\varepsilon_n}),$$

for $\varepsilon_0\ldots\varepsilon_n \in \{0,1\}^{n+1}$, $n \geq 0$. The preimage of any interval in $[-2, 2(d-1)]$ by G is a union of two intervals: one included in I_0 and another in I_1. We can use this fact to inductively verify that none of these intervals is empty. In addition, we have the inclusions

$$I_{\varepsilon_0} \supset I_{\varepsilon_0\varepsilon_1} \supset \cdots \supset I_{\varepsilon_0\ldots\varepsilon_n} \supset \ldots$$

Hence, for any sequence $(\varepsilon_n)_{n\geq 0}$, we have a decreasing sequence of closed, non-empty intervals, which implies that the intersection

$$\bigcap_{n\geq 0} I_{\varepsilon_0\ldots\varepsilon_n}$$

is non-empty. By construction, all points in this intersection must have $(\varepsilon_n)_{n\geq 0}$ as associated sequence, but by injectivity there can only be one. Therefore, the map $\Lambda \to \{0,1\}^{\mathbb{N}}$ defined above is a bijection. □

Lemma 5.4.9 enables us now to use the following version of the Ruelle-Perron-Frobenius Theorem, whose proof can be found in [59, §2.2]. For $I \subset \mathbb{R}$, let $C(I)$ denote the space of continuous functions on I, endowed with the topology of uniform convergence, and let $\mathbb{R}_+^*$ denote the set of strictly positive real numbers.

Theorem 5.4.10. *Let $\kappa : \Lambda \to \mathbb{R}_+^*$ be a Hölder function. Consider $L_{q,\kappa}$ as an operator on $C(\Lambda)$, and let λ_κ be its spectral radius. Then there exists a unique probability measure ν_κ on Λ and a unique strictly positive function $l_\kappa \in C(\Lambda)$ such that*

$$L_{q,\kappa} l_\kappa = \lambda_\kappa l_\kappa, \qquad L_{q,\kappa}^* \nu_\kappa = \lambda_\kappa \nu_\kappa \qquad \textit{and} \qquad \int_\Lambda l_\kappa d\nu_\kappa = 1.$$

Moreover, the spectral radius of $L_{q,\kappa}$ restricted to the space of functions with zero integral with respect to ν_κ is strictly smaller than λ_κ. Also, for every $g \in C(\Lambda)$, we have

$$\lim_{n\to\infty} \frac{1}{\lambda_\kappa^n} L_{q,\kappa} g = \int_\Lambda g d\nu_\kappa.$$

Let us now define the subspace Φ, whose orthogonal complement in ℓ^2_π will be the direct product of all eigenspaces of Δ_π. Let p_i denote the vertex $((d-1)^{\mathbb{N}}, i) \in \Gamma_\pi$, for $i \in X$, and let $\varphi_i \in \ell^2_\pi$ be the function which takes value 1 at p_i, -1 at p_{i-1} and 0 everywhere else (see Figure 5.8), for $i \in X$. We define Φ as the following subspace:

$$\Phi = \langle \Delta_\pi^n \varphi_i \mid n \geq 0, i \in X \rangle.$$

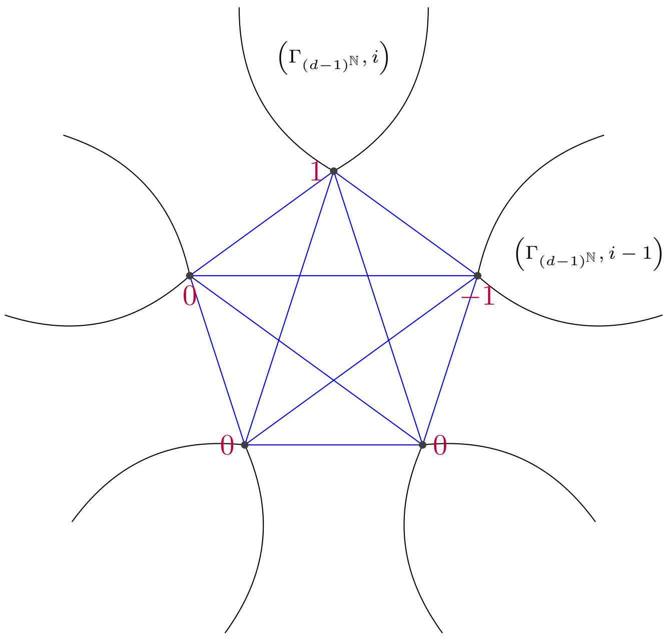


Figure 5.8: The functions $\varphi_i \in \ell^2_\pi$.

Lemma 5.4.11. *For every $i \in X$,*

$$\Pi^* \varphi_i = (\Delta_\pi + 2)\varphi_i.$$

Proof. Notice that $\Delta_\pi \varphi_i$ is supported in the A-pieces containing p_i and p_{i-1}. It takes value 1 at the A-neighbors of p_i and p_{i-1}, and value -1 at the A-neighbors of p_{i-1} and p_i. On the other hand, $\Pi^*\varphi_i$ takes value 1 on the full A-piece containing p_i and -1 on that containing p_{i-1}. This proves the claim. □

Recall that we set $\theta(x) = \frac{x+2}{2}$, and let $k(x) = x + 2$, $h(x) = 2(d-1) - x$ and $\rho(x) = \frac{1}{2}$. Notice that $\theta = k\rho$ and that $hk = h \circ G$. From now on, for any measurable function $\kappa : \Lambda \to \mathbb{R}$, we shall write $L_\kappa = L_{G,\kappa}$.

Proposition 5.4.12. *Let ν_ρ be the probability measure on Λ from Theorem 5.4.10, so that $L_\rho^*\nu_\rho = \nu_\rho$. Then, for every $i \in X$, the spectral measure of Δ_π associated with φ_i is $\mu_{\varphi_i} = \frac{2}{d} h\nu_\rho$. Moreover, μ_{φ_i} is singular continuous.*

Proof. Let $\mu = \mu_{\varphi_i}$ be the spectral measure of Δ_π associated with φ_i, and let $\mu' = \mu_{\Pi^*\varphi_i}$ be the spectral measure of Δ_π associated with $\Pi^*\varphi_i$. By Lemma 5.4.11, we know that $\Pi^*\varphi_i = k(\Delta_\pi)\varphi_i$, so $\mu' = k^2\mu$. Using Proposition 5.4.8, we know moreover that $\mu(\{d-2\}) = 0$ and $\mu' = L_\theta^*\mu$. Combining these two results, we obtain that $L_\theta^*\mu = k^2\mu$.

As a consequence of Lemma 4.2.7, we know that $\mu(\{-2\}) = 0$. Therefore, for any measurable function $\alpha : \Lambda \to \mathbb{R}$, we have, using that $\theta = k\rho$,

$$\int_\Lambda L_{\rho/k}\alpha \, d\mu = \int_\Lambda L_\theta \frac{\alpha}{k^2} \, d\mu \overset{L_\theta^*\mu = k^2\mu}{=} \int_\Lambda k^2 \frac{\alpha}{k^2} \, d\mu = \int_\Lambda \alpha \, d\mu,$$

which implies that $L_{\rho/k}^*\mu = \mu$. Furthermore, by Lemma 4.2.6, we know that $\mu(\{2(d-1)\}) = 0$ as well. Consequently, using the relation $hk = h \circ G$,

$$\int_\Lambda L_{\rho/k}\frac{\alpha}{h} \, d\mu = \int_\Lambda L_\rho \frac{\alpha}{hk} \, d\mu = \int_\Lambda L_\rho \frac{\alpha}{h \circ G} \, d\mu = \int_\Lambda \frac{1}{h} L_\rho\alpha \, d\mu.$$

Equivalently, that $L_\rho^*(\frac{1}{h}\mu) = \frac{1}{h}L_{\rho/k}^*\mu$, and hence that $L_\rho^*(\frac{1}{h}\mu) = \frac{1}{h}\mu$.

By Proposition 5.4.4, any eigenfunction of Δ_π must be constant on the B-piece formed by p_i, for $i \in X$, and therefore it must be orthogonal to φ_i, for $i \in X$. We then conclude that $\mu\left(\bigcup_{n\geq 0} G^{-n}(d-2)\right) = 0$, so μ has to be concentrated on Λ, in light of of Theorem 4.2.8.

The spectral radius of L_ρ is $\lambda_\rho = 1$. Indeed, since $L_\rho 1 = 1$ by an elementary calculation, we have

$$\lim_{n\to\infty} \frac{1}{\lambda_\rho^n} = \lim_{n\to\infty} \frac{1}{\lambda_\rho^n} L_\rho^n 1 = \int_\Lambda d\nu_\rho = \nu_\rho(\Lambda) = 1.$$

This implies that, for every $g \in C(\Lambda)$,

$$\lim_{n\to\infty} L_\rho^n g = \int_\Lambda g \, d\nu_\rho.$$

We have proven that $L_\rho^*(\frac{1}{h}\mu) = \frac{1}{h}\mu$. Our goal now is to show that $\frac{1}{h}\mu$ is a finite measure, and then by unicity of ν_ρ, they must be multiples of one another.

Let $g \in C(\Lambda)$ such that g vanishes on a neigborhood of $2(d-1)$ and $0 < \int_\Lambda \frac{g}{h} \, d\mu < \infty$. We know that $L_\rho^n g$ converges uniformly to the constant function $\int_\Lambda g \, d\nu_\rho$. Using this and $L_\rho^*(\frac{1}{h}\mu) = \frac{1}{h}\mu$, we find, for $n \geq 0$ big enough,

$$\int_\Lambda \frac{g}{h} \, d\mu = \int_\Lambda \frac{1}{h} L_\rho^n g \, d\mu = \int_\Lambda \frac{1}{h} \left(\int_\Lambda g \, d\nu_\rho \right) d\mu = \int_\Lambda g \, d\nu_\rho \int_\Lambda \frac{1}{h} \, d\mu.$$

Therefore, the measure $\frac{1}{h}\mu$ must be finite, so it must be a multiple of ν_ρ. Let us show that they differ by a factor of $\frac{2}{d}$.

Notice that $\mu(\Lambda) = \|\varphi_i\|^2 = 2$. We also remark that $L_\rho h = d$. Indeed, for any $y \in \Lambda$,

$$L_\rho h(y) = \sum_{x\in G^{-1}(y)} \rho(x)h(x) = \frac{1}{2}\sum_{x\in G^{-1}(y)} (2(d-1) - x) =$$

$$= \frac{1}{2}\left(2d - 2 - (d-2) - \sqrt{(d-1)^2 + 1 + y}\right)$$

$$+\frac{1}{2}\left(2d - 2 - (d-2) + \sqrt{(d-1)^2 + 1 + y}\right) =$$

$$= \frac{1}{2}(d + d) = d.$$

Since $L_\rho^*\nu_\rho = \nu_\rho$, we have

$$\int_\Lambda h\, d\nu_\rho = \int_\Lambda L_\rho h\, d\nu_\rho = d\int_\Lambda d\nu_\rho = d.$$

We conclude

$$\int_\Lambda d\mu = \mu(\Lambda) = 2 = \frac{2}{d}\int_\Lambda h\, d\nu_\rho.$$

Hence $\mu = \frac{2}{d}h\nu_\rho$.

Finally, let us show that μ is singular continuous. We already know that it is supported on Λ, a Cantor set of zero Lebesgue measure, so its absolutely continuous part is trivial. In addition, the measure ν_ρ is atomless. Indeed, Theorem 5.4.10 implies that the action of G on Λ preserves the measure ν_ρ. If $x \in \Lambda$ was such that $\nu_\rho(\{x\}) > 0$, then $\nu_\rho(\{G^n(x)\}) = \nu_\rho(\{x\})$ for any $n \geq 0$, which is absurd as ν_ρ is a probability measure. In conclusion, μ has trivial absolutely continuous and discrete parts, so it is indeed singular continuous. □

In order to prove the decomposition of the space ℓ^2_π into the direct sum of the eigenspaces of Δ_π and Φ, we will need to introduce another function besides the φ_i. Before that, recall that $\Phi = \langle \Delta_\pi^n\varphi_i \mid n \geq 0, i \in X\rangle$ or, equivalently, $\Phi = \langle p(\Delta_\pi)\varphi_i \mid p \in \mathbb{C}[x], i \in X\rangle$.

Lemma 5.4.13. *The subspace Φ is invariant under the operators Δ_π, Π^* and Π.*

Proof. First, notice that Φ is invariant under Δ_π by definition. Let $p \in \mathbb{C}[x]$, then, by Lemmas 4.2.1 and 5.4.11,

$$\Pi^* p(\Delta_\pi)\varphi_i = p(G(\Delta_\pi))\Pi^*\varphi_i = p(G(\Delta_\pi))(\Delta_\pi + 2)\varphi_i \in \Phi,$$

so Φ is also invariant under Π^*.

We observe that $\Pi\varphi_i = \varphi_i$. In addition, by Lemma 5.4.11,

$$\Pi\Delta_\pi\varphi_i = \Pi(\Pi^* - 2)\varphi_i = (d-2)\varphi_i.$$

Once again by Lemma 4.2.1, we have

$$\Pi G(\Delta_\pi)^n \varphi_i = \Delta_\pi^n \Pi \varphi_i = \Delta_\pi^n \varphi_i$$

and

$$\Pi G(\Delta_\pi)^n \Delta_\pi \varphi_i = \Delta_\pi^n \Pi \Delta_\pi \varphi_i = (d-2)\Delta_\pi^n \varphi_i.$$

Let $p \in \mathbb{C}[x]$, and let us decompose it as $p(x) = \sum_{n=0}^N a_n G(x)^n + \sum_{n=0}^N b_n G(x)^n x$, for some coefficients $a_n, b_n \in \mathbb{C}$. We then conclude

$$\Pi p(\Delta_\pi)\varphi_i = \sum_{n=0}^N a_n \Pi G(\Delta_\pi)^n \varphi_i + \sum_{n=0}^N b_n \Pi G(\Delta_\pi)^n \Delta_\pi \varphi_i =$$

$$= \sum_{n=0}^N a_n \Delta_\pi^n \varphi_i + (d-2)\sum_{n=0}^N b_n \Delta_\pi^n \varphi_i \in \Phi,$$

so Φ is invariant under Π. □

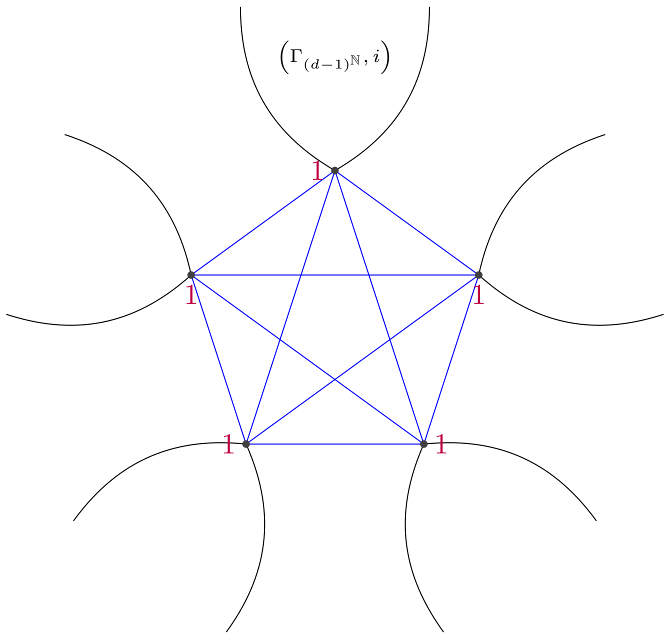


Figure 5.9: The function $\psi \in \ell^2_\pi$.

We now introduce the function $\psi \in \ell^2_\pi$, defined by $\psi(p_i) = 1$ for every $i \in X$, and zero everywhere else. Observe that ψ is orthogonal to φ_i for all $i \in X$.

Lemma 5.4.14. *We have*

$$\Pi^* \psi = (\Delta_\pi - (d-2))\psi.$$

Proof. On the one hand, $\Pi^*\psi$ is the function which takes value 1 on the A-pieces to which p_i belong, for every $i \in X$, and zero everywhere else. On the other hand, $\Delta_\pi\psi$ takes value $d-1$ on p_i and value 1 on its A-neighbors, for every $i \in X$. Hence $(\Delta_\pi - \Pi^*)\psi = (d-2)\psi$. □

We briefly recall the functions $\theta(x) = \frac{x+2}{2}$, $k(x) = x+2$, $h(x) = 2(d-1)-x$, $\rho(x) = \frac{1}{2}$, and the relations $\theta = k\rho$ and $hk = h \circ G$. Let us define as well the functions $l(x) = x-(d-2)$, $\tau = \frac{\theta}{l^2}$ and $\sigma = \frac{\tau}{k} = \frac{\rho}{l^2}$.

Proposition 5.4.15. *The spectral measure of Δ_π associated with ψ is discrete. More precisely, ψ is contained in the direct sum of the eigenspaces of Δ_π.*

Proof. Let $\mu = \mu_\psi|_\Lambda$ be the restriction to Λ of the spectral measure of Δ_π associated with ψ. By Theorem 4.2.8, we have to show that $\mu = 0$. Let $\mu' = \mu_{\Pi^*\psi}|_\Lambda$ be the restriction to Λ of the spectral measure of Δ_π associated with $\Pi^*\psi$.

By Lemma 5.4.14, $\mu' = l^2\mu$, and by Proposition 5.4.8, $\mu' = L_\theta^*\mu$, so we obtain $L_\theta^*\mu = l^2\mu$. As a consequence of Lemma 4.2.6, $\mu(\{2(d-1)\}) = 0$, so, for any measurable function $\alpha : \Lambda \to \mathbb{R}$,

$$\int_\Lambda L_\theta \frac{\alpha}{l^2}\, d\mu = \int_\Lambda l^2 \frac{\alpha}{l^2}\, d\mu = \int_\Lambda \alpha\, d\mu.$$

Therefore $\frac{1}{l^2}L_\theta^*\mu = \mu$. Furthermore,

$$\int_\Lambda L_\tau \alpha\, d\mu = \int_\Lambda L_{\theta/l^2}\alpha\, d\mu = \int_\Lambda L_\theta \frac{\alpha}{l^2}\, d\mu = \int_\Lambda \alpha\, d\mu,$$

which shows that $L_\tau^*\mu = \mu$.

Notice that $L_\tau^n h = hL_\sigma^n 1$ for any $n \geq 0$. Indeed, inductively on n, the statement is trivial for $n = 0$ and, for any $y \in \Lambda$,

$$L_\tau^n h(y) = \sum_{x\in G^{-1}(y)} \tau(x)L_\tau^{n-1}h(x) = \sum_{x\in G^{-1}(y)} \tau(x)h(x)L_\sigma^{n-1}1(x) =$$

$$= \sum_{x\in G^{-1}(y)} \sigma(x)k(x)h(x)L_\sigma^{n-1}1(x) = \sum_{x\in G^{-1}(y)} \sigma(x)h(G(x))L_\sigma^{n-1}1(x) =$$

$$= h(y)\sum_{x\in G^{-1}(y)} \sigma(x)L_\sigma^{n-1}1(x) = h(y)L_\sigma^n 1(y),$$

where we used the relations $\tau = \sigma k$ and $hk = h \circ G$. Notice also that $L_\sigma 1 < 1$ on Λ, as for every $y \in \Lambda$ we have

$$L_\sigma 1(y) = \sum_{x\in G^{-1}(y)} \sigma(x) =$$

$$= \frac{1}{2}\left(\frac{1}{\left(d-2-(d-2)-\sqrt{(d-1)^2+1+y}\right)^2}\right) +$$

$$+\frac{1}{2}\left(\frac{1}{\left(d-2-(d-2)+\sqrt{(d-1)^2+1+y}\right)^2}\right)=$$

$$=\frac{1}{2}\left(\frac{1}{(d-1)^2+1+y}+\frac{1}{(d-1)^2+1+y}\right)=\frac{1}{(d-1)^2+1+y},$$

which is strictly smaller than 1, provided that $d > 2$.

Let now λ_σ be the spectral radius of L_σ, and let ν_σ be the probability measure on Λ from Theorem 5.4.10. Since $L_\sigma^*\nu_\sigma = \lambda_\sigma\nu_\sigma$, we have

$$\lambda_\sigma = \int_\Lambda \lambda_\sigma \, d\nu_\sigma = \int_\Lambda L_\sigma 1 \, d\nu_\sigma < \int_\Lambda d\nu_\sigma = 1.$$

Because $\frac{1}{\lambda_\sigma^n} L_\sigma^n 1$ uniformly converges to $\int_\Lambda d\nu_\sigma = 1$, $L_\sigma^n 1$ uniformly converges to 0. But since $L_\tau^n h = h L_\sigma^n 1$, we conclude that $L_\tau^n h$ must also converge to 0. Finally, as $L_\tau^*\mu = \mu$,

$$\int_\Lambda h \, d\mu = \int_\Lambda L_\tau^n h \, d\mu \to 0.$$

Hence, $\mu(\Lambda \setminus \{2(d-1)\}) = 0$, but we already proved that $\mu(\{2(d-1)\}) = 0$, so, in fact, $\mu = 0$. $\square$

Recall that for $f, g \in \ell^2_\pi$, the spectral measure $\mu_{f,g}$ is the measure whose moments are $\langle \Delta_\pi^n f, g\rangle$, for every $n \geq 0$, so that $\mu_f = \mu_{f,f}$.

Lemma 5.4.16. *For every $f, g \in \ell^2_\pi$, we have $\mu_{\Pi f, g} = G_* \mu_{f, \Pi^* g}$.*

Proof. For every $p \in \mathbb{C}[x]$, and using Lemma 4.2.1,

$$\int_{\mathbb{R}} p \, d\mu_{\Pi f, g} = \langle p(\Delta_\pi)\Pi f, g\rangle = \langle \Pi p(G(\Delta_\pi)) f, g\rangle =$$

$$= \langle p(G(\Delta_\pi)) f, \Pi^* g\rangle = \int_{\mathbb{R}} p \circ G \, d\mu_{f, \Pi^* g}.$$

$\square$

We are now ready to prove the decomposition of the space ℓ^2_π into the direct sum of eigenspaces of Δ_π and Φ.

Proposition 5.4.17. *For every $f \in \Phi^\perp$, the spectral measure μ_f of Δ_π associated with f is discrete. The eigenvalues of $\Delta_\pi|_{\Phi^\perp}$ are $\bigcup_{n\geq 0} G^{-n}(d-2)$.*

Proof. First, we know by Corollary 5.4.5 that the eigenspaces E_x are nontrivial for every $x \in \bigcup_{n\geq 0} G^{-n}(d-2)$. Let P be the orthogonal projector to $\Phi^\perp$ in ℓ^2_π. Lemma 5.4.13 implies that P commutes with Δ_π, Π^* and Π. Let us show that, for every $f, g \in \ell^2_\pi$, the measure $\mu_{Pf,g}$ is discrete and concentrated on $\bigcup_{n\geq 0} G^{-n}(d-2)$. It suffices to consider f finitely supported.

Let $\Gamma_\pi^n = (\Gamma^n_{(d-1)^{\mathbb{N}}}, X)$ be the subgraph of Γ_π consisting of vertices of the form $(X^n(d-1)^{\mathbb{N}}, X)$, for every $n \geq 0$. Notice that only d vertices in Γ_π^n have edges to $\Gamma_\pi \setminus \Gamma_\pi^n$, more precisely those having the form $((d-1)^{n-1}0(d-1)^{\mathbb{N}}, i)$, $i \in X$. Let us call these vertices q_i^n, and recall that p_i were the vertices in the central B-piece, of the form $((d-1)^{\mathbb{N}}, i)$, $i \in X$.

For every $i \in X$, q_i^n has exactly $d-1$ B-neighbors, which lie outside Γ_π^n. Let us call them $r_{i,j}^n$, for $j = 1, \dots, d-1$. Intuitively, the subgraph Γ_π^n is the same as $\bigcup_{i\in X} \mathcal{B}_{p_i}(2^n - 1)$, and $\bigcup_{i\in X} \mathcal{B}_{p_i}(2^n)$ is the subgraph spanned by the union of Γ_π^n and $\{r_{i,j}^n \mid i \in X, j = 1, \dots, d-1\}$. Let us call this other subgraph $\bar{\Gamma}_\pi^n$. See Figure 5.10 for a picture of these graphs.

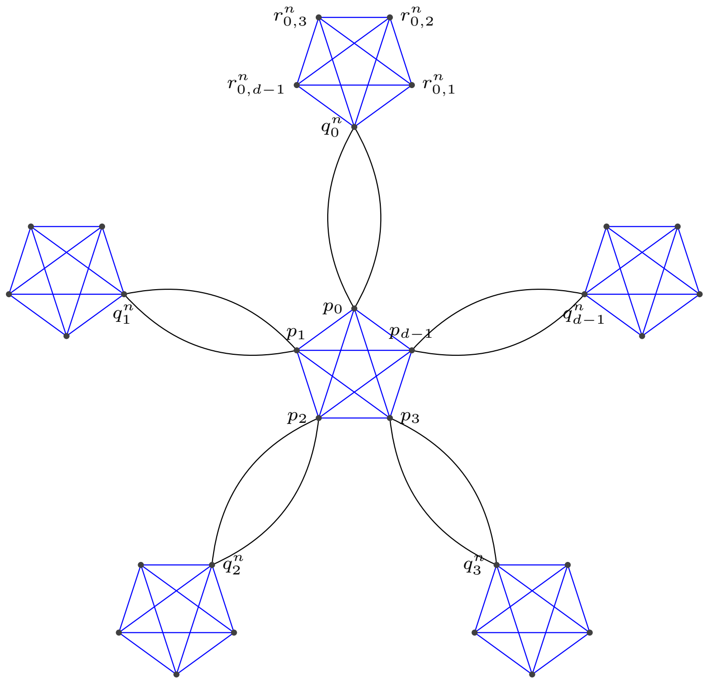


Figure 5.10: The subgraph Γ_π^n contains q_i^n, for $i \in X$. The subgraph $\bar{\Gamma}_\pi^n$ contains also the vertices $r_{i,j}^n$, for $i \in X$ and $j = 1, \dots, d-1$.

Consider now the subspace L_n of functions vanishing outside $\bar{\Gamma}_\pi^n$ which are constant on $r_{i,j}^n$, for $j = 1, \dots, d-1$. Namely,

$$L_n = \left\{ f \in \ell^2_\pi \,\middle|\, \mathrm{supp}(f) \subset \bar{\Gamma}_\pi^n, \quad \forall i \in X, \forall j, j' = 1, \dots, d-1,\ f(r_{i,j}^n) = f(r_{i,j'}^n) \right\}.$$

For every $n \geq 0$, notice that both ΠL_{n+1} and $\Pi \Delta_\pi L_{n+1}$ are contained in L_n. The former is immediate, and so is, for the latter, the inclusion of the support in $\bar{\Gamma}_\pi^n$. If $f \in L_{n+1}$

and we set α to be its value at q_i^{n+1} and β its value at $r_{i,j}^{n+1}$, for all $j = 1, \dots, d-1$, then the value of $\Pi\Delta_\pi f$ at $r_{i,j}^n$ is equal to $\alpha + (2d-3)\beta$ for all $j = 1, \dots, d-1$, so it does not depend on j. See Figure 5.11 for an illustration of these facts.

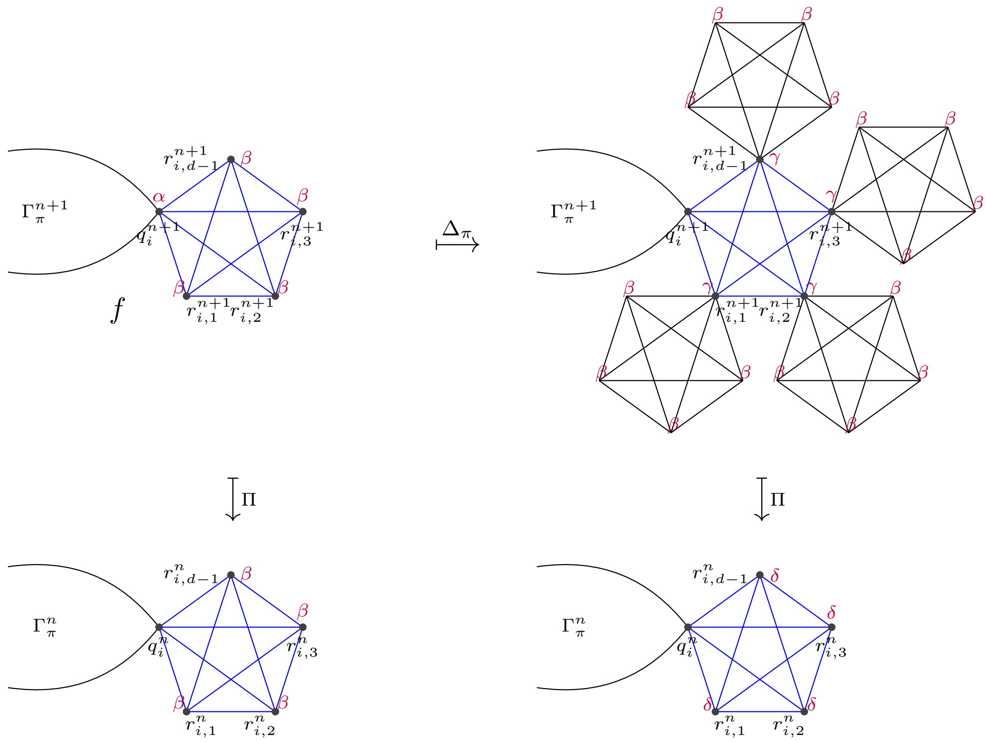


Figure 5.11: For any $f \in L_{n+1}$, both Πf and $\Pi\Delta_\pi f$ belong to L_n. For better readability, we set $\gamma = \alpha + (d-2)\beta$ and $\delta = \alpha + (2d-3)\beta$.

We will show, by induction on $n \geq 0$, that for every $f \in L_n$ and every $g \in \ell_\pi^2$, the measure $\mu_{Pf,g}$ is discrete and concentrated on $\bigcup_{n\geq 0} G^{-n}(d-2)$.

For $n = 0$, we can see that the subspace L_0 is generated by $\{\varphi_i, \Delta_\pi\varphi_i, \psi, \Delta_\pi\psi \mid i \in X\}$. If $f = \varphi_i$ or $f = \Delta_\pi\varphi_i$, we have $Pf = 0$, and so $\mu_{Pf,g} = 0$ for every $g \in \ell_\pi^2$. If $f = \psi$ or $f = \Delta_\pi\psi$, Proposition 5.4.15 implies that $\mu_{Pf,g}$ is discrete and concentrated on $\bigcup_{n\geq 0} G^{-n}(d-2)$ for every $g \in \ell_\pi^2$.

Now assume the claim to be true for some $n \geq 0$, and let $f \in L_{n+1}$. Since both Πf and $\Pi\Delta_\pi f$ belong to L_n, we know that the measures $\mu_{P\Pi f,g} = \mu_{\Pi Pf,g}$ and $\mu_{P\Pi\Delta_\pi f,g} = \mu_{\Pi\Delta_\pi Pf,g}$ are discrete and concentrated on $\bigcup_{n\geq 0} G^{-n}(d-2)$ for every $g \in \ell_\pi^2$. By Lemma 5.4.16, the measures $\mu_{Pf,\Pi^* g}$ and $\mu_{Pf,\Delta_\pi\Pi^* g}$ are discrete and concentrated on $\bigcup_{n\geq 0} G^{-n}(d-2)$. Equivalently, if H is the image of the operators Π^* and $\Delta_\pi\Pi^*$, the measure $\mu_{Pf,g}$ is discrete and concentrated on $\bigcup_{n\geq 0} G^{-n}(d-2)$ for every $g \in H$. However, by Lemma 4.2.5 we know that $\mathrm{sp}(\Delta_\pi|_{H^\perp}) = \{d-2, -2\}$, so the measure $\mu_{Pf,g}$ must be discrete and concentrated on $\bigcup_{n\geq 0} G^{-n}(d-2) \cup \{-2\}$ for every $g \in \ell^2_\pi$. Finally, by Lemma 4.2.7, we know that the eigenspace E_{-2} is trivial, so $\mu_{Pf,g}(\{-2\}) = 0$ and therefore $\mu_{Pf,g}$ is discrete and concentrated on $\bigcup_{n\geq 0} G^{-n}(d-2)$. □

Combining the results of Propositions 5.4.12, 5.4.15 and 5.4.17, we obtain the following Theorem:

Theorem 5.4.18. *Let G_ω be a spinal group with $d \geq 3$, $m = 1$ and $\omega \in \Omega_{d,1}$, and let Δ_π be the adjacency operator on the Schreier graph Γ_π from Theorem 3.7.3, for $\pi \in \mathrm{Epi}(B, A)$ occurring infinitely often in ω. The space ℓ^2_π can be decomposed as $\Phi \oplus \Phi^\perp$, with Φ being the subspace spanned by the functions $\Delta^n_\pi \varphi_i$, for $n \geq 0$ and $i \in X$. $\Phi^\perp$ is the direct sum of the eigenspaces of Δ_π. The spectrum of $\Delta_\pi|_\Phi$ is singular continuous and the spectrum of $\Delta_\pi|_{\Phi^\perp}$ is pure point.*

Remark 5.4.19. This Theorem together with the results of Sections 3.7 and 5.3 allows to observe that the spectral type of the adjacency operator is not preserved under the Gromov-Hausdorff convergence in the space of marked graphs. Indeed, let us consider the example of Schreier graphs of the Fabrykowski-Gupta group. Using the notation from Section 3.7, we have

$$(\Gamma_\pi, p_i) = \lim_{n\to\infty} (\Gamma_{1^{\mathbb{N}}}, (d-1)^n 01^{\mathbb{N}}),$$

with $\pi \in \mathrm{Epi}(B, A)$ mapping b to a and $p_i = ((d-1)^{\mathbb{N}}, i) \in \Gamma_\pi$. The graph $\Gamma_{1^{\mathbb{N}}}$ has pure point spectrum, by Theorem 5.3.9, and the marking of the graph does not change the spectral measures. Nonetheless, Theorem 5.4.18 implies that Γ_π has a Kesten spectral measure with nontrivial singular continuous part. A possible explanation for the appearance of the singular continuous component might be the fact that Γ_π exhibits a natural nontrivial symmetry, given by the maps $(\xi, i) \mapsto (\xi, j)$, for any $i, j \in X$. On the other hand, no marking of $\Gamma_{1^{\mathbb{N}}}$ yields a similar type of symmetry.

Chapter 6

Complexity

Linear subshifts constitute a remarkable subject of study in the area of symbolic dynamics, and spinal groups have previously been related to their Schreier graphs [38, 63, 39], although only for the binary case. In this chapter we review some notions of low complexity related to linear subshifts and extend them to the context of Schreier dynamical systems. We moreover characterize when they are satisfied by the dynamical systems arising from spinal groups.

6.1 Linear subshifts

Let $\mathcal{A}$ be a finite alphabet, $\mathcal{A}^* = \cup_{n\in\mathbb{N}}\mathcal{A}^n$ the free monoid of finite words on $\mathcal{A}$, and let $\mathcal{A}^{\mathbb{Z}}$ be the set of all two-ended sequences of symbols in $\mathcal{A}$, equipped with the product topology. For $\omega \in \mathcal{A}^{\mathbb{Z}}$ and $n \in \mathbb{N}$, we call $W_n(\omega) = \{\omega_i \dots \omega_{i+n-1} \mid i \in \mathbb{Z}\} \subset \mathcal{A}^n$ the set of all possible subwords of ω of length n, and we write $W(\omega) = \cup_{n\in\mathbb{N}} W_n(\omega) \subset \mathcal{A}^*$. If $u \in \mathcal{A}^*$ or $u \in \mathcal{A}^{\mathbb{N}}$, we also denote by $W_n(u)$ the set of all subwords of u of length n and $W(u) = \cup_{n\in\mathbb{N}} W_n(u) \subset \mathcal{A}^*$.

Consider the shift operator $T : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$, given by $(T\omega)_n = \omega_{n+1}$. A *subshift* is a pair (Ω, T), where $\Omega \subset \mathcal{A}^{\mathbb{Z}}$ is closed and invariant under T. For $n \in \mathbb{N}$, we set $W_n = W_n(\Omega) = \cup_{\omega\in\Omega} W_n(\omega) \subset \mathcal{A}^n$, the set of all subwords in Ω of length n. The shift T defines an action of $\mathbb{Z}$ on Ω. We call a subshift *minimal* if this action is minimal, i.e., if all orbits are dense. There exist various measures of complexity of minimal subshifts, let us introduce two of them.

Definition 6.1.1. We say that a subshift (Ω, T) is *linearly repetitive* if

$$\exists C \geq 1,\ \forall n \geq 0,\ \forall u \in W_n,\ \forall w \in W_{Cn},\ u \in W_n(w).$$

In other words, if there is a constant $C \geq 1$ for which every subword of length n occurs in every subword of length Cn.

Linear repetitivity is a strong form of minimality, and it has been widely studied (see [17], [25] and [50]).

Definition 6.1.2. We say that a subshift (Ω, T) satisfies the *Boshernitzan condition* (B) if there exists a T-invariant ergodic probability measure ν on Ω such that

$$\limsup_{n\to\infty} n\varepsilon(n) > 0,$$

where $\varepsilon(n) = \min\{\nu(C(u)) \mid u \in W_n\}$ and $C(u) = \{\omega \in \Omega \mid \omega_0 \dots \omega_{n-1} = u\}$. Here $C(u)$ is the cylinder of u, i.e., the set of sequences having u at the positions 0 to $n-1$, and $\varepsilon(n)$ is the minimum measure of these sets over all subwords u. This value represents the minimal probability among all the possible subwords of length n.

This condition was introduced by Boshernitzan in [13], and it was shown to imply unique ergodicity for minimal subshifts in [14].

Toeplitz subshifts constitute a remarkable family of subshifts which provides examples of aperiodic subshifts but yet is simple enough to be studied in detail. A typical example of the so-called *simple Toeplitz subshifts* is the substitutional subshift which encodes the Schreier graphs of Grigorchuk's group. It was introduced in [55] and has been studied in a slightly different form in [38] and [37]. Some very explicit results about combinatorics and complexity of simple Toeplitz subshifts are proven in [63].

Definition 6.1.3. Let $(a_k)_k \in \mathcal{A}^{\mathbb{N}}$ be a sequence of letters and $(n_k)_k \in \mathbb{N}^{\mathbb{N}}$ such that $n_k \geq 2$ for all $k \in \mathbb{N}$. We define $p^{(0)} \in \mathcal{A}^*$ as the empty word and, recursively, for $k \geq 0$,

$$p^{(k+1)} = p^{(k)} a_k p^{(k)} \dots a_k p^{(k)},$$

where the word $p^{(k)}$ appears n_k times and the letter a_k appears $n_k - 1$ times. Because $p^{(k)}$ is a prefix of $p^{(k+1)}$ for every $k \geq 0$, the sequence $(p^{(k)})_k$ converges to a one-ended infinite word $p^{(\infty)} \in \mathcal{A}^{\mathbb{N}}$. The *simple Toeplitz subshift* associated with the coding sequences $(a_k)_k$ and $(n_k)_k$ is

$$\Omega = \{\omega \in \mathcal{A}^{\mathbb{Z}} \mid W(\omega) \subset W(p^{(\infty)})\},$$

i.e., the set of all two-ended words whose subwords all occur as subwords of $p^{(\infty)}$.

There is an equivalent definition of simple Toeplitz sequences via a hole-filling procedure, which can be found in [63]. By construction, we can verify that for any $\omega \in \Omega$ in a simple Toeplitz subshift and for any $i \in \mathbb{Z}$, there exists $n \in \mathbb{Z}$ such that ω_{i+nk} are all equal, for all $k \in \mathbb{Z}$.

Proposition 6.1.4 ([52]). *Simple Toeplitz subshifts are minimal and uniquely ergodic.*

We bring our attention to the simple Toeplitz subshifts defined by $(a_k)_k \in \mathcal{A}^{\mathbb{N}}$ and $(n_k)_k \in \mathbb{N}^{\mathbb{N}}$ such that $n_k = 2$ for every $k \in \mathbb{N}$. In this case, for $k \geq 0$ we have

$$p^{(k+1)} = p^{(k)} a_k p^{(k)}.$$

By a result in [63], we can characterize when simple Toeplitz subshifts are linearly repetitive or satisfy (B). Even though this characterization is given in general, here we will consider only the aforementioned subfamily of simple Toeplitz subshifts with $n_k = 2$ for every $k \geq 0$. Let $\tilde{\mathcal{A}}$ be the eventual alphabet of $(a_k)_k$, i.e., the set of letters which occur infinitely often in $(a_k)_k$. Let also $m_k \geq 1$ be the smallest m for which $\{a_k, \ldots, a_{k+m-1}\} = \tilde{\mathcal{A}}$, for every $k \in \mathbb{N}$.

Proposition 6.1.5 ([63]). *Let $(a_k)_k \in \mathcal{A}^{\mathbb{N}}$ and $(n_k)_k \in \mathbb{N}^{\mathbb{N}}$ such that $n_k = 2$ for every $k \in \mathbb{N}$ and consider the simple Toeplitz subshift (Ω, T) they define. Then*

- (Ω, T) *is linearly repetitive if and only if* $(m_k)_k$ *is bounded.*
- (Ω, T) *satisfies* (B) *if and only if* $(m_k)_k$ *has a bounded subsequence.*

As illustrated by this result, in the context of simple Toeplitz subshifts, the Boshernitzan condition (B) is a weaker analog of linear repetitivity.

Example 6.1.6. The Schreier graphs associated with Grigorchuk's group ($d = 2$, $m = 2$) can be encoded as a simple Toeplitz subshift, which can also be obtained via a substitution. Let $\mathcal{A} = \{a, x, y, z\}$ and $(a_k)_k \in \mathcal{A}^{\mathbb{N}}$ defined as $a_0 = a$, $a_k = x, y, z$ whenever k is congruent with $1, 2, 3$ modulo 3, respectively, for every $k \geq 1$. Let (Ω, T) be the simple Toeplitz subshift associated with this sequence and $(n_k)_k$, with $n_k = 2$ for every $k \geq 0$. In [38], this subshift was used in order to study spectral properties on the Schreier graphs of Grigorchuk's group. We can alternatively construct this subshift with the following substitution on $\mathcal{A}$:

$$\tau(a) = axa, \quad \tau(x) = y, \quad \tau(y) = z, \quad \tau(z) = x.$$

The sequence $(\tau^k(a))_k$ converges to a one-ended infinite word $\eta \in \mathcal{A}^{\mathbb{N}}$. The set of all two-ended words $\omega \in \mathcal{A}^{\mathbb{Z}}$ whose subwords are subwords of η is equal to Ω. The resulting simple Toeplitz subshift (Ω, T) is minimal, uniquely ergodic, linearly repetitive and satisfies (B).

It is moreover shown in [39] that Schreier graphs of all spinal groups with $d = 2$ can be encoded by simple Toeplitz subshifts, however only few of them are substitutional.

6.2 Schreier dynamical systems

Following the similarities found between linear subshifts and Schreier graphs of spinal groups acting on the binary tree, it is our goal now to extend the notions studied for linear subshifts to more general dynamical systems arising from Schreier graphs, not necessarily linear.

Let G be a group generated by a finite set S, and let G act on a topological space $\mathcal{X}$ by homeomorphisms. Recall from Section 3.7 that $\mathrm{Sch} : \mathcal{X} \to \mathcal{G}_{*,S}$ is the map assigning to each point $x \in \mathcal{X}$ its marked Schreier graph (Γ_x, x), and let $\mathcal{X}_0 \subset \mathcal{X}$ be the set of continuity points of Sch. We defined the space of Schreier graphs by

$$\mathcal{G}_{G,\mathcal{X}} = \overline{\mathrm{Sch}(\mathcal{X}_0)},$$

with G acting on $\mathcal{G}_{G,\mathcal{X}}$ by shifting the marked vertex. Recall also that in the space of marked graphs $\mathcal{G}_{*,S}$ we consider the topology of local convergence, a basis of which is given by the cylinder sets

$$C_{(\Gamma,x)}(r) = \{(\Gamma', x') \in \mathcal{G}_{*,S} \mid \mathcal{B}_{(\Gamma,x)}(r) \cong \mathcal{B}_{(\Gamma',x')}(r)\},$$

for given $(\Gamma, x) \in \mathcal{G}_{*,S}$ and $r \geq 0$.

Definition 6.2.1. We call the pair $(\mathcal{G}_{G,\mathcal{X}}, G)$ a *Schreier dynamical system.*

Remark 6.2.2. Schreier dynamical systems are similar to subshifts in the sense that

$$\mathcal{G}_{G,\mathcal{X}} = \{(\Gamma, x) \in \mathcal{G}_{*,S} \mid \forall r \geq 0,\ \exists x' \in \mathcal{X}_0,\ (\Gamma, x) \in C_{(\Gamma_{x'},x')}(r)\}.$$

Recall that $(\mathcal{G}_{G,\mathcal{X}}, G)$ is *minimal* if the action of G on $\mathcal{G}_{G,\mathcal{X}}$ is minimal, i.e., if all orbits are dense. Equivalently, if there is an isomorphic copy of any ball in the orbit of any graph in Ω. Formally, if

$$\forall r \geq 0,\ \forall (\Gamma, x), (\Gamma', x') \in \mathcal{G}_{G,\mathcal{X}},\ \exists g \in G \ \text{ s.t. } g(\Gamma', x') \in C_{(\Gamma,x)}(r).$$

Proposition 6.2.3. *Any Schreier dynamical system $(\mathcal{G}_{G,\mathcal{X}}, G)$ is minimal.*

Proof. By continuity, it suffices to show that, for every $x, x' \in \mathcal{X}_0$ and $r \geq 0$, there exists $g \in G$ such that $(\Gamma_{x'}, gx') \in C_{(\Gamma_x,x)}(r)$. Let $U = \mathrm{Sch}^{-1}(C_{(\Gamma_x,x)}(r)) \cap \mathcal{X}_0$, which is open since cylinders are open and Sch is continuous on $\mathcal{X}_0$. We have $x \in U$, because $(\Gamma_x, x) \in C_{(\Gamma_x,x)}(r)$. Since the action of G on $\mathcal{X}$ is minimal, there exists $g \in G$ such that $gx' \in U$. Hence, $(\Gamma_{x'}, gx') \in C_{(\Gamma_x,x)}(r)$. □

Let us now establish some notions of complexity for Schreier dynamical systems, which are inspired by the analogy with linear subshifts.

Definition 6.2.4. We say that $(\mathcal{G}_{G,\mathcal{X}}, G)$ is *linearly repetitive* if there is an isomorphic copy of any ball at a linear distance from the marked vertex of any graph in $\mathcal{G}_{G,\mathcal{X}}$. Formally, if

$$\exists C \geq 0 \text{ s.t. } \forall r \geq 0,\ \forall (\Gamma, x), (\Gamma', x') \in \mathcal{G}_{G,\mathcal{X}},$$

$$\exists g \in G,\ |g| \leq Cr, \text{ s.t. } g(\Gamma', x') \in C_{(\Gamma,x)}(r).$$

Notice that this definition is a strengthening of minimality. If we compare both conditions, for linear repetitivity, we need g not only to exist but also to have length bounded linearly on r.

Definition 6.2.5. We say that $(\mathcal{G}_{G,\mathcal{X}}, G)$ satisfies the *Boshernitzan condition* (B) if there is a G-invariant, ergodic probability measure ν on $\mathcal{G}_{G,\mathcal{X}}$ such that

$$\limsup_{r\to\infty} r\varepsilon(r) > 0$$

where $\varepsilon(r) = \min\{\nu(C_{(\Gamma,x)}(r)) \mid (\Gamma, x) \in \mathcal{G}_{G,\mathcal{X}}\}$.

Notice that the set $D(r) = \{C_{(\Gamma,x)}(r) \mid (\Gamma, x) \in \mathcal{G}_{G,\mathcal{X}}\}$ is finite, so $\varepsilon(r)$ is well defined as a minimum. Let $d_r = |D(r)|$ be the number of possible r-balls that occur in $\mathcal{G}_{G,\mathcal{X}}$. Notice also that because of minimality, all r-balls are found in any orbit. Since ν is an invariant measure, $\nu(C_{(\Gamma,x)}(r)) = 1/d_r$ for any $(\Gamma, x) \in \mathcal{G}_{G,\mathcal{X}}$. We can hence rewrite the Boshernitzan condition as

$$\limsup_{r\to\infty} \frac{r}{d_r} > 0.$$

Definition 6.2.6. We say that $(\mathcal{G}_{G,\mathcal{X}}, G)$ satisfies the condition (B') if it satisfies the same condition as for linear repetitivity but, instead of for every $r \geq 0$, for an increasing sequence going to infinity. Formally, if

$$\exists C \geq 0,\ \exists (r_n)_n \nearrow \infty \text{ s.t. } \forall n \geq 0,\ \forall (\Gamma, x), (\Gamma', x') \in \mathcal{G}_{G,\mathcal{X}},$$

$$\exists g \in G,\ |g| \leq Cr_n, \text{ s.t. } g(\Gamma', x') \in C_{(\Gamma,x)}(r_n).$$

Notice that linear repetitivity implies (B').

Remark 6.2.7. We may now consider any minimal linear subshift (Ω, T) over an alphabet $\mathcal{A}$ as a Schreier dynamical system. Let $G = \langle t_a \mid a \in \mathcal{A}\rangle$ be the free product of $|\mathcal{A}|$ copies of $\mathbb{Z}$. The group G acts on Ω as follows

$$t_a\omega = \begin{cases} T\omega & a = \omega_0 \\ \omega & a \neq \omega_0 \end{cases}.$$

Under this action, each $\omega \in \Omega$ is stabilized by all t_a but t_{ω_0}. Assuming the subshift (Ω, T) is aperiodic, the associated Schreier graphs are as linear as in Figure 6.1. We can

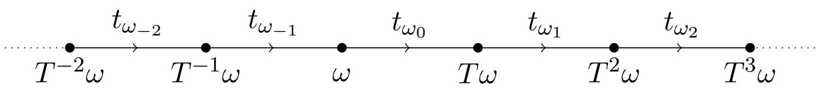


Figure 6.1: Schreier graph of the action of G on Ω, for the orbit of a point $\omega \in \Omega$. Loops are omitted.

verify that the dynamical system given by (Ω, T) is equivalent to the Schreier dynamical system $(\mathcal{G}_{G,\Omega}, G)$.

In this setting, the conditions for linear repetitivity and (B) for $(\mathcal{G}_{G,\Omega}, G)$ coincide with the classical ones for the linear subshift (Ω, T).

Extending the notion of simple Toeplitz is more involved. Its definition involves the concatenation of words, which should be interpreted and redefined for graphs. Additionally, the group acting on a linear subshift is always $\mathbb{Z}$, which is no longer the case for Schreier dynamical systems. We now extend the notion of concatenation to graphs. Let $\mathcal{G}_{d*,\Sigma}$ denote the space of graphs with edge labels in an alphabet Σ and d marked vertices, for $d \geq 1$.

Definition 6.2.8. Let Σ be a finite alphabet and $d_1, d_2 \geq 1$. Let $\Gamma_1 \in \mathcal{G}_{d_1*,\Sigma}$ and $\Gamma_2 \in \mathcal{G}_{d_2*,\Sigma}$, with marked vertices $(v_0, \ldots, v_{d_1-1})$ and $(w_0, \ldots, w_{d_2-1})$, respectively. The *concatenation* of Γ_1 and Γ_2 at the vertices v_i and w_j, for $0 \leq i \leq d_1 - 1$ and $0 \leq j \leq d_2 - 1$, is the graph

$$\Gamma_1 \stackrel{v_i|w_j}{\text{---}} \Gamma_2 = (\Gamma_1 \sqcup \Gamma_2)/\{v_i = w_j\} \in \mathcal{G}_{(d_1+d_2-2)*,\Sigma},$$

whose marked vertices are $(v_0, \ldots, \hat{v}_i, \ldots, v_{d_1-1}, w_0, \ldots, \hat{w}_j, \ldots, w_{d_2-1})$.

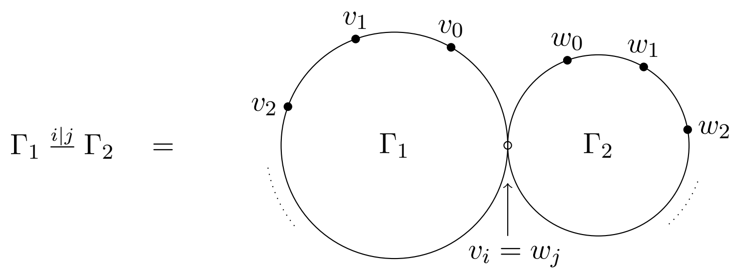


Figure 6.2: Concatenation of Γ_1 and Γ_2 at v_i and w_j.

This definition of graph concatenation will allow us to extend the notion of simple Toeplitz subshifts to Schreier dynamical systems.

Definition 6.2.9. Let $d \geq 2$ and let S be a finite, symmetric, generating set of a group G. Let $\Xi \in \mathcal{G}_{1*,S}$ be a finite graph, marked at the vertex ξ, and let $\mathcal{A} \subset \mathcal{G}_{d*,S}$ be a finite alphabet of finite graphs, whose elements will be called letters. Let $(\Lambda_k)_k \in \mathcal{A}^{\mathbb{N}}$ be a sequence of letters. We recursively define $\Gamma^{(k)} \in \mathcal{G}_{2*,S}$ as follows:

- $\Gamma^{(0)}$ is the graph with one vertex and no edges. We consider it marked twice at that vertex.

- For $k \geq 0$, let $\Gamma_0, \ldots, \Gamma_{d-1}$ be d disjoint copies of $\Gamma^{(k)} \in \mathcal{G}_{2*,S}$, with each Γ_i marked at the vertices (γ_i, γ_i'). Let also $\Xi_1, \ldots, \Xi_{d-2}$ be $d-2$ disjoint copies of the graph $\Xi \in \mathcal{G}_{1*,S}$, with each Ξ_i marked at the vertex ξ_i. We set

$$\tilde{\Gamma}_i = \begin{cases} \Gamma_i \xrightarrow{\gamma_i'|\xi_i} \Xi_i & 1 \leq i \leq d-2 \\ \Gamma_i & i = 0, d-1 \end{cases}.$$

  Let $(\lambda_0, \ldots, \lambda_{d-1})$ be the marked vertices of Λ_k. We define $\Gamma^{(k+1)} \in \mathcal{G}_{2*,S}$ as

$$\Gamma^{(k+1)} = \Lambda_k \xrightarrow{\lambda_0|\gamma_0} \tilde{\Gamma}_0 \xrightarrow{\lambda_1|\gamma_1} \tilde{\Gamma}_1 \xrightarrow{\lambda_2|\gamma_2} \ldots \xrightarrow{\lambda_{d-1}|\gamma_{d-1}} \tilde{\Gamma}_{d-1},$$

  with marked vertices $(\gamma_0', \gamma_{d-1}')$.

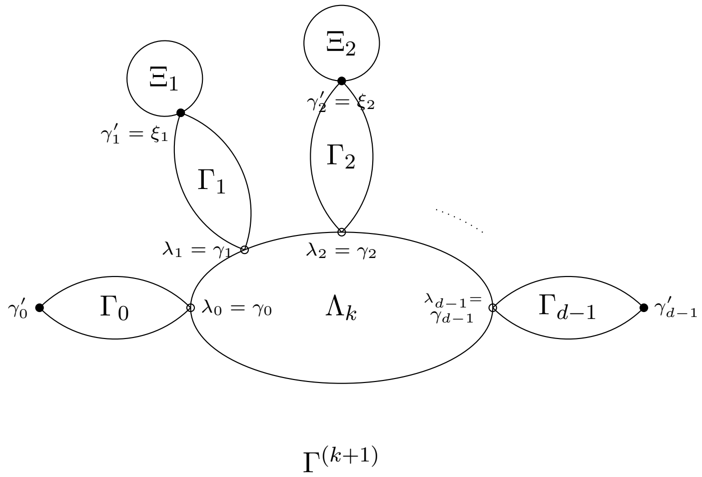


Figure 6.3: Construction of the graph $\Gamma^{(k+1)}$.

Notice that this construction is well defined, since $\Gamma^{(k)} \in \mathcal{G}_{2*,S}$ for every $k \geq 0$. Indeed, this is true by definition for $k = 0$. For $k \geq 1$, the graphs $\tilde{\Gamma}_i$ belong to $\mathcal{G}_{1*,S}$ for $1 \leq i \leq d-2$, and to $\mathcal{G}_{2*,S}$ if $i = 0, d-1$. We concatenate a graph with d marked vertices with $d-2$ graphs with one marked vertex and 2 graphs with two marked vertices, so the resulting graph has two marked vertices.

Now, once the sequence $(\Gamma^{(k)})_k$ of finite graphs is defined, let (v_k, v_k') be the marked vertices of $\Gamma^{(k)} \in \mathcal{G}_{2*,S}$. For any $r \geq 0$, the ball $\mathcal{B}_{\Gamma^{(k)}, v_k'}(r)$ will be isomorphic for k big enough. Hence, we have a sequence of marked graphs $(\Gamma^{(k)}, v_k') \in \mathcal{G}_{1*,S}$ which converges to a marked graph $(\tilde{\Gamma}^{(\infty)}, v_\infty) \in \mathcal{G}_{1*,S}$ in the local convergence topology. Let $\hat{\Gamma}_0, \ldots, \hat{\Gamma}_{d-1}$

be d disjoint copies of $\tilde{\Gamma}^{(\infty)}$, marked at the vertices $v_0, \dots, v_{d-1}$, and let $\Lambda \in \mathcal{A}$ occurring infinitely often in $(\Lambda_k)_k$, marked at the vertices $\lambda_0, \dots, \lambda_{d-1}$. We finally define

$$\Gamma^{(\infty)} = \Lambda \xrightarrow{\lambda_0|v_0} \hat{\Gamma}_0 \xrightarrow{\lambda_1|v_1} \hat{\Gamma}_1 \xrightarrow{\lambda_2|v_2} \dots \xrightarrow{\lambda_{d-1}|v_{d-1}} \hat{\Gamma}_{d-1}.$$

Notice that not every graph constructed in such a way is necessarily a Schreier graph. Whenever it is, we will call it a *simple Toeplitz graph*. The coding information of a simple Toeplitz graph is the tuple $(d, \Xi, (\Lambda_k)_k)$. We set

$$\mathcal{G}_{\Gamma^{(\infty)}} = \{(\Gamma, x) \in \mathcal{G}_{*,S} \mid \forall r \geq 0,\ \exists v \in \Gamma^{(\infty)},\ (\Gamma, x) \in C_{(\Gamma^{(\infty)}, v)}(r)\}.$$

A *simple Toeplitz Schreier dynamical system* is a Schreier dynamical system $(\mathcal{G}_{G,\mathcal{X}}, G)$ such that $\mathcal{G}_{G,\mathcal{X}} = \mathcal{G}_{\Gamma^{(\infty)}}$ for some simple Toeplitz graph $\Gamma^{(\infty)}$.

Remark 6.2.10. Let (Ω, T) be a simple Toeplitz subshift defined by an alphabet $\mathcal{A}$ and the sequences $(a_k)_k \in \mathcal{A}^{\mathbb{N}}$ and $(n_k)_k \in \mathbb{N}^{\mathbb{N}}$, with $n_k = 2$ for every $k \geq 0$. Suppose that $a_k \neq a_0$ for every $k \geq 1$. We may recover this subshift as a simple Toeplitz Schreier dynamical system by setting $d = 2$ as follows. For $a_0 \in \mathcal{A}$, we set Λ_{a_0} to be the graph with two vertices u and v and an undirected a_0-edge between them, marked at (u, v). For $a \in \mathcal{A} \setminus \{a_0\}$, we set Λ_a to be the graph with two vertices u and v and an undirected a-edge between them, plus a'-loops on u and v for every $a' \in \mathcal{A} \setminus \{a_0, a\}$, marked at (u, v). Let $\mathcal{A}' = \{\Lambda_a \mid a \in \mathcal{A}\} \subset \mathcal{G}_{2*,\mathcal{A}}$. The sequence $(\Lambda_{a_k})_k \in \mathcal{A}'^{\mathbb{N}}$ defines a simple Toeplitz graph $\Gamma^{(\infty)}$. If we let $G = \langle t_a \mid t_a^2 = 1, \forall a \in \mathcal{A} \rangle$ be the free product of $|\mathcal{A}|$ copies of $\mathbb{Z}/2\mathbb{Z}$, acting on Ω as

$$t_a \omega = \begin{cases} T\omega & a = \omega_0 \\ T^{-1}\omega & a = \omega_{-1} \\ \omega & \text{otherwise} \end{cases},$$

then $\mathcal{G}_{\Gamma^{(\infty)}} = \mathcal{G}_{G,\Omega}$, and the orbits are the same as for (Ω, T). Notice that for $d = 2$ the graph Ξ does not play any role in the construction of $\Gamma^{(\infty)}$.

Proposition 6.2.11. *Let G_ω be a spinal group with $d \geq 2$, $m \geq 1$ and $\omega \in \Omega_{d,m}$. The Schreier dynamical system $(\mathcal{G}_{G_\omega, X^{\mathbb{N}}}, G_\omega)$ is simple Toeplitz.*

Proof. Let Ξ, Θ and Λ_π, for $\pi \in \mathrm{Epi}(B, A)$, be as in Proposition 3.1.3. We can verify that the construction from Definition 6.2.9 with parameters $(d, \Xi, (\Lambda_k)_k)$ given by $\Lambda_0 = \Theta$ and $\Lambda_k = \Lambda_{\omega_{k-1}}$, for $k \geq 1$, yields the same space of graphs as $\mathcal{G}_{G_\omega, X^{\mathbb{N}}}$. □

In Proposition 6.1.5, a characterization for linear repetitivity and (B) for linear simple Toeplitz subshifts was provided. We now want to give similar statements for the simple Toeplitz Schreier dynamical systems defined by spinal groups.

Let $(\mathcal{G}_{G,\mathcal{X}}, G)$ be a simple Toeplitz Schreier dynamical system, defined by $(d, \Xi, (\Lambda_k)_k)$. We define a sequence of integers $(m_k)_k$ as follows. For every $k \geq 0$, m_k is the smallest integer such that $\{\Lambda_k, \ldots, \Lambda_{k+m_k-1}\} = \{\Lambda_n\}_{n\geq k}$. For the Schreier dynamical system $(\mathcal{G}_{G_\omega,X^{\mathbb{N}}}, \mathcal{G}_\omega)$ associated with a spinal group G_ω, for every $k \geq 0$, m_k is the smallest integer such that $\{\omega_k, \ldots, \omega_{k+m_k-1}\} = \{\omega_n\}_{n\geq k}$.

Theorem 6.2.12. *Let G_ω be a spinal group with $d \geq 2$, $m \geq 1$ and $\omega \in \Omega_{d,m}$, and let $(\mathcal{G}_{G_\omega,X^{\mathbb{N}}}, G_\omega)$ be its associated Schreier dynamical system. Then $(\mathcal{G}_{G_\omega,X^{\mathbb{N}}}, G_\omega)$ is linearly repetitive if and only if $(m_k)_k$ is bounded.*

Proof. Suppose that $m_k \leq M$ for every $k \geq 0$. Let $r \geq 0$. As linear repetitivity is a local condition and the action is minimal, it suffices to consider two points $\xi, \xi' \in \Gamma_\xi$, for $\xi \in X^{\mathbb{N}} \setminus \mathrm{Cof}((d-1)^{\mathbb{N}})$. We have to find $g \in G_\omega$, of length not greater than Cr, such that the balls $\mathcal{B} = \mathcal{B}_{(\Gamma_\xi,\xi)}(r)$ and $\mathcal{B}_{(\Gamma_\xi,g\xi')}(r)$ are isomorphic, for a constant C independent of ξ, ξ' and r.

Let us consider the foldings $\varphi_k : \Gamma_\xi \to \Gamma_k$, for every $k \geq 0$, given by $\eta \mapsto \eta_0 \ldots \eta_{d-1}$, and let $R \geq 0$ be the smallest such that φ_R is injective on $\mathcal{B}$. We will assume $R \geq 2$, as the possibilities for smaller R are not many and can be checked directly or absorbed into C.

Recall that $\Gamma_\xi^R = X^R\sigma^R(\xi)$ is the copy of Γ_R which contains ξ. By definition of R, it contains $\mathcal{B}$. Moreover, we set $\mathcal{B}_i = \mathcal{B} \cap \varphi_R^{-1}(X^{R-1}i)$, so that $\mathcal{B} = \cup_{i=0}^{d-1}\mathcal{B}_i$. Let s_i be the smallest such that φ_{s_i} is injective on $\mathcal{B}_i$, and set $s = \max_{i\in X} s_i$. Notice that $s_i \leq s \leq R-1$.

By minimality of R, $\varphi_R(\mathcal{B})$ contains the central piece $\Lambda_{\omega_{R-2}}$ of Γ_R, and by definition of s we have $\mathrm{diam}(\varphi_R(\mathcal{B})) \leq 2\,\mathrm{diam}(\Gamma_s) + 1$. In addition, by minimality of s and s_i, we find that $\mathrm{diam}(\varphi_R(\mathcal{B})) \geq \mathrm{diam}(\Gamma_{s-1}) + 2$. We then have

$$\mathrm{diam}(\Gamma_{s-1}) + 2 \leq 2r \leq 2\,\mathrm{diam}(\Gamma_s) + 1,$$

$$2^{s-1} \leq 2r \leq 2^{s+1} - 1.$$

Let us assume first that $s \leq R-2$. In this case, $\varphi_R(\mathcal{B})$ is contained in the subgraph $X^s(d-1)^{R-2-s}0X$ of Γ_R, and so $\mathcal{B}$ and $\varphi_R(\mathcal{B})$ are isomorphic. By assumption, there exists some t, with $s \leq t \leq s+M-1$, such that $\omega_t = \omega_{R-2}$. By choosing t minimal, we may assume $s \leq t \leq R-2$. We define the following injective map

$$\begin{array}{rccc} \psi: & X^s(d-1)^{R-2-s}0X & \to & \Gamma_{t+2} \\ & w(d-1)^{R-2-s}0i & \mapsto & w(d-1)^{t-s}0i \end{array}.$$

Notice that since $\omega_t = \omega_{R-2}$, ψ preserves the edges of the graphs, and so it is in fact an isomorphism with its image. In particular, since $\mathcal{B}$ is isomorphic to $\varphi_R(\mathcal{B})$, it is also isomorphic to $\psi(\varphi_R(\mathcal{B}))$.

Define now $g \in G_\omega$ to be the labeling of a shortest path in Γ_{t+2} from $\varphi_{t+2}(\xi')$ to $\psi(\varphi_R(\xi))$, and set $\mathcal{B}' = \mathcal{B}_{(\Gamma_\xi, g\xi')}(r)$. We have

$$\varphi_{t+2}(\mathcal{B}') = \mathcal{B}_{(\Gamma_\xi, \varphi_{t+2}(g\xi'))}(r) = \mathcal{B}_{(\Gamma_\xi, g\varphi_{t+2}(\xi'))}(r) =$$

$$= \mathcal{B}_{(\Gamma_\xi, \psi(\varphi_R(\xi)))}(r) \cong \psi(\varphi_R(\mathcal{B})) \cong \mathcal{B}.$$

This implies that $\varphi_{t+2}(\mathcal{B}')$ is contained in the subgraph $X^t 0 X$ of Γ_{t+2}, and so that $\mathcal{B}'$ and $\varphi_{t+2}(\mathcal{B}')$ are isomorphic. Hence, $\mathcal{B} \cong \mathcal{B}'$. Moreover,

$$|g| \leq \operatorname{diam}(\Gamma_{t+2}) = 2^{t+2} - 1 \leq 2^{s+M+1} - 1 \leq 2^{M+3} r.$$

Assume now $s = R - 1$. For every $g \in G_\omega$ such that $g\varphi_R(\xi') = \varphi_R(\xi)$ in Γ_R, we set $\mathcal{B}' = \mathcal{B}_{(\Gamma_\xi, g\xi')}(r)$, and we have

$$\varphi_R(\mathcal{B}') = \mathcal{B}_{(\Gamma_\xi, \varphi_R(g\xi'))}(r) = \mathcal{B}_{(\Gamma_\xi, g\varphi_R(\xi'))}(r) =$$

$$= \mathcal{B}_{(\Gamma_\xi, \varphi_R(\xi))}(r) = \varphi_R(\mathcal{B}).$$

Notice also that φ_R must also be injective on $\mathcal{B}'$, as otherwise we would have $2r = \operatorname{diam}(\mathcal{B}') > \operatorname{diam}(\varphi_R(\mathcal{B}')) = \operatorname{diam}(\varphi_R(\mathcal{B})) = 2r$. Let u be the vertex $(d-1)^R$ in Γ_R. We have either $u \in \varphi_R(\mathcal{B})$ or $u \notin \varphi_R(\mathcal{B})$.

If $u \notin \varphi_R(\mathcal{B})$, then let $g \in G_\omega$ be the labeling of a shortest path from $\varphi_R(\xi')$ to $\varphi_R(\xi)$ in Γ_R. In this case, we trivially have $\mathcal{B} \cong \mathcal{B}'$, and furthermore

$$|g| \leq \operatorname{diam}(\Gamma_R) = 2^R - 1 = 2^{s+1} - 1 \leq 8r.$$

If $u \in \varphi_R(\mathcal{B})$, let v be its unique preimage in $\mathcal{B}$. We proceed differently depending on whether B fixes v or not.

Assume first that v is fixed by B, which may only happen if $d \geq 3$. Let g be the labeling of a shortest path from $\varphi_{R+1}(\xi')$ to $\varphi_R(\xi)1$ in Γ_{R+1}. Since φ_R is injective on both $\mathcal{B}$ and $\mathcal{B}'$, so is φ_{R+1}. Moreover, $\varphi_{R+1}(\mathcal{B}) \subset X^R i$ for some $i \in X$ while by definition $\varphi_{R+1}(\mathcal{B}') \subset X^R 1$. The unique preimage v' of u in $\mathcal{B}'$ is also fixed by B, which implies that $\mathcal{B} \cong \mathcal{B}'$. In addition,

$$|g| \leq \operatorname{diam}(\Gamma_{R+1}) = 2^{R+1} - 1 = 2^{s+2} - 1 \leq 16r.$$

Suppose now that v is not fixed by B, and belongs to a copy of Λ_{ω_N}, for some $N \geq R$. By assumption, there exists some t, with $R \leq t \leq R + M - 1$, such that $\omega_t = \omega_N$. We may choose t minimal so that $t \leq N$. Define $g \in G_\omega$ as the labeling of a shortest path from $\varphi_{t+1}(\xi')$ to $\varphi_R(\xi)(d-1)^{t-R}0$ in Γ_{t+1}. Again as φ_R is injective on both $\mathcal{B}$ and $\mathcal{B}'$, so is φ_{t+1}. Moreover, $\varphi_{t+1}(\mathcal{B}) \subset X^R(d-1)^{t-R+1}$ and $\varphi_{t+1}(\mathcal{B}') \subset X^R(d-1)^{t-R}0$ by definition. Notice that $\mathcal{B}$ and $\mathcal{B}'$ are isomorphic if and only if the loops at v and v' are labeled by the same generators. The former has loops labeled by generators in $\mathrm{Ker}(\omega_N)$, while the latter has loops labeled by generators in $\mathrm{Ker}(\omega_t)$. Since we chose t such that $\omega_t = \omega_N$, $\mathcal{B} \cong \mathcal{B}'$. Furthermore,

$$|g| \leq \mathrm{diam}(\Gamma_{t+1}) = 2^{t+1} - 1 \leq 2^{R+M} - 1 = 2^{s+M+1} - 1 \leq 2^{M+3} r.$$

This concludes the proof of the implication $\Leftarrow$, as in all cases we found an element $g \in G_\omega$ for which the ball of radius r around $g\xi'$ is isomorphic to the ball of radius r around ξ, and if we take $C = 2^{M+4}$, we have $|g| \leq Cr$.

Conversely, assume that the Schreier dynamical system is linearly repetitive and $(m_k)_k$ is not bounded. Let C be the linear repetitivity constant, and set M such that $C \leq 2^{M-3} - 1$. Since $(m_k)_k$ is not bounded, there exist $n \geq 0$ and $N \geq n + M$ such that $\omega_N \notin \{\omega_n, \ldots, \omega_{n+M-1}\}$. We assume moreover that N is minimal and set $r = 2^{n+2} - 1$.

Let now $\xi \in X^{\mathbb{N}} \setminus \mathrm{Cof}((d-1)^{\mathbb{N}})$ and $\xi' \in \mathrm{Cof}(\xi)$ be two vertices of Γ_ξ, such that $\varphi_{N+2}(\xi) = (d-1)^N 00 =: v$ and $\varphi_{N+2}(\xi') = (d-1)^{N-1}000 =: v'$. Because of linear repetitivity, there exists $g \in G_\omega$ with $|g| \leq Cr$ such that $\mathcal{B} := \mathcal{B}_{(\Gamma_\xi, \xi)}(r) \cong \mathcal{B}_{(\Gamma_\xi, g\xi')}(r) =: \mathcal{B}'$.

Notice that $r = \mathrm{diam}(\Gamma_{n+2}) \leq \mathrm{diam}(\Gamma_{N+1})$. As $\varphi_{N+2}(\xi)$ belongs to the subgraph $X^N 00$ of Γ_{N+2}, this implies that φ_{N+2} is injective on $\mathcal{B}$, and hence that it must be also injective on $\mathcal{B}'$. We had that $\mathcal{B} \cong \mathcal{B}'$, so in particular

$$\mathcal{B}_{(\Gamma_{N+2}, v)}(r) = \varphi_{N+2}(\mathcal{B}) \cong \varphi_{N+2}(\mathcal{B}') = \mathcal{B}_{(\Gamma_{N+2}, gv')}(r).$$

At the same time,

$$|g| + r \leq (C+1)r \leq 2^{M-3}(2^{n+2} - 1) \leq 2^{n+M-1} - 1 < 2^N - 1 = d(v, v').$$

Therefore $\varphi_{N+2}(\mathcal{B}')$ must be confined to the subgraph $X^{N+1}0$ of Γ_{N+2}. However, the central piece of Γ_{N+2}, a copy of Λ_{ω_N}, is contained in $\varphi_{N+2}(\mathcal{B})$, so its image by the isomorphism must be mapped to a copy of Λ_{ω_s} within $X^{N+1}0$, for some $s \leq N - 1$. If $n \leq s \leq N - 1$, we have $\omega_s = \omega_N$, which is a contradiction by minimality of N, and if $s < n$ then φ_{N+2} cannot be injective on $\mathcal{B}'$, which is also a contradiction. □

Theorem 6.2.13. *Let G_ω be a spinal group with $d \geq 2$, $m \geq 1$ and $\omega \in \Omega_{d,m}$, and let $(\mathcal{G}_{G_\omega, X^{\mathbb{N}}}, G_\omega)$ be its associated Schreier dynamical system. Then $(\mathcal{G}_{G_\omega, X^{\mathbb{N}}}, G_\omega)$ satisfies (B') if and only if $(m_k)_k$ has a bounded subsequence.*

Proof. The proof is analogous to the proof of Theorem 6.2.12. For the implication $\Leftarrow$, let $(m_{k_n})_n$ be the bounded subsequence of $(m_k)_k$. We call a radius $r \geq 0$ admissible if there exists some $n \geq 0$ such that $2^{k_n - 1} \leq 2r \leq 2^{k_n} - 1$, and define the sequence of radii from the definition of condition (B') as the increasing sequence of admissible radii, which is clearly unbounded.

Let r be an admissible radius, with $2^{k-1} \leq 2r \leq 2^k - 1$ with $k = k_n$ for some $n \geq 0$, and define the sequence of radii from the definition of condition (B') as the increasing sequence of admissible radii, which is clearly unbounded. We proceed as for linear repetitivity and define R and s. The inequality $2^{s-1} \leq 2r \leq 2^{s+1} + 1$ then implies $k - 1 \leq s \leq k$.

For the case $s \leq R - 2$, we precise that the t is chosen such that $k \leq t \leq k + M - 1$, instead, but we obtain the inequality $s \leq t \leq s + M$, which later allows us to conclude that $|g| \leq 2^{M+4} r$. If $s = R - 1$, we proceed again as for linear repetitivity. The inequality $k - 1 \leq R - 1 \leq k$ allows us to find the same bounds for the length of g in each case.

The proof of the converse is analogous to the proof of Theorem 6.2.12 too. Assume that the Schreier dynamical system satisfies (B'), which sets the constant C and defines the sequence of admissible radii, and let M be as above. If $(m_k)_k$ does not have a bounded subsequence, then there exists some $n \geq 0$ such that, for every $k \geq n$, $\{\omega_k, \ldots, \omega_{k+M-1}\} \neq \{\omega_l\}_{l \geq k}$. Now possibly the radius $2^{n+2} - 1$ is not admissible, but in that case we let r be an admissible radius such that $r \geq 2^{n+2} - 1$ and set n' such that $r \leq 2^{n'+2} - 1$. As $n' \geq n$, there exists some minimal $N \geq n' + M - 1$ such that $\omega_N \notin \{\omega_{n'}, \ldots, \omega_{n'+M-1}\}$. From this point on, we proceed as in the proof of Theorem 6.2.12, but writing n' in the place of n. □

Corollary 6.2.14. *Let $d \geq 2$ and $m \geq 2$, and let Ω_{LR} and $\Omega_{B'}$ be the subsets of $\Omega_{d,m}$ for which $(\mathcal{G}_{G_\omega, X^{\mathbb{N}}}, G_\omega)$ is linearly repetitive and satisfies (B'), respectively. If μ is the uniform Bernoulli measure on* $\operatorname{Epi}(B, A)^{\mathbb{N}}$*, we have*

$$\mu(\Omega_{LR}) = 0, \quad \mu(\Omega_{B'}) = 1.$$

Proof. First notice that $\omega \in \Omega_{LR}$ if and only if $(m_k)_k$ is bounded. If we set

$$\Omega_{LR}^M = \{\omega \in \Omega_{d,m} \mid \forall k \geq 0,\ \{\omega_k, \ldots, \omega_{k+M-1}\} = \{\omega_n\}_{n \geq k}\},$$

then $\Omega_{LR} = \cup_{M \geq 1} \Omega_{LR}^M$. Because Ω_{LR}^M cannot contain any sequence with constant M-prefix, we have

$$\mu\left(\Omega_{LR}^M\right) \leq \mu\left(\bigsqcup_{\substack{\omega_0 \cdots \omega_{M-1} \\ \text{not constant}}} \omega_0 \ldots \omega_{M-1} \Omega_{LR}^M\right) = \frac{E^M - E}{E^M} \mu\left(\Omega_{LR}^M\right),$$

where E denotes $|\operatorname{Epi}(B, A)|$. Hence, $\mu\left(\Omega_{LR}^M\right)$ must be zero, and as Ω_{LR} is the countable union of sets of measure zero, it must have measure zero itself.

Similarly, notice that $\omega_{B'}$ if and only if $(m_k)_k$ has a bounded subsequence. Setting

$$\Omega_{B'}^M = \{\omega \in \Omega_{d,m} \mid \forall k \geq 0,\ \exists l \geq k,\ \{\omega_l, \ldots, \omega_{l+M-1}\} = \{\omega_n\}_{n \geq l}\},$$

we have $\Omega_{B'} = \cup_{M\geq 1}\Omega_{B'}^{M}$. In this case, for $M \geq E$, we can bound the measure of $\Omega_{B'}^{M}$ from below by adding the restriction that the M-prefix contains all epimorphisms in $\mathrm{Epi}(B, A)$. Hence, we reduce the problem to finding the probability that a given M-tuple of $\mathrm{Epi}(B, A)^M$ contains all the epimorphisms. Using the union bound, we see that the probability that the tuple does not contain all of them is not greater than $(|E| - 1)\left(\frac{|E|-1}{|E|}\right)^{M-1}$, and therefore

$$\mu\left(\Omega_{B'}^{M}\right) \geq \mu\left(\bigsqcup_{\substack{\{\omega_0,\ldots,\omega_{M-1}\}=\\ =\mathrm{Epi}(B,A)}} \omega_0 \ldots \omega_{M-1}\Omega_{B'}^{M}\right) \geq$$

$$\geq \left(1 - (|E| - 1)\left(\frac{|E| - 1}{|E|}\right)^{M-1}\right)\mu\left(\Omega_{B'}^{M}\right).$$

This coefficient tends to 1 as M increases, which shows that $\mu(\Omega_{B'}) = 1$. □

Theorems 6.2.12 and 6.2.13 are intended as analogous statements of Proposition 6.1.5 for the Schreier dynamical systems arising from spinal groups. For linear repetitivity, we obtain indeed a parallel result. Nevertheless, while in Proposition 6.1.5 the weaker analog for linear simple Toeplitz subshifts is condition (B), in Theorem 6.2.13 it is condition (B'). These conditions are in fact equivalent in the context of linear simple Toeplitz subshifts. However, we do not know whether this is the case for simple Toeplitz Schreier dynamical systems in general. A reasonable starting point to solve this question would be finding out whether this is true for spinal groups.

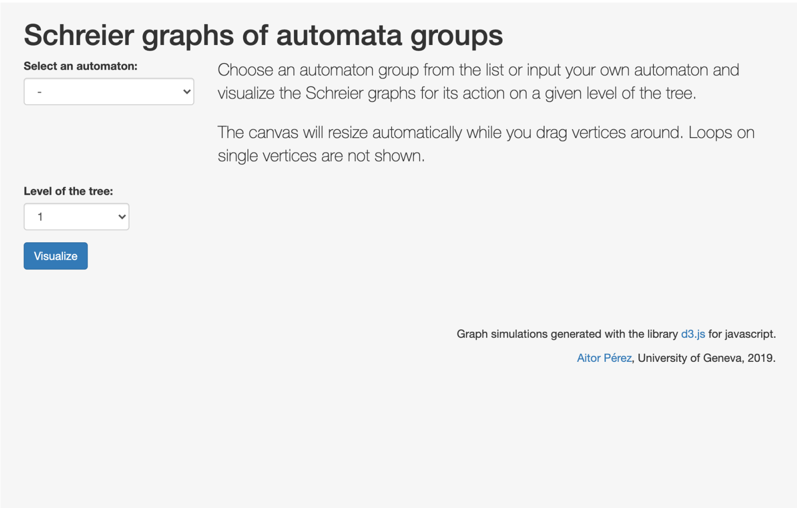
Appendix A

Visualization tool for Schreier graphs of automata groups

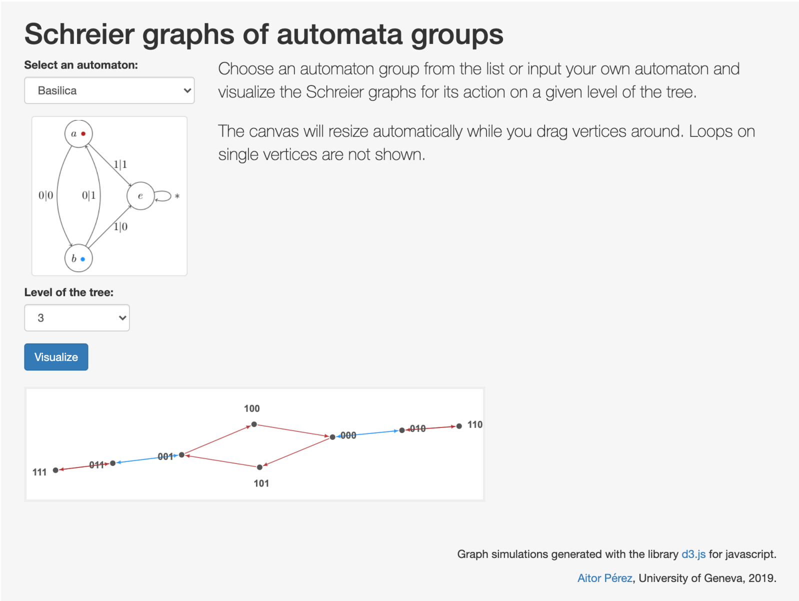
In the study of Schreier graphs of groups acting on rooted trees, the first step is always to understand those associated with the action on the finite levels of the tree. These finite graphs can be computed algorithmically with little effort. However, understanding their underlying structure is key if one wants to extrapolate to other possibly infinite Schreier graphs.

During the realization of this thesis, and with the aim of making this task simpler, an interactive graphical tool to visualize Schreier graphs of automata groups was developed. It can be found at `https://unige.ch/~perezper`.

In this appendix, we provide a brief description of the tool as well as some screenshots to illustrate it, but we actively encourage the reader to visit the URL and play around with the examples that are included.

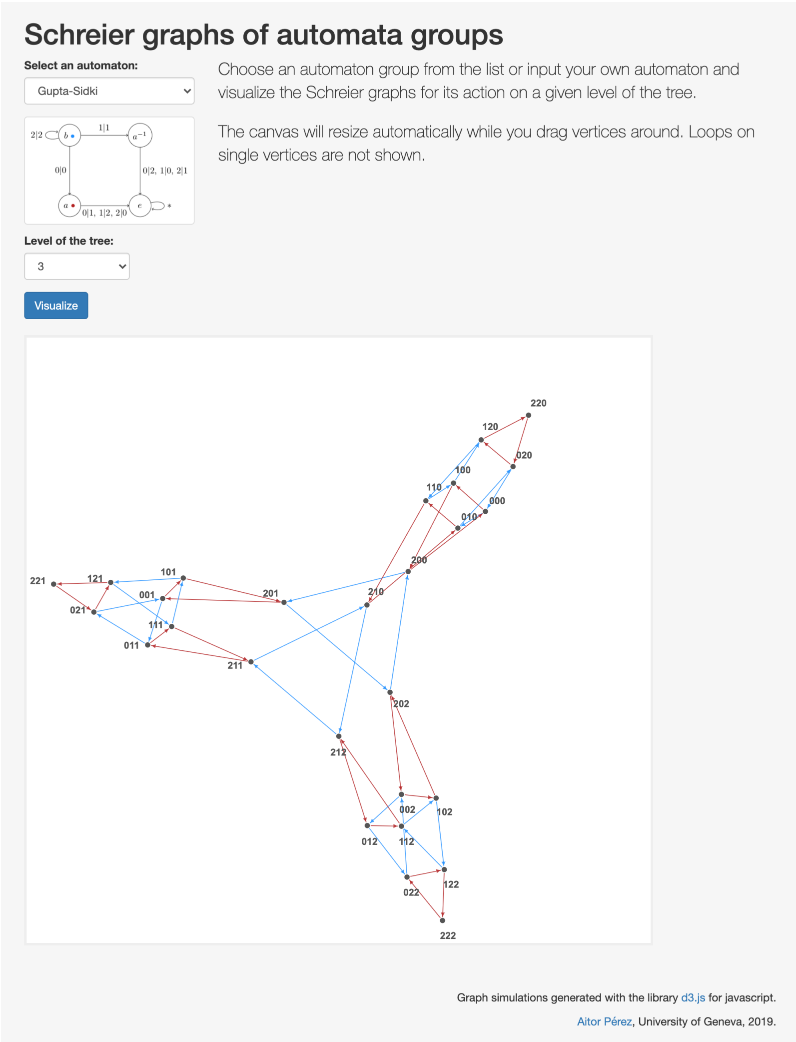


The first dropdown menu contains a list of examples of automata groups. After choosing one, and selecting a level of the tree from the second dropdown, clicking on `Visualize` will create a view where the Schreier graph of the chosen automata group with respect to the stabilizer of any vertex of the selected level will be displayed.

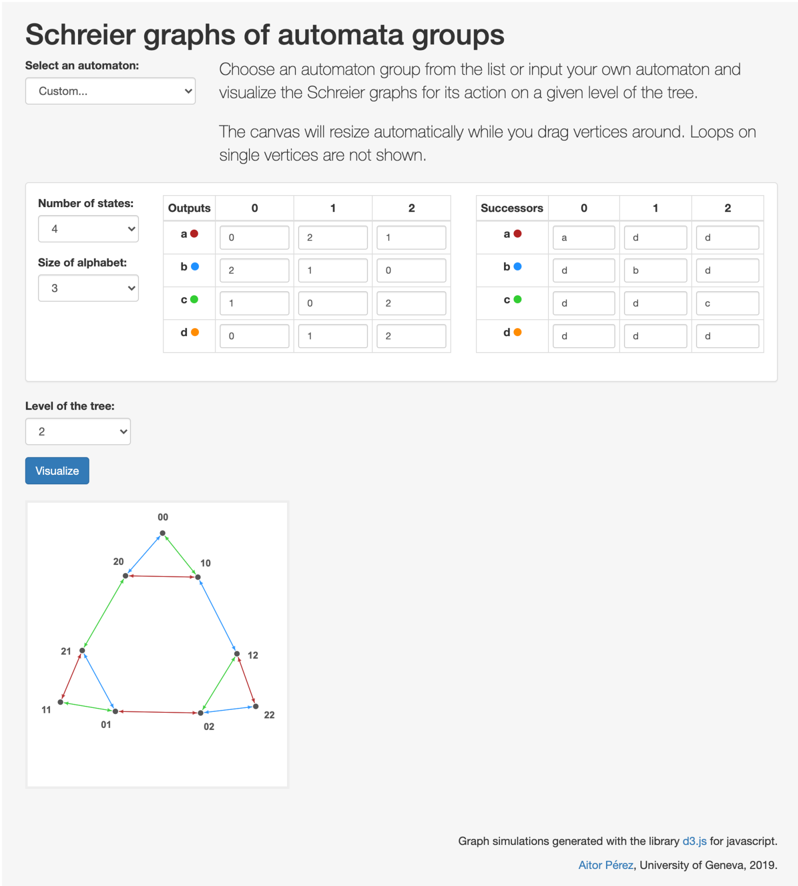


The vertices will rearrange themselves in a way that minimizes the distance between neighbors but forcing a separation so that the graph is readable. Notice that loops are not drawn.

Any vertex can be dragged and dropped from their position, and the rest of the graph will adapt consequently.



In the first dropdown menu, the option labeled `Custom...` allows to specify any valid automaton within the size bounds and visualize its associated Schreier graphs. When selected, a new panel appears. After choosing the number of states and the size of the alphabet, two tables must be filled, in order to specify the output and successor of each state and input. The graph will be displayed after clicking on `Visualize`. We provide here an example where the Hanoi towers group on 3 pegs is specified as custom automaton.


Schreier graphs of automata groups
Select an automaton:
Custom...
Choose an automaton group from the list or input your own automaton and visualize the Schreier graphs for its action on a given level of the tree.
The canvas will resize automatically while you drag vertices around. Loops on single vertices are not shown.
Number of states:
4
Size of alphabet:
3
Outputs 0 1 2
a 0 2 1
b 2 1 0
c 1 0 2
d 0 1 2
Successors 0 1 2
a a d d
b d b d
c d d c
d d d d
Level of the tree:
2
Visualize
00
20
10
21
12
11
01
02
22
Graph simulations generated with the library d3.js for javascript.
Aitor Pérez, University of Geneva, 2019.

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