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# Greedy Routing on Virtual Raw Anchor Coordinate (VRAC) System

Pierre Leone and Kasun Samarasinghe
Centre Universitaire d'Informatique
University of Geneva
Switzerland
{pierre.leone,kasun.wijesiriwardana}@unige.ch

***Abstract*—Geographic routing is an appealing routing strategy that uses the location information of the nodes to route the data. This technique uses only local information of the communication graph topology and does not require computational effort to build routing table or equivalent data structures. A particularly efficient implementation of this paradigm is greedy routing, where along the data path the nodes forward the data to a neighboring node that is closer to the destination. The decreasing distance to the destination implies the success of the routing scheme. A related problem is to consider an abstract graph and decide whether there exists an embedding of the graph in a metric space, called a greedy embedding, such that greedy routing guarantees the delivery of the data. A common approach to assign geographic coordinates is to measure distances (for instance the distances between neighboring nodes) and compute (virtual) coordinates.**

**The rationale of the Virtual raw Anchor Coordinate System (VRAC) is to use the (raw) measured distances as coordinates in order to avoid further computations. More precisely, each node needs to measure three distances. In this paper, we investigate the existence of greedy routing in the VRAC coordinate system using a metric free characterization of greedy paths that is more general than in previous works. We show that if the graph is saturated (see definition in the text) then the greedy algorithm guarantees delivery. Interestingly, the approach of greediness here applies to Schnyder drawings of planar triangulations. Indeed, by choosing the measured distances appropriately Schnyder drawings of planar triangulations are always saturated and hence our greedy routing algorithm succeeds.**

**The VRAC coordinates have conditions to satisfy to make greedy routing successful. These conditions can be inferred from geometric considerations. However, we formulate these conditions in an abstract way in order to avoid geometric considerations and in order to make possible further derivation of virtual VRAC coordinate systems, i.e. using only the abstract graph description. In particular using only local information would lead to distributed algorithm.**

## I. Introduction

Geometric routing is an appealing routing technique that uses geographic positions of nodes for routing data in communication networks. Its most attractive form, *greedy routing* forwards messages along a distance decreasing path towards the destination. Greedy routing decision is completely based on the local neighborhood information (positions of the neighboring nodes) and the position of the destination node, hence considered a local routing algorithm. Nevertheless, greedy routing does not always guarantee the delivery of messages, hence alternative mechanisms are required to guarantee the delivery.

Geometric routing requires an underlying location service for assigning coordinates to nodes. Such a service can be expensive, especially considering the wireless networks with power constrained wireless devices. As an alternative, in [1] geometric routing based on virtual coordinates instead of physical coordinates has proposed. In the same spirit, the more important problem of virtual coordinate assignments such that greedy routing guaranteeing delivery, namely *greedy embedding* has introduced [2]. In subsequent research, algorithms for greedy embedding on different geometric spaces [3] for different classes of graphs were proposed. Although, computation of greedy embeddings in a distributed environment is expensive in terms of the message complexity.

Virtual Raw Anchor Coordinate System (VRAC) uses raw distances (Euclidean distance) from a set of anchor nodes as the coordinate of a node [4], [5] (see Figure 1). In this paper, we consider the problem of greedy routing on Virtual Raw Anchor Coordinate (VRAC) system. We propose a greedy routing algorithm, which does not rely on an underlying metric, rather based on a metric free definition of a greedy path. When compared to [4], [5], [6], [7], [8], we provide a more general definition of greedy paths. This new approach of greediness can also be used as a building block of geographical routing algorithms of the type Greedy-Face-Greedy as we propose for VRAC in [8]. Alternatively, we develop here on the new connexion of VRAC with the problem of greedy embedding that makes possible the new approach of greediness. Actually, our greedy routing algorithm guarantees delivery on all Schnyder drawings of a planar triangulation (see the discussion at the end of Section IV) and this extents previous results of existential flavor[1].

[1]We refer to [12] for description of Schnyder drawings. What is relevant here is that given a planar triangulation and a set of weights of the triangular faces we obtain a Schnyder drawing (embedding) on the plane of the graph. Existential result says that there exists a set of weights such that the Schnyder drawing is greedy. Application of the greedy routing algorithm shows that every set of weights leads to successful greedy routing. We emphasize that in this second case there is no embedding of the graph. The result is useless for graph drawing but relevant to routing.

## II. BACKGROUND

A greedy embedding is an embedding of a graph on a metric geometric space such that, greedy routing always succeeds. In other words, between every node pair $u, v$ there is another node $w$ adjacent to $u$, such that $d(u, v) > d(w, v)$, where $d(.)$ is the underlying metric on the geometric space. Papadimitrou & Ratajczak [2] were the first to study the existence of *greedy embeddings*. In particular, they constructively proved the existence of a greedy embedding of a 3-connected graph in $\mathbb{R}^3$. Moreover, it is *conjectured* that any 3-connected planar graph admits a greedy embedding in $\mathbb{R}^2$.

The conjecture was proved affirmatively for different classes of graphs. In [9] it is proved for 3-connected graphs, in [10] for Delaunay triangulations, in [11] for graphs that satisfy conditions with respect to the power diagram. In [3], a greedy embedding in the hyperbolic plane of a connected finite graphs is constructed. More related to our approach, in [12] the conjecture is proven for planar triangulations (maximal planar graphs), see also [13], [14], [15]. Another direction of greedy embedding research considers the efficient representation of coordinates. In [16] authors proposed a greedy embedding on a hyperbolic space with $\mathcal{O}\big(log(n)\big)$ bit complexity. Such a coordinate representation is called *succinct*, which is important for the design of scalable routing schemes. In subsequent research, succinct greedy routing schemes are proposed in [17], [14].

Common approach in above proposals is to compute an embedding given a graph and to use the underlying metric of the respective space to perform greedy routing. However, our approach is to avoid the definition of a metric and the computation of the planar embedding, see for instance [13] for an algorithm to compute the greedy embedding of planar triangulations. We rely on the metric-free definition of greedy paths in [18] - without embedding the graph. Moreover, the computation of coordinate systems used in [12], [14], [15] requires global topology knowledge, as opposed to the local approach in our work. In [19], a local routing algorithm is proposed for $half-\theta_6$ graphs, which is a special case of a Delaunay triangulation. It utilizes the geometric properties of a $half-\theta_6$ graph and proves a strict upper bound on the path stretch. This approach does not compute a greedy embedding, yet based on the underlying geometry on the plane.

In Schnyder's work [20], planar graphs are characterized by the existence of three total order relations on the vertex set of the graph (and extra conditions). Using these order relations Schnyder builds three spanning[2] directed trees called the *realizer* and the coordinates are computed using these trees. This coordinate system has relevant properties, for graph drawing, see for instance [21], that we do not need for routing[3]. In [12], [14], [15], Schnyder's characterization of (maximal) planar graphs [20] is used. This characterization is discussed in section III and is also used in this work.

[2]Actually, spanning internal nodes of the triangulation

[3]Although it is relevant to ask if these properties are necessary for constructing a greedy embedding

In this article, we propose a greedy routing algorithm based on virtual raw anchor coordinate (VRAC) localization paradigm [4], [5], see Figure 1. In VRAC, nodes use the distance from three distinguished nodes as the coordinate, see Figure 1, i.e. the usual scheme for localizing the nodes is 1) measure distances and, 2) compute the coordinates. In this work our goal is to avoid using geometry. Our algorithm uses three order conditions $<_i^*$ to perform routing. Note that the order relations come in an abstract way, although they can be derived from geometric properties as well.

**Our contribution** In this paper we;

- Provide a metric-free definition of greedy paths.
- Design a greedy routing algorithm that guarantees delivery if the communication graph is saturated, see definition 5.
- Provide a greedy routing algorithm for all schnyder drawings of a planar triangulations.

The rationale that motivate this work is to extend the capabilities of the VRAC coordinate system by proposing a definition of greediness as general as possible. We hope that further investigations will lead to the design of distributed algorithm that compute the VRAC coordinates using only the abstract form of the communication graph, i.e. the graph $G = (V, E)$. In a sense, by avoiding the computation of the greedy embedding, our approach decouples the problems of greedy graph embedding and the design of successful greedy algorithms.

The algorithm is numerically validated with a simple Java simulator. Results are not reported here since the simulations bring nothing more than confirmation that the algorithm is successful. The simulator can be obtained under request to the authors.

In sections IV and III, we present Schnyder's characterization of planar maximal graphs. In section V we prove the properties that we need to build a greedy path between any two nodes. Finally, in section VII we state the main result of the paper about the existence of greedy paths.

## III. VIRTUAL RAW ANCHOR COORDINATES (VRAC)

Virtual Raw Anchor Coordinate System (VRAC), is a cost effective localization scheme, which uses raw distances (Euclidean distance) from a set of anchor nodes as the coordinate of a node [4], [5]. The motivation for using VRAC is to avoid the expensive computations required to compute the geographic coordinates. Indeed, most localization schemes measure the distances from a set of anchors and embed the communication graph in a metric space. In VRAC coordinate system such an embedding is avoided and raw distances are maintained as coordinates.

Despite being computationally efficient, VRAC does not have any underlying geometric properties to perform geometric routing. Although, we can define three order relations on the set of nodes $V$.

**Definition 1.** *The three order relations $<_i$, $i = 1, 2, 3$ on*

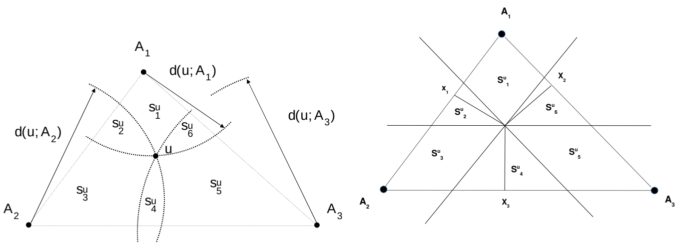



Fig. 1: Two realistic ways of constructing the VRAC. Coordinate assignment with raw distances from anchors on the left and perpendicular distances from heights of the triangles on the right. The coordinates of u are $(d(u, A_1), d(u, A_2), d(u, A_3))$ on the left and $(x_1, x_2, x_3)$ on the right.

*$V \times V$ are defined by*

$$\forall u, v \in V \quad u <_i v \Longleftrightarrow d(u, A_i) > d(v, A_i) \Longleftrightarrow u_1 > v_i.$$

It is important to note that these order relations are total[4](see [5] for details). Due to the totality of orders, a minimum element with respect to each total order can be defined, which is denoted as $\min_i$ for $i = 1, 2, 3$. Furthermore, based on the total orders, we define sectors associated with a node $u$ as follows.

**Definition 2.** *(Sectors) We define the following sectors associated to a node $u \in V$, see Figure 1. Note that the reference node $u$ does not belong to the sectors.*

$$\begin{aligned}
s_1^u &= \{v \mid u <_1 v,\ u >_2 v,\ u >_3 v\}.\\
s_2^u &= \{v \mid u <_1 v,\ u <_2 v,\ u >_3 v\}.\\
s_3^u &= \{v \mid u >_1 v,\ u <_2 v,\ u >_3 v\}.\\
s_4^u &= \{v \mid u >_1 v,\ u <_2 v,\ u <_3 v\}.\\
s_5^u &= \{v \mid u >_1 v,\ u >_2 v,\ u <_3 v\}.\\
s_6^u &= \{v \mid u <_1 v,\ u >_2 v,\ u <_3 v\}.
\end{aligned}$$

The sector $s_i^u$ is also referred to as *the sector $i$ of $u$*[5].

**Definition 3.** *Given a node $D$, we also use the convenient notation $s_D^u$ to denote the sector $j$ of $u$ such that $D \in s_j^u$, i.e. $D \in s_D^u$.*

We use the definition of partial orders $<_1^*, <_2^*, <_3^*$ below (see Lemma 3.1 in [20]) to further distinguish the edges.

**Definition 4.** *For each $i \in \{1, 2, 3\}$ we define*

$$\begin{aligned}
(u, v) \in <_i^* \quad &\Longleftrightarrow \quad (u, v) \in <_i, (v, u) \in <_{i+1}, (v, u) \in <_{i-1}\\
&\Longleftrightarrow \quad v \in s_{2i-1}^u,
\end{aligned}$$

[4]A total order is a binary relation which is valid for *all the pairs* in a set

[5]We use the notation $i + 1 \equiv i \bmod 6 + 1$ if $i$ is the index of a sector, i.e. $s_i^u$.

According to the definition, $<_1^*, <_2^*, <_3^*$ partial orders correspond to sectors $s_1, s_3$ *and* $s_5$ respectively. Based on these partial orders, a node can distinguish its neighboring nodes.

## IV. Schnyder Characterization and Saturated Communication Graph

We consider an ad-hoc network with VRAC localization infrastructure. Let $G = (V, E)$ represent the communication graph, where $V, E$ are the vertex and edge sets respectively. We assume that three anchor nodes $A_i, i = 1, 2, 3$ are placed such that the network lies entirely in the interior of the triangle $A_1A_2A_3$. This ensures that the property $a)$ of (1) is satisfied. Each node $u$ is able to measure the distances $d(u, A_i), i = 1, 2, 3$, and uses $(d(u(A_1), d(u, A_2), d(u, A_3))$ as coordinate and the order relation $<_i, i = 1, 2, 3$ are naturally defined using these coordinates.

We concentrate on the special class of graphs, which satisfies certain connectivity conditions. These conditions are due to the schnyder characterization of planar graphs based on order relations. Given a planar graph $G = (V, E)$, it is proven in [20] that there exist three total order relations on $V \times V$, denoted $<_1, <_2, <_3$ such that

$$\begin{aligned}
&a)\ \ \textstyle\bigcap_{i=1,2,3} <_i = \emptyset, \text{ and}\\
&b)\ \ \forall (x, y) \in E, \forall z \notin \{x, y\}\ \exists i \in \{1, 2, 3\}\\
&\ \text{s.t. } (x, z) \in <_i \text{ and } (y, z) \in <_i .
\end{aligned} \qquad (1)$$

This is called a *(3-dimensional) representation* of a planar graph. Such representation of a planar graph does not use a (planar) embedding and applies to an abstract graph. Using this representation we get a characterization of a greedy path that applies to abstract graphs as well and ignore the embedding. We also use the notation $x <_i z$ for $(x, z) \in <_i$ and we say $v$ is a neighboring node of $u$ to say that $(u, v) \in E$. It is important to note that $<_i^*$ partial orders defined in definition 4 follow the conditions stated in equation 1, hence these orders can be used in the construction of planar graphs on VRAC (see [4] for details). We observe following properties of these orders, which are useful in the greedy path construction in Section V.

**Property 1.** *The empty intersection property $a)$ in (1) implies that for each $u, v \in V$ there exists exactly one $i \in 1, 2, 3$ such that $(u, v) \in <_i^*$ or $(v, u) \in <_i^*$ (equivalently $v \in s_{2i-1}^u$ or $u \in s_{2i-1}^v$). It is convenient to remember that if $(u, v) <_i^*$ then $v \in s_{2i-1}^u$, i.e. $s_1^u$ or $s_3^u$ or $s_5^u$, the indexes are odd and even otherwise.*

**Property 2.** *A node $u$ has at most one edge $(u, v) \in E$ such that $v \in s_{2i-1}^u$. Moreover, such a node $v$ satisfies that $v <_i z\ \forall z \in s_{2i-1}^u, i = 1, 2, 3$, i.e. $v = min_i\{z \mid z \in s_{2i-1}^u\}$, see Lemma 3.1 of [20], this follow from condition $b)$ in (1). These*

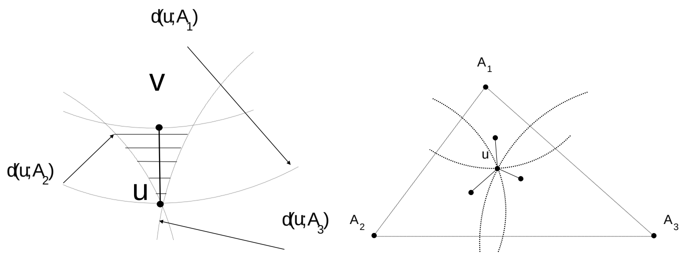



Fig. 2: **Left:** The geometric interpretation of Property 2. The hatched region is void. **Right:** Locally a node $u$ of a maximal planar graph as exactly three edges in the sectors $s_1^u, s_3^u, s_u^5$ if the representation is standard. The undetermined (0,1,2,...) number of edges in the sectors $s_2^u, s_4^u, s_6^u$ are not represented.

*properties can be written*

$$\textit{If } \mathbf{v} \in \mathbf{s_1^u},\ \mathbf{z} \neq \mathbf{v} \textit{ we have } \quad \left.\begin{array}{l} z <_2 u \\ z <_3 u \end{array}\right\} \Rightarrow z >_1 v. \tag{2}$$

$$\textit{If } \mathbf{v} \in \mathbf{s_3^u},\ \mathbf{z} \neq \mathbf{v} \textit{ we have } \quad \left.\begin{array}{l} z <_1 u \\ z <_3 u \end{array}\right\} \Rightarrow z >_2 v. \tag{3}$$

$$\textit{If } \mathbf{v} \in \mathbf{s_5^u},\ \mathbf{z} \neq \mathbf{v} \textit{ we have } \quad \left.\begin{array}{l} z <_1 u \\ z <_2 u \end{array}\right\} \Rightarrow z >_3 v. \tag{4}$$

Property 2 is from [20] and the proof uses part $b)$ of the graph representation $(1)$. There is a nice geometric *void* condition associated to this property (see Figure 2). Indeed, if we assume that the edge $(u, v)$ belongs to $s_1^u$ then the existence of a node $u <_1 w <_1 v$ violates the second condition of $(1)$, see Figure 2 and Figure 3c of [12]. It is interesting to compare this void region with the corresponding ones of the planar Relative Neighborhood Graphs (RNG) or Gabriel Graphs (GG) [22].

In order to construct a planarized subgraph of a given graph, each node $u$ decides to keep or remove the edges shared with neighboring nodes in the following way. For a neighboring node $v$, i.e. $(u, v) \in E$, $u$ keeps the edge if $\exists i \in \{1, 2, 3\}$ such that $(u, v) <_i^*$. Equivalently, only the shortest edges in $s_{2i-1}^u$ is kept. Notice that there must be at most one edge in sectors $s_{2i-1}^u$. When the given graph complies with the geometric conditions, that can be checked locally as described in [5] are met the resulting graph is planar. An immediate saturation condition leads to the following definition.

**Definition 5.** *A planar graph is saturated if there exists exactly one edge in each sector* $s_{2i-1}^u, i = 1, 2, 3$ *for each node* $u$.

The definition holds for planar graphs given in an abstract way. Indeed, such a planar graph admits a Schnyder representation and the definition refers to this representation. Our greedy routing algorithm is performed on this planar saturated graph and in the rest of the text all references to the graph are implicitly to this graph. We propose a metric free characterization of a greedy path[6] and show that it guarantees delivery when the graph is saturated, see Definition 5. We use the combinatorial properties (partial orders) to reason on the delivery guarantees of our algorithm. These combinatorial properties are derived from geometric properties of a saturated graph, yet the greedy path construction is also valid if we assume that the order relation $<_i$ are given in another (abstract) way. This is why in the rest of the paper we avoid direct reference to VRAC coordinate system and make only use of the order relations.

An important relationship can be drawn between saturated graphs and classical schnyder drawings of planar triangulations. A Schnyder drawing embeds a planar triangulation on the plane. Given a Schynder drawing VRAC coordinates can be derived as in the right of Figure 1. Resulting coordinates satisfy the saturation conditions due to Lemma 4 in [12][7]. Hence our greedy algorithm is successful in any Schnyder drawings. Comparatively, in [12] authors show the existence of a (greedy) Schnyder drawing if the graph is a planar triangulation. In [14] authors show a routing algorithm successful on any Schnyder drawing as well. However, their algorithm needs the computation of Schnyder drawing coordinates, where as our algorithm is not restricted to this setting.

## V. Greedy Routing

### A. Overview of Routing Strategy

If the (planarized) graph is saturated then each internal node $u$, has exactly one edge in each sector $s_1^u, s_3^u, s_5^u$ and an indeterminate $(0, 1, 2, \ldots)$ number in the remaining sectors $s_2^u, s_4^u, s_6^u$. It is helpful to look at the geometric visualization in the right of Figure 2.

For routing from a node $u$ to a destination $D \in s_1^u$ (or $s_3^u$ or $s_5^u$) the natural option is to follow the existing edge $(u, v)$ such that $v \in s_D^u$ ($= s_1^u$ or $s_3^u$ or $s_5^u$ respectively). Next, from $v$, if $D \in s_1^v$ (or $s_3^v$ or $s_5^v$) we repeat the same strategy. However, it may happen that $D \notin s_1^v$ (or $\notin s_3^v$ or $\notin s_5^v$). In this case $D \in s_2^v \cup s_6^u$ (or $s_2^v \cup s_4^v$ or $s_4^v \cup s_6^u$, see Proposition 3) and the existence of an edge in the sector $s_D^v$ is not provided by the Schnyder's characterization $(1)$. Nevertheless, in Proposition 2 we that saturation implies the existence of an edge in the sector $S_D^v$.

### B. Characterization of Greedy Paths

Our greedy routing technique differs from the classical ones as we do not assume that an underlying metric exists. Instead, we use metric-free axioms, which characterize greedy paths provided in [18], i.e. given a destination $D$ we have 1. (**transitivity**) if node $v$ is greedy for $u$ and $w$ is greedy for $v$ then $w$ is greedy for $u$ as well and 2. (**odd symmetry**) if $v$ is greedy for $u$ then $u$ is not greedy for $v$. Moreover, the coordinate system that we use differs from previous work, that are based on Schnyder's characterization of planar graphs, for example in [23], [12]. They use Schnyder drawing as the coordinate system in [20] that is more complex to compute than VRAC. This difference is possible here because initially

[6]Given two nodes $u$ and $v$ we don't assume that we can compute the distance $d(u, v)$.

[7]It is an open question, whether the saturation conditions are satisfied if the VRAC coordinates are obtained as on the left of Figure 1, i.e using the raw distance to the anchors.

the coordinate system was designed to draw the planar graph, while we limit our purpose to route the data.

**Definition 6.** *For destination node $D$, a path $\{u^k\}$ is a greedy path if there exists $i \in \{1, 2, 3\}$ such that*

$$\forall k \ \ u^{k+1} <_i u^k, \ \text{or} \ \forall k \ \ u^{k+1} >_i u^k. \tag{5}$$

*For a greedy path there is a coordinate that changes monotonically.*

In the following, we build greedy paths from $u$ to $D$ such that $u <_i u^k <_i D$ the fact that $D$ is an upper bound and the construction continues while $u^k <_i D$ implies the convergence of the sequence to $D$. In the proofs of Propositions 2 and 3 we use the assumption that the graph is saturated, to say that given a node $u$ there exists neighboring nodes in the sectors $s_1^u, s_3^u, s_5^u$. We must rule out the case where the node $u$ is one of the distinguished nodes $A_1, A_2, A_3$ since these nodes may not have any neighboring nodes in these sectors. Actually, these nodes do not cause any trouble because there is a path from any internal nodes to them with increasing coordinate $<_i$ respectively [20]. Hence, in order to make our best to simplify the exposition we no longer make any reference to these particular nodes in the proofs.

**Proposition 1.** *If $D' \in s_i^D$ and $D'' \in s_i^{D'}$ then $D'' \in s_i^D$*

*Proof.* This property follows directly from the transitivity of the inequalities in the definition of the sectors $(2)$. ☐

**Proposition 2.** *We assume that the graph $G$ is saturated. Then provided that the destination $D$ belongs to $s_2^u$ (or $s_4^u$, or $s_6^u$) then there is a path $\{u^i\}$ in $G$ with $u^0 = u$ such that $u^{i+1} \in s_2^{u^i}$ ($u^{i+1} \in s_4^{u^i}$ or $u^{i+1} \in s_6^{u^i}$ respectively), and the path converges to $D$.*

*Along the path, the order $<_3$ ($<_1$, $<_2$) decreases monotonically if $D \in s_2^u$ ($D \in s_4^u$, $D \in s_6^u$ respectively).*

*Proof.* For concreteness we consider $D \in s_4^u$. If $u$ is connected to $D$ we define $u^1 = D$ and the proposition is true. Otherwise, we prove below that there exists a neighboring node of $u$, $u^1$ such that $D \in s_4^{u^1}$ and $D <_1 u^1 <_1 u$. Hence, by applying the construction iteratively we construct the sequence of points that satisfy $u^{i+1} \in s_4^{u^i}$, lower bounded by $D$ and decreases with respect to $<_1$, i.e. $D <_1 u^{i+1} <_1 u^i$. Such a sequence converges to $D$.

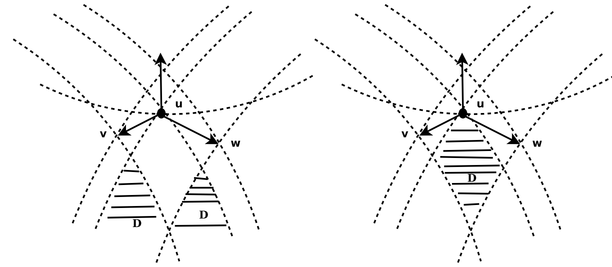
**Let us prove that given $u$ such that $D \in s_4^u$ there exists $x$ such that $(u, x) \in E$, $D \in s_4^x$ and $D <_1 x <_1 u$.** $u$ is internal, by the assumption on saturation, there exists two neighboring nodes of $u$ such that $v \in s_3^u$ and $w \in s_5^u$. we then have

$$D <_1 u, \ D >_2 u, \ D >_3 u \ \Leftrightarrow D \in s_4^u \tag{6}$$

$$v <_1 u, \ v >_2 u, \ v <_3 u \ \Leftrightarrow v \in s_3^u \tag{7}$$

$$w <_1 u, \ w <_2 u, \ w >_3 u \ \Leftrightarrow w \in s_5^u \tag{8}$$

If $v$ (or $w$) is such that $D \in s_4^v$ (or $D \in s_4^w$) the next point on the path is $u^1 = v$ (or $u^1 = w$) and $(7)$ shows that $v = u^1 <_1 u$, and $D \in s_4^v \Rightarrow v >_1 D$ (or $(8)$ shows that $w = u^1 <_1 u$ , and $D \in s_4^w \Rightarrow w >_1 D$).




(a) Destination $D \in s_{2i}^u$ and ($D \in s_{2i}^v$ *or* $s_{2i}^w$)

(b) Destination $D \in s_{2i}^u$ and $\neg(D \in s_{2i}^v$ *or* $s_{2i}^w)$

Fig. 3: Two cases to consider in greedy path construction

Otherwise, we have to prove that there exists a neighboring node of $u$ in the sector $s_4^u$ that satisfies the conditions. We have that $D >_2 u >_2 w$, and $D >_3 u >_3 v$ (using $(6, 7, 8)$) and $D \notin s_4^v$ and $D \notin s_4^w$ imply

$$D \notin s_4^v \Rightarrow \left.\begin{array}{l} D >_1 v \ D <_2 v \ D >_3 v \quad \text{or} \\ D <_1 v \ D <_2 v \ D >_3 v \end{array}\right\} \Rightarrow D <_2 v \tag{9}$$

$$D \notin s_4^w \Rightarrow \left.\begin{array}{l} D >_1 w \ D >_2 w \ D <_3 w \quad \text{or} \\ D <_1 w \ D >_2 w \ D <_3 w \end{array}\right\} \Rightarrow D <_3 w \tag{10}$$

Next, because $D \in s_4^u \Rightarrow u \in s_1^D$ and the assumption of saturation, there exists an edge $(D, D')$ with $D' \in s_1^D$. If $D' = u$ we are done.

Otherwise, by the property $(2)$ and $u \in s_1^D$ we have that $\boxed{D' <_1 u}$.

By gathering the inequalities corresponding to $u \in s_1^D$ with the ones deduced from $(9)$,$(10)$ we obtain $D <_1 D', v >_2 D >_2 D', w >_3 D >_3 D'$. Using $D' <_1 u$, $D' <_2 v$ with property $(3)$ we obtain $\boxed{D' >_3 u}$.

Last from $D' <_1 u$, $D' <_3 w$ and property $(4)$ (with edge $(u, w)$ instead of $(u, v)$) we obtain $\boxed{D' >_2 u}$. Finally, we have proved that $D' \in s_4^u$ with the boxes equations and $D <_1 D' <_1 u$. The node $D'$ plays the same role as $D$ in the statement of the proposition but with an increasing $<_1$ order position. Because of the bound $D' <_1 u$ we see that by applying iteratively the construction we obtain a sequence $D', D'', \ldots$ that converges to $u$ and such that all the points belong to $s_4^u$. Moreover, along the sequence we have $D' \in s_1^D$, $D'' \in s_1^{D'}$,... and Proposition 1 implies that all the points in the sequence belong to $s_1^D$. In particular, for the point $x$ that is connected to $u$ $x \in s_1^D \Leftrightarrow D \in s_4^x$. We have then proved the existence of a point $x \in s_4^u$ that satisfies $D \in s_4^x$ and such that $D <_1 x <_1 u$. ☐

**Remark 1. Construction of the greedy path if $D \in s_{2i}^u$**
*In order to route from $u$ to $D \in s_{2i}^u$ the node $u$ must first check whether for $v \in s_{2i+1}^u$ and $w \in s_{2i-1}^u$ one of the condition $D \in s_{2i}^v$ or $D \in s_{2i}^w$ is satisfied and if yes sends the message accordingly, see the left of Figure 3. Otherwise, the message is forwarded to (the existing by Proposition 2) neighboring node*

*in $x \in s^u_{2i}$ such that $D \in s^x_{2i}$, see the right of Figure 3. This routing scheme converges because the coordinate $i$ decreases along the path and the path doesn't step over D, as all the points in the path are $>_i D$.*

**Proposition 3.** *Let us assume that $(u, v) \in E$ and $D, v \in s^u_1$ (or $s^u_3$ or $s^u_5$). Then, $D \notin s^v_3 \cup s^v_4 \cup s^v_5$ (or $s^v_1 \cup s^v_5 \cup s^v_6$ or $s^v_1 \cup s^v_2 \cup s^v_3$).*

*Proof.* Let us consider $v, D \in^u_1$ the other cases are proved similarly by a permutation of the indices. We have

$$v \in s^u_1 \Leftrightarrow u <_1 v \quad u >_2 v \quad u >_3 v$$
$$D \in s^u_1 \Leftrightarrow u <_1 D \quad u >_2 D \quad u >_3 D$$

Part b) of the Schnyder's conditions (1) implies that $D$ must be larger than $u$ and $v$ for one order and we see on the two inequalities above that it can only be $<_1$. The condition $D \in s^v_3 \cup s^v_4 \cup s^v_5$ implies that $v >_1 D$ and hence there is no $i \in 1, 2, 3$ such that $u, v <_i D$ and the result in proved. □

**Remark 2. Construction of the greedy path if $D \in s^u_{2i-1}$** *The practical implication of Proposition 3 for routing is to prove the existence of a greedy path from u to $D \in s^u_{2i-1}$. We decompose the construction in two parts and for concreteness we consider $D \in s^u_1$.*
**Part 1.** *The maximality assumption implies the existence of a node $v \in s^u_1$ such that $(u, v) \in E$. If $v = D$ we are done. Else, $u$ sends the message to $v$ and the first coordinate $<_1$ increases, the second $<_2$ and third ones $<_3$ decrease. If $D \in s^v_1$ then $v$ repeats the same procedure and the coordinates continue to be updated monotonically and $D >_1 v$ because $D \in s^v_1$ and this implies that the first part of the construction converges to $D$ or switches to the second part.*
**Part 2.** *If the path reaches a node $v$ such that $D \notin s^v_1$ the construction of the path continues with this second part. In this case $D \in s^v_2$ or $D \in s^v_6$ must be satisfied because of Proposition 3. In both cases we have $D >_1 v$ and we can apply Proposition 2 that shows the existence of a sequence of nodes $v'$ with $D \in s^{v'}_2$ or $D \in s^{v'}_6$ respectively and this sequence eventually reaches $D$. If $D \in s^{v'}_2$ then by Proposition 2 the coordinate $<_3$ continues to decrease along the second part of the construction. If $D \in s^{v'}_6$ the coordinate $<_2$ continues to decrease. In both cases we have shown that along the two parts of the construction one coordinate ($<_2$ or $<_3$) decreases monotonically and the resulting path is then greedy. We point out that this second part is similar to* **Remark 1**. *However, in the present case we have to prove that one coordinate is decreasing monotonically although the path construction follows* **Part 1**.

## VI. Description of Algorithm 1

The Pseudo-code of the routing algorithm is provided in Figure 1. We provide here a complete description. The algorithm is executed at node $u$ to destination node $D$.

- $D$ is a neighbouring node of $u$ (Line 3), the routing terminates.
- $D$ belongs to an 'odd' sector of $u$ (Line 5), by the saturation assumption, Definition 5, there exists a unique neighboring node of $u$, called $v$ in the same sector of $u$ than $D$. $v$ is the next hop. This corresponds to **Part 1** of the construction of greedy paths in **Remark 2**.
- $D$ belongs to an 'even' sector of $u$ (Line 7), in this case saturation alone does not ensures the existence of a neighboring node of $u$ in this sector. The algorithm considers both neighboring nodes of $v$ and $w$ of $u$ that belong to the 'odd' sector of $u$ by saturation. If destination $D$ belongs to the same sector of $v$ or $w$ than $D$ the node is the next hop (Line 8 and 11). If not (Line 14) Proposition 2 proves the existence of a neighboring node of $u$ in th same sector as D. This corresponds to **Remark 1** and **Part 2** of **Remark 2**.

## VII. Routing in Maximal Planar Graph

In [12], it is proven using Schnyder's characterization of planar graphs (1) that there exists an embedding of the graph in the plane[8] such that greedy routing is successful (using the natural metric). The embedding is a particular instance of a familly of Schnyder drawings. In [23], [14] the authors use a similar coordinate system and design a routing algorithm. In our approach we avoid the computation of the embedding. In the setting of unit Disk Graph (UDG) it is shown in [24] that Schnyder's characterization is proved to be useful for planarizing and routing on the communication graph.
Our results are summarized in the Theorem 1, where the proofs of the two forms are apparent from the previous sections. The pseudo-code of the algorithm is provided in Algorithm 1 and the correctness of the algorithm is proved in the remarks 1 and 2 of the construction of the path if $D \in s^u_{2i}$ or $D \in s^u_{2i+1}$ that follow the Propositions 2 and 3.

**Theorem 1.** *The first formulation of the Theorem refers to an abstract graph and the second one to an embedded graph.*

1) *There is a greedy routing algorithm on every saturated planar graph.*
2) *Every Schnyder drawing of a planar triangulation is a greedy embedding.*

## VIII. Conclusion

In this article we propose a definition of greedy routing that is independent of a graph embedding in a metric space. We use a new localization paradigm (VRAC) which maintains raw distances from distinguished nodes as coordinates. The goal is to avoid complex computations of coordinates, hence appealing in real network settings. The proposed greedy routing strategy is based on VRAC and the respective greedy path characterization. We emphasize that our greedy routing strategy does not require to embed the graph in a metric space. The next step towards the development of a general practical routing algorithm is to formulate a distributed algorithm that extract virtual VRAC coordinates of the communication graph (or a subgraph) using only the abstract graph, i.e. without

[8] Actually in the plane in $\mathbb{R}^3$ such that $x + y + z = 1$.

**Algorithm 1** Pseudo-code of the greedy routing

INPUT Source $u$, Destination $D$
**repeat**
    **if** $D \in \mathcal{N}_u$ **then** $u = D$ $\triangleright$ $\mathcal{N}_u$ is the set of neighbors of $u$
    **else**
        **if** $D \in s^u_{2i-1}$ **then**
            $u = v \in s^u_{2i-1}$ s.t. $(u, v) \in E$ $\triangleright$ $v$ is unique
        **else** $\triangleright$ $D \in s^u_{2i}$ consider $v \in s^u_{2i-1}$ and $w \in s^u_{2i+1}$ s.t. $(u, v), (u, w) \in E$
            **if** $D \in s^v_{2i}$ **then**
                $u = v$
            **else**
                **if** $D \in s^w_{2i}$ **then**
                    $u = w$
                **else**
                    $u = x \in s^u_{2i}$ s.t. $D \in s^x_{2i}$ $\triangleright$ must exist by Proposition 2
                **end if**
            **end if**
        **end if**
    **end if**
**until** u=D

measuring distances nor using geometry. Such an algorithm would generalize the approach in [6], [4], [5], [24] where the coordinates are measured and the techniques could be merged to overcome the situation where the graph is not maximal. With our approach here we decouple the problems of greedy graph embedding and the design of a successful greedy algorithm. An interesting question is whether there exists a graph that does not admit a greedy embedding but on the top of which there exists a successful greedy algorithm.

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