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Complete list of tight Bell inequalities for two parties with four binary settings

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We give the complete list of 175 facets of the local polytope for the case where Alice and Bob each choose their measurements from a set of four binary outcome measurements. For each inequality we compute the maximum quantum violation for qubits, the resistance to noise, and the minimal detection efficiency required for closing the detection loophole with maximally entangled qubit states, in the case where both detectors have the same efficiency (symmetric case).

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I. INTRODUCTION

Bell inequalities are central to the study of nonlocality, but finding the complete list of Bell inequalities for a given Bell scenario can be a very difficult task [1]. A Bell scenario is specified by a number of measurement settings and a number of measurement outcomes for each party. In the case of two parties with two measurement choices each (the simplest case), there is only one Bell inequality, the Clauser-Horne-Shimony-Holt (CHSH) inequality [2]. The local polytope has two facets, CHSH and positivity. If one allows both parties to choose between three binary outcome measurements, there is only one new relevant inequality besides CHSH. For four settings on each side, the number of facet inequalities grows to 175, where 169 of these inequalities genuinely use the four settings. The list of 175 inequalities has been found in [3–5], however the realization that the list is complete is due to [6]. Therefore, one could find an almost complete list of the inequalities distributed in the literature. As a service to the community, we present the complete list in a single document. In addition, we study the basic quantum properties of these inequalities by computing the local and two-qubit quantum bounds, the state which attains the quantum bound, the resistance to noise for both the state that violates maximally the inequality and the maximally entangled state, and finally the minimum detector efficiency required to close the detection loophole assuming Alice and Bob’s detectors have the same efficiency and the maximally entangled two-qubit state.

In Sec. II we review all Bell inequalities for scenarios with fewer binary-outcome measurements. In Sec. III we describe the computation of the local and quantum bounds, the resistance to noise and minimal detection efficiency to close the detection loophole, and present the main results.

II. REVIEW OF ALL BELL INEQUALITIES WITH LESS SETTINGS

In any bipartite Bell scenario, the statistics are fully specified by the joint probability distribution $p(ab|xy)$, where

a, b and x, y are the outputs and inputs of Alice and Bob, respectively. For binary outcome scenarios XY_{22} , where X is the number of settings of Alice and Y is the number of settings of Bob, there are $4XY$ probability elements that specify $p(ab|xy)$. Some of these elements are not necessary to fully specify the statistics though, as they are not independent of each other due to the normalization and nonsignalling conditions. Taking these into account, one finds that there are only $4XY - XY - X(Y-1) - Y(X-1) = XY + X + Y$ independent probability elements. Therefore one can fully specify the statistics of a binary outcome Bell test using $XY + X + Y$ elements, this is the idea behind the Collins-Gisin (CG) notation [7]. As we deal only with binary outcome measurement, in the rest of the paper we shall denote a XY_{22} scenario simply by XY .

Using the CG notation, the probability distributions of 22 are specified by a table with the following elements:

$$\begin{array}{c|cc} & p^B(0|0) & p^B(0|1) \\ \hline p^A(0|0) & p(00|00) & p(00|01) \\ p^A(0|1) & p(00|10) & p(00|11) \end{array} \leq b$$

where $p^A(a|x)$ and $p^B(b|y)$ are the marginals of Alice and Bob, respectively.

Such a table defines a joint probability distribution $p(ab|xy)$. One can also describe an inequality using such a table, in this case each element of the table instead represents the coefficient that multiplies the probability element indicated in the probability table. We introduce coefficients for the joint probability distribution d_{xy} , the marginals of Alice c_x , and those of Bob e_y . In general, a Bell inequality for a XY scenario can be written as

$$I = \sum_{xy} d_{xy} p(00|xy) + \sum_x c_x p^A(0|x) + \sum_y e_y p^B(0|y) \leq b. \tag{1}$$

In the 22 scenario the only relevant inequality is the CHSH inequality

$$CHSH = \begin{array}{c|cc} & -1 & 0 \\ \hline -1 & 1 & 1 \\ 0 & 1 & -1 \end{array} \leq 0$$

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As described in the Introduction, even for the case of binary outcomes few is known. All tight Bell inequalities are known for the following cases: X2 and 2X ($X \geq 2$), 33 and 43 and 34. For the cases X2 and 2X ($X \geq 2$), there is only one Bell inequality [7–9]: the CHSH inequality.

The 33 scenario has one new inequality [7].

$$I_{3322} = \begin{array}{c|ccc} & -1 & 0 & 0 \\ \hline -2 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 0 & 1 & -1 & 0 \end{array} \leq 0$$

If Alice’s third setting and Bob’s first setting are not used, I_{3322} reduces to CHSH. Therefore in terms of minimal detection efficiency (symmetric) to close the detection loophole, I_{3322} and CHSH perform the same.

In the 34 scenario there are three new inequivalent inequalities:

$$I_{3422}^{(1)} = \begin{array}{c|cccc} & 1 & 0 & 0 & 1 \\ \hline 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 \\ -2 & 1 & 1 & 1 & -1 \end{array} \leq 2$$

(a)

$$I_{3422}^{(2)} = \begin{array}{c|cccc} & -1 & 0 & -1 & 1 \\ \hline 0 & -1 & 0 & 1 & -1 \\ 1 & 1 & -1 & 0 & -1 \\ -1 & 1 & 1 & 1 & 0 \end{array} \leq 1$$

(b)

$$I_{3422}^{(3)} = \begin{array}{c|cccc} & 0 & 0 & -1 & 2 \\ \hline 1 & -2 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & 1 & -1 \end{array} \leq 2$$

(c)

The optimal settings for $I_{3422}^{(1)}$ assuming the maximally entangled state are the same optimal settings of the elegant Bell inequality [10,11]. Alice’s optimal settings are three orthogonal measurements X, Y, and Z, while Bob’s settings form the vertices of a regular tetrahedron on the Bloch sphere. With $I_{3422}^{(1)}$ one can have a larger quantum violation using partially entangled two-qubit states. In this case, the optimal settings of Alice are still X, Y, and Z, but Bob’s settings become an irregular tetrahedron.

All the inequalities we present in this section can be lifted to the 44 scenario [12], therefore we already know five inequalities of 44. These five liftings correspond to inequalities 1 to 5 in Table I. The first 31 inequalities of Table I are the 31 inequalities published in [3], given in the same order. The list of coefficients in the CG notation for each inequality is provided separately as a file [13], and the inequalities are listed in it in the same order.

III. BELL INEQUALITIES WITH FOUR SETTINGS FOR EACH PARTY

We give the complete list of 175 facets of the local polytope in the case of four binary outcome measurements for both Alice and Bob. We present all the inequalities in a file [13], except the trivial one [positivity: $p(ab|xy) \geq 0$ for all a, b, x, y], in the case of four outcomes for Alice and Bob. The main results are given in Table I. Along with each inequality, we provide the local bound L , quantum bound Q , the resistance to noise λ , and the minimal detection efficiency η required for closing the detection loophole in the symmetric case. Note that six of the 175 inequalities correspond to liftings: positivity, CHSH, I_{3322} , $I_{3422}^{(1)}$, $I_{3422}^{(2)}$, and $I_{3422}^{(3)}$. Therefore there are 169 inequalities which use the four settings on both sides.

A. Quantum violation

The local bound L is computed by finding the optimal strategy using only shared randomness and local operations. Joint probability distributions $p(ab|xy)$ which are local in this sense can be decomposed in the following way:

$$p(ab|xy) = \int q(\lambda) p^A(a|x) p^B(b|y) d\lambda, \tag{2}$$

where λ is the shared random variable and $q(\lambda)$ is its probability distribution.

The quantum bound Q gives the maximal quantum violation for a two-qubit pure entangled state of the form

$$|\psi(\theta)\rangle = \cos \theta |00\rangle + \sin \theta |11\rangle. \tag{3}$$

Note that we compute the maximal quantum bound for qubits. Nevertheless, it is known that using higher-dimensional states one can in some cases achieve a better quantum bound [14].

We optimize over projective nondegenerate von Neumann measurements. Each measurement setting of Alice (Bob) is described by a vector \vec{a}_x (\vec{b}_y) on the Bloch sphere. One has

$$p(00|xy) = \text{Tr}(A_x \otimes B_y |\psi(\theta)\rangle \langle \psi(\theta)|), \tag{4}$$

where $A_x := \frac{\mathbb{1} + \vec{a}_x \cdot \vec{\sigma}}{2}$, and similarly for B_y .

All the inequalities for scenarios 22, 32, 23, and 33 are maximally violated by maximally entangled qubit states. This is not the case for scenarios 34 and 44, where the maximal violation can be given by a partially entangled state. In the 44 scenario we find inequalities which are maximally violated by partially entangled states for nondegenerate measurements, which can be in some cases very far from the maximally entangled state. An example of such an inequality is facet number 15, which is maximally violated by the state:

$$|\psi(\theta_{\max})\rangle = 0.402|00\rangle + 0.916|11\rangle. \tag{5}$$

The violation is small compared to other inequalities of this table, and we see that the resistance to noise is bad. However, as described in [3], using degenerate measurements this inequality (I_{4422}^4 in [3]) is maximally violated by the maximally entangled state. Indeed using degenerate measurements this inequality becomes equivalent to CHSH.

TABLE I. Quantum properties of all Bell inequalities with four binary-outcome settings for both parties. For each inequality we indicate the name under which they can be found in the literature, we give the quantum state that achieves the largest violation $|\psi(\theta_{\max})\rangle = \cos \theta_{\max}|00\rangle + \sin \theta_{\max}|11\rangle$, and we give its resistance to noise λ . For the maximally entangled state, we provide the resistance to noise λ_{ME} , as well as the detection efficiency η required to close the symmetric detection loophole. All quantities are computed for two-qubit systems and nondegenerate measurements. The boldface numbers correspond to the best values for noise resistance and detection efficiency, in the case of the state it corresponds to the most partially entangled state.

No.	Name	L	Q	θ_{\max}/π	λ	λ_{ME}	η_{sym}
1	CHSH	1	1.2071	0.25	0.7071	0.7071	0.8284
2	I_{3322}	1	1.25	0.25	0.8	0.8	0.8284
3	I_{4322}^1	1	1.2361	0.2332	0.864	0.866	0.8761
4	I_{4322}^2	1	1.2596	0.2251	0.828	0.8333	0.8685
5	I_{4322}^3	1	1.4365	0.25	0.7746	0.7746	0.8514
6	I_{4422}^1	1	1.197	0.2356	0.8988	0.9	0.8571
7	I_{4422}^2	2	2.6214	0.2476	0.763	0.763	0.8443
8	A_5	1	1.4353	0.2447	0.7751	0.7752	0.8214
9	A_6	1	1.2321	0.25	0.8829	0.8829	0.8373
10	AS_1	3	3.5412	0.25	0.7348	0.7348	0.8472
11	AS_2	4	4.8785	0.25	0.74	0.74	0.8506
12	AI_1	3	3.6056	0.2435	0.7676	0.7679	0.8323
13	AI_2	2	2.5	0.25	0.8	0.8	0.8508
14	I_{4422}^3	1	1.238	0.2257	0.863	0.866	0.8761
15	I_{4422}^4	1	1.056	0.1315	0.9728	1	1
16	I_{4422}^5	2	2.4365	0.25	0.7746	0.7746	0.8514
17	I_{4422}^6	1	1.4495	0.25	0.8165	0.8165	0.8697
18	I_{4422}^7	2	2.4548	0.2379	0.7937	0.7949	0.8405
19	I_{4422}^8	3	3.4207	0.2456	0.856	0.8561	0.8858
20	I_{4422}^9	2	2.4617	0.2352	0.8441	0.8455	0.8392
21	I_{4422}^{10}	2	2.6139	0.2461	0.8175	0.8176	0.8458
22	I_{4422}^{11}	3	3.6384	0.2444	0.779	0.7792	0.8474
23	I_{4422}^{12}	3	3.6188	0.2404	0.7843	0.7849	0.8382
24	I_{4422}^{13}	1	1.25	0.2069	0.8889	0.8889	0.8944
25	I_{4422}^{14}	2	2.4103	0.238	0.8298	0.831	0.8523
26	I_{4422}^{15}	1	1.25	0.25	0.8889	0.8889	0.8944
27	I_{4422}^{16}	1	1.2407	0.219	0.8791	0.8829	0.9009
28	I_{4422}^{17}	3	3.6714	0.2488	0.7883	0.7883	0.8611
29	I_{4422}^{18}	2	2.1812	0.2064	0.9508	0.9623	0.9575
30	I_{4422}^{19}	3	3.4307	0.25	0.8745	0.8745	0.887
31	I_{4422}^{20}	2	2.3056	0.25	0.9075	0.9231	0.899
32	J_{4422}^{44}	1	1.4145	0.2279	0.8085	0.8122	0.8607
33	J_{4422}^{25}	2	2.4459	0.2393	0.797	0.7977	0.8573
34	J_{4422}^{60}	1	1.5923	0.25	0.7715	0.7715	0.8351
35	J_{4422}^{43}	2	2.4414	0.2287	0.8192	0.8221	0.8788
36	J_{4422}^{69}	2	2.4693	0.2423	0.8099	0.8104	0.8581
37	J_{4422}^{74}	1	1.4332	0.2432	0.8219	0.8223	0.8603
38	J_{4422}^6	2	2.4158	0.2407	0.8279	0.8284	0.8409
39	J_{4422}^1	2	2.3871	0.2393	0.8378	0.8386	0.8498
40	J_{4422}^{26}	3	3.6402	0.2436	0.7785	0.7788	0.8409
41	J_{4422}^{63}	2	2.6035	0.2433	0.7885	0.789	0.8589
42	J_{4422}^{75}	3	3.627	0.2405	0.7821	0.7827	0.8546
43	J_{4422}^{32}	3	3.5902	0.2386	0.7922	0.7931	0.8565
44	J_{4422}^{23}	3	3.609	0.2403	0.787	0.7875	0.847
45	J_{4422}^{66}	3	3.6186	0.2422	0.7843	0.7848	0.8451
46	J_{4422}^{81}	3	3.5996	0.2397	0.7896	0.7904	0.8547

TABLE I. (Continued.)

No.	Name	L	Q	θ_{\max}/π	λ	λ_{ME}	η_{sym}
47	J_{4422}^{71}	3	3.5823	0.2471	0.7944	0.7945	0.8574
48	J_{4422}^{88}	2	2.616	0.2468	0.7851	0.7851	0.8454
49	J_{4422}^{33}	3	3.6151	0.2423	0.7853	0.7857	0.8458
50	J_{4422}^{54}	2	2.393	0.2307	0.8513	0.8539	0.8783
51	J_{4422}^{111}	2	2.7576	0.2455	0.7674	0.7676	0.856
52	J_{4422}^{118}	3	3.5283	0.2295	0.8255	0.8286	0.8829
53	J_{4422}^{14}	3	3.6023	0.2424	0.8059	0.8063	0.864
54	J_{4422}^{37}	2	2.3909	0.2278	0.8648	0.8671	0.8942
55	J_{4422}^{20}	2	2.5356	0.2353	0.8236	0.8251	0.8624
56	N_{4422}^{10}	2	2.5	0.25	0.8333	0.8333	0.8633
57	N_{4422}^4	2	2.3956	0.2373	0.8634	0.8646	0.8887
58	J_{4422}^9	2	2.3423	0.222	0.8796	0.8835	0.8927
59	J_{4422}^{82}	2	2.7308	0.241	0.79	0.7908	0.8608
60	J_{4422}^{67}	2	2.6731	0.2455	0.8034	0.8035	0.8609
61	J_{4422}^{51}	3	3.6678	0.2419	0.8046	0.8051	0.8573
62	N_{4422}^7	3	3.6244	0.2425	0.815	0.8155	0.8645
63	J_{4422}^2	3	3.614	0.2353	0.8175	0.8189	0.8526
64	J_{4422}^{102}	3	3.6651	0.2385	0.8053	0.8062	0.8433
65	J_{4422}^{87}	3	3.6849	0.2419	0.8006	0.801	0.8545
66	J_{4422}^{83}	3	3.6692	0.2414	0.8043	0.8048	0.8571
67	J_{4422}^{112}	2	2.6248	0.2352	0.8149	0.8163	0.8607
68	J_{4422}^{94}	2	2.5686	0.238	0.8287	0.8298	0.8707
69	J_{4422}^{24}	3	3.6943	0.2432	0.7984	0.7987	0.8523
70	J_{4422}^{35}	3	3.6933	0.2464	0.7987	0.7987	0.8379
71	J_{4422}^{36}	3	3.6706	0.2393	0.8039	0.8048	0.8422
72	J_{4422}^{22}	5	5.8156	0.2426	0.7862	0.7866	0.8637
73	J_{4422}^{61}	5	5.8176	0.2495	0.7858	0.7858	0.8632
74	J_{4422}^{27}	3	3.9643	0.25	0.7568	0.7568	0.8508
75	J_{4422}^{72}	4	4.7878	0.2401	0.792	0.7929	0.8639
76	J_{4422}^{41}	4	4.7596	0.2356	0.798	0.7993	0.8606
77	J_{4422}^{76}	4	4.8291	0.2413	0.7835	0.784	0.8581
78	J_{4422}^{62}	4	4.75	0.25	0.8	0.8	0.861
79	J_{4422}^{106}	4	4.8382	0.2415	0.7816	0.7821	0.8568
80	J_{4422}^{126}	4	4.8024	0.2441	0.789	0.7894	0.8616
81	J_{4422}^{77}	4	4.8406	0.2408	0.7811	0.7817	0.8601
82	J_{4422}^{116}	2	2.7652	0.2429	0.7968	0.7972	0.8632
83	J_{4422}^{50}	3	3.8556	0.2438	0.7781	0.7784	0.8503
84	J_{4422}^{16}	4	4.5674	0.2305	0.8409	0.8434	0.8844
85	J_{4422}^{19}	4	4.6742	0.25	0.8165	0.8165	0.8648
86	J_{4422}^4	4	4.6862	0.2452	0.8138	0.814	0.8630
87	J_{4422}^{42}	2	2.6012	0.2362	0.8331	0.8342	0.8777
88	J_{4422}^{90}	4	4.8398	0.2449	0.7813	0.7815	0.8444
89	J_{4422}^{58}	4	4.8814	0.2457	0.7729	0.773	0.8386
90	J_{4422}^{17}	3	3.4288	0.2364	0.8749	0.8756	0.888
91	J_{4422}^{34}	2	2.4075	0.2377	0.8804	0.8808	0.9016
92	J_{4422}^{121}	3	3.5971	0.2274	0.834	0.8374	0.8713
93	J_{4422}^{59}	2	2.427	0.2372	0.8754	0.876	0.8848
94	J_{4422}^{31}	2	2.2133	0.2017	0.9336	0.9443	0.9462
95	J_{4422}^7	2	2.3642	0.2304	0.8917	0.8942	0.8859
96	J_{4422}^8	2	2.2657	0.1837	0.9186	0.9333	0.9254

TABLE I. (Continued.)

No.	Name	L	Q	θ_{\max}/π	λ	λ_{ME}	η_{sym}
97	J_{4422}^{30}	2	2.2459	0.1832	0.9242	0.9443	0.937
98	J_{4422}^{11}	2	2.2229	0.1865	0.9309	0.9428	0.9355
99	J_{4422}^{110}	5	5.9457	0.2447	0.7746	0.7748	0.8571
100	J_{4422}^{70}	5	5.9539	0.2411	0.7731	0.7737	0.8526
101	J_{4422}^{93}	5	5.9627	0.2423	0.7715	0.7719	0.8552
102	J_{4422}^{107}	5	5.934	0.2478	0.7768	0.7768	0.8584
103	J_{4422}^{45}	3	3.7572	0.2378	0.811	0.812	0.8682
104	J_{4422}^{47}	4	4.7645	0.2408	0.8096	0.8101	0.87
105	J_{4422}^{28}	4	4.75	0.25	0.8125	0.8125	0.861
106	J_{4422}^{86}	3	3.75	0.25	0.8125	0.8125	0.8571
107	J_{4422}^{48}	4	4.745	0.2409	0.8135	0.8142	0.8697
108	J_{4422}^{15}	3	3.6133	0.2433	0.8412	0.8417	0.8664
109	J_{4422}^{122}	2	2.6275	0.2381	0.8382	0.8391	0.8732
110	N_{4422}^3	3	3.6093	0.2425	0.8421	0.8427	0.8671
111	N_{4422}^2	3	3.6135	0.2338	0.8412	0.8429	0.8673
112	J_{4422}^{129}	3	4.0523	0.2434	0.7688	0.7692	0.851
113	J_{4422}^{80}	4	4.9051	0.2464	0.7945	0.7946	0.8618
114	J_{4422}^{64}	4	4.9167	0.2466	0.7924	0.7926	0.8639
115	J_{4422}^{68}	4	5.0179	0.2416	0.7747	0.7753	0.8521
116	J_{4422}^{78}	3	3.8134	0.2406	0.8114	0.812	0.8675
117	S_{242}^{51}	5	6.0135	0.2417	0.7754	0.776	0.8526
118	J_{4422}^{99}	3	3.8264	0.2388	0.809	0.8099	0.8587
119	S_{242}^{52}	4	4.8704	0.2364	0.8008	0.8021	0.8456
120	J_{4422}^{124}	4	4.941	0.2455	0.7881	0.7882	0.8474
121	J_{4422}^{21}	4	4.5441	0.2397	0.8655	0.8662	0.8872
122	J_{4422}^{127}	3	3.6147	0.2269	0.8506	0.8537	0.8683
123	J_{4422}^5	3	3.5007	0.2404	0.8749	0.8754	0.8734
124	N_{4422}^6	3	3.5971	0.2496	0.8543	0.8543	0.8546
125	J_{4422}^{46}	5	5.9717	0.2436	0.7942	0.7946	0.8661
126	J_{4422}^{108}	5	5.9676	0.2403	0.7949	0.7955	0.8581
127	J_{4422}^{89}	5	6.0036	0.239	0.7889	0.7898	0.8603
128	J_{4422}^{96}	3	3.9417	0.2413	0.7993	0.7999	0.8642
129	J_{4422}^{117}	5	5.9721	0.2447	0.7941	0.7944	0.866
130	J_{4422}^{39}	4	4.8265	0.2306	0.8194	0.8221	0.8793
131	J_{4422}^{53}	4	4.7994	0.2428	0.8243	0.8246	0.8653
132	J_{4422}^{57}	4	4.8053	0.2415	0.8232	0.8237	0.8646
133	J_{4422}^{55}	3	3.7066	0.2393	0.8415	0.8422	0.8752
134	J_{4422}^{56}	4	4.7491	0.2419	0.8335	0.834	0.872
135	J_{4422}^3	4	4.7249	0.2448	0.838	0.8382	0.8647
136	J_{4422}^{49}	4	4.8089	0.2392	0.8226	0.8234	0.861
137	N_{4422}^9	4	4.7399	0.2445	0.8352	0.8354	0.8626
138	J_{4422}^{98}	5	6.1497	0.2396	0.7767	0.7776	0.8622
139	J_{4422}^{85}	4	4.9763	0.2362	0.8038	0.805	0.8694
140	J_{4422}^{79}	5	6.0156	0.2396	0.7975	0.7982	0.8529
141	J_{4422}^{119}	5	5.8489	0.2421	0.8249	0.8253	0.8714
142	J_{4422}^{125}	5	6	0.25	0.8	0.8	0.8541
143	J_{4422}^{105}	5	6.0742	0.2418	0.7883	0.7888	0.8465
144	J_{4422}^{65}	4	5.111	0.2466	0.7826	0.7827	0.8484
145	J_{4422}^{113}	3	3.8195	0.2399	0.83	0.8306	0.8698
146	J_{4422}^{101}	5	6.0296	0.2429	0.7953	0.7956	0.8439

TABLE I. (*Continued.*)

No.	Name	L	Q	θ_{\max}/π	λ	λ_{ME}	η_{sym}
147	J_{4422}^{128}	5	6.0096	0.2497	0.7985	0.7985	0.8531
148	N_{4422}^{11}	3	3.4917	0.2246	0.8905	0.8947	0.8914
149	J_{4422}^{13}	3	3.5629	0.2422	0.8766	0.8772	0.8615
150	J_{4422}^{91}	7	8.2993	0.2422	0.7659	0.7663	0.8554
151	J_{4422}^{92}	4	5.0648	0.2405	0.7997	0.8002	0.8622
152	J_{4422}^{52}	3	4.0999	0.2442	0.7944	0.7947	0.8586
153	J_{4422}^{115}	3	3.9802	0.2394	0.8126	0.8134	0.8687
154	J_{4422}^{38}	4	4.9295	0.2342	0.8205	0.8225	0.8687
155	J_{4422}^{40}	6	6.902	0.2441	0.833	0.8333	0.883
156	J_{4422}^{84}	4	5.067	0.2407	0.8083	0.8089	0.8675
157	J_{4422}^{95}	5	5.9418	0.2401	0.8269	0.8277	0.8697
158	J_{4422}^{120}	3	3.7931	0.2385	0.8502	0.8509	0.8899
159	J_{4422}^{18}	5	5.7018	0.2416	0.8651	0.8653	0.8892
160	J_{4422}^{29}	5	6.0246	0.2461	0.8145	0.8146	0.8515
161	J_{4422}^{114}	5	6.0653	0.2463	0.8086	0.8087	0.8562
162	N_{4422}^8	5	6.0189	0.2408	0.8154	0.816	0.8614
163	J_{4422}^{103}	5	6.052	0.2385	0.8105	0.8114	0.8491
164	N_{4422}^1	5	6.0076	0.2467	0.8171	0.8171	0.8622
165	J_{4422}^{123}	5	6.0641	0.2422	0.8088	0.8092	0.8475
166	J_{4422}^{10}	3	3.3738	0.2138	0.9233	0.9281	0.9155
167	J_{4422}^{12}	3	3.4198	0.25	0.9147	0.9147	0.8893
168	J_{4422}^{73}	4	5.1205	0.2478	0.8091	0.8092	0.8714
169	J_{4422}^{100}	6	7.2675	0.2416	0.7894	0.7899	0.8534
170	N_{4422}^{12}	7	8.325	0.2431	0.7905	0.7909	0.8654
171	J_{4422}^{97}	4	5.1584	0.2477	0.8119	0.8119	0.8657
172	N_{4422}^5	7	8.4377	0.2403	0.785	0.7857	0.8516
173	J_{4422}^{104}	6	7.5876	0.2398	0.7836	0.7844	0.8574
174	J_{4422}^{109}	9	10.7261	0.2417	0.7766	0.7771	0.8552

B. Resistance to noise

Let $|\psi\rangle$ be the state that maximally violates a specific bell inequality. Then the resistance to noise $1 - \lambda$ is defined as the amount of white noise that can be mixed with $|\psi\rangle$ in order for the bell inequality not to be violated anymore:

$$\rho = \lambda|\psi\rangle\langle\psi| + (1 - \lambda)\frac{\mathbb{1}}{4}. \quad (6)$$

The best bipartite inequality XY for $X, Y \leq 4$ in terms of resistance to noise using qubits is by far CHSH.

C. Minimum detection efficiency for closing the detection loophole

There are different possible strategies to close the detection loophole. All of them involve preventing the attacker (Eve) from exploiting the nondetections. How should the experimenter handle a nondetection event? One possible strategy, which was implemented in [3], is that Alice and Bob output $a = 0$ or $b = 0$, respectively, in order to deal with a nondetection. However, they could also output 1, giving four possibilities in total. One can also consider strategies specific to each input, in which case one has 256 possible strategies.

We have also optimized the detection efficiencies over these no-click strategies.

A Bell inequality with detector efficiency η for both Alice and Bob can be written [15]:

$$I_{\eta,\eta} = \eta^2 Q + \eta(1 - \eta)(M_A + M_B) + (1 - \eta)^2 X \leq L, \quad (7)$$

where M_A is the value that the Bell inequality yields when only Alice's detector fires, M_B is the bound when only Bob's detector fires, and X accounts for when both detectors do not fire.

We find that the smallest value for the symmetric detection efficiency using a maximally entangled two-qubit state is 82.14%, which is already the best result of [3]. This inequality, labeled A_5 in [3], is number 8 on our table.

D. Correlation inequalities

In this section we present the two inequalities of this list which can be cast into correlator-only form. A correlator $E(x, y)$ is defined as

$$E(x, y) = p(a = b|xy) - p(a \neq b|xy). \quad (8)$$

CHSH for example can be put into full-correlation form

$$E(0, 0) + E(0, 1) + E(1, 0) - E(1, 1) \stackrel{L}{\leq} 2 \stackrel{Q}{\leq} 2\sqrt{2}, \quad (9)$$

where L and Q relate to the local and quantum bounds, respectively.

We find that only three inequalities can be put into full-correlation form: inequalities 1 (CHSH), 10, and 11. Facet number 10 is the inequality named AS_1 in [3], while facet number 11 is AS_2 . These inequalities are the only ones which can be put in full-correlation form for the 44 scenario, which confirms the result of Avis *et al.* [16].

IV. CONCLUSION

We have studied the complete list of four binary-outcome settings Bell inequalities. We give the full list and a table with the local and quantum bounds of all inequalities, the two-qubit

state that maximally violates each inequality, the resistance to noise and the minimal detection efficiency for maximally entangled qubit states to close the detection loophole in a Bell experiment where both detectors have the same efficiency. We find several inequalities which are maximally violated by partially entangled states, which is interesting for the study of nonlocality. It is also confirmed that the minimum detection efficiency is 82.14%, and is found for the inequality A_5 published in [3].

Note added. Recently, the complete list of inequalities was presented in [17], although the inequalities were not studied. In order not to confuse readers, we have added the same name convention for the inequalities that are used in [17].

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