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Non-commutative differential calculus and its applications in low-dimensional topology

THÈSE

présentée à la Faculté des Sciences de l'Université de Genève
pour obtenir le grade de Docteur ès sciences, mention mathématiques

par

Florian Naef

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intitulée :

«Non-commutative Differential Calculus and its Applications in Low-dimensional Topology»

La Faculté des sciences, sur le préavis de Monsieur A. ALEXEEV, professeur ordinaire et directeur de thèse (Section de mathématiques), Monsieur P. SEVERA, docteur (Section de mathématiques), Monsieur T. SCHEDLER, docteur (Department of mathematics, Faculty of Natural Sciences, Imperial College London, United Kingdom), autorise l'impression de la présente thèse, sans exprimer d'opinion sur les propositions qui y sont énoncées.

Genève, le 18 décembre 2017

Thèse - 5160 -

Le Doyen

N.B. - La thèse doit porter la déclaration précédente et remplir les conditions énumérées dans les "Informations relatives aux thèses de doctorat à l'Université de Genève".

Résumé

Cette thèse est consacrée à l'étude du calcul différentiel non-commutatif et à ses applications à certaines questions de topologie de basse dimension. La connexion entre ces deux domaines est faite via le problème de Kashiwara-Vergne, qui prend ses sources en théorie de Lie et est lié aux groupes de tresse et aux associateurs de Drinfeld.

Cette thèse contient les parties et résultats suivants:

- On définit des notions de structures de Poisson non-commutatives et applications moment étroitement liées à la notion de double crochet de Poisson de van den Bergh. Le résultat principal obtenu est que sous certaines conditions techniques, l'application moment détermine de manière unique la structure de Poisson, ce qui contraste avec la situation dans le monde commutatif, où ceci est possible uniquement si l'espace est «petit» par rapport à l'image de l'application moment. Une des applications est la linéarisation de l'espace des modules de connexions plates, ou de manière équivalente de l'algèbre de Lie de Goldman.

- On étudie le problème de linéarisation pour la bigèbre de Lie de Goldman-Turaev. Plus précisément, on montre que dans le genre 0, ce problème de linéarisation est essentiellement équivalent au problème de Kashiwara-Vergne. Les deux équations du problème de Kashiwara-Vergne correspondent respectivement au crochet de Goldman et au co-crochet de Turaev. Le point clé est l'identification du co-crochet de Turaev avec une généralisation de l'application divergence apparaissant dans la formulation du problème de Kashiwara-Vergne.

- A la lumière de l'équivalence précédente, on définit de nouveaux problèmes de Kashiwara-Vergne, associés à une surface de Riemann de genre arbitraire avec un nombre arbitraire de composantes de bord. A nouveau, trouver des solutions à ces problèmes résout le problème de linéarisation de la bigèbre de Lie de Goldman-Turaev correspondante. On montre ensuite que tous ces

problèmes ont en effet des solutions, en utilisant des techniques provenant de la théorie des associateurs elliptiques.

- Une courte démonstration du problème de linéarisation dans le genre 0 sur le corps des nombres complexes est donnée, utilisant la monodromie de la connexion Knizhnik-Zamolodchikov. De plus, la dépendance à la structure complexe est rendue explicite.

- Une autre application du calcul différentiel non commutatif est donné en théorie de jauge. On montre que la trivialisation universelle de la première classe de Pontryagin est en fait le même problème que la recherche de solutions à la première équation de Kashiwara-Vergne.

Abstract

This thesis studies non-commutative differential calculus and its applications to questions in low-dimensional topology. The connection is through the Kashiwara-Vergne problem which originates in Lie theory and has close ties to braid groups and Drinfeld associators.

This thesis contains the following main parts and results:

- Notions of non-commutative Poisson structures and moment maps are defined, closely related to van den Bergh's notion of double Poisson brackets. The main result is that under some technical conditions, the moment map uniquely determines the Poisson structure, in contrast to the situation in the commutative world, where this is only possible if the space is "small" compared with the target of the moment map. One application is linearization of the moduli space of flat connections, or equivalently the Goldman Lie algebra.

- The linearization problem for the Goldman-Turaev Lie bialgebra is studied. In more detail, it is shown that in genus 0 this linearization problem is essentially equivalent to the Kashiwara-Vergne problem. The two equations in the Kashiwara-Vergne problem are seen to correspond to the Goldman bracket and Turaev cobracket, respectively. The key fact is that one can identify the Turaev cobracket with a generalization of the divergence map that appears in the formulation of the Kashiwara-Vergne problem.

- Motivated by the previous equivalence, new Kashiwara-Vergne problems are defined, associated to a Riemann surface of arbitrary genus with arbitrary number of boundary components. Finding solutions to these problems again solves the linearization problem for the corresponding Goldman-Turaev Lie bialgebra. Then it is shown that all these problems indeed have solutions, using techniques from the theory of elliptic associators.

- A short proof of the linearization problem in genus 0 over the field of complex numbers is carried out, using the monodromy of the Knizhnik-Zamolodchikov

connection. Moreover, the dependence on the complex structure is made explicit.

- One further application of non-commutative differential calculus is in gauge theory. It is shown that universally trivializing the first Pontryagin class is the same problem as finding solutions to the first Kashiwara-Vergne equation.

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Contents

Chapter 1

Introduction

Non-commutative differential calculus

This thesis discusses several applications of non-commutative differential calculus to questions related to low-dimensional topology. The main motivating idea of non-commutative differential calculus is the following principle usually attributed to Kontsevich. Let A be an associative algebra.

Definition 1.1. The non-commutative analog of a structure on A should induce the commutative version of that structure on the representation space $\mathrm{Hom}(A, \mathrm{End}(V))$ for any vector space V.

For instance, a non-commutative 1-form should induce a 1-form on the space $\mathrm{Hom}(A, \mathrm{End}(V))$ for any V. This principle suggests, that we ought to think of elements in A as matrix-valued functions. Another instance of such a notion is van den Bergh's theory of double Poisson brackets ([9]). A double Poisson bracket is an operations of type

$$A \otimes A \xrightarrow{\{\cdot,\cdot\}} A \otimes A,$$

which might look bewildering at first, however, it adheres to the principle 1.1. Namely, if one computes the Poisson bracket of two matrix-valued functions, one naturally gets a function with values that have four indices, which one can distribute into two matrices.

Throughout this thesis, we assume that A is a free and complete algebra. By freeness it follows that the spaces $\mathrm{Hom}(A, \mathrm{End}(V))$ are smooth. And the completeness assumption is equivalent to saying that we only study properties of the representation spaces that are in a formal neighborhood around the trivial representation, that is we work with power series without worrying about their convergence properties.

Non-commutative Poisson structures

The notion of van den Bergh's double Poisson bracket in this setting then boils down to the following: Let $M = \mathrm{End}(V)^{\times n}$, and study classical (formal) Poisson brackets on M that have the property, that they can be defined by a formula only involving the associative algebra structure of $\mathrm{End}(V)$ and the trace $\mathrm{End}(V) \to \mathbb{K}$. For that reason, let us call such a Poisson structure *universally defined*.

The two key observations in chapter 3 are the following:

- Most naturally occuring universally defined Poisson structures only use the induced Lie bracket on $\mathrm{End}(V)$ and the induced quadratic pairing. Namely, the formula makes sense for any Lie algebra $\mathfrak{g}$ that carries a non-degenerate invariant scalar product. In this case, we say that the Poisson bracket is of *Lie type*.
- There are far fewer universally defined Poisson brackets than all Poisson brackets. In particular, to check for equality is easier if we know that the Poisson brackets we wish to compare are universally defined.

The second observation was already made by Massuyeau-Turaev in [8], which we restate in our language.

Theorem 1.2. *Let Π be a universally defined Poisson bracket on A. Assume that Π is non-degenerate, then it is uniquely determined by a certain element $\mu \in [A, A]$.*

Here the element μ is called the *moment map*, and has the property that $\{\mu, \cdot\}_\Pi$ encodes the $\mathrm{GL}(V)$ action on $\mathrm{Hom}(A, \mathrm{End}(V))$. The main result of chapter 3 can then be seen as a generalization of theorem 1.2, where the non-degeneracy condition is dropped. In that case, one needs the additional condition, that the two Poisson structures one wishes to compare, (universally) have the same symplectic leaves. This last condition states that the Poisson structures are gauge equivalent, or b-transforms of each other. The main result of chapter 3 can then be stated as follows.

Theorem 1.3. *There is a one-to-one correspondence between moment maps and Poisson structures in a given b-transformation class. Moreover, any two Poisson structures in a given b-transformation class are isomorphic to each other.*

The second statement is an application of a non-commutative version of Moser's lemma.

(Universal) moduli spaces of flat connections

The main application that we have in mind is the following situation. Let Σ be a surface of genus g with $n+1$ marked boundary components. It follows from [1] that the moduli space of connection on that surface $\mathcal{M}(\Sigma, \mathfrak{g}) = \mathrm{Hom}(\pi_1(\Sigma), G) = G^{2g+n}$ carries a quasi-Poisson structure. Here quasi-Poisson should be thought of as the multiplicative analogue of Poisson. This has moreover the property that it is indeed universally defined. In fact, it comes from a simple geometric operation

$$\kappa : \mathbb{K}\pi_1 \otimes \mathbb{K}\pi_1 \longrightarrow \mathbb{K}\pi_1 \otimes \mathbb{K}\pi_1,$$

which is defined in terms of intersections of the two immersed curves. This is the based loop version of an observation made by Goldman [6]. Namely, any free loop γ in Σ defines a function $\mathrm{Tr}\,\mathrm{Hol}_\gamma$ on $\mathcal{M}(\Sigma, \mathrm{GL}(N))$ which is given by evaluating the holonomy on the loop γ and taking trace of the result. Goldman shows that the Poisson bracket $\{\mathrm{Tr}\,\mathrm{Hol}_\gamma, \mathrm{Tr}\,\mathrm{Hol}_{\gamma'}\}$ is again a sum of function of the form $\mathrm{Tr}\,\mathrm{Hol}_{\gamma''}$, where one can compute the loops γ'' geometrically by looking at the intersections of γ and γ'. Let us define the vector space $|\mathbb{K}\pi_1| = \mathbb{K}[S^1, \Sigma] = \mathbb{K}\pi_1/_{\mathrm{conj}}$, with basis given by free homotopy classes of loops in Σ. Then Goldman's considerations show

Theorem 1.4. $|\mathbb{K}\pi_1|$ *carries a natural bracket* $[\cdot,\cdot]$. $(|\mathbb{K}\pi_1|, [\cdot,\cdot])$ *is called the* Goldman Lie algebra.

Additionally, there is a universal procedure to take the logarithm of a quasi-Poisson structure and turn it into a Poisson structure. One concludes that studying the quasi-Poisson structure on $\mathcal{M}(\Sigma, \mathfrak{g})$ is the same as studying the double quasi-Poisson structure on $\mathbb{K}\pi_1$, which after taking logarithm is a double Poisson structure on $\widehat{\mathbb{K}\pi_1}$, the completion of $\mathbb{K}\pi_1$, which in turn boils down to studying its moment map $\mu \in \widehat{\mathbb{K}\pi_1}$, which in this case is given by the logarithm of the element in π_1 representing a chosen boundary component.

Let us now restrict for the moment to the case where the surface is a three punctured sphere. Choosing appropriate generators of π_1, we can identify $\widehat{\mathbb{K}\pi_1} \cong \mathbb{K}\langle\langle x, y\rangle\rangle$. The moment map μ is then given by

$$\mu = \log(e^x e^y) = x + y + \frac{1}{2}[x, y] + \cdots,$$

namely the Baker-Campbell-Hausdorff formula. In view of the above theorem it is clear that we can linearize the Poisson structure on $\mathcal{M}(\Sigma, G)$, by linearizing μ. Or more precisely, restoring some technicalities, the linearizaton problem for κ is equivalent to the first two equations of the following problem.

Definition 1.5 (Kashiwara-Vergne problem)**.** Find $F \in \mathrm{Aut}(\widehat{\mathrm{Lie}}(x,y))$ such that

$$F(x) = F_x x F_x^{-1}, \quad F(y) = F_y x F_y^{-1}, \quad \text{for some } F_x, F_y, \tag{KV0}$$

$$F(\log(e^x e^y)) = x + y, \tag{KVI}$$

$$j(F) = h(x+y) - h(x) - h(y) \quad \text{for some } h \in \mathbb{K}[[s]]. \tag{KVII}$$

The Goldman-Turaev Lie bialgebra and the Kashiwara-Vergne problem

This is the Kashiwara-Vergne problem, which first appeared in [7], in a slightly different form. The original purpose to consider these equations was to show the Duflo isomorphism theorem. In that context $j(F)$ is a non-commutative version of the log-Jacobian. Solutions to the Kashiwara-Vergne problem were constructed in [2] and later in [3] using Drinfeld associators. In our case, it turns out that the first equation KVI is equivalent to the problem of linearizing κ. One of the main results of chapter 4 is to give a topological interpretation of equation KVII. More precisely, on $\mathbb{K}\pi_1$ there is additional structure given by counting self-intersections of loops. The same geometric procedure that gives us κ, now applied to a single loops defines (ignoring some technical difficulties) an operation of the following type

$$\delta : |\mathbb{K}\pi_1| \longrightarrow |\mathbb{K}\pi_1| \otimes |\mathbb{K}\pi_1|,$$

that is, when one reconnects a loop at a self-intersection point one ends up two loops. This operation was considered by Turaev in [15]. It is compatible with the Goldman bracket.

Theorem 1.6. $(|K\pi_1|, [\cdot,\cdot], \delta)$ *is an involutive Lie bialgebra, called the* Goldman-Turaev Lie bialgebra*.*

It turns out that this operation is intimately related to the KVII equation. In more detail, one can think of the non-commutative log-Jacobian j as being associated to a non-commutative volume form. The volume form only shows its presence through the existence of a divergence map. It is shown in chapter 4, that there exists a divergence map on $\mathbb{K}\pi_1$, generalizing the type of divergence that appears in equation KVII. Using this divergence, the Turaev cobracket δ is then the non-commutative analogue of a classical construction in BV-geometry, that is, δ of a function is the divergence of the Hamiltonian vector field it generates. This fact is the principal connection between the Turaev cobracket and the equation KVII.

The following is then the main result of 4, stated colloquially.

Theorem 1.7. *An element $F \in \mathrm{Aut}(\widehat{\mathrm{Lie}}(x,y))$ induces a linearization of the structures κ and δ on a three punctured sphere, if and only if F is a solution to the Kashiwara-Vergne problem.*

This theorem gives a complete topological interpretation of the Kashiwara-Vergne problem. However, the linearization problem for the Goldman-Turaev Lie bialgebra can certainly be stated for a surface of arbitrary genus, with an arbitrary number of additional punctures. The problems in genus 0 with arbitrary number of punctures are straightforward generalizations of the original KV problem. Namely, they are related to the properties of Baker-Campbell-Hausdorff formulas of the type

$$\log(e^{x_1}e^{x_2}\cdots e^{x_n}),$$

and can be solved by gluing together solutions of the original KV problem. This corresponds to gluing together pair of pants according to a chosen parenthesization on n symbols, i.e. something like writing

$$\log(e^{x_1}e^{x_2}\cdots e^{x_n}) = \log(\cdots((e^{x_1}e^{x_2})e^{x_3})\cdots e^{x_n}).$$

The genus 0 case is thus entirely handled in chapter 4.

The higher genus case is discussed, in less detail, in chapter 5. The story is very similar, and the above theorem still holds, for the generalized definition of the Kashiwara-Vergne problem, which is as follows.

Definition 1.8 (KV Problem of type $(g, n+1)$)**.** Find an element $F \in \mathrm{TAut}^{(g,n+1)}$ such that

$$F\Big(\sum_{i=1}^{g}[x_i,y_i] + \sum_{j=1}^{n} z_j\Big) = \log\Big(\prod_{i=1}^{g}(e^{x_i}e^{y_i}e^{-x_i}e^{-y_i})\prod_{j=1}^{n} e^{z_j}\Big) =: \xi \qquad (\mathrm{KVI}^{(g,n+1)})$$

$$j(F) = \sum_{i=1}^{n} \mathrm{tr}\ h(z_i) - \mathrm{tr}\ h(\xi) - \mathbf{r} \quad \text{for some Duflo function } h \in \mathbb{Q}[[s]], \qquad (\mathrm{KVII}^{(g,n+1)})$$

where $\mathbf{r} = \sum_{i=1}^{g} \mathrm{tr}(\log(\frac{x_i}{e^{x_i}-1}) + \log(\frac{y_i}{e^{y_i}-1}))$.

In this formulation, the KV0 equation is absorbed into the definition of TAut. The first KV equation is still some sort of generalized Baker-Campbell-Hausdorff formula. Further, there is a fixed element $\mathbf{r}$ which encodes the Haar measure on G.

The existence of solutions of these problems is addressed again by a gluing formula. However, one can only glue solutions along pair of pants. Thus, using that the original KV problem has solution, one can reduce the general problem to the case of the once puncture torus. The solution of this remaining case is motivated by the following facts. Firstly, the only known solutions of the original KV problem come from Drinfeld associators. In the theory of elliptic associators (see [5]), there is a family of elliptic associators depending on the complex structure on the torus, which one can in particular degenerate to a sphere with two points identified, where it becomes an ordinary associator. This degeneration phenomenon allows to write an elliptic associator in terms of an ordinary associator. We cannot show at the moment that every elliptic associator gives a solution to the elliptic KV problem, however, the same formula that turns an ordinary associator into an elliptic associator also works in the context of KV. Namely, there is a map from solutions of the ordinary KV problem to solutions of the elliptic KV problem. This implies the main result of chapter 5.

Theorem 1.9.

$$\mathrm{Sol}\,\mathrm{KV}^{(g,n+1)} \neq \varnothing.$$

The Knizhnik-Zamolodchikov approach

Another approach to the problem in genus 0 can be motivated as follows. The first known Drinfeld associator is constructed using the monodromy of the Knizhnik-Zamolodchikov (KZ) connection,

$$d + \frac{x}{z}dz + \frac{y}{z-1}dz,$$

where x, y are the generators of a free Lie algebra. One can see this equation as parametrizing all flat connections on $\mathbb{C} \setminus \{0,1\}$ up to gauge transformations. This equation gives an affine chart on $\mathcal{M}(\Sigma, \mathfrak{g})$ around the trivial connection. The question is whether this linearizes the Goldman-Turaev Lie bialgebra structure. If one follows the construction of an associator out of this connection, takes the corresponding solution of Kashiwara-Vergne and uses this in our machinery to construct a linearization map of the Goldman-Turaev Lie bialgebra, then one obtains the same as directly taking the (universal) monodromy representation of the KZ connection. Thus we know that the monodromy of the KZ connection linearizes the Goldman-Turaev Lie bialgebra. However, it turns out that one can give a much shorter and more elementary proof of this fact, which is carried out in chapter 6.

Descent equations and universal trivializations of the first Pontryagin class

One last application of the non-commutative differential calculus is in gauge theory. The starting point is usually a connection 1-form $A \in \Omega^1(P)$ on a principal G-bundle P. One can see this as an $T[1]G$-equivariant map of graded manifolds

$$T[1]P \xrightarrow{A} \mathfrak{g}[1],$$

where the right $T[1]G$-action on $\mathfrak{g}[1]$ comes from identifying $\mathfrak{g}[1] \cong G$ $T[1]G$. Promoting it to a dg-map, one obtains a map

$$C^\infty(T[1]\mathfrak{g}[1]) \cong W(\mathfrak{g}) \longrightarrow \Omega^\bullet(P),$$

where the left hand side is identified with the Weil algebra as G-differential algebras, i.e. dg-algebras with a dg-$T[1]G$-action. Doing the bar construction we obtain a map of simplicial dg-manifolds,

$$B(T[1]P, T[1]G, \{*\}) \longrightarrow B(\mathfrak{g}[1], T[1]G, \{*\}).$$

Functions on the target space are what is know as the Bott-Shulmann complex. It roughly encodes information about how a connection 1-form and its exterior derivatives transform with respect to gauge transformations. If we replace G by its Lie algebra equipped with the Baker-Campbell-Hausdorff formula, we see that the Bott-Shulmann complex is given by functions on the representation space of a certain non-commutative algebra, and all the structure is induced by structure on that algebra, which we call the universal Bott-Shulmann complex. By replacing the group with its Lie algebra, we made all the characteristic classes exact. The most interesting one is the first Pontryagin class

$$p_1 = \langle F, F \rangle.$$

It turns out that if we want to universally trivialize this characteristic class we again stumble across the first Kashiwara-Vergne equation, and we obtain the following theorem, again stated colloquially.

Theorem 1.10. *Universal trivializations of the first Pontryagin class p_1 are in one-to-one correspondence with solutions of the first Kashiwara-Vergne equation (KVI).*

Apart from the first two chapter, every chapter corresponds to one of the following publications:

F. Naef, *Poisson Brackets in Kontsevich's "Lie World"*, preprint arXiv:1608.08886.

Alekseev, A., Kawazumi, N., Kuno, Y. and Naef, F., 2017. *The Goldman-Turaev Lie bialgebra in genus zero and the Kashiwara-Vergne problem*, preprint arXiv:1703.05813.

Alekseev, A., Kawazumi, N., Kuno, Y. and Naef, F., 2017. *Higher genus Kashiwara–Vergne problems and the Goldman–Turaev Lie bialgebra*, Comptes Rendus Mathematique, 355(2), pp.123-127.

Anton Alekseev, Florian Naef, *Goldman–Turaev formality from the Knizhnik–Zamolodchikov connection*, In Comptes Rendus Mathematique, Volume 355, Issue 11, 2017, Pages 1138-1147.

A. Alekseev, N. Florian, X. Xu and C. Zhu, Chern-Simons, *Wess-Zumino and other cocycles from Kashiwara-Vergne and associators*, Lett Math Phys (2017). https://doi.org/10.1007/s11005-017-0985-4

Chapter 2

Background Material

2.1 Introduction to non-commutative differential calculus

The purpose of this section is to give a gentle introduction to the type of non-commutative differential calculus that is used in chapter 3. To that purpose let us recall the relevant notions and properties of commutative differential calculus we wish to generalize.

Let V be a finite dimensional vector space, say $\mathbb{R}^n$. One can define the following objects.

- $C^\infty(V)$: the algebra of functions.
- $\mathfrak{X}(V)$: the space of vector fields with the following propertie:
  - It carries the structure of a Lie algebra.
  - It naturally acts on $C^\infty(V)$.
  - It is a free C^∞-module generated by $\frac{\partial}{\partial x_i}$, where x_i is a basis of V^*.
- $\Omega^\bullet(V)$: the graded algebra of differential forms, carrying a differential $d : \Omega^\bullet \to \Omega^{\bullet+1}$.

In the non-commutative setting, we replace the commutative algebra of functions with a non-commutative algebra A. In our affine example let $A = \mathbb{C}\langle\langle x_1, \dots, x_n \rangle\rangle$ be the algebra of non-commutative power series in the symbols $x_1, \dots, x_n$.

2.1.1 Vector fields

We wish to find a notion of vector fields that closely matches the three properties satisfied by ordinary vector fields. However, the properties will be satisfied by two different definitions of a vector fields. Our first attempt is to take the space of all derivations of A, namely

$$\mathrm{Der}(A) := \{\phi : A \to A \mid \phi \text{ is } \mathbb{C}\text{-linear}, \ \phi(ab) = a\phi(b) + \phi(a)b \ \forall a, b \in A\}.$$

This definitions clearly satisfies the first two properties. It is a Lie algebra as the commutator of two derivations is again a derivation, and by definition it acts on A. However, it is not naturally an A-module, and hence it is also not clear in what way it should be generated by partial derivatives.

This leads us to our second attempt. Here we wish to satisfy the third property, that is we try to give some meaning to the symbol $\frac{\partial}{\partial x_i}$. More precisely, let us try to evaluate $\frac{\partial}{\partial x_i}$ on a function $f \in A$. The naive definition is to set $\frac{\partial}{\partial x_i} f$ to the part of $f(x_1, \ldots, x_i + T, \ldots, x_n)$ that is linear in T.

Let us illustrate this on an example. Let $f = x_1 x_2 x_1$. Then to evaluate $\frac{\partial}{\partial x_2}$ one computes

$$f(x_1, x_2 + T) = x_1 x_2 x_1 + x_1 T x_1.$$

So the result is given by $x_1 T x_2$, or

$$\frac{\partial}{\partial x_2} f = x_1 \otimes x_2,$$

if we write $\otimes$ instead of T. From this discussion it is clear that $\frac{\partial}{\partial x_i}$ naturally lie in the following space of *double derivations*

$$D_A := \mathrm{Der}(A, A \otimes A) = \{\phi : A \to A \otimes A \mid \phi(ab) = (a \otimes 1)\phi(b) + \phi(a)(1 \otimes b)\}.$$

The space $A \otimes A$ has two commuting A-bimodule structures, and outer and an inner one, whose definition is clear from its name. Then D_A is the space of derivations $A \to A \otimes A$ with respect to the outer bimodule structure. Clearly, the inner bimodule structure can be used to turn D_A into am A-bimodule. The following lemma shows that this definition of vector fields fulfills the other half of our desired properties.

Lemma 2.1. *D_A is a free A-bimodule generated by $\frac{\partial}{\partial x_i}$.*

A feature that is not present in the commutative world is the following. There exists a canonical element $\phi_0 \in D_A$, given by

$$\phi_0(a) = 1 \otimes a - a \otimes 1.$$

In coordinates this can be expressed as

$$\phi_0 = \sum_i [x_i, \tfrac{\partial}{\partial x_i}].$$

There is a natural map from our second definition of vector field to the first one.

$$\begin{aligned} D_A &\longrightarrow \mathrm{Der}(A) \\ \phi &\longmapsto a \mapsto \phi^{'}(a)\phi^{''}(a), \end{aligned}$$

that is one just multiplies the two A factors. We then have the following lemma.

Lemma 2.2.

$$D_A \otimes_{A\otimes A^{op}} A \cong \mathrm{Der}(A).$$

Remark 2.3. A consequence of this lemma is that we can think of derivations of A as cyclic words in the symbols x_i and $\frac{\partial}{\partial x_i}$ that contain exactly one of the $\frac{\partial}{\partial x_i}$ symbols.

Constructions like this will appear again, we use the notation $|M| := M \otimes_{A\otimes A^{\mathrm{op}}} A$ and denote the canonical projection by $m \mapsto |m|$. In this notation $|A|$ denotes the space of cyclic words in A.

It is clear that $\mathrm{Der}(A) = |D_A|$ is a Lie algebra, however, it is not apparent what the appropriate structure should be on D_A. For this we consider the following construction. Let A^T be the algebra A with a symbol T freely adjoined, and consider the Lie algebra $\mathrm{Der}_T(A^T)$ of derivations of A^T that are T-linear. The space D_A can be canonically identified with the T-degree 1 part of $\mathrm{Der}_T(A^T)$. It is now clear that there is a bracket

$$\mathrm{Der}_{T_1}(A^{T_1}) \otimes \mathrm{Der}_{T_2}(A^{T_2}) \to \mathrm{Der}_{T_1,T_2}(A^{T_1,T_2}),$$

which satisfies a version of the Jacobi identity. When restricted to D_A the target space can be identified with $A \otimes D_A \oplus D_A \otimes A$. Since the construction was compatible with setting $T = 1$, one gets the following diagram of operations, where in passing from each row to the next another symbol T is set to 1.

$$\begin{array}{ccc}
D_A \otimes D_A & \longrightarrow & A \otimes D_A \oplus D_A \otimes A \\
\downarrow & & \downarrow \\
\mathrm{Der}(A) \otimes D_A & \longrightarrow & D_A \\
\downarrow & & \downarrow \\
\mathrm{Der}(A) \otimes \mathrm{Der}(A) & \longrightarrow & \mathrm{Der}(A) \\
\downarrow & & \downarrow \\
\mathrm{Der}(A) \otimes \mathrm{End}(|A|) & \longrightarrow & \mathrm{End}(|A|) \\
\downarrow & & \downarrow \\
\mathrm{End}(|A|) \otimes \mathrm{End}(|A|) & \longrightarrow & \mathrm{End}(|A|)
\end{array}$$

Each row represents a bracket operations satisfying a version of the Jacobi identity.

2.1.2 Differential forms

Here we take as motivation remark 2.3, and define the space of differential forms to be cyclic words in x_i and dx_i, where dx_i have degree 1. More formally, let us define $T[1]A$ as the free algebra generated by the symbols x_i, dx_i, and set

$$\Omega^\bullet := |T[1]A|,$$

that is the space of cyclic words in $T[1]A$. This space is graded by the number of d symbols, and has a canonical differential d. The forms of low degree can be described more explicitly. Zero forms are just cyclic words in A, that is $|A|$. For 1-forms we have the following Lemma.

Lemma 2.4. *The map $A^{\times n} \to \Omega^1; (a_1, \cdots, a_n) \mapsto \sum |a_i dx_i|$ is a bijection.*

The following lemma is the basic underlying fact of the result in chapter 3.

Lemma 2.5. *There is a bijection $\Omega^{2,cl} \cong [A, A] \in A$ making the following diagram commute.*

$$\begin{array}{ccc}
A^{\times n} \ni (a_1, \cdots, a_n) & \longrightarrow & \sum |a_i dx_i| \in \Omega^1 \\
\downarrow & & \downarrow d \\
[A, A] \ni [a_i, x_i] & \longrightarrow & \sum |da_i dx_i| \in \Omega^{2,cl}
\end{array}$$

This lemma follows once we give a description of the maps in both directions. In one direction it follows from the fact, that one can contract forms with double derivations to get elements in $T[1]A$. Then the map $\Omega^{2,\mathrm{cl}} \to [A,A]$ is given by contracting with ϕ_0 and taking the d-primitive. The computation goes as follows:

$$\begin{aligned}\iota_{\phi_0}|da_i dx_i| &= |\iota_{\phi_0} da_i dx_i| - |da_i \iota_{\phi_0} dx_i| \\ &= |\phi_0(a_i)dx_i| - da_i\phi_0(x_i)| \\ &= |[\otimes, a_i]dx_i| - |da_i[\otimes, x_i]| \\ &= |\otimes d[a_i, x_i]|.\end{aligned}$$

The end result can be identified with $d[a_i, x_i]$, of which one can recover the primitive $[a_i, x_i]$ by contracting for instance with the Euler vector field.

In the other direction the map can be described as follows: Take an element in $[a_i, x_i] \in [A, A]$. Now apply the (even) derivation $D : A \to \mathbb{D}A := \mathbb{K}\langle\langle x_1, \cdots, x_n, Dx_1, \cdots, Dx_n\rangle\rangle$ defined by $D(x_i) = Dx_i$ and $D(Dx_i) = 0$. Note that D^2 is not identically zero, as now everything is of even degree. We apply D^2 to $[a_i, x_i]$ and then project it to $\mathbb{D}A \otimes_{A\otimes A^{\mathrm{op}}} A$ which in turn projects to $\Omega^\bullet$. The computation looks as follows:

$$\begin{aligned}|D^2[a_i, x_i]| &= |D([Da_i, x_i] + [a_i, Dx_i])| \\ &= |[D^2 a_i, x_i] + 2[Da_i, Dx_i]| \\ &= 2|Da_i Dx_i - Dx_i Da_i|.\end{aligned}$$

As a corollary we get for instance that symplectic forms are in one-to-one correspondence with elements in $[A, A]$ that are in a certain sense non-degenerate.

2.2 quasi-Poisson Manifolds

Let $\mathfrak{g}$ be a finite dimensional Lie algebra with a symmetric invariant inner pairing. Let e_a be an orthonormal basis of $\mathfrak{g}$. Then one can define the Cartan 3-form as

$$\phi = \tfrac{1}{2} f^{abc} e_a \wedge e_b \wedge e_c \in \left(\bigwedge^3 \mathfrak{g}\right)^{\mathfrak{g}}.$$

Recall that a Poisson manifold is a manifold M with a bivector field $\Pi \in T^2_{\mathrm{poly}}(M)$ such that $[\Pi, \Pi] = 0$. Let now M be a $\mathfrak{g}$-manifold. That is a manifold together with an action of $\mathfrak{g}$ given by an element $\rho \in \mathfrak{g} \otimes \mathfrak{X}(M) \cong \mathfrak{g}^* \otimes \mathfrak{X}(M)$, where we used that $\mathfrak{g}^* \cong \mathfrak{g}$.

Definition 2.6. A quasi-Poisson structure is a $\mathfrak{g}$-invariant bivector field $\Pi \in T^2_{\mathrm{poly}}(M)$ such that

$$[\Pi, \Pi] = \rho(\phi).$$

Remark 2.7. It is straightforward to check that M/G is a Poisson manifold, provided it is a manifold. More concretely, the subalgebra $C^\infty(M)^{\mathfrak{g}}$ of $\mathfrak{g}$-invariant functions is a Poisson algebra.

A moment map for a Poisson manifolds is a $\mathfrak{g}$-equivariant map $\mu : M \to \mathfrak{g}^*$ such that $\{\mu, \cdot\}_\Pi = \rho$. The quasi analogue of a moment map is the following.

Definition 2.8. A $\mathfrak{g}$-equivariant map $\mu : M \to G$, where G is equipped with the adjoint action, is a moment map if

$$\{\mu, \cdot\}_\Pi = \tfrac{1}{2}(\mu\hat{\rho} + \hat{\rho}\mu),$$

where $\hat{\rho}$ is the $\mathfrak{g}$-valued vector field obtained from ρ by identifying $\mathfrak{g}^*$ with $\mathfrak{g}$.

The notation is straightforward to interpret in the case where G is a matrix group, and in general one needs to see TG as a group, canonically containing $\mathfrak{g}$ and G as subgroups.

The reason we are interested in this notion is that the moduli space of flat connections on a surface with a marked point on the boundary naturally has the structure of a quasi-Poisson manifold with moment map.

2.2.1 Moduli spaces of flat connection

Let Σ be a surface of genus g, with $n+1$ boundary components and a marked point $*$ on the boundary. Then one can consider the space of all flat $\mathfrak{g}$-connections on Σ up to gauge transformations that are trivial at the point $*$. Assuming G is simply connected, or restricting to the case where the bundle is trivial we get

$$\mathcal{M}(\Sigma, \mathfrak{g}) := \{A \in \Omega^1(\Sigma) \otimes \mathfrak{g} \mid dA + \tfrac{1}{2}[A, A] = 0\}/\{g : \Sigma \to G \mid g(*) = e\}.$$

Remark 2.9. After choosing generators of $\pi_1(\Sigma, *)$, taking monodromy gives the following identification.

$$\mathcal{M}(\Sigma, \mathfrak{g}) \cong G^{2g+n}.$$

The main theorem is the following

Theorem 2.10 ([1])**.** *The space $\mathcal{M}(\Sigma, \mathfrak{g})$ carries a natural quasi-Poisson structure. It moreover has a moment map which is given by the monodromy around the boundary component that contains $*$.*

In particular, the space of $\mathfrak{g}$-invariant functions is a Poisson algebra. We can see this algebra as the algebra of functions on the space of flat connections modulo all gauge transformations. In the case of $G = \mathrm{GL}(N)$ there is a particularly nice subspace of invariant functions given by a free loop γ in Σ, namely one can take the function $A \mapsto \operatorname{Tr}\operatorname{Hol}_A(\gamma)$. Then we have the following theorem.

Theorem 2.11 (Goldman's theorem, see [8])**.**

$$\{\mathrm{Tr}\,\mathrm{Hol}(\gamma), \mathrm{Tr}\,\mathrm{Hol}(\gamma^{'})\} = \sum_{p\in\gamma\cap\gamma^{'}} \epsilon_p \,\mathrm{Tr}\,\mathrm{Hol}(\gamma \star_p \gamma^{'}),$$

where $\epsilon_p \in \{\pm 1\}$ *compares the orientation of the tangent vectors to* γ *and* $\gamma^{'}$ *with the orientation of the surface, and* $\gamma \star_p \gamma^{'}$ *is the loops given by traversing* γ *starting from* p *and then traversing* $\gamma^{'}$*.*

2.3 Kashiwara-Vergne problem

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then one can form the following two algebras. The symmetric algebra and the universal enveloping algebra of $\mathfrak{g}$.

$$\begin{aligned} S(\mathfrak{g}) &= T\mathfrak{g}/(x \otimes y - y \otimes x = 0), \\ U(\mathfrak{g}) &= T\mathfrak{g}/(x \otimes y - y \otimes x = [x, y]). \end{aligned}$$

One can interpret $S(\mathfrak{g})$ as either the algebra of constant coefficient differential operators on $\mathfrak{g}$ or as distributions on $\mathfrak{g}$ supported at 0 where the algebra stucture is given by the convolution with respect to addition. Similarly, $U(\mathfrak{g})$ is the algebra of left invariant differential operators on G, or distributions with support at the identity with the convolution over the group multiplication as the product. From this identification with distributions it is clear, that the exponential map induces an isomorphism of vector spaces. This isomorphism is called the *Poincaré-Birkhoff-Witt* (PBW) isomorphism,

$$\begin{aligned} S(\mathfrak{g}) &\overset{\mathrm{PBW}}{\longrightarrow} U(\mathfrak{g}) \\ x^n &\longmapsto x^n, \quad \text{for } x \in \mathfrak{g} \ . \end{aligned}$$

Unless $\mathfrak{g}$ is abelian, this is not an isomorphism of algebras, as in that case $U\mathfrak{g}$ is a non-commutative algebra. In any case, this map respects the $\mathfrak{g}$-actions on both sides. Thus it makes sense to restrict to $\mathfrak{g}$-invariants and ask the question about the compatibility with the algebra structure. Note that $\mathfrak{g}$-invariants on $U\mathfrak{g}$ is its center $Z(U\mathfrak{g})$. Even restricted to $(S\mathfrak{g})^{\mathfrak{g}}$, the map PBW is still not an isomorphism of algebras, but one can change the map as follows to correct that defect. Let us define the following element $h \in \hat{S}(\mathfrak{g}^*)$, which we call the *Duflo function*

$$h(x) = \det\left(\frac{1 - e^{\mathrm{ad}_x}}{\mathrm{ad}_x}\right)^{\frac{1}{2}}.$$

It being a formal function, one can rewrite this element in a form that will more closely resemble its appearance later in this thesis. Let $g \in \mathbb{Q}[[s]]$ be the power series

$$g(s) = \tfrac{1}{2} \log \left(\frac{1 - e^{-s}}{s} \right).$$

Then

$$h(x) = e^{\mathrm{tr}(g(\mathrm{ad}_x))}.$$

Elements of $\mathfrak{g}^*$ act on $S\mathfrak{g}$ by constant vector fields. Thus we can see h as a differential operator on $S\mathfrak{g}$ of infinite order. Let us denote this operator again by h. Then the following is true.

Theorem 2.12 (Duflo's isomorphism theorem [4]). *The map* $\mathrm{Duf} := \mathrm{PBW} \circ h$ *restricts to an isomorphism of algebras* $(S\mathfrak{g})^{\mathfrak{g}} \to Z(U\mathfrak{g})$*.*

Duflo's original proof is not algebraic in nature, he writes:

> *"Il devrait exister une démonstration algébrique du théorème 2, mais je n'en connais pas."*

In [5], Kashiwara and Vergne found that a certain property of the Baker-Campbell-Hausdorff formula would indeed give such an algebraic proof. Their approach is roughly as follows. Try to find an automorphism (not necessarily of algebras) $\tilde{F} : S\mathfrak{g} \otimes S\mathfrak{g} \to S\mathfrak{g} \otimes S\mathfrak{g}$ compensating for the non-commutativity of

$$\begin{array}{ccc} \tilde{F} \circlearrowright \; S\mathfrak{g} \otimes S\mathfrak{g} & \longrightarrow & S\mathfrak{g} \\ \downarrow{\scriptstyle \mathrm{Duf} \otimes \mathrm{Duf}} & & \downarrow{\scriptstyle \mathrm{Duf}} \\ U\mathfrak{g} \otimes U\mathfrak{g} & \longrightarrow & U\mathfrak{g} \end{array}$$

with the property that $\tilde{F}$ is the identity when restricted to $\mathfrak{g}$-invariants. Let us think of $S\mathfrak{g}$ and $U\mathfrak{g}$ as distributions acting on a space of test function. Then the non-commutativity of the above square is expressed by the following. Let $\phi \in C_0^\infty(G)$ be a test function, and $\alpha, \beta \in S(\mathfrak{g})$. Then the map $\tilde{F}$ should satisfy

$$\langle \tilde{F}(\alpha \otimes \beta), h(x)h(y)\phi(e^x e^y) \rangle = \langle \alpha \otimes \beta, h(x+y)\phi(e^{x+y}).$$

Let us make the Ansatz that $\tilde{F} = F \circ \frac{h(x+y)}{h(x)h(y)}$, where F is now induced by a (formal) diffeomorphism of $\mathfrak{g} \times \mathfrak{g}$. Then we have

$$\langle \tilde{F}(\alpha \otimes \beta), h(x)h(y)\phi(e^x e^y) \rangle = \langle \alpha \otimes \beta, h(x+y) \frac{h(F(x))h(F(y))}{h(x)h(y)} \phi(e^{F(x)} e^{F(y)}) \rangle,$$

thus we need to make sure that $h(F(x)) = h(x)$ and that $e^{F(x)}e^{F(y)} = e^{x+y}$. Since j is $\mathfrak{g}$-invariant, the first condition is satisfied if we have the following,

$$F(x,y) = (F_1(x,y)xF_1(x,y)^{-1}, F_2(x,y)xF_2(x,y)^{-1}), \tag{KV0}$$

for some function $F_i : \mathfrak{g} \times \mathfrak{g} \to G$ for $i = 1, 2$. The second condition we ask for is

$$F(e^x e^y) = e^{x+y}. \tag{KVI}$$

These two conditions will clearly make the above diagram commute. The last thing to ensure is that $\tilde{F}$ is the identity on $\mathfrak{g} \times \mathfrak{g}$-invariant distributions on $\mathfrak{g} \times \mathfrak{g}$. For this we need the following result.

Lemma 2.13. *Let $F : \mathfrak{g} \to \mathfrak{g}$ be defined by $x \mapsto f(x)xf(x)^{-1}$ for some $f : \mathfrak{g} \to G$. Then for any $\mathfrak{g}$-invariant distribution α on $\mathfrak{g}$ one has*

$$(F \circ J(F))\alpha = \alpha,$$

where

$$J(F) := \det(\mathrm{Ad}_f)^{-1} \det(DF).$$

Illustration. Let us illustrate this result in the case where $\alpha = \varphi(x)dx$ is given by a smooth differential form, that is φ is a smooth function and dx is the translation invariant top form on $\mathfrak{g}$. By assumption α is $\mathfrak{g}$-invariant. Hence for any $g \in G$ we have

$$\varphi(x)dx = \varphi(gxg^{-1})d(gxg^{-1}) = \varphi(gxg^{-1}) \det(\mathrm{Ad}_g)dx,$$

and hence

$$\varphi(gxg^{-1}) = \varphi(x) \det(\mathrm{Ad}_g)^{-1}.$$

Now it follows that

$$F^*\alpha = \varphi(f(x)xf(x)^{-1})d(f(x)xf(x)^{-1}) = \det(\mathrm{Ad}_f)^{-1} \det(DF)\alpha.$$

In terms of distributions, we then have

$$\begin{aligned} \langle \alpha, \phi \rangle &= \langle F^*\alpha, F^*\phi \rangle \\ &= \langle J(F)\alpha, F^*\phi \rangle \\ &= \langle (F \circ J(F))\alpha, \phi \rangle. \end{aligned}$$

□

Applying this lemma to the Lie algebra $\mathfrak{g} \times \mathfrak{g}$, it is now clear that $\tilde{F}$ is the identity on invariants if we impose the following.

$$J(F) = \frac{h(x+y)}{h(x)h(y)} \tag{KVII}$$

Definition 2.14. A solution of the Kashiwara-Vergne problem is an $F \in \mathrm{Aut}(S\mathfrak{g} \otimes S\mathfrak{g})$ satisfying the conditions KV0, KVI and KVII.

The previous discussions culminate into the following theorem.

Theorem 2.15. *From the existence of a solution of the Kashiwara-Vergne problem it follows that* $\mathrm{Duf} : S\mathfrak{g}^{\mathfrak{g}} \to Z(U\mathfrak{g})$ *is an isomorphism of algebras.*

Instead of asking for a solution of the Kashiwara-Vergne problem, one can instead try and find a universal solution, that is a solution that is expressed solely in terms of the Lie bracket. More precisely, we can formulate the following.

Definition 2.16 ((Universal) Kashiwara-Vergne problem)**.** A solution of the universal Kashiwara-Vergne problem is an $F \in \mathrm{Aut}(\mathrm{Lie}(x, y))$ satisfying equations KV0, KVI and KVII.

In the following the word universal will be dropped.

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Chapter 3

Poisson Brackets in Kontsevich's "Lie World"

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Abstract: In this note we develop the theory of double brackets in the sense of van den Bergh [2] in Kontsevich's non-commutative"Lie World". These double brackets can be thought of as Poisson structures defined by formal expressions only involving the structure maps of a quadratic Lie algebra. The basic example is the Kirillov-Kostant-Souriau (KKS) Poisson bracket.

We introduce a notion of non-degenerate double brackets. Surprisingly, in this framework the KKS bracket turns out to be non-degenerate. The main result of the paper is the uniqueness theorem for double brackets with a given moment map. As applications, we establish a monoidal equivalence between Hamiltonian quasi-Poisson spaces and Hamiltonian spaces and give a new proof of the theorem by L. Jeffrey in [1] on symplectic structure on the moduli space of flat $\mathfrak{g}$-connections on a surface of genus 0.

Keyword: Kontsevich's non-commutative differential calculus, double Poisson structures
MSC: 17B63,14A22.

1 Introduction

In [5], Kontsevich defined a version of non-commutative geometry based on a free associative or free Lie algebra. The latter notion is referred to as the "Lie World". The coordinate algebra is identified with a free Lie algebra L with generators $x_1, \ldots, x_n$ and the space of functions is defined as the space of co-invariants under the adjoint action $(\mathrm{Sym}^2 L)_L$ where the following notation is used to denote elements,

$$\langle [x_1, x_2], x_3 \rangle = \langle x_1, [x_2, x_3] \rangle.$$

In this note, we develop some differential calculus on such non-commutative spaces and introduce the notion of a non-commutative Poisson structure. Our notion is essentially equivalent to the notion of double Poisson brackets in the sense of van den Bergh [2]. Defined in this framework, a non-commutative Poisson structure gives rise to Poisson structures on representation varieties $\mathrm{Hom}(L, \mathfrak{g})$ for all quadratic Lie algebras $\mathfrak{g}$. In particular, for $n = 1$ (the free Lie algebra with one generator), we define a non-commutative Poisson bracket which gives rise to Kirillov-Kostant-Souriau (KKS) Poisson structures on $\mathrm{Hom}(L, \mathfrak{g}) = \mathfrak{g} \cong \mathfrak{g}^*$. We define a non-degeneracy condition for non-commutative Poisson structures. It is interesting that the KKS Poisson structures (and their direct sums for $n > 1$) turn out to be non-degenerate (while this condition usually fails in the geometric context). We also define a natural notion of a moment map which gives rise to moment maps defined on the representation varieties. The central result of the paper is the theorem which states that (under certain conditions) the moment map uniquely defines a non-commutative Poisson structure. In the geometric context, such results hold only in very special cases such as coadjoint orbits or toric varieties.

Our theory finds several applications. For instance, we show that the equivalence between the categories of formal Hamiltonian quasi-Poisson spaces and formal Hamiltonian spaces, can be upgraded to a monoidal equivalence where one has to put a non-trivial associator on the category of Hamiltonian spaces. Another application is a new proof of Theorem 6.6 in [1] which states that the moduli space of flat $\mathfrak{g}$-connections on a surface of genus 0 with prescribed monodromy around the punctures is symplectomorphic to the symplectic reduction of the product of coadjoint orbits if the monodromies are sufficiently close to the identity. The proof of this statement reduces to finding an automorphism F of L which maps to each other the corresponding moment maps:

$$F(x_1 + \cdots + x_n) = \log(e^{x_1} \ldots e^{x_n}).$$

The isomorphism of Poisson structures follows by the theorem relating Poisson structures and moment maps.

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2 Lie spaces

We recall some definitions from [5]. Let **Lie** denote the category of free complete graded (super-)Lie algebras, where morphisms are continuous Lie algebra morphisms. We define the category **LieSp** of (formal affine) Lie spaces as the opposite category of Lie. This definition is very much in analogy with the equivalence of (commutative) affine schemes and $\mathbf{Ring}^{\mathrm{op}}$. Much of the language that follows is motivated by this analogy. By definition, there is a canonical contravariant functor

$$\mathcal{O} : \mathbf{LieSp} = \mathbf{Lie}^{\mathrm{op}} \longrightarrow \mathbf{Lie}.$$

More concretely, a Lie space $\mathcal{L}$ is nothing but a (graded) Lie algebra, which we choose to call the coordinate Lie algebra of the space and denote it by $\mathcal{O}(\mathcal{L})$. Morphisms between Lie spaces are maps of Lie algebras in the opposite direction. A choice of free homogenous generators of $\mathcal{O}(\mathcal{L})$ shall be called a coordinate system, or just coordinates. Let us denote by

$$L(z_1, \cdots, z_n) \in \mathbf{Lie},$$

the completed free graded (super-)Lie algebra in generators $z_1, \cdots, z_n$, where each generator has possibly non-zero degree, and the completion is taken with respect to the lower central series. Let furthermore

$$\mathcal{L}(z_1, \cdots, z_n) \in \mathbf{LieSp},$$

denote the Lie space whose coordinate Lie algebra is $L(z_1, \cdots, z_n)$. Thus $L(z_1, \cdots, z_n)$ and $\mathcal{L}(z_1, \cdots, z_n)$ are the same objects, the only difference is in the direction we choose to write morphisms, and of course in our interpretation. Using the language introduced above, the $z_1, \cdots, z_n$ are coordinates on the space $\mathcal{L}(z_1, \cdots, z_n)$. And elements of $\mathcal{O}(\mathcal{L}(z_1, \cdots, z_n))$ are Lie series in the coordinates $z_1, \cdots, z_n$. Let now

$$\mathcal{L}_n := \mathcal{L}(x_1, \cdots, x_n)$$
$$L_n := L(x_1, \cdots, x_n)$$

denote the above with all generators x_i of degree 0.

In this context, $\mathcal{L}_n$ is nothing but the product of n copies of the affine line $\mathcal{L}_1$, since products in **LieSp** are coproducts in **Lie** that is completed free products. In what follows, we wish to do differential geometry on these Lie spaces. Guiding our intuition is the fact, that each element of L_n induces a formal $\mathfrak{g}$-valued function on $\mathfrak{g}^{\times n}$. Abstractly this follows from the fact that $\mathfrak{g}^{\times n} = \mathrm{Hom}(L_n, \mathfrak{g})$, but more concretely is it seen by just interpreting elements in L_n as formulae. Take for instance $[x_1, x_2]$, it can be seen as a function taking as inputs two elements $x_1, x_2 \in \mathfrak{g}$ and giving as output another element of $\mathfrak{g}$. In this sense, the space $\mathcal{L}_n$ can be thought of as a "universal version" of $\mathfrak{g}^{\times n}$. If we want to produce a $\mathbf{k}$-valued function, one possibility is to take the product of two $\mathfrak{g}$-valued functions with respect to some inner product on $\mathfrak{g}$. Let us from now on assume that $\mathfrak{g}$ is a quadratic Lie algebra, i.e. there is a chosen non-degenerate invariant inner product. The definition of functions on a Lie space is then chosen such that it induces $\mathbf{k}$-valued functions on $\mathfrak{g}^{\times n}$, that is

$$F(\mathcal{L}) := \mathcal{O}(\mathcal{L}) \otimes \mathcal{O}(\mathcal{L}) / \left\{ a \otimes b - \pm b \otimes a,\ [a,b] \otimes c - a \otimes [b,c] \right\},$$

or in other words the object in vector spaces representing the functor of symmetric invariant inner products on L. We will denote the universal inner product by

$$\begin{aligned} \mathcal{O}(\mathcal{L}) \otimes \mathcal{O}(\mathcal{L}) &\longrightarrow F(\mathcal{L}) \\ a \otimes b &\longmapsto \langle a, b \rangle \,. \end{aligned}$$

Remark 3.1. Note that there is a difference between the space of functions and the coordinate algebra. Whereas the latter carries the structure of a Lie algebra, the former is merely a vector space, that is functions cannot be multiplied. To get an algebra, one might choose instead to work with the symmetric algebra over $F(\mathcal{L})$, however, we choose not to do so.

Remark 3.2. In terms of graphical calculus, elements of the coordinate Lie algebra can be seen as rooted Jacobi tree, whereas functions are simply Jacobi trees, where the leaves are labeled by generators of the Lie algebra. This picture will in particular explain later, why we cannot contract arbitrary forms with polyvector fields, since this would generate loops, and thus leave the world we choose to work in.

Remark 3.3 (Ass). As in [5] everything works analogously if one replaces Lie algebras by associative algebra. Instead of developing the theory in parallel, the differences are pointed out in remarks. In the associative world F also goes under the name of $HH_0(A) = A/[A, A]$, that is the zero-th Hochschild homology. Moreover, by embedding a free Lie algebra into its universal enveloping algebra, which is a free associative algebra, all "Lie" functions embed into "Ass" functions. The last part can be seen from the Cartan-Eilenberg isomorphism $HH(U(\mathfrak{g})) = H_{\mathrm{Lie}}(\mathfrak{g}, (U\mathfrak{g})^{\mathrm{ad}})$, which applied to our case says $HH_0(U(L_n)) = (U(L_n))_{L_n} \cong S(L_n)_{L_n}$, namely that "Ass" functions are the L_n-coinvariants of the symmetric algebra over L_n. In particular, we see that the quadratic part coincides with the definition of "Lie" functions. Graphically, we are replacing Jacobi trees with ribbon trees.

In order to get forms and polyvector fields, we introduce the odd tangent and cotangent bundle, respectively,

$$\begin{aligned} T[1]\left(\mathcal{L}(z_1,\cdots,z_n)\right) &:= \mathcal{L}(z_1,\cdots,z_n,dz_1,\cdots,dz_n), \quad |dz_i| = |z_i|+1, \\ T^*[1]\left(\mathcal{L}(z_1,\cdots,z_n)\right) &:= \mathcal{L}(z_1,\cdots,z_n,\partial_1,\cdots,\partial_n), \qquad |\partial_i| = -|z_i|+1, \end{aligned}$$

and

$$\begin{aligned} T[1]\mathcal{L}_n &:= \mathcal{L}(x_1,\cdots,x_n,dx_1,\cdots,dx_n), \quad |dx_i| = 1, \\ T^*[1]\mathcal{L}_n &:= \mathcal{L}(x_1,\cdots,x_n,\partial_1,\cdots,\partial_n), \qquad |\partial_i| = 1, \end{aligned}$$

in the non graded case. Their functions are then denoted by

$$\begin{aligned} \Omega(\mathcal{L}) &:= F(T[1]\mathcal{L}), \\ \mathfrak{X}(\mathcal{L}) &:= F(T^*[1]\mathcal{L}). \end{aligned}$$

Both are graded vector spaces and by the usual formulae $\Omega(\mathcal{L}_n)$ can be endowed with a differential of degree 1. After some preparation, the usual formulae can be used to define a Lie bracket on $\mathfrak{X}(\mathcal{L}_n)$ with a Lie bracket analogous to the Schouten bracket. The Schouten bracket can be interpreted as induced by the canonical odd symplectic stucture on $T^*[1]\mathcal{L}_n$. It will be shown that the bracket also defines an action of polyvector fields on the coordinate Lie algebra of $T^*[1]\mathcal{L}_n$. These structures are compatible with specialization, that is for any quadratic Lie algebra $\mathfrak{g}$ we get canonical maps

$$\begin{aligned} \Omega(\mathcal{L}_n) &\to \Omega(\mathfrak{g}^{\times n}), \\ \mathfrak{X}(\mathcal{L}_n) &\to \mathfrak{X}(\mathfrak{g}^{\times n}), \end{aligned}$$

of complexes and Lie algebras, respectively. More concretely, let e_α be a basis of $\mathfrak{g}$. Let $t_{\alpha\beta} = \langle e_\alpha, e_\beta\rangle$ be the coefficients of the inner product and $t^{\alpha\beta}$ its inverse. Let x^α denote the dual basis of e_α and hence a coordinate system on $\mathfrak{g}$. The above maps are then induced by

$$\begin{aligned} \mathcal{O}(T[1]\mathcal{L}_n) &\longrightarrow \Omega(\mathfrak{g}^{\times n}) \otimes \mathfrak{g} \\ x_i &\longmapsto x_i^\alpha \otimes e_\alpha \\ dx_i &\longmapsto dx_i^\alpha \otimes e_\alpha \end{aligned}$$

and

$$\begin{aligned}\mathcal{O}(T^*[1]\mathcal{L}_n) &\longrightarrow \mathfrak{X}(\mathfrak{g}^{\times n}) \otimes \mathfrak{g}\\ x_i &\longmapsto x_i^\alpha \otimes e_\alpha\\ \partial_i &\longmapsto t^{\alpha\beta} \tfrac{\partial}{\partial x_i^\alpha} \otimes e_\beta.\end{aligned}$$

To descend to functions, the inner product on $\mathfrak{g}$ is applied on the $\mathfrak{g}$ factor. The invertibility of the inner product on $\mathfrak{g}$ is only used in the second map. A form, polyvector field or $\mathfrak{g}$-valued function on $g^{\times n}$ induced by an object on $\mathcal{L}_n$ will be called *universal.* For example, the KKS Poisson bivector on $\mathfrak{g}$, $\langle x, [\partial_x, \partial_x]\rangle$, is a universal bivector field. Let us explicitly compute the image of this bivector field under the above map as follows,

$$\begin{aligned}\left\langle x^\alpha \otimes e_\alpha, \left[t^{\beta\gamma} \tfrac{\partial}{\partial x_i^\beta} \otimes e_\gamma, t^{\delta\epsilon} \tfrac{\partial}{\partial x_i^\delta} \otimes e_\epsilon\right]\right\rangle &= t_{\alpha\eta} c^\eta_{\gamma\epsilon} t^{\beta\gamma} t^{\delta\epsilon} x^\alpha \tfrac{\partial}{\partial x^\beta} \tfrac{\partial}{\partial x^\delta}\\ &= c^{\beta\delta}_\alpha x^\alpha \tfrac{\partial}{\partial x^\beta} \tfrac{\partial}{\partial x^\delta},\end{aligned}$$

where $c^\eta_{\gamma\epsilon}$ are the structure constants of $\mathfrak{g}$ and in the last step we raised and lowered indices using the inner product. The adjoint action on $\mathfrak{g}$, seen as a $\mathfrak{g}$-valued vector field using the inner product, is universal, as it is induced by $[x, \partial_x]$. Moreover, these objects get represented faithfully that way, as shown by

Lemma 3.4. *Let $f \in \Omega(\mathcal{L}_n), \mathfrak{X}(\mathcal{L}_n)$ or $\mathcal{L}_n$. If $f \neq 0$ then f induces a non-zero object on $\mathfrak{sl}(N)$ with its Killing form for N sufficiently large.*

Proof. Using polarization, one can reduce to the case where f is linear in each coordinate. Any form or vector field that is multi-linear in the odd variables, can be seen as an ordinary multi-linear function on twice as many variables, by identifying $T\mathfrak{g}^{\times n} \cong \mathfrak{g}^{\times 2n}$ and $T^*\mathfrak{g}^{\times n} \cong \mathfrak{g}^{\times 2n}$. By embedding into the associative world, the problem is reduced to showing that the set of functions on $\mathfrak{sl}(N)^n$ of the form

$$\operatorname{tr}(\operatorname{ad}_{x_{\sigma(1)}} \cdots \operatorname{ad}_{x_{\sigma(n)}}) + (-1)^n \operatorname{tr}(\operatorname{ad}_{x_{\sigma(n)}} \cdot \operatorname{ad}_{x_{\sigma(1)}})$$

for$\sigma \in S_n$, are linearly independent. This can be seen by direct computation. □

Remark 3.5. One can also use the double of the truncated free Lie algebra (with zero cobracket) to show faithfulness.

It is clear that on any given quadratic Lie algebra we only get a comparatively small amount of functions, forms and vector fields, in particular all the objects are $\mathfrak{g}$-invariant. The way one can use lemma 3.4 is that whenever we have a construction or operation on a Lie space that induces a corresponding construction or operation on a concrete Lie algebra, one can use lemma 3.4 to show that identities that hold on all lie algebras also hold on the Lie space. The theory of Lie spaces can thus be thought of studying structures on $\mathfrak{g}^{\times n}$ that are of a particularly natural type, that is in the image of the above specialization maps.

The usual yoga using contraction with the Euler vector field shows that $\Omega(\mathcal{L}_n)$ is acyclic. Moreover, there is a simple description $\Omega^1(\mathcal{L}_n)$ and $\Omega^2(\mathcal{L}_n)$.

Lemma 3.6. *The following maps are isomorphisms of vector spaces.*

$$\begin{aligned} {L_n}^{\times n} &\longrightarrow \Omega^1(\mathcal{L}_n) \\ (\alpha_i) &\longmapsto \Sigma \langle dx_i, \alpha_i \rangle \end{aligned}$$

$$\begin{aligned} \mathfrak{u}\,(n, \mathcal{U}(L_n)) = \{(a_{ij}) \in U(L_n)\,,\, a_{ij} + *a_{ji} = 0\} &\longrightarrow \Omega^2(\mathcal{L}_n) \\ (a_{ij}) &\longmapsto \Sigma \left\langle dx_i, \mathrm{Ad}_{a_{ij}}\, dx_j \right\rangle, \end{aligned}$$

where $*$ is the canonical antipode on the universal enveloping algebra of L_n.

In words, the lemma says that the space of 1-forms is given by an n-tuple of Lie series, whereas the space of 2-forms is given by a skew-symmetric matrix of associative series, where the antipode is used for the skew-symmetry.

Using the lemma we define the maps $\frac{\partial}{\partial x_i} : F(\mathcal{L}_n) \to L_n$ as the composition of d, the inverse of the above map, and projection onto the ith component, or equivalently such that

$$d\alpha = \Sigma \left\langle dx_i, \tfrac{\partial \alpha}{\partial x_i} \right\rangle \quad \text{for } \alpha \in F(\mathcal{L}_n).$$

Remark 3.7 (Ass). The same is true in the "associative" world, where the role of $\mathcal{U}(L_n)$ is now played by $A \otimes A$, where A is the underlying free associative algebra, because these objects encode functions that are linear in two additional "separator" variables.

Proof. Surjectivity follows easily from the defining relations in both cases. For injectivity one defines the operation of contracting with the coordinate vector fields ∂_i as follows. Let ι_{∂_i} denote the derivation of degree -1 on $\mathcal{O}(T[1]\mathcal{L}_n)$ with values in its universal enveloping algebra with module structure given my left multiplying by specifying

$$\iota_{\partial_i}(dx_j) = \delta_{ij}, \quad \iota_{\partial_i}(x_j) = 0.$$

One checks that the following map is well-defined

$$\begin{aligned} \Omega(\mathcal{L}_n) &\xrightarrow{\iota_{\partial_i}} \mathcal{O}(T[1]\mathcal{L}_n) = L(x_1, \cdots, x_n, dx_1, \cdots, dx_n) \\ \langle \alpha, \beta \rangle &\longmapsto \mathrm{ad}^*_{\iota_{\partial_i}\alpha}\, \beta + (-1)^{|\alpha||\beta|}\, \mathrm{ad}^*_{\iota_{\partial_i}\beta}\, \alpha. \end{aligned}$$

This operation has the following defining property. Let ϕ be the derivation of degree -1 on $\mathcal{O}(T[1]\mathcal{L}_n)$ with values in $L\left(x_1,\ldots,x_n,dx_1,\ldots,dx_n,t\right)$, where t has degree 0, defined by

$$\phi\left(dx_j\right)=\delta_{ij}t,\quad \phi\left(x_j\right)=0,$$

which straightforwardly extends to

$$\Omega(\mathcal{L}_n)\xrightarrow{\phi}F(\mathcal{L}(x_1,\cdots,x_n,dx_1,\cdots,dx_n,t)).$$

The maps ι_{∂_i} and ϕ are then adjoint to each other in the following sense, that for any $\omega\in\Omega(\mathcal{L}_n)$ we have

$$\phi(\omega)=\left\langle t,\iota_{\partial_i}\left(\omega\right)\right\rangle .$$

Now the lemma follows by applying the operator ι_{∂_i} to the elements of the specified form. More precisely, for one forms the map $(\iota_{\partial_i})_{i=1,\ldots,n}$ is an inverse to the map in the lemma. And in the case of 2-forms it follows from the observation, that any degree 1 element in $L(x_1,\cdots,x_n,dx_1,\cdots,dx_n)$ is of the form $\sum\mathrm{ad}_{\pi_i}dx_i$ for uniquely determined associative series π_i. □

Using the above lemma we can construct the Schouten bracket. We define the following map of degree -1

$$\begin{aligned}\mathfrak{X}^\bullet&\xrightarrow{[\cdot,\cdot]}\mathrm{Der}(T^*[1]L_n)\\ \alpha&\longmapsto[\alpha,x_i]=(-1)^{|\partial_i|(|\alpha|-|\partial_i|)}\frac{\partial\alpha}{\partial(\partial_i)}\\ &\qquad[\alpha,\partial_i]=-(-1)^{|x_i|(|\alpha|-|x_i|)}\frac{\partial\alpha}{\partial x_i}.\end{aligned}$$

Remark 3.8. Note that the gory signs would disappear if we choose right partial derivatives instead of left partial derivatives. The formula can then be seen as induced by

$$[\cdot,\cdot]=\overleftarrow{\frac{\partial}{\partial(\partial_i)}}\overrightarrow{\frac{\partial}{\partial x_i}}-(-1)^{|x_i||\partial_i|}\overleftarrow{\frac{\partial}{\partial x_i}}\overrightarrow{\frac{\partial}{\partial(\partial_i)}}$$

It is straightforward to check that this operation descends to a bracket on $\mathfrak{X}^\bullet$ and the above map intertwines this bracket with the commutator of derivations. In particular, the bracket on $\mathfrak{X}^\bullet$ satisfies the Jacobi identity.

Moreover, the degree one part identifies vector fields with derivations, that is

$$\begin{aligned}\mathfrak{X}^1&\longrightarrow\mathrm{Der}(L_n)\\ \sum\left\langle\alpha_i,\partial_i\right\rangle&\longmapsto(x_i\mapsto\alpha_i).\end{aligned}$$

As the construction of $\Omega^\bullet$ is covariant, vector fields also act on forms, we denote this action by the usual symbol for Lie derivatives, that is by L_X for $X\in\mathfrak{X}^1$

2.1 Poisson bivector fields

Definition 3.9. A bivector field $\Pi \in \mathfrak{X}^2$ is called a *Poisson structure* if $[\Pi, \Pi] = 0$. It is called *non-degenerate* if the matrix representing it by lemma 3.6 is not a zero-divisor.

Any bivector field Π induces a bracket by the formula $\{\alpha, \beta\}_\Pi = [\alpha, [\Pi, \beta]]$, where α, β are either both functions, or one of them is a function and the other one an element of the coordinate algebra. If Π is Poisson, then this defines a Lie bracket on the space of functions and an action of this Lie algebra of functions onto the underlying Lie algebra. However, the bivector field Π carries more information than these operations, and more precisely, there exist $\Pi_1 \neq \Pi_2$ distinct bivector fields that induce the same operations, but only Π_1 is Poisson, whereas Π_2 is not (see examples of quasi-Poisson structures later).

Example 3.10. Any bivector field of the form

$$\left\langle \alpha, \sum_j [x_j, \partial_j] \right\rangle$$

for arbitrary α induces a trivial Lie bracket on functions. In the associative case, one can show the converse.

Remark 3.11. It is possible to view a bivector field as a kind of biderivation on the Lie algebra with values in its universal enveloping algebra. To make this precise, one can add two "placeholder" symbols s,t as generators of the Lie algebra. The formula $(a, b) \mapsto \{\langle a, s\rangle, \langle b, t\rangle\}_\Pi$ defines a bracket with values in functions linear in s and t, which we can identify with the universal enveloping algebra similar as in lemma 3.6 and in particular in its proof. Bivector fields are in one-to-one correspondence with such mappings that are biderivations in a suitable sense.

Remark 3.12 (Ass). Using the same formula $(a, b) \mapsto \{\langle a, s\rangle, \langle b, t\rangle\}_\Pi$ for $a, b \in A$ in the associative world, gives rise to a map $A \otimes A \to A \otimes A$. One checks that the theory of Poisson brackets in the associative world is equivalent to the theory of double Poisson brackets of van den Bergh [2] in the case where the underlying algebra is free. Moreover, the embedding of the "Lie" world into the "Ass" world preserves non-degeneracy as will be clear from the proof of lemma 3.16 below.

Remark 3.13. Note that the notion of non-degeneracy does not descend to specialization and is easier to satisfy.

Example 3.14. The following two bivector fields can readily be seen to be non-degenerate Poisson structures.

$$\Pi_{\mathrm{KKS}} = \langle \partial_x, [x, \partial_x] \rangle$$
$$\Pi_{\mathrm{symp}} = \langle \partial_x, \partial_y \rangle$$

More examples are constructed by taking direct sums of those, which will also be denoted by the same symbol if only one type is used.

Lemma 3.15 ("Weinstein's splitting theorem"). *Given a Poisson structure* Π*, one finds coordinates, i.e. free generators* x_i, y_i, z_j *of the underlying Lie algebra, such that* $\Pi = \sum \langle \partial_{x_i}, \partial_{y_i} \rangle + \tilde{\Pi}$*, where* $\tilde{\Pi}$ *does only contain terms in the variables* z_j *and does not contain any constant terms.*

Proof. Let Π_0 denote the constant part of Π. We are trying to classify deformations of Π_0 that are governed by the dg Lie subalgebra of the Schouten Lie algebra $(\mathfrak{X}^\bullet, [\cdot,\cdot], d = [\Pi_0, \cdot])$ where vector fields have components at least quadratic and bivector fields are at least linear. One checks that this is actually a direct summand. One checks that $(\mathfrak{X}^\bullet, d)$ deformation retracts onto $(\mathfrak{X}(z_1, \cdots, z_n), d = 0)$ where the z_j form a basis of the null space of Π_0. The result now follows since the gauge group in this case is pronilpotent.

□

Each bivector field $\Pi = \Sigma \left\langle \partial_i, \operatorname{Ad}_{\Pi_{ij}} \partial_j \right\rangle \in \mathfrak{X}^2$ defines a map $T^*[1]\mathcal{L} \overset{(-)^\Pi}{\to} T[1]\mathcal{L}$ by the formula

$$(dx_i)^\Pi = [\Pi, x_i] = \iota_{dx_i}\Pi = 2 \operatorname{Ad}_{\Pi_{ij}} \partial_j$$

Lemma 3.16. *Let* $\Pi \in \mathfrak{X}^2$ *be non-degenerate. Then the induced map*

$$\Omega^n \xrightarrow{(-)^\Pi} \mathfrak{X}^n$$

is injective for $n \geq 2$.

Proof. Let A denote the universal enveloping algebra of the free Lie algebra, that is the free associative algebra. Let DA denote the free A-bimodule generated by the symbols ∂_i. By symmetrizing over the permutation group in n symbols the space of n-forms Ω^n can be embedded into $DA^{\otimes n} \otimes_{A \otimes A^{\mathrm{op}}} A$. The map $(-)^\Pi$, for simplicity seen as an endomorphism of Ω^n, extends as follows to an injective map. The non-degeneracy assumption is equivalent to Π defining an injective map of a free right-A module and consequently (using skew-adjointness of Π) also an injective map ϕ of a free left-A module A^m where m is the dimension. More concretely, let e_i be the canonical basis of A^m, then

$$\begin{aligned} A^m &\xrightarrow{\phi} A^m \\ \alpha_i e_i &\longmapsto \alpha_i \Pi_{ij} e_j \end{aligned}$$

is an injective map of free left A-modules. Consider $A \otimes A$ as a right-A module using the following A-bimodule structure

$$(\alpha \otimes \beta).a = \alpha a^{'} \otimes *a^{''}\beta,$$

where Sweedler's notation is used, and $*$ denotes the antipode. This defines a free A-module, as the invertible map $\alpha \otimes \beta \mapsto \alpha^{'} \otimes \beta^{''}$ gives a map to an obviously free A-module. The left-$A \otimes A^{\mathrm{op}}$ module structure coming from the outer A-bimodule structure is left as it is. The natural map $DA \to DA$ determined by Π is now given by $\mathrm{id} \otimes \phi$, that is

$$\begin{aligned} DA \cong (A \otimes A) \otimes_A A^m &\xrightarrow{\mathrm{id} \otimes \phi} && (A \otimes A) \otimes_A A^m \cong DA \\ \alpha \partial_i \beta \cong (\alpha \otimes \beta) \otimes e_i &\longmapsto (\alpha \otimes \beta) \otimes \Pi_{ij} e_j = && (\alpha \Pi_{ij}^{'} \otimes *\Pi_{ij}^{''} \beta) \otimes e_j \cong \alpha \, \mathrm{ad}_{\Pi_{ij}}(\partial_j)\beta, \end{aligned}$$

and is injective since we tensor an injective map with a free module. To finish the proof, one notices that $DA^{\otimes k}$ is a free A-bimodule for all $k \geq 1$ and $DA^{\otimes n} \otimes_{A \otimes A^{\mathrm{op}}} A \cong DA^{\otimes n-k} \otimes_{A \otimes A^{\mathrm{op}}} DA^{\otimes k}$ for any k (here $DA^{\otimes n-k}$ being an A-bimodule is naturally a right-$A \otimes A^{\mathrm{op}}$ module). □

Lemma 3.17. *If Π is Poisson, there is a canonical Lie bracket on $\Omega^\bullet$ such that $(-)^\Pi$ is a map of dgLAs.*

Proof. By lemma 3.4 it is enough to define the bracket on the left hand side that specializes to the classical one. To that purpose we define an odd Poisson bivector field on $T^*[1]\mathcal{L}_n$ as follows

$$\widetilde{\Pi} = 2 \left\langle \tfrac{\partial}{\partial dx_i}, \mathrm{ad}_{\Pi_{ij}} \tfrac{\partial}{\partial x_j} \right\rangle + \left\langle \tfrac{\partial}{\partial x_i}, \mathrm{ad}_{d\Pi_{ij}} \tfrac{\partial}{\partial x_j} \right\rangle,$$

which satisfies the required properties. □

For later convenience we spell out the bracket on 1-forms, that is of the Lie algebra of 1-forms denoted by Ω^1_Π. Let $\alpha = \langle \alpha_i, dx_i \rangle$ $\beta = \langle \beta_j dx_j \rangle$ be 1-forms, then

$$[\langle \alpha_i, dx_i \rangle, \langle \beta_j, dx_j \rangle]_\Pi = \left\langle [\alpha^\Pi, \beta_i] - [\beta^\Pi, \alpha_i], dx_i \right\rangle + 2 \left\langle \alpha_i, \mathrm{Ad}_{d\Pi_{ij}} \beta_j \right\rangle.$$

The dgLA $\mathfrak{X}^\bullet$ is canonically filtered by (polynomial degree -1 + form degree -1). This also defines a filtration on $\Omega^\bullet_\Pi$, and by the splitting lemma it is clear that the zeroth associated graded component is a copy of a $\mathfrak{gl}_{2k}$, where $2k$ is the dimension of the symplectic vector space determined by the constant term of Π. In particular, Ω^1_Π is then an extension of a $\mathfrak{gl}_{2k}$ by a pronilpotent Lie algebra, and hence it's easily integrated to a group $Exp(\Omega^1_\Pi)$ together with its actions on forms and polyvector fields. Moreover, $Exp(\Omega^1_\Pi)$ is a central extension of a Lie algebra of derivations of $\mathcal{L}_n$ by a factor, which in the non-degenerate case can be identified with the space of Casimir functions, that is $\{f \in F(L_n) \mid [\Pi, f] = 0\}$.

Remark 3.18. The bivector field $\tilde{\Pi}$ is actually Poisson and the map $(-)^\Pi$ is a Poisson map, for a straightforward definition of Poisson maps. However, we shall have no use for this slightly stronger fact.

Remark 3.19. A more geometric construction goes as follows. Let $\mathcal{C} = T^*[2]T[1]\mathcal{L}_n = T[1]T^*[1]\mathcal{L}_n$ denote the standard Courant algebroid. Let $Q \in F(\mathcal{C})$ denote the Euler vector field on $T[1]M$, which is a Hamiltonian for the de Rham differential on $\mathcal{C}$. The map $(-)^\Pi$ can be seen as the composition

$$T^*[1]\mathcal{L}_n \to \mathcal{C} \overset{\exp(\{\Pi,\cdot\})}{\to} \mathcal{C} \to T[1]\mathcal{L}_n,$$

where the first and third maps are canonical. Thus the Courant bracket $\{\cdot, \{Q, \cdot\}\}$ gets twisted to $\{\cdot, \{\widetilde{Q}, \cdot\}\}$, where $\widetilde{Q} = \exp(\{\Pi, \cdot\})(Q)$. This is indeed a Poisson bracket on $T[1]\mathcal{L}_n$ iff it is at most quadratic, i.e. iff $\{\Pi, \{\Pi, Q\}\} = 0$, i.e. iff Π is Poisson. In that case $\widetilde{Q} = Q + \tilde{\Pi}$.

Maurer-Cartan elements in $\Omega^\bullet$ thus inject to the ones in $\mathfrak{X}^\bullet$. Namely, a Maurer-Cartan element $\sigma \in \Omega^2$ defines a new Poisson bracket $\Pi + \sigma^\Pi$. In terms of matrices (cf lemma 3.6) this Poisson structure is given by $\Pi - \Pi\sigma\Pi = \Pi(1 - \Pi\sigma)$. We call $\sigma \in \Omega^2$ non-degenerate if the matrix $(1 - \Pi\sigma)$ is invertible. This is in particular sufficient for $\Pi + \sigma^\Pi$ to be a non-degenerate Poisson structure. Let us denote by Π_0 and σ_0 the constant terms, then the condition is equivalent to $(1 - \Pi_0\sigma_0)$ being non-degenerate, i.e. invertible.

Corollary 3.20. *The set of non-degenerate Maurer-Cartan elements in $\Omega^\bullet$ is an $\mathcal{E}xp(\Omega^1_\Pi)$ homogeneous space isomorphic to $\mathcal{P} := \{\Pi + \sigma^\Pi \mid [\Pi + \sigma^\Pi, \Pi + \sigma^\Pi] = 0, (1 - \Pi_0\sigma_0)$ invertible$\}$ where the action on the latter is by automorphism of the free Lie algebra. The stabilizer Lie algebra at σ is isomorphic to Ω^0 with Lie bracket induced by the Poisson bracket $\Pi + \sigma^\Pi$.*

Proof. The isomorphism alluded to is given by $(-)^\Pi$, which is injective on forms of degree ≥ 2. Transitivity of the $\mathcal{E}xp(\Omega^1_\Pi)$ action essentially follows from acyclicity of

$\Omega^\bullet$. More precisely, by the splitting lemma we can write $\mathcal{E}xp(\Omega^1_\Pi)$ as a prounipotent extension of GL_k. The action of $\mathcal{E}xp(\Omega^1_\Pi)$ descends to this GL_k by only considering the constant terms. It is then just the usual action of GL_k on symplectic forms on a vector space. The invertibility condition of the theorem ensures that we get a symplectic form again, that is it does not become degenerate. Let now $\sigma \in \Omega^2$ be any non-degenerate Maurer-Cartan element. By applying an element from GL_k we can assume that $\Pi + \sigma^\Pi$ has the same constant term as Π, and thus σ lies in the pronilpotent part of $\Omega^\bullet$. Now the standard argument in an acyclic pronilpotent dgLA shows that σ is gauge equivalent to 0.

For the statement about the stabilizer, we only need to determine it at $\sigma = 0$ by the transitivity of the $\mathcal{E}xp(\Omega^1_\Pi)$ action, where it is clear. □

Remark 3.21. If Π has no constant part, then the assumption on non-degeneracy is void.

Lemma 3.22. *The set $\mathcal{P}$ is in bijection with non-degenerate closed 2-forms. More precisely,*

$$\begin{aligned}\mathcal{P} = \{\Pi^\omega = (1-\Pi\omega)^{-1}\Pi \mid d\omega = 0,\ (1-\Pi_0\omega_0)\ \textit{invertible}\ \} \\ \cong \{\omega \in \Omega^{2,cl} \mid (1-\Pi_0\omega_0)\ \textit{invertible}\ \}.\end{aligned}$$

Proof. One checks that each element in $\Pi \in \mathcal{P}$ is of the form $(1-\Pi\omega)^{-1}\Pi$ for a unique $\omega \in \Omega^2$, by solving the equation of matrices with entries in the free associative algebra,

$$\Pi - \Pi\sigma\Pi = (1-\Pi\omega)^{-1}\Pi,$$

for ω, namely one gets

$$\omega = \sigma(1-\Pi\sigma)^{-1}. \tag{3.1}$$

Uniqueness follows since one uses non-degeneracy of Π. Moreover, this is well-defined, since by definition of $\mathcal{P}$ the matrix $(1-\Pi\sigma)$ is invertible. Moreover, since $(1-\Pi\sigma) = (1-\Pi\omega)^{-1}$, we see that the invertiblity condition is the same as in the definition of $\mathcal{P}$. It remains to check that $d\sigma + [\sigma,\sigma]_\Pi$ is equivalent to $dw = 0$. For this purpose, let us define the map $C : \mathcal{E}xp(\Omega^1_\Pi) \to \Omega^2$, by the assignment $C(\phi) = \omega$ for $\phi.\Pi = \Pi^\omega$ and $\phi \in \mathcal{E}xp(\Omega^1_\Pi)$. One checks that this is a group 1-cocyle. Since $\mathcal{E}xp(\Omega^1_\Pi)$ acts transitively on $\mathcal{P}$, the image of C is exactly the image of $\mathcal{P}$ under the map defined by the formula (3.1). The associated Lie algebra 1-cocycle c is given by $\lambda \mapsto d\lambda$, which implies that any ω defined by (3.1) is indeed closed. To show that we get any non-degenerate closed 2-form, we first reduce to the case where $\Pi_0\omega_0 = 0$

by using the GL_{2k} action. By solving a Moser flow type equation one shows that there is a one-parameter family $\phi_t \in \mathcal{E}xp(\Omega^1_\Pi)$ such that $C(\phi_t) = t\omega$. More precisely let $\omega = d\lambda$ and $\dot{\phi}\phi^{-1} = \alpha_t$, the Moser equation is then

$$c(\alpha_t) + (\alpha_t).(t\omega) = \omega,$$

with a solution given by $\alpha_t = \lambda(1 + t\Pi\omega)^{-1}$.

□

Remark 3.23. The cocycle $C : \mathcal{E}xp(\Omega^1_\Pi) \to \Omega^{2,cl}$ can be lifted to a group cocycle with values in Ω^1 if there is a Liouville vector field for the Poisson structure, that is if there exists X such that $[\Pi, X] = \Pi$. This is in particular true for any homogeneous Poisson structure. For the KKS Poisson structure this is used in [9].

Remark 3.24. A different proof for the last steps can be obtained by showing that the formula

$$[\Pi^\omega, \Pi^\omega] = 2(d\omega)^{\Pi^\omega}$$

holds for all non-degenerate 2-forms ω. One quick way of seeing this is by using lemma 3.4 it can be reduced to the same formula in "ordinary" differential geometry, where it follows for symplectic Π by the usual formula

$$[\Pi, \Pi] = -2(d(\Pi^{-1}))^\Pi$$

and for general Π by a density argument.

2.2 Moment Maps

Let $\rho = \sum[x_i, \partial_i] \in T^*[1]\mathcal{L}_n$ denote the canonical action vector field. Note that it is independent of the choice of coordinates.

Remark 3.25. Viewed under the natural embedding of Lie into associative algebras, ρ corresponds to the canonical element in $\mathrm{Der}(A, A \otimes A)$ that maps $a \mapsto 1 \otimes a - a \otimes 1$.

One defines the operator $\iota_\rho : \Omega^\bullet \to \mathcal{O}(T^*[1]\mathcal{L}_n)$ by the following procedure. Adjoin an extra variable t and define $\rho^t := \langle t, \rho\rangle$ as in the proof of lemma 3.6. Now write ρ^t as $\langle \partial_i, \rho^t_i\rangle$ and define a derivation ι_{ρ^t} of degree -1 on $T^*[1](\mathcal{L}_n \times \mathcal{L}(t)$ by sending $dx_i \mapsto \rho^t_i$, which descends to a derivation on $\Omega^\bullet(\mathcal{L}_n \times \mathcal{L}(t))$. The operator ι_ρ is now defined by the formula

$$\iota_{\rho^t}\alpha = \langle t, \iota_\rho(\alpha)\rangle \quad \text{for } \forall\alpha \in \Omega^\bullet(\mathcal{L}_n)$$

Lemma 3.26.

$$\begin{aligned} \iota_{\rho^t}(d\alpha) &= [\alpha, t] && \forall \alpha \in F(\mathcal{L}_n) \\ \iota_\rho \left\langle dx_i, \mathrm{Ad}_{\omega_{ij}} dx_j \right\rangle &= 2 \left\langle dx_i, \mathrm{Ad}_{x_i} \mathrm{Ad}_{\omega_{ij}} dx_j \right\rangle && \forall \left\langle dx_i, \mathrm{Ad}_{\omega_{ij}} dx_j \right\rangle \in \Omega^2(\mathcal{L}_n) \end{aligned}$$

Definition 3.27. An element $\mu \in \mathcal{L}_n$ is called a *moment map* for Π if

$$[\Pi, \mu] = \rho$$

Remark 3.28. The existence of a moment map implies non-degeneracy.

Remark 3.29. Any linear Poisson structure, for instance the linear part of a splitting of an arbitrary Poisson structure, defines a Lie algebra in the "Lie world", which by Lazard duality is a commutative algebra C. Non-degeneracy is equivalent to the absence of elements $\beta \in C$ such that $\alpha\beta = 0\ \forall \alpha \in C$, and existence of a moment map is equivalent to the existence of a unit element in C. Moreover, the Poisson structure is rigid, if C is semi-simple, i.e. does not contain nilpotents, i.e. is the product of finite field extensions.

Lemma 3.30. *Let $\Pi \in \mathfrak{X}^2$ be a non-degenerate Poisson structure. If it admits a moment map, then it is unique.*

Proof. This can be seen by rewriting the moment map condition as $d\mu^\Pi = \rho$ and using non-degeneracy of Π. □

The next theorem is the main result of this note, showing that Poisson structures are essentially uniquely determined by their moment maps. Let $\Pi \in \mathfrak{X}^2$ be non-degenerate Poisson with moment map $\mu \in \mathcal{L}_n$. Let Π_0 be the constant terms of Π, in particular Π_0 defines a pairing on an n-dimensional vector space V. Let Z denote the kernel of this pairing. One checks that the degree 2 part of μ, denoted by μ_2, endows V/Z with a linear symplectic structure.

Theorem 3.31. *Given $\Pi \in \mathfrak{X}^2$ a non-degenerate Poisson structure with moment map $\mu \in \mathcal{L}_n$. Then we have the following.*

i) All elements in $\mathcal{P}$ admit a unique moment map.

ii) Assigning the moment map to a given Poisson structure in $\mathcal{P}$ defines an injective map $\mathcal{P} \to L_n$, compatible with the transitive $\mathcal{E}xp(\Omega^1_\Pi)$-action.

iii) The image of this map is

$$\{\phi.\mu \ \mid \phi \in \mathcal{E}xp(\Omega^1_\Pi)\} = \{\tilde{\mu} \in \mu + L_n^{\geq 2} \mid \tilde{\mu}_2 \textit{ is non-degenerate on } V/Z\}$$

.

The theorem gives us a well-defined map

$$\begin{aligned} \{\phi.\mu \ \mid \phi \in \mathcal{E}xp(\Omega^1_\Pi)\} &\longrightarrow \Omega^{2,cl} \\ \eta &\longmapsto \omega^\eta, \end{aligned}$$

with the property that $\Pi^{\omega^\eta} = (1 - \Pi\omega^\eta)^{-1}\Pi$ is Poisson with moment map η, which is a bijection in the case where Π has no constant terms.

Proof. i) Note that since each element in $\mathcal{P}$ is of the form $\phi.\Pi$ for some $\phi \in \mathrm{Aut}_n$ there is at least one corresponding moment map, namely $\phi.\mu$, which is unique by the previous lemma.

ii) For a $\mu + \tilde{\mu}$ and a 2-form ω, one can rewrite the equation $[\Pi^\omega, \mu + \tilde{\mu}] = \rho$ as $d\tilde{\mu} = \iota_\rho \omega$, or

$$\mathrm{Ad}_{\tilde{\mu}_i} = -\sum_j \mathrm{Ad}_{x_j} \mathrm{Ad}_{\omega_{ji}}, \tag{3.2}$$

where $d\tilde{\mu} = \sum \mathrm{Ad}_{\tilde{\mu}_i} dx_i$, $\omega = \sum \left\langle dx_i, \mathrm{Ad}_{\omega_{ij}} dx_j \right\rangle$. This shows injectivity, namely the equation uniquely determines ω_{ij}. The compatibility with the $\mathcal{E}xp(\Omega^1_\Pi)$-action is clear.

iii) For surjectivity one can check directly that the form ω defined by equation (3.2) is indeed closed and has the desired property. A more geometric construction suggested by Ševera goes as follows. Let M denote the central extension of $(T[1]\mathcal{L}_n)^{\deg \leq 1}$ by Ω^2, that is $[\alpha, \beta] = \langle \alpha, \beta \rangle$ for 1-forms α, β. One checks that elements of the form $\tilde{\mu} + d\tilde{\mu} + \omega$ with ω closed form a subalgebra. Thus ω can be constructed by taking the degree 2 part of $\tilde{\mu}(x_1 + dx_1, \ldots, x_n + dx_n)$. To write down this element one needs to write $\tilde{\mu} = \sum [\mu_1^k, \mu_2^k]$ to determine

$$\omega = \left\langle d\mu_1^k, d\mu_2^k \right\rangle.$$

One computes

$$\begin{aligned} \iota_{\rho^t}\omega &= \left\langle \iota_{\rho^t} d\mu_1^k, d\mu_2^k \right\rangle - \left\langle d\mu_1^k, \iota_{\rho^t} d\mu_2^k \right\rangle \\ &= \left\langle [\mu_1^k, t], d\mu_2^k \right\rangle - \left\langle d\mu_1^k, [\mu_2^k, t] \right\rangle \\ &= -\left\langle t, [\mu_1^k, d\mu_2^k] + [d\mu_1^k, \mu_2^k] \right\rangle \\ &= -\langle t, d\tilde{\mu} \rangle. \end{aligned}$$

By computing the constant terms one checks that the condition on the mompent map is equivalent to ω being non-degenerate.

□

Remark 3.32 (Ass). There is also a version of the theorem in the associative case. Here the coefficients ω_{ij} lie in $A \otimes A^{op}$. Similar as in lemma 3.16, one can choose an automorphism of $A \otimes A$ such that Ad_{x_j} is represented by left multiplying on the left factor. Then one checks that for 3.2 to have a solution, necessarily $\tilde{\mu} \in [A, A]$, which is also sufficient for the rest of the proof to go through.

Example 3.33. The theorem states, that a moment map uniquely determines a Poisson bracket in a given gauge class. However, there exist distinct Poisson brackets with the same moment map. To construct an example, consider $\Phi \in \mathrm{Aut}(L_3)$ given by $x_1 \mapsto x_1 + [x_2, x_3], x_2 \mapsto x_2 - [x_2, x_3], x_3 \mapsto x_3$. Clearly, $\Phi(x_1 + x_2 + x_3) = x_1 + x_2 + x_3$, however, one easily checks that Φ does not preserve $\langle \partial_i, [x_i, \partial_i] \rangle$.

Remark 3.34. In the symplectic case, the theorem is essentially equivalent to the result of Massuyeau-Turaev about non-degenerate Fox pairings (cf. [6]), which strengthens an earlier result of Kawazumi-Kuno (cf. [7]).

2.3 Kirillov-Kostant-Souriau Poisson structure

In this section the results are spelled out for the case of the Kirillov-Kostant-Souriau bivector field given by $\Pi := \frac{1}{2} \Sigma \langle x_i, [\partial_i, \partial_i] \rangle$. This induces the map

$$T^*[1]\mathcal{L}_n \xrightarrow{(-)^\Pi} T[1]\mathcal{L}_n\,,\; dx_i = [x_i, \partial_i],\; x_i = x_i.$$

The space of Casimir functions is determined by the following

Lemma 3.35. *The kernel of the map* $(-)^\Pi : \Omega(\mathcal{L}_n) \to \mathfrak{X}(\mathcal{L}_n)$ *is linearly spanned by* $\langle x_i, dx_i \rangle$.

Identifying vector fields with derivations we get a map

$$\begin{aligned} \Omega^1(\mathcal{L}_n) &\longrightarrow \mathrm{Der}(\mathcal{L}_n) \\ \langle \alpha_i, dx_i \rangle &\longmapsto (x_i \mapsto [x_i, \alpha_i]). \end{aligned}$$

Let us denote the Lie algebra $\Omega^1(\mathcal{L}_n)$ by tder_n.

The bracket on tder_n can be computed as follows. Let $\alpha = \langle \alpha_i, dx_i \rangle, \beta = \langle \beta_i, dx_i \rangle \in \Omega^1(\mathcal{L}_n)$ then

$$[\alpha, \beta] = \left\langle \alpha^\sharp(\beta_i) - \beta^\sharp(\alpha_i) + [\alpha_i, \beta_i], dx_i \right\rangle.$$

Remark 3.36. Our definition of tder_n differs by an n-dimensional abelian direct summand from the one in [8]. The same remark applies to TAut_n.

Let us denote the integrating Lie group of tder_n by TAut_n. The above map exponentiates to

$$\mathrm{TAut}_n \longrightarrow \{F \in \mathrm{Diff}(\mathcal{L}_n) = \mathrm{Aut}(L_n) \mid F(x_i) = e^{A_i} x_i e^{-A_i} \text{ for some } A_i \in L_n\}$$

Using the e^{A_i} as components of a map, TAut_n can be thought of as $\mathrm{Map}(\mathcal{L}_n, \mathcal{E}xp(\mathcal{L}_n))$.

The closed one forms $\Omega^1_{\mathrm{cl}}(\mathcal{L}_n) \cong \Omega^0(\mathcal{L}_n)$ form a Lie subalgebra, whose Lie group $\mathcal{H}\mathrm{am}(\mathcal{L}_n)$ can be characterized by the following lemma that appears in [10],

Lemma 3.37 (Drinfeld)**.** *Let* $\phi \in \mathcal{E}xp(\Omega^1(\mathcal{L}_n))$. *Then*

$$\phi \in \mathcal{H}\mathrm{am}(\mathcal{L}_n) \iff \phi(\Sigma x_i) = \Sigma x_i$$

Proof. This follows from theorem 3.31. For convenience we give a direct proof. It is enough to show that for $\alpha = \langle \alpha_i dx_i \rangle$, $\alpha(\Sigma x_i) = 0$ implies that α is closed. We are going to use the fact that any Lie series α can be written as

$$\alpha = \frac{\partial \alpha}{\partial x_i} x_i = x_i \left(\frac{\partial \alpha}{\partial x_i} \right)^* \in \mathbf{k} \langle x_1, \ldots, x_n \rangle ,$$

where $*$ denotes the antipode in $\mathbf{k}\langle x_1, \ldots, x_n \rangle$. Using this we get

$$\begin{aligned}
\alpha(\Sigma x_i) = 0 \quad &\iff \quad [\alpha_i, x_i] = 0 \\
&\iff \quad x_i \alpha_i = \alpha_j x_j \\
&\iff \quad x_i \frac{\partial \alpha_i}{\partial x_j} x_j = x_i \left(\frac{\partial \alpha_j}{\partial x_i} \right)^* x_j \\
&\iff \quad \frac{\partial \alpha_i}{\partial x_j} = \left(\frac{\partial \alpha_j}{\partial x_i} \right)^* \quad \forall i, j \\
&\iff \quad d\alpha = 0
\end{aligned}$$

□

Lemma 3.38. TAut_n *acts transitively on* $\Sigma x_i + L_n^{\geq 2}$.

Proof. Also follows from our main theorem 3.31. □

3 Hamiltonian spaces as a TAut-algebra

As was shown above the groups TAut_n act transitively on $\sum x_i + L_n^{\geq 2}$. The corresponding groupoids fit together to form an operad in groupoids which we denote again by TAut_n. Instead of giving the operadic compositions, a faithful (after taking a suitable limit) action on a category is constructed, from which the operadic structure can be infered.

Let $\mathcal{C}_n$ denote the category of formal g^n-Hamiltonian spaces, that is Poisson manifolds with a Poisson map into g^n (recall that g is quadratic). Using the canonical map $\mathcal{C}_n \to \mathcal{C}$, one sees that $\mathrm{Fun}(\mathcal{C}_n, \mathcal{C})$ form an operad in groupoids.

A map of operads $TAut_n \to \mathrm{Fun}(\mathcal{C}_n, \mathcal{C})$ is defined as follows.

$$\begin{aligned} \sum x_i + L_n^{\geq 2} &\longrightarrow \mathrm{Fun}(\mathcal{C}_n, \mathcal{C}) \\ \mu &\longmapsto (M, \Pi, h) \mapsto (M, \Pi^{\omega^\mu}, \mu \circ h) \end{aligned}$$

where (M, Π, h) is a Hamiltonian g-space with Π its Poisson structure and $h : M \to \mathfrak{g}^n$ its moment map. On arrows it is defined as follows.

$$\begin{aligned} \mathrm{TAut}_n &\longrightarrow \mathrm{End}(\mathrm{Fun}(\mathcal{C}_n, \mathcal{C})) \\ g : \mathfrak{g}^n \to G^n &\longmapsto M \to M \ ; \ m \mapsto g(h(m)).m \end{aligned}$$

where some abuse of notation is committed and an element $g \in TAut_n$ is viewed as its induced map $\mathfrak{g}^n \to G^n$.

Theorem 3.39. *The above map is a well-defined map of operads $TAut_n \to$ $\mathrm{Fun}(\mathcal{C}_n, \mathcal{C})$*

Proof. The first point to notice is that $\mu \in \sum x_i + L_n^{\geq 2}$ give well-defined functors $\mathcal{C}_n \to \mathcal{C}$. Here a Poisson map $M \to \mathfrak{g}^n$ is gauge transformed by a closed 2-form on $\mathfrak{g}^n$ and then composed with a Poisson map $\mu : \mathfrak{g}^n \to \mathfrak{g}$. For the second part one needs to check that a $g \in \mathrm{TAut}_n$ indeed intertwines the respective Poisson structures. Note that the above formula defines an action of the group TAut_n on M by diffeomorphisms. It remains to check that g intertwines

$$(M, \Pi^{\omega^\mu}, \mu \circ h) \xrightarrow{g} (M, \Pi^{\omega^{g \cdot \mu}}, (g.\mu) \circ h).$$

The moment map part is obvious. Moreover, the statement can be reduced to the case $\mu = \sum x_i$ by transitivity of the action. Thus the statement becomes

$$g.\Pi - \Pi = \Pi^{\omega^{g \cdot \mu}} - \Pi.$$

Since both sides define group cocycle with values in bivector fields on M, it is enough to verify that the corresponding Lie cocycles coincide. Let $(u_1, \cdots, u_n) \in \mathrm{tder}_n$ and let moreover $\rho_i = [h_i, \Pi]$ denote the i-th g-valued action vector field on M, the Lie cocycle of the left hand side computes to

$$\begin{aligned} L_{\sum \langle u_i \circ h, \rho_i \rangle} \Pi &= \sum \langle [u_i \circ h, \Pi], \rho_i \rangle \\ &= \sum \left\langle \tfrac{\partial u_i}{\partial h_j} [h_j, \Pi], \rho_i \right\rangle \\ &= \sum \left\langle \tfrac{\partial u_i}{\partial h_j} \rho_j, \rho_i \right\rangle, \end{aligned}$$

which is by definition the same as the right hand side. □

Remark 3.40. Not that for each $\mu \in \sum x_i + L_n^{\geq 2}$ we get a product of Hamiltonian spaces. The resulting G-action, however, is always the diagonal action.

Remark 3.41. One can extend beyond formal Hamiltonian spaces by restricting to suitably convergent elements.

3.1 Application to Hamiltonian quasi-Poisson spaces

Let us briefly recall the relevant definitions from [3]. Let $\phi \in \Lambda^3 \mathfrak{g}$ denote the Cartan three-form of the quadratic Lie algebra $\mathfrak{g}$.

Definition 3.42. A pair (M, Π) of a $\mathfrak{g}$-manifold together with a bivector field $\Pi \in \Gamma(\Lambda^2 TM)$ is called *quasi-Poisson* if

$$[\Pi, \Pi] = \phi_M,$$

where ϕ_M denotes the tri-vector field on M induced by ϕ and the $\mathfrak{g}$-action on M.

Definition 3.43. A map $\mu : M \to G$ is called a moment map if

$$(1 \otimes \mu^* df)\Pi = (1 \otimes \mu^*)\langle \rho \otimes df, Z \rangle$$

where $Z \in \mathfrak{g} \otimes \mathfrak{X}(G)$ is the element $v \mapsto \frac{1}{2}(v^L + v^R)$, that is the mean of the right and the left action $\mathfrak{g}$ on G, where $\mathfrak{g}$ and $\mathfrak{g}^*$ are identified.

Equivalently, we write

$$\{\mu, \cdot\}_\Pi = \tfrac{1}{2}(\mu \hat{\rho} + \hat{\rho} \mu),$$

where $\hat{\rho}$ is the $\mathfrak{g}$-valued vector field obtained from the $\mathfrak{g}$-action by identifying $\mathfrak{g}^*$ with $\mathfrak{g}$.

A tuple (M, Π, μ) is called a $\mathfrak{g}$-*Hamiltonian quasi-Poisson space*

The category of $\mathfrak{g}$-Hamiltonian quasi-Poisson spaces admits a monoidal structure given by the following

Definition 3.44 (Fusion)**.** Let (M, Π) be a $\mathfrak{g} \times \mathfrak{g}$-Hamiltonian quasi-Poisson space, then

$$\Pi_{\text{fus}} = \Pi - \psi_M$$

gives a $\mathfrak{g}$-Hamiltonian quasi-Poisson space with the diagonal $\mathfrak{g}$-action and moment map defined by multiplying the two factors.

For two $\mathfrak{g}$-Hamiltonian quasi-Poisson spaces M and N, we define their fusion product by

$$M \circledast N := (M \times N, \Pi_M + \Pi_N - \psi_{M\times N}, \mu_1 \cdot \mu_2).$$

Example 3.45. The moduli space of flat $\mathfrak{g}$-connections on a surface $\Sigma_{g,n}$ of genus g with n boundary components, and a marked point on the boundary is given by $\mathrm{Hom}(\pi_1, G) \cong G^{2g+n-1}$, and carries a natural quasi-Poisson structure. It can be constructed by viewing it as $DG^{\circledast g} \circledast G^{\circledast n-1}$.

A quasi-Poisson bivector can in general be turned into a Poisson bivector by adding an r-matrix term. One particular (dynamical) r-matrix is the Alekseev-Meinrenken dynamical r-matrix. Thus the constructions goes as follows. Let $\nu(z) := \frac{1}{z} - \frac{1}{2}\coth(\frac{z}{2}) = -\frac{z}{12} + \frac{z^3}{720} + \cdots$ and define the following universal 2-form on $\mathfrak{g}$,

$$T = \langle dx, \nu(\mathrm{ad}_x)dx \rangle .$$

Then recall (cf. [3])

Proposition 3.46 (Exponentiation)**.** *Let (M, Π) be a $\mathfrak{g}$-Poisson manifold with moment map $\mu : M \to \mathfrak{g}$. Consider T as a map $\mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$. Then $(M, \Pi - (\mu^*T)_M)$ is a $\mathfrak{g}$-Hamiltonian quasi-Poisson manifold with moment map* $\exp \circ \mu$.

The bivector field can also be written as $\Pi - (\mu^*T)^\sharp$ by considering T as a 2-form on $\mathfrak{g}$ and using the morphism $\sharp$ between forms and polyvector fields induced by Π.

Let $\mathcal{E}xp$ denote the functor sending a $\mathfrak{g}$-Hamiltonian Poisson space to the $\mathfrak{g}$-Hamiltonian quasi-Poisson space given by the last proposition.

Example 3.47. The standard $\mathfrak{g}$-Hamiltonian quasi-Poisson space G with moment map the identity, corresponds to $\mathfrak{g}$ with its KKS structure under this functor.

Remark 3.48. Using Lemma 3.16 one can reduce Proposition 3.46 to the fact the following universal bivector field is quasi-Poisson with moment map e^x, which one can easily check,

$$\Pi_{\text{KKS}} - T^\sharp = \langle \frac{\partial}{\partial e^x} e^x, e^x \frac{\partial}{\partial e^x} \rangle$$

where $\frac{\partial}{\partial e^x} e^x = \frac{\mathrm{ad}_x}{e^{\mathrm{ad}_x}-1}\partial_x$ and $e^x \frac{\partial}{\partial e^x} = \frac{\mathrm{ad}_x}{1-e^{-\mathrm{ad}_x}}\partial_x$. A direct consequence (using 3.16 is that $T \in \Omega^2(\mathcal{L}_1)$ satisfies the classical dynamical Yang-Baxter equation

$$-2dT + [T,T]_{\Pi_{\mathrm{KKS}}} = \frac{1}{6}\langle dx, [dx, dx]\rangle \,.$$

Pulling back the fusion product along the functor $\mathcal{E}xp$ one gets a second monoidal structure on the category of $\mathfrak{g}$-Hamiltonian spaces, which we denote again by $\circledast$. It is given by

$$(M, \Pi_M, \mu_M) \circledast (N, \Pi_N, \mu_N) = (M \times N, \Pi_M + \Pi_N + ((\mu_M \times \mu_N)^*\sigma)^\sharp, \log(e^{\mu_M} e^{\mu_N}))$$

for

$$\sigma = T_{12} - T_1 - T_2 + \langle dx, dy\rangle \in \Omega^2(\mathcal{L}_n).$$

This σ is thus a Maurer-Cartan element in $\Omega^2(\mathcal{L}_n)$ since it is so for any $\mathfrak{sl}_k$. Setting $\omega = \sigma(1 + \Pi\sigma)^{-1} \in \Omega^2\mathcal{L}_n$, the above Poisson structure can be written as $(\Pi_M \times \Pi_N)^{((\mu_M\times\mu_N)^*\omega)}$ and $\omega = \omega^{\log(e^{x_1}e^{x_2})}$. In particular, there two products are induced by $x_1 + x_2$ and $\log(e^{x_1}e^{x_2})$, respectively.

Let now $F \in \mathrm{TAut}_2$ be an element intertwining those two structures, that is such that

$$F(\log(e^{x_1}e^{x_2})) = x_1 + x_2$$

Remark 3.49. As shown in [4] one particular source of such F is Drinfeld associators. Namely, let $\Phi = \exp(\phi)$ for $\phi \in \hat{Lie}(x, y)$ be a Drinfeld associator. Then we associate to it the F_Φ with components

$$\left(\Phi(x, -x-y), e^{-\frac{x+y}{2}}\Phi(y, -x-y)\right).$$

As a consequence of the above discussion, we get the following.

Proposition 3.50. *Let M, N be two g-Hamiltonian Poisson spaces. Then the following map is Poisson.*

$$\begin{array}{ccc} M \times N & \overset{F_{M,N}}{\to} & \mathcal{E}xp^{-1}\left(\mathcal{E}xp(M) \circledast \mathcal{E}xp(N)\right) \\ (a, b) & \mapsto & (F_1(\mu_M(a), \mu_N(b)).a, F_2(\mu_M(a), \mu_N(b)).b) \end{array}$$

Moreover, the $F_{M,N}$ are a natural transformation.

The maps $F_{M,N}$ can now be interpreted as a monoidal structure on the functor $\mathcal{E}xp$. Let $\mathcal{C}$ denote the category of g-Hamiltonian Poisson spaces with monoidal product given by the product of Poisson spaces. Instead of the trivial associator isomorphism, let $\mathcal{C}$ be endowed with the associator derived from F. More precisely, define

$$\Phi^F = F_{1,23}F_{2,3}F_{1,2}^{-1}F_{12,3}^{-1} \in \mathrm{TAut}_3,$$

and use it to define a diffeomorphism $\Phi^F_{X,Y,Z}$ for any triple X, Y, Z of $\mathfrak{g}$-Hamiltonian Poisson spaces. Let $\mathcal{D}$ denote the category of g-Hamiltonian quasi-Poisson spaces. Then we get

Proposition 3.51. *An $F \in \mathrm{TAut}_2$ such that $F(\log(e^{x_1}e^{x_2})) = x_1 + x_2$ promotes the functor $\mathcal{E}xp$ to a monoidal equivalence*

$$(\mathcal{C}, \times, \Phi^F) \quad \xrightarrow{\mathcal{E}xp} \quad (\mathcal{D}, \circledast, \mathrm{id})$$

Corollary 3.52. *An $F \in \mathcal{E}xp(\Omega^1(\mathcal{L}_2))$ satisfying* (3.1) *gives a Poisson map*

$$O_{\lambda_1} \times \cdots \times O_{\lambda_n} /\!\!/_0 G \to \mathcal{M}(\Sigma_{0,n}, C_1, \cdots, C_n)$$

where O_{λ_i} are coadjoint orbits for given $\lambda_i \in \mathfrak{g} \cong \mathfrak{g}^*$, and $\mathcal{M}(\Sigma_{0,n}, C_1, \cdots, C_n)$ is the moduli space of flat connections on a surface of genus 0 with n punctures and monodromies around the punctures prescribed by conjugacy classes $C_i = G.exp(\lambda_i)$.

Remark 3.53. Taking $F = F_{\Phi_{KZ}}$ to be associated to the Knizhnik-Zamolodchikov associator, the previous map is given by

$$a_1, a_2 \mapsto d - \left(\frac{a_1}{z} + \frac{a_2}{z-1}\right) dz$$

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Chapter 4

The Goldman-Turaev Lie bialgebra in genus zero and the Kashiwara-Vergne problem

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Abstract: In this paper, we describe a surprising link between the theory of the Goldman-Turaev Lie bialgebra on surfaces of genus zero and the Kashiwara-Vergne (KV) problem in Lie theory. Let Σ be an oriented 2-dimensional manifold with non-empty boundary and $\mathbb{K}$ a field of characteristic zero. The Goldman-Turaev Lie bialgebra is defined by the Goldman bracket $\{-,-\}$ and Turaev cobracket δ on the $\mathbb{K}$-span of homotopy classes of free loops on Σ.

Applying an expansion $\theta : \mathbb{K}\pi \to \mathbb{K}\langle x_1, \ldots, x_n\rangle$ yields an algebraic description of the operations $\{-,-\}$ and δ in terms of non-commutative variables $x_1, \ldots, x_n$. If Σ is a surface of genus $g = 0$ the lowest degree parts $\{-,-\}_{-1}$ and δ_{-1} are canonically defined (and independent of θ). They define a Lie bialgebra structure on the space of cyclic words which was introduced and studied by Schedler [31]. It was conjectured by the second and the third authors that one can define an expansion θ such that $\{-,-\} = \{-,-\}_{-1}$ and $\delta = \delta_{-1}$. The main result of this paper

Keyword: Goldman bracket, Turaev cobracket, Kashiwara-Vergne conjecture

states that for surfaces of genus zero constructing such an expansion is essentially equivalent to the KV problem. In [24], Massuyeau constructed such expansions using the Kontsevich integral.

In order to prove this result, we show that the Turaev cobracket δ can be constructed in terms of the double bracket (upgrading the Goldman bracket) and the non-commutative divergence cocycle which plays the central role in the KV theory. Among other things, this observation gives a new topological interpretation of the KV problem and allows to extend it to surfaces with arbitrary number of boundary components (and of arbitrary genus, see [2]).

1 Introduction

Let Σ be a 2-dimensional oriented manifold of genus g with non empty boundary and let $\mathbb{K}$ be a field of characteristic zero. In [15], Goldman introduced a Lie bracket on the $\mathbb{K}$-span of homotopy classes of free loops on Σ. One can also think of this space as a space spanned (over $\mathbb{K}$) by conjugacy classes in the fundamental group $\pi := \pi_1(\Sigma)$. The notation that we use for this space is

$$|\mathbb{K}\pi| = \mathbb{K}\pi/[\mathbb{K}\pi, \mathbb{K}\pi].$$

One of the *raisons d'être* of the Goldman bracket is the finite dimensional description of the Atiyah-Bott symplectic structure on moduli spaces of flat connections on Σ [7]. In [34], Turaev constructed a Lie cobracket on the quotient space $|\mathbb{K}\pi|/\mathbb{K}\,\mathbf{1}$, where $\mathbf{1}$ stands for the homotopy class of a trivial (contractible) loop. Given a framing (that is, a trivialization of the tangent bundle) of Σ, the cobracket δ can be lifted to a cobracket δ^+ on $|\mathbb{K}\pi|$ which together with the Goldman bracket defines a Lie bialgebra structure. In more detail, this means that δ^+ verifies the co-Jacobi identity, and that it is a 1-cocycle with respect to the Goldman bracket. This structure was studied in detail in the works of Chas and Sullivan, and it was one of the motivations for introducing the string topology program [11].

The Goldman bracket admits an upgrade to the double bracket in the sense of van den Bergh, $\kappa : \mathbb{K}\pi \otimes \mathbb{K}\pi \to \mathbb{K}\pi \otimes \mathbb{K}\pi$, and the cobracket δ^+ (for a given framing) can be upgraded to a map $\mu : \mathbb{K}\pi \to |\mathbb{K}\pi| \otimes \mathbb{K}\pi$. All the operations described above: the Goldman bracket, the Turaev cobracket, the cobracket δ^+, κ and μ are defined in terms of intersections and self-intersections of curves on Σ.

Since $\partial\Sigma \neq \emptyset$, the fundamental group π is a free group and one can consider expansions (that is, Hopf algebra homomorphisms) $\theta : \mathbb{K}\pi \to A$, where A is a degree completed free Hopf algebra with generators $x_1, \ldots, x_n$. An example of such an expansion is the exponential expansion $\theta^{\exp}(\gamma_i) = e^{x_i}$ which depends on the choice of the basis $\gamma_1, \dots, \gamma_n$ in π. An expansion allows to transfer the topologically defined operations κ and μ (as well as the Goldman bracket and Turaev cobracket) to the free Hopf algebra A and to the space of cyclic words in letters $x_1, \dots x_n$

$$|A| = A/[A, A].$$

We use the notation $a \mapsto |a|$ for the natural projection $A \to |A|$. We will denote the operations transferred to A and $|A|$ by $\kappa_\theta, \mu_\theta, \{-,-\}_\theta, \delta_\theta^+$ and δ_θ, respectively.

Assume that Σ is a surface of genus $g = 0$. Consider the grading on A defined by assignment $\deg(x_i) = 1$ and the induced grading on $|A|$. The Lie bracket $\{-,-\}_\theta : |A| \otimes |A| \to |A|$ contains contributions of different degrees starting with degree (-1). This lowest degree part is canonically defined and does not depend on the choice of θ. The Goldman bracket being related to the Atiyah-Bott symplectic structures on moduli of flat connections, its lowest degree part comes from the Kirillov-Kostant-Souriau (KKS) Poisson structure on coadjoint orbits. Hence, we use the notation $\{-,-\}_{\mathrm{KKS}}$ for this bracket. For all expansions θ we have

$$\{-,-\}_\theta = \{-,-\}_{\mathrm{KKS}} + \text{ higher order terms.}$$

It turns out that there is a class of expansions, called *special expansions*, for which all higher order terms vanish and the equality

$$\{-,-\}_\theta = \{-,-\}_{\mathrm{KKS}}$$

holds without extra corrections. Originally, the problem of finding such expansions was solved in the case of Σ of genus g with one boundary component in [20]. This result was extended to the case of surfaces with arbitrary number of boundary components in [22], [27].

In the case of Σ of genus $g = 0$ with 3 boundary components, the fundamental group π is a free group with 2 generators and A is the degree completion of the free Hopf algebra with two generators x_1 and x_2. In this case, special expansions are of the form $\theta_F = F^{-1} \circ \theta^{\exp}$, where F is an automorphism of a free Lie algebra with two generators (naturally extended to A) with the property

$$F(x_i) = e^{-a_i} x_i e^{a_i} \tag{4.1}$$

for $i = 1, 2$ and

$$F(x_1 + x_2) = \log(e^{x_1} e^{x_2}). \tag{4.2}$$

Similarly to the Goldman bracket, the transferred cobracket δ_θ^+ admits a decomposition

$$\delta_\theta^+ = \delta^{\mathrm{alg}} + \text{ higher order terms.}$$

For a surface of genus zero, δ^{alg} is again an operator of degree (-1). On the space of cyclic words, there is a Lie bilagebra structure defined by the KKS bracket and the cobracket δ^{alg}. This Lie bialgebra structure was introduced and studied by Schedler [31].

It is natural to ask a question of whether there exists a special expansion θ such that

$$\delta_\theta^+ = \delta^{\text{alg}}.$$

In this paper, we answer this question for surfaces of genus zero. The key observation is that equation (4.2) plays an important role in the Kashiwara-Vergne (KV) problem in Lie theory [17]. In more detail, the reformulation of the KV problem in [6] requires to find an automorphism F of a free Lie algebra in two generators which satisfies (4.1), (4.2) and the mysterious equation

$$j(F^{-1}) = |h(x_1) + h(x_2) - h(x_1 + x_2)|. \tag{4.3}$$

Here $h(z) \in \mathbb{K}[[z]]$ is a formal power series in one variable (also called *the Duflo function*). The map j is the group 1-cocycle integrating the non-commutative divergence 1-cocycle $\text{div} : \text{tDer}(A) \to |A|$, and $\text{tDer}(A) \subset \text{Der}(A)$ is the Lie algebra of tangential derivations on A which plays an important role in the Kashiwara-Vergne theory [6]. The divergence cocycle admits an upgrade $\text{tDiv} : \text{tDer}(A) \to |A| \otimes |A|$.

The following theorem establishes a surprising link between the Kashiwara-Vergne theory and the properties of the Goldman-Turaev Lie bialgebra:

Theorem 4.1. *Let F be a solution of equations* (4.1) *and* (4.2)*. Then, $\delta_\theta^+ = \delta^{\text{alg}}$ for the expansion $\theta = F^{-1} \circ \theta^{\exp}$ if and only if F verifies equation* (4.3) *up to linear terms.*

We also define a version of the KV problem for surfaces of genus zero and an arbitrary number of boundary components $n \geq 3$. It turns out that all these problems can be easily solved using the $n = 3$ case. Moreover, a result analogous to Theorem 4.1 holds true for any n.

The operations κ_θ and μ_θ also acquire a nice form for expansions θ defined by solutions of the Kashiwara-Vergne problem. In more detail, we have

$$\kappa_\theta = \kappa_{\text{KKS}} + \kappa_s,$$

where κ_{KKS} is again the degree (-1) part of κ_θ. The double bracket κ_s contains certain higher order terms encoded in the power series $s(z) \in \mathbb{K}[[z]]$ related to the even part h_{even} of the Duflo function by the following formula

$$\frac{dh_{\text{even}}}{dz} = \frac{1}{2}\left(s(z) + \frac{1}{2}\right) = -\frac{1}{2}\sum_{k\geq 1} \frac{B_{2k}}{(2k)!} z^{2k-1},$$

where B_{2k} are Bernoulli numbers.

There is a similar description for μ:

$$\mu_\theta = \mu^{\mathrm{alg}} + \mu_s + \tilde{\mu}_g,$$

where μ_s is defined by the formal power series $s(z)$. The new contribution $\tilde{\mu}_g$ is defined by the formal power series

$$g(z) = \frac{dh}{dz} = -\frac{1}{2}\sum_{k\geq 1}\frac{B_{2k}}{(2k)!}z^{2k-1} + \sum_{k\geq 1}(2k+1)h_{2k+1}z^{2k},$$

where h_{2k+1} are the odd Taylor coefficients of the Duflo function. Hence, $g(z)$ encodes the information on the full Duflo function (and not only on its even part).

Remark 4.2. In [24], Massuyeau constructed expansions θ verifying the condition $\delta^+_\theta = \delta^{\mathrm{alg}}$ using the Kontsevich integral. Moreover, he obtained a description of an operation which is closely related (and essentially equivalent) to μ. His description works for any choice of a Drinfeld associator Φ which one needs for the construction of the Kontsevich integral, and it involves the Γ-function of Φ; see [13].

Our results are based on the following theorem:

Theorem 4.3. *Let F be an automorphism of the free Lie algebra with two generators which satisfies equation* (4.1) *and let $\theta = F^{-1} \circ \theta^{\exp}$ be the corresponding expansion. Then,*

$$\delta^+_\theta(|a|) = \mathrm{tDiv}(\{|a|, -\}_\theta), \tag{4.4}$$

if $j(F^{-1}) \in |A|$ belongs to the center of the Lie algebra $(|A|, \{-,-\}_\theta)$.

In proving Theorem 4.3 we first show it for $\theta = \theta^{\exp}$ and then extend it for F's solving the KV problem. In the course of the proof, we need the description of the center of the Lie bracket $\{-,-\}_{\mathrm{KKS}}$. We give a direct algebraic proof of the fact that the center is spanned by elements of the form

$$Z(|A|, \{-,-\}_{\mathrm{KKS}}) = \sum_{i=0}^{n} |\mathbb{K}[[x_i]]|,$$

where $x_0 = -\sum_{i=1}^n x_i$. Equation (4.4) shows that the cocycle property of the divergence cocycle implies the cocycle property of the cobracket δ^+_θ. However, it does not shed light on the co-Jacobi identity for δ^+_θ.

The structure of the paper is as follows. In Section 2, we recall the notions of tangential and double derivations and introduce the space of cyclic words. In Section 3, we discuss various versions of the divergence map and derive their multiplicative properties. In Section 4, we consider double brackets (in the sense of van den Bergh) and study properties of composition maps such as (4.4). In Section 5, we recall the topological definitions of the Goldman bracket, Turaev cobracket and of the maps κ and μ. In Section 6, we introduce the notion of expansions and define the transfer of structures to the Hopf algebra A. In Section 7, we show that solutions of the Kashiwara-Vergne problem give rise to expansions with property $\delta_\theta^+ = \delta^{\rm alg}$. In Section 8, we prove Theorem 4.1. Appendix A contains an algebraic proof of the description of the center of the KKS Lie bracket.

Our results were announced in [2]. This paper focuses on the case of genus zero. The higher genus case is more involved, and it will be considered in the forthcoming paper [3].

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2 Derivations on free associative/Lie algebras

In this section, we give basic definitions of derivations and double derivations on free associative and Lie algebras.

2.1 Tangential derivations

Let $\mathbb{K}$ be a field of characteristic 0. For a positive integer n, let $A = A_n$ be the degree completion of the free associative algebra over $\mathbb{K}$ generated by indeterminates $x_1, \dots, x_n$. The algebra A has a complete Hopf algebra structure whose coproduct Δ, augmentation ε, and antipode ι are defined by

$$\Delta(x_i) := x_i \otimes 1 + 1 \otimes x_i, \quad \varepsilon(x_i) := 0, \quad \iota(x_i) := -x_i.$$

Here and throughout the paper, we simply use $\otimes$ for the complete tensor product $\hat{\otimes}$.

We denote by $A_{\geq 1} = \ker(\varepsilon)$ the positive degree part of A_n. The primitive part $L = L_n := \{a \in A \mid \Delta(a) = a \otimes 1 + 1 \otimes a\}$ of A is naturally identified with the degree completion of the free Lie algebra over $\mathbb{K}$ generated by $x_1, \dots, x_n$. Sometimes we drop n from the notation and simply use A and L.

Following [35] §2.1, we consider the following natural A-bimodule structures on the algebra $A \otimes A$. The *outer* bimodule structure is given by

$$c_1(a \otimes b)c_2 := c_1 a \otimes bc_2 = (c_1 \otimes 1)(a \otimes b)(1 \otimes c_2), \tag{4.5}$$

while the *inner* bimodule structure is given by

$$c_1 * (a \otimes b) * c_2 := ac_2 \otimes c_1 b = (1 \otimes c_1)(a \otimes b)(c_2 \otimes 1). \tag{4.6}$$

Here, $a, b, c_1, c_2 \in A$.

Let $\mathrm{Der}(A)$ be the Lie algebra of (continuous) derivations on the algebra A. Note that an element of $\mathrm{Der}(A)$ is completely determined by its values on generators $x_1, \ldots, x_n$. We set

$$\mathrm{tDer}(A) := A^{\oplus n},$$

and define a map $\rho : \mathrm{tDer}(A) \to \mathrm{Der}(A)$ such that for $u = (u_1, \ldots, u_n) \in \mathrm{tDer}(A)$ we have

$$\rho(u)(x_i) := [x_i, u_i].$$

Then, we can equip $\mathrm{tDer}(A)$ with a Lie bracket given as follows: for $u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_n) \in \mathrm{tDer}(A)$, we have $[u, v] = w$, where $w = (w_1, \ldots, w_n)$ and

$$w_i := \rho(u)(v_i) - \rho(v)(u_i) + [u_i, v_i].$$

The map $\rho : \mathrm{tDer}(A) \to \mathrm{Der}(A), u \mapsto \rho(u)$ becomes a Lie algebra homomorphism.

Remark 4.4. The kernel of the map ρ is spanned by elements of the form $u = (h_1(x_1), \ldots, h_n(x_n))$, where $h_i(z) \in \mathbb{K}[[z]]$ are formal power series in one variable.

Remark 4.5. The map ρ lifts to an injective Lie algebra homomorphism $\hat{\rho} : \mathrm{tDer}(A) \to A^{\oplus n} \rtimes \mathrm{Der}(A)$ defined by $\hat{\rho}(u) := (u, \rho(u))$. The target is a semi-direct product of Lie algebras $A^{\oplus n}$ and $\mathrm{Der}(A)$, where $\mathrm{Der}(A)$ acts diagonally on the n-fold direct product $A^{\oplus n}$.

We define Lie algebras $\mathrm{Der}(L)$ and $\mathrm{tDer}(L)$ by replacing A in the preceding paragraphs with the Lie algebra L. In particular, $\mathrm{tDer}(L) = L^{\oplus n}$ as a set. In an obvious way, $\mathrm{Der}(L)$ and $\mathrm{tDer}(L)$ are Lie subalgebras of $\mathrm{Der}(A)$ and $\mathrm{tDer}(A)$, respectively. We use the same notation $\rho : \mathrm{tDer}(L) \to \mathrm{Der}(L)$ for the restriction of ρ to $\mathrm{tDer}(L)$. The kernel of this map is spanned by the elements

$$q_i = (0, \ldots, x_i, \ldots, 0) \in \mathrm{tDer}(L), \tag{4.7}$$

where x_i is placed in the ith position.

Remark 4.6. Here we follow the notation in [1]. In [6], the notation is different and elements in the image of ρ are called *tangential* derivations.

2.2 Double derivations

Definition 4.7. A *double derivation* on A is a $\mathbb{K}$-linear map $\phi : A \to A \otimes A$ satisfying

$$\phi(ab) = \phi(a)b + a\phi(b)$$

for any $a, b \in A$.

In the definition, we make use of the outer bimodule structure (4.5). Let $\mathrm{D}_A = \mathrm{Der}(A, A \otimes A)$ be the set of double derivations on A. It has a natural A-bimodule structure given by

$$(c_1 \phi c_2)(a) = c_1 * \phi(a) * c_2$$

for $c_1, c_2 \in A$ and $\phi \in \mathrm{D}_A$. Here we use the inner bimodule structure (4.6).

As an A-bimodule, D_A is free of rank n. In fact, the map $\mathrm{D}_A \to (A \otimes A)^{\oplus n}, \phi \mapsto (\phi(x_1), \ldots, \phi(x_n))$ is an isomorphism of A-bimodules where the A-bimodule structure on $A \otimes A$ is the inner one. There is a basis of D_A given by double derivations $\partial_i, i = 1, \ldots, n$ such that

$$\partial_i(x_j) = \delta_{i,j}(1 \otimes 1),$$

where $\delta_{i,j}$ is the Kronecker δ-symbol. It is convenient to use the Sweedler convention for the action of double derivations: $\phi(a) = \phi'(a) \otimes \phi''(a)$. For instance, we write $\phi(x_i) = \phi_i' \otimes \phi_i''$ which implies $\phi = \sum_i \phi_i'' \partial_i \phi_i'$. We also write $\partial_i a = \partial_i' a \otimes \partial_i'' a$. Using this notation, we obtain

$$\phi(a) = \sum_i \phi_i'' \partial_i \phi_i'(a) = \sum_i (\partial_i' a)\phi_i' \otimes \phi_i''(\partial_i'' a). \tag{4.8}$$

The Lie algebra of derivations $\mathrm{Der}(A)$ naturally acts on $A \otimes A$ as follows: for $u \in \mathrm{Der}(A), a, b \in A$ we have $u(a \otimes b) = u(a) \otimes b + a \otimes u(b)$. This implies that the space of double derivations D_A carries a natural action of $\mathrm{Der}(A)$,

$$u : \phi \mapsto [u, \phi] = u \circ \phi - \phi \circ u = (u \otimes 1 + 1 \otimes u) \circ \phi - \phi \circ u.$$

The double derivation $[u, \phi]$ is completely determined by the n-tuple $[u, \phi](x_j)$ for $j = 1, \ldots, n$. For example, for $\phi = \partial_i$, we have

$$[u, \partial_i](x_j) = \delta_{i,j} u(1 \otimes 1) - \partial_i u(x_j) = -\partial_i' u(x_j) \otimes \partial_i'' u(x_j). \tag{4.9}$$

Let tD_A denote the A-bi-submodule of D_A generated by $\mathrm{ad}_{x_i} \partial_i = x_i \partial_i - \partial_i x_i$, $i = 1, \ldots, n$. Note that

$$(\mathrm{ad}_{x_i} \partial_i)(x_j) = \delta_{i,j}(1 \otimes x_i - x_i \otimes 1).$$

Sometimes it is convenient to have a special notation for the injective map $i : \mathrm{tD}_A \to \mathrm{D}_A$ which simply sends an element $\phi \in \mathrm{tD}_A$ to the same tangential double derivation ϕ but viewed as an element of D_A.

In what follows we will need the following Lemma:

Lemma 4.8. *Let $C \in A \otimes A$ and assume that $Cx_i - x_iC = 0$ for some i. Then, $C = 0$.*

Proof. Introduce a bi-degree corresponding to the degrees in the first and second copies of A in $A \otimes A$. Since the equation $Cx_i - x_iC = 0$ is homogeneous, we can assume that C is of some total degree l. Then, $C = \sum_{k=0}^{l} C^{k,l-k}$ splits into a sum of terms with given bi-degrees. The equation $Cx_i - x_iC = 0$ reads

$$C^{k+1,l-k-1}x_i - x_iC^{k,l-k} = 0.$$

In particular, for $k = l$ the equation yields $x_iC^{l,0} = 0$ which implies $C^{l,0} = 0$. For $k = l-1$ we obtain $0 = C^{l,0}x_i - x_iC^{l-1,1}$ which implies $C^{l-1,1} = 0$. By induction, we get $C^{k,l-k} = 0$ for all k and $C = 0$, as required. □

Lemma 4.9. $\mathrm{tD}_A \subset \mathrm{D}_A$ *is a free A-bimodule of rank n with a basis* $\{\mathrm{ad}_{x_i}\partial_i\}_{i=1}^n$.

Proof. Assume that $\sum_i C_i'(\mathrm{ad}_{x_i}\partial_i)C_i'' = 0$ for some $C_i = C_i' \otimes C_i'' \in A \otimes A, i = 1, \dots, n$. By applying this double derivation to x_j, we obtain $(C_j)^\circ x_j - x_j(C_j)^\circ = 0$, where $(a \otimes b)^\circ = b \otimes a$ for $a, b \in A$. Hence, $C_j = 0$ for all j, as required. □

As a sample computation, we introduce an element of tD_A which will be used later.

Lemma 4.10. *Let $\phi_0 := \sum_i \mathrm{ad}_{x_i}\partial_i \in \mathrm{tD}_A$. Then, $\phi_0(a) = 1 \otimes a - a \otimes 1$ for all $a \in A$.*

Proof. Both $\phi_0 := \sum_i \mathrm{ad}_{x_i}\partial_i$ and the map $a \mapsto 1 \otimes a - a \otimes 1$ are double derivations on A. Since $\phi_0(x_i) = 1 \otimes x_i - x_i \otimes 1$ they agree on generators of A, and therefore they coincide. □

Through the map ρ, the Lie algebra $\mathrm{tDer}(A)$ acts on D_A.

Lemma 4.11. *The action of* $\mathrm{tDer}(A)$ *on* D_A *preserves* tD_A.

Proof. Let $u \in \mathrm{tDer}(A)$ and $\phi = \sum_i c_i'(\mathrm{ad}_{x_i}\partial_i)c_i'' \in \mathrm{tD}_A$. Then, by a straightforward computation, we see that $(u \cdot \phi)(x_i) = [\rho(u), \phi](x_i) = u \cdot \phi(x_i) - \phi(u(x_i))$ is equal to the value at x_i of the following double derivation

$$u(c_i')(\mathrm{ad}_{x_i}\partial_i)c_i'' + c_i'(\mathrm{ad}_{x_i}\partial_i)u(c_i'') + \phi''(u_i)(\mathrm{ad}_{x_i}\partial_i)\phi'(u_i) + c_i'(\mathrm{ad}_{x_i}\partial_i)u_ic_i'' - c_i'u_i(\mathrm{ad}_{x_i}\partial_i)c_i''.$$

This proves the assertion. □

2.3 Cyclic words and extra parameters

For a (topological) A-bimodule N, let $[A, N]$ be (the closure of) the $\mathbb{K}$-linear subspace of A spanned by elements of the form $an - na$, where $a \in A$ and $n \in N$. Set $|N| := N/[A, N]$ and let $|\ |: N \to |N|$ be the natural projection. Since A is naturally an A-bimodule, we can apply the construction described above to obtain

$$|A| := A/[A, A].$$

Note that $|A|$ is also isomorphic to the $\mathbb{K}$-span of cyclic words in letters $x_1, \dots, x_n$. Examples of such words are $|x_1|$, $|x_1x_2| = |x_2x_1|$ *etc.*

The space of double derivations D_A is also an A-bimodule.

Lemma 4.12. *The map*

$$\phi \mapsto |\phi| : a \mapsto \phi'(a)\phi''(a)$$

establishes an isomorphism $|\mathrm{D}_A| \cong \mathrm{Der}(A)$.

Proof. First, observe that

$$|b\phi - \phi b| : a \mapsto \phi'(a)(b\phi''(a)) - (\phi'(a)b)\phi''(a) = 0$$

for any $\phi \in \mathrm{D}_A$ and $b \in A$. Therefore, the map $\phi \mapsto |\phi|$ descends to a map $|\mathrm{D}_A| \to \mathrm{Der}(A)$. This map is surjective since $\sum_i |a_i\partial_i| : x_i \mapsto a_i$ for arbitrary $a_i \in A$, and hence we obtain all derivations of A in that way. Finally, this map is injective since D_A is a free A-bimodule of rank n with generators ∂_i and $\{\sum_i a_i\partial_i;\ a_i \in A\} \subset \mathrm{D}_A$ is a section of the canonical projection $\mathrm{D}_A \to |\mathrm{D}_A|$. □

Example 4.13. Let $a \in A$ and consider the derivation $u = |a\phi_0| = |a \sum_i \mathrm{ad}_{x_i}\partial_i|$. By computing it on generators, we obtain

$$u(x_j) = [a, x_j].$$

Hence, it is an inner derivation with generator a.

For the A-bimodule tD_A, we define the map

$$\psi = \sum_i c_i'(\mathrm{ad}_{x_i}\partial_i)c_i'' \mapsto |\psi| = -(c_1''c_1', \dots, c_n''c_n') \in \mathrm{tDer}(A)$$

which defines an isomorphism $|\mathrm{tD}_A| \to \mathrm{tDer}(A)$. It is easy to check that $\rho(|\psi|) = |i(\psi)|$ for all $\psi \in \mathrm{tD}_A$.

Lemma 4.14. *The map* $\mathrm{tD}_A \to \mathrm{tDer}(A), \psi \mapsto |\psi|$ *is* $\mathrm{tDer}(A)$*-equivariant.*

Proof. Let $u \in \mathrm{tDer}(A)$ and $\phi = \sum_i c_i'(\mathrm{ad}_{x_i}\partial_i)c_i'' \in \mathrm{tD}_A$. The coefficient of $\mathrm{ad}_{x_i}\partial_i$ in $u \cdot \phi \in \mathrm{tD}_A$ is computed in the proof of Lemma 4.11. Then the ith component of $|u \cdot \phi| \in \mathrm{tDer}(A)$ is given by

$$\begin{aligned} & -(c_i''u(c_i') + u(c_i'')c_i' + \phi'(u_i)\phi''(u_i) + u_i c_i'' c_i' - c_i'' c_i' u_i) \\ =& u(-c_i''c_i') - |\phi|(u_i) + [u_i, -c_i''c_i'] \\ =& u(|\phi|_i) - |\phi|(u_i) + [u_i, |\phi|_i] \\ =& [u, |\phi|]_i. \end{aligned}$$

This proves that $|u \cdot \phi| = [u, |\phi|]$, as required. □

Let A^T be the degree completion of the free associative algebra over $\mathbb{K}$ generated by $x_1, \ldots, x_n$, and T. The algebra A^T is isomorphic to A_{n+1}, but in what follows the extra generator T will play a different role from the other generators. Define an injective map $\mathrm{D}_A \to \mathrm{Der}(A^T)$ defined by formula

$$\phi \mapsto \phi^T : a \mapsto \phi'(a)T\phi''(a)$$

for $a \in A$ and $\phi^T(T) = 0$. The image of this map consists of derivations of A^T which are linear in T and which vanish on the generator T. The map $\phi \mapsto \phi^T$ commutes with the natural actions of $\mathrm{Der}(A)$ on D_A and on $\mathrm{Der}(A^T)$. By composing with the evaluation map ev_1 (which puts the extra generator T equal 1), we obtain a derivation of A: $a \mapsto ev_1(\phi^T(a))$ which coincides with $|\phi|$.

Remark 4.15. Let $\phi \in \mathrm{D}_A$ be a double derivation. The derivation $\phi^T \in \mathrm{Der}(A^T)$ descends to an endomorphism of $|A^T|$ and it maps $|A| \subset |A^T|$ to cyclic words linear in T. But the space of cyclic words linear in T is isomorphic to A via the map $a \mapsto |Ta|$. Hence, the double derivation ϕ defines a map $\phi : |A| \to A$ by formula

$$\phi : |a| \mapsto \phi^T(|a|) = |\phi'(a)T\phi''(a)| \mapsto \phi''(a)\phi'(a).$$

Similarly, we define a map $\mathrm{tD}_A \to \mathrm{tDer}(A^T)$ by formula

$$\psi = \sum_i c_i'(\mathrm{ad}_{x_i}\partial_i)c_i'' \mapsto \psi^T := -(c_1''Tc_1', \ldots, c_n''Tc_n', 0).$$

As before, it is equivariant under the action of $\mathrm{tDer}(A)$ on tD_A and under the adjoint action of $\mathrm{tDer}(A)$ on itself. Again, it is easy to check that $\rho(\psi^T) = i(\psi)^T$.

2.4 Integration to groups

Let $\mathfrak{g}$ be a non-negatively graded Lie algebra and assume that the degree zero part $\mathfrak{g}_0$ is contained in the center of $\mathfrak{g}$. Then, one can use the Baker-Campbell-Hausdorff formula to define a group law on $\mathfrak{g}$. We denote the corresponding group by $\mathcal{G}$. It is convenient to use the multiplicative notation for the group elements: to $u \in \mathfrak{g}$ corresponds the group element $e^u \in \mathcal{G}$ and $e^u e^v = e^{u*v}$, where

$$u * v = \mathrm{bch}(u, v) = \log(e^u e^v) = u + v + \frac{1}{2}[u, v] + \dots$$

Lie algebras $\mathrm{Der}^+(A)$ (derivations of A of positive degree), L and $\mathrm{tDer}(L)$ are positively graded pro-nilpotent Lie algebras. Lie algebras A (with Lie bracket the commutator) and $\mathrm{tDer}(A)$ are non-negatively graded, and their zero degree parts are contained in the center. Hence, the BCH formula defines a group law on all of them. We denote the groups integrating A and L by $\exp(A)$ and $\exp(L)$, respectively. For the integration of $\mathrm{tDer}(A)$ and $\mathrm{tDer}(L)$ we use the notation $\mathrm{TAut}(A)$ and $\mathrm{TAut}(L)$.

As a set, the group $\mathrm{TAut}(A)$ is isomorphic to

$$\mathrm{TAut}(A) \cong (\exp(A))^{\times n} .$$

Elements of $\mathrm{TAut}(A)$ are n-tuples $F = (F_1, \dots, F_n)$, where $F_i \in \exp(A)$. The map $\rho : \mathrm{tDer}(A) \to \mathrm{Der}(A)$ lifts to a map (denoted by the same letter) $\rho : \mathrm{TAut}(A) \to \mathrm{Aut}(A)$ described in the following way. First of all, note that any element $F = \exp(f) \in \exp(A)$ with $f \in A$ decomposes as $F = F_0 F_{\geq 1}$, where $f = f_0 + f_{\geq 1}$ is the decomposition into the degree 0 part $f_0 \in \mathbb{K}$ and the positive degree part $f_{\geq 1} \in A_{\geq 1}$, $F_0 = \exp(f_0)$ and $F_{\geq 1} = \exp(f_{\geq 1})$. Then, $F_{\geq 1}$ makes sense as an invertible element of A, by the exponential series $\exp(f_{\geq 1}) = \sum_{k=0}^{\infty}(1/k!){f_{\geq 1}}^k$. With this notation, the whole group $\exp(A)$ acts on A by inner automorphisms:

$$F.a := (F_{\geq 1})^{-1} a F_{\geq 1}, \quad \text{for } F \in \exp(A),\, a \in A.$$

We will use the shorthand notation $F^{-1}aF$ for $(F_{\geq 1})^{-1} a F_{\geq 1}$. Now, the map $\rho : \mathrm{TAut}(A) \to \mathrm{Aut}(A)$ is defined by

$$\rho(F) : x_i \mapsto F_i^{-1} x_i F_i.$$

The group law on $\mathrm{TAut}(A)$ is given by

$$(F \cdot G)_i = F_i(\rho(F) G_i),$$

where $F \cdot G$ stands for the product in $\mathrm{TAut}(A)$ and the product $F_i(\rho(F)G_i)$ takes place in $\exp(A)$. A similar description applies to the group $\mathrm{TAut}(L)$. Note that if $F \in \mathrm{TAut}(L)$, then $F_{\geq 1} = F$.

All Lie algebra actions described in the previous section integrate to group actions. For instance, the adjoint action of $\mathrm{tDer}(A)$ on itself integrates to the adjoint action Ad of $\mathrm{TAut}(A)$ on $\mathrm{tDer}(A)$, and the natural action of $\mathrm{tDer}(A)$ on tD_A (Lemma 4.11) integrates to a group action of $\mathrm{TAut}(A)$ on tD_A. By Lemma 4.14, the map $\psi \mapsto |\psi|$ from tD_A to $\mathrm{tDer}(A)$ is $\mathrm{TAut}(A)$-equivariant: for any $F \in \mathrm{TAut}(A)$, we have

$$|F.\psi| = \mathrm{Ad}_F\,|\psi|.$$

3 Divergence maps

In this section, we describe various versions of the non-commutative divergence map and describe their properties.

3.1 Double divergence maps

Consider $|A| := A/[A,A]$ and let $\mathbf{1} \in |A|$ denote the image of the unit $1 \in A$. The natural action of $\mathrm{Der}(A)$ on A descends to an action on $|A|$. By using the map ρ, we can view A and $|A|$ as left $\mathrm{tDer}(A)$-modules. In the following proposition, we introduce the *divergence map* $\mathrm{Div} : \mathrm{Der}(A) \to |A| \otimes |A|$. By abuse of notation, we use the letter $|\ |$ for the map $|\ |^{\otimes 2} : A \otimes A \to |A| \otimes |A|$.

Proposition 4.16. *The map* $\mathrm{Div} : \mathrm{Der}(A) \to |A| \otimes |A|$ *defined by formula*

$$\mathrm{Div}(u) := \sum_i |\partial_i(u(x_i))|$$

for $u \in \mathrm{Der}(A)$ *is a* 1*-cocycle on the Lie algebra* $\mathrm{Der}(A)$*. That is, for all* $u, v \in \mathrm{Der}(A)$*, we have*

$$\mathrm{Div}([u,v]) = u \cdot \mathrm{Div}(v) - v \cdot \mathrm{Div}(u).$$

Proof. Consider a map $Z_i : A \to A \otimes A$ defined as follows:

$$Z_i := \partial_i \circ [u,v] - u \circ \partial_i \circ v + v \circ \partial_i \circ u = [v, \partial_i] \circ u - [u, \partial_i] \circ v.$$

When applied to the generator x_i, it yields

$$\begin{aligned} Z_i(x_i) &= [v,\partial_i](u(x_i)) - [u,\partial_i](v(x_i)) \\ &= \textstyle\sum_k \big(-(\partial'_k u(x_i))(\partial'_i v(x_k)) \otimes (\partial''_i v(x_k))(\partial''_k u(x_i)) \\ &\qquad + (\partial'_k v(x_i))(\partial'_i u(x_k)) \otimes (\partial''_i u(x_k))(\partial''_k v(x_i)) \big), \end{aligned}$$

where we have used equation (4.9). Note that

$$\mathrm{Div}([u,v]) - u \cdot \mathrm{Div}(v) + v \cdot \mathrm{Div}(u) = \sum_i |Z_i(x_i)|,$$

and combining with the previous formula we obtain

$$\begin{array}{rcl}\sum_i |Z_i(x_i)| & = & \sum_{i,k} (-|(\partial'_k u(x_i))(\partial'_i v(x_k))| \otimes |(\partial''_i v(x_k))(\partial''_k u(x_i))| \\ & & \qquad + |(\partial'_k v(x_i))(\partial'_i u(x_k))| \otimes |(\partial''_i u(x_k))(\partial''_k v(x_i))|) = 0.\end{array}$$

Here the two terms cancel each other after using the cyclic property of $|\cdot|$ and renaming i and k. ☐

The Lie algebra $\mathrm{tDer}(A)$, which is related to $\mathrm{Der}(A)$ by the map $\rho : \mathrm{tDer}(A) \to \mathrm{Der}(A)$, carries additional 1-cocycles as explained in the following lemma.

Lemma 4.17. *For $1 \leq i \leq n$, the map $c_i : \mathrm{tDer}(A) \to |A|$ defined by formula $c_i(u) := |u_i|$ for $u = (u_1, \ldots, u_n) \in \mathrm{tDer}(A)$ is a 1-cocycle on the Lie algebra $\mathrm{tDer}(A)$.*

Proof. Let $u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_n) \in \mathrm{tDer}(A)$. The ith component of $[u, v]$ is $w_i = \rho(u)v_i - \rho(v)u_i + [u_i, v_i]$. Since $|[u_i, v_i]| = 0$, we obtain

$$c_i([u, v]) = |w_i| = |\rho(u)v_i| - |\rho(v)u_i| = u \cdot |v_i| - v \cdot |u_i| = u \cdot c_i(v) - v \cdot c_i(u).$$

☐

We define the *tangential divergence* cocycle $\mathrm{tDiv} : \mathrm{tDer}(A) \to |A| \otimes |A|$ as a linear combination of the divergence cocycle and of the cocycles c_i.

Definition 4.18. For $u \in \mathrm{tDer}(A)$, set

$$\mathrm{tDiv}(u) = \mathrm{Div}(\rho(u)) + \sum_i (c_i(u) \otimes \mathbf{1} - \mathbf{1} \otimes c_i(u)).$$

Since ρ is a Lie algebra homomorphism and Div and c_i's are 1-cocycles, it is obvious that tDiv is a 1-cocylce as well. In the following lemma we give a more explicit formula for tDiv.

Lemma 4.19. *For $u = (u_1, \ldots, u_n) \in \mathrm{tDer}(A)$, we have*

$$\mathrm{tDiv}(u) = \sum_i |x_i(\partial_i u_i) - (\partial_i u_i)x_i|.$$

Proof. We compute

$$\begin{array}{rcl}\mathrm{tDiv}(u) & = & \sum_i |\partial_i[x_i, u_i]| + \sum_i(|u_i| \otimes \mathbf{1} - \mathbf{1} \otimes |u_i|) \\ & = & \sum_i |1 \otimes u_i + x_i(\partial_i u_i) - (\partial_i u_i)x_i - u_i \otimes 1| + \sum_i(|u_i| \otimes \mathbf{1} - \mathbf{1} \otimes |u_i|) \\ & = & \sum_i |x_i(\partial_i u_i) - (\partial_i u_i)x_i|.\end{array}$$

☐

3.2 Relation to the ordinary divergence

Recall the *divergence map* introduced in [6] §3.3. Any element $a \in A$ can be uniquely written as $a = a^0 + \sum_i a^i x_i$, where $a^0 \in \mathbb{K}$ and $a^i \in A$. One defines the map $\mathrm{div} : \mathrm{tDer}(L) \to |A|$ by formula

$$\mathrm{div}(u) := \sum_i |x_i (u_i)^i|,$$

where $u = (u_1, \dots, u_n) \in \mathrm{tDer}(L)$. The map div is a 1-cocycle on the Lie algebra $\mathrm{tDer}(L)$ with values in $|A|$.

Consider an algebra homomorphism from A to $A \otimes A^{\mathrm{op}}$ defined by formula

$$\widetilde{\Delta} := (1 \otimes \iota)\Delta : A \to A \otimes A.$$

It induces a map from $|A|$ to $|A| \otimes |A|$, for which we use the same letter $\widetilde{\Delta}$. In the following proposition, we show that $\widetilde{\Delta}$ intertwines the divergence maps div and tDiv.

Proposition 4.20. *The following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{tDer}(L) & \xrightarrow{\mathrm{div}} & |A| \\ \big\downarrow & & \big\downarrow \widetilde{\Delta} \\ \mathrm{tDer}(A) & \xrightarrow{\mathrm{tDiv}} & |A| \otimes |A| \end{array}$$

Lemma 4.21. *For any $a \in L$ and $1 \leq i \leq n$, we have $\widetilde{\Delta}(a^i) = \partial_i a$.*

Proof. Both maps $L \to A \otimes A$, $a \mapsto \partial_i a$ and $a \mapsto \widetilde{\Delta}(a^i)$ have the property

$$\phi([a, b]) = \phi(a)b + a\phi(b) - b\phi(a) - \phi(b)a.$$

Since $\partial_i(x_j) = \widetilde{\Delta}((x_j)^i) = \delta_{i,j}(1 \otimes 1)$, we conclude that $\widetilde{\Delta}(a^i) = \partial_i a$ for all $a \in L$, as required. $\square$

Proof of Proposition 4.20. Note that we have the following expression for $\mathrm{tDiv}(u)$:

$$\mathrm{tDiv}(u) = \sum_i |\widetilde{\Delta}(x_i)(\partial_i u_i)| \tag{4.10}$$

for any $u \in \mathrm{tDer}(A)$, where the product of $\widetilde{\Delta}(x_i) = x_i \otimes 1 - 1 \otimes x_i$ and $\partial_i u_i$ is taken in $A \otimes A^{\mathrm{op}}$. Now let $u \in \mathrm{tDer}(L)$. Then, using equation (4.10) and Lemma 4.21, we compute

$$\widetilde{\Delta}(\mathrm{div}(u)) = \widetilde{\Delta}\left(\sum_i |x_i (u_i)^i|\right) = \sum_i |\widetilde{\Delta}(x_i)\widetilde{\Delta}((u_i)^i)| = \sum_i |\widetilde{\Delta}(x_i)(\partial_i u_i)| = \mathrm{tDiv}(u).$$

$\square$

The following lemma will be used later.

Lemma 4.22. *The maps* $\widetilde{\Delta} : A \to A \otimes A$ *and* $\widetilde{\Delta} : |A| \to |A| \otimes |A|$ *are injective. They are maps of* $\mathrm{Der}(L)$*-modules.*

Proof. The injectivity follows from $(1 \otimes \varepsilon)\widetilde{\Delta}(a) = (1 \otimes \varepsilon)\Delta(a) = a$ for any $a \in A$. Since the coproduct Δ and the antipode ι are morphisms of $\mathrm{Der}(L)$-modules, so is the map $\widetilde{\Delta}$. □

3.3 Divergence maps with extra parameters

Next we introduce a certain refinement of tDiv. Recall that A^T is the degree completion of the free associative algebra over $\mathbb{K}$ generated by $x_1, \dots, x_n$, and T. We have an embedding

$$j_T : (|A| \otimes A) \oplus (A \otimes |A|) \to |A^T| \otimes |A^T|, \quad (|a| \otimes b, c \otimes |d|) \mapsto |a| \otimes |Tb| + |Tc| \otimes |d|,$$

through which we identify the source and the image. Let $p_1 : \mathrm{im}(j_T) \to |A| \otimes A$ and $p_2 : \mathrm{im}(j_T) \to A \otimes |A|$ be projections on the first and the second factors in the tensor product, respectively. The map $\mathrm{tD}_A \to \mathrm{tDer}(A^T), \phi \mapsto \phi^T$ has the property that $\mathrm{tDiv}(\phi^T) \in \mathrm{im}(j_T)$ since every component of ϕ^T (with the exception of the last one which vanishes) is linear in T.

Definition 4.23. Define the maps $\mathrm{tDiv}^T : \mathrm{tD}_A \to |A| \otimes A$ and $\underline{\mathrm{tDiv}}^T : \mathrm{tD}_A \to A \otimes |A|$ by $\mathrm{tDiv}^T(\phi) := p_1(\mathrm{tDiv}(\phi^T))$ and $\underline{\mathrm{tDiv}}^T(\phi) := p_2(\mathrm{tDiv}(\phi^T))$ for $\phi \in \mathrm{tD}_A$.

Let $\phi = \sum_i c'_i(\mathrm{ad}_{x_i}\partial_i)c''_i \in \mathrm{tD}_A$. By (4.10), $\mathrm{tDiv}(\phi^T) = -\sum_i |\widetilde{\Delta}(x_i)\partial_i(c''_i T c'_i)|$, and the terms where T appears to the right of the symbol $\otimes$ are

$$-\sum_i |\widetilde{\Delta}(x_i)((\partial_i c''_i)Tc'_i)| = -\sum_i |(x_i \otimes 1)(\partial_i c''_i)(1 \otimes Tc'_i) - (\partial_i c''_i)(1 \otimes Tc'_i)(1 \otimes x_i)|.$$

Therefore, we obtain

$$\mathrm{tDiv}^T(\phi) = (|\ | \otimes 1)\left(\sum_i (1 \otimes c'_i x_i - x_i \otimes c'_i)(\partial_i c''_i)\right). \tag{4.11}$$

In a similar way, we obtain

$$\underline{\mathrm{tDiv}}^T(\phi) = (1 \otimes |\ |)\left(\sum_i (\partial_i c'_i)(c''_i \otimes x_i - x_i c''_i \otimes 1)\right). \tag{4.12}$$

Proposition 4.24. *For any $a \in A$ and $\phi \in \mathrm{tD}_A$,*

$$\begin{aligned}
\mathrm{tDiv}^T(a\phi) =&(1 \otimes a)\mathrm{tDiv}^T(\phi), \\
\mathrm{tDiv}^T(\phi a) =&\mathrm{tDiv}^T(\phi)(1 \otimes a) + (|\ | \otimes 1)\phi(a), \\
\underline{\mathrm{tDiv}}^T(a\phi) =&(a \otimes 1)\underline{\mathrm{tDiv}}^T(\phi) + (1 \otimes |\ |)\phi(a), \\
\underline{\mathrm{tDiv}}^T(\phi a) =&\underline{\mathrm{tDiv}}^T(\phi)(a \otimes 1).
\end{aligned}$$

Proof. The first equation is an immediate consequence of (4.11). For the second one, we compute,

$$\begin{aligned}
\mathrm{tDiv}^T(\phi a) =&(|\ | \otimes 1)\left(\sum_i (1 \otimes c_i' x_i - x_i \otimes c_i')\partial_i(c_i'' a)\right) \\
=&(|\ | \otimes 1)\left(\sum_i (1 \otimes c_i' x_i - x_i \otimes c_i')(\partial_i c_i'')a\right) \\
&+ (|\ | \otimes 1)\left(\sum_i (1 \otimes c_i' x_i - x_i \otimes c_i')(c_i'' \otimes 1)\partial_i a\right) \\
=&\mathrm{tDiv}^T(\phi)(1 \otimes a) + (|\ | \otimes 1)\left(\sum_i \phi(x_i)\partial_i a\right).
\end{aligned}$$

Since $(|\ |\otimes 1)\phi(a) = (|\ |\otimes 1)(\sum_i \phi(x_i)(\partial_i a))$, the second equation follows. The formulas for $\underline{\mathrm{tDiv}}^T$ can be obtained similarly (we use $(1\otimes|\ |)\phi(a) = (1\otimes|\ |)(\sum_i(\partial_i a)\phi(x_i))$). □

Proposition 4.25. *For any $\phi \in \mathrm{tD}_A$, we have*

$$\mathrm{tDiv}(|\phi|) = (1 \otimes |\ |)\mathrm{tDiv}^T(\phi) + (|\ | \otimes 1)\underline{\mathrm{tDiv}}^T(\phi).$$

Proof. By (4.11) and (4.12), we compute

$$\begin{aligned}
&(1 \otimes |\ |)\mathrm{tDiv}^T(\phi) + (|\ | \otimes 1)\underline{\mathrm{tDiv}}^T(\phi) \\
=&(|\ | \otimes |\ |)\left(\sum_i (1 \otimes x_i - x_i \otimes 1)(\partial_i c_i'')(1 \otimes c_i') + \sum_i (1 \otimes x_i - x_i \otimes 1)(c_i'' \otimes 1)(\partial_i c_i')\right) \\
=&(|\ | \otimes |\ |)\left(\sum_i (x_i \otimes 1 - 1 \otimes x_i)\partial_i(-c_i'' c_i')\right) \\
=&\mathrm{tDiv}(|\phi|).
\end{aligned}$$

□

3.4 Integration of divergence cocycles

Let $\mathfrak{g}$ be non-negatively graded Lie algebra with the zero degree part $\mathfrak{g}_0$ contained in the center of $\mathfrak{g}$. Let $\mathcal{G}$ be the group integrating $\mathfrak{g}$ (using the BCH series) and M a graded module over $\mathfrak{g}$. Assume that the action of $\mathfrak{g}_0$ on M is trivial. Then, the Lie algebra action of $\mathfrak{g}$ integrates to a group action of $\mathcal{G}$ on M. Let $c : \mathfrak{g} \to M$ be a Lie algebra 1-cocycle. Under the conditions listed above, it integrates to a group 1-cocycle $C : \mathcal{G} \to M$ which satisfies the following defining properties: the first one is the group cocycle condition

$$C(fg) = C(f) + f \cdot C(g), \tag{4.13}$$

for all $f, g \in \mathcal{G}$. The second one is the relation between c and C:

$$\frac{d}{dt} C(\exp(tu))|_{t=0} = c(u), \tag{4.14}$$

for all $u \in \mathfrak{g}$. These two conditions imply

$$C(\exp(u)) = \frac{e^u - 1}{u} \cdot c(u) \tag{4.15}$$

for $u \in \mathfrak{g}$ and $\exp(u) \in \mathcal{G}$.

The 1-cocycles $\mathrm{Div} : \mathrm{Der}^+(A) \to |A| \otimes |A|$ (the restriction of the 1-cocycle Div to derivations of positive degree), $\mathrm{tDiv} : \mathrm{tDer}(A) \to |A| \otimes |A|$ and $\mathrm{div} : \mathrm{tDer}(L) \to |A|$ integrate to group 1-cocycles $\mathrm{J} : \exp(\mathrm{Der}^+(A)) \to |A| \otimes |A|, \mathrm{tJ} : \mathrm{TAut}(A) \to |A| \otimes |A|$ and $j : \mathrm{TAut}(L) \to |A|$. The statement of Proposition 4.20 and Lemma 4.22 gives rise to the following relation between group cocycles:

$$\widetilde{\Delta}(j(F)) = \mathrm{tJ}(F) \tag{4.16}$$

for all $F \in \mathrm{TAut}(L)$. The following lemma establishes a property of Lie algebra and group cocycles which will be important later in the text:

Lemma 4.26. *Let M be a graded $\mathfrak{g}$-module, $c : \mathfrak{g} \to M$ a Lie algebra 1-cocycle and $C : \mathcal{G} \to M$ the group 1-cocycle integrating c. Then, for all $u \in \mathfrak{g}, f \in \mathcal{G}$ we have*

$$c(\mathrm{Ad}_f u) = f \cdot \left(c(u) + u \cdot C(f^{-1}) \right). \tag{4.17}$$

Proof. Consider the expression $C(fe^{tu}f^{-1})$ and compute the t-derivative at $t = 0$ in two ways. On the one hand,

$$\frac{d}{dt} C(fe^{tu}f^{-1})|_{t=0} = \frac{d}{dt} C(e^{t\mathrm{Ad}_f u})|_{t=0} = c(\mathrm{Ad}_f u).$$

On the other hand,

$$\begin{array}{rcl}\frac{d}{dt}\, C(fe^{tu}f^{-1})|_{t=0} & = & \frac{d}{dt}\,\left(C(f)+f\cdot C(e^{tu})+(fe^{tu})\cdot C(f^{-1})\right)|_{t=0} \\ & = & f\cdot c(u)+f\cdot(u\cdot C(f^{-1})) \\ & = & f\cdot\left(c(u)+u\cdot C(f^{-1})\right).\end{array}$$

□

Among other things, equation (4.17) implies that $c(\mathrm{Ad}_f u) = f \cdot c(u)$ if and only if $u \cdot C(f^{-1}) = 0$.

By applying equation (4.17) to the cocycle tDiv, we obtain the following equation which will be of importance in the next sections:

$$\mathrm{tDiv}(\mathrm{Ad}_F u) = F \cdot (\mathrm{tDiv}(u) + u \cdot \mathrm{tJ}(F^{-1})),$$

where $u \in \mathrm{tDer}(A), F \in \mathrm{TAut}(A)$. It is convenient to introduce a "pull-back" of the tDiv cocycle by the automorphism F: $F^*\mathrm{tDiv}(u) = F^{-1} \cdot \mathrm{tDiv}(\mathrm{Ad}_F u)$. It is again a 1-cocycle on $\mathrm{tDer}(A)$, and it satisfies the formula

$$F^*\mathrm{tDiv}(u) = \mathrm{tDiv}(u) + u \cdot \mathrm{tJ}(F^{-1}). \tag{4.18}$$

There is a similar transformation property for the map $\mathrm{tDiv}^T : \mathrm{tD}_A \to |A| \otimes A$. We compute

$$\begin{array}{rcl}F^*\mathrm{tDiv}^T(\phi) & = & (F^{-1}\otimes F^{-1})\cdot \mathrm{tDiv}^T((F\otimes F)\circ\phi\circ F^{-1}) \\ & = & (F^{-1}\otimes F^{-1})\cdot p_1(\mathrm{tDiv}(\mathrm{Ad}_F\phi^T)) \\ & = & p_1(\mathrm{tDiv}(\phi^T)) + p_1(\phi^T\cdot \mathrm{tJ}(F^{-1})) \\ & = & \mathrm{tDiv}^T(\phi) + (1\otimes\phi)\cdot \mathrm{tJ}(F^{-1}).\end{array} \tag{4.19}$$

Here F is uniquely extended to a tangential automorphism of A^T preserving T, and in the last line we use the map $\phi : |A| \to A$ defined in Remark 4.15.

4 Double brackets and auxiliary operations

In this section, we introduce double brackets (in the sense of van den Bergh) and discuss various operations which can be built from double brackets and divergence maps.

4.1 Definition and basic properties

Definition 4.27. A *double bracket* on A is a $\mathbb{K}$-linear map $\Pi : A\otimes A \mapsto A\otimes A, a\otimes b \mapsto \Pi(a,b)$ such that for any $a, b, c \in A$,

$$\Pi(a, bc) = \Pi(a, b)c + b\Pi(a, c), \tag{4.20}$$
$$\Pi(ab, c) = \Pi(a, c) * b + a * \Pi(b, c). \tag{4.21}$$

One often uses the notation

$$\{a, b\}_\Pi = \Pi(a, b) = \Pi(a, b)' \otimes \Pi(a, b)''.$$

Remark 4.28. We use the terminology "double bracket" in a wider sense than [35], where a double bracket is defined to be a map satisfying (4.20) and the skew-symmetry condition: for any $a, b \in A$,

$$\Pi(b, a) = -\Pi(a, b)^\circ. \tag{4.22}$$

Here, $(c \otimes d)^\circ = d \otimes c$ for $c, d \in A$. As is easily seen, (4.20) and (4.22) imply (4.21).

For a given double bracket, there are several auxiliary operations.

Lemma 4.29. *Let $\Pi : A \otimes A \to A \otimes A$ be a double bracket. Then, for every $a \in A$ the map*

$$\{a, -\}_\Pi : b \mapsto \Pi(a, b)' \otimes \Pi(a, b)''$$

is a double derivation on A.

Proof. Indeed, by equation (4.20) we have

$$\{a, bc\}_\Pi = \Pi(a, b)\, c + b\, \Pi(a, c) = \{a, b\}_\Pi\, c + b\, \{a, c\}_\Pi,$$

as required. ☐

By applying the map $|\ | : \mathrm{D}_A \to \mathrm{Der}(A)$, we obtain a derivation on A of the form

$$|\{a, -\}_\Pi| : b \mapsto \Pi(a, b)'\Pi(a, b)''.$$

Lemma 4.30. *The map $A \to \mathrm{Der}(A)$ given by $a \mapsto |\{a, -\}_\Pi|$ descends to $|A|$.*

Proof. We have,

$$|\{ac, -\}_\Pi| : b \mapsto \Pi(a, b)'c\Pi(a, b)'' + \Pi(c, b)'a\Pi(c, b)''.$$

The right hand side is symmetric under exchange of a and c. Hence, it vanishes on commutators, as required. ☐

Motivated by this lemma, we often use the notation

$$|\{a,-\}_{\Pi}| = \{|a|,-\}_{\Pi}.$$

By equation (4.20), the double bracket induces a map $\{-,-\}_{\Pi} : |A| \otimes |A| \to |A|$ defined by formula

$$|a| \otimes |b| \mapsto |\{|a|,b\}_{\Pi}| =: \{|a|,|b|\}_{\Pi}. \tag{4.23}$$

Furthermore, by Remark 4.15, we obtain a map $\{-,-\}_{\Pi} : A \otimes |A| \to A$ defined by

$$a \otimes |b| \mapsto \{a,-\}_{\Pi}(|b|) = \Pi(a,b)''\Pi(a,b)' =: \{a,|b|\}_{\Pi}. \tag{4.24}$$

All the three operations above are called the *bracket* associated with Π.

Lemma 4.31. *Let* Π *be a skew-symmetric double bracket. Then, for any* $a, b \in A$,

$$\{a,|b|\}_{\Pi} = -\{|b|,a\}_{\Pi}.$$

Proof. Since Π is skew-symmetric, $\Pi(b,a) = -\Pi(a,b)^{\circ} = -\Pi(a,b)'' \otimes \Pi(a,b)'$. Then

$$\{|b|,a\}_{\Pi} = \Pi(b,a)'\Pi(b,a)'' = -\Pi(a,b)''\Pi(a,b)' = -\{a,|b|\}_{\Pi}.$$

□

In order to give concrete examples of double brackets we consider the space $\mathrm{D}_A \otimes_A \mathrm{D}_A$ and its A-bimodule structure given by $c_1(\phi_1 \otimes_A \phi_2)c_2 := (c_1\phi_1) \otimes_A (\phi_2 c_2)$ for $c_1, c_2 \in A$ and $\phi_1, \phi_2 \in \mathrm{D}_A$. Set

$$(DA)_2 := |\mathrm{D}_A \otimes_A \mathrm{D}_A|.$$

For $\phi_1, \phi_2 \in \mathrm{D}_A$, we define a map $\Pi_{\phi_1 \otimes \phi_2} : A \otimes A \to A \otimes A$ by

$$\Pi_{\phi_1 \otimes \phi_2}(a,b) := \phi_2'(b)\phi_1''(a) \otimes \phi_1'(a)\phi_2''(b), \tag{4.25}$$

where we write $\phi_1(a) = \phi_1'(a) \otimes \phi_1''(a)$ and $\phi_2(b) = \phi_2'(b) \otimes \phi_2''(b)$. In other words,

$$\Pi_{\phi_1 \otimes \phi_2}(a,-) = \phi_1'(a)\phi_2\phi_1''(a) \in \mathrm{D}_A \tag{4.26}$$

for any $a \in A$. One can check that $\Pi_{\phi_1 \otimes \phi_2}$ is a double bracket on A. In [35] §4, it is proved that this assignment induces a $\mathbb{K}$-linear isomorphism

$$(DA)_2 \xrightarrow{\cong} \{\text{double brackets on } A\}.$$

It is convenient to describe double brackets on A through this isomorphism.

4.2 Tangential double brackets

In this section, we consider a class of double brackets of particular interest.

Definition 4.32. A double bracket Π on A is called *tangential* if $\{a, -\}_\Pi \in \mathrm{tD}_A$ for any $a \in A$.

Notice that Π is tangential if and only if $\{x_i, -\}_\Pi \in \mathrm{tD}_A$ for every i. Recall that to a tangential double derivation $\{a, -\}_\Pi \in \mathrm{tD}_A$ one can associate an element $|\{a, -\}_\Pi| \in \mathrm{tDer}(A)$ such that

$$\rho\left(|\{a, -\}_\Pi|\right) = |i(\{a, -\}_\Pi)| = \{|a|, -\}_\Pi \in \mathrm{Der}(A).$$

For simplicity, with the assumption that Π is tangential, we often use the same notation $\{|a|, -\}_\Pi$ for the lift $|\{a, -\}_\Pi| \in \mathrm{tDer}(A)$.

Definition 4.33. Let Π be a tangential double bracket on A.

(i) Define the map $\mathrm{tDiv}_\Pi : |A| \to |A| \otimes |A|$ by $\mathrm{tDiv}_\Pi(|a|) := \mathrm{tDiv}(\{|a|, -\}_\Pi)$ for $|a| \in |A|$.

(ii) Define the maps $\mathrm{tDiv}^T_\Pi : A \to |A| \otimes A$ and $\underline{\mathrm{tDiv}}^T_\Pi : A \to A \otimes |A|$ by $\mathrm{tDiv}^T_\Pi(a) := \mathrm{tDiv}^T(\Pi(a, -))$ and $\underline{\mathrm{tDiv}}^T_\Pi(a) := \underline{\mathrm{tDiv}}^T(\Pi(a, -))$ for $a \in A$.

Another description of the map tDiv^T_Π is as follows. First, we consider the unique extension of Π to a tangential double bracket $\Pi^T : A^T \otimes A^T \to A^T \otimes A^T$ by imposing the condition $\Pi^T(T, -) = \Pi^T(-, T) = 0$. Note that $\Pi^T(Ta, -) = T\,\Pi(a, -)$ for any $a \in A$. To simplify the notation, we drop T and denote the corresponding bracket by $\{-, -\}_\Pi$. Then, it is easy to check that

$$\mathrm{tDiv}^T_\Pi(a) = p_1(\mathrm{tDiv}(\{|Ta|, -\}_\Pi)),$$

where $\{|Ta|, -\}_\Pi \in \mathrm{Der}(A^T)$ is lifted to an element of $\mathrm{tDer}(A^T)$, and $\mathrm{tDiv} : \mathrm{tDer}(A^T) \to |A^T| \otimes |A^T|$ is the tangential divergence on $\mathrm{tDer}(A^T)$.

Note that tDiv_Π, tDiv^T_Π, and $\underline{\mathrm{tDiv}}^T_\Pi$ are $\mathbb{K}$-linear in Π, i.e., $\mathrm{tDiv}_{\Pi_1+\Pi_2} = \mathrm{tDiv}_{\Pi_1} + \mathrm{tDiv}_{\Pi_2}$ for Π_1 and Π_2 tangential, etc.

Proposition 4.34. *Let Π be a tangential double bracket on A. Then for any $a, b \in A$,*

$$\mathrm{tDiv}^T_\Pi(ab) = \mathrm{tDiv}^T_\Pi(a)(1 \otimes b) + (1 \otimes a)\mathrm{tDiv}^T_\Pi(b) + (|\ | \otimes 1)\Pi(a, b), \tag{4.27}$$

$$\underline{\mathrm{tDiv}}^T_\Pi(ab) = \underline{\mathrm{tDiv}}^T_\Pi(a)(b \otimes 1) + (a \otimes 1)\underline{\mathrm{tDiv}}^T_\Pi(b) + (1 \otimes |\ |)\Pi(b, a), \tag{4.28}$$

and for any $a \in A$,

$$\mathrm{tDiv}_\Pi(|a|) = (1 \otimes |\ |)\mathrm{tDiv}^T_\Pi(a) + (|\ | \otimes 1)\underline{\mathrm{tDiv}}^T_\Pi(a). \tag{4.29}$$

Proof. By Proposition 4.24, we compute

$$\begin{aligned}
&\mathrm{tDiv}^T_\Pi(ab) = \mathrm{tDiv}^T(\Pi(ab,-))\\
=&\mathrm{tDiv}^T(\Pi(a,-)b + a\Pi(b,-))\\
=&\mathrm{tDiv}^T(\Pi(a,-))(1\otimes b) + (|\ |\otimes 1)\Pi(a,b) + (1\otimes a)\mathrm{tDiv}^T(\Pi(b,-))\\
=&\mathrm{tDiv}^T_\Pi(a)(1\otimes b) + (1\otimes a)\mathrm{tDiv}^T_\Pi(b) + (|\ |\otimes 1)\Pi(a,b).
\end{aligned}$$

This proves (4.27). To prove (4.28), we compute

$$\begin{aligned}
&\underline{\mathrm{tDiv}}^T_\Pi(ab) = \underline{\mathrm{tDiv}}^T(\Pi(ab,-))\\
=&\underline{\mathrm{tDiv}}^T(\Pi(a,-)b + a\Pi(b,-))\\
=&\underline{\mathrm{tDiv}}^T(\Pi(a,-))(b\otimes 1) + (a\otimes 1)\underline{\mathrm{tDiv}}^T(\Pi(b,-)) + (1\otimes|\ |)\Pi(b,a)\\
=&\underline{\mathrm{tDiv}}^T_\Pi(a)(b\otimes 1) + (a\otimes 1)\underline{\mathrm{tDiv}}^T_\Pi(b) + (|\ |\otimes 1)\Pi(b,a).
\end{aligned}$$

Finally, (4.29) follows from Proposition 4.25. □

Remark 4.35. The notion of a double bracket (Definition 4.27) and its auxiliary operations make sense for any associative $\mathbb{K}$-algebra. In §5, we consider a topologically defined double bracket on the group ring $\mathbb{K}\pi$, where π is the fundamental group of an oriented surface.

4.3 The KKS double bracket

In this section, we consider an important example of a double bracket: the Kirillov-Kostant-Souriau (KKS) double bracket. We will use the map (4.25) to identify double brackets with elements of $(DA)_2$. In this identification, the symbol $\otimes$ stands for $\otimes_A$. The KKS double bracket is given by the following formula:

$$\Pi_{\mathrm{KKS}} = \sum_{i=1}^n |\partial_i \otimes \mathrm{ad}_{x_i}\partial_i|. \tag{4.30}$$

Lemma 4.36. *The double bracket* Π_{KKS} *is tangential and skew-symmetric.*

Proof. The skew-symmetry (4.22) is obvious from the following computation:

$$\Pi_{\mathrm{KKS}} = \sum_i |\partial_i \otimes (x_i\partial_i - \partial_i x_i)| = -\sum_i |(x_i\partial_i - \partial_i x_i)\otimes \partial_i|,$$

where we have used the cyclicity property for x_i under the $|\cdot|$ sign. The fact that Π_{KKS} is tangetial can be checked on generators:

$$\Pi_{\mathrm{KKS}}(x_i,-) = \mathrm{ad}_{x_i}\partial_i \in \mathrm{tD}_A.$$

□

Denote $x_0 = -\sum_{i=1}^n x_i$.

Lemma 4.37. *For the KKS double bracket, we have*

$$\begin{aligned}
&\{|h(x_i)|, a\}_{\mathrm{KKS}} = \{a, |h(x_i)|\}_{\mathrm{KKS}} = 0,\\
&\{x_0, a\}_{\mathrm{KKS}} = -\phi_0(a),\\
&\{x_0, |a|\}_{\mathrm{KKS}} = \{|a|, x_0\}_{\mathrm{KKS}} = 0,\\
&\{|h(x_0)|, a\}_{\mathrm{KKS}} = -\{a, |h(x_0)|\}_{\mathrm{KKS}} = [a, \dot{h}(x_0)],
\end{aligned}$$

where $a \in A$, $i = 1, \ldots, n$, $h(z) \in \mathbb{K}[[z]]$ and $\dot{h}(z)$ its derivative.

Proof. First of all, note that Π_{KKS} is skew-symmetric by Lemma 4.36 so that we can use Lemma 4.31.

For the first equality, note that $\{x_i, -\}_{\mathrm{KKS}} = [x_i, \partial_i]$. Then, for any formal power series $h(z) \in \mathbb{K}[[z]]$, we have

$$\{|h(x_i)|, -\}_{\mathrm{KKS}} = |\dot{h}(x_i)[x_i, \partial_i]| = 0,$$

where we have used the cyclic property under the $|\cdot|$ sign.

For the second equality, we have

$$\{x_0, -\}_{\mathrm{KKS}} = -\{\sum_i x_i, -\}_{\mathrm{KKS}} = -\sum_i [x_i, \partial_i] = -\phi_0.$$

This also proves $\{|a|, x_0\}_{\mathrm{KKS}} = 0$.

Finally, for the last equality we compute

$$\{|h(x_0)|, -\}_{\mathrm{KKS}} = -|\dot{h}(x_0)\phi_0|$$

which is the inner derivation with generator $-\dot{h}(x_0)$ (see Example 4.13), as required. □

Lemma 4.38. *For $h(z) \in \mathbb{K}[[z]]$ and $a \in A$, let $h' \otimes h'' = \widetilde{\Delta} h(x_0)$. Then*

$$\begin{aligned}
&\{|h'|, ah''\}_{\mathrm{KKS}} = -\dot{h}(\mathrm{ad}_{x_0})(a) + \dot{h}(0)a,\\
&\{|h'|, h''a\}_{\mathrm{KKS}} = \dot{h}(-\mathrm{ad}_{x_0})(a) - \dot{h}(0)a.
\end{aligned}$$

Proof. One computes

$$\begin{aligned}
\{|h'|, ah''\}_{\mathrm{KKS}} &= \{|h'|, a\}_{\mathrm{KKS}}\, h'' + a\{|h'|, h''\}_{\mathrm{KKS}}\\
&= [a, (\dot{h})']\, (\dot{h})''\\
&= a(\dot{h})'(\dot{h})'' - (\dot{h})'a(\dot{h})''\\
&= a\dot{h}(0) - \dot{h}(\mathrm{ad}_{x_0})a,
\end{aligned}$$

where in the second line we have used Lemma 4.37 and in the last line we have used that $(\dot{h})'(\dot{h})'' = \varepsilon(\dot{h}(x_0)) = \dot{h}(0)$ and $(\dot{h})'a(\dot{h})'' = \dot{h}(\mathrm{ad}_{x_0})a$. The other equality can be proven in a similar way. □

One can obtain an explicit formula for the KKS double bracket on a pair of words $z = z_1 \cdots z_l, w = w_1 \cdots w_m$, where $z_j, w_k \in \{x_i\}_{i=1}^n$. Indeed,

$$\begin{aligned}\Pi_{\mathrm{KKS}}(z,w) &= \textstyle\sum_k w_1 \ldots w_{k-1} \Pi_{\mathrm{KKS}}(z, w_k) w_{k+1} \ldots w_m \\ &= \textstyle\sum_{j,k} w_1 \cdots w_{k-1} \left((z_1 \cdots z_{j-1}) * \Pi_{\mathrm{KKS}}(z_j, w_k) \right. \\ &\qquad \left. * (z_{j+1} \cdots z_l) \right) w_{k+1} \cdots w_m .\end{aligned}$$

Since $\Pi_{\mathrm{KKS}}(z_j, w_k) = \delta_{z_j, w_k}(1 \otimes z_j - z_j \otimes 1)$, we obtain

$$\begin{aligned}\Pi_{\mathrm{KKS}}(z,w) &= \textstyle\sum_{j,k} \delta_{z_j,w_k} \left(w_1 \cdots w_{k-1} z_{j+1} \cdots z_l \otimes z_1 \cdots z_j w_{k+1} \cdots w_m \right. \\ &\qquad \left. - w_1 \cdots w_{k-1} z_j \cdots z_l \otimes z_1 \cdots z_{j-1} w_{k+1} \cdots w_m \right) .\end{aligned}$$

Remark 4.39. The KKS double bracket is an algebraic counterpart of the version of the homotopy intersection form κ introduced in [21], [26]. To align the notation, we sometimes use the symbol $\kappa^{\mathrm{alg}} : A \otimes A \to A \otimes A$ for the KKS double bracket. Double brackets can be naturally associated to quivers [35]. In this formalism, the KKS double bracket is associated to the star-shaped quiver with one internal vertex and n external vertices.

Remark 4.40. The double bracket Π_{KKS} verifies the van den Bergh version of the Jacobi identity [35]. In particular, this implies that

$$(|a|, |b|) \mapsto \{|a|, |b|\}_{\mathrm{KKS}}$$

is a Lie bracket on $|A|$. In the framework of double brackets associated to quivers, this Lie algebra is known as the necklace Lie algebra [9], [14]. See also [28] for a related structure in higher genus.

Theorem 4.41. *The center of the Lie algebra* $(|A|, \{-,-\}_{\mathrm{KKS}})$ *is spanned by* $|\mathbb{K}[[x_i]]|$, $0 \leq i \leq n$,

$$Z(|A|, \{-,-\}_{\mathrm{KKS}}) = \sum_{i=0}^n |\mathbb{K}[[x_i]]|.$$

Proof. A proof using Poisson geometry of representation varieties can be obtained with the techniques of [12]. In Appendix A, we give a direct algebraic proof of this statement. □

Next, we compute the maps tDiv_Π and tDiv_Π^T for the KKS double bracket. We will use the following shorthand notation: $\mathrm{tDiv}_{\mathrm{KKS}}$ and $\mathrm{tDiv}_{\mathrm{KKS}}^T$. A similar style notation will be later used for other double brackets.

Proposition 4.42. *For $z = z_1 z_2 \cdots z_m$ with $z_j \in \{x_i\}_{i=1}^n$, we have*

$$\begin{aligned}\mathrm{tDiv}_{\mathrm{KKS}}(|z|) = -\sum_{j<k} \delta_{z_j,z_k} \big(&|z_j \cdots z_{k-1}| \wedge |z_{k+1} \cdots z_m z_1 \cdots z_{j-1}| \\ &+|z_k \cdots z_m z_1 \cdots z_{j-1}| \wedge |z_{j+1} \cdots z_{k-1}|\big)\,,\end{aligned}$$

$$\begin{aligned}\mathrm{tDiv}^T_{\mathrm{KKS}}(z) = \sum_{j<k} \delta_{z_j,z_k} \big(&|z_{j+1} \cdots z_{k-1}| \otimes z_1 \cdots z_j z_{k+1} \cdots z_m \\ &-|z_j \cdots z_{k-1}| \otimes z_1 \cdots z_{j-1} z_{k+1} \cdots z_m\big)\,.\end{aligned}$$

Proof. Since $\{z,-\}_{\mathrm{KKS}} = \sum_j z_1 \cdots z_{j-1}(\mathrm{ad}_{z_j}\partial_{z_j}) z_{j+1} \cdots z_m$, the ith component of $\{|z|,-\}_{\mathrm{KKS}} \in \mathrm{tDer}(A)$ is $-\sum_j \delta_{x_i,z_j} z_{j+1} \cdots z_m z_1 \cdots z_{j-1}$. Therefore,

$$\begin{aligned}\mathrm{tDiv}_{\mathrm{KKS}}(|z|) &= -\sum_j |\widetilde{\Delta}(z_j)\partial_{z_j}(z_{j+1} \cdots z_m z_1 \cdots z_{j-1})| \\ &= -\sum_{j<k} \delta_{z_j,z_k} \big(|z_j \cdots z_{k-1}| \otimes |z_{k+1} \cdots z_m z_1 \cdots z_{j-1}| \\ &\qquad -|z_{j+1} \cdots z_{k-1}| \otimes |z_{k+1} \cdots z_m z_1 \cdots z_j|\big) \\ &\quad - \sum_{k<j} \delta_{z_j,z_k} \big(|z_j \cdots z_m z_1 \cdots z_{k-1}| \otimes |z_{k+1} \cdots z_{j-1}| \\ &\qquad -|z_{j+1} \cdots z_m z_1 \cdots z_{k-1}| \otimes |z_{k+1} \cdots z_j|\big) \\ &= -\sum_{j<k} \delta_{z_j,z_k} \big(|z_j \cdots z_{k-1}| \wedge |z_{k+1} \cdots z_m z_1 \cdots z_{j-1}| \\ &\qquad +|z_k \cdots z_m z_1 \cdots z_{j-1}| \wedge |z_{j+1} \cdots z_{k-1}|\big)\,.\end{aligned}$$

For $\mathrm{tDiv}^T_{\mathrm{KKS}}$, using (4.11) we compute

$$\begin{aligned}\mathrm{tDiv}^T_{\mathrm{KKS}}(z) &= (|\ | \otimes 1)\left(\sum_j (1 \otimes z_1 \cdots z_j - z_j \otimes z_1 \cdots z_{j-1})\partial_{z_j}(z_{j+1} \cdots z_m)\right) \\ &= \sum_{j<k} \delta_{z_j,z_k} \big(|z_{j+1} \cdots z_{k-1}| \otimes z_1 \cdots z_j z_{k+1} \cdots z_m \\ &\qquad -|z_j \cdots z_{k-1}| \otimes z_1 \cdots z_{j-1} z_{k+1} \cdots z_m\big)\,.\end{aligned}$$

□

Remark 4.43. The operations $\mathrm{tDiv}_{\mathrm{KKS}}$ and $\mathrm{tDiv}^T_{\mathrm{KKS}}$ are algebraic counterparts of the Turaev cobracket δ and of the operation μ (a self-intersection map introduced in [21]). To align the notation, we sometimes use the following symbols: $\delta^{\mathrm{alg}} = -\mathrm{tDiv}_{\mathrm{KKS}}, \mu^{\mathrm{alg}} = \mathrm{tDiv}^T_{\mathrm{KKS}}$.

Remark 4.44. The vector space $|A|$ endowed with the Lie bracket induced by Π_{KKS} and with the operation $\delta^{\mathrm{alg}} : |A| \to \wedge^2 |A|$ is in fact a Lie bialgebra. This Lie bialgebra was introduced and studied in [31] (see also [28] for a similar structure in higher genus). It can be viewed as an algebraic counterpart of the Goldman-Turaev Lie bialgebra for a surface of genus zero defined in the next section.

4.4 Double brackets Π_s

Another family of examples are the double brackets of the form

$$\Pi_s = |\phi_0 \otimes s(-\mathrm{ad}_{x_0})\phi_0| = |s(\mathrm{ad}_{x_0})\phi_0 \otimes \phi_0|,$$

for $s = s(z) \in \mathbb{K}[[z]]$ a formal power series. In particular, for $s = 1$ we obtain $\Pi_1 = |\phi_0 \otimes \phi_0|$. For the later use, we will need the following Sweedler style notation:

$$\widetilde{\Delta}(s(-x_0)) = (1 \otimes \iota)\Delta(s(-x_0)) = s' \otimes s'' \in A \otimes A.$$

Proposition 4.45. *For any s, the double bracket Π_s is tangential. For any $a, b \in A$, we have*

$$\Pi_s(a, b) = s''a \otimes s'b - bs''a \otimes s' - s'' \otimes as'b + bs'' \otimes as'.$$

The tangential derivation $\{|a|, -\}_s \in \mathrm{tDer}(A)$ vanishes for all $a \in A$. All the bracket operations $\{-, -\}_s : |A| \otimes A \to A$, $|A| \otimes |A| \to |A|$ and $A \otimes |A| \to A$ are identically zero.

Proof. Let $a \in A$. By Lemma 4.10 and (4.26), we have

$$\Pi_s(a, -) = (s(-\mathrm{ad}_{x_0})\phi_0)a - a(s(-\mathrm{ad}_{x_0})\phi_0) = s'\phi_0 s''a - as'\phi_0 s''. \qquad (4.31)$$

On the one hand, this shows that Π_s is tangential. On the other hand, for $b \in A$ we have

$$\Pi_s(a, b) = s' * (1 \otimes b - b \otimes 1) * (s''a) - (as') * (1 \otimes b - b \otimes 1) * s''.$$

Expanding the right hand side of this equation, we obtain the second statement in the proposition.

The equation (4.31) shows that the tangential derivation $\{|a|, -\}_s \in \mathrm{tDer}(A)$ is an inner derivation with generator $s''as' - s''as' = 0$. The last statement on the bracket follows from the second statement. □

Proposition 4.46. *For any $a \in A$, $\mathrm{tDiv}_s(|a|) = 0$ and*

$$\mathrm{tDiv}_s^T(a) = |s''| \otimes as' - |s''a| \otimes s'.$$

Proof. By Proposition 4.45, the tangential derivation $\{|a|, -\}_s \in \mathrm{tDer}(A)$ vanishes. Hence, $\mathrm{tDiv}_s(|a|) = 0$ for all $a \in A$. By Lemma 4.10 and Proposition 4.24,

$$\begin{aligned}
\mathrm{tDiv}_s^T(a) =& \mathrm{tDiv}^T(s'\phi_0 s'' a - a s'\phi_0 s'') \\
=& \mathrm{tDiv}^T(s'\phi_0)(1 \otimes s''a) + (|\ | \otimes 1)(s'\phi_0)(s''a) \\
& - \mathrm{tDiv}^T(as'\phi_0)(1 \otimes s'') - (|\ | \otimes 1)(as'\phi_0)(s'') \\
=& (|\ | \otimes 1)\left(s' * (1 \otimes s''a - s''a \otimes 1) - (as') * (1 \otimes s'' - s'' \otimes 1)\right) \\
=& (|\ | \otimes 1)(1 \otimes s's''a - s''a \otimes s' - 1 \otimes as's'' + s'' \otimes as') \\
=& |s''| \otimes as' - |s''a| \otimes s'.
\end{aligned}$$

Here we have used the facts that $\mathrm{tDiv}^T(\phi_0) = 0$ (this can be seen directly from (4.11)), and that $s's'' = \varepsilon(s(-x_0)) \in \mathbb{K}$. □

5 The Goldman-Turaev Lie bialgebra

In this section, we recall the notions of Goldman bracket, Turaev cobracket and their upgrades to the group algebra of the fundamental group. All these operations are defined in terms of intersections and self-intersections of curves on a surface Σ.

Let Σ be a connected oriented smooth 2-dimensional manifold with non-empty boundary $\partial\Sigma$. Fix a point $* \in \partial\Sigma$, and let $\pi := \pi_1(\Sigma, *)$ be the fundamental group of Σ with base point $*$. The group ring $\mathbb{K}\pi$ has a Hopf algebra structure whose coproduct Δ, augmentation ε, and antipode ι are defined by the following formulas: for $\alpha \in \pi$,

$$\Delta(\alpha) := \alpha \otimes \alpha, \quad \varepsilon(\alpha) := 1, \quad \iota(\alpha) := \alpha^{-1}.$$

By a *curve*, we mean a path or a loop on Σ. In what follows, we consider several operations on (homotopy classes of) curves on Σ. Let I be the unit interval $[0, 1]$ or the circle S^1, and let α be a smoothly immersed curve, i.e., a C^∞-immersion $\alpha : I \to \Sigma$. For distinct points $s, t \in I$, where we further assume that $s < t$ if $I = [0, 1]$, we define the curve α_{st} to be the restriction of α to the interval $[s, t]$. For $t \in I$, let $\dot\alpha(t) \in T_{\alpha(t)}\Sigma$ denote the velocity vector of α at t. If $p \in \alpha(I)$ is a simple point of α and $t = \alpha^{-1}(p)$, we also write $\dot\alpha_p$ for $\dot\alpha(t)$. For paths $\alpha, \beta : [0, 1] \to \Sigma$, their concatenation $\alpha\beta$ is defined if and only if $\alpha(1) = \beta(0)$, and is the path first traversing α, then β. If there is no fear of confusion, we use the same letter for a curve and its (regular) homotopy class.

For a path $\gamma : [0, 1] \to \Sigma$, its inverse $\overline{\gamma} : [0, 1] \to \Sigma$ is defined by $\overline{\gamma}(t) := \gamma(1 - t)$.

Let $p \in \Sigma$ and let $\vec{a}, \vec{b} \in T_p\Sigma$ be linearly independent tangent vectors. The *local intersection number* $\varepsilon(\vec{a}, \vec{b}) \in \{\pm 1\}$ of $\vec{a}$ and $\vec{b}$ is defined as follows. If the ordered pair $(\vec{a}, \vec{b})$ is a positively oriented frame of Σ, then $\varepsilon(\vec{a}, \vec{b}) := +1$; otherwise, $\varepsilon(\vec{a}, \vec{b}) := -1$.

Take an orientation preserving embedding $\nu : [0,1] \to \partial\Sigma$ with $\nu(1) = *$, and set $\bullet := \nu(0)$. Using ν, we obtain the following isomorphism of groups:

$$\pi \xrightarrow{\cong} \pi_1(\Sigma, \bullet), \quad \alpha \mapsto \nu\alpha\overline{\nu}. \tag{4.32}$$

5.1 The Goldman bracket and the double bracket κ

We recall an operation which measures the intersection of two based loops on Σ (see [21] §4.2 and [26] §7). Let α and β be loops on Σ based at $\bullet$ and $*$, respectively. We assume that α and β are C^∞-immersions with their intersections consisting of transverse double points. Then set

$$\kappa(\alpha, \beta) := -\sum_{p \in \alpha\cap\beta} \varepsilon(\dot{\alpha}_p, \dot{\beta}_p)\, \beta_{*p}\alpha_{p\bullet}\nu \otimes \overline{\nu}\alpha_{\bullet p}\beta_{p*} \in \mathbb{K}\pi \otimes \mathbb{K}\pi. \tag{4.33}$$

The result only depends on the homotopy classes of α and β, and using the isomorphism (4.32), we obtain a $\mathbb{K}$-linear map $\kappa : \mathbb{K}\pi \otimes \mathbb{K}\pi \to \mathbb{K}\pi \otimes \mathbb{K}\pi$.

Proposition 4.47. *The map $\kappa : \mathbb{K}\pi \otimes \mathbb{K}\pi \to \mathbb{K}\pi \otimes \mathbb{K}\pi$ is a double bracket on $\mathbb{K}\pi$ in the sense of Remark 4.35. For any $x, y \in \mathbb{K}\pi$, we have*

$$\kappa(y, x) = -\kappa(x, y)^\circ + xy \otimes 1 + 1 \otimes yx - x \otimes y - y \otimes x, \tag{4.34}$$
$$\kappa(\iota(x), \iota(y)) = (\iota \otimes \iota)\kappa(x, y)^\circ. \tag{4.35}$$

Here, $(u \otimes v)^\circ = v \otimes u$ for $u, v \in \mathbb{K}\pi$.

Proof. For the proof of the first statement, see [21] Lemma 4.3.1. (Note that we use a different convention here. See also Remark 4.53 below.) The formula (4.34) can be found in the proof of Lemma 7.2 in [26]. The formula (4.35) can be seen directly from the defining formula (4.33). □

Remark 4.48. Essentially the same operations as the map κ were independently introduced by Papakyriakopoulos [30] and Turaev [33]. Yet another version is the *homotopy intersection form* $\eta : \mathbb{K}\pi \otimes \mathbb{K}\pi \to \mathbb{K}\pi$ introduced by Massuyeau and Turaev [25].

Let $\hat{\pi} := [S^1, \Sigma]$ be the set of homotopy classes of free loops on Σ. For any $p \in \Sigma$ (in particular, for $p = *$), let $|\ | : \pi_1(\Sigma, p) \to \hat{\pi}$ be the map obtained by forgetting the base point of a based loop. Notice that the map $|\ | : \pi \to \hat{\pi}$ induces a $\mathbb{K}$-linear isomorphism

$$\mathbb{K}\hat{\pi} \cong |\mathbb{K}\pi|,$$

where $|\mathbb{K}\pi| = \mathbb{K}\pi/[\mathbb{K}\pi, \mathbb{K}\pi]$. In this paper, we mainly use the notation $|\mathbb{K}\pi|$.

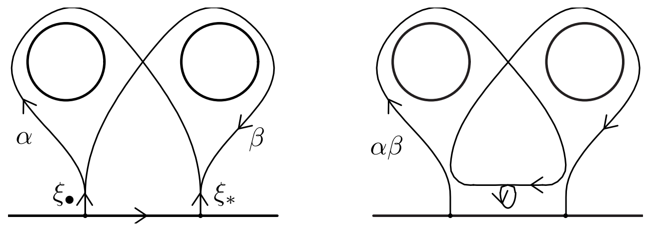

Figure 4.1: the group structure on $\pi^+ = \pi^+_{\bullet *}$

By equation (4.23), the double bracket κ induces a bracket $\{-,-\} = \{-,-\}_\kappa$ on $\mathbb{K}\hat{\pi} \cong |\mathbb{K}\pi|$. To be more explicit, suppose that α and β are smoothly immersed loops on Σ, such that their intersections consist of transverse double points. For each intersection $p \in \alpha \cap \beta$, let α_p and β_p be the loops α and β *based at* p, and let $\alpha_p\beta_p \in \pi_1(\Sigma, p)$ be their concatenation. Then from the formula (4.33), we obtain

$$\{\alpha, \beta\} = -\sum_{p \in \alpha\cap\beta} \varepsilon(\dot{\alpha}_p, \dot{\beta}_p)|\alpha_p\beta_p|.$$

Note that $\{-,-\}$ is *minus* the *Goldman bracket* defined in [15].

5.2 Turaev cobracket and the self-intersection map μ

Choose inward non-zero tangent vectors $\xi_\bullet \in T_\bullet\Sigma$ and $\xi_* \in T_*\Sigma$, respectively. Let $\pi^+ = \pi^+_{\bullet *}$ be the set of regular homotopy classes relative to the boundary $\{0,1\}$ of C^∞-immersions $\gamma : ([0,1], 0, 1) \to (\Sigma, \bullet, *)$ such that $\dot{\gamma}(0) = \xi_\bullet$ and $\dot{\gamma}(1) = -\xi_*$. We define a group structure on π^+ as follows: for $\alpha, \beta \in \pi^+$, their product $\alpha\beta$ is the insertion of a *positive* monogon to (a suitable smoothing of) $\alpha\overline{\nu}\beta$ (see Figure 4.1). The map $\pi^+ \to \pi, \gamma \mapsto \overline{\nu}\gamma$ is a surjective group homomorphism, and its kernel is an infinite cyclic group.

Let $\gamma : ([0,1], 0, 1) \to (\Sigma, \bullet, *)$ be a C^∞-immersion such that $\dot{\gamma}(0) = \xi_\bullet$ and $\dot{\gamma}(1) = -\xi_*$. We assume that the self-intersections of γ consist of transverse double points. For each self-intersection p of γ, let $\gamma^{-1}(p) = \{t^p_1, t^p_2\}$ with $t^p_1 < t^p_2$. Then we define $\mu(\gamma) = \mu_{\bullet *}(\gamma) \in |\mathbb{K}\pi| \otimes \mathbb{K}\pi$ by

$$\mu(\gamma) := -\sum_p \varepsilon(\dot{\gamma}(t^p_1), \dot{\gamma}(t^p_2))\, |\gamma_{t^p_1 t^p_2}| \otimes \overline{\nu}\gamma_{0 t^p_1}\gamma_{t^p_2 1}, \tag{4.36}$$

where the sum runs over all the self-intersections of γ. Using the same argument as in [21] Proposition 3.2.3, we can show that the result depends only on the regular homotopy class of γ. Since in this statement we deal with *regular homotopy* classes,

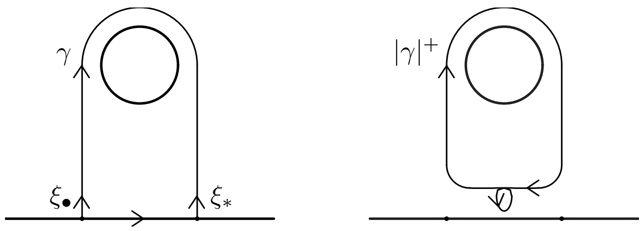

Figure 4.2: the closing operation $|\ |^+$

we only have to check the invariance under the second and third Reidemeister moves $(\omega 2)$ and $(\omega 3)$ (see [15] §5). In this way, we obtain a $\mathbb{K}$-linear map $\mu = \mu_{\bullet *} : \mathbb{K}\pi^+ \to |\mathbb{K}\pi| \otimes \mathbb{K}\pi$.

Proposition 4.49. *For any $\alpha, \beta \in \pi^+$,*

$$\mu(\alpha\beta) = \mu(\alpha)(1 \otimes \overline{\nu}\beta) + (1 \otimes \overline{\nu}\alpha)\mu(\beta) + (|\ | \otimes 1)\kappa(\overline{\nu}\alpha, \overline{\nu}\beta).$$

Proof. The proof is essentially the same as the proof of Lemma 4.3.3 in [21], so we omit the detail. □

Remark 4.50. In [33], Turaev introduced essentially the same operation as μ.

We denote by $\hat{\pi}^+$ the set of regular homotopy classes of immersed free loops on Σ. Let $\alpha : S^1 \to \Sigma$ be a C^∞-immersion whose self-intersections consist of transverse double points. Setting $D_\alpha := \{(t_1, t_2) \in S^1 \times S^1 \mid t_1 \neq t_2, \alpha(t_1) = \alpha(t_2)\}$, we define

$$\delta^+(\alpha) := \sum_{(t_1,t_2)\in D_\alpha} \varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2))\, |\alpha_{t_1 t_2}| \otimes |\alpha_{t_2 t_1}| \in |\mathbb{K}\pi| \otimes |\mathbb{K}\pi|. \tag{4.37}$$

The right hand side only depends on the regular homotopy class of γ, and we obtain a $\mathbb{K}$-linear map $\delta^+ : \mathbb{K}\hat{\pi}^+ \to |\mathbb{K}\pi| \otimes |\mathbb{K}\pi|$.

Remark 4.51. To show that $\delta^+(\alpha)$ is well-defined, we can use a similar argument to [34] §8. A regular homotopy version of the cobracket δ was first introduced by Turaev in [34] §18. Another version which is closer to our definition was introduced in [18].

Consider the closing operation $|\ |^+ : \pi^+ \to \hat{\pi}^+$, which maps γ to an immersed loop obtained by inserting a positive monogon to $\overline{\nu}\gamma$ (see Figure 4.2). Let $\mathbf{1} \in \hat{\pi}$ denote the class of a constant loop. Then for any $\gamma \in \pi^+$,

$$\delta^+(|\gamma|^+) = -\mathrm{Alt}(1 \otimes |\ |)\mu(\gamma) + |\gamma| \wedge \mathbf{1}. \tag{4.38}$$

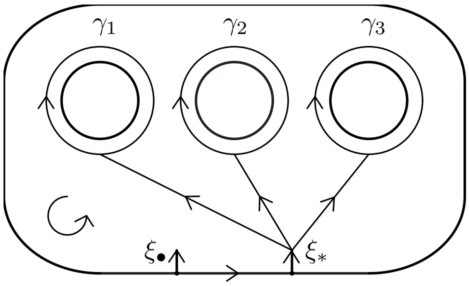

Figure 4.3: a surface of genus 0, embedded in $\mathbb{R}^2$ ($n = 3$)

Here, $\mathrm{Alt} : |\mathbb{K}\pi|\otimes|\mathbb{K}\pi| \to |\mathbb{K}\pi|\otimes|\mathbb{K}\pi|$ is the map sending $\alpha\otimes\beta$ to $\alpha\wedge\beta = \alpha\otimes\beta-\beta\otimes\alpha$.

The original *Turaev cobracket* [34] is a map $\delta : |\mathbb{K}\pi|/\mathbb{K}\mathbf{1} \to (|\mathbb{K}\pi|/\mathbb{K}\mathbf{1}) \otimes (|\mathbb{K}\pi|/\mathbb{K}\mathbf{1})$. It is defined using the formula (4.37), where the loop α represents an element of $\hat{\pi}$ (not of $\hat{\pi}^+$) and the right hand side is regarded as an element of $(|\mathbb{K}\pi|/\mathbb{K}\mathbf{1}) \otimes (|\mathbb{K}\pi|/\mathbb{K}\mathbf{1})$.

Remark 4.52. The Goldman bracket $[-,-]_{\mathrm{G}} = -\{-,-\}$ is a Lie bracket on $\mathbb{K}\hat{\pi}$, and $\mathbb{K}\mathbf{1}$ is a Lie ideal [15]. The map δ is a Lie cobracket on the quotient Lie algebra $\mathbb{K}\hat{\pi}/\mathbb{K}\mathbf{1}$, and the triple $(\mathbb{K}\hat{\pi}/\mathbb{K}\mathbf{1}, [-,-]_{\mathrm{G}}, \delta)$ is a *Lie bialgebra* [34]. As was shown by Chas [10], it is *involutive* in the sense that $[-,-]_{\mathrm{G}} \circ \delta = 0$.

Remark 4.53. The operations μ and κ in this paper are different from the original versions in [21]. One can convert one version into the other by switching the first and second components of the target.

5.3 The case of genus zero

In this subsection, let Σ be a surface of genus 0 with $n+1$ boundary components. We label the boundary components of Σ as $\partial_0\Sigma, \partial_1\Sigma, \dots, \partial_n\Sigma$, so that $* \in \partial_0\Sigma$. Take a free generating system $\gamma_1, \dots, \gamma_n$ for π, such that each γ_i is freely homotopic to the positively oriented boundary component $\partial_i\Sigma$ and the product $\gamma_1 \cdots \gamma_n$ is the negatively oriented boundary component $\partial_0\Sigma$ (see Figure 4.3).

Figure 4.4 shows the following:

Proposition 4.54.

$$\begin{aligned}
&\kappa(\gamma_i, \gamma_i) = 1 \otimes \gamma_i^2 - \gamma_i \otimes \gamma_i \quad \textit{for any } i,\\
&\kappa(\gamma_i, \gamma_j) = 0 \quad \textit{if } i < j,\\
&\kappa(\gamma_i, \gamma_j) = 1 \otimes \gamma_i\gamma_j + \gamma_j\gamma_i \otimes 1 - \gamma_i \otimes \gamma_j - \gamma_j \otimes \gamma_i \quad \textit{if } i > j.
\end{aligned}$$

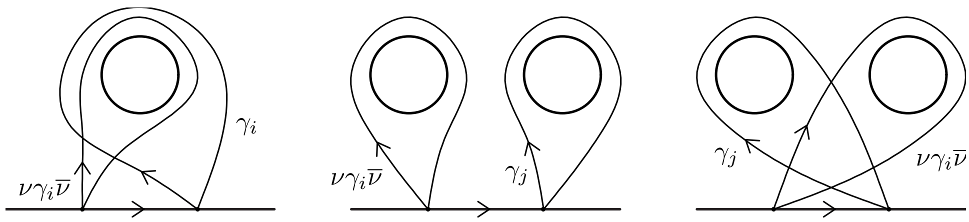

Figure 4.4: the proof of Proposition 4.54

Choose an orientation preserving embedding $\Sigma \to \mathbb{R}^2$ such that $\partial_0\Sigma$ is the outermost boundary component of Σ, as in Figure 4.3. Note that such an embedding is unique up to isotopy. The standard framing of $\mathbb{R}^2$ induces a framing of Σ. We may assume that the velocity vector of ν does not rotate with respect to this framing. Then we can define the rotation number function

$$\mathrm{rot} = \mathrm{rot}_{\bullet *} : \pi^+ \to \frac{1}{2} + \mathbb{Z}.$$

It satisfies the product formula $\mathrm{rot}(\alpha\beta) = \mathrm{rot}(\alpha) + \mathrm{rot}(\beta) + (1/2)$ for any $\alpha, \beta \in \pi^+$.

The surjective homomorphism $\pi^+ \to \pi, \gamma \mapsto \overline{\nu}\gamma$ has a section $i = i_{\bullet *} : \pi \to \pi^+$ defined by the condition that $\mathrm{rot}(i(\gamma)) = -1/2$ for any $\gamma \in \pi$. Composing $\mu : \mathbb{K}\pi^+ \to |\mathbb{K}\pi| \otimes \mathbb{K}\pi$ with the section i, we obtain a $\mathbb{K}$-linear map $\mu = \mu_{\bullet *} : \mathbb{K}\pi \to |\mathbb{K}\pi| \otimes \mathbb{K}\pi$ (again denoted by μ).

Proposition 4.55. *The map $\mu : \mathbb{K}\pi \to |\mathbb{K}\pi| \otimes \mathbb{K}\pi$ satisfies the product formula*

$$\mu(xy) = \mu(x)(1 \otimes y) + (1 \otimes x)\mu(y) + (|\ | \otimes 1)\kappa(x, y)$$

for any $x, y \in \mathbb{K}\pi$, and we have $\mu(\gamma_i) = 0$ for any i. Moreover, these two properites characterise the map μ.

Proof. The product formula follows from Proposition 4.49. Notice that a suitable smoothing of $\nu\gamma_i$ has no self-intersections and that its rotation number is $-1/2$. Thus $i(\gamma_i) = \nu\gamma_i$ and $\mu(\gamma_i) = 0$. The last statement follows from the fact that π is generated by $\{\gamma_i\}_i$. □

Proposition 4.56. *The composition map*

$$\pi \xrightarrow{i} \pi^+ \xrightarrow{|\ |^+} \hat{\pi}^+ \xrightarrow{\delta^+} |\mathbb{K}\pi| \otimes |\mathbb{K}\pi|.$$

descends to a map $\delta^+ : \hat{\pi} \to |\mathbb{K}\pi| \otimes |\mathbb{K}\pi|$. For any $\gamma \in \pi$, we have

$$\delta^+(|\gamma|) = -\mathrm{Alt}(1 \otimes |\ |)\mu(\gamma) + |\gamma| \wedge \mathbf{1}. \tag{4.39}$$

Its $\mathbb{K}$*-linear extension* $\delta^+ : |\mathbb{K}\pi| \cong \mathbb{K}\hat{\pi} \to |\mathbb{K}\pi| \otimes |\mathbb{K}\pi|$ *is a lift of the Turaev cobracket in the sense that* $\varpi^{\otimes 2} \circ \delta^+ = \delta \circ \varpi$*, where* $\varpi : |\mathbb{K}\pi| \to |\mathbb{K}\pi|/\mathbb{K}\mathbf{1}$ *is the natural projection.*

Proof. To prove the first statement, we need to show $\delta^+(|i(\alpha\beta)|^+) = \delta^+(|i(\beta\alpha)|^+)$ for any $\alpha, \beta \in \pi$. We write $\mu(\alpha) = |\alpha'| \otimes \alpha''$ and $\mu(\beta) = |\beta'| \otimes \beta''$. Then, by (4.38) and Proposition 4.55,

$$\begin{aligned}
\delta^+(|i(\alpha\beta)|^+) =& - \operatorname{Alt}(1 \otimes |\ |)\mu(i(\alpha\beta)) + |i(\alpha\beta)| \wedge \mathbf{1} \\
=& - \operatorname{Alt}(1 \otimes |\ |)(\mu(\alpha)(1 \otimes \beta) + (1 \otimes \alpha)\mu(\beta) \\
&\quad + (|\ | \otimes 1)\kappa(\alpha, \beta)) + |\alpha\beta| \wedge \mathbf{1} \\
=& - \operatorname{Alt}(|\alpha'| \otimes |\alpha''\beta| + |\beta'| \otimes |\alpha\beta''| + (|\ | \otimes |\ |)\kappa(\alpha, \beta)) + |\alpha\beta| \wedge \mathbf{1} \\
=& |\alpha''\beta| \wedge |\alpha'| + |\alpha\beta''| \wedge |\beta'| + |\alpha\beta| \wedge \mathbf{1} - \operatorname{Alt}(|\ | \otimes |\ |)\kappa(\alpha, \beta).
\end{aligned}$$

Also, by (4.34),

$$\operatorname{Alt}(|\ | \otimes |\ |)\kappa(\beta, \alpha) = -\operatorname{Alt}(|\ | \otimes |\ |)(\kappa(\alpha, \beta)^\circ) = \operatorname{Alt}(|\ | \otimes |\ |)\kappa(\alpha, \beta).$$

These two computations prove $\delta^+(|i(\alpha\beta)|^+) = \delta^+(|i(\beta\alpha)|^+)$, as required. The formula (4.39) follows from (4.38). Finally, by (4.39), $\varpi^{\otimes 2}\delta^+(|\gamma|) = -\operatorname{Alt}(1 \otimes |\ |)\mu(\gamma) = \delta\varpi(|\gamma|)$, where the last equality follows from (4.36) and (4.37) (the latter being interpreted as the defining formula for δ). □

Remark 4.57. In fact, the $\mathbb{K}$-vector space $|\mathbb{K}\pi|$ is a Lie bialgebra with respect to the Goldman bracket and the map δ^+.

By interchanging the roles of $*$ and $\bullet$, we obtain an operation $\mu_{*\bullet} : \mathbb{K}\pi \to \mathbb{K}\pi \otimes |\mathbb{K}\pi|$ which will be used in the next section. This map satisfies the product formula

$$\mu_{*\bullet}(xy) = \mu_{*\bullet}(x)(y \otimes 1) + (x \otimes 1)\mu_{*\bullet}(y) + (1 \otimes |\ |)\kappa(y, x) \tag{4.40}$$

for any $x, y \in \mathbb{K}\pi$, and we have $\mu_{*\bullet}(\gamma_i^{-1}) = 0$ for all i. Moreover, these two properties characterise $\mu_{*\bullet}$ (c.f. Proposition 4.55).

One can show that $(|\ | \otimes 1)(\mu_{\bullet *}(\gamma)^\circ) + |\gamma| \wedge \mathbf{1} = -(|\ | \otimes 1)\mu_{*\bullet}(\gamma)$ for any $\gamma \in \pi$. Therefore, (4.39) is rephrased as

$$\delta^+(|\gamma|) = -(1 \otimes |\ |)\mu_{\bullet *}(\gamma) - (|\ | \otimes 1)\mu_{*\bullet}(\gamma). \tag{4.41}$$

5.4 Completions

Let $I\pi := \ker(\varepsilon)$ be the augmentation ideal of $\mathbb{K}\pi$. The powers $(I\pi)^p$, $p \geq 0$, give a descending filtration of two-sideded ideals of $\mathbb{K}\pi$. The projective limit $\widehat{\mathbb{K}\pi} :=$

$\varprojlim_{p\to\infty} \mathbb{K}\pi/(I\pi)^p$ naturally has the structure of a complete Hopf algebra. Also, put $|\widehat{\mathbb{K}\pi}| := \varprojlim_{p\to\infty} |\mathbb{K}\pi|/|(I\pi)^p|$. Since π is a free group of finite rank, the natural map $\mathbb{K}\pi \to \widehat{\mathbb{K}\pi}$ is injective and the image is dense with respect to the filtration $\widehat{I\pi}^p := \ker(\widehat{\mathbb{K}\pi} \to \mathbb{K}\pi/(I\pi)^p)$, $p \geq 0$. The natural map $|\mathbb{K}\pi| \to |\widehat{\mathbb{K}\pi}|$ has a similar property.

As shown in [21] §4 and [22] §4, the operations κ, the Goldman bracket, μ, and δ^+ on $\mathbb{K}\pi$ and $\mathbb{K}\hat{\pi}$ have natural extensions to the completions $\widehat{\mathbb{K}\pi}$ and $|\widehat{\mathbb{K}\pi}|$. Therefore, we have continuous maps $\kappa : \widehat{\mathbb{K}\pi}\hat{\otimes}\widehat{\mathbb{K}\pi} \to \widehat{\mathbb{K}\pi}\hat{\otimes}\widehat{\mathbb{K}\pi}$, $\{-,-\} : |\widehat{\mathbb{K}\pi}|\hat{\otimes}|\widehat{\mathbb{K}\pi}| \to |\widehat{\mathbb{K}\pi}|\hat{\otimes}|\widehat{\mathbb{K}\pi}|$, $\delta^+ : |\widehat{\mathbb{K}\pi}| \to |\widehat{\mathbb{K}\pi}|\hat{\otimes}|\widehat{\mathbb{K}\pi}|$, and $\mu : \widehat{\mathbb{K}\pi} \to |\widehat{\mathbb{K}\pi}|\hat{\otimes}\widehat{\mathbb{K}\pi}$.

6 Expansions and transfer of structures

In this section, we introduce a notion of expansions which allow to transfer the operations (*e.g.* the double bracket and cobracket) from the group algebra $\mathbb{K}\pi$ to the free associative algebra A.

Keep the notation in §5.3 and identify the homology class of γ_i with the ith generator x_i of $A = A_n$. Then $H_1(\Sigma;\mathbb{K})$ is isomorphic to the degree 1 part of A. Recall the notation $x_0 = -\sum_{i=1}^n x_i \in H_1(\Sigma;\mathbb{K}) \subset A$.

Definition 4.58. A *group-like* expansion of π is a map $\theta : \pi \to A$ such that

(i) $\theta(\alpha\beta) = \theta(\alpha)\theta(\beta)$ for any $\alpha, \beta \in \pi$,

(ii) for any $\alpha \in \pi$, $\theta(\alpha) = 1 + [\alpha] + (\text{higher terms})$, where $[\alpha] \in H_1(\Sigma;\mathbb{K})$ is the homology class of α,

(iii) for any $\alpha \in \pi$, $\theta(\alpha) \in \exp(L)$.

A group-like expansion θ is called *tangential* if

(iv) for any $1 \leq i \leq n$, there exists an element $g_i \in L$ such that $\theta(\gamma_i) = e^{g_i}e^{x_i}e^{-g_i}$.

A tangential expansion is called *special* if

(v) $\theta(\gamma_1 \cdots \gamma_n) = \exp(-x_0)$.

Example 4.59. There is a distinguished tangential expansion $\theta^{\exp}$ defined on generators $\theta^{\exp}(\gamma_i) = e^{x_i}$. This expansion is not special since $\theta^{\exp}(\gamma_1 \cdots \gamma_n) = e^{x_1} \cdots e^{x_n} \neq e^{-x_0}$.

Remark 4.60. The notion of a group-like expansion and that of a special expansion were introduced by Massuyeau in [23] and [24], respectively. The first example of special expansions was given by [16] by using the Kontsevich integral. Special expansions appear implicitly in [1] §4.1.

Any group-like expansion θ induces an injective map $\theta : \mathbb{K}\pi \to A$ of Hopf algebras and an isomorphism $\theta : \widehat{\mathbb{K}\pi} \xrightarrow{\cong} A$ of complete Hopf algebras ([23] Proposition 2.10). It also induces an isomorphism $\theta : |\widehat{\mathbb{K}\pi}| \xrightarrow{\cong} |A|$ of filtered $\mathbb{K}$-vector spaces ([22] Proposition 7.1). By the discussion in §5.4, there exist unique continuous maps $\kappa_\theta : A \otimes A \to A \otimes A, \mu_\theta : A \to |A| \otimes A$ and $\delta_\theta^+ : |A| \to |A| \otimes |A|$ such that the following diagrams are commutative:

$$\begin{array}{ccc} \mathbb{K}\pi \otimes \mathbb{K}\pi & \xrightarrow{\kappa} & \mathbb{K}\pi \otimes \mathbb{K}\pi \\ {\scriptstyle \theta\otimes\theta}\downarrow & & \downarrow{\scriptstyle \theta\otimes\theta} \\ A \otimes A & \xrightarrow{\kappa_\theta} & A \otimes A, \end{array}$$

$$\begin{array}{ccc} \mathbb{K}\pi & \xrightarrow{\mu} & |\mathbb{K}\pi| \otimes \mathbb{K}\pi \\ {\scriptstyle \theta}\downarrow & & \downarrow{\scriptstyle \theta\otimes\theta} \\ A & \xrightarrow{\mu_\theta} & |A| \otimes A, \end{array} \qquad \begin{array}{ccc} |\mathbb{K}\pi| & \xrightarrow{\delta^+} & |\mathbb{K}\pi| \otimes |\mathbb{K}\pi| \\ {\scriptstyle \theta}\downarrow & & \downarrow{\scriptstyle \theta\otimes\theta} \\ |A| & \xrightarrow{\delta_\theta^+} & |A| \otimes |A|. \end{array}$$

We will first study the maps $\kappa_\theta, \mu_\theta$ and δ_θ^+ for the expansion $\theta^{\rm exp}$. In order to proceed, we need the following technical lemma:

Lemma 4.61. *Let N be an A-bimodule and let $\phi : A \to N$ be a $\mathbb{K}$-linear continuous map such that $\phi(ab) = \phi(a)b + a\phi(b)$ for any $a, b \in A$. Then, for any $x \in A_{\geq 1}$,*

$$\phi(e^x) = e^x \left(\frac{1 - e^{-\mathrm{ad}_x}}{\mathrm{ad}_x} \phi(x) \right).$$

Proof. Note that the expression $I(s) = \phi(e^{sx})$ satisfies the ordinary differential equation

$$\frac{dI(s)}{ds} = \phi(xe^{sx}) = \phi(x)e^{sx} + x\phi(e^{sx}) = \phi(x)e^{sx} + xI(s)$$

with initial condition $I(0) = 0$ (since $\phi(1) = 0$). It is easy to see that the function

$$e^{sx} \left(\frac{1 - e^{-s\,\mathrm{ad}_x}}{\mathrm{ad}_x} \phi(x) \right)$$

satisfies both the differential equation and the initial condition. Hence, by uniqueness of solutions of ordinary differential equation, the expression above gives a formula for $I(s)$. Putting $s = 1$ yields the desired result. □

Proposition 4.62. *The double bracket* $\Pi_{\mathrm{mult}} := \kappa_{\theta^{\exp}}$ *is tangential, and it is given by the following formula:*

$$\Pi_{\mathrm{mult}} = \sum_{i=1}^{n} \left| \partial_i \otimes \left(\frac{1}{1 - e^{-\mathrm{ad}_{x_i}}} \right) \mathrm{ad}_{x_i}^2 \partial_i \right| - \sum_{1 \leq j < i \leq n} |\mathrm{ad}_{x_i} \partial_i \otimes \mathrm{ad}_{x_j} \partial_j|. \tag{4.42}$$

Remark 4.63. The expression Π_{mult} is well known in quasi-Poisson geometry. The first term defines a canonical (non skew-symmetric) bracket on a connected Lie group G with quadratic Lie algebra, see Example 1 in [32]. The second term is the so-called fusion term, see Section 4 in [32].

Proof. It suffices to show that $(\theta^{\exp} \otimes \theta^{\exp})\kappa(\gamma_i, \gamma_j) = \Pi_{\mathrm{mult}}(e^{x_i}, e^{x_j})$ for any $1 \leq i, j \leq n$. By (4.26),

$$\Pi_{\mathrm{mult}}(x_i, -) = \left(\frac{1}{1 - e^{-\mathrm{ad}_{x_i}}} \right) \mathrm{ad}_{x_i}^2 \partial_i + \sum_{k<i} \mathrm{ad}_{x_i}(\mathrm{ad}_{x_k} \partial_k).$$

This proves that Π_{mult} is tangential. By Lemma 4.61, we compute

$$\begin{aligned} \Pi_{\mathrm{mult}}(e^{x_i}, -) &= e^{x_i} \left(\frac{1 - e^{-\mathrm{ad}_{x_i}}}{\mathrm{ad}_{x_i}} \Pi_{\mathrm{mult}}(x_i, -) \right) \\ &= e^{x_i}(\mathrm{ad}_{x_i} \partial_i) + \sum_{k<i} \left(e^{x_i}(\mathrm{ad}_{x_k} \partial_k) - (\mathrm{ad}_{x_k} \partial_k) e^{x_i} \right). \end{aligned} \tag{4.43}$$

From this equation, we obtain

$$\begin{aligned} &\Pi_{\mathrm{mult}}(e^{x_i}, e^{x_i}) = 1 \otimes e^{x_i} e^{x_i} - e^{x_i} \otimes e^{x_i} \quad \text{for any } i, \\ &\Pi_{\mathrm{mult}}(e^{x_i}, e^{x_j}) = 0 \quad \text{if } i < j, \\ &\Pi_{\mathrm{mult}}(e^{x_i}, e^{x_j}) = 1 \otimes e^{x_i} e^{x_j} + e^{x_j} e^{x_i} \otimes 1 - e^{x_i} \otimes e^{x_j} - e^{x_j} \otimes e^{x_i} \quad \text{if } i > j. \end{aligned} \tag{4.44}$$

Comparing this with Proposition 4.54, we obtain the result. □

Proposition 4.64. *We have*

$$\mu_{\theta^{\exp}} = \mathrm{tDiv}_{\mathrm{mult}}^T, \quad (\mu_{*\bullet})_{\theta^{\exp}} = \underline{\mathrm{tDiv}}_{\mathrm{mult}}^T, \quad \delta_{\theta^{\exp}}^+ = -\mathrm{tDiv}_{\mathrm{mult}}.$$

Proof. For the computation of $\mu_{\theta^{\exp}}$, Proposition 4.55 and equation (4.27) show that μ and $\mathrm{tDiv}_{\mathrm{mult}}^T$ satisfy the same product formulas, and Proposition 4.62 shows that $\theta^{\exp}$ intertwines the double brackets κ and Π_{mult}. Hence, it remains to show that $\theta^{\exp}$ intertwines μ and $\mathrm{tDiv}_{\mathrm{mult}}^T$ on some set of generators of $\mathbb{K}\pi$. By Proposition 4.55, $\mu(\gamma_i) = 0$ for all i. The fact that $\mathrm{tDiv}_{\mathrm{mult}}^T(e^{x_i}) = 0$ can be checked directly from (4.43) using (4.11) and the fact that $\partial_k(e^{x_i}) = 0$ for $k < i$.

The formula for $(\mu_{*\bullet})_{\theta^{\exp}}$ can be proved similarly: we use the fact that $\mu_{*\bullet}(\gamma_i^{-1}) = 0$ for any i, and the computation of $\Pi_{\mathrm{mult}}(e^{-x_i}, e^{-x_j})$ obtained from (4.44). The formula for $\delta_{\theta^{\exp}}^+$ follows from equations (4.29) and (4.41). □

Remark 4.65. Explicit expressions for $\mu_{\theta^{\exp}}$ and $\delta^{+}_{\theta^{\exp}}$ were obtained in [19].

For arbitrary special expansions, Massuyeau and Turaev proved the following theorem:

Theorem 4.66 ([27], see also [24] Theorem 5.2)**.** *For any special expansion θ, we have*

$$\kappa_\theta = \Pi_{\rm add} := \Pi_{\rm KKS} + \Pi_s,$$

where

$$s(z) = \frac{1}{z} - \frac{1}{1-e^{-z}}.$$

Remark 4.67. Actually, Massuyeau and Turaev work mainly with the homotopy intersection form $\eta : \mathbb{K}\pi \otimes \mathbb{K}\pi \to \mathbb{K}\pi$. Since κ can be recovered from η ([22] Proposition 4.7) and vice versa, it is easy to translate their result into the form presented in Theorem 4.66.

Remark 4.68. Theorem 4.66 admits an interpretation in terms of the exponentiation construction in quasi-Poisson geometry, see Section 7 in [4]. The KKS double bracket is a double Poisson bracket, and adding the term Π_s turns it into a quasi-Poisson double bracket in the sense of van den Bergh.

Remark 4.69. Note that the function $s(z)$ admits the following presentation in terms of Bernoulli numbers:

$$s(z) = -\frac{1}{2} - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\, z^{2k-1}.$$

It defines a solution of the classical dynamical Yang-Baxter equation (CDYBE).

As a consequence of Theorem 4.66, for a special expansion θ, we have the following commutative diagram:

$$\begin{array}{ccc} |\mathbb{K}\pi| \otimes |\mathbb{K}\pi| & \xrightarrow{\{-,-\}} & |\mathbb{K}\pi| \\ {\scriptstyle \theta\otimes\theta}\downarrow & & \downarrow{\scriptstyle \theta} \\ |A| \otimes |A| & \xrightarrow{\{-,-\}_{\rm KKS}} & |A|. \end{array} \tag{4.45}$$

This is obtained independently by Massuyeau and Turaev [27] and by the second and third authors [22]. Note that Π_s does not contribute in the Lie bracket on $|A|$ (Proposition 4.45).

Remark 4.70. By combining Theorem 4.66 and Proposition 4.62 we obtain the following result: let $F \in {\rm TAut}(L)$ be such that $F(x_1 + \cdots + x_n) = \log(e^{x_1} \ldots e^{x_n})$. Then,

$$(F^{-1} \otimes F^{-1})\Pi_{\rm mult}(F \otimes F) = \Pi_{\rm add}. \tag{4.46}$$

This statement does not involve topology of surfaces, and it admits a direct algebraic proof [29].

Remark 4.71. Let $F \in \mathrm{TAut}(L)$ and $\theta_F = F^{-1} \circ \theta^{\exp}$. Then, the map F induces an isomorphism of Lie algebras

$$(|A|, \{-,-\}_{\mathrm{mult}}) \cong (|A|, \{-,-\}_\theta).$$

In particular, this isomorphism restricts to an isomorphism of the centers. Since we know the center of the Lie algbera $(|A|, \{-,-\}_{\mathrm{KKS}})$, we conclude that the center of the Lie algebra $(|A|, \{-,-\}_\theta)$ is spanned by the elements $|(F^{-1}(x_i))^k| = |x_i^k|$ for $i = 1, \ldots, n$ and by $|(F^{-1}\log(e^{x_1} \ldots e^{x_n}))^k|$.

7 The Kashiwara-Vergne problem and homomorphic expansions

In this section, we discuss the connection between the Kashiwara-Vergne (KV) problem in Lie theory and the transfered structures μ_θ and δ_θ^+.

7.1 KV problems for surfaces of genus zero

We start with the formulation of the KV problem:

Kashiwara-Vergne problem: Find an element $F \in \mathrm{TAut}(L_2)$ satisfying the conditions

$$F(x_1 + x_2) = \log(e^{x_1} e^{x_2}), \tag{KVI}$$

$$\exists\, h(z) \in \mathbb{K}[[z]] \text{ such that } j(F^{-1}) = |h(x_1) + h(x_2) - h(x_1 + x_2)|. \tag{KVII}$$

Remark 4.72. The KV problem was originally discovered as a property of the Baker-Campbell-Hausdorff series which implies the Duflo theorem on the center of the universal enveloping algebra. The formulation above follows [6].

Recall the following result:

Theorem 4.73 ([5],[6])**.** *The KV problem admits solutions.*

As will be shown in our main theorems, the Kashiwara-Vergne problem is closely related to the topology of a surface of genus 0 with 3 boundary components. Regarding the topology of a surface of genus 0 with $n + 1$ boundary components for $n \geq 2$, we introduce a generalization of the Kashiwara-Vergne problem.

Kashiwara-Vergne problem of type $(0,n+1)$: An element $F\in \mathrm{TAut}(L_n)$ is a solution to the KV problem of type $(0,n+1)$ if

$$F(x_1+x_2+\cdots+x_n)=\log(e^{x_1}e^{x_2}\cdots e^{x_n}), \qquad (\mathrm{KV}^{(0,n+1)}\mathrm{I})$$

$$\exists h(z)\in \mathbb{K}[[z]] \text{ such that } j(F^{-1})=\left|\sum_{i=1}^{n} h(x_i)-h\left(\sum_{i=1}^{n} x_i\right)\right|. \qquad (\mathrm{KV}^{(0,n+1)}\mathrm{II})$$

The conditions $(\mathrm{KV}^{(0,3)}\mathrm{I})$ and $(\mathrm{KV}^{(0,3)}\mathrm{II})$ coincide with the conditions (KVI) and (KVII), respectively. The existence of a solution to the problem of type $(0,n+1)$ follows from that to the original problem as follows.

Let $F\in \mathrm{TAut}(L_2)$ and write $F=\exp(u)$ with $u=(u_1,u_2)\in \mathrm{tDer}(L_2)$. For $n\geq 2$, we introduce the following elements of $\mathrm{tDer}(L_n)=L^{\oplus n}$:

$$\begin{aligned}
u^{n-1,n} :=&(0,...,0,u_1(x_{n-1},x_n),u_2(x_{n-1},x_n)),\\
u^{n-2,(n-1)n} :=&(0,...,0,u_1(x_{n-2},x_{n-1}+x_n),u_2(x_{n-2},x_{n-1}+x_n),u_2(x_{n-2},x_{n-1}+x_n)),\\
&\vdots\\
u^{2,3\cdots n} :=&(0,u_1(x_2,x_3+\cdots+x_n),u_2(x_2,x_3+\cdots+x_n),...,u_2(x_2,x_3+\cdots+x_n)),\\
u^{1,2\cdots n} :=&(u_1(x_1,x_2+\cdots+x_n),u_2(x_1,x_2+\cdots+x_n),...,u_2(x_1,x_2+\cdots+x_n)).
\end{aligned}$$

Then, set

$$F^{(n)}:=F^{n-1,n}\circ F^{n-2,(n-1)n}\circ\cdots\circ F^{2,3\cdots n}\circ F^{1,2\cdots n}\in \mathrm{TAut}(L_n),$$

where $F^{n-1,n}=\exp(u^{n-1,n})$ *etc.* For more details about this operadic notation, see [1], [6]. For our purpose, the *naturality* of this construction is important. For example, consider the map $A_2\to A_n, a\mapsto a^{1,2\cdots n}=a(x_1,x_2+\cdots+x_n)$ which maps x_1 to x_1 and x_2 to $x_2+\cdots+x_n$. Then, $F(a)^{1,2\cdots n}=F^{1,2\cdots n}(a^{1,2\cdots n})$ for any $a\in A_2$. The divergence map is natural in this sense ([6] §3), and so is its integrating cocycle j. For example, $j(F)^{1,2\cdots n}=j(F^{1,2\cdots n})$ for any $F\in \mathrm{TAut}(L_2)$.

Lemma 4.74. *Let* $F\in \mathrm{TAut}(L_2)$ *be a solution of* (KVI) *and* (KVII). *Then,* $F^{(n)}\in \mathrm{TAut}(L_n)$ *is a solution to the Kashiwara-Vergne problem of type* $(0,n+1)$.

Proof. We first show that $F^{(n)}$ satisfies $(\mathrm{KV}^{(0,n+1)}\mathrm{I})$. We have $F^{i,(i+1)\cdots n}(x_i+x_{i+1}+\cdots+x_n)=\mathrm{bch}(x_i,x_{i+1}+\cdots+x_n)$ for $i=1,\dots,n-1$. Successive application of this equality yields

$$\begin{aligned}
F^{(n)}(x_1+\cdots+x_n)=&\mathrm{bch}(x_1,\mathrm{bch}(x_2,\mathrm{bch}(x_3,\cdots,\mathrm{bch}(x_{n-1},x_n)\cdots)))\\
=&\log(e^{x_1}e^{x_2}\cdots e^{x_n}).
\end{aligned}$$

To prove that $F^{(n)}$ satisfies $(\mathrm{KV}^{(0,n+1)}\mathrm{II})$, put $G_i := (F^{i,(i+1)\cdots n})^{-1}$ for $i = 1,\dots,n-1$. Then $j(G_i) = |h(x_i) + h(x_{i+1}+\cdots+x_n) - h(x_i+\cdots+x_n)|$ and G_i acts on the last $n-i$ generators $x_{i+1},\dots,x_n$ as an inner automorphism; there exists $g_i \in \exp(L_n)$ such that $G_i(x_k) = g_i^{-1}x_k g_i$ for $i < k \le n$. Therefore, $G_1\cdots G_{i-1}$ acts trivially on $j(G_i)$. By (4.13), we compute

$$\begin{aligned} j((F^{(n)})^{-1}) =& j(G_1G_2\cdots G_{n-1}) \\ =& j(G_1) + G_1\cdot j(G_2) + \cdots + G_1G_2\cdots G_{n-2}\cdot j(G_{n-1}) \\ =& \sum_{i=1}^{n-1} |h(x_i) + h(x_{i+1}+\cdots+x_n) - h(x_i+\cdots+x_n)| \\ =& \left|\sum_{i=1}^{n} h(x_i) - h(x_1+\cdots+x_n)\right|. \end{aligned}$$

This completes the proof. □

7.2 Homomorphic expansions

In this section, we define the notion of homomorphic expansion and show that expansions defined by solutions of the KV problem are in fact homomorphic.

Lemma 4.75. *Assume that $F \in \mathrm{TAut}(L_n)$ satisfies $(\mathrm{KV}^{(0,n+1)}\mathrm{I})$. Then, the map $\theta_F := F^{-1}\circ\theta^{\exp}$ is a special expansion.*

Proof. Referring to the definition of special expansions, the condition (i) is satisfied since F^{-1} and $\theta^{\exp}$ are algebra homomorphisms. Since $F^{-1}\in\mathrm{TAut}(L_n)$, for each i there exists an element $g_i \in L$ such that $F^{-1}(x_i) = e^{g_i}x_ie^{-g_i}$. Therefore, $\theta_F(\gamma_i) = F^{-1}(e^{x_i}) = e^{g_i}e^{x_i}e^{-g_i}$, proving (iv). The condition (v) follows from the condition $(\mathrm{KV}^{(0,n+1)}\mathrm{I})$: $\theta_F(\gamma_1\cdots\gamma_n) = F^{-1}(e^{x_1}\cdots e^{x_n}) = \exp(-x_0)$. The conditions (ii) and (iii) follow from (i) and (iv). □

We introduce a class of expansions which transform the Goldman bracket and the Turaev cobracket δ^+ in a nice way:

Definition 4.76. A special expansion $\theta : \pi \to A$ is called *homomorphic* if $\delta^+_\theta = \delta^{\mathrm{alg}}$.

Our terminology "homomorphic" follows the work of Bar-Natan and Dancso [8]. Homomorphic expansions are closely related to solutions of the generalized Kashiwara-Vergne problem:

Theorem 4.77. *Let $F \in \mathrm{TAut}(L_n)$ be a solution to the Kashiwara-Vergne problem of type $(0,n+1)$.*

(i) Let us write $\widetilde{\Delta}(s(-x_0)) = s' \otimes s''$ *where* $s(z)$ *is the function in Theorem 4.66, let* $g(z) := \dot{h}(z) \in \mathbb{K}[[z]]$ *be the derivative of* $h(z)$ *and write* $\widetilde{\Delta}(g(-x_0)) = g' \otimes g'' \in A \otimes A$*. Then, for any* $a \in A$*,*

$$\mu_{\theta_F}(a) = \mu^{\mathrm{alg}}(a) + |s''| \otimes as' - |s''a| \otimes s' + |g'| \otimes [a, g'']. \tag{4.47}$$

(ii) We have $\delta^+_{\theta_F} = \delta^{\mathrm{alg}}$*.*

Remark 4.78. Massuyeau [24] proved that for any special expansion θ arising from the Kontsevich integral, $\delta^+_\theta = \delta^{\mathrm{alg}}$. He also considered an operation which is closely related to our μ, and showed its tensorial description similar to (4.47).

Proof of Theorem 4.77. We split the proof into several steps:

Step 1. Let $F \in \mathrm{TAut}(L_n)$ be a solution of the KV problem of type $(0, n+1)$. Consider the expansion $\theta_F = F^{-1} \circ \theta^{\exp}$ and the corresponding map $\mu_F = \mu_{\theta_F}$. By Proposition 4.64, $\mu_{\theta^{\exp}} = \mathrm{tDiv}^T_{\Pi_{\mathrm{mult}}}$. Hence,

$$\begin{aligned}
\mu_F(a) &= (F^{-1} \otimes F^{-1}) \cdot \mathrm{tDiv}^T(\{F(a), -\}_{\mathrm{mult}}) \\
&= F^* \mathrm{tDiv}^T(\{a, -\}_{\mathrm{add}}) \\
&= \mathrm{tDiv}^T(\{a, -\}_{\mathrm{add}}) + (1 \otimes \{a, -\}_{\mathrm{add}}) \mathrm{tJ}(F^{-1}) \\
&= \mathrm{tDiv}^T_{\mathrm{add}}(a) + (1 \otimes \{a, -\}_{\mathrm{add}}) \mathrm{tJ}(F^{-1}).
\end{aligned}$$

Here in the second line we used equation (4.46), and in the third line the transformation property (4.19) of the map tDiv^T. We now discuss the two terms on the right hand side separately.

Step 2. By Proposition 4.46, we have

$$\mathrm{tDiv}^T_{\mathrm{add}}(a) = \mathrm{tDiv}^T_{\mathrm{KKS}}(a) + \mathrm{tDiv}^T_s(a) = \mu^{\mathrm{alg}}(a) + |s''| \otimes as' - |s''a| \otimes s'. \tag{4.48}$$

Note that an explicit formula for $\mu^{\mathrm{alg}} = \mathrm{tDiv}^T_{\mathrm{KKS}}$ is given in Proposition 4.42.

Step 3. We now turn to the expression $(1 \otimes \{a, -\}_{\mathrm{add}}) \mathrm{tJ}(F^{-1})$. It is convenient to use the Sweedler notation for $\mathrm{tJ}(F^{-1}) = \widetilde{\Delta}(j(F^{-1})) = |f'| \otimes |f''|$. Again, we represent the double bracket as a sum of two terms, $\Pi_{\mathrm{add}} = \Pi_{\mathrm{KKS}} + \Pi_s$. The term coming from Π_s is equal to zero,

$$(1 \otimes \{a, -\}_s) \mathrm{tJ}(F^{-1}) = |f'| \otimes \{a, |f''|\}_s = 0$$

since Π_s vanishes on $|A|$ by Proposition 4.45.

Step 4. We now turn to the analysis of the term $(1 \otimes \{a, -\}_{\mathrm{KKS}}) \mathrm{tJ}(F^{-1})$. By the second KV equation $(\mathrm{KV}^{(0,n+1)}\mathrm{II})$, we have $j(F^{-1}) = |\sum_{i=1}^n h(x_i) - h(-x_0)|$. Note that $\widetilde{\Delta}(|{x_i}^k|)$ belongs to the span of elements of the form $|{x_i}^l| \otimes |{x_i}^m|$. By Lemma 4.37, $\{a, |h(x_i)|\}_{\mathrm{KKS}} = 0$ for all $a \in A$. Hence,

$$(1 \otimes \{a, -\}_{\mathrm{KKS}})\widetilde{\Delta}(\sum_i |h(x_i)|) = 0.$$

Step 5. The remaining contribution is as follows:

$$\begin{aligned} -(1 \otimes \{a, -\}_{\mathrm{KKS}})\widetilde{\Delta}(|h(-x_0)|) &= -(1 \otimes \{a, -\}_{\mathrm{KKS}})|h(1 \otimes x_0 - x_0 \otimes 1)| \\ &= (|\ | \otimes 1)[1 \otimes a, \dot{h}(1 \otimes x_0 - x_0 \otimes 1)]. \end{aligned}$$

In the third line, we used Lemma 4.37. Applying the Sweedler notation

$$\widetilde{\Delta}(g(-x_0)) = \widetilde{\Delta}(\dot{h}(-x_0)) = \dot{h}(1 \otimes x_0 - x_0 \otimes 1) = g' \otimes g'',$$

we can rewrite the result as $|g'| \otimes [a, g'']$. This concludes the proof of equation (4.47).

Step 6. From the defining formulas for μ^{alg} and δ^{alg}, we see that $\mathrm{Alt}(1 \otimes |\ |)\mu^{\mathrm{alg}}(a) = -\delta^{\mathrm{alg}}(|a|)$ for any $a \in A$. Since the constant term of $s(z)$ is $-1/2$ and all the other terms are of odd degree, we have $s'' \otimes s' = -1 \otimes 1 - s' \otimes s''$. Therefore, from (4.47) we get $\mathrm{Alt}(1 \otimes |\ |)\mu_{\theta_F}(a) = -\delta^{\mathrm{alg}}(|a|) + |a| \wedge \mathbf{1}$ for any $a \in A$. By (4.39), we have

$$(\theta_F \otimes \theta_F)\delta^+(|\gamma|) = -\mathrm{Alt}(1 \otimes |\ |)\mu_{\theta_F}(\theta_F(\gamma)) + |\theta(\gamma)| \wedge \mathbf{1} = \delta^{\mathrm{alg}}(|\theta_F(\gamma)|)$$

for any $\gamma \in \pi$. Since the image $\theta_F(|\mathbb{K}\pi|)$ is dense in $|A|$, we conclude $\delta^+_{\theta_F} = \delta^{\mathrm{alg}}$.
This completes the proof of Theorem 4.77. □

8 Topological interpretation of the Kashiwara-Vergne theory

Theorem 4.77 shows that the Kashiwara-Vergne equations for an element $F \in \mathrm{TAut}(L_n)$ are sufficient to prove the equality $\delta^+_{\theta_F} = \delta^{\mathrm{alg}}$ for the expansion $\theta_F = F^{-1} \circ \theta^{\exp}$. In this section, our goal is to show that assuming the first KV equation, the second one becomes a necessary condition for $\delta^+_{\theta_F} = \delta^{\mathrm{alg}}$.

8.1 Special derivations

As a technical preparation, we discuss Lie algebras of special derivations and Kashiwara-Vergne Lie algebras.

Definition 4.79. An element $u \in \mathrm{tDer}(A)$ is called *special* if $\rho(u)(x_0) = 0$.

We denote by sDer(A) the set of special elements in tDer(A). Namely,

$$\mathrm{sDer}(A) = \{u = (u_1, \dots, u_n) \in \mathrm{tDer}(A) \mid \sum_i [x_i, u_i] = 0\}.$$

This is a Lie subalgebra of tDer(A). Likewise, we define sDer(L) to be the set of special elements in tDer(L).

Consider the symmetrization map $N : |A| \to A$. For $a = z_1 \cdots z_m$ with $z_j \in \{x_i\}_{i=1}^n$, we have

$$N(|a|) = \sum_{j=1}^{m} z_j \cdots z_m z_1 \cdots z_{j-1},$$

and $N(|a|) = 0$ for $a \in \mathbb{K}$. If a is homogeneous of degree m, then $|N(|a|)| = m|a|$.

Remark 4.80. The symmetrization map N coincides with the map $|A| \to A$ induced by the double derivation $\sum_i x_i \partial_i$.

Lemma 4.81. *For any* $|a| \in |A|$*, the derivation* $\{|a|, -\} \in \mathrm{sDer}(A)$ *is special, and it is given by*

$$\{|a|, -\}_{\mathrm{KKS}} = -(u_1, \dots, u_n),$$

where $u_i \in A$ *are uniquely defined by formula* $N(|a|) = \sum_{i=1}^n x_i u_i$*. Furthermore, the map* $|a| \mapsto \{|a|, -\}_{\mathrm{KKS}}$ *induces an isomorphism*

$$|A|/\mathbb{K}\mathbf{1} \cong \mathrm{sDer}(A). \tag{4.49}$$

Proof. Lemma 4.37 implies that $\{|a|, x_0\}_{\mathrm{KKS}} = 0$ for all $a \in A$. Hence, $\{|a|, -\}_{\mathrm{KKS}} \in \mathrm{sDer}(A)$. In order to prove the formula for the tangential derivation $\{|a|, -\}_{\mathrm{KKS}}$, we may assume that $a = z_1 \cdots z_m$ with $z_j \in \{x_i\}_{i=1}^n$. Then the ith component of $\{|a|, -\}_{\mathrm{KKS}}$ is of the form

$$-\sum_j \delta_{x_i, z_j} z_{j+1} \cdots z_m z_1 \cdots z_{j-1}$$

(see the proof of Proposition 4.42), and this is equal to $-u_i$.

Let $|a| \in |A|$ be a homogeneous element of degree $m \geq 1$ in the kernel of the map $|a| \mapsto \{|a|, -\}_{\mathrm{KKS}}$. The formula for $\{|a|, -\}_{\mathrm{KKS}}$ shows that $|a| = 0$. Clearly, $\mathbf{1}$ is in the kernel of the map $|A| \to \mathrm{sDer}(A)$. Hence, the map $|A|/\mathbb{K}\mathbf{1} \to \mathrm{sDer}(A)$ is injective.

To show the surjectivity, let $v = (v_1, \dots, v_n)$ be a special derivation and assume that v_i is homogeneous of degree m for any i. Then, $a := -(1/(m+1)) \sum_i x_i v_i$ is cyclically invariant since $\sum_i [x_i, v_i] = 0$, and $N(|a|) = (m+1)a = -\sum_i x_i v_i$. Therefore, $\{|a|, -\}_{\mathrm{KKS}} = v$. ◻

8.2 The Lie algbera krv_n

Definition 4.82 ([6] §4)**.** The *Kashiwara-Vergne Lie algebra* krv_2 is defined by

$$\mathrm{krv}_2 := \{u \in \mathrm{sDer}(L_2) \mid \exists h(z) \in \mathbb{K}[[z]], \mathrm{div}(u) = |h(x) + h(y) - h(x+y)|\}.$$

Similarly, for any $n \geq 2$, we introduce a generalization of krv_2,

$$\mathrm{krv}_n := \{u \in \mathrm{sDer}(L) \mid \exists h(z) \in \mathbb{K}[[z]],\ \mathrm{div}(u) = |\sum_{i=1}^{n} h(x_i) - h(-x_0)|\}.$$

For our purposes, it is convenient to introduce a slightly different version of the Kashiwara-Vergne Lie algebra,

$$\mathrm{krv}_n^{\mathrm{KKS}} = \{u \in \mathrm{sDer}(L) \mid \mathrm{div}(u) \in Z(|A|, \{-,-\}_{\mathrm{KKS}})\}.$$

Recall that the kernel of the map $\rho : \mathrm{tDer}(L) \to \mathrm{Der}(L)$ is spanned by the elements

$$q_i = -\frac{1}{2}\{|x_i^2|, -\}_{\mathrm{KKS}} = (0, \dots, 0, \overset{i}{\check{x}_i}, 0, \dots, 0) \in \mathrm{sDer}(L) \subset \mathrm{tDer}(L).$$

The relation between the two versions of the Kashiwara-Vergne Lie algebra is given by the following proposition:

Proposition 4.83.

$$\mathrm{krv}_n^{\mathrm{KKS}} = \mathrm{krv}_n \oplus \left(\bigoplus_{i=1}^{n} \mathbb{K} q_i\right),$$

For the proof, we need the following lemma:

Lemma 4.84. *Consider the substitution map $\varpi_k : |A_n| \to |A_1| \cong \mathbb{K}[[z]], a \mapsto a|_{x_k=z, x_{k'}=0 (k'\neq k)}$, given by $x_k \mapsto z$ and $x_{k'} \mapsto 0$ for $k' \neq k$. Let $u = (u_1, \dots, u_n) \in \mathrm{tDer}(L_n)$ and assume that the degree 1 part of u_k is in $\bigoplus_{k'\neq k} \mathbb{K} x_{k'}$. Then we have $\varpi_k(\mathrm{div}(u)) = 0$. Furthermore, we have $\varpi_k(j(\exp(u))) = 0$.*

Proof. If $v = [z_1, [z_2, \dots, [z_{m-1}, z_m] \cdots]]$, where $z_j \in \{x_i\}_{i=1}^n$, is a Lie monomial of degree $m \geq 2$, then $\varpi_k(v^k) = 0$. By linearity, we see that $\varpi_k(v^k) = 0$ if the degree 1 part of $v \in L$ is in $\bigoplus_{k'\neq k} \mathbb{K} x_{k'}$. Now, since $\varpi_k(x_{k'}) = 0$ for $k' \neq k$, we obtain

$$\varpi_k(\mathrm{div}(u)) = \varpi_k |x_k (u_k)^k| + \sum_{k'\neq k} \varpi_k |x_{k'} (u_{k'})^{k'}| = |z \cdot 0| = 0.$$

Finally, by (4.15), we obtain $j(\exp(u))|_{x_k=z, x_{k'}=0(k'\neq k)} = 0$. ☐

Proof of Proposition 4.83. For the inclusion $\mathrm{krv}_n \oplus (\bigoplus_{i=1}^n \mathbb{K}q_i) \subset \mathrm{krv}_n^{\mathrm{KKS}}$, we first consider the elements q_i. Since

$$\mathrm{div}(q_i) = |x_i| \in Z(|A|, \{-,-\}_{\mathrm{KKS}}),$$

we conclude that $q_i \in \mathrm{krv}_n^{\mathrm{KKS}}$. If $u \in \mathrm{krv}_n$, then $\mathrm{div}(u) = |\sum_{i=1}^n h(x_i) - h(-x_0)|$ for some $h(z) \in \mathbb{K}[[z]]$. In particular, $\mathrm{div}(u)$ is in the center with respect to the Lie bracket $\{-,-\}_{\mathrm{KKS}}$. Therefore, $\mathrm{krv}_n \subset \mathrm{krv}_n^{\mathrm{KKS}}$.

Note that expressions of the form $|\sum_{i=1}^n h(x_i) - h(-x_0)|$ contain no linear terms. Therefore, $\mathrm{krv}_n \cap \bigoplus_{i=1}^n \mathbb{K}q_i = \{0\}$ and $\mathrm{krv}_n \oplus (\bigoplus_{i=1}^n \mathbb{K}q_i) \subset \mathrm{krv}_n^{\mathrm{KKS}}$, as required.

To prove the inclusion in the opposite direction, let $u \in \mathrm{krv}_n^{\mathrm{KKS}}$. By Theorem 4.41, there exist elements $h_i(z) \in z^2\mathbb{K}[[z]]$, $0 \le i \le n$, and $c_i \in \mathbb{K}$, $0 \le i \le n$, such that

$$\mathrm{div}(u) = |c_0 + \sum_{i=1}^n (c_i x_i + h_i(x_i)) - h_0(-x_0)|.$$

From the definition of the divergence, we have $c_0 = 0$. Hence,

$$\mathrm{div}(u - \sum_{i=1}^n c_i q_i) = |\sum_{i=1}^n h_i(x_i) - h_0(-x_0)|.$$

By Lemma 4.84, we obtain $h_i(z) = h_0(z)$. Therefore, $u - \sum_{i=1}^n c_i q_i \in \mathrm{krv}_n$ and $u \in \mathrm{krv}_n \oplus (\bigoplus_{i=1}^n \mathbb{K}q_i)$. □

The Lie algebras $\mathrm{sDer}(A)$, $\mathrm{sDer}(L)$, krv_n and $\mathrm{krv}_n^{\mathrm{KKS}}$ integrate to the groups $\mathrm{SAut}(A)$, $\mathrm{SAut}(L)$, KRV_n and $\mathrm{KRV}_n^{\mathrm{KKS}}$, respectively. Clearly, $\mathrm{KRV}_n \subset \mathrm{KRV}_n^{\mathrm{KKS}} \subset \mathrm{SAut}(L)$. A more explict description of the group KRV_n is as follows:

$$\mathrm{KRV}_n = \{F \in \mathrm{SAut}(L) \mid \exists h(z) \in \mathbb{K}[[z]], j(F) = |\sum_{i=1}^n h(x_i) - h(-x_0)|\}.$$

To see this, let $u \in \mathrm{krv}_n$ and $F = \exp(u) \in \mathrm{KRV}_n$. We have $\mathrm{div}(u) = |\sum_{i=1}^n h(x_i) - h(-x_0)|$ for some $h(z) \in \mathbb{K}[[z]]$. Since u is a special derivation, $u \cdot \mathrm{div}(u) = u \cdot |\sum_{i=1}^n h(x_i) - h(-x_0)| = 0$. By (4.15), $j(F) = \mathrm{div}(u) + (1/2)u \cdot \mathrm{div}(u) + \cdots = \mathrm{div}(u)$. The other inclusion can be proved similary (note that the operator $(e^{\mathrm{ad}_u} - 1)/\mathrm{ad}_u$ is invertible).

The group $\mathrm{KRV}_n^{\mathrm{KKS}}$ admits a similar description:

$$\mathrm{KRV}_n^{\mathrm{KKS}} = \{F \in \mathrm{SAut}(L) \mid j(F) \in Z(|A|, \{-,-\}_{\mathrm{KKS}})\}.$$

As before, the fact that $j(F) \in Z(|A|, \{-,-\}_{\mathrm{KKS}})$ is equivalent to the existence of $h(z) \in \mathbb{K}[[z]]$ such that

$$j(F) = |\sum_{i=1}^n h(x_i) - h(-x_0)| \bmod \bigoplus_{i=1}^n \mathbb{K}|x_i|.$$

8.3 Topological interpretation of the KV problem

The following theorem gives a topological interpretation of the condition $(\mathrm{KV}^{(0,n+1)}\mathrm{II})$.

Theorem 4.85. *Suppose that* $F \in \mathrm{TAut}(L)$ *satisfies* $(\mathrm{KV}^{(0,n+1)}\mathrm{I})$ *and set* $\theta_F := F^{-1} \circ \theta^{\exp}$. *Then,* F *satisfies* $\delta^+_{\theta_F} = \delta^{\mathrm{alg}}$ *if and only if*

$$j(F^{-1}) \in Z(|A|, \{-,-\}_{\mathrm{KKS}}).$$

Equivalently, there exists an element $G \in \ker(\rho) \subset \mathrm{SAut}(L)$ *such that* FG *satisfies* $(\mathrm{KV}^{(0,n+1)}\mathrm{II})$.

Proof. Let $F \in \mathrm{TAut}(L)$ be a solution of $(\mathrm{KV}^{(0,n+1)}\mathrm{I})$. Consider the following diagram:

$$\begin{array}{ccc}
|\mathbb{K}\pi| & \xrightarrow{\delta^+} & |\mathbb{K}\pi| \otimes |\mathbb{K}\pi| \\
{\scriptstyle \theta^{\exp}}\downarrow & & \downarrow{\scriptstyle \theta^{\exp}\otimes\theta^{\exp}} \\
|A| & \xrightarrow{-\mathrm{tDiv}_{\mathrm{mult}}} & |A| \otimes |A| \\
{\scriptstyle F^{-1}}\downarrow & & \downarrow{\scriptstyle F^{-1}\otimes F^{-1}} \\
|A| & \xrightarrow{\delta^+_{\theta_F}} & |A| \otimes |A|.
\end{array}$$

The upper square is commutative by Proposition 4.64, and the lower square is commutative by definition of the transfered cobracket. For $a \in A$ we use equations (4.46), (4.18), (4.16) and Proposition 4.46 to compute

$$\begin{aligned}
\delta^+_{\theta_F}(|a|) &= -(F^{-1} \otimes F^{-1})\mathrm{tDiv}_{\mathrm{mult}}(F.|a|) \\
&= -F^{-1}\mathrm{tDiv}(F(\{|a|,-\}_{\mathrm{add}})) \\
&= -\mathrm{tDiv}(\{|a|,-\}_{\mathrm{add}}) - \{|a|,-\}_{\mathrm{add}} \cdot \mathrm{tJ}(F^{-1}) \\
&= \delta^{\mathrm{alg}}(|a|) - \{|a|,-\}_{\mathrm{KKS}} \cdot \widetilde{\Delta}(j(F^{-1})).
\end{aligned}$$

The equality $\delta^+_{\theta_F} = \delta^{\mathrm{alg}}$ holds true if and only if $\{|a|,-\}_{\mathrm{KKS}} \cdot \widetilde{\Delta}(j(F^{-1})) = 0$ for every $a \in A$. Looking at the $|A| \otimes \mathbf{1}$-component of this equation, we see that the equation is satisfied only if $j(F^{-1}) \in |A|$ is central with respect to $\{-,-\}_{\mathrm{KKS}}$. In the other direction, recall that the center is a sub-coalgebra of $|A|$. Hence, if $j(F^{-1})$ is central, the expression $\{|a|,-\}_{\mathrm{KKS}} \cdot \widetilde{\Delta}(j(F^{-1}))$ vanishes as required.

For the second statement, if $G \in \ker(\rho)$ and FG is a solution of $(\mathrm{KV}^{(0,n+1)}\mathrm{II})$, then

$$\begin{aligned}
j(F^{-1}) &= j(G) + G.j((FG)^{-1}) \\
&= j(G) + j((FG)^{-1}) \\
&\in \{|\textstyle\sum_i (h(x_i) + c_i x_i) - h(-x_0)|\, c_i \in \mathbb{K}, h(z) \in \mathbb{K}[[z]]\} \\
&\subset Z(|A|, \{-,-\}_{\mathrm{KKS}}).
\end{aligned}$$

In the other direction, one uses the argument similar to the proof of Proposition 4.83. □

Remark 4.86. One can prove Theorem 4.3 in Introduction by using the same argument as above. Indeed, for $F \in \mathrm{TAut}(L)$ and $\theta_F = F^{-1} \circ \theta^{\exp}$ we have

$$\delta^+_{\theta_F}(|a|) = -\mathrm{tDiv}(\{|a|, -\}_\theta) - \{|a|, -\}_\theta \cdot \mathrm{tJ}(F^{-1}).$$

If $j(F^{-1})$ is central in the Lie algebra $(|A|, \{-, -\}_\theta)$, it is spanned by elements of the form $|x_i^k|, |(F^{-1}\log(e^{x_1}\dots e^{x_n}))^k|$ (see Remark 4.71). Since the center is a sub-coalgebra of $|A|$, the expression $\{|a|, -\}_\theta \cdot \mathrm{tJ}(F^{-1})$ vanishes, as required.

Remark 4.87. Define the set $\Theta^{(0,n+1)}$ of homomorphic expansions and the set $\mathrm{SolKV}^{(0,n+1)}$ of solutions of the Kashiwara-Vergne problem. There is a bijection between these two sets established by the formula $F \mapsto \theta_F = F^{-1} \circ \theta^{\exp}$. Furthermore, the group KRV_n acts freely and transitively on the set $\mathrm{SolKV}^{(0,n+1)}$, and the action is given by formula $F \mapsto FH$. The set $\Theta^{(0,n+1)}$ carries a transitive action of the group $\mathrm{KRV}_n^{\mathrm{KKS}}$ given by $\theta \mapsto H^{-1} \circ \theta$. The stabilizer of this action is $\ker(\rho)$ for all homomorphic expansions θ. The bijection $\Theta^{(0,n+1)} \cong \mathrm{SolKV}^{(0,n+1)}$ is equivariant under the action of KRV_n (the action on $\Theta^{(0,n+1)}$ is induced by the inclusion $\mathrm{KRV}_n \subset \mathrm{KRV}_n^{\mathrm{KKS}}$).

8.4 Topological interpretation of Duflo functions

In this section, we apply Theorem 4.77 to establish a relation between the function $s(z)$ coming from the topologically defined double bracket κ and the Duflo function $h(z)$ entering solutions of the Kashiwara-Vergne equations.

Recall that the function $g(z)$ in equation (4.47) is the derivative of the Duflo function $h(z)$ and $h(z)$ can be chosen to lie in $z^2\mathbb{K}[[z]]$. Decomposing both g and h into a sum of even and odd parts, we obtain $g_{\mathrm{even}} = \dot{h}_{\mathrm{odd}}, g_{\mathrm{odd}} = \dot{h}_{\mathrm{even}}$.

Proposition 4.88. *Let F be a solution of the Kashiwara-Vergne problem of type $(0, n+1)$ for $n \geq 1$ and $h(z)$ the corresponding Duflo function. Then, modulo the linear part,*

$$g_{\mathrm{odd}}(z) = \dot{h}_{\mathrm{even}}(z) \equiv \frac{1}{2}\left(\frac{1}{2} + s(z)\right). \tag{4.50}$$

Remark 4.89. In the original Kashiwara-Vergne problem, the function $h(z)$ satisfies

$$h_{\mathrm{even}}(z) = -\frac{1}{2}\sum_{k\geq 1} \frac{B_{2k}}{2k \cdot (2k)!} z^{2k} \tag{4.51}$$

by [6] Proposition 6.1. This matches the expression obtained from (4.50) by substituting $s(z) = z^{-1} - (1 - e^{-z})^{-1}$.

Remark 4.90. In fact, by inspection of the lower degree terms of the KV equations, one can also show that the linear term of $g(z)$ is $-(B_2/4)z = -(1/24)z$.

Proof of Proposition 4.88. Recall that the double bracket κ and operation μ satisfy the involutivity property (see [21] Proposition 3.2.7):

$$\{-,-\} \circ \mu = 0 : \mathbb{K}\pi \to |\mathbb{K}\pi| \otimes \mathbb{K}\pi \to \mathbb{K}\pi.$$

Let F be a solution of the Kashiwara-Vergne problem of type $(0, n+1)$. By applying expansion θ_F, we obtain

$$\{-,-\}_{\mathrm{add}} \circ \mu_{\theta_F} = 0 : A \to |A| \otimes A \to A. \tag{4.52}$$

Since $\{-,-\}_s$ vanishes on $|A| \otimes A$, we can replace the bracket $\{-,-\}_{\mathrm{add}}$ by the bracket $\{-,-\}_{\mathrm{KKS}}$. Furthermore, the graded quotient of the equation (4.52) implies $\{-,-\}_{\mathrm{KKS}} \circ \mu^{\mathrm{alg}} = 0$. By using (4.47), we obtain

$$\{|s''|, as'\}_{\mathrm{KKS}} - \{|s''a|, s'\}_{\mathrm{KKS}} + \{|g'|, ag''\}_{\mathrm{KKS}} - \{|g'|, g''a\}_{\mathrm{KKS}} = 0$$

for any $a \in A$. The contribution $\{|s''a|, s'\}_{\mathrm{KKS}}$ vanishes since $\{|s''a|, -\}_{\mathrm{KKS}}$ is a special derivation and special derivations annihilate x_0. The remaining terms read

$$\begin{aligned} &- \dot{s}(-\mathrm{ad}_{x_0})(a) + \dot{s}(0)a + \dot{g}(-\mathrm{ad}_{x_0})(a) - \dot{g}(0)a - (-\dot{g}(\mathrm{ad}_{x_0})(a) + \dot{g}(0)a) \\ =&(2\dot{g}_{\mathrm{odd}}(\mathrm{ad}_{x_0}) - \dot{s}(\mathrm{ad}_{x_0}))(a) + (\dot{s}(0) - 2\dot{g}(0))a. \end{aligned}$$

Here we use that $s'' \otimes s' = -1 \otimes 1 - s' \otimes s''$ and Lemma 4.38. Since the last expression vanishes for all a, we conclude that $2g_{\mathrm{odd}}(z) - s(z) = 0$ modulo the terms of degree ≤ 1. This proves the proposition. □

A Proof of Theorem 4.41

In this appendix we deduce Theorem 4.41 from the following theorem. Most of this appendix will be spent for its proof.

Theorem 4.91. *Suppose that $u \in \mathrm{Der}(A)$ satisfies $|u(a)| = 0$ for any $a \in A$. Then u is an inner derivation, i.e., there exists an element $w \in A$ such that $u(a) = [a, w]$ for any $a \in A$.*

Let H be the linear subspace of A spanned by x_i, $1 \leq i \leq n$. The linear subspace $H^{\otimes m} \subset A$ is nothing but the set of homogeneous elements of degree m. In order to prove Theorem 4.91, we need Proposition 4.92 and Lemma 4.95 stated below.

Proposition 4.92. *Let $x \in H \setminus \{0\}$ and let $a \in H^{\otimes m}$ with $m \geq 1$. If $|ax^l| = 0$ for some $l \geq m-1$, then $a \in [x, A]$.*

To prove Proposition 4.92, we may assume that $n \geq 2$ and $x = x_1$.

For a multi-index $I = (i_1, \dots, i_m)$, where $i_k \in \{1, \dots, n\}$, set $p_I := x_{i_1} \cdots x_{i_m} \in H^{\otimes m}$. We introduce the subspaces of $H^{\otimes m}$ defined by

$$P := \mathrm{Span}_{\mathbb{K}}\{p_I \mid i_1 \neq 1, i_m = 1\},$$
$$Q := \mathrm{Span}_{\mathbb{K}}\{p_I \mid i_1 \neq 1, i_m \neq 1\}.$$

Clearly, $H^{\otimes m} = xH^{\otimes(m-1)} \oplus P \oplus Q$. Note that $P = 0$ if $m = 1$.

Lemma 4.93. *Let $m \geq 2$. If $b \in Q$ and $|bx^l| = 0$ for some $l \geq m-1$, then $b = 0$.*

Proof. Let $\pi : H \to \mathbb{K}x$ be the projection defined by $x_1 \mapsto x$ and $x_i \mapsto 0$ for $i = 2, \dots, n$. Let $\sigma := (1, 2, \dots, l+m) \in \mathfrak{S}_{l+m}$ be the cyclic permutation. Since $|bx^l| = 0$, we have $\sum_{i=0}^{l+m-1} \sigma^i(bx^l) = 0 \in H^{\otimes(l+m)}$. Applying $1_H^{\otimes m} \otimes \pi^{\otimes l}$ to this equation, we obtain $bx^l = 0$, since $b \in Q$ implies $(1_H^{\otimes m} \otimes \pi^{\otimes l})\sigma^i(bx^l) = 0$ for $i = 1, \dots, l+m-1$. Hence $b = 0$. □

Proof of Proposition 4.92. The assertion is clear if $m = 1$. In fact, $|ax^l| = 0$ implies $a = 0$.

Let $m \geq 2$ and suppose that the assertion holds if $\deg(a) = m-1$. Let $a \in H^{\otimes m}$ and assume that $|ax^l| = 0$ for some $l \geq m-1$. There exist uniquely determined elements $a' \in H^{\otimes(m-1)}$, $b' \in H^{\otimes(m-1)}$ with $b'x \in P$, and $b'' \in Q$ such that $a = xa' + b'x + b''$. We have

$$0 = |ax^l| = |(xa' + b'x + b'')x^l| = |(a' + b')x^{l+1}| + |b''x^l|.$$

Notice that the first term $|(a'+b')x^{l+1}|$ is in the span of monomials $|p_I|$ containing a sequence of $l+1$ consecutive x's, while the second term $|b''x^l|$ is in the span of $|p_I|$'s containing no such a sequence. Therefore, $|b''x^l| = 0$. Applying Lemma 4.93, we obtain $b'' = 0$. Also, we have $|(a'+b')x^{l+1}| = 0$.

Now we have $l + 1 \geq (m-1) - 1$ since $l \geq m-1$. By the inductive assumption, $a' + b' = [x, a_1]$ for some $a_1 \in H^{\otimes(m-2)}$. Then $a = xa' + b'x = [x, a'] + (a'+b')x = [x, a'] + [x, a_1]x = [x, a' + a_1 x] \in [x, A]$, as required. □

Corollary 4.94. *Let $x \in H \setminus \{0\}$ and $a \in A$. Then $a \in [x, A]$ if and only if $|ax^l| = 0$ for any $l \geq 1$.*

Proof. The 'only if'-part is trivial. In fact, it is clear that $|[x, b]x^l| = -|b[x, x^l]| = 0$. Here we remark that it suffices to show the 'if'-part for any homogeneous $a \in A$, since $[A, A] \subset A$ is homogeneous. Hence Proposition 4.92 implies the 'if'-part. □

Lemma 4.95. *Let x and $y \in H$ be linearly independent. Suppose that an element $a \in A$ satisfies $[x^l, a] \in [y, A]$ and $[y^l, a] \in [x, A]$ for any $l \geq 0$. Then we have $a \in \mathbb{K}[[x]] + \mathbb{K}[[y]]$.*

Proof. We may assume $a \in H^{\otimes d}$ for some $d \geq 1$. Since $\mathbb{K}x^d \cap yH^{\otimes(d-1)} = 0$, we have a decomposition

$$H^{\otimes d} = \mathbb{K}x^d \oplus yH^{\otimes(d-1)} \oplus V$$

for some linear subspace $V \subset H^{\otimes d}$. In particular, we have $a = \lambda x^d + ya' + a''$ for some $\lambda \in \mathbb{K}$, $a' \in H^{\otimes(d-1)}$ and $a'' \in V$. From the assumption, there exists some $b \in H^{\otimes(2d-1)}$ such that $[x^d, a] = [y, b]$. We have $b = -x^d b_0 + yb' + b''$ for some $b_0 \in H^{\otimes(d-1)}$, $b' \in H^{\otimes(2d-2)}$ and $b'' \in V \otimes H^{\otimes(d-1)}$. Then the equation $[x^d, a] = [y, b]$ implies

$$x^d(ya' + a'') - (ya' + a'')x^d = yb + x^d b_0 y - (yb' + b'')y$$

Looking at its $x^d H^{\otimes d}$-component in the decomposition $H^{\otimes 2d} = x^d H^{\otimes d} \oplus yH^{\otimes(2d-1)} \oplus V \otimes H^{\otimes d}$, we obtain $x^d(ya' + a'') = x^d b_0 y$, so that $ya' + a'' = b_0 y$. So we conclude $a = \lambda x^d + b_0 y$. Similarly the assumption $[y^d, a] \in [x, A]$ implies $a = \mu y^d + c_0 x$ for some $\mu \in \mathbb{K}$ and $c_0 \in H^{\otimes(d-1)}$. Hence we obtain $a = \lambda x^d + \mu y^d$. This proves Lemma 4.95. □

Remark 4.96. Note that if $a \in \mathbb{K}[[x]] + \mathbb{K}[[y]]$ then $[x^l, a] \in [y, A]$ and $[y^l, a] \in [x, A]$ for any $l \geq 0$. For example,

$$[x^l, y^m] = \left[x, \sum_{i=0}^{l-1} x^i y^m x^{l-1-i}\right].$$

Proof of Proposition 4.91. If $n = 1$, then $|A| = A$, so that the theorem is trivial. Hence we may assume $n \geq 2$. Since $0 = |u({x_i}^{l+1})| = \sum_{k=0}^{l} |{x_i}^k u(x_i) {x_i}^{l-k}| = (l+1)|u(x_i){x_i}^l|$ for any $l \geq 0$, we have some $a_i \in A_{\geq 1}$ such that $u(x_i) = [x_i, a_i]$ by Corollary 4.94. Let i and $j \in \{1, 2, \ldots, n\}$ be distinct. Then, for any $p, q \geq 1$, we have

$$0 = |u({x_i}^p {x_j}^q)| = |[{x_i}^p, a_i]{x_j}^q| + |{x_i}^p[{x_j}^q, a_j]| = |[{x_i}^p, a_i - a_j]{x_j}^q| = |{x_i}^p[a_i - a_j, {x_j}^q]|.$$

This implies $[{x_i}^p, a_i - a_j] \in [x_j, A]$ and $[{x_j}^q, a_i - a_j] \in [x_i, A]$ by Corollary 4.94. Hence, by Lemma 4.95, we have $a_i - a_j \in \mathbb{K}[[x_i]] + \mathbb{K}[[x_j]]$, namely, we have unique formal series $f_{ij}(x_j) \in \mathbb{K}[[x_j]]$ and $f_{ji}(x_i) \in \mathbb{K}[[x_i]]$ such that $a_i - a_j = f_{ij}(x_j) - f_{ji}(x_i)$ and $f_{ij}(0) = f_{ji}(0) = 0$. If $n = 2$, then $w := a_1 + f_{21}(x_1) = a_2 + f_{12}(x_2)$ satisfies $u(a) = [a, w]$ for any $a \in A$.

For the rest of this proof, we assume $n \geq 3$. Let i, j and $k \in \{1, 2, \ldots, n\}$ be distinct. Then

$$\begin{aligned} 0 &= (a_i - a_j) + (a_j - a_k) + (a_k - a_i) \\ &= f_{ij}(x_j) - f_{ji}(x_i) + f_{jk}(x_k) - f_{kj}(x_j) + f_{ki}(x_i) - f_{ik}(x_k) \end{aligned}$$

Since $f_{ji}(0) = f_{ki}(0) = 0$, we have $f_{ji}(x_i) = f_{ki}(x_i)$. Hence $w := a_i + f_{ji}(x_i) \in A$ is independent of the choice of i and j, so that $u(a) = [a, w]$ for any $a \in A$. □

Proof of Theorem 4.41. From Lemma 4.37, the center includes $|\mathbb{K}[[x_i]]|$ for any $i = 0, 1, \ldots, n$.

Let $a \in A$ be a homogeneous element of degree $m \geq 1$ satisfying $\{|a|, |b|\}_{\text{KKS}} = 0$ for any $|b| \in |A|$. This means $|\{|a|, b\}_{\text{KKS}}| = 0$ for any $b \in A$. From Theorem 4.91, there exists an element $w \in A$ such that $\{|a|, b\}_{\text{KKS}} = [b, w]$ for any $b \in A$. By Lemma 4.37, we have $[x_0, w] = \{|a|, x_0\}_{\text{KKS}} = 0$. It is easy to check that for any non-zero element $r \in A$ of degree 1, any element of A commuting with r is a formal power series in r. Hence we have $w \in \mathbb{K}[[x_0]]$, and so $\{|a|, -\}_{\text{KKS}} = (c_1 {x_1}^{m-1} + w, \ldots, c_n {x_n}^{m-1} + w) \in \text{sDer}(A)$ for some $c_i \in \mathbb{K}$. By Lemma 4.81, we obtain $m|a| = |N(|a|)| = -\sum_i |c_i {x_i}^m + x_i w| = |x_0 w| - \sum_i |c_i {x_i}^m| \in \sum_{i=0}^n |\mathbb{K}[[x_i]]|$. This proves the theorem. □

Remark 4.97. The same method allows to compute the center of the *completed* Goldman Lie algebra for any compact connected oriented surface with non-empty boundary. The key point is to use expansions to transfer the Goldman bracket to a Lie algebra structure on the space of cyclic words, as we saw in the case of genus zero (see diagram (4.45)). For surfaces of genus $g \geq 1$, one can also use the technique of [12], see Corollary 8.6.2.

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Chapter 5

Higher genus Kashiwara-Vergne problems and the Goldman-Turaev Lie bialgebra

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Abstract: We define a family $\mathrm{KV}^{(g,n+1)}$ of Kashiwara-Vergne problems associated with compact connected oriented 2-manifolds of genus g with $n+1$ boundary components. The problem $\mathrm{KV}^{(0,3)}$ is the classical Kashiwara-Vergne problem from Lie theory. We show the existence of solutions of $\mathrm{KV}^{(g,n+1)}$ for arbitrary g and n. The key point is the solution of $\mathrm{KV}^{(1,1)}$ based on the results by B. Enriquez on elliptic associators. Our construction is motivated by applications to the formality problem for the Goldman-Turaev Lie bialgebra $\mathfrak{g}^{(g,n+1)}$. In more detail, we show that every solution of $\mathrm{KV}^{(g,n+1)}$ induces a Lie bialgebra isomorphism between $\mathfrak{g}^{(g,n+1)}$ and its associated graded $\mathrm{gr}\,\mathfrak{g}^{(g,n+1)}$. For $g = 0$, a similar result was obtained by G. Massuyeau using the Kontsevich integral. For $g \geq 1, n = 0$, our results imply that the obstruction to surjectivity of the Johnson homomorphism provided by the Turaev cobracket is equivalent to the Enomoto-Satoh obstruction.

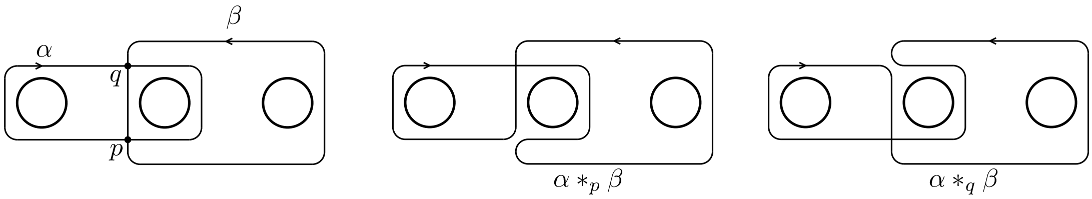

Figure 5.1: Example of the Goldman bracket. In this figure, $[\alpha, \beta] = \alpha *_p \beta - \alpha *_q \beta$.

1 The Goldman-Turaev Lie bialgebra

Let $\Sigma = \Sigma_{g,n+1}$ be an oriented surface of genus g with $n+1$ boundary components. We fix a framing (that is, a trivialization of the tangent bundle) of Σ and choose a base point $* \in \partial\Sigma$. We denote by $\pi = \pi_1(\Sigma, *)$ the fundamental group of Σ and define $\mathfrak{g}^{(g,n+1)} = \mathbb{Q}[S^1, \Sigma] = \mathbb{Q}\pi/[\mathbb{Q}\pi, \mathbb{Q}\pi]$ to be the vector space spanned by homotopy classes of free loops. When no confusion arises, we shorten the notation to $\mathfrak{g}$.

The vector space $\mathfrak{g}^{(g,n+1)}$ carries a canonical Lie bialgebra structure defined in terms of intersections of loops. The Lie bracket on $\mathfrak{g}^{(g,n+1)}$ is called the *Goldman bracket* [4] and is defined as follows. Let α and β be loops on Σ whose intersections are transverse double points. Then, the Lie bracket $[\alpha, \beta]$ is given by

$$[\alpha, \beta] = \sum_{p \in \alpha \cap \beta} \varepsilon_p \, \alpha *_p \beta,$$

where $\varepsilon_p \in \{\pm 1\}$ is the local intersection number of α and β at p, and $\alpha *_p \beta$ is the homotopy class of the concatenation of the loops α and β based at p (see Fig. 1). The Lie cobracket on $\mathfrak{g}^{(g,n+1)}$ is called the *Turaev cobracket* [13] and is defined as follows. Let γ be a loop on Σ. By a suitable homotopy, we can deform γ to an immersion with transverse double points whose rotation number with respect to the framing of Σ is zero. For each self-intersection p of γ, one can divide γ into two branches γ_p^1 and γ_p^2, where the pair of the tangent vectors of γ_p^1 and γ_p^2 forms a positive basis for $T_p\Sigma$. Then, the Lie cobracket $\delta(\gamma)$ is given by

$$\delta(\gamma) = \sum_p \gamma_p^1 \otimes \gamma_p^2 - \gamma_p^2 \otimes \gamma_p^1,$$

where the sum is taken over all the self-intersections of γ (see Fig. 2).

The group algebra $\mathbb{Q}\pi$ carries a canonical filtration with the following property. Choose a set of generators $\alpha_i, \beta_i, \gamma_j \in \pi$ with $i = 1, \ldots, g, j = 1, \ldots, n$ such that

$$\prod_{i=1}^{g} [\alpha_i, \beta_i] \prod_{j=1}^{n} \gamma_j = \gamma_0,$$

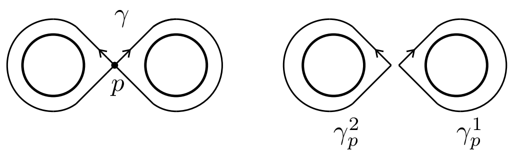

Figure 5.2: Example of the Turaev cobracket. In this figure, $\delta(\gamma) = \gamma_p^1 \otimes \gamma_p^2 - \gamma_p^2 \otimes \gamma_p^1$.

where γ_0 is the homotopy class of the boundary component which contains $*$. Then, the elements $(\alpha_i - 1), (\beta_i - 1) \in \mathbb{Q}\pi$ are of filtration degree 1 and $(\gamma_j - 1) \in \mathbb{Q}\pi$ have filtration degree 2. This filtration on π induces a two step filtration on $H = H_1(\Sigma, \mathbb{Q})$ with $H^{(1)} = H$ and $H^{(2)}$ the kernel of the intersection pairing. The associated graded of $\widehat{\mathbb{Q}\pi}$ with respect to this filtration is the complete Hopf algebra given by $\operatorname{gr}\widehat{\mathbb{Q}\pi} \cong T(\operatorname{gr} H) = U(\mathrm{L}(\operatorname{gr} H))$, where $T(\operatorname{gr} H)$ is the completed tensor algebra of the graded vector space $\operatorname{gr} H$ and $U(\mathrm{L}(\operatorname{gr} H))$ is the completed universal enveloping algebra of the free Lie algebra $\mathrm{L}(\operatorname{gr} H)$.

Let $\hat{\mathfrak{g}}^{(g,n+1)}$ be the completion of $\mathfrak{g}^{(g,n+1)}$. The completed associated graded vector space $\operatorname{gr}\mathfrak{g}$ can be canonically identified with the space of formal series in cyclic tensor powers of the graded vector space $\operatorname{gr} H = H/H^{(2)} \oplus H^{(2)}$. A choice of a basis in π induces a basis in $\operatorname{gr} H : x_i, y_i, z_j$ with $i = 1, \dots, g, j = 1, \dots n$, where x_i, y_i are of degree 1 and z_j are of degree 2. The graded vector space $\operatorname{gr}\mathfrak{g}$ carries a canonical Lie bialgebra structure induced by the Goldman-Turaev Lie bialgebra structure on $\mathfrak{g}$. It turns out that both the Lie bracket and Lie cobracket on $\operatorname{gr}\mathfrak{g}$ are of degree (-2).

Let $\mathcal{M}(\Sigma)$ be the mapping class group of the surface Σ fixing the boundary $\partial\Sigma$ pointwise. There are a subset $\mathcal{M}(\Sigma)^\circ \subset \mathcal{M}(\Sigma)$ and an embedding $\tau : \mathcal{M}(\Sigma)^\circ \to \hat{\mathfrak{g}}^{(g,n+1)}$ such that any Dehn twist is in $\mathcal{M}(\Sigma)^\circ$, $\mathcal{M}(\Sigma_{g,1})$ includes the Torelli group and the graded quotient of τ is the classical Johnson homomorphism [7]. In [6], the second and third authors proved that $\delta \circ \tau = 0 : \mathcal{M}(\Sigma)^\circ \to \hat{\mathfrak{g}}\hat{\otimes}\hat{\mathfrak{g}}$. This is one of the motivations to study the graded version of the Goldman-Turaev Lie bialgebra and the corresponding formality problem.

Definition 1. A *group-like expansion* is an isomorphism $\theta : \widehat{\mathbb{Q}\pi} \to \operatorname{gr}\widehat{\mathbb{Q}\pi}$ of complete filtered Hopf algebras with the property $\operatorname{gr}\theta = \mathrm{Id}$.

It is easy to see that group-like expansions exist. For instance, every choice of a basis in π described above induces a group-like expansion $\theta^{\exp}$ defined by its values on generators:

$$\theta^{\exp}(\alpha_i) = e^{x_i}, \quad \theta^{\exp}(\beta_i) = e^{y_i}, \quad \theta^{\exp}(\gamma_j) = e^{z_j}.$$

In fact, group-like expansions are a torsor under the group of automorphisms of the complete Hopf algebra $\operatorname{gr}\widehat{\mathbb{Q}\pi} \cong T(\operatorname{gr} H)$ with associated graded the identity. That is, every group-like expansion is of the form $\theta = F \circ \theta^{\exp}$ for some $F \in \mathrm{Aut}(\mathrm{L}(\mathrm{gr}\, H))$. Furthermore, every group-like expansion defines an isomorphism of filtered vector spaces $\mathfrak{g} \to \mathrm{gr}\, \mathfrak{g}$ with associated graded the identity map.

Definition 2. A group-like expansion θ is called *homomorphic* if it induces an isomorphism of Lie bialgebras $\mathfrak{g} \to \mathrm{gr}\, \mathfrak{g}$.

It is easy to check that expansions $\theta^{\exp}$ are not homomorphic. Existence of homomorphic expansions is one of the main results of this paper. Our strategy is to reformulate the problem in terms of properties of the automorphism $F \in \mathrm{Aut}(\mathrm{L}(\mathrm{gr}\, H))$. This leads us to a generalization of the Kashiwara-Vergne problem in the theory of free Lie algebras.

2 The Kashiwara-Vergne problem in higher genus

Denote by $\mathrm{L}^{(g,n+1)} = \mathrm{L}(\mathrm{gr}\, H)$ the completed free Lie algebra in the generators x_i, y_i, z_j with $\deg x_i = \deg y_i = 1, \deg z_j = 2$. Define the completed graded Lie algebra of *tangential derivations* (in the sense of [1]):

$$\mathrm{tder}^{(g,n+1)} = \{u \in \mathrm{Der}^+(\mathrm{L}^{(g,n+1)}) \mid u(z_j) = [z_j, u_j] \text{ for some } u_j \in \mathrm{L}^{(g,n+1)}\},$$

This Lie algebra integrates to a pro-unipotent group

$$\mathrm{TAut}^{(g,n+1)} = \{F \in \mathrm{Aut}^+(\mathrm{L}^{(g,n+1)}) \mid F(z_j) = F_j^{-1} z_j F_j \text{ for some } F_j \in \exp(\mathrm{L}^{(g,n+1)})\}.$$

Also following [1], let $\mathrm{tr}^{(g,n+1)}$ denote the vector space of series in cyclic words in the x_i, y_i and z_j's and tr the natural projection from associative words to cyclic words. The space $\mathrm{tr}^{(g,n+1)}$ carries an action of $\mathrm{tder}^{(g,n+1)}$ coming from a natural action of $\mathrm{Der}^+(\mathrm{L}^{(g,n+1)})$. Recall the definition of the map $\partial_{x_i} : \mathrm{L}^{(g,n+1)} \to U(\mathrm{L}^{(g,n+1)})$ given by the formula

$$\begin{aligned}\alpha(x_1, \cdots, x_i + \epsilon\xi, \cdots, x_g, y_1, \cdots, y_g, z_1, \cdots z_n)\\ = \alpha + \epsilon\, \mathrm{ad}(\partial_{x_i}\alpha)\, \xi + O(\epsilon^2) \quad \text{for } \alpha \in \mathrm{L}^{(g,n+1)},\end{aligned}$$

and the same definition for y_i and z_i. One defines the non-commutative divergence map $\mathrm{div} : \mathrm{tder}^{(g,n+1)} \to \mathrm{tr}^{(g,n+1)}$ as follows:

$$u \longmapsto \sum_{i=1}^{g} \mathrm{tr}(\partial_{x_i}(u(x_i)) + \partial_{y_i}(u(y_i))) + \sum_{j=1}^{n} \mathrm{tr}(z_j \partial_{z_j}(u_j)),$$

where u_j is chosen such that it has no linear terms in z_j. In the case of $n = 0$, the divergence coincides with the Enomoto-Satoh obstruction [2] for surjectivity of the Johnson homomorphism.

Proposition 3. The map div is a Lie algebra 1-cocycle on $\mathrm{tder}^{(g,n+1)}$ with values in $\mathrm{tr}^{(g,n+1)}$. It integrates to a group 1-cocycle $j : \mathrm{TAut}^{(g,n+1)} \to \mathrm{tr}^{(g,n+1)}$.

Let $r(s)$ be the power series $\log(\frac{s}{e^s-1}) \in \mathbb{Q}[[s]]$, and let $\mathbf{r} = \sum_{i=1}^g \mathrm{tr}(r(x_i)+r(y_i)) \in \mathrm{tr}^{(g,n+1)}$. Below we define a new family of *Kashiwara-Vergne problems* associated to a surface of genus g with $n+1$ boundary components.

Definition 4 (KV Problem of type $(g, n+1)$). Find an element $F \in \mathrm{TAut}^{(g,n+1)}$ such that

$$F(\sum_{i=1}^{g}[x_i, y_i] + \sum_{j=1}^{n} z_j) = \log(\prod_{i=1}^{g}(e^{x_i}e^{y_i}e^{-x_i}e^{-y_i}) \prod_{j=1}^{n} e^{z_j}) =: \xi \qquad (\mathrm{KVI}^{(g,n+1)})$$

$$j(F) = \sum_{i=1}^{n} \mathrm{tr}\, h(z_i) - \mathrm{tr}\, h(\xi) - \mathbf{r} \quad \text{for some Duflo function } h \in \mathbb{Q}[[s]]. \qquad (\mathrm{KVII}^{(g,n+1)})$$

Let $\mathrm{Sol\,KV}^{(g,n+1)}$ denote the set of solutions of the KV problem of type $(g, n+1)$. Note that $\mathrm{KV}^{(0,3)}$ is the classical KV problem [5] in the formulation of [1]. The following is the first main result of this note:

Theorem 5. Let $F \in \mathrm{Sol\,KV}^{(g,n+1)}$. Then, the group-like expansion $F^{-1} \circ \theta^{\exp}$ is homomorphic. Moreover, if $F \in \mathrm{TAut}^{(g,n+1)}$ satisfies $(\mathrm{KV\,I}^{(g,n+1)})$, then $F^{-1} \circ \theta^{\exp}$ is homomorphic if and only if $F \in \mathrm{Sol\,KV}^{(g,n+1)}$.

This result holds true for the framing adapted to the set of generators $\alpha_i, \beta_i, \gamma_j \in \pi$. A framing on Σ can be specified by the values of its rotation number function on simple closed curves which are freely homotopic to $\alpha_i, \beta_i, \gamma_j$. The adapted framing takes value 0 on α_i, β_i, and -1 on γ_j. Other framings require more detailed discussion. Our proof uses the theory of van den Bergh double brackets [14], their moment maps [10] and their relation to the Goldman bracket [9].

3 Solving $\mathrm{KV}^{(g,n+1)}$

Our second main result is the following theorem:

Theorem 6. For all $g \geq 0, n \geq 0$, $\mathrm{Sol\,KV}^{(g,n+1)} \neq \varnothing$.

Together, Theorem 5 and Theorem 6 imply existence of homomorphic expansions for any g and n (for $g = 0$, an independent proof was given by G. Massuyeau [8]). Among other things, it follows that the obstruction to surjectivity of the Johnson homomorphism provided by the Turaev cobracket is equivalent to the Enomoto-Satoh obstruction.

In what follows, we sketch a proof of this statement. Let us denote the variables appearing in the definition of $\mathrm{tder}^{(g_1+g_2,n_1+n_2+1)}$ by x_i^1, y_i^1, z_j^1 and x_i^2, y_i^2, z_j^2 respectively and define the following map

$$\begin{aligned} \mathcal{P} : \mathrm{tder}^{(0,3)} &\longrightarrow \mathrm{tder}^{(g_1+g_2,n_1+n_2+1)} \\ (u_1, u_2) &\longmapsto \Big(w^k \mapsto [w^k, u_k(\phi_1, \phi_2)] \Big) \text{, with } w^k \in \{x_i^k, y_i^k, z_j^k\}, k = 1, 2, \end{aligned}$$

where

$$\phi_1 = \sum_i [x_i^1, y_i^1] + \sum_j z_j^1, \quad \phi_2 = \sum_i [x_i^2, y_i^2] + \sum_j z_j^2.$$

This map is a Lie algebra homomorphism. It lifts to a group homomorphism $\mathrm{TAut}^{(0,3)} \to \mathrm{TAut}^{(g_1+g_2,n_1+n_2+1)}$ (also denoted by $\mathcal{P}$). Denote by $t \in \mathrm{tder}^{(0,3)}$ the tangential derivation $t : z_1 \mapsto [z_1, z_2], z_2 \mapsto [z_2, z_1]$. Recall that for $F \in \mathrm{Sol\,KV}^{0,3}$ there is a family of solutions $F_\lambda = F \exp(\lambda t)$ for $\lambda \in \mathbb{Q}$.

Proposition 7. Let $F_1 \in \mathrm{Sol\,KV}^{(g_1,n_1+1)}, F_2 \in \mathrm{Sol\,KV}^{(g_2,n_2+1)}, F \in \mathrm{Sol\,KV}^{(0,3)}$ such that their Duflo functions coincide, $h_1 = h_2 = h$. If $n_1 = n_2 = 0$ or $g_1 = 0$ or $g_2 = 0$, then there is $\lambda \in \mathbb{Q}$ such that

$$\tilde{F} := (F_1 \times F_2) \circ \mathcal{P}(F_\lambda) \in \mathrm{Sol\,KV}^{(g_1+g_2,n_1+n_2+1)},$$

and the corresponding Duflo function $\tilde{h} = h_1 = h_2 = h$.

Proposition 7 reduces the proof of Theorem 6 to the cases $\mathrm{KV}^{(0,3)}$ and $\mathrm{KV}^{(1,1)}$. By [1], the problem $\mathrm{KV}^{(0,3)}$ does admit solutions. The remaining case is thus $(g, n+1) = (1, 1)$. Let $\mathrm{TAut}^{(0,3)}_{z_1-z_2} \subset \mathrm{TAut}^{(0,3)}$ denote the subgroup which preserves $z_1 - z_2$ up to quadratic terms. One defines the following group homomorphism $\mathrm{TAut}^{(0,3)}_{z_1-z_2} \to \mathrm{TAut}^{(1,1)}$:

$$F \longmapsto F^{\mathrm{ell}} : \begin{cases} e^{x_1} \mapsto F_1(\psi_1, \psi_2)^{-1} e^{x_1} F_2(\psi_1, \psi_2) \\ e^{y_1} \mapsto F_2(\psi_1, \psi_2)^{-1} e^{y_1} F_2(\psi_1, \psi_2), \end{cases} \quad \text{where } \psi_1 = e^{x_1} y_1 e^{-x_1},\ \psi_2 = -y_1.$$

Furthermore, let $\varphi \in \mathrm{TAut}^{(1,1)}$ be an automorphism defined by $\varphi(x_1) = x_1,\ \varphi(y_1) = \frac{e^{\mathrm{ad}_{x_1}} - 1}{\mathrm{ad}_{x_1}}(y_1)$.

Proposition 8. Let $F \in \mathrm{Sol\,KV}^{(0,3)}$, then there is a unique $\lambda \in \mathbb{Q}$, such that $(Fe^{\lambda t})^{\mathrm{ell}} \circ \varphi \in \mathrm{Sol\,KV}^{(1,1)}$.

The proof of Proposition 8 is based on the results of [3]. Together with Proposition 7, it settles in the positive the existence issue for solutions of Kashiwara-Vergne problems $\mathrm{KV}^{(g,n+1)}$. We now turn to the uniqueness problem. Recall the notation $\phi = \sum_i [x_i, y_i] + \sum_j z_j$.

Definition 9. The Kashiwara-Vergne Lie algebra $\mathfrak{krv}^{(g,n+1)}$ is defined as

$$\mathfrak{krv}^{(g,n+1)} = \left\{ u \in \mathrm{tder}^{(g,n+1)} \,\middle|\, \begin{array}{l} u(\phi) = 0 \\ \mathrm{div}(u) = \sum_j \mathrm{tr}\, h(z_j) - \mathrm{tr}\, h(\phi) \text{ for some } h \in \mathbb{Q}[[s]] \end{array} \right\}.$$

Note that there are other definitions of Kashiwara-Vergne Lie algebras in the literature, see [11, 12] for an alternative definition of $\mathfrak{krv}^{(1,1)}$ based on the theory of moulds and [15] for a graph theoretic definition for arbitrary manifolds. At this point, we do not know what is the relation of these approaches to our considerations.

The pro-nilpotent Lie algebra $\mathfrak{krv}^{(g,n+1)}$ integrates to a group $\mathrm{KRV}^{(g,n+1)}$ which acts freely and transitively on the set of solutions of the Kashiwara-Vergne problem $\mathrm{KV}^{(g,n+1)}$, $G : F \mapsto FG$. Of particular interest are the group $\mathrm{KRV}^{(1,1)}$ and the Lie algebra $\mathfrak{krv}^{(1,1)}$:

$$\mathfrak{krv}^{(1,1)} = \{u \in \mathrm{tder}^{(1,1)} = \mathrm{der}^+ L(x, y); u([x, y]) = 0, \mathrm{div}(u) = -\mathrm{tr}\, h([x, y])\}.$$

The following result gives partial information on its structure:

Proposition 10. There is an injective Lie homomorphism of the Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}_1$ into $\mathfrak{krv}^{(1,1)}$.

Proposition 11. The elements $\delta_{2n} \in \mathrm{Der}^+ L(x, y), n = 1, \ldots$ uniquely defined by conditions $\delta_{2n}([x, y]) = 0, \delta_{2n}(x) = \mathrm{ad}_x^{2n}(y)$ belong to $\mathfrak{krv}^{(1,1)}$.

In [3], it is conjectured that $\mathfrak{grt}_1$ together with δ_{2n}'s form a generating set for the elliptic Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}_{\mathrm{ell}}$. In view of Propositions 10 and 11, we conjecture that $\mathfrak{grt}_{\mathrm{ell}}$ injects in $\mathfrak{krv}^{(1,1)}$.

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Chapter 6

Goldman-Turaev formality from the Knizhnik-Zamolodchikov connection

Anton Alekseev, Florian Naef

Abstract: For an oriented 2-dimensional manifold Σ of genus g with n boundary components the space $\mathbb{C}\pi_1(\Sigma)/[\mathbb{C}\pi_1(\Sigma), \mathbb{C}\pi_1(\Sigma)]$ carries the Goldman-Turaev Lie bialgebra structure defined in terms of intersections and self-intersections of curves. Its associated graded Lie bialgebra (under the natural filtration) is described by cyclic words in $H_1(\Sigma)$ and carries the structure of a necklace Schedler Lie bialgebra. The isomorphism between these two structures in genus zero has been established in [13] using Kontsevich integrals and in [2] using solutions of the Kashiwara-Vergne problem.

In this note we give an elementary proof of this isomorphism over $\mathbb{C}$. It uses the Knizhnik–Zamolodchikov connection on $\mathbb{C}\backslash\{z_1, \dots z_n\}$. We show that the isomorphism naturally depends on the complex structure on the surface. The proof of the isomorphism for Lie brackets is a version of the classical result by Hitchin [9]. Surprisingly, it turns out that a similar proof applies to cobrackets.

Furthermore, we show that the formality isomorphism constructed in this note coincides with the one defined in [2] if one uses the solution of the

Kashiwara-Vergne problem corresponding to the Knizhnik-Zamolodchikov associator.

1 Holonomy maps

In this section we recall the definition of the Knizhnik-Zamolodchikov connection and define the holonomy map using iterated integrals.

1.1 The Knizhnik-Zamolodchikov connection

For $n \in \mathbb{Z}_{\geq 1}$ let $\mathfrak{t}_{n+1}$ be the Drinfeld-Kohno Lie algebra of infinitesimal braids. It has generators $t_{ij} = t_{ji}$ with $i \neq j$ for $i, j = 1, \ldots, n+1$ and relations $[t_{ij}, t_{kl}] = 0$ for all quadruples of distinct labels i, j, k, l and $[t_{ij} + t_{ik}, t_{jk}] = 0$ for all triples i, j, k. The Lie subalgebra of $\mathfrak{t}_{n+1}$ generated by $a_i = t_{i(n+1)}$ for $i = 1, \ldots, n$ is a free Lie algebra with n generators.

There is a canonical flat Knizhnik-Zamolodchikov (KZ) connection on the configuration space $\mathrm{Conf}_{n+1}(\mathbb{C})$ of $n+1$ points $z_1, \ldots, z_{n+1} \in \mathbb{C}$ with values in $\mathfrak{t}_{n+1}$:

$$d + A_{\mathrm{KZ}} = d + \frac{1}{2\pi i} \sum_{i<j} t_{ij} \, d\log(z_i - z_j).$$

The fiber of the forgetful map $\mathrm{Conf}_{n+1}(\mathbb{C}) \to \mathrm{Conf}_n(\mathbb{C})$ (by forgetting the point $z = z_{n+1}$) over $(z_1, \ldots, z_n) \in \mathrm{Conf}_n(\mathbb{C})$ is given by $\Sigma = \mathbb{C} \setminus \{z_1, \ldots, z_n\}$. The restriction of the KZ connection to the fiber is of the form:

$$d + A = d + \frac{1}{2\pi i} \sum_{i=1}^{n} a_i \, d\log(z - z_i). \tag{1}$$

Note that the 1-forms $\frac{1}{2\pi i} \, d\log(z - z_i)$ form a basis in the cohomology $H^1(\Sigma)$. Then, we can naturally view a_i's as a basis of $H_1 = H_1(\Sigma)$. In fact, this is a natural basis of H_1 given by the cycles around the marked points. It is convenient to use the notation

$$A(z) = \frac{1}{2\pi i} \sum_{i=1}^{n} a_i d\log(z - z_i) \in \Omega^1(\Sigma) \otimes H_1.$$

1.2 Free algebras and iterated integrals

The degree completed Hopf algebra TH_1 over $\mathbb{C}$ is naturally isomorphic to the completed universal enveloping algebra of the free Lie algebra $\mathbb{L}(a_1, \ldots, a_n)$ with generators $a_1, \ldots, a_n$. Group-like elements in TH_1 form a group $G_n = \exp(\mathbb{L}(a_1, \ldots, a_n))$

isomorphic to $\mathbb{L}(a_1, \ldots, a_n)$ equipped with the group law defined by the Baker-Campbell-Hausdorff series.

For a path γ parametrized by $z : [0,1] \to \Sigma$, we define the holonomy of the connection $d + A$ using interated integrals:

$$\begin{array}{rcl} \mathrm{Hol}_\gamma & = & \sum_{k=0}^{\infty} \int_{1 \geq t_1 \geq \cdots \geq t_k \geq 0} A(z(t_1)) \ldots A(z(t_k)) \\ & = & \sum_{\mathbf{i}} a_{i_1} \ldots a_{i_k} \int_\gamma \frac{dz(t_1)}{z(t_1) - z_{i_1}} \circ \cdots \circ \frac{dz(t_k)}{z(t_k) - z_{i_k}} \quad \in \quad G_n, \end{array}$$

where $\mathbf{i} = (i_1, \ldots, i_k)$. Since the connection $d + A$ is flat, the holonomy Hol_γ is independent of the homotopy deformations of γ with fixed end points. Hence, for small deformations of γ one can view Hol_γ as a functions of its endpoints. The de Rham differential of this function is given by

$$d\,\mathrm{Hol}_\gamma = A(z(1))\,\mathrm{Hol}_\gamma - \mathrm{Hol}_\gamma\,A(z(0)).$$

Let γ be a closed path and consider the path $\gamma(s)$ which starts at $z(s)$, follows γ and ends at $z(s)$. The corresponding holonomy is denoted by $\mathrm{Hol}_\gamma(s) = \mathrm{Hol}_{\gamma(s)}$ and its de Rham differential has the form

$$d\,\mathrm{Hol}_\gamma(s) = A(z(s))\,\mathrm{Hol}_\gamma - \mathrm{Hol}_\gamma\,A(z(s)). \tag{2}$$

For a closed path γ and two points $s \neq t \in S^1$ we denote by $\gamma(t \leftarrow s)$ the oriented path starting at $z(s)$, following γ and ending at $z(t)$. The corresponding holonomy is denoted $\mathrm{Hol}_\gamma(t \leftarrow s) = \mathrm{Hol}_{\gamma(t \leftarrow s)}$ and its de Rham differential is given by

$$d\,\mathrm{Hol}_\gamma(t \leftarrow s) = A(z(t))\,\mathrm{Hol}_\gamma(t \leftarrow s) - \mathrm{Hol}_\gamma(t \leftarrow s)\,A(z(s)).$$

Choose a base point $p \in \Sigma$ and denote by $\pi_1 = \pi_1(\Sigma, p)$ the fundamental group of Σ with base point p. Let γ be a closed path based at p. The map

$$W : \gamma \mapsto \mathrm{Hol}_\gamma$$

descends to a group homomorphism $\pi_1 \to G_n$ and induces an isomorphism of completed Hopf algebras $\mathbb{C}\pi_1 \to TH_1$.

Denote by

$$|\mathbb{C}\pi_1| = \mathbb{C}\pi_1 / [\mathbb{C}\pi_1, \mathbb{C}\pi_1]$$

the space spanned by conjugacy classes in π_1 and similarly

$$|TH_1| = TH_1 / [TH_1, TH_1]$$

the space spanned by cyclic words in H_1. Note that $|\mathbb{C}\pi_1|$ is also isomorphic to the vector space spanned by free homotopy classes of loops in Σ. The map W induces a map $|W| : |\mathbb{C}\pi_1| \to |TH_1|$ which is independent of the base point p. In what follows we will study properties of this map.

Remark 6.1. For higher genus surfaces, iterated integrals of the harmonic Magnus connection were studied in [10]. It is not known whether the harmonic Magnus expansion gives rise to a Goldman-Turaev formality map.

2 Goldman-Turaev formality

The space $|\mathbb{C}\pi_1|$ carries the canonical Goldman-Turaev Lie bialgebra structure which depends on the framing of Σ. Moreover, it is canonically filtered by powers of the augmentation ideal of the group algebra $\mathbb{C}\pi_1$. The space $|TH_1|$ carries the necklace Schedler Lie bialgebra structure (see [14]) which depends on the choice of a basis in H_1. The main result of this note is an elementary proof of the following theorem:

Theorem 6.2. *The map $|W| : |\mathbb{C}\pi_1| \to |TH_1|$ is an isomorphism of Lie bialgebras for the Goldman-Turaev Lie bialgebra structure on (completed) $|\mathbb{C}\pi_1|$ defined by the blackboard framing and the necklace Schedler Lie bialgebra structure on $|TH_1|$ defined by the natural basis $\{a_1, \ldots, a_n\} \subset H_1$.*

2.1 Kirillov-Kostant-Souriau double bracket

Recall the Kirillov-Kostant-Souriau (KKS) double bracket (in the sense of van den Bergh [16]) on TH_1 which is completely determined by its values on generators:

$$a_i \otimes a_j \mapsto \{a_i, a_j\} = \delta_{ij}(1 \otimes a_i - a_i \otimes 1) = \delta_{ij}\,(1 \wedge a_i).$$

One of the key observations is the following lemma (reinterpreting [6]):

Lemma 6.3.

$$\{A(z), A(w)\} = \frac{1}{2\pi i}(1 \wedge (A(z) - A(w)))\, d\log(z-w)$$

Proof. The proof is by the direct computation:

$$\begin{aligned}
\{A(z), A(w)\} &= \frac{1}{(2\pi i)^2}\,\{\sum_i a_i d\log(z-z_i), \sum_j a_j d\log(w-z_j)\}\\
&= \frac{1}{(2\pi i)^2}\,\sum_i (1 \wedge a_i) d\log(z-z_i) d\log(w-z_i)\\
&= \frac{1}{(2\pi i)^2}\,\sum_i (1 \wedge a_i)(d\log(z-z_i) - d\log(w-z_i)) d\log(z-w)\\
&= \frac{1}{2\pi i}(1 \wedge (A(z) - A(w))) d\log(z-w).
\end{aligned}$$

□

2.2 Variations of the holonomy map

In order to proceed we will need some notation from non-commutative differential geometry. Let $\partial_i : TH_1 \to TH_1 \otimes TH_1$ for $i = 1, \ldots, n$ denote double derivations with the property $\partial_i a_j = \delta_{ij}(1 \otimes 1)$. They induce maps (denoted by the same letter) $\partial_i : |TH_1| \to TH_1$, and by composition $\partial_i \partial_j : |TH_1| \to TH_1 \otimes TH_1$. In what follows we will use the formulas for the first and second derivatives of the elements $|W_\gamma|$ for γ a closed path parametrized by a map $z : S^1 \to \Sigma$:

$$\begin{aligned} \partial_j |W_\gamma| &= \int_{S^1} \mathrm{Hol}_\gamma(s)\, d\log(z(s) - z_j), \\ \partial_i \partial_j |W_\gamma| &= \int_{S_1 \times S_1} (\mathrm{Hol}_\gamma(t \leftarrow s) \otimes \mathrm{Hol}_\gamma(s \leftarrow t))\; d\log(z(s) - z_i) d\log(z(t) - z_j), \end{aligned} \tag{3}$$

where in the expression for the second derivative the torus $S^1 \times S^1$ is oriented by the volume form $dsdt$.

2.3 Necklace Schedler Lie bialgebra

We will also make use of the maps $\mathrm{Tr} : TH_1 \otimes TH_1 \to |TH_1|$ given by $\mathrm{Tr}(a \otimes b) = |ab|$ and $\mathrm{Tr}^{12} : TH_1 \otimes TH_1 \to |TH_1| \otimes |TH_1|$ which is the component-wise projection $\mathrm{Tr}^{12}(a \otimes b) = |a| \otimes |b|$. In terms of the KKS double bracket, the necklace Schedler Lie bialgebra structure looks as follows: for $\psi, \psi' \in |TH_1|$ we have

$$\begin{aligned} [\psi, \psi'] &= \sum_{ij} \mathrm{Tr}\left((\partial_i \psi \otimes \partial_j \psi')\{a_i, a_j\}\right) \\ \delta\psi &= \sum_{ij} \mathrm{Tr}^{12}\left((\partial_i \partial_j \psi)\{a_i, a_j\}\right). \end{aligned}$$

Remark 6.4. The necklace Lie algebra structure corresponding to the KKS double bracket first appeared in [5]. It was then generalized to other double brackets and other quivers (a quiver is a part of a general definition of a necklace Lie algebra) in [12, 7, 4]. The first description of the cobracket is in [14]. The formulas above represent the KKS necklace Schedler Lie bialgebra structure in a form convenient for our purposes.

2.4 Proof of Theorem 6.2

We are now ready to give a proof of Theorem 6.2:

Proof of the Lie homomorphism. Let γ and γ' be two closed loops in Σ with a finite number of transverse intersections parametrized by $z, w : [0, 1] \to \Sigma$. Denote by

(s_i, t_i) pairs of parameters corresponding to intersection points, $p_i = z(s_i) = w(t_i)$ and by $D_i(\varepsilon) \subset S^1 \times S^1$ small discs of radius ε around (s_i, t_i) with boundaries small circles $S_i(\varepsilon)$ positively oriented under the orientation defined by the volume form $dsdt$ on the torus $S^1 \times S^1$. Let $\epsilon_i = +1$ if the orientation induced by the form $dsdt$ on $T_{p_i}\Sigma$ coincides with the blackboard orientation and $\epsilon_i = -1$ otherwise.

We compute the necklace Lie bracket of the elements $|W_\gamma|$ and $|W_{\gamma'}|$:

$$
\begin{aligned}
&[|W_\gamma|, |W_{\gamma'}|] = \\
&= \operatorname{Tr}\left((\partial_i |W_\gamma| \otimes \partial_j |W_{\gamma'}|)\{a_i, a_j\}\right) \\
&= \int_{S^1\times S^1} \operatorname{Tr}\left((\operatorname{Hol}_\gamma(s) \otimes \operatorname{Hol}_{\gamma'}(t))\{A(z(s)), A(w(t))\}\right) \\
&= \frac{1}{2\pi i}\int_{S^1\times S^1} \operatorname{Tr}\Big(\Big(\operatorname{Hol}_\gamma(s)\otimes \operatorname{Hol}_{\gamma'}(t)\Big)\Big(1\wedge (A(z(s)) - A(w(t))\Big)\Big)\, \operatorname{dlog}(z-w) \\
&= \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{S^1\times S^1\setminus \cup_i D_i(\varepsilon)} d|\operatorname{Hol}_\gamma(s)\operatorname{Hol}_{\gamma'}(t)|\, \operatorname{dlog}(z-w) \\
&= -\sum_i |\operatorname{Hol}_\gamma(s_i)\operatorname{Hol}_{\gamma'}(t_i)| \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{S_i(\varepsilon)} d\,\log(z-w) \\
&= \sum_i |\operatorname{Hol}_\gamma(s_i)\operatorname{Hol}_{\gamma'}(t_i)|\epsilon_i.
\end{aligned}
$$

Here in the first line we use the definition of the necklace Schedler Lie bracket, in the second line the expression (3) for the first derivative of $|W_\gamma|$ and the definition of $A(z)$, in the third line Lemma 6.3, in the fourth line formula (2) for the differential of the holonomy map, in the fifth line the Stokes formula and in the sixth line the residue formula. Note that the result of the calculation is exactly the expression for the image under the map W of the Goldman bracket [8] of the loops γ and γ'.

□

Proof of the cobracket homomorphism. Next, consider a closed path γ parametrized by $z : S^1 \to \Sigma$. Assume that γ is an immersion with finitely many transverse self-intersections $p_i = z(s_i) = z(t_i)$ (it is convenient to have each point appear twice in the count with $s_j = t_i$ and $t_j = s_i$). Denote $\alpha^\pm(\varepsilon)$ the circles on $S^1 \times S^1$ corresponding to the parameters $t = s \pm \varepsilon$ and by β the strip between them. Similar to the proof above, let $D_i(\varepsilon)$ be small discs of radius ε around the self-intersection points, $S_i(\varepsilon)$ their positively oriented boundaries and ϵ_i signs of self-intersections defined as above. Note that $\epsilon_j = -\epsilon_i$ for i and j representing the same self-intersection point.

We compute the necklace cobracket of the element $|W_\gamma|$:

$$
\begin{aligned}
&\delta|W_\gamma| = \\
&= \mathrm{Tr}^{12}\left(\partial_i\partial_j|W_\gamma|\,\{a_i,a_j\}\right) \\
&= \int_{S^1\times S^1}\mathrm{Tr}^{12}(\mathrm{Hol}_\gamma(t\leftarrow s)\otimes\mathrm{Hol}_\gamma(s\leftarrow t))\{A(z(s)),A(z(t))\}) \\
&= \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{S^1\times S^1\backslash\beta(\varepsilon)\cup(\cup_i D_i(\varepsilon))} d(\mathrm{Tr}^{12}(\mathrm{Hol}_\gamma(t\leftarrow s)\otimes\mathrm{Hol}_\gamma(s\leftarrow t))\,\mathrm{dlog}(z(s)-z(t)) \\
&= -\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\cup_i S_i(\varepsilon)\cup\alpha^+(\varepsilon)\cup\alpha^-(\varepsilon)}(|\mathrm{Hol}(t\leftarrow s)|\otimes|\mathrm{Hol}(s\leftarrow t)|)\,\mathrm{dlog}(z(s)-z(t)) \\
&= \sum_i|\mathrm{Hol}(t_i\leftarrow s_i)|\otimes|\mathrm{Hol}(s_i\leftarrow t_i)|\epsilon_i+(1\wedge|W_\gamma|)\frac{1}{2\pi i}\int_\gamma d\,\log(\dot z(t)) \\
&= \sum_i|\mathrm{Hol}(t_i\leftarrow s_i)|\otimes|\mathrm{Hol}(s_i\leftarrow t_i)|\epsilon_i+(|W_\gamma|\wedge|1|)\,\mathrm{rot}(\gamma).
\end{aligned}
$$

Here in the first line we use the definition of the necklace Schedler cobracket, in the second line the expression (3) for the second derivative of $|W_\gamma|$, in the third line Lemma 6.3 and the expression (2) for the de Rham differential of the holonomy map, in the fourth line the Stokes formula and in the fifth line the definition of the rotation number $\mathrm{rot}(\gamma)$. The resulting expression is exactly the Turaev cobracket of the path γ with respect to the blackboard framing which defines the rotation number. □

Remark 6.5. In the proof above, the calculation of the Goldman bracket of the holonomies follows the standard scheme, see [9], [3]. The calculation of the Turaev cobracket uses the same techniques, but it seems to be new.

Remark 6.6. In [15], the cobracket is defined on the space $|\mathbb{C}\pi_1|/\mathbb{C}|1|$, where $|1|$ is the homotopy class of the trivial loop on Σ. If one fixes a framing on Σ, this definition can be lifted to $|\mathbb{C}\pi_1|$ by the formula above (for details, see [11]).

3 The z-dependence of the holonomy map

In this section, we study the dependence of the holonomy map $|W|:|\mathbb{C}\pi_1|\to|TH_1|$ on the positions of the poles $z_1,\ldots,z_n$ in the connection (1). We will use the notation $|W^{(z)}|$ to make this z-dependence more explicit.

3.1 The bundle $\mathcal{G}_n$

Consider the natural fibration $\mathrm{Conf}_{n+1}\to\mathrm{Conf}_n$ and assign to a point of the base $(z_1,\ldots.z_n)$ the Goldman-Turaev Lie algebra of the fiber $|\mathbb{C}\pi_1(\Sigma)|$ for $\Sigma=\mathbb{C}\backslash\{z_1,\ldots,z_n\}$. This defines a bundle of Lie bialgebras $\mathcal{G}_n$ over Conf_n.

In a neighborhood of a point $z = (z_1, \ldots, z_n) \in \mathrm{Conf}_n$ the bundle $\mathcal{G}_n$ can be trivialized as follows. Choose small open disks D_i centered at z_i such that $D_i \cap D_j = \emptyset$ for all $i \neq j$ and define a local chart $U^{(z)}$ around z which consists of points $w = (w_1, \ldots, w_n) \in \mathrm{Conf}_n$ such that $w_i \in D_i$. Note that the surface with boundary $\tilde{\Sigma} = \mathbb{C} \backslash \cup_i D_i$ is homotopically equivalent to $\Sigma^{(w)} = \mathbb{C} \backslash \{w_1, \ldots, w_n\}$ for all $w \in U^{(z)}$ via the inclusion $\tilde{\Sigma} \subset \Sigma^{(w)}$. The canonical isomorphism $|\mathbb{C}\pi_1(\tilde{\Sigma})| \cong |\mathbb{C}\pi_1(\Sigma^{(w)})|$ defines a trivialization of $\mathcal{G}_n$ in this chart.

The bundle $\mathcal{G}_n$ carries a canonical flat connection defined by the following family of local flat sections. Let $\gamma : S^1 \to \mathbb{C}$ be a closed curve. Then, the homotopy class $[\gamma]$ defines a flat section over the complement of the subset of Conf_n where $z_i \in \gamma(S^1)$ for some i. Moreover, it is easy to see that there is a unique local flat section passing through each point of $\mathcal{G}_n$.

The canonical flat connection is compatible with the Goldman-Turaev Lie bialgebra structure on the fibers. Indeed, on flat sections over $U^{(z)}$ defined by closed curves in $\tilde{\Sigma}$, the Goldman bracket and Turaev cobracket take the same value independent of w (sufficiently close to z).

The flat bundle $\mathcal{G}_n$ admits the following description: the fundamental group of Conf_n is the pure braid group PB_n. The bundle $\mathcal{G}_n$ is induced by the natural representation of PB_n on the free group $\pi_1(\Sigma)$.

3.2 The bundle $\mathcal{H}_n$

The natural action of PB_n on the homology $H_1(\Sigma)$ is trivial. This trivial action lifts to $|TH_1|$ and gives rise to the trivial vector bundle $\mathcal{H}_n \to \mathrm{Conf}_n$ with fiber $|TH_1|$.

The bundle $\mathcal{H}_n$ carries a canonical flat connection. Recall that the Drinfeld-Kohno Lie algebra $\mathfrak{t}_n$ acts on the free Lie algebra $\mathbb{L}(a_1, \ldots, a_n)$ via the identification $a_i = t_{i(n+1)} \in \mathfrak{t}_{n+1}$. Since this action is induced by the adjoint action of $\mathfrak{t}_{n+1}$ on itself, we will use the notation ad. For instance, $\mathrm{ad}(t_{12})a_1 = [t_{12}, t_{1(n+1)}] = -[t_{2(n+1)}, t_{1(n+1)}] = -[a_2, a_1]$. In particular, the central element of $\mathfrak{t}_n$

$$c = \sum_{i<j} t_{ij}$$

acts by the inner derivation $\mathrm{ad}(c)a = [a, a_1 + \cdots + a_n]$.

The canonical connection on $\mathcal{H}_n$ is (a version of) the KZ connection:

$$\nabla_n = d + \frac{1}{2\pi i} \sum_{i<j} \mathrm{ad}(t_{ij}) d \log(z_i - z_j). \tag{4}$$

It is flat because the original KZ connection is flat and because the map ad is a Lie algebra homomorphism.

Remark 6.7. The image of the Lie algbera $\mathfrak{t}_n$ under the map ad is contained in the Kashiwara-Vergne Lie algebra $\mathfrak{krv}_n$ acting on $|TH_1|$ by derivations of the necklace Schedler Lie bialgebra structure [2]. Hence, the flat connection (4) is compatible with the Lie bialgebra structure on the fibers of $\mathcal{H}_n$.

3.3 The holonomy is an isomorphism of flat bundles with connection

For each $z = (z_1, \ldots, z_n) \in \mathrm{Conf}_n$ the holonomy map $|W^{(z)}|$ maps the fiber $|\mathbb{C}\pi_1(\Sigma^{(z)})|$ of $\mathcal{G}_n$ to the fiber $|TH_1|$ of $\mathcal{H}_n$. This collection of maps for $z \in \mathrm{Conf}_n$ defines a smooth bundle map $|W| : \mathcal{G}_n \to \mathcal{H}_n$.

Theorem 6.8. *The bundle map $|W|$ intertwines the canonical flat connection on $\mathcal{G}_n$ and the connection ∇_n on $\mathcal{H}_n$.*

Proof. It is sufficient to compare the canonical connections and ∇_n on local charts. In more detail, let $\gamma : S^1 \to \tilde{\Sigma} = \sigma \backslash \cup_i D_i$ be a closed curve. Then, $[\gamma]$ defines a flat section of $\mathcal{G}_n$ over $U^{(z)}$. It is then sufficient to show that $|W_\gamma^{(w)}|$ is a ∇_n-flat section of $\mathcal{H}_n$ for $w \in U^{(z)}$.

Let $w : (-1, 1) \to \mathrm{Conf}_n(\mathbb{C})$ be a smooth path in $U^{(z)}$ with $w(0) = z$ and the first derivative $w'(0) = u = (u_1, \ldots, u_n)$. Choose a starting point on the curve γ and denote the corresponding map by $\widehat{\gamma} : [0, 1] \to \tilde{\Sigma}$.

Consider the KZ connection on $\mathrm{Conf}_{n+1}(\mathbb{C})$ and define the map $\mu = w \times \widehat{\gamma} : (-1, 1) \times S^1 \to \mathrm{Conf}_{n+1}(\mathbb{C})$. The pull-back of the KZ connection under μ is flat. For a given $s \in (-1, 1)$ the holonomy of the induced connection along $\{s\} \times \widehat{\gamma}$, denoted by $\mathrm{Hol}_\gamma(s)$, takes values in TH_1 and its projection to $|TH_1|$ coincides with $|W^{(w(s))}|$ (under the identification $t_{i(n+1)} = a_i$).

Since the pull-back connection is flat, the holonomies for different values of s can be compared using the following formula:

$$\mathrm{Hol}_\gamma(s) = \mathrm{Hol}_w(s \leftarrow 0)\mathrm{Hol}_\gamma(0)\mathrm{Hol}_w(0 \leftarrow s),$$

where $\mathrm{Hol}_w(0 \leftarrow s) = \mathrm{Hol}_w(s \leftarrow 0)^{-1}$ is the holonomy of the KZ connection for $n + 1$ points along the part of the path w between 0 and s. Differentiating this formula in s at $s = 0$, we obtain

$$\mathrm{Hol}'_\gamma(0) = \frac{1}{2\pi i} \sum_{1 \leq i < j \leq n+1} \frac{u_i - u_j}{z_i - z_j} \mathrm{ad}(t_{ij})\mathrm{Hol}_\gamma(0),$$

where $u_{n+1} = 0$ and $z_{n+1} = \hat{\gamma}(0)$. Projecting this equation to $|TH_1|$ yields

$$(\partial_s |W^{(w(s))}|)_{s=0} = \frac{1}{2\pi i} \sum_{1 \leq i < j \leq n} \frac{u_i - u_j}{z_i - z_j} \mathrm{ad}(t_{ij})|W^{(w(0))}|.$$

Here we have used the fact that operators $\mathrm{ad}(t_{i(n+1)})$ are inner derivations of TH_1 and they act by zero on $|TH_1|$. The formula above shows that $|W_\gamma^{(w(s))}|$ is a flat section over $(-1,1)$. Since $w(s)$ was an arbitrary path, this shows that $|W_\gamma^{(w)}|$ is a flat section for ∇_n. □

Remark 6.9. The KZ connection on $\mathcal{H}_n$ is induced by the flat connection on the trivial bundle $\mathcal{L}_n = \mathrm{Conf}_n \times \mathbb{L}(a_1, \ldots, a_n)$ defined by the same formula (4).

Let $\mathfrak{g}$ be a Lie algebra (possibly $\mathbb{L}(a_1, \ldots, a_n)$ itself) and consider the bundle of fiberwise Lie algebra homomorphisms of $\mathcal{L}_n$ into $\mathfrak{g}$. This is the trivial bundle $\mathrm{Conf}_n \times \mathfrak{g}^n$, where we identify a Lie algebra homomorphism $x : \mathbb{L}(a_1, \ldots, a_n) \to \mathfrak{g}$ with the n-tuple $(x_1, \ldots, x_n) = (x(a_1), \ldots, x(a_n))$. The KZ connection induces a connection on this bundle which is computed as follows:

$$\begin{aligned}
((d + A_{\mathrm{KZ}})x)(a_k) &= dx_k + \frac{1}{2\pi i}\sum_{i<j}(t_{ij}.x)(a_k)d\log(z_i - z_j) \\
&= dx_k + \frac{1}{2\pi i}\sum_{i<j} x(-[t_{ij}, a_k])d\log(z_i - z_j) \\
&= dx_k + \frac{1}{2\pi i}\sum_{i} x([a_i, a_k])d\log(z_i - z_k) \\
&= dx_k + \frac{1}{2\pi i}\sum_{i} [x_i, x_k]d\log(z_i - z_k).
\end{aligned}$$

Flat sections of this connection are solutions of the Schlesinger equations for isomonodromic deformations:

$$\frac{\partial x_k}{\partial z_i} = \frac{1}{2\pi i}\frac{[x_k, x_i]}{z_k - z_i}, \quad \frac{\partial x_k}{\partial z_k} = -\frac{1}{2\pi i}\sum_{i \neq k}\frac{[x_k, x_l]}{z_k - z_l}. \tag{5}$$

For $\mathfrak{g} = \mathbb{L}(a_1, \ldots, a_n)$, let $x_k = x_k(a_1, \ldots, a_n, z_1, \ldots, z_n) = x_k(z)$ be local solutions to (5), such that they generate $\mathbb{L}(a_1, \ldots, a_n)$ in each fiber. We call $(x_1, \ldots, x_n)$ isomonodromic coordinates.

Let $\gamma \subset \Sigma^{(z)}$ be a closed curve. It defines a local flat section $[\gamma]$ of $\mathcal{G}_n$ and by Theorem 6.8 it gives rise to a local flat section $|W_\gamma|$ of $\mathcal{H}_n$. By the considerations above, $|W_\gamma|$ is represented by a constant (independent of $z_1, \ldots, z_n$) function of the isomonodromic coordinates $(x_1, \ldots, x_n)$.

3.4 The moduli space of curves

Recall that the moduli space of genus zero curves with $n+1$ marked points $\mathcal{M}_{0,n+1}$ can be defined as the quotient of the configuration space of $n+1$ points on $\mathbb{C}P^1$

modulo the natural PSL_2-action by Möbius transformations:

$$\mathcal{M}_{0,n+1} = \mathrm{Conf}_{n+1}(\mathbb{C}P^1)/\mathrm{PSL}_2.$$

Equivalently, one can fix one of the points to be at infinity of $\mathbb{C}P^1$ and consider:

$$\mathcal{M}_{0,n+1} \cong \mathrm{Conf}_n(\mathbb{C})/\Gamma,$$

where Γ is the group of translations and dilations

$$\Gamma = \{z \mapsto az + b; a \in \mathbb{C}^*, b \in \mathbb{C}\}.$$

The action of the group Γ lifts to the bundles $\mathcal{G}_n$ and $\mathcal{H}_n$ as follows. The bundle $\mathcal{H}_n = \mathrm{Conf}_n \times |TH_1|$ is trivial and one extends the action of Γ on Conf_n by the trivial action on the fibers. For the bundle $\mathcal{G}_n$, the group Γ acts by diffeomorphisms $\Sigma^{(z)} \to \Sigma^{(az+b)}$, where $(az + b) = (az_1 + b, \ldots, az_n + b)$. The action of Γ on $\mathcal{G}_n$ is the induced action on the homotopy classes of curves

$$[\gamma(s)] \mapsto [\gamma_{a,b}(s) = a\gamma(s) + b].$$

In conclusion, $\mathcal{G}_n$ and $\mathcal{H}_n$ become equivariant vector bundles under the action of Γ and they give rise to vector bundles over $\mathcal{M}_{0,n+1}$ (that we again denote by $\mathcal{G}_n$ and $\mathcal{H}_n$).

The canonical flat connection on $\mathcal{G}_n$ descends from Conf_n to $\mathcal{M}_{0,n+1}$ because the action of Γ maps flat sections to flat sections. It turns out that the KZ connection on $\mathcal{H}_n$ also descends to the moduli space:

Proposition 6.10. *The flat connection*

$$\nabla_n = d + \frac{1}{2\pi i} \sum_{i<j} \mathrm{ad}(t_{ij}) d \log(z_i - z_j)$$

descends to a flat connection on $\mathcal{M}_{0,n+1}$.

Proof. It is obvious that the connection 1-form is basic under the action of the group of translations $z \mapsto z + b$ and invariant under the action of dilations $z \mapsto \lambda z$. In order to see that it is horizontal under the action of dilations, define the Euler vector field $v = \sum_{i=1}^n z_i \partial_i$. Contracting it with the connection 1-form yields

$$\iota(v) \sum_{i<j} \mathrm{ad}(t_{ij}) d \log(z_i - z_j) = \sum_{i<j} \mathrm{ad}(t_{ij}) = \mathrm{ad}(c) = 0,$$

where we have used the fact that c acts by zero on $|TH_1|$. $\square$

Finally, the holonomy map descends to a bundle map over the moduli space as well:

Proposition 6.11. *The holonomy map $|W|$ descends to $\mathcal{M}_{0,n+1}$.*

Proof. The group Γ acts on the complex plane by diffeomorphisms and the connection

$$A^{(z_1,\dots,z_n)} = \frac{1}{2\pi i}\sum_{i=1}^{n} a_i d\log(z - z_i)$$

is mapped to $A^{(az+b)}$. Hence, the holonomy map $W^{(z)}$ evaluated on a curve $\gamma : [0,1] \to \Sigma^{(z)}$ coincides with the holonomy map $W^{(az+b)}$ evaluated on the curve $\gamma_{a,b} : [0,1] \to \Sigma^{(az+b)}$. Applying projection $TH_1 \to |TH_1|$, we conclude that $|W|$ intertwines the actions of Γ on $\mathcal{G}_n$ and $\mathcal{H}_n$.

□

We can summarize the statements above as follows:

Theorem 6.12. *The flat bundles with connections $\mathcal{G}_n$ and $\mathcal{H}_n$ and the bundle map $|W|$ over the configuration space* Conf_n *descend to the moduli space of genus zero curves $\mathcal{M}_{0,n+1}$.*

Remark 6.13. By Remark 6.7, the pure braid group PB_n (the fundamental group of the configuration space Conf_n) acts on $|TH_1|$ by automorphisms of the necklace Schedler Lie bialgebra structure. Theorem 6.12 shows that this action only depends on the complex structure on the surface (that is, on the point of $\mathcal{M}_{0,n+1}$).

4 Relation to the KZ associator and the Kashiwara-Vergne problem

In this section we establish the relation between the map $|W| : |\mathbb{C}\pi_1| \to TH_1$ and the solution of the Kashiwara-Vergne problem defined by the KZ associator.

Recall the following statements:

Theorem 6.14 (Theorem 5 in [1])**.** *For every Drinfeld associator $\Phi(x,y)$ the automorphism F_Φ of the free Lie algebra $\mathcal{L}(x,y)$ defined by formulas*

$$x \mapsto \Phi(x,-x-y)x\Phi(x,-x-y)^{-1},$$
$$y \mapsto e^{-(x+y)/2}\Phi(y,-x-y)y\Phi(y,-x-y)^{-1}e^{(x+y)/2}$$

is a solution of the Kashiwara-Vergne problem.

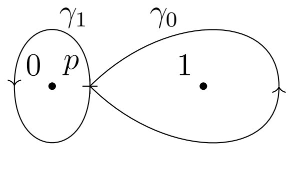

Figure 6.1: Definition of γ_0 and γ_1

For the second statement we need the following notation: let $\Sigma = \mathbb{C}\backslash\{0,1\}$, choose a base point $p \in [0,1]$ and two generators $\gamma_0, \gamma_1 \in \pi_1$ as shown on Fig. 6.1.

Theorem 6.15 (Theorem 7.6 in [2])**.** *Let F be an automorphism of the free Lie algebra $\mathcal{L}(x,y)$ which solves the Kashiwara-Vergne problem. Then, the map $\rho_F : \mathbb{C}\pi_1 \to TH_1$ defined on generators by*

$$\rho_F(\gamma_0) = \exp(F(x)), \quad \rho_F(\gamma_1) = \exp(F(y))$$

induces an isomorphism of Lie bialgebras $|\mathbb{C}\pi_1| \to |TH_1|$.

We will now prove the following proposition:

Proposition 6.16. *For $\Sigma = \mathbb{C}\backslash\{0,1\}$ the Lie bialgebra isomorphism $|W|$ coincides with the one induced by the solution of the Kashiwara-Vergne problem corresponding to the KZ associator $\Phi_{\mathrm{KZ}}(x,y)$.*

Let $n = 2$ and $z_1 = 0, z_2 = 1$. Denote $x = a_1, y = a_2$. The connection $d + A$ acquires the form

$$d + A = d + \frac{1}{2\pi i}\left(x\, d\, \log(z) + y\, d\, \log(z-1)\right).$$

Following Drinfeld [5], define fundamental solutions of the equation $(d + A)\Psi = 0$ with asymptotics

$$\Psi_0(z) = (1 + O(z))z^{\frac{x}{2\pi i}}, \quad \Psi_1(z) = (1 + O(z))(z-1)^{\frac{y}{2\pi i}}.$$

Choose the base point $p \in [0,1]$ and a basis in π_1 represented by the curves γ_0 which surrounds 0 and γ_1 which surrounds 1. Define the representation $\rho_{\mathrm{KZ}} : \pi_1 \to G_2$ as follows:

$$\rho_{\mathrm{KZ}}(\gamma_0) = e^x, \quad \rho_{\mathrm{KZ}}(\gamma_1) = \Phi_{\mathrm{KZ}}(x,y)^{-1} e^y \Phi_{\mathrm{KZ}}(x,y),$$

where Φ_{KZ} is the KZ associator.

Lemma 6.17. *The representations ρ_{KZ} and W are equivalent under conjugation by $\Psi_0(p)$. The corresponding maps $|\mathbb{C}\pi_1| \to TH_1$ coincide.*

Proof. By definition of the holonomy,

$$\Psi_0(\gamma_0 \cdot p) = \mathrm{Hol}_{\gamma_0} \Psi_0(p), \quad \Psi_1(\gamma_1 \cdot p) = \mathrm{Hol}_{\gamma_1} \Psi_1(p).$$

In combination with

$$\Psi_0(\gamma_0 \cdot p) = \Psi_0(p)e^x, \quad \Psi_1(\gamma_1 \cdot p) = \Psi_1(p)e^y, \quad \Psi_1(p) = \Psi_0(p)\Phi_{\mathrm{KZ}}(x, y)$$

we obtain

$$\rho_{\mathrm{KZ}}(\gamma_0) = e^x = \Psi_0(p)^{-1}\Psi_0(\gamma_0 \cdot p) = \Psi_0(p)^{-1}\mathrm{Hol}_{\gamma_0}\Psi_0(p)$$

and

$$\begin{aligned}\rho_{\mathrm{KZ}}(\gamma_1) &= \Phi_{\mathrm{KZ}}^{-1} e^y \Phi_{\mathrm{KZ}} = \Phi_{\mathrm{KZ}}^{-1}\Psi_1(p)^{-1}\Psi_1(\gamma_1 \cdot p)\Phi_{\mathrm{KZ}} \\ &= (\Psi_1(p)\Phi_{\mathrm{KZ}})^{-1}\mathrm{Hol}_{\gamma_1}(\Psi_1(p)\Phi_{\mathrm{KZ}}) = \Psi_0(p)^{-1}\mathrm{Hol}_{\gamma_1}\Psi_0(p),\end{aligned}$$

as required.

Since the two representations of π_1 are equivalent, they descend to the same map on $|\mathbb{C}\pi_1|$.

□

Proof of Proposition. To complete the proof of Proposition, we remark that the representation ρ_{KZ} is equivalent to the representation

$$\begin{aligned}\rho(\gamma_0) &= \Phi(x, -x-y)e^x\Phi(x, -x-y)^{-1}, \\ \rho(\gamma_1) &= e^{-(x+y)/2}\Phi(y, -x-y)e^y\Phi(y, -x-y)^{-1}e^{(x+y)/2}.\end{aligned}$$

The equivalence is given by conjugation with $e^{x/2}\Phi(-x-y, x)$ which corresponds to moving the base point from the neighborhood of 0 to the neighborhood of ∞.

The map $|\mathbb{C}\pi_1| \to |TH_1|$ induced by ρ coincides with the one induced by the solution of the Kashiwara-Vergne problem corresponding to the KZ associator $\Phi_{\mathrm{KZ}}(x, y)$. □

Remark 6.18. G. Massuyeau explained to us that for $n = 2$ the Lie bialgebra isomorphism $|\mathbb{C}\pi_1| \to |TH_1|$ constructed in [13] using the Kontsevich integral coincides with the map described above if one chooses the associator $\Phi = \Phi_{\mathrm{KZ}}$.

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Chapter 7

Chern-Simons, Wess-Zumino and other cocycles from Kashiwara-Vergne and associators

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To the memory of Ludwig Faddeev

Abstract: Descent equations play an important role in the theory of characteristic classes and find applications in theoretical physics, *e.g* in the Chern-Simons field theory and in the theory of anomalies. The second Chern class (the first Pontrjagin class) is defined as $p = \langle F, F \rangle$ where F is the curvature 2-form and $\langle \cdot, \cdot \rangle$ is an invariant scalar product on the corresponding Lie algebra $\mathfrak{g}$. The descent for p gives rise to an element $\omega = \omega_3 + \omega_2 + \omega_1 + \omega_0$ of mixed degree. The 3-form part ω_3 is the Chern-Simons form. The 2-form part ω_2 is known as the Wess-Zumino action in physics. The 1-form component ω_1 is related to the canonical central extension of the loop group LG.

In this paper, we give a new interpretation of the low degree components ω_1 and ω_0. Our main tool is the universal differential calculus on free Lie algebras due to Kontsevich. We establish a correspondence between solutions of the first Kashiwara-Vergne equation in Lie theory and universal solutions of the descent equation for the second Chern class p. In more detail, we define a 1-cocycle C which maps automorphisms of the free Lie algebra to one forms. A solution of the Kashiwara-Vergne

equation F is mapped to $\omega_1 = C(F)$. Furthermore, the component ω_0 is related to the associator Φ corresponding to F. It is surprising that while F and Φ satisfy the highly non-linear twist and pentagon equations, the elements ω_1 and ω_0 solve the linear descent equation.

1 Motivation: descent equations

Le G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let $P \to M$ be a principal G-bundle with connection $A \in \Omega^1(P, \mathfrak{g})$. Without loss of generality, one can assume that G is a matrix Lie group (by Ado Theorem, G always admits a faithful representation). Then, gauge transformations can be written in the form

$$A \mapsto A^g = g^{-1}Ag + g^{-1}dg, \tag{1}$$

and the curvature of A is defined by

$$F = dA + \frac{1}{2}[A, A] = dA + A^2.$$

Polynomials of A and F form the Weil algebra $W\mathfrak{g}$ (for details see *e.g.* [9]). The defining equation of the curvature F and the Bianchi identity give rise to the definition of the Weil differential:

$$d_W A = F - \frac{1}{2}[A, A], \quad d_W F = -[A, F].$$

Elements of $W\mathfrak{g}$ basic under the G-action (that is, G-invariant and horizontal) give rise to differential forms on the total space of the bundle P which descend to the base M. Moreover, it turns out that such forms are automatically closed and gauge invariant. Their cohomology classes in $H^\bullet(M, \mathbb{R})$ are characteristic classes of the bundle P.

Assume that the Lie algebra $\mathfrak{g}$ carries an invariant scalar product $\langle \cdot, \cdot \rangle$. Then, the element of $W\mathfrak{g}$

$$p = \langle F, F \rangle$$

is basic. Its cohomology class is the second Chern class (or the first Pontrjagin class) of P if we choose a rescaled inner product with a suitable coefficient. In $W\mathfrak{g}$, the element p admits a primitive:

$$p = d\,\mathrm{CS},$$

where $\mathrm{CS} \in W\mathfrak{g}$ is the Chern-Simons form

$$\mathrm{CS} = \langle A, dA \rangle + \frac{1}{3}\langle A, [A, A] \rangle. \tag{2}$$

Note that p does not admit a basic primitive within $W\mathfrak{g}$. This is why the corresponding characteristic class is non vanishing in general. However, if one removes the requirement of being basic, the Weil algebra is acyclic and all closed elements of non vanishing degree admit primitives.

The Chern-Simons form (2) plays a major role in applications to Quantum Topology and Quantum Field Theory (QFT). In this paper, we will study its version in the complex

$$\mathcal{C}^n = \oplus_{i+j+k=n} W^i\mathfrak{g} \otimes \Omega^j(G^k). \tag{3}$$

Here $W^i\mathfrak{g}$ is the subspace of $W\mathfrak{g}$ spanned by the elements of degree i, and G^k is a direct product of k copies of G. The graded vector space $C = \oplus_n C^n$ admits a differential

$$D = d_W + d_{dR} + \Delta,$$

where d_W is the Weil differential on $W\mathfrak{g}$, d_{dR} is the de Rham differential on $\Omega(G^k)$, $\deg_{\mathrm{dR}}$ the de Rham degree $i + j$, and Δ is the group cohomology differential given by

$$\begin{aligned}&(\Delta\alpha)(A, g_1, \ldots, g_{k+1})\\ &\quad = \alpha(A^{g_1}, g_2, \ldots, g_{k+1}) - \alpha(A, g_1g_2, \ldots, g_{k+1}) + \cdots + (-1)^{k+1}\alpha(A, g_1, \ldots, g_k).\end{aligned}$$

Here $\alpha \in W\mathfrak{g} \otimes \Omega(G^k)$, and the notation $\alpha(A, g_1g_2, \ldots, g_{k+1})$ stands for the pull-back of α under the co-face map $G^{k+1} \to G^k$ defined by

$$(g_1, g_2, g_3, \ldots, g_{k+1}) \mapsto (g_1g_2, g_3, \ldots, g_{k+1}).$$

In the complex C, we can ask again for the primitive of the element p. That is, for a solution of the equation $p = D\omega$. Solutions of this equation serve to define the Chern-Simons functional on simplicial spaces, and also in the theory of anomalies in QFT (see [7], [8], [19], [10]).

If we denote $d = d_W + d_{dR}$, we obtain a system of equations known as descent equations:

$$\begin{array}{rcl} d\omega_3 &=& p,\\ -\Delta\omega_3 + d\omega_2 &=& 0,\\ \Delta\omega_2 + d\omega_1 &=& 0,\\ -\Delta\omega_1 + d\omega_0 &=& 0,\\ \Delta\omega_0 &=& 0. \end{array} \tag{4}$$

Here ω_i is an element of $W\mathfrak{g} \otimes \Omega(G^{3-i})$ of degree i (the sum of the Weil degree in $W\mathfrak{g}$ and the de Rham degree in $\Omega(G^{3-i})$. In particular, $\omega_3 \in W\mathfrak{g}$ and the first equation reads

$$d_W\omega_3 = p.$$

Hence, we conclude that $\omega_3 = \mathrm{CS}$. In order to understand the next equation, we write

$$\Delta\omega_3 = \mathrm{CS}(A^g) - \mathrm{CS}(A) = d\langle A, g^{-1}dg\rangle - \frac{1}{6}\langle g^{-1}dg, [g^{-1}dg, g^{-1}dg]\rangle.$$

Finding the primitive of this expression under the differential d depends on vanishing of the cohomology class of the Cartan 3-form on G:

$$\eta = \frac{1}{6}\langle g^{-1}dg, [g^{-1}dg, g^{-1}dg]\rangle.$$

The cohomology class $[\eta]$ is non vanishing in general. In particular, it is non vanishing on all semisimple compact Lie groups. Hence, in this case descent equations admit no solution. There are several ways to address this difficulty. For QFT applications, one should replace differential forms on G by differential characters, and consider exponentials of periods of these forms.

In a geometric setting, one realises p on a specific G-principal bundle P over a manifold X. The geometric data corresponding to p is the Chern-Simon 2-gerbe. When $p(P)$ vanishes, one is expected to find trivialisations of the Chern-Simon 2-gerbe, which are known as String structures over P [16]. As shown in [14], descent equations (4) govern the connection data of String principal 2-bundles.

In this paper, we choose another approach: we replace the group G by the corresponding formal group G_{formal}. Since formal manifolds are modeled on one chart (which is itself a formal vector space), the cohomology of G_{formal} is trivial and the cohomology class of the corresponding Cartan 3-form vanishes. In more detail, the exponential map

$$\exp : \mathfrak{g}_{\mathrm{formal}} \to G_{\mathrm{formal}}$$

establishes an isomorphism between G_{formal} and its Lie algebra. The primitive of the Cartan 3-form can be computed using the Poincaré homotopy operator h_P:

$$\eta = dh_P(\eta).$$

In order to make this approach more precise, we will use a version of the Kontsevich universal differential calculus. This leads to an (almost) unique solution for the component ω_2 of the extended Chern-Simons form which is called the Wess-Zumino action in the physics literature:

$$\mathrm{WZ}(A, g) = \langle A, dgg^{-1}\rangle - \frac{1}{6}\, h_P(\langle g^{-1}dg, [g^{-1}dg, g^{-1}dg]\rangle).$$

The goal of this paper is to give an interpretation of the components ω_1 and ω_0 of the extended Chern-Simons form. In more detail, we will define a 1-cocycle

C mapping the group of tangential automorphisms of the free Lie algebra with n generators TAut_n (see the next sections for a precise definition) to the space of universal 1-forms $\Omega^1(G^n_{\mathrm{formal}})$. We will then establish the following surprising relation between certain special elements in TAut_2 and TAut_3 and universal solutions of descent equations.

It turns out that the equation $\delta\omega_2 + d\omega_1 = 0$, with ω_2 given by the Wess-Zumino action, admits solutions of the form $\omega_1 = C(g)$, where $g \in \mathrm{TAut}_2$ with the property

$$g(x_1 + x_2) = \log(e^{x_1}e^{x_2}).$$

Here the right hand side is the Baker-Campbell-Hausdorff series. Furthermore, the next descent equation $\Delta\omega_1 + d\omega_0 = 0$ translates into the twist equation from the theory of quasi-Hopf algebras:

$$g^{1,2}g^{12,3} = g^{2,3}g^{1,23}\Phi^{1,2,3},$$

where $g^{1,2}$, $g^{12,3}$, etc. are images of g under various co-face maps, and $\Phi \in \mathrm{TAut}_3$ is related to ω_0 via $\omega_0 = (g^{2,3}g^{1,23}).h_P(C(\Phi))$. Last but not least, the descent equation $\Delta\omega_0 = 0$ translates into the pentagon equation for Φ:

$$\Phi^{12,3,4}\Phi^{1,2,34} = \Phi^{2,3,4}\Phi^{1,23,4}\Phi^{1,2,3}.$$

The structure of the paper is as follows: the second chapter contains a recollection of non-commutative differential calculus and the construction of the universal Bott-Shulman double complex ([4], [5], [15]) in the abelian and non-abelian cases. In particular, the relevant cohomology groups are computed and their relation with the classical Bott-Shulman complex is discussed. The third chapter contains a short discussion of the Kashiwara-Vergne theory and the construction of the 1-cocycle C which is the main new object in this note. After establishing some of its properties, we prove the formulas for ω_1 and ω_0.

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2 Universal calculus for gauge theory

2.1 Non-commutative differential calculus

Let $\mathbb{k}$ be a field of characteristic zero, and let $\mathrm{Lie}(x_1, \ldots, x_n)$ be a free graded Lie superalgebra with generators $x_1, \ldots, x_n$. We assume that the degrees $|x_i|$ are non-negative for all i. If all the degrees vanish, we use the shorthand notation Lie_n instead of $\mathrm{Lie}(x_1, x_2, \ldots, x_n)$. One considers $\mathrm{Lie}(x_1, x_2, \ldots, x_n)$ as a coordinate algebra of a non-commutative space. Note that this free Lie algebra carries two gradings: the first one is induced by the degrees of generators, the other one is obtained by counting the number of letters in a Lie word.

Following Drinfeld and Kontsevich, we define functions on this non-commutative space in the following way:

$$\begin{aligned} \mathcal{F}\langle x_1, \ldots, x_n\rangle &= \mathrm{Lie}(x_1, \ldots, x_n) \otimes \mathrm{Lie}(x_1, \ldots, x_n)/ \\ &\quad \mathrm{Span}\{a \otimes b - (-1)^{|a||b|} b \otimes a,\ a \otimes [b, c] - [a, b] \otimes c\}. \end{aligned}$$

The natural projection is denoted by $\langle \cdot, \cdot \rangle$. Again, we use the notation $\mathcal{F}_n$ when all generators have vanishing degree.

Example 1. The space $\mathcal{F}_1$ is a line with generator $\langle x_1, x_1\rangle$. The space $\mathcal{F}_2$ is infinite dimensional. In low degrees, we have the following elements which contain two letters: $\langle x_1, x_1\rangle$, $\langle x_2, x_2\rangle$ and $\langle x_1, x_2\rangle$. The space $\mathcal{F}_3$ contains a unique line of multilinear elements spanned by $\langle x_1, [x_2, x_3]\rangle$.

The space of differential forms is defined as functions on the shifted tangent bundle. More concretely,

$$\Omega\langle x_1, \ldots, x_n\rangle = \mathcal{F}\langle x_1, \ldots, x_n, dx_1, \ldots, dx_n\rangle,$$

where we have $|dx_i| = |x_i| + 1$. Equivalently, this is the space of functions on the differential envelope of $\mathrm{Lie}(x_1, \cdots, x_n)$ which is $\mathrm{Lie}(x_1, \ldots, x_n, dx_1, \ldots, dx_n)$ with obvious differential. The differential descends to $\Omega\langle x_1, \ldots, x_n\rangle$ making it into a complex of vector spaces.

Example 2. Let A be a generator of degree 1. The corresponding space of differential forms $\Omega\langle A\rangle = \mathcal{F}\langle A, dA\rangle$ will be referred to as the universal Weil algebra. Sometimes it is more convenient to use the following generator of degree 2

$$F := dA + \frac{1}{2}[A, A]$$

instead of dA. Its differential is given by $dF = -[A, F]$. The low degree elements of $\Omega\langle A\rangle$ are as follows. In degree 2: $\langle A, A\rangle$, in degree 3: $\langle A, F\rangle$ and $\langle A, [A, A]\rangle$, and in degree 4: $\langle F, F\rangle$ and $\langle F, [A, A]\rangle$.

Lemma 7.1. *The complex* $(\Omega\langle x_1, \ldots, x_k\rangle, d)$ *is acyclic for* $k \geq 1$.

Proof. Consider the derivation e of $\mathrm{Lie}(x_1, \ldots, x_k, dx_1, \ldots, dx_k)$ defined on generators as $e(dx_i) = x_i, e(x_i) = 0$. A direct calculation shows that $de + ed = n$, where n is the Euler derivation counting the number of generators in a Lie word. The derivations e and n descend to operators on $\Omega\langle x_1, \ldots, x_k\rangle$. If $\alpha \in \Omega\langle x_1, \ldots, x_k\rangle$ is a cocycle, we have $d(e(\alpha)) = n\alpha$. Since all elements of $\Omega\langle x_1, \ldots, x_k\rangle$ contain at least two generators, $n \neq 0$ and we obtain an explicit homotopy which provides primitives of all cocycles.

□

2.2 Universal abelian gauge transformations

In this section, we define a universal version of abelian connections and gauge transformations. Let A denote a generator of degree 1, and $x_1, \ldots, x_n$ generators of degree 0. We define a co-simplicial complex of the following form:

$$\Omega\langle A\rangle \overset{\delta_0,\delta_1}{\Longrightarrow} \Omega\langle A, x_1\rangle \overset{\delta_0,\delta_1,\delta_2}{\Longrightarrow} \Omega\langle A, x_1, x_2\rangle \ldots, \tag{5}$$

where the co-face maps are defined as follows:

$$\begin{array}{llll} \delta_0: & A \mapsto A + dx_1, & x_1 \mapsto x_2 & \ldots \quad x_n \mapsto x_{n+1}, \\ \delta_1: & A \mapsto A, & x_1 \mapsto x_1 + x_2 & \ldots \quad x_n \mapsto x_{n+1}, \\ & \vdots & & \\ \delta_n: & A \mapsto A, & x_1 \mapsto x_1 & \ldots \quad x_n \mapsto x_n + x_{n+1}, \\ \delta_{n+1}: & A \mapsto A, & x_1 \mapsto x_1 & \ldots \quad x_n \mapsto x_n. \end{array} \tag{6}$$

The requirement that co-face maps commute with the differential determines them on $dA, dx_1, \ldots, dx_n$. In particular, we have $\delta_i(dA) = dA$ for all i. The co-simplicial differential is defined by

$$\delta = \sum_i (-1)^i \delta_i = \delta_0 - \delta_1 + \delta_2 - \ldots$$

The differentials d and δ commute hence we get a double complex. Let $D = d + \delta$ be the differential on its total complex:

$$\Omega^\bullet\langle A, x_\bullet\rangle = \oplus_{n=0}^\infty \Omega^\bullet\langle A, x_1, \ldots, x_n\rangle. \tag{7}$$

Note that in $D = d + \delta$ there is a sign suppressed in the second term that depends on the de Rham degree. It will be clear from the context which sign applies.

Remark 7.2. A more geometric construction of the complex (5) can be obtained as follows. Consider the opposite category of (differential) graded Lie algebras as a category of (dg) spaces. Let $G = \mathrm{Lie}(x)$ be the free Lie algebra in one generator considered as an object in this category. There are two non-isomorphic group structures on G: the one induced by linear function $x_1 + x_2$ and the one induced by the Baker-Campbell-Hausdorff Lie series $\log(e^{x_1} e^{x_2})$. The first one turns G into an abelian group.

Let $T[1](-)$ denote the functor of taking universal differential envelopes. The space of left G-orbits in $T[1]G$ denoted by $V := G\backslash T[1]G$ inherits a right action of $T[1]G$, which we call the gauge action. One identifies $V = \mathrm{Lie}(A)$ for a generator A of degree 1, and the action is given by

$$A \mapsto A + dx$$

for the first group structure on G, and

$$A \mapsto e^{-x} A e^{x} + e^{-x} d(e^{x})$$

for the second one. This action lifts to $W = T[1]V$ as a right dg action. Functions on W form the universal Weil algebra.

Note that in the non-abelian case V carries a canonical Chevalley-Eilenberg differential $d_{\mathrm{CE}} : A \mapsto -\frac{1}{2}[A, A]$. Hence, $W = T[1]V$ carries the de Rham differential d and the twisted differential $d + d_{\mathrm{CE}}$ (which corresponds to the standard Weil differential). In this way, one gets the two geometrically equivalent coordinate systems on $T[1]V$: the first one is $\{A, dA\}$ and the second one is $\{A, F = dA + \frac{1}{2}[A, A]\}$.

The space $* = \mathrm{Lie}(\varnothing) = 0$ carries a trivial left $T[1]G$ action. The complex (7) is then given by functions on the two-sided bar construction of $T[1]G$ acting on $T[1]V$ and on $*$:

$$\mathcal{F}(B(T[1]V, T[1]G, *)).$$

The result is a simplicial dg-complex. Note that this description is completely categorical. In particular, if one replaces the category opposite to dg Lie algebras (which we were using above) by the category of dg manifolds, we recover the classical Bott-Shulman complex.

In the abelian case, the construction can be simplified:

$$\begin{aligned} B(T[1]W, T[1]G, *) &= B(\mathrm{Lie}(A, dA), \mathrm{Lie}(x, dx), *) \\ &= B(\mathrm{Lie}(A), \mathrm{Lie}(dx), *) \times \mathrm{Lie}(dA) \times B(*, \mathrm{Lie}(x), *), \end{aligned}$$

where the first factor is contractible and the last factor serves as a counterpart of BG. This analogy will be made more precise below.

In what follows we present several calculations for the complex $\Omega^\bullet\langle A, x_\bullet\rangle$ and study its properties.

Lemma 7.3. *The complex* $(\Omega^\bullet\langle A, x_\bullet\rangle, D = d + \delta)$ *is acyclic.*

Proof. Recall that the complex $(\Omega^\bullet\langle A, x_1, \ldots, x_n\rangle, d)$ is acyclic for all $n \geq 0$. We can compute the cohomology of the complex $(\Omega^\bullet\langle A, x_\bullet\rangle, D = d + \delta)$ by using the spectral sequence with the first page $H^{\geq 0}(\Omega^\bullet\langle A, x_1, \ldots, x_n\rangle, d) = 0$. Since the first page vanishes, the cohomology of the total complex vanishes as well.

□

The Lemma above makes use of the acyclicity of the universal de Rham complex (which we will consider as columns of the double complex). In what follows, we will also need information about the row direction given by the cosimplicial differential δ.

Lemma 7.4. *The injective chain map* $(\mathcal{F}\langle dA, x_\bullet\rangle, \delta) \to (\Omega\langle A, x_\bullet\rangle, \delta)$ *induces an isomorphism in cohomology.*

Proof. Note that $\Omega\langle A, x_\bullet\rangle$ is the diagonal part of the bi-cosimplicial complex

$$C_{m,n} = \mathcal{F}\langle A, dx_1, \ldots, dx_m, dA, x_1, \ldots, x_n\rangle,$$

where the coface maps of the first cosimplicial component act on generators $A, dx_1, \ldots, dx_m$ and of the second cosimplicial component on generators $dA, x_1, \ldots, x_n$. By the Eilenberg-Zilber Theorem [18, Section 8.5], the cosimplicial cohomology of the diagonal $\oplus_n C_{n,n}$ is isomorphic to the bi-cosimplicial cohomology of the total complex $\oplus_{m,n} C_{m,n}$ with differential $\delta = \delta' + \delta''$. Here δ' acts on generators $A, dx_1, \ldots, dx_m$ and δ'' acts on generators $dA, x_1, \ldots, x_n$. The following operator

$$(h\alpha)(A, dx_1, \ldots, dx_{m-1}, dA, x_1, \ldots, x_n) = \alpha(0, A, dx_1, \ldots, dx_{m-1}, dA, x_1, \ldots, x_n)$$

provides a homotopy between the identity and the projection to constant functions of A in $C_{0,\bullet}$ for the differential δ'. Hence, it defines a deformation retraction to the subcomplex $\mathcal{F}\langle dA, x_\bullet\rangle$ and the injection of $\mathcal{F}\langle dA, x_\bullet\rangle$ in $\Omega\langle A, x_\bullet\rangle$ induces an isomorphism in cohomology, as required. □

Let $\mathcal{H}\langle dA, x_1, \ldots, x_n\rangle \subset \mathcal{F}\langle dA, x_1 \ldots, x_n\rangle$ be the subspace spanned by the elements linear with respect to $x_1, \ldots, x_n$ and completely skew-symmetric under the action of the permutation group S_n.

Example 3. Here are some examples of elements in $\mathcal{H}\langle dA, x_\bullet\rangle$: $\langle dA, [x_1, x_2]\rangle$ and $\varphi = \langle x_1, [x_2, x_3]\rangle$. The class φ plays an important role the following section.

Lemma 7.5. *Elements $\alpha \in \mathcal{H}\langle dA, x_\bullet \rangle$ are δ-closed.*

Proof. We give an example of a calculation for $\mathcal{H}\langle dA, x_1 \rangle$:

$$\begin{aligned}(\delta\alpha)(dA, x_1, x_2) &= \alpha(dA, x_1) - \alpha(dA, x_1 + x_2) + \alpha(dA, x_2) \\ &= \alpha(dA, x_1) - \alpha(dA, x_1) - \alpha(dA, x_2) + \alpha(dA, x_2) \\ &= 0.\end{aligned}$$

Here we have used the linearity of α with respect to the argument x. The calculation works in the same way in higher degrees. □

Lemma 7.6. *The injective chain map $(\mathcal{H}\langle dA, x_1, ..., x_n \rangle, 0) \subset (\mathcal{F}\langle dA, x_1..., x_n \rangle, \delta)$ induces an isomorphism in cohomology.*

Proof. Standard (see [13], [3], [17], [6]). □

The lemma above implies that δ-closed elements in $\Omega\langle A \rangle$ must be functions of dA. Such a function is unique up to a multiple, and it is given by the abelian second Chern class:

$$p = \langle dA, dA \rangle.$$

Since the total double complex $(\Omega\langle dA, x_\bullet \rangle, D)$ is acyclic, one can ask for a primitive (the cochain of transgression) of the function p which is given by the following formula

$$\omega = \langle A, dA \rangle + \langle A, dx_1 \rangle - \langle x_1, dx_2 \rangle.$$

Since the differentials d and δ both preserve the number of letters, the primitive can be chosen in the same graded component as the class $\langle dA, dA \rangle$. The form $\langle A, dA \rangle$ is the abelian Chern-Simons element, and $\langle A, dx_1 \rangle$ is the abelian Wess-Zumino action, the expression $\langle x_1, dx_2 \rangle$ stands for the Kac-Peterson cocycle on the current algebra. The primitive ω is unique up to exact terms

$$\omega' = \omega + D(a_1 \langle A, x_1 \rangle + a_2 \langle x_1, x_2 \rangle + a_3 \langle x_1, x_1 \rangle + a_4 \langle x_2, x_2 \rangle),$$

where a_i's are arbitrary coefficients.

2.3 Non-abelian descent equations

In this section, we consider a more complicated cosimplicial structure on $\Omega\langle A, x_\bullet \rangle$ which captures the features of non-abelian gauge theory. In more detail, we define new coface maps:

$$
\begin{array}{llll}
\Delta_0: & A \mapsto e^{-x_1}Ae^{x_1} + e^{-x_1}de^{x_1}, & x_1 \mapsto x_2 & \dots \quad x_n \mapsto x_{n+1}, \\
\Delta_1: & A \mapsto A, & x_1 \mapsto \log(e^{x_1}e^{x_2}) & \dots \quad x_n \mapsto x_{n+1} \\
 & \vdots & & \\
\Delta_n: & A \mapsto A, & x_1 \mapsto x_1 & \dots \quad x_n \mapsto \log(e^{x_n}e^{x_{n+1}}), \\
\Delta_{n+1}: & A \mapsto A, & x_1 \mapsto x_1 & \dots \quad x_n \mapsto x_n.
\end{array}
\tag{8}
$$

Here the formulas

$$
\begin{aligned}
e^{-x_1}Ae^{x_1} &= \exp(-\mathrm{ad}_{x_1})A = A - [x_1, A] + \frac{1}{2}[x_1, [x_1, A]] - \dots, \\
e^{-x_1}de^{x_1} &= f(\mathrm{ad}_{x_1})dx_1, \quad \text{for } f(z) = \frac{1-e^{-z}}{z}
\end{aligned}
\tag{9}
$$

define the non-abelian gauge action and

$$\log(e^{x_1}e^{x_2}) = x_1 + x_2 + \frac{1}{2}[x_1, x_2] + \dots$$

is the Baker-Campbell-Hausdorff series. The cosimplicial differential is defined as the alternated sum of coface maps,

$$\Delta = \sum_i (-1)^i \Delta_i.$$

As before, the de Rham and cosimplicial differentials commute and let again $D = d + \Delta$ denote the total differential (with suppressed de Rham sign). The complex $(\Omega^\bullet\langle A, x_\bullet\rangle, D = d+\Delta)$ is again acyclic since it is acyclic under the de Rham differential d.

Recall that in the non-abelian framework it is convenient to use the generator

$$F = dA + \frac{1}{2}[A, A]$$

instead of dA since it has a nice transformation law under gauge transformations: $F \mapsto e^{-x_1}Fe^{x_1}$. In what follows we will be interested in cohomology of the cosimiplical differential Δ. It is convenient to introduce a decreasing filtration of the complex $\Omega\langle A, x_\bullet\rangle$ by the number of generators in the given expression (the elements of filtration degree k contain at least k generators). It is clear that the associated graded complex coincides with the cosimplicial complex for abelian gauge transformations.

Lemma 7.7. *Let $\alpha \in \Omega\langle A\rangle$ be a Δ-cocycle. Then, $\alpha = \langle F\rangle = \lambda\langle F, F\rangle$ for $\lambda \in \Bbbk$.*

Proof. Assume that $\Delta\alpha = 0$. Then its principal part α_{low} (containing the lowest number of generators) is a δ-cocycle. Hence, $\alpha_{\text{low}} = \lambda\langle dA, dA\rangle$ for some $\lambda \in k$. Note that $\alpha' = \lambda\langle F, F\rangle$ is a Δ-cocycle. By the argument above, the lowest degree part of $\alpha - \alpha'$ vanishes. Hence, $\alpha = \alpha' = \lambda\langle F, F\rangle$, as required. □

Lemma 7.8. *There is an element* $\phi = \langle x_1, [x_2, x_3]\rangle + \cdots \in \mathcal{F}\langle x_1, x_2, x_3\rangle$ *such that* $\Delta\phi = 0$. *Its cohomology class is the generator of the cohomology group* $H(\mathcal{F}\langle x_\bullet\rangle, \Delta) \cong k[\phi]$.

Proof. Consider the decreasing filtration on $\mathcal{F}\langle x_\bullet\rangle$ defined by the number of generators in the expression. Recall that the associated graded complex of $(\mathcal{F}\langle x_\bullet\rangle, \Delta)$ is $(\mathcal{F}\langle x_\bullet\rangle, \delta)$. By Lemma 7.6, the cohomology of the latter complex is spanned by the class of $\varphi = \langle x_1, [x_2, x_3]\rangle$. That is, $[\varphi]$ is the only class on the first page of the spectral sequence defined by the filtration. Hence, it cannot be killed at any later page and it lifts to a cohomology class which spans $H(\mathcal{F}\langle x_\bullet\rangle, \Delta)$. □

Recall that the element $\langle F, F\rangle$ is also a cocycle under the de Rham differential d. Hence, $DF = dF + \Delta F = 0$. Since the complex $(\Omega\langle A, x_\bullet\rangle, D)$ is acyclic, there is a primitive $\omega = \omega_3 + \omega_2 + \omega_1 + \omega_0$ such that $D\omega = \langle F, F\rangle = 0$, with $\omega_3 \in \Omega^3\langle A\rangle$, $\omega_2 \in \Omega^2\langle A, x_1\rangle$, $\omega_1 \in \Omega^1\langle A, x_1, x_2\rangle$, $\omega_0 \in \Omega^0\langle A, x_1, x_2, x_3\rangle$. The following diagram visualizes the lower left corner of the double complex that we are using:

$$
\begin{array}{ccccccccc}
\uparrow d & & \uparrow d & & & & & & \\
\Omega^3\langle A\rangle & \xrightarrow{\Delta} & \Omega^3\langle A, x_1\rangle & \xrightarrow{\Delta} & \dots & & & & \\
\uparrow d & & \uparrow d & & \uparrow d & & & & \\
\Omega^2\langle A\rangle & \xrightarrow{\Delta} & \Omega^2\langle A, x_1\rangle & \xrightarrow{\Delta} & \Omega^2\langle A, x_1, x_2\rangle & \xrightarrow{\Delta} & \dots & & \\
\uparrow d & & \uparrow d & & \uparrow d & & \uparrow d & & \\
\Omega^1\langle A\rangle & \xrightarrow{\Delta} & \Omega^1\langle A, x_1\rangle & \xrightarrow{\Delta} & \Omega^1\langle A, x_1, x_2\rangle & \xrightarrow{\Delta} & \Omega^1\langle A, x_1, x_2, x_3\rangle & \xrightarrow{\Delta} & \dots \\
\uparrow d & & \uparrow d & & \uparrow d & & \uparrow d & & \\
\Omega^0\langle A\rangle & \xrightarrow{\Delta} & \Omega^0\langle A, x_1\rangle & \xrightarrow{\Delta} & \Omega^0\langle A, x_1, x_2\rangle & \xrightarrow{\Delta} & \Omega^0\langle A, x_1, x_2, x_3\rangle & \xrightarrow{\Delta} & \dots
\end{array}
\tag{10}
$$

which simplifies by degree reasons to

$$
\begin{array}{ccccccccc}
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & & & & & \\
\Omega^3\langle A\rangle & \xrightarrow{\Delta} & \Omega^3\langle A, x_1\rangle & \xrightarrow{\Delta} & \dots & & & & \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & & & \\
0 & \xrightarrow{\Delta} & \Omega^2\langle A, x_1\rangle & \xrightarrow{\Delta} & \Omega^2\langle A, x_1, x_2\rangle & \xrightarrow{\Delta} & \dots & & \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
0 & \xrightarrow{\Delta} & \mathbb{k}\langle dx_1, x_1\rangle \oplus \mathbb{k}\langle A, x_1\rangle & \xrightarrow{\Delta} & \Omega^1\langle A, x_1, x_2\rangle & \xrightarrow{\Delta} & \Omega^1\langle A, x_1, x_2, x_3\rangle & \xrightarrow{\Delta} & \dots \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
0 & \xrightarrow{\Delta} & \mathbb{k}\langle x_1, x_1\rangle & \xrightarrow{\Delta} & \mathcal{F}\langle x_1, x_2\rangle & \xrightarrow{\Delta} & \mathcal{F}\langle x_1, x_2, x_3\rangle & \xrightarrow{\Delta} & \dots
\end{array}
\tag{11}
$$

The main result of this section is the following theorem:

Theorem 7.9. *Let $\omega_1 \in \Omega^1\langle A, x_1, x_2\rangle$ such that $d\Delta\omega_1 = 0$. Then, there exists a unique element $\omega = \omega_0 + \omega_1 + \omega_2 + \omega_3$ such that $D\omega = \lambda\langle F, F\rangle$ for some $\lambda \in \mathbb{k}$. Moreover, there exists an element ω_1 which yields $\lambda = 1$.*

To prove this result, we need the following lemma:

Lemma 7.10. *The complex $(\Omega^k\langle A, x_\bullet\rangle, \Delta)$ is exact for k odd and for $k = 2$.*

Proof. Recall that the associated graded of the complex $(\Omega^k\langle A, x_\bullet\rangle, \Delta)$ with respect to filtration defined by the number of generators yields the complex $(\Omega^k\langle A, x_\bullet\rangle, \delta)$. By Lemma 7.4, this latter complex is acyclic for k odd. Hence, so is the complex $(\Omega^k\langle A, x_\bullet\rangle, \Delta)$. Among other things, this implies that the third row is exact in $\Omega^3\langle A, x_1\rangle$.

For $k = 2$, the associated graded complex has non-trivial cohomology. The non-trivial cohomology classes are represented by $\langle dA, x_1\rangle \in \Omega^2\langle A, x_1\rangle$ and $\langle dA, [x_1, x_2]\rangle \in \Omega^2\langle A, x_1, x_2\rangle$. Choose the lift $\langle F, x_1\rangle$ of the class of $\langle dA, x_1\rangle$ (they only differ by higher degree terms) and compute:

$$
\begin{aligned}
\Delta\langle F, x\rangle =& \langle e^{-x_1} F e^{x_1}, x_2\rangle - \langle F, \log(e^{x_1} e^{x_2})\rangle + \langle F, x_1\rangle \\
=& \langle F, e^{x_1} x_2 e^{-x_1}\rangle - \langle F, \log(e^{x_1} e^{x_2})\rangle + \langle F, x_1\rangle \\
=& \langle F, x_2 + [x_1, x_2]\rangle - \langle F, x_1 + x_2 + \tfrac{1}{2}[x_1, x_2]\rangle + \langle F, x_1\rangle, \quad \text{up to degree 3} \\
=& \tfrac{1}{2}\langle F, [x_1, x_2]\rangle \\
=& \tfrac{1}{2}\langle dA, [x_1, x_2]\rangle, \quad \text{up to degree 3}
\end{aligned}
$$

Hence, the two cohomology classes kill each other in the Δ-complex, and the $k = 2$ row is exact □

Now we are ready to prove Theorem 7.9.

Proof. Let $\omega_1 \in \Omega^1\langle A, x_1, x_2\rangle$ such that $d\Delta\omega_1 = 0$. We now perform a zig-zag process in order to find the remaining terms in ω. First, since the columns of the double complex are exact with the respect to the de Rham differential d, there is a unique element $\omega_0 \in \Omega^0\langle A, x_1, x_2, x_3\rangle$ such that $d\omega_0 = -\Delta\omega_1$. This implies $d\Delta\omega_0 = -\Delta d\omega_0 = \Delta^2\omega_1 = 0$, and by exactness of the columns $\Delta\omega_0 = 0$.

Next, note that $\Delta d\omega_1 = -d\Delta\omega_1 = 0$. Hence, by Lemma 7.10 there is an element $\omega_2 \in \Omega^2\langle A, x_1\rangle$ such that $\Delta\omega_2 = -d\omega_1$. The element ω_2 is unique by exactness of the 2nd row in the term $\Omega^2\langle A, x_1\rangle$. Finally, since $\Delta d\omega_2 = -d\Delta\omega_2 = d^2\omega_1 = 0$, by Lemma 7.10 there is an element $\omega_3 \in \Omega^3\langle A\rangle$ such that $\Delta\omega_3 = -d\omega_2$. Since the only Δ-closed element in $\Omega\langle A\rangle$ is $\langle F, F\rangle$ (which is in degree 4), the choice of ω_3 is unique.

Observe that $\Delta d\omega_3 = -d\Delta\omega_3 = 0$. Hence, $d\omega_3 \in \mathrm{Span}\{\langle F, F\rangle\}$ and $d\omega_3 = \lambda\langle F, F\rangle$ for some $\lambda \in \Bbbk$. In summary, for $\omega = \omega_3 + \omega_2 + \omega_1 + \omega_0$ we have $D\omega = \lambda\langle F, F\rangle$. On the other hand, since the double complex $\Omega\langle A, x_\bullet\rangle$ is exact under the total differential D, the equation $D\omega = \langle F, F\rangle$ admits solutions and there is an element ω_1 which yields $\lambda = 1$.

□

2.4 From universal calculus to finite dimensional Lie algebras

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over the field $\Bbbk$. Recall that the Weil algebra of $\mathfrak{g}$ is a differential graded commutative algebra (together with a $T[1]G$-action) defined on

$$W\mathfrak{g} = S\mathfrak{g}^* \otimes \wedge\mathfrak{g}^*.$$

Note that the space $W\mathfrak{g} \otimes \mathfrak{g}$ is naturally a differential graded Lie algebra, being a product of a Lie algebra with a ring. Define an element $a \in W\mathfrak{g} \otimes \mathfrak{g}$ as the canonical element (with respect to the bilinear form) in $\wedge^1\mathfrak{g}^* \otimes \mathfrak{g} \subset W\mathfrak{g} \otimes \mathfrak{g}$.

Lemma 7.11. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then, the assignment $A \mapsto a$ defines a homomorphism of differential graded Lie algebras $\mathcal{P}_\mathfrak{g} : \mathrm{Lie}\langle A, dA\rangle \to W\mathfrak{g}\otimes\mathfrak{g}$. If $\mathfrak{g}$ carries an invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$, this homomorphism induces a chain map $\Omega\langle A\rangle \to W\mathfrak{g}$.*

Proof. The first statement follows from the fact that $\mathrm{Lie}\langle A, dA\rangle$ is a free dg Lie algebra in one generator of degree 1. The second statement follows directly from the definition of $\mathcal{F}$. □

In a similar fashion, consider the cosimplicial complex $W\mathfrak{g} \otimes \Omega^\bullet(G_{\text{formal}})$, where G_{formal} is the formal group integrating the Lie algebra $\mathfrak{g}$. Recall that in the formal group the logarithm $\log : G_{\text{formal}} \to \mathfrak{g}$ is well-defined. We have the following simple result:

Lemma 7.12. *The assignment $A \mapsto a, x_i \mapsto \log(g_i)$ defines a morphism of co-simplicial complexes*

$$\mathcal{P}_{\mathfrak{g}} : (\Omega\langle A, x_\bullet\rangle, D = d + \Delta) \to (W\mathfrak{g} \otimes \Omega(G^\bullet_{\text{formal}}), D = d + \Delta).$$

Proof. The definitions of d and Δ on the two sides match. □

This implies that calculations in the universal complex $(\Omega\langle A, x_\bullet\rangle, D = d + \Delta)$ automatically carry over to all co-simplicial complexes $(W\mathfrak{g} \otimes \Omega(G^\bullet_{\text{formal}}), D = d + \Delta)$. In particular, solving descent equations in the universal framework gives rise to solutions for all finite dimensional Lie algebras with an invariant symmetric bilinear form.

3 The Kashiwara-Vergne theory

3.1 Derivations of free Lie algebras and the pentagon equation

Let der_n be the Lie algebra of continuous derivations of Lie_n and Aut_n the group of continuous automorphisms of Lie_n. Let tder_n be the vector space $\mathrm{Lie}_n^{\times n}$ equipped with the map

$$\begin{aligned} \mathrm{tder}_n &\xrightarrow{\rho} \mathrm{Der}_n \\ (u_1, \cdots, u_n) &\mapsto u : (x_i \mapsto [u_i, x_i]). \end{aligned}$$

It is easy to see that the formula

$$[(u_1, \cdots, u_n), (v_1, \cdots, v_n)]_i := \rho(u)v_i - \rho(v)u_i - [u_i, v_i]. \tag{12}$$

defines a Lie bracket on tder_n and makes ρ into a Lie algebra homomorphism. This homomorphism defines an action of tder_n on Lie_n, $\mathcal{F}\langle x_1, \ldots, x_n\rangle$, $\Omega\langle x_1, \ldots, x_n\rangle$ and other spaces where Der_n acts. In particular, the action on $\Omega\langle x_1, \ldots, x_n\rangle$ commutes with the de Rham differential:

$$\rho(u)d\alpha = d(\rho(u)\alpha).$$

We will often use a notation $u.\alpha = \rho(u)\alpha$ for actions of tder_n on various spaces.

Equipped with the Lie bracket (12), tder_n is a pro-nilpotent Lie algebra which readily integrates to a group denoted TAut_n together with the group homomorphism (again denoted by ρ):

$$\rho : \mathrm{TAut}_n \to \{g \in \mathrm{Aut}_n \mid g(x_i) = e^{\alpha_i} x_i e^{-\alpha_i}, \text{ for some } \ \alpha_i \in \mathrm{Lie}_n\}.$$

In what follows we will need a Lie subalgebra of special derivations

$$\mathrm{sDer}_n = \{u \in \mathrm{tder}_n \mid u.(x_1 + \cdots + x_n) = 0\}.$$

This Lie algebra integrates to a group

$$\mathrm{SAut}_n = \{g \in \mathrm{TAut}_n \mid g.(x_1 + \ldots x_n) = x_1 + \ldots x_n\}.$$

Define a vector space isomorphism $\gamma : \mathrm{tder}_n \to \Omega^1 \langle x_1, \ldots, x_n \rangle$

$$\gamma : u = (u_1, \ldots, u_n) \mapsto \sum_i \langle u_i, dx_i \rangle.$$

The following Lemma is due to Drinfeld ([6]):

Lemma 7.13. *An element $u \in \mathrm{tder}_n$ is in sDer_n if and only if $d\gamma(u) = 0$.*

The construction described above has the following naturality property. Every partially defined map $f : \{1, \ldots, n\} \to \{1, \ldots, m\}$ induces a homomorphism of free Lie algebras $\mathrm{Lie}_m \mapsto \mathrm{Lie}_n$ defined by

$$f^* : x_i \mapsto \sum_{f(l)=i} x_l.$$

Furthermore, this map induces a map on differential forms

$$f^* : \Omega\langle x_1, \ldots, x_m \rangle \to \Omega\langle x_1, \ldots, x_n \rangle.$$

Composed with the inverse of γ, this map defines a Lie homomorphism $f^* : \mathrm{tder}_m \to \mathrm{tder}_n$;

$$(u_i)_{i\in\{1,\ldots,m\}} \longmapsto \left(u_{f(k)}(x_i = \sum_{f(l)=i} x_l) \right)_{k\in\{1,\ldots,n\}}.$$

The standard notation $f^*(u) = u^{f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(n)}$ can be used. For instance,

$$u^{12,3} = (u_1(x_1 + x_2, x_3), u_1(x_1 + x_2, x_3), u_2(x_1 + x_2, x_3)).$$

These maps easily integrate to maps between TAut_n. For example, for $g \in \mathrm{TAut}_2$ one can define an element $\Phi_g \in \mathrm{TAut}_3$

$$\Phi_g = (g^{12,3})^{-1}(g^{1,2})^{-1} g^{2,3} g^{1,23}. \tag{13}$$

Then part of Proposition 7.1 in [2] reads:

Lemma 7.14. *Let $g \in TAut_2$ and assume that*

$$g(x_1 + x_2) = \log(e^{x_1} e^{x_2}).$$

Then, $\Phi_g \in \mathrm{SAut}_3$ and it satisfies the pentagon equation

$$\Phi_g^{12,3,4} \Phi_g^{1,2,34} = \Phi_g^{1,2,3} \Phi_g^{1,23,4} \Phi_g^{2,3,4}. \tag{$\pentagon$}$$

3.2 The 1-cocycle $c : \mathrm{tder}_n \to \Omega_n^1$

In this section we define and study a 1-cocycle $c : \mathrm{tder}_n \to \Omega\langle x_1, \ldots, x_n\rangle$ which plays a key role in constructing solutions of descent equations.

Theorem 7.15. *The map $c : \mathrm{tder}_n \to \Omega^1\langle x_1, \ldots, x_n\rangle$ defined by formula*

$$c : u = (u_1, \ldots, u_n) \mapsto \sum_{i=1}^{n} \langle x_i, du_i\rangle \tag{14}$$

is a 1-cocycle. That is,

$$c([u, v]) = u.c(v) - v.c(u) \quad \forall u, v \in \mathrm{tder}_n.$$

Moreover, c is natural in the sense that $c(f^*(u)) = f^*(c(u))$ for any partially defined map $f : \{1, \ldots, m\} \to \{1, \ldots, n\}$ of finite sets.

Proof. First, we have $u.c(v) = \sum_{i=1}^{n} u.\langle x_i, dv_i\rangle = \sum(\langle [u_i, x_i], dv_i\rangle + \langle x_i, d(u.v_i)\rangle)$. Similarly, $v.c(u) = \sum v.\langle x_i, du_i\rangle = \sum_{i=1}^{n}(\langle [v_i, x_i], du_i\rangle + \langle x_i, d(v.u_i)\rangle)$. Therefore,

$$\begin{aligned} u.c(v) - v.c(u) &= \sum_{i=1}^{n}(\langle x_i, d(u.v_i) - d(v.u_i)\rangle + \langle [u_i, x_i], dv_i\rangle - \langle [v_i, x_i], du_i\rangle) \\ &= \sum_{i=1}^{n}(\langle x_i, d(u.v_i) - d(v.u_i) + d([v_i, u_i])\rangle \\ &= c([u, v]), \end{aligned}$$

here we use the fact $\langle [u_i, x_i], dv_i\rangle - \langle [v_i, x_i], du_i\rangle = \langle x_i, [dv_i, u_i]\rangle - \langle x_i, [du_i, v_i]\rangle = \langle x_i, d[v_i, u_i]\rangle$.

For the naturality statement, let $f : \{1, \ldots, m\} \to \{1, \ldots, n\}$ be a partially defined map. On the one hand, we have

$$f^*(c(u)) = \sum_{i=1}^{n} f^*\langle x_i, du_i\rangle = \sum_{i=1}^{n}\left(\sum_{k\in f^{-1}(i)} \langle x_k, du_i(x_j = \sum_{f(l)=j} x_l)\rangle\right).$$

Note that the summation is actually over k, and $i = f(k)$.

On the other hand, by the definition of $f^* : \mathrm{tder}_m \longrightarrow \mathrm{tder}_n$, we have

$$c(f^*(u)) = \sum_{k=1}^{n} \langle x_k, du_{f(k)}(x_j = \sum_{f(l)=j} x_l) \rangle,$$

which implies the identity $c(f^*(u)) = f^*(c(u))$. □

Remark 7.16. The map c defines a nontrivial class in the Lie algebra cohomology

$$[c] \in H^1(\mathrm{tder}_n, \Omega^1\langle x_1, \ldots, x_n\rangle).$$

Indeed, the degree of c is equal to $+1$. If c were a trivial 1-cocycle, it would have been of the form $c(u) = u.\alpha$ for some $\alpha \in \Omega^1\langle x_1, \ldots, x_n\rangle$ of degree $+1$. However, such elements do not exist.

Lemma 7.17. *The cocycle c is a bijection. Furthermore, it restricts to a bijection*

$$c : \mathrm{sDer}_n \to \Omega^{1,closed}\langle x_1, \ldots, x_n\rangle \cong \Omega^0\langle x_1, \ldots, x_n\rangle.$$

Proof. Consider the map $c \circ \gamma^{-1} : \Omega^1_n \to \Omega^1_n$. It is equal to $c \circ \gamma = d \circ e - \mathrm{Id}$ (where e is the derivation of Ω_n defined by $e(dx_i) = x_i, e(x_i) = 0$). Indeed,

$$(d \circ e - \mathrm{Id}) \sum_{i=1}^{n} \langle u_i, dx_i\rangle = d \sum_{i=1}^{n} \langle u_i, x_i\rangle - \sum_{i=1}^{n} \langle u_i, dx_i\rangle = \sum_{i=1}^{n} \langle x_i, du_i\rangle.$$

Note that

$$(d \circ e - \mathrm{Id})(\mathrm{Id} - e \circ d) = n - \mathrm{Id},$$

where $n = de + ed$ is the Euler derivation counting the number of generators. Since, the number of generators is always greater or equal to two, $n - \mathrm{Id}$ is an invertible operator. Hence, c is a bijection.

Note that

$$\gamma(u) + c(u) = d \sum_{i=1}^{n} \langle x_i, u_i\rangle.$$

Hence, $c(u)$ is closed if and only if $\gamma(u)$ is closed, and $\gamma(u)$ is closed if and only if $u \in \mathrm{sDer}_n$ as follows from Drinfeld's Lemma 7.13. □

The Lie algebra 1-cocycle c integrates to a Lie group cocycle, which is the map $C : \mathrm{TAut}_n \to \Omega^1_n$ uniquely determined by the following properties:

$$C(g \circ f) = C(g) + g.C(f),$$

$$\left.\frac{d}{dt}\right|_{t=0} C(e^{tu}) = c(u).$$

The map C is given by the explicit formula,

$$C(e^u) = \left(\frac{e^u - 1}{u}\right).c(u), \ \forall u \in \mathrm{tder}_n.$$

Proposition 7.18. *The Lie group cocycle* $C : \mathrm{TAut}_n \to \Omega^1$ *is a bijection. It restricts to a bijection* $C : \mathrm{SAut}_n \to \Omega_n^{1,closed} \cong \Omega_n^0$.

Proof. This follows directly from the fact that the action of tder_n on Ω_n^1 is of positive degree and from the bijectivity of c. □

3.3 Solving descent equations with Kashiwara-Vergne theory

Let us define $S = \{g \in \mathrm{TAut}_2 \mid g(x_1 + x_2) = \log(e^{x_1}e^{x_2})\}$, the set of solutions to the first equation in the Kashiwara-Vergne problem.

Lemma 7.19. *Any* $g \in S$ *satisfies* $d\Delta C(g) = 0$.

Proof. Since $g \in S$ is a solution of the first Kashiwara-Vergne equation, its associator (13) satisfies $\Phi_g \in \mathrm{SAut}_3$. Set $\omega = C(g)$ and apply the cocycle C to the equation $g^{2,3}g^{1,23} = g^{1,2}g^{12,3}\Phi_g$ to get

$$\begin{aligned}
0 &= C(g^{2,3}g^{1,23}) - C(g^{1,2}g^{12,3}) \\
&= (\omega^{2,3} + g^{2,3}.\omega^{1,23}) - (\omega^{1,2} + g^{1,2}.\omega^{12,3} + (g^{1,2}g^{12,3}).C(\Phi_g)) \\
&= (\omega(x_2, x_3) + g^{2,3}.\omega(x_1, x_2 + x_3)) \\
&\qquad - (\omega(x_1, x_2) + g^{1,2}.\omega(x_1 + x_2, x_3) + (g^{1,2}g^{12,3}).C(\Phi_g)) \\
&= (\omega(x_2, x_3) + \omega(x_1, \log(e^{x_2}e^{x_3}))) \\
&\qquad - (\omega(x_1, x_2) + \omega(\log(e^{x_1}e^{x_2}), x_3) + (g^{1,2}g^{12,3}).C(\Phi_g)) \\
&= \Delta(\omega) - (g^{1,2}g^{12,3}).C(\Phi_g).
\end{aligned}$$

Note that the universal 1-form $(g^{1,2}g^{12,3}).C(\Phi_g)$ is closed since C maps $\Phi_g \in \mathrm{SAut}_3$ to $\Omega_3^{1,\mathrm{closed}}$, and $\Omega^{1,\mathrm{closed}}$ is preserved by every automorphism (in particular, by $g^{1,2}g^{12,3}$). Hence,

$$d\Delta\omega = d\Big((g^{1,2}g^{12,3}).C(\Phi_g)\Big) = (g^{1,2}g^{12,3}).dC(\Phi_g) = 0. \tag{15}$$

□

For $g \in S$, Lemma 7.19 implies that the element $\omega_1 = C(g) \in \Omega^1\langle x_1, x_2\rangle$ verifies the conditions of Theorem 7.9. Hence, there is a unique $\omega = \omega_0 + \omega_1 + \omega_2 + \omega_3$ such that its components are solutions of the descent equations. In particular,

$\omega_0 \in \Omega^0\langle x_1, x_2, x_3\rangle$ is a Δ-cocycle. By Lemma 7.8 the cohomology $H^3(\Omega^0, \Delta)$ is spanned by the generator $\phi = \langle x_1, [x_2, x_3]\rangle + \dots$. The following lemma shows that the cohomology class $[\omega_0] \in H^3(\Omega^0, \Delta)$ is independent of the choice of $g \in S$:

Lemma 7.20. *The cohomology class* $[\omega_0] \in H^3(\Omega^0, \Delta)$ *is independent of* g*. More precisely,*

$$[\omega_0] = \frac{1}{12}[\phi].$$

Proof. Recall that S is a right torsor under the action of the group SAut_2. If one chooses a base point $g \in S$, all other solutions of the first Kashiwara-Vergne equation are of the form $g' = g \circ f$ for $f \in \mathrm{SAut}_2$. The cocycle condition for C yields

$$C(g \circ f) - C(g) = g.C(f).$$

The 1-form $C(f)$ is closed (since $f \in \mathrm{SAut}_2$) and therefore exact. Hence, $g.C(f)$ is also exact, and we will denote it by $d\nu$. Applying the differential Δ to the equation above, we obtain

$$\Delta C(g \circ f) - \Delta C(g) = \Delta d\nu = d\Delta\nu.$$

This implies

$$d\omega_0(g') - d\omega_0(g) = \Delta C(g') - \Delta C(g) = d\Delta\nu.$$

Since the kernel of d in degree 0 is trivial, we obtain

$$\omega_0(g') - \omega_0(g) = \Delta\nu,$$

as required.

In order to compute the missing coefficient, we use equation (15) to conclude

$$d\omega_0 = \Delta\omega_1 = (g^{1,2}g^{12,3}).C(\Phi_g).$$

If Φ_g is a Drinfeld associator (*e.g.* the Knizhnik-Zamolodchikov associator), we have $\Phi_g = \exp(u_2 + \dots)$, where $u_2 \in \mathrm{tder}_3$ is the following tangential derivation of degree 2:

$$u_2 = \frac{1}{24}\left([x_2, x_3], [x_3, x_1], [x_2, x_3]\right)$$

and $\dots$ stand for higher degree terms (see Propositions 7.3 and 7.4 in [2]). Up to degree 2, the equation for ω_0 reads

$$\begin{aligned} d\omega_0 &= c(u_2) + \dots \\ &= \tfrac{1}{24}\left(\langle d[x_2, x_3], x_1\rangle + \langle d[x_3, x_1], x_2\rangle + \langle d[x_1, x_2], x_3\rangle\right) + \dots \\ &= \tfrac{1}{12}\, d\langle x_1, [x_2, x_3]\rangle + \cdots \end{aligned}$$

which implies

$$\omega_0 = \frac{1}{12} \langle x_1, [x_2, x_3] \rangle + \cdots ,$$

as required. □

Proposition 7.21. *The cocycle C defines a bijection*

$$\begin{aligned} S = \{g \in \mathrm{TAut}_2 \mid g(x_1 + x_2) = \log(e^{x_1} e^{x_2})\} \\ \xrightarrow{C} \{\omega^1 \in \Omega^1\langle x_1, x_2\rangle \mid \Delta(\omega^1) = d\omega_0, [\omega_0] = \tfrac{1}{12}[\phi] \in H^3(\mathcal{F}\langle x_\bullet\rangle, \Delta)\}. \end{aligned}$$

Proof. The cocycle C is an injective map. Hence, so is its restriction to S. It remains to show the surjectivity.

Let $\omega_1, \tilde{\omega}_1 \in \Omega^1\langle x_1, x_2\rangle$ be two elements which satisfy conditions of the Proposition, and assume that $\omega_1 = C(g)$ for some $g \in S$. The corresponding elements $\omega_0, \tilde{\omega}_0 \in \Omega^0\langle x_1, x_2\rangle$ belong to the same cohomology class in $H^3(\Omega^0, \Delta)$, and therefore $\tilde{\omega}_0 - \omega_0 = \Delta\mu$ for some $\mu \in \Omega^0\langle x_1, x_2\rangle$. This implies

$$\Delta(\tilde{\omega}_1 - \omega_1) = d(\tilde{\omega}_0 - \omega_0) = d\Delta\mu = \Delta d\mu.$$

We know that the kernel of $\Delta : \Omega^1\langle A, x_1, x_2\rangle \to \Omega^1\langle A, x_1, x_2, x_3\rangle$ is spanned by the elements $\Delta d\langle x_1, x_1\rangle, \Delta\langle A, x_1\rangle$. The first of them is d-exact, and the second one contains A. Therefore, the kernel of the map $\Delta : \Omega^1\langle x_1, x_2\rangle \to \Omega^1\langle x_1, x_2, x_3\rangle$ is spanned by the d-exact element $d\Delta\langle x_1, x_1\rangle$.

We conclude that $\tilde{\omega}_1 - \omega_1 = d\sigma$ is d-exact. Choosing $f \in \mathrm{SAut}_2$ such that $C(f) = d(g^{-1}.\sigma) = g^{-1}.d\sigma$, we obtain

$$C(g \circ f) = C(g) + g.C(f) = \omega_1 + d\sigma = \tilde{\omega}_1,$$

as required. □

Remark 7.22. The proof furthermore shows that elements of S are up to the action of $e^{\Bbbk d\langle x_1, x_1\rangle}$ in bijection with the set of representatives of the cohomology class $\frac{1}{12}[\phi]$. Moreover, one can use results from [2] to conclude that there is a bijection between normalized associators in SAut and normalized solutions to $\Delta\omega_0 = 0$.

Together with theorem 7.9 one concludes

Theorem 7.23. *Primitives of $\frac{1}{2}\langle F, F\rangle \in \Omega^\bullet\langle A, x_\bullet\rangle$ with respect to $d + \Delta$ are of the form $\omega_0 + \omega_1 + \omega_2 + \omega_3$, where*

$$\begin{aligned} \omega_3 &= \tfrac{1}{2} CS(A) \\ \omega_2 &= \tfrac{1}{2} WZ(A, e^{x_1}) + sd\langle A, x_1\rangle \\ \omega_1 &= -C(g) - s\Delta\langle A, x_1\rangle \end{aligned}$$

for uniquely determined $g \in S$ and $s \in \Bbbk$. Moreover, $[\omega_0] = -\frac{1}{12}[\phi]$.

Proof. By Proposition 7.21, every element $g \in S$ gives rise to $\omega_1 = C(g)$ which satisfies the conditions of Theorem 7.9. Hence, it defines a unique element $\omega = \omega_0 + \omega_1 + \omega_2 + \omega_3$ with the property $D\omega = \lambda\langle F, F\rangle$. Consider $g = \exp(u_1 + \dots)$, where $u_1 = \frac{1}{2}(x_2, 0)$ is the tangential derivation of degree one, and $\dots$ stand for higher order terms. It is sufficient to keep the terms up to degree 2 which yields

$$\omega_0 = 0 + \dots, \omega_1 = -\frac{1}{2}\langle x_1, dx_2\rangle + \dots, \omega_2 = \frac{1}{2}\langle A, dx_1\rangle + \dots, \omega_3 = \frac{1}{2}\langle A, dA\rangle + \cdots$$

which shows that $\lambda = 1/2$, as required.

The primitives of $\frac{1}{2}\langle F, F\rangle$ form an affine space over space of D-exact elements of total degree 3. This space is of the form

$$D(\Bbbk d\langle x_1, x_1\rangle \oplus \langle A, x_1\rangle) + D\Omega^0\langle x_1, x_2\rangle.$$

The sum in not direct since $d\Delta\langle x_1, x_1\rangle$ spans the intersection of the two subspaces. We have already classified all solutions with $\omega_1 \in \Omega^1\langle x_1, x_2\rangle$. Hence, a general solution for $\omega_1 \in \Omega^1\langle A, x_1, x_2\rangle$ which verifies $d\Delta\omega_1 = 0$ is of the form

$$\omega_1 = -C(g) - s\Delta\langle A, x_1\rangle.$$

Formulas for ω_2 and ω_3 follow from the descent equations. □

Remark 7.24. Let $g \in S$. Then, there is the following interesting identity relating the pentagon equation for Φ_g to the last descent equation $\Delta\omega_0 = 0$ for ω_0 defined by g:

$$g^{1,2}g^{12,3}g^{123,4}.\Big(C(\Phi_g^{12,3,4}\Phi_g^{1,2,34}) - C(\Phi_g^{1,2,3}\Phi_g^{1,23,4}\Phi_g^{2,3,4})\Big) = \Delta d\omega_0 = d\Delta\omega_0.$$

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