On maximal subgroups and other aspects of branch groups

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On Maximal Subgroups And Other Aspects Of Branch Groups

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Résumé

Les groupes branchés, une classe de groupes définis par leur action sur un arbre enraciné, occupent une place importante dans la théorie des groupes, puisqu'ils possèdent souvent des propriétés rares qui sont impossibles à trouver parmi les groupes plus classiques. Ils sont ainsi une riche source d'exemples et de contre-exemples à plusieurs questions notables. Malgré leur importance, de nombreux aspects de ces groupes restent encore mal compris.

Dans cette thèse, nous étudions l'un de ces aspects en particulier, à savoir l'indice des sous-groupes maximaux. Plus précisément, nous tentons de mieux cerner les conditions sous lesquelles tous les sous-groupes maximaux d'un groupe branché sont d'indice fini. Nous essayons notamment de déterminer s'il existe un lien entre cette propriété et la croissance d'un groupe branché.

Comme l'étude de cette question requiert une bonne compréhension de la croissance des groupes branchés, nous commençons par démontrer une version plus forte d'un critère sur la croissance développé par Bartholdi et Pochon. Nous utilisons ensuite ce critère pour obtenir de nouveaux exemples de groupes à croissance intermédiaire.

Par la suite, nous cherchons à développer une méthode générale pour étudier les sous-groupes maximaux des groupes branchés. À cette fin, inspiré par les travaux pionniers de Pervova sur le sujet, nous démontrons que si un groupe faiblement branché et auto-répliquant agit primitivement sur le premier niveau d'un arbre enraciné, alors les projections de tout sous-groupe prodense et propre sont aussi des sous-groupes prodenses et propres. Comme l'existence de sous-groupes maximaux d'indice infini est fortement liée à l'existence de sous-groupes prodenses et propres, ce résultat technique nous fournit un moyen d'exploiter l'autosimilarité et des arguments de réduction de longueur pour étudier cette question dans de nombreux groupes branchés.

Muni de cet outil, nous étudions ensuite les sous-groupes maximaux d'une classe particulière de groupes branchés, les groupes de Šunić agissant sur l'arbre binaire et contenant un élément d'ordre infini. Cette partie de la thèse est issue d'un travail en collaboration avec Alejandra Garrido. Nous démontrons d'abord que tous ces groupes possèdent une propriété connue sous le nom de propriété de congruence pour les groupes branchés. Grâce à ce résultat, l'étude des sous-groupes prodenses est grandement simplifiée, ce qui nous permet de démontrer l'existence de sous-groupes maximaux d'indice infini dans ces groupes. Ceci montre entre autres qu'il est possible pour des groupes branchés à croissance intermédiaire d'admettre de tels sous-groupes maximaux. Nous complétons notre étude en obtenant une classification complète de tous les sous-groupes maximaux de ces groupes. Nous prouvons en particulier que tous les sous-

groupes maximaux d'indice infini sont conjugués au groupe qui les contient par un automorphisme de l'arbre.

Toujours dans le cadre d'une collaboration avec Alejandra Garrido, nous nous interrogeons ensuite sur l'existence d'un possible lien entre la présence d'un élément d'ordre infini et celle d'un sous-groupe maximal d'indice infini dans la famille des groupes de Šunić en général. Nous démontrons que la situation observée pour les groupes agissant sur l'arbre binaire est exceptionnelle et que pour les arbres de degré plus élevé, il existe des groupes possédant un élément d'ordre infini mais dont tous les sous-groupes maximaux sont d'indice fini.

Par la suite, nous étudions encore le cas d'un dernier groupe, le groupe Basilique. Ce groupe est très différent des autres groupes considérés dans cette thèse sous plusieurs aspects. En particulier, il est faiblement branché mais pas branché, il n'est pas juste infini, il est à croissance exponentielle et il est sans torsion. Néanmoins, nous montrons que tous ses sous-groupes maximaux sont d'indice fini, ce qui illustre la diversité des groupes possédant cette propriété.

Enfin, pour conclure cette thèse, nous élargissons notre étude des sous-groupes des groupes branchés à une autre famille importante de sous-groupes, les sous-groupes paraboliques. Bien qu'ils ne soient jamais maximaux, ces derniers sont faiblement maximaux, ce qui fait de leur étude un prolongement naturel de celle des sous-groupes maximaux. Nous démontrons que, sous certaines conditions, ces sous-groupes ne sont jamais de type fini, et nous étudions les conditions sous lesquelles ils sont isomorphes.

Abstract

Branch groups, a class of groups defined through their action on a rooted tree, play an important role in group theory, since they often possess rare properties that cannot be found in more classical groups. They are therefore a rich source of examples and counterexamples to many notable questions. Despite their importance, many aspects of these groups remain poorly understood.

In this thesis, we study one of those aspects in particular, namely the index of maximal subgroups. More precisely, we try to better understand the conditions under which every maximal subgroup of a branch group is of finite index. Among other things, we try to determine whether or not this property is linked to the growth of the group.

As the study of this question requires a good understanding of the growth of branch groups, we begin by establishing a strong version of a criterion on growth developed by Bartholdi and Pochon. We then apply this criterion to obtain new examples of groups of intermediate growth.

Afterwards, we aim to develop a general method to help in the study of maximal subgroups of branch groups. To this end, inspired by the pioneering work of Pervova on the subject, we prove that if a self-replicating weakly branch group acts primitively on the first level of a rooted tree, then the projections of every proper prodense subgroup are also proper prodense subgroups. As the existence of maximal subgroups of infinite index is closely linked with the existence of proper prodense subgroups, this technical result allows us to exploit self-similarity and length reduction arguments to study this question in many branch groups.

Equipped with this tool, we then turn our attention to the study of maximal subgroups of a particular class of branch groups, namely the Šunić groups acting on the binary rooted tree and containing an element of infinite order. The results in this part of the thesis were obtained in a joint work with Alejandra Garrido. We first show that these groups all possess what is known as the congruence subgroup property for branch groups. This property greatly simplifies the study of prodense subgroups, which allows us to show that these groups admit maximal subgroups of infinite index. In particular, this shows that it is possible for a branch group of intermediate growth to contain such maximal subgroups. We then complete our investigation by obtaining a complete classification of every maximal subgroups of these groups. In particular, we show that every maximal subgroup of infinite index is a conjugate of the whole group by an automorphism of the rooted tree.

Also as part of a joint work with Alejandra Garrido, we then investigate the possibility of a link between the existence of an element of infinite order

and the existence of a maximal subgroup of infinite index in the entire family of Šunić groups. However, we show that the behaviour observed for the binary rooted tree is exceptional, and that for trees of higher degrees, there can exist groups with elements of infinite order that contain only maximal subgroups of finite index.

We then study maximal subgroups in one last group, the Basilica group. This group is very different from the others considered in this thesis. Indeed, among other things, it is weakly branch but not branch, it is not just-infinite, it is of exponential growth, and it is torsion-free. Nevertheless, we show that all of its maximal subgroups are of finite index, which illustrate the great diversity of groups with this property.

Finally, we conclude this thesis by broadening our study of subgroups of branch groups to another important family of subgroups, namely the parabolic subgroups. Although they are never maximal, these subgroups are weakly maximal, which makes studying them a natural continuation of our investigation of maximal subgroups. We show that, under certain conditions, parabolic subgroups are never finitely generated. We also obtain conditions under which they are all isomorphic.

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Table of Contents

Résumé **i**

Abstract **iii**

Remerciements **v**

Table of Contents **vi**

1 Introduction **1**

2 Background **9**
2.1 Primitive actions 9
2.2 Permutational wreath product 10
2.3 Alphabets and words 11
2.4 Graphs 12
2.5 Growth of groups 16
2.6 Rooted trees and their automorphisms 20
2.7 Branch and weakly branch groups 29
2.8 Spinal groups 33
2.9 Šunić groups 35

3 Groups of intermediate growth **41**
3.1 Preliminaries 42
3.2 Incompressible elements and growth 47
3.3 Growth of spinal groups 53
3.4 Open questions 57

4 Congruence subgroup property **59**
4.1 Definitions and basic properties 60
4.2 CSP and the LERF property 62
4.3 Congruence subgroup property for Šunić groups 63
4.4 Just-infiniteness and the congruence subgroup property 67

5 The class $\mathcal{MF}$ and dense subgroups **71**
5.1 The class $\mathcal{MF}$ 71
5.2 The class $\mathcal{MF}$ and profinite topology 72
5.3 Pro-$\mathcal{MF}$-dense and prodense subgroups 73
5.4 Weakly branch groups and proper prodense subgroups 77

6 Groups of intermediate growth not in $\mathcal{MF}$ 85
6.1 Dense subgroups 86
6.2 Proper dense subgroups 88
6.3 Maximal subgroups of infinite index 91
6.4 Finding all maximal subgroups of infinite index 99
6.5 Maximal subgroups of finite index 106
6.6 Open questions 107

7 Non-torsion branch groups in $\mathcal{MF}$ 109
7.1 Generalised Fabrykowski-Gupta groups 109
7.2 Maximal subgroups of Fabrykowski-Gupta groups 111
7.3 Open questions 116

8 Maximal subgroups of the Basilica group 119
8.1 The Basilica group 120
8.2 The commutator subgroup of the Basilica group 121
8.3 Quotients of the Basilica group 123
8.4 Maximal subgroups of the Basilica group 127

9 Parabolic subgroups 133
9.1 Infinitely generated parabolic subgroups 134
9.2 Stabilisers of regular points 135
9.3 Isomorphism classes of parabolic subgroups 143
9.4 Index of parabolic subgroups 147

Bibliography 153

Chapter 1

Introduction

The main goal of this thesis is to further our understanding of the structure of groups acting on rooted trees. Such groups have attracted a significant amount of attention in the last few decades, mostly thanks to the unusual properties that they can possess.

This interest in groups acting on rooted trees can in large parts be traced back to Grigorchuk's discovery in 1980 [41] of a very important group, now known as the first Grigorchuk group. It was originally introduced as an additional answer to a famous question in group theory, known as the general Burnside problem.

In 1902, William Burnside asked in [16] if a finitely generated group can be infinite but periodic[i]. Three years later, he proved in [17] that a finitely generated subgroup of $GL(n, \mathbb{C})$ where the order of every element is bounded must be finite. In 1911, Schur improved on Burnside's result and showed in [79] that every finitely generated periodic subgroup of $GL(n, \mathbb{C})$ must be finite. This meant that, if they exist, infinite periodic groups can only be found outside of the classical realm of groups of matrices.

Grigorchuk showed in [41] that the group that bears his name is an infinite finitely generated periodic group. It was not the first example of such a group. Indeed, the general Burnside problem was first solved in 1964 by Golod [39], building up on his work with Shafarevich in [40]. Novikov and Adian had also shown in 1968 [69, 70, 71] that there exist infinite finitely generated periodic groups with bounded exponent, thus solving a question known as the restricted Burnside problem. Nevertheless, the Grigorchuk group remained interesting, as it was one of the few known examples of infinite finitely generated torsion groups. It was also remarkably simple to define and study. For instance, it has a solvable word problem (see [43]).

Were it confined to its role as an additional solution to Burnside's problem, the first Grigorchuk group would still have been notable, but this group also turned out to play a crucial role in the theory of growth. The growth of a finitely generated group is a quasi-isometry invariant that measures in some sense how large an infinite group is. It was first studied by Švarc in 1955

[i] A *periodic* group is a group where every element has finite order. We will also use the term *torsion* for the same notion.

[84] and independently by Milnor in 1968 [61] in connection with volume and curvature in Riemannian manifolds.

The growth of a finitely generated group can belong to one of three broad classes: polynomial, exponential or intermediate. Whereas examples of groups of polynomial or exponential growth are readily found, no example of a group of intermediate growth was known at the time. This led Milnor to ask in 1968 [62] if it was true that every finitely generated group is either of polynomial or of exponential growth. That same year, Milnor and Wolf proved in [86] and [60] that this is true in the case of solvable groups. In conjunction with the Tits alternative [82], their theorem also implies that the same holds for finitely generated linear groups.

These partial results made it all the more surprising when, in 1983, Grigorchuk showed in [42, 43] that the group he studied in [41] was of intermediate growth, thus finally solving Milnor's question. This additionally had consequences in the theory of amenability[ii]. Indeed, being a group of intermediate growth, Grigorchuk's group is amenable, but it cannot be elementary amenable by a theorem of Chou [20], thus settling an open question mentioned by Day in 1957 [23].

Although it was first defined as a group of transformations of an interval, it was soon realised that the best way to understand the Grigorchuk group was as a group of automorphisms of a rooted tree, and the remarkable breakthroughs of Grigorchuk led to a surge of interest in such groups. Very quickly, many new examples of infinite finitely generated periodic groups, such as the Gupta-Sidki p-groups [54], as well as new examples of groups of intermediate growth, such as the Grigorchuk groups [43, 44], were discovered and studied. These examples were later generalised to even larger families of groups, such as GGS groups, spinal groups [13, 9] and multi-edge spinal groups [1, 58].

Another direction of generalisation of the Grigorchuk group was obtained by trying to capture its structural properties. This led Grigorchuk to define the notion of branch groups in 1997. However, branch groups had already implicitly appeared earlier in a different context, namely in the work of Wilson on just-infinite groups [85]. Indeed, they are one of three classes into which just-infinite groups naturally split. Branch groups thus seem of great importance, and in fact most of the interesting examples of groups acting on rooted trees belong to this class.

Over the years, in addition to torsion and growth, various other aspects of the Grigorchuk groups and its many generalisations were investigated, such as width [6], automorphisms [50], abstract commensurators [78], subgroup separability [49] and spectral properties of quasi-regular representations [7], to name but a few. All these results confirmed that automorphisms of rooted trees are a fertile ground for groups with remarkable properties.

In this thesis, we will be mainly concerned with one specific aspect of groups acting on rooted trees, namely maximal subgroups. As their study will form a central part of this text, let us first give a definition.

Definition 1.0.1. Let G be a group. A proper subgroup $M < G$ is a *maximal subgroup* of G if it is a maximal element of the set of all proper subgroups of

[ii] We will not discuss the notion of amenability in this text, but we refer the interested reader to [72] (for example) for the definition.

G, partially ordered by inclusion. In other words, M is maximal if for every subgroup L such that $M \leq L \leq G$, we have either $L = M$ or $L = G$. ❧

It is important to note that maximal subgroups do not always exist. For instance, it is well-known that the group $\mathbb{Q}$ of rational numbers contains no maximal subgroups. However, in a finitely generated group, every proper subgroup is contained in a maximal one[iii] [67]. Therefore, as we will mostly be concerned with finitely generated groups, this needs not worry us.

The study of maximal subgroups can yield important information about the structure of a group, since they sit at the top of the subgroup lattice. Furthermore, they can also be used to compute the Frattini subgroup, which is the set of all non-generators of the group[iv]. Additionally, the study of maximal subgroups of a group is equivalent to the study of primitive actions of this group. As primitive actions are in some sense the building blocks out of which every action is constructed, maximal subgroups have attracted a lot of attention, especially in the case of finite groups, where efforts have been made to classify them (see for instance [2, 55]).

For infinite discrete groups, there is no hope of obtaining a general classification, but one can still try to understand maximal subgroups in different classes of groups. One of the most basic question that one can ask in this setting is whether or not a group contains maximal subgroups of infinite index. This was solved for linear groups by Margulis and Soĭfer in 1981 [59]. They showed that a finitely generated linear group contains a maximal subgroup of infinite index if and only if it is not virtually solvable, in which case it contains in fact uncountably many such subgroups.

The result of Margulis and Soifer inspired Gelander and Glasner to study maximal subgroups in different families of groups of geometric origin, such as mapping class groups, hyperbolic groups and groups acting on (non-rooted) trees. They obtained in [36] characterisations of groups containing maximal subgroups with trivial normal core (and so, in particular, of infinite index) in those settings.

In the context of branch groups, this question was first studied by Pervova in 2000. She showed in [73, 74] that every maximal subgroup of torsion Grigorchuk 2-groups, torsion GGS groups and torsion EGS groups is of finite index. In fact, she even showed that the maximal subgroups of those groups are all normal. It was then asked by Bartholdi, Grigorchuk and Šunić ([9], Question 14) if there could exist a finitely generated branch group containing a maximal subgroup of infinite index. This was settled by Bondarenko in 2010, when he constructed in [14] finitely generated branch groups with maximal subgroups of infinite index.

Although Pervova's and Bondarenko's results showed that the class of finitely generated branch groups can contain both groups with and without maximal subgroups of infinite index, they also gave rise to many questions. Chief among them is the question of the existence of an algebraic characterisation of branch groups with maximal subgroups of infinite index, à la Margulis-Soifer.

One could also wonder if there exists a link between growth and the existence of maximal subgroups of infinite index. Indeed, it follows from Gromov's theorem on groups of polynomial growth [53] that a group of polynomial growth cannot contain maximal subgroups of infinite index. The same is also true of

[iii]It is interesting to note that the proof of this result does not require the axiom of choice.
[iv]Recall that a non-generator is an element that is redundant in every generating set.

every group of intermediate growth studied by Pervova. It is thus natural to wonder if there can exist a finitely generated group of subexponential growth with maximal subgroups of infinite index, and this question was explicitly asked in [24] (Example 3.10 (8)). In complete generality, this was answered very recently by Nekrashevych. Indeed, in his 2018 paper [66], he constructed a simple group of intermediate growth. As it is simple, this group contains no proper subgroup of finite index, so all its maximal subgroups must be of infinite index. However, being simple, Nekrashevych's group is far from being a branch group, so one could still ask the same question in the more restricted class of branch groups, or even in the larger class of groups acting on rooted trees.

Finally, one could also ask how far Pervova's results and techniques can be generalised. Alexoudas, Klopsch and Thillaisundaram [1], and later Klopsch and Thillaisundaram [58], extended them to a large class of branch groups that they call multi-edge spinal groups. However, like Pervova, they only manage to obtain results for periodic groups in this family. It is thus natural to ask if there is a deeper connection between torsion and the index of maximal subgroups in those families, or if it is just a technical condition that could be removed through a more careful analysis.

The main objective of this thesis is to study these questions and to, at least partially, answer some of them. We begin in **Chapter 2** by reviewing some of the definitions, concepts and basic results that will appear in the rest of the text. Included in this chapter are the basics of the theory of growth of groups, of groups acting on rooted trees and more specifically of branch and weakly branch groups. A special attention is given to spinal groups (Section 2.8) and in particular to Šunić groups (Section 2.9), since we will heavily focus on those at many points in the text.

In **Chapter 3**, as a prelude to our study of the link between growth and the index of maximal subgroups, we study growth in finitely generated groups acting on rooted trees. Although this is not strictly necessary, since the growth of the groups whose maximal subgroups we study in the rest of this thesis was either already known prior to our result or is still currently unknown, we nevertheless think it worthwhile to include this work, as we believe that the results contained in this chapter could be very useful in the study of growth in groups acting on rooted trees. Furthermore, the techniques we use to study growth also share many similarities with the ones we employ to study maximal subgroups, although there are some key differences. The main result of this chapter is Theorem 3.2.1, which links the growth of some groups to the growth of what we call *incompressible elements*. This criterion is in fact a slightly generalised version of the one developed by Bartholdi and Pochon in [10]. We then apply this criterion to obtain Theorem 3.3.6, which states that many spinal groups acting on the 3-regular rooted tree are of intermediate growth. To the best of our knowledge, this was not known before for all but one of the non-torsion groups, known as the Fabrykowski-Gupta group.

In **Chapter 4**, we study a property of groups acting on rooted trees, called the *congruence subgroup property* (CSP, for short), which is of great importance in the study of maximal subgroups of infinite index. After reviewing basic facts and definitions in Section 4.1, we study in Section 4.2 a link between the CSP and another well-known property of groups connected to maximal subgroups, the LERF property. More precisely, we make the following observation.

Proposition (4.2.2)**.** *Let G be a group acting on a rooted tree T of bounded degree. If G has the congruence subgroup property and contains an element of infinite order, then G is not LERF.*

This proposition shows that torsion plays an important role for the LERF property. By contrast, Grigorchuk and Wilson showed that the Grigorchuk group is LERF [49], and Garrido obtained the same result for the Gupta-Sidki 3-group [34].

We then study in Section 4.3 the congruence subgroup property in the special case of Šunić groups and show the following theorem.

Theorem (4.3.8)**.** *Let G be a Šunić group different from the infinite dihedral group. Then, G has the congruence subgroup property.*

As a corollary, we observe in Section 4.4 that every Šunić group is just-infinite (Theorem 4.4.3). We then show that this is in fact true of every finitely generated branch group with the congruence subgroup property.

Theorem (4.4.4)**.** *Let X be a finite alphabet, and let G be a finitely generated branch group acting on X^*. If G has the congruence subgroup property, then G is just-infinite.*

In **Chapter 5**, we develop the technical tools that we will need in our study of the index of maximal subgroups. In Section 5.1, we define the class $\mathcal{MF}$ of groups containing no maximal subgroups of infinite index, and we prove that this class is well-behaved with respect to extensions.

Theorem (5.1.2)**.** *Let G be a finitely generated group and $N \trianglelefteq G$ be a finitely generated normal subgroup of G. If N and G/N are in $\mathcal{MF}$, then G is also in $\mathcal{MF}$.*

Then, in Section 5.2, we review the well-known link between the class $\mathcal{MF}$ and profinite topology. In Section 5.3, we introduce pro-$\mathcal{MF}$-dense subgroups and their link with prodense subgroups and maximal subgroups of infinite index. Finally, in Section 5.4, we prove the main result of this chapter, namely Theorem 5.4.3. This theorem is a generalisation to a much larger class of groups of a result of Pervova proved in [73, 74], and will be the main technical tool upon which our investigation of maximal subgroups relies.

Our study of maximal subgroups in branch group begins properly in **Chapter 6**, where we study maximal subgroups of non-torsion Šunić groups acting on the binary rooted tree. The main results of this chapter can be summarised in the following theorem, which is an amalgam of Theorem 6.2.7, Theorem 6.4.17 and Proposition 6.1.3.

Theorem 1.0.2. *Let G be a non-torsion Šunić group acting on the binary rooted tree. If G is different from the infinite dihedral group, then G contains countably many maximal subgroups of infinite index, all isomorphic to G. In particular, there exist branch groups of intermediate growth with maximal subgroups of infinite index.*

Thus, we see that even in the class of branch group, having intermediate growth is not sufficient to ensure that every maximal subgroup is of finite index. The proof of Theorem 1.0.2 occupies Sections 6.1 to 6.4. In Section 6.5, we briefly describe, for completeness, the maximal subgroups of finite index of Šunić groups, so that we have a complete description of all maximal subgroups of those groups.

After Theorem 1.0.2 and Proposition 4.2.2, one might expect that a Šunić group which is not periodic always contain a maximal subgroup of infinite index. However, in **Chapter 7**, we show that this is not the case. More precisely, we show in Theorem 7.2.7 that a special class of non-torsion Šunić groups, that we call the *generalised Fabrykowski-Gupta* groups, belong to the class $\mathcal{MF}$. In particular, this shows that there are branch groups of intermediate growth which are in $\mathcal{MF}$ but are not periodic and not LERF. Thus, even in the class of Šunić groups, the relationship between torsion and maximal subgroups of infinite index seems to be more subtle.

In **Chapter 8**, in order to have a better idea of the range of behaviours that groups in $\mathcal{MF}$ can exhibit, we investigate the maximal subgroups of a group known as the Basilica group, which is in many respects quite different from the groups that were considered up until now. The Basilica group was first studied by Grigorchuk and Żuk in [52] as an interesting example of a group generated by an automaton with three states and two letters. It was later shown by Bartholdi and Virág [12] to be amenable, thus making it, thanks to a result in [52], the first example of an amenable but not subexponentially amenable group. In contrast to the branch groups that were shown to be in $\mathcal{MF}$, the Basilica group is torsion-free, of exponential growth, does not have the congruence subgroup property, is not just-infinite, is not just-non-nilpotent and is not branch, but only weakly branch. Nevertheless, we show in Theorem 8.4.12 that every maximal subgroup of the Basilica group is of finite index. This shows that none of the above properties are necessary conditions for a weakly branch group to belong to the class $\mathcal{MF}$. Therefore, any algebraic characterisation of weakly branch groups in this class, if it exists, must be more subtle.

Finally, we conclude in **Chapter 9** by expanding our investigations to a different but no less important class of subgroups in groups acting on rooted trees, namely *parabolic* subgroups. Parabolic subgroups are the stabilisers of points on the boundary of the rooted tree. Although they can never be maximal, in branch groups they are weakly maximal [8], which makes their study a natural generalisation of the study of maximal subgroups of infinite index. Furthermore, parabolic subgroups also play an important role in the theory of representations of branch groups (see for example [8, 26, 47]), so it seems important to have a better understanding of them. In this chapter, we investigate various algebraic aspects of these subgroups. In Section 9.1, we study the rank of parabolic subgroups. We show in Theorem 9.1.1 that parabolic subgroups of weakly maximal groups are never finitely generated, as long as the rigid stabilisers satisfy a technical uniformity condition.

In Sections 9.2 and 9.3, we study the isomorphism classes of parabolic subgroups and show that such subgroups are frequently isomorphic. The following theorem summarises some of the main results of these sections.

Theorem 1.0.3. *Let G be a finitely generated self-similar regular branch group, and let $P_1, P_2 \leq G$ be two parabolic subgroups. If both P_1 and P_2 have trivial groups of germs, then they are isomorphic. On the other hand, if the groups of germs of P_1 and P_2 are not isomorphic, then P_1 and P_2 are not isomorphic.*

Note that although we stated the result here for self-similar regular branch groups, some of the results in Sections 9.2 and 9.3 hold for much larger classes of groups. Interestingly, we also apply them to Thompson's group F in Proposition 9.2.14. Although the result we obtain in this case is not new, this suggests that these techniques could be used to study stabilisers in a large class of groups, even if they do not necessarily act on a rooted tree.

In the last section, Section 9.4, we study the index of parabolic subgroups in groups acting on rooted trees. It is easy to construct examples of infinite groups acting on rooted trees in such a way that all of their parabolic subgroups are of finite index. It is thus natural to ask under which conditions the existence of a parabolic subgroup of infinite index is guaranteed. In the case of automata groups, it was asked by D'Angeli, Rodaro and Wächter ([22], Open Problem 4.3) if the existence of a parabolic subgroup of infinite index is equivalent to the infiniteness of the group. In Theorem 9.4.8, we show that this is indeed the case.

A note about previously published results

We should note that the results of Chapter 3 were already published in [31] and that most of this chapter was taken from this article with minor modifications. Similarly, the results of Section 4.3 and Chapter 6 come from a joint work with Alejandra Garrido and were already published in [32]. Again, large parts of the text were borrowed from that article and modified or expanded upon as necessary.

Lastly, we would like to mention that parts of the results of Chapter 7 were obtained by working in collaboration with Alejandra Garrido, although they have not (at least at the time of writing) been written up and submitted for publication.

Chapter 2

Background

In this chapter, we collect various basic definitions and results which we will assume to be known throughout this text. We try to keep everything as self-contained as possible, but proofs will frequently be omitted when they are simple or present little interest.

2.1 Primitive actions

In this section, we give the definition of a primitive action, a notion that is closely linked with maximal subgroups, as we will explain.

Let us first define the notion of a *block* for a group action.

Definition 2.1.1. Let X be a set and let G be a group acting on X. A non-empty subset $B \subseteq X$ is called a *block* if for all $g \in G$, we have either $gB = B$ or $gB \cap B = \emptyset$.

Remark 2.1.2. Obviously, for any set X and any group G acting on X, we always have that the whole set X and singletons $\{x\}$ are blocks. We will call these *trivial blocks*.

A primitive action is an action that admits no non-trivial blocks.

Definition 2.1.3. Let X be a set of cardinality at least two and let G be a group acting on X. We say that the action of G on X is *primitive* if G acts transitively on X and admits no non-trivial blocks.

In other words, a primitive action is an action that does not preserve any non-trivial partition.

Remark 2.1.4. Note that the hypothesis of transitivity is only required for the case where $|X| = 2$. Indeed, if $|X| \geq 3$, then the non-existence of non-trivial blocks implies transitivity.

Let us see a few examples of primitive actions.

Example 2.1.5. Let $m \geq 2$ be an integer and let $X = G = \mathbb{Z}/m\mathbb{Z}$. Then, G acts naturally on X by left-multiplication. It is easy to show that this action is primitive if and only if m is a prime number.

Example 2.1.6. Let X be a set and let G be a group acting on X. If the action of G on X is 2-transitive (meaning that it is transitive on pairs of distinct points), then it is primitive. ❧

Primitive actions are closely linked with maximal subgroups[i], since the latter are exactly the stabilisers of points of primitive actions, as the next proposition shows.

Proposition 2.1.7. *Let G be a group and let $M < G$ be a proper subgroup. Then, M is maximal if and only if the action of G on G/M is primitive.*

2.2 Permutational wreath product

In this section, we will briefly define the restricted permutational wreath product of two groups and discuss its order in the finite case. This construction is of great importance in the theory of group acting on rooted trees and will thus make regular appearances throughout this text.

We begin by giving the definition.

Definition 2.2.1. Let H be a group, let X be a set and let G be a group acting on X on the left. The *restricted permutational wreath product* of H by G with respect to X, denoted by $H \wr_X G$, is the group

$$H \wr_X G = G \ltimes K,$$

where K is the group of finitely supported maps from X to H and the (right) action of G on K is given by

$$(f \cdot g)(x) = f(g \cdot x).$$

Thus, for $(g, f), (g', f') \in H \wr_X G$, we have

$$(g, f)(g', f') = (gg', (f \cdot g')f').$$

❧

Remark 2.2.2. The definition of the permutational wreath product is often given in terms of right action, but we give here the definition in terms of left actions, as in general, throughout this text, we will mainly consider left actions. ❧

Notation 2.2.3. If the space X on which the group G acts is clear from the context, we will often write simply $H \wr G$ instead of $H \wr_X G$.

Moreover, we will frequently drop the words *restricted* and *permutational* and only write *wreath product* to talk about $H \wr_X G$. ❧

Given a group G acting on a set X, one can in particular take the wreath product of G with itself. One can then take the wreath product of this wreath product with G, and so on. This gives rise to the notion of an iterated wreath product.

[i] see Definition 1.0.1 for the definition of a maximal subgroup

Definition 2.2.4. Let X be a set, and let G be a group acting on X. For $n \in \mathbb{N}$, the n^{th} *iterated wreath product of* G is the group

$$\wr^n G = (\wr^{n-1} G) \wr_X G,$$

where $\wr^0 G = \{1\}$ (and thus $\wr^1 G = G$). ☙

The notion of iterated wreath product will be especially important later when we discuss groups acting on regular rooted trees.

If G is a finite group acting on a finite set X, and if H is also a finite group, then the wreath product $H \wr_X G$ is also finite and its order is easily determined in terms of the sizes of G, H and X.

Proposition 2.2.5. *Let X be a finite set, let G be a finite group acting on X and H be a finite group. Then, we have*

$$|H \wr_X G| = |H|^{|X|}|G|.$$

In particular, assuming that $|X| \geq 2$, we have

$$|\wr^n G| = |G|^{\frac{|X|^n - 1}{|X| - 1}}.$$

2.3 Alphabets and words

In this section, we will define the notion of words over a finite alphabet, which will be used throughout this text.

Definition 2.3.1. An *alphabet* is a finite set X. A *letter* is an element of an alphabet. A *word over the alphabet* X is a finite sequence $w = a_1 a_2 \dots a_n$ of letters in X, and its *length*, which we will denote by $|w|$, is the length of the sequence. The word of length 0 will be called the *empty word* and be denoted by ε. ☙

Notation 2.3.2. Let X be a finite set and let $n \in \mathbb{N}$ be an integer. We will denote by X^n the set of words of length n over the alphabet X. We will denote by $X^* = \bigcup_{n=0}^{\infty} X^n$ the set of all words in the alphabet X. ☙

There is a natural operation on words, called concatenation.

Definition 2.3.3. Let X be an alphabet and $w = a_1 \dots a_n$, $v = b_1 \dots b_m$ be two words in X^*. The word $wv = a_1 \dots a_n b_1 \dots b_m$ is called the *concatenation* of w and v. ☙

Proposition 2.3.4. *Let X be a finite set. Then, X^* with the operation of* concatenation *is a monoid, called the* free monoid on X.

Using the notion of concatenation, we can define the notion of prefix. This induces a partial order on the set of words over an alphabet.

Definition 2.3.5. Let X be an alphabet and let $v, w \in X^*$ be two words over X. We say that v is a *prefix* of w, and we write $v \leq w$, if there exists $w' \in X^*$ such that $w = vw'$. ☙

Proposition 2.3.6. *Let X be an alphabet. Then, $\leq$ defines a partial order on the set X^*.*

In addition to finite words, one can also define right-infinite words over the alphabet X.

Definition 2.3.7. Let X be an alphabet. A *(right-)infinite word over X* is a right-infinite sequence $w = a_1a_2a_3\dots$ of elements of X. We will denote the set of all right-infinite words by X^∞. ☙

We cannot concatenate two infinite word, but we can still define an operation of concatenation between a finite word and an infinite word.

Definition 2.3.8. Let X be an alphabet, $w = a_1 \dots a_n \in X^*$ be a word over X and $\xi = b_1b_2\dots \in X^\infty$ be an infinite word over X. The *concatenation* of w and ξ, written $w\xi$, is the infinite word $w\xi = a_1\dots a_nb_1b_2\dots \in X^\infty$. ☙

Remark 2.3.9. Let X be an alphabet. Concatenation between finite and infinite words over X is associative, in the sense that if $v, w \in X^*$ are two finite words and $\xi \in X^\infty$ is an infinite word, we have $v(w\xi) = (vw)\xi$. ☙

Thanks to this concatenation, the notion of prefix still makes sense for infinite words.

Definition 2.3.10. Let X be an alphabet, $w \in X^*$ be a word over X and $\xi \in X^\infty$ be an infinite word over X. We say that w is a prefix of ξ, and write $w \leq \xi$, if there exists $\xi' \in X^\infty$ such that $\xi = w\xi'$. ☙

This allows us to define a topology on $X^* \sqcup X^\infty$ that makes it a compact space.

Proposition 2.3.11. *Let X be an alphabet and let $\mathcal{T}$ be the topology on $X^* \sqcup X^\infty$ defined by the basis of open sets $\{\{w\} \mid w \in X^*\} \cup \{C_w \mid w \in X^*\}$, where*

$$C_w = \{\zeta \in X^* \sqcup X^\infty \mid w \leq \zeta\}.$$

Then, $X^* \sqcup X^\infty$ with the topology $\mathcal{T}$ is a compact space. Furthermore, if the cardinality of X is at least two, then X^∞, equipped with the subspace topology, is a Cantor space (i.e. homeomorphic to the Cantor set).

In this topology, a sequence $(w_i)_{i\in\mathbb{N}}$ converges to an infinite word $\xi \in X^\infty$ if and only if w_i and ξ share larger and larger common prefixes as i goes to infinity.

2.4 Graphs

In this section, we will briefly recall a few basic notions related to graphs.

Definition of a graph

There are various definitions of the notion of a graph. Here, we will use the one given by Jean-Pierre Serre in his book [80].

Definition 2.4.1. A *graph* is a tuple $\Gamma = (V, E, \bar{\ }, o, t)$, where V and E are sets called respectively the set of *vertices* and the set of *(oriented) edges*, $o, t \colon E \to V$ are maps from the set of edges to the set of vertices and $\bar{\ } \colon E \to E$ is an involution without fixed points such that $o(\bar{e}) = t(e)$ for all $e \in E$. The vertex $o(e)$ is called the *origin* of e and the vertex $t(e)$ is called the *terminus* of e. The edge $\bar{e}$ is called the *inverse* edge. An *unoriented edge* is a subset of E of the form $\{e, \bar{e}\}$.

An *orientation* of the graph Γ is a subset $E_+ \subset E$ such that $E_+ \cap E_- = \emptyset$ and $E_+ \cup \bar{E}_+ = E$. An *oriented graph* is a graph Γ together with an orientation.

Notation 2.4.2. In what follows, we will frequently omit the maps $\bar{\ }, o$ and t from the notation and write simply $\Gamma = (V, E)$.

Remark 2.4.3. While the definition of a graph given above is very general, it is sometimes cumbersome to define graphs using this formalism. For this reason, we will frequently define graphs by giving only a set V and a set E whose elements are subsets of V of size 1 or 2. This is an abuse of notation, since the set E in this case is not really the set of edges, but the set of *unoriented* edges, but this should significantly simplify the presentation without introducing any confusion.

Graphs are frequently represented diagrammatically as points connected by lines, such as in Figure 2.4. An edge in such a diagram corresponds to an unoriented edge in the graph. One can also represent oriented graphs by adding arrows pointing from the origin to the terminus of each edge according to the orientation of the graph.

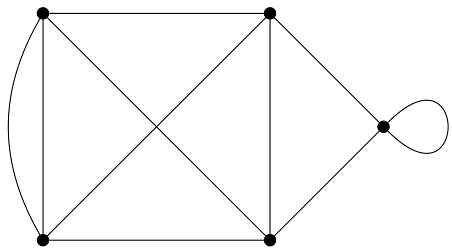


Figure 2.1: A pictorial representation of a graph.

To each vertex of a graph, one can associate a (potentially infinite) number called the degree of the vertex.

Definition 2.4.4. Let $\Gamma = (V, E)$ be a graph and let $v \in V$ be a vertex. The *degree* of v is $\deg(v) = |\{e \in E \mid o(e) = v\}|$. The graph Γ is said to be *locally finite* if $\deg(v) < \infty$ for all $v \in V$, and is said to be k-regular if $\deg(v) = k$ for all $v \in V$.

Morphisms

There is a natural definition of morphism between graphs.

Definition 2.4.5. Let $\Gamma_1 = (V_1, E_1, \bar{\cdot}, o_1, t_1)$ and $\Gamma_2 = (V_2, E_2, \tilde{\cdot}, o_2, t_2)$ be two graphs. A morphism $f\colon \Gamma_1 \to \Gamma_2$ between Γ_1 and Γ_2 is a pair of maps (f_V, f_E), where $f_V\colon V_1 \to V_2$ and $f_E\colon E_1 \to E_2$, such that for all $e \in E_1$, we have

(i) $f_E(\bar{e}) = \widetilde{f_E(e)}$,

(ii) $f_V(o(e)) = o(f_E(e))$.

If $\Gamma_2 = \Gamma_1$, f is said to be an *endomorphism*.

The morphism f is said to be *injective* if both f_V and f_E are injective. ☙

Notice that if $f = (f_V, f_E)$ is a morphism from Γ_1 to Γ_2 and $g = (g_V, g_E)$ is a morphism from Γ_2 to Γ_3, their composition $g \circ f := (g_V \circ f_V, g_E \circ f_E)$ is a morphism from Γ_1 to Γ_3.

Definition 2.4.6. Let $\Gamma_1 = (V_1, E_1, \bar{\cdot}, o_1, t_1)$ and $\Gamma_2 = (V_2, E_2, \tilde{\cdot}, o_2, t_2)$ be two graphs and $f\colon \Gamma_1 \to \Gamma_2$ be a morphism. If there exists a morphism $g\colon \Gamma_2 \to \Gamma_1$ such that $g \circ f = \mathrm{id}_{\Gamma_1}$ and $f \circ g = \mathrm{id}_{\Gamma_2}$, where id_Γ is the identity morphism of the graph Γ, then f is an *isomorphism*. In the case where $\Gamma_1 = \Gamma_2$, the isomorphism f is called an *automorphism*. ☙

Using morphisms, we can define the notion of subgraphs.

Definition 2.4.7. Let $\Gamma_1 = (V_1, E_1, \bar{\cdot}, o_1, t_1)$ and $\Gamma_2 = (V_2, E_2, \tilde{\cdot}, o_2, t_2)$ be two graphs. We say that Γ_1 is a *subgraph* of Γ_2 if there exists an injective morphism $f\colon \Gamma_1 \to \Gamma_2$. ☙

Paths and connectedness

Given a graph, one can define define the notion of a path.

Definition 2.4.8. Let $\Gamma = (V, E)$ be a graph and $n \in \mathbb{N}$ be an integer. For $n \geq 1$, a *(finite) path (of length n)* is a finite sequence $p = (e_1, \dots, e_n)$ of edges such that $t(e_i) = o(e_{i+1})$ for all $1 \leq i < n$. The vertex $o(p) := o(e_1)$ is called the *origin* of p, the vertex $t(p) := t(e_n)$ is called the *terminus* of p and we say that p is a path from $o(p)$ to $t(p)$.

If $n = 0$, for every $v \in V$, we define the *path of length* 0 *at* v, or *empty path at* v, as the empty sequence of edges $p = ()$ where we set $o(p) = t(p) = v$.

A *(right-)infinite path* is a right-infinite sequence of edges $p = (e_1, e_2, \dots)$ such that $t(e_i) = o(e_{i+1)}$ for all $i \geq 1$. In this case, we define the *origin* of p as $o(e_1)$.

A *bi-infinite path* is a bi-infinite sequence of edges $p = (\dots, e_{-1}, e_0, e_1, \dots)$ such that $t(e_i) = o(e_{i+1})$ for all $i \in \mathbb{Z}$.

Let (e_i, e_{i+1}) be a subsequence of length 2 of a (possibly infinite) path p. If $e_{i+1} = \bar{e_i}$, this subsequence is called a *backtracking*. A path is said to be *reduced* if it contains no backtrackings. ☙

Notation 2.4.9. In what follows, if p is a finite path in a graph Γ, we will denote the length of p by $|p|$. ⚘

Remark 2.4.10. If $p = (e_1, \ldots, e_n)$ is a finite path from v to w, there also exists a (possibly empty) reduced path between the same two vertices, since removing a backtracking from a path gives us a shorter path between the same vertices. ❧

Using the notion of a path, we can define the connected components of a graph.

Definition 2.4.11. Let $\Gamma = (V, E)$ be a graph and $v \in V$ be a vertex. The *connected component of* v is the set of all $w \in V$ such that there exists a path p from v to w. ❧

Proposition 2.4.12. *The relation on V given by $v \sim w$ if there exists a path p such that $o(p) = v$ and $t(p) = w$ is an equivalence relation. Therefore, the set V can be partitioned into connected components.*

Definition 2.4.13. A graph $\Gamma = (V, E)$ is *connected* if V consists of only one connected component. In other words, Γ is connected if for all $v, w \in V$, there exists a path p from v to w. ❧

Paths and strong connectedness for oriented graphs

In the case of oriented graphs, one usually wants paths to consist of edges that are oriented consistently with the given orientation of the graph. This then gives rise to the notion of strongly connected components.

Definition 2.4.14. Let $\Gamma = (V, E)$ be a graph with a given orientation E_+. A path p is said to respect the orientation if $e_i \in E_+$ for every edge e_i in the path p.

For every $v \in V$, the *strongly connected component of* v is the set of all $w \in V$ such that there exist paths p and q from v to w and w to v, respectively, respecting the orientation.

The oriented graph Γ is said to be strongly connected if all the vertices of V belong to the same strongly connected component. ❧

Metric

Paths allow us to define a natural metric on the vertex set of connected graph.

Definition 2.4.15. Let $\Gamma = (V, E)$ be a connected graph, and for $v, w \in V$, let $P_{v,w}$ be the set of paths from v to w. The *distance* between v and w is

$$d_\Gamma(v, w) = \min \{|p| \in \mathbb{N} \mid p \in P_{v,w}\}.$$

A path $p \in P_{v,w}$ such that $|p| = d_\Gamma(v, w)$ is called a *geodesic*. ❧

Notice that $d_\Gamma \colon V \times V \to \mathbb{N}$ is a metric on V.

Cycles and trees

Cycles are paths in a graph that begin and end with the same vertex.

Definition 2.4.16. Let $\Gamma = (V, E)$ be a graph. A *cycle* in Γ is a path c of length $n \geq 1$ such that $o(c) = t(c)$. If c is of length 1, then it is called a *loop*.

A *reduced cycle* is a cycle c without backtrackings. ❧

Graphs without reduced cycles form a special class of graphs.

Definition 2.4.17. Let $\Gamma = (V, E)$ be a graph without reduced cycles. If Γ is connected, it is called a *tree*. If Γ is not connected, it is called a *forest*. ❧

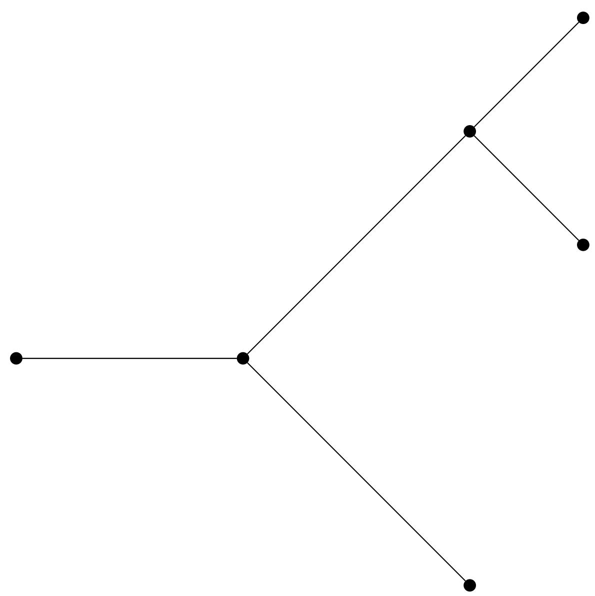


Figure 2.2: A tree.

Trees possess many interesting properties. Here, we present a few that will be useful later on.

Proposition 2.4.18. *In a tree, there exists a unique reduced path between two vertices. In particular, geodesics in trees are unique.*

Proposition 2.4.19. *Every connected subgraph of a tree is also a tree.*

2.5 Growth of groups

In this section, we will introduce a very important asymptotic invariant in geometric group theory called the *growth* of a group. According to [53] and [46], this concept was first studied by Švarc in [84] and was also independently studied by Milnor in [61].

Growth type of non-decreasing functions

Before we define growth in groups, we first need to define the notion of growth type for non-decreasing functions.

Definition 2.5.1. Given two non-decreasing functions $f, g\colon \mathbb{N} \to \mathbb{N}$, we say that g *dominates* f, and write $f \lesssim g$, if there exist $C, D \in \mathbb{N}^*$ such that $f(n) \leq Cg(Dn)$ for all $n \in \mathbb{N}^*$. The functions f and g are said to be *equivalent* or of the same *growth type*, written $f \sim g$, if $f \lesssim g$ and $g \lesssim f$. We say that

- f is of *polynomial growth* if there exists $d \in \mathbb{N}$ such that $f \lesssim n^d$
- f is of *superpolynomial growth* if $n^d \lnsim f$ for all $d \in \mathbb{N}$
- f is of *exponential growth* if $f \sim e^n$
- f is of *subexponential growth* if $f \lnsim e^n$
- f is of *intermediate growth* if f is of superpolynomial growth and of subexponential growth.

☙

We will call the equivalence class of a non-decreasing function under this equivalence relation its *growth type*.

Word metric

Given a finite and symmetric (i.e. closed under the operation of taking the inverse) generating set of a group, one can construct a natural metric on the group called the *word metric*.

Definition 2.5.2. Let G be a finitely generated group and S be a symmetric finite generating set. The *word norm* on G (with respect to S) is the map

$$\begin{aligned} |\cdot|_S\colon G &\to \mathbb{N} \\ g &\mapsto \min\{k \in \mathbb{N} \mid g = s_1 \dots s_k, s_i \in S\}\,. \end{aligned}$$

The *word metric* on G (with respect to S) is the metric

$$\begin{aligned} d_S\colon G \times G &\to \mathbb{N} \\ (g, h) &\mapsto |g^{-1}h|_S \end{aligned}$$

induced by the word norm on G. ☙

There is a nice geometric interpretation to this metric. Indeed, to any group G with a given finite symmetric generating set S, one can associate a graph called the *Cayley graph* of G.

Definition 2.5.3. Let G be a group and let S be a finite symmetric generating set of G. The *(right) Cayley graph of G with respect to S* is the graph $\mathrm{Cay}(G, S) = (G, E)$, where

$$E = \{\{g, gs\} \subset G \mid g \in G, s \in S\}\,.$$

☙

It is easy to see that the word metric is simply the metric coming from the Cayley graph of G with the generating set S (see Definition 2.4.15).

Growth of groups

Given a group G with a finite symmetric generating set S, it is clear that for each $n \in \mathbb{N}$, there is only a finite number of elements at distance at most n from the identity. The growth function of G can then be defined as the function counting the number of elements in balls centred at the identity.

Definition 2.5.4. Let G be a group generated by a finite and symmetric set S. The map

$$\begin{aligned} \gamma_{G,S} \colon \mathbb{N} &\to \mathbb{N} \\ n &\mapsto |B_{G,S}(n)|, \end{aligned}$$

where $B_{G,S}(n)$ is the ball of radius n centred at the identity in the word metric of G with respect to S, is the *growth function* of G (with respect to S). ☙

The growth function of a group depends on the generating set we choose. However, as we will see below, the *growth type* of the growth function does not depend on the choice of generating set.

Proposition 2.5.5. *Let G be a finitely generated group and S, T be two finite symmetric generating set. Then, $\gamma_{G,S} \sim \gamma_{G,T}$.*

Proof. Let

$$C = \max_{t \in T}\{|t|_S\}.$$

Then, clearly, for all $g \in G$, we have $|g|_S \leq C|g|_T$. Hence,

$$\gamma_{G,T}(n) \leq \gamma_{G,S}(Cn),$$

so $\gamma_{G,T} \lesssim \gamma_{G,S}$. The result follows from the symmetry of the argument. □

Thanks to proposition 2.5.5, we see that the growth type of the growth function is independent of the choice of generating set and is thus a property of the group. This allows us to define the growth of a group.

Definition 2.5.6. Let G be a finitely generated group. The *growth* of G is the growth type of the growth function $\gamma_{G,S}$ for any finite symmetric generating set S.

The group G is said to be of *polynomial, superpolynomial, exponential, subexponential or intermediate growth* if $\gamma_{G,S}$ is of polynomial, superpolynomial, exponential, subexponential or intermediate growth, respectively. ☙

Notation 2.5.7. In what follows, for a group G with a finite symmetric generating set S, we will frequently drop the S from the notation and write simply γ_G for the growth function of G with respect to S. Furthermore, we will generally not make any distinction between the growth of the group G and the growth function associated to a specific generating set and rely on context to distinguish between the two. ❧

As one would expect, the growth of subgroups and quotients is bounded from above by the growth of the group.

Proposition 2.5.8. *Let G be a finitely generated group, $H \leq G$ be a finitely generated subgroup and $N \trianglelefteq G$ be a normal subgroup of G. Then,* *(i)* $\gamma_H \lesssim \gamma_G$,

(ii) $\gamma_{G/N} \lesssim \gamma_G$.

By the above proposition, since the growth of a non-abelian free group is exponential, a finitely generated group cannot have a growth faster than exponential. Therefore, the growth of a finitely generated group must belong to one of three broad classes: polynomial, intermediate or exponential. Thanks to a result of Gromov, we have an algebraic characterisation of groups of polynomial growth.

Theorem 2.5.9 (Gromov's Theorem, [53])**.** *A finitely generated group G is of polynomial growth if and only if it is virtually nilpotent.*

Exponential growth rate

In what follows, we will be mainly interested in distinguishing between groups of exponential and subexponential growth. For this purpose, it will be convenient to study a quantity called the exponential growth rate of the group.

Proposition 2.5.10. *Let G be a finitely generated group and S be a finite symmetric generating set. The limit*

$$\kappa_{G,S} = \lim_{n\to\infty} \gamma_{G,S}(n)^{\frac{1}{n}}$$

exists.

Proof. It follows easily from the definition that for all $n, m \in \mathbb{N}$,

$$\gamma_{G,S}(n+m) \leq \gamma_{G,S}(n)\gamma_{G,S}(m).$$

Hence, $\ln \gamma_{G,S}$ is a subadditive function, so according to Fekete's subadditive lemma (see [76], Problem 98), the limit

$$\lim_{n\to\infty} \frac{\ln \gamma_{G,S}(n)}{n}$$

exists. The result follows. □

Definition 2.5.11. The limit

$$\kappa_{G,S} = \lim_{n\to\infty} \gamma_{G,S}(n)^{\frac{1}{n}}$$

is the *exponential growth rate* of the group G with respect to the generating set S. ☙

The exponential growth rate of a group depends on the choice of generating set. However, whether it is equal to 1 or strictly greater than 1 depends only on whether the group is of subexponential or of exponential growth.

Proposition 2.5.12. *Let G be a finitely generated group with a finite symmetric generating set S. Then, $\kappa_{G,S} > 1$ if and only if G is of exponential growth.*

It will sometimes be more convenient to consider the growth of spheres instead of the growth of balls. In the case of infinite finitely generated groups, the exponential growth rate can also be calculated from the size of spheres.

Proposition 2.5.13. *Let G be an infinite finitely generated group with finite symmetric generating set S. Then,*

$$\kappa_{G,S} = \lim_{n\to\infty} |\Omega_{G,S}(n)|^{\frac{1}{n}},$$

where $\Omega_{G,S}(n)$ is the sphere of radius n in the word metric on G with respect to S.

Proof. It follows from the definition of the word metric that $\Omega_{G,S}(n+m) \leq \Omega_{G,S}(n)\Omega_{G,S}(m)$, so by Fekete's subadditive lemma (see [76], Problem 98), the limit exists.

Since $\Omega_{G,S}(n) \leq \gamma_{G,S}(n)$, we have $\lim_{n\to\infty} |\Omega_{G,S}(n)|^{\frac{1}{n}} \leq \kappa_{G,S}$. On the other hand, we have

$$\gamma_{G,S}(n) = \sum_{i=0}^{n} |\Omega_{G,S}(i)|.$$

If there exists $M \in \mathbb{N}$ such that $|\Omega_{G,S}(n)| \leq M$ for all $n \in \mathbb{N}$, then we have

$$\gamma_{G,S}(n) \leq (n+1)M,$$

so that

$$1 \leq \lim_{n\to\infty} |\Omega_{G,S}(n)|^{\frac{1}{n}} \leq \kappa_{G,S} \leq \lim_{n\to\infty} ((n+1)M)^{\frac{1}{n}} = 1.$$

If $|\Omega_{G,S}(n)|$ is not bounded, then there exists a subsequence $(i_n)_{n\in\mathbb{N}}$ such that $|\Omega_{G,S}(i_n)| \geq |\Omega_{G,S}(j)|$ for all $j \leq i_n$. We then have

$$\begin{aligned}
\kappa_{G,S} = \lim_{n\to\infty} \gamma_{G,S}(i_n)^{\frac{1}{i_n}} &= \lim_{n\to\infty} \left(\sum_{j=0}^{i_n} |\Omega_{G,S}(j)| \right)^{\frac{1}{i_n}} \\
&\leq \lim_{n\to\infty} ((i_n+1)|\Omega_{G,S}(i_n)|)^{\frac{1}{i_n}} \\
&= \lim_{n\to\infty} |\Omega_{G,S}(i_n)|^{\frac{1}{i_n}} = \lim_{n\to\infty} |\Omega_{G,S}(n)|^{\frac{1}{n}}.
\end{aligned}$$

In both cases, we have $\kappa_{G,S} \leq \lim_{n\to\infty} |\Omega_{G,S}(n)|^{\frac{1}{n}} \leq \kappa_{G,S}$, which concludes the proof. □

2.6 Rooted trees and their automorphisms

We will now define rooted trees and their automorphisms, which are central objects of study of this thesis.

Rooted trees

Definition 2.6.1. A *rooted tree* is a tree $T = (V, E)$ along with a distinguished vertex $v_0 \in V$ called the *root*.

A vertex $v \in V$ different from v_0 is called a *leaf* if $\deg(v) = 1$. ☙

Notation 2.6.2. Let $T = (V, E, v_0)$ be a rooted tree. We will frequently abuse the notation and write $v \in T$ instead of $v \in V$ for a vertex of the tree.

Notation 2.6.3. Let $T = (V, E, v_0)$ be a rooted tree. For any vertex $v \in V$, we will denote by $|v|$ the distance $d_T(v, v_0)$ between v and the root v_0 and call it the *length* of the vertex v.

Length of vertices partition the vertex set of a rooted tree into levels.

Definition 2.6.4. Let $T = (V, E, v_0)$ be a rooted tree and $n \in \mathbb{N}$ be an integer. The n^{th}-level of the tree is the set

$$L_n = \{v \in V \mid |v| = n\}.$$

The fact that every vertex is connected to the root by a unique geodesic allows us to define a partial order on the set of vertices of a rooted tree.

Definition 2.6.5. Let $T = (V, E, v_0)$ be a rooted tree and $v, w \in V$ be two vertices. If the unique geodesic from v_0 to w passes through v, we will say that w is a *descendant* of v and write $v \leq w$. If furthermore we have $|w| = |v| + 1$, we will say that w is a *child* of v and that v is a *parent* of w.

Proposition 2.6.6. *The descendance relation forms a partial order on the set of vertices of a rooted tree.*

In what follows, we will be interested in studying the automorphisms of rooted trees, and therefore will mostly want to restrict our attention to highly symmetric trees. This leads us to the definition of spherically homogeneous and regular rooted trees.

Definition 2.6.7. Let $T = (V, E, v_0)$ be a rooted tree. We will say that T is *spherically homogeneous* if it has no leaves[ii] and if for all $n \in \mathbb{N}$, we have $\deg(v) = \deg(w)$ for all $v, w \in L_n$.

If there exists $k \geq 1$ such that $\deg(v_0) = k$ and $\deg(v) = k + 1$ for all $v \in V \setminus \{v_0\}$, the rooted tree T is said to be *k-regular.*

Notice that a k-regular rooted tree is not a k-regular tree in the usual sense of the word.

Subtrees

There exists a natural notion of rooted subtrees of a rooted tree.

Definition 2.6.8. Let $T = (V, E, v_0)$ and $T' = (V', E', v_0')$ be rooted trees. We say that T' is a *rooted subtree* of T if there exists an injective morphism of graphs $f \colon (V', E') \to (V, E)$ such that $f(v_0') \leq f(v')$ for all $v' \in V'$.

To any vertex of a rooted tree T, one can associate a natural subtree of T rooted at v.

[ii] Please note that the notion of spherically homogeneous rooted trees could also make sense for trees with leaves. However, such a tree would necessarily be finite. As we will mainly be concerned with infinite trees in this text, we choose for convenience to exclude the finite case from the definition.

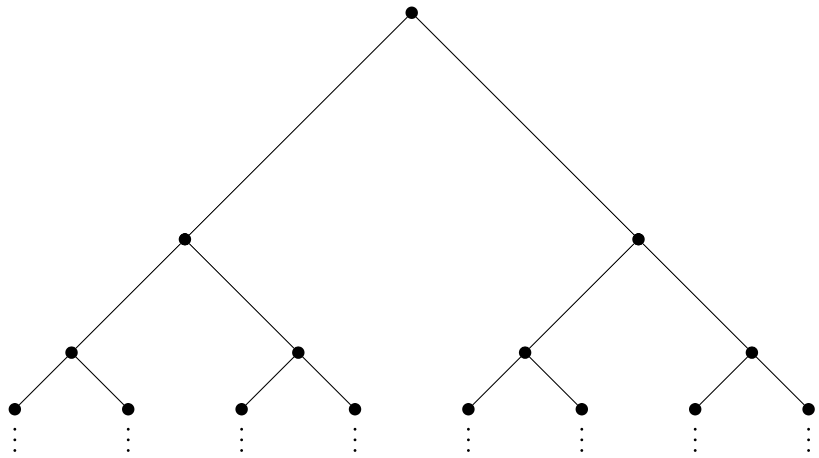


Figure 2.3: The binary (2-ary) rooted tree.

Proposition 2.6.9. *Let $T = (V, E, v_0)$ be a rooted tree and let $v \in V$ be a vertex. Let*

$$V_v = \{w \in V \mid v \leq w\}$$

and let

$$E_v = \{e \in E \mid o(e) \in V_v \text{ and } t(e) \in V_v\}.$$

Then, $T_v = (V_v, E_v, v)$ is a rooted subtree of T.

Proof. It is sufficient to show that T_v is a rooted tree, but this follows from the fact that it is connected and Proposition 2.4.19. ☐

Definition 2.6.10. Let T be a rooted tree and $v \in T$ be a vertex. The rooted subtree T_v of T described in Proposition 2.6.9 is called the *subtree rooted at* v. ❧

Automorphisms

We will now introduce the group of automorphisms of a rooted tree and some of its important subgroups. We begin by defining an automorphism of rooted trees.

Definition 2.6.11. Let $T = (V, E, v_0)$ be a rooted tree. An *automorphism* of T is an automorphism of graphs $f : (V, E) \to (V, E)$ such that $f(v_0) = v_0$. ❧

Clearly, the set of all automorphisms of a rooted tree T with the operation of composition forms a group.

Notation 2.6.12. If T is a rooted tree, we will denote the group of automorphisms of T by $\mathrm{Aut}(T)$. For $g \in \mathrm{Aut}(T)$ and $v \in T$, we will denote the image of v by g by $g \cdot v$, or simply by gv when no confusion is possible. ☙

Note that, as there is at most one edge between two vertices of a rooted tree $T = (V, E, v_0)$, an automorphism $f: T \to T$ is uniquely determined by the way it acts on the set of vertices V. Therefore, we will sometimes abuse the notation and refer to maps $f_V: V \to V$ as automorphisms of T and vice-versa.

Using this slight abuse of notation, automorphisms of rooted trees can be characterised as bijections on the vertex set that preserve length and the partial order on vertices.

Proposition 2.6.13. *Let $T = (V, E, v_0)$ be a rooted tree. A map $f: V \to V$ is an automorphism of T if and only if it satisfies the following conditions:*

(i) f is bijective,

(ii) $|f(v)| = |v|$ for all $v \in V$,

(iii) for all $v, w \in V$, we have $v \leq w$ if and only if $f(v) \leq f(w)$.

We now introduce a few important subgroups of groups of automorphisms of a rooted tree.

Definition 2.6.14. Let T be a rooted tree and let $G \leq \mathrm{Aut}(T)$ be a group of automorphisms of T.

(i) For $v \in T$ a vertex, the subgroup $\mathrm{St}_G(v) = \{g \in G \mid gv = v\}$ is called the *stabiliser of v*.

(ii) For $n \in \mathbb{N}$, the subgroup $\mathrm{St}_G(n) = \bigcap_{v \in L_n} \mathrm{St}_G(v)$ is called the *stabiliser of level n*.

(iii) For $v \in T$, the subgroup $\mathrm{Rist}_G(v) = \{g \in G \mid gw = w \quad \forall w \notin T_v\}$ is called the *rigid stabiliser of v*.

(iv) For $n \in \mathbb{N}$, the subgroup $\mathrm{Rist}_G(n) = \prod_{v \in L_n} \mathrm{Rist}_G(v)$ is called the *rigid stabiliser of level n*.

Remark 2.6.15. If $v, w \in L_n$ are such that $v \neq w$, then $\mathrm{Rist}_G(v) \cap \mathrm{Rist}_G(w) = 1$ and it is easy to check that the elements of $\mathrm{Rist}_G(v)$ commute with the elements of $\mathrm{Rist}_G(w)$. Therefore, $\mathrm{Rist}_G(n)$ is a well-defined subgroup of G.

Notation 2.6.16. Let T be a rooted tree, $v \in T$ be a vertex and $n \in \mathbb{N}$ be an integer. We will write $\mathrm{St}(v)$, $\mathrm{St}(n)$, $\mathrm{Rist}(v)$ and $\mathrm{Rist}(n)$ for $\mathrm{St}_{\mathrm{Aut}(T)}(v)$, $\mathrm{St}_{\mathrm{Aut}(T)}(n)$, $\mathrm{Rist}_{\mathrm{Aut}(T)}(v)$ and $\mathrm{Rist}_{\mathrm{Aut}(T)}(n)$, respectively.

Proposition 2.6.17. *Let T be a rooted tree and $G \leq \mathrm{Aut}(T)$ be a group of automorphisms of T. For all $n \in \mathbb{N}$, the subgroups $\mathrm{St}_G(n)$ and $\mathrm{Rist}_G(n)$ are normal in G.*

Regular rooted trees and their automorphisms

We will now consider in more details the special case of regular rooted trees and their automorphisms. Most of what we present here can be generalised in some form to spherically homogeneous rooted trees, but we will not need such generality here.

Words over alphabets and regular rooted trees

We will see that every regular rooted tree can be represented by the set of words over a finite alphabet.

Proposition 2.6.18. *Let $k \in \mathbb{N}$ be an integer with $k \geq 1$ and let X be a set of cardinality k. Let X^* be the set of words over X (see Section 2.3) and let*

$$E = \{\{v, w\} \subset X \mid v \leq w, |w| = |v| + 1\}.$$

Then, $T = (X^*, E, \varepsilon)$ is a k-regular rooted tree, and every k-regular rooted tree is isomorphic to T.

Notation 2.6.19. If X is a finite set, we will frequently denote the rooted tree (X^*, E, ε) defined above simply by X^*. Note, therefore, that depending on the context, X^* could mean one of three things: the set of words over the alphabet X, the free monoid on X, or the $|X|$-regular rooted tree. We will usually rely on context to differentiate between those three different uses. Note that in what follows, unless otherwise specified, when we write $\mathrm{Aut}(X^*)$, we will always mean the group of automorphisms of the *rooted tree* X^*, not the group of automorphisms of the free monoid X^*. ☙

Let X be a finite alphabet. In Section 2.3, we defined a partial order on X^*. Now, if we view X^* as a $|X|$-regular rooted tree, Proposition 2.6.6 also gives us a partial order on X^*. It is easy to see that these two relations coincide.

For any $v \in X^*$, the subtree T_v rooted at v is simply the subtree whose vertex set is the set of words having v as a prefix. It is easy to see that the map from T_v to X^* that simply deletes the prefix v gives us a canonical isomorphism between T_v and X^*.

Proposition 2.6.20. *Let X be a finite set and let $T = X^*$ be the $|X|$-regular rooted tree. For $v \in X^*$, the map*

$$\begin{aligned} T_v &\to T \\ vw &\mapsto w \end{aligned}$$

is an isomorphism of rooted trees.

Projection to a vertex

Let X be a finite set of cardinality $d \in \mathbb{N}$, and let us consider the d-regular rooted tree X^*. Let us choose $g \in \mathrm{Aut}(X^*)$. As g is an automorphism of the rooted tree X^*, its action must be compatible with prefixes, meaning that if $u, v \in X^*$ are such that $u \leq v$, then $g \cdot u \leq g \cdot v$. In particular, this means that for all $v, w \in X^*$, there exists $w_v \in X^*$ such that

$$g \cdot (vw) = (g \cdot v) w_v.$$

Therefore, for all $v \in X^*$, we can define the map

$$\begin{aligned} g|_v \colon X^* &\to X^* \\ w &\mapsto w_v. \end{aligned}$$

Proposition 2.6.21. *Let X be a finite alphabet and let $g \in Aut(X^*)$ be an automorphism of the rooted tree X^*. Then, for every $v \in X^*$, the map $g|_v \colon X^* \to X^*$ defined above is an automorphism of X^*.*

Proof. It is easy to see that $g|_v$ must preserve length and the partial order, since g does. Therefore, it follows from Proposition 2.6.13 that $g|_v \in Aut(X^*)$. □

This allows us to define, for any $v \in X^*$, a map from $\mathrm{Aut}(X^*)$ to itself.

Definition 2.6.22. Let X be a finite set. For any $v \in X^*$, we define the *projection to v* as the map

$$\begin{aligned} \varphi_v \colon \mathrm{Aut}(X^*) &\to \mathrm{Aut}(X^*) \\ g &\mapsto g|_v. \end{aligned}$$

For $g \in \mathrm{Aut}(X^*)$, we will call $\varphi(g) = g|_v$ the *projection of g at v*. ☙

Remark 2.6.23. Note that the projection of an element is sometimes called by other names in the literature, such as *section* or *restriction*. ☙

Notice that the projection map to a vertex v is in general not a homomorphism, unless $v = \varepsilon$, in which case it is simply the identity. However, it becomes a homomorphism when restricted to the stabiliser of v.

Proposition 2.6.24. *Let X be a finite set and let $v \in X^*$ be a vertex in the rooted tree X^*. Then, for any $G \leq \mathrm{Aut}(X^*)$, the restriction of φ_v to $\mathrm{St}_G(v)$, which we will also denote by φ_v, is a homomorphism.*

Proof. Clearly, φ_v sends the identity to the identity. Furthermore, if $g, h \in \mathrm{St}_G(v)$, then for all $w \in X^*$, we have

$$v(gh)|_v w = (gh)vw = g(hvw) = gvh|_v w = vg|_v h|_v w,$$

so $\varphi_v(gh) = \varphi_v(g)\varphi_v(h)$. □

This allows us to define the projection of a group on a vertex.

Definition 2.6.25. Let X be a finite set, let $G \leq \mathrm{Aut}(X^*)$ be a group of automorphisms of the rooted tree X^* and let $v \in X^*$ be a vertex. The group $\varphi_v(\mathrm{St}_G(v)) \leq \mathrm{Aut}(X^*)$ is called the *projection of G at v*, and we will denote it by G_v. ☙

Remark 2.6.26. Notice that G_v is not the image of G under φ_v, but the image of the stabiliser of v. This is necessary in order to ensure that G_v is a group. ☙

Rooted and finitary automorphisms

We will now define a special class of automorphisms of a regular rooted tree: those whose projections to all but a finite number of vertices are the identity.

Definition 2.6.27. Let X be a finite set and let $g \in \mathrm{Aut}(X^*)$ be an automorphism of the rooted tree X^*. We say that g is *finitary* if there exists a finite set $F \subset X^*$ such that $\varphi_v(g) = 1$ for all $v \in X^* \setminus F$. If $F = \{\varepsilon\}$, then g is said to be a *rooted* automorphism. ☙

As we will see in Proposition 2.6.29, finitary automorphisms form a locally finite subgroup of $\mathrm{Aut}(T)$ and should be thought of as coming from automorphisms of finite rooted trees. In order to better understand this subgroup, we first need to introduce a bit of notation.

For a finite set X, recall from Definition 2.2.4 that

$$\wr^n \mathrm{Sym}(X) = \left(\wr^{n-1} \mathrm{Sym}(X)\right) \wr \mathrm{Sym}(X),$$

where the wreath product is taken over the set X. If we write $X = \{a_1, a_2, \dots, a_k\}$, we have that every element $\tau \in \wr^n \mathrm{Sym}(X)$ can be uniquely written as

$$\tau = \sigma(\tau_{a_1}, \tau_{a_2}, \dots, \tau_{a_k}),$$

where $\sigma \in \mathrm{Sym}(X)$ and $\tau_{a_i} \in \wr^{n-1} \mathrm{Sym}(X)$ for every $i \in \{1, 2, \dots, k\}$. We can inductively define an action of $\wr^n \mathrm{Sym}(X)$ on X^n by

$$\tau(v_1 \dots v_n) = \sigma(v_1)\tau_{v_1}(v_2 \dots v_n),$$

and it is easy to check that this action is faithful. Thus, we can consider $\wr^n \mathrm{Sym}(X)$ as a subgroup of $\mathrm{Sym}(X^n)$, which we will do from now on. This subgroup is exactly the subgroups of permutations of X^n coming from automorphisms of the rooted tree X^*.

Proposition 2.6.28. *Let X be a finite set and $n \in \mathbb{N}$ be a natural number. For every $g \in \mathrm{Aut}(X^*)$, the permutation of X^n given by $(v \mapsto gv)$ is an element of $\wr^n \mathrm{Sym}(X)$.*

We can now better describe the structure of finitary automorphisms.

Proposition 2.6.29. *Let X be a finite set, let $FAut(X^*)$ be the set of finitary automorphisms of X^* and let us define*

$$FAut_n(X^*) = \{g \in FAut(X^*) \mid \varphi_v(g) = 1 \text{ for all } v \in X^* \text{ with } |v| \geq n\}$$

for all $n \in \mathbb{N}$. Then, we have that $FAut(X^*) = \bigcup_{n \geq 0} FAut_n(X^*)$ is a locally finite subgroup of $\mathrm{Aut}(X^*)$ and that, for all $n \in \mathbb{N}$, the map

$$\begin{aligned} FAut_n(X^*) &\to \wr^n \mathrm{Sym}(X) \leq \mathrm{Sym}(X^n) \\ g &\mapsto (v \mapsto gv) \end{aligned}$$

is an isomorphism. In particular, for the subgroup of rooted isomorphisms $FAut_1(X^*)$, we have $FAut_1(X^*) \cong \mathrm{Sym}(X)$.

In what follows, we will frequently identify $\wr^n \mathrm{Sym}(X)$ and $\mathrm{FAut}_n(X^*)$ thanks to the canonical isomorphism given in the above proposition.

Wreath decomposition

To any automorphism of a regular rooted tree, we can associate a unique finitary automorphism which has the same action up to a specific level.

Proposition 2.6.30. *Let X be a finite set, let $n \in \mathbb{N}$ be an integer and let $g \in \mathrm{Aut}(X^*)$ be an automorphism. Then, there exists a unique $\tau_{g,n} \in FAut_n(X^*)$ such that $\tau_{g,n}^{-1} g \in \mathrm{St}(n)$.*

Proof. It follows from Proposition 2.6.29 that there is a unique $\tau_{g,n} \in \mathrm{FAut}_n(X^*)$ such that $\tau_{gn} v = gv$ for all $v \in X^n$. □

Using this finitary automorphism, we can define what is known as the *wreath decomposition* of an automorphism.

Proposition 2.6.31. *Let X be a finite set. For every $n \in \mathbb{N}$, the map*

$$\begin{aligned} \psi_n \colon \mathrm{Aut}(X^*) &\to \mathrm{Aut}(X^*) \wr (\wr^n \mathrm{Sym}(X)) \\ g &\mapsto \tau_{g,n} \left(g|_{v_1}, g|_{v_2}, \dots, g|_{v_{|X|^n}} \right) \end{aligned}$$

is an isomorphism, where the wreath product is taken over the set $X^n = \{v_1, v_2, \dots, v_{|X|^n}\}$ and where $\left(g|_{v_1}, g|_{v_2}, \dots, g|_{v_{|X|^n}}\right)$ should be understood as the map from X^n to $\mathrm{Aut}(X^*)$ sending v to $g|_v$. The image of g by ψ_n is called the wreath decomposition of g on the n^{th} level.

Notation 2.6.32. Strictly speaking, an element of $\mathrm{Aut}(X^*) \wr (\wr^n \mathrm{Sym}(X))$ should be written as τf, where $\tau \in \wr^n \mathrm{Sym}(X)$ and $f \colon X^n \to \mathrm{Aut}(X^*)$. However, in what follows, we will always implicitly assume that we have a canonical order on the elements of X (which can then be extended to X^n by the lexicographic order), which means that we can unambiguously write functions from X^n to $\mathrm{Aut}(X^*)$ simply as a $|X|^n$-tuple of elements of $\mathrm{Aut}(X^*)$, as we have done above.

Notation 2.6.33. In what follows, we will frequently omit the ψ_n from the notation. This is harmless, since this map is an isomorphism. Therefore, if $\psi_n(g) = \tau(g_1, \dots, g_{|X|^n})$ for some $\tau \in \wr^n \mathrm{Sym}(X)$ and $g_1, \dots, g_{|X|^n} \in \mathrm{Aut}(X^*)$, we will often simply write

$$g = \tau \left(g_1, g_2, \dots, g_{|X|^n} \right).$$

Furthermore, if $g \in \mathrm{St}(n)$, or in other words, if τ is the trivial permutation, we will omit τ from the notation and write simply

$$g = \left(g_1, g_2, \dots, g_{|X|^n} \right).$$

Self-similar groups

We will now introduce a special class of groups acting on regular rooted trees, called *self-similar groups*. These groups will be one of the central object of study of this thesis. Briefly, a group is self-similar if the projections of each of its elements belong to the same group.

Definition 2.6.34. Let X be a finite alphabet and let $G \leq \mathrm{Aut}(X^*)$ be a group of automorphisms of the rooted tree X^*. We say that G is *self-similar* if for all $g \in G$ and all $v \in X^*$, we have $g_v \in G$.

If, furthermore, we have that $G_v = G$ for all $v \in X^*$ (see Definition 2.6.25), we then say that G is *self-replicating*.

Remark 2.6.35. For G to be self-similar, it is sufficient that $g_a \in G$ for all $a \in X$. Likewise, it suffices to check that $G_a = G$ for all $a \in X$ to prove that G is self-replicating.

More generally, following Bartholdi [3] one can also define a self-similar family of groups.

Definition 2.6.36. Let X be a finite alphabet, let Ω be a set and let $\sigma\colon \Omega \to \Omega$ be a map from this set to itself. For each $\omega \in \Omega$, let $G_\omega \leq \mathrm{Aut}(X^*)$ be a group of automorphisms of the rooted tree X^*. The family $(G_\omega)_{\omega\in\Omega}$ is called a *self-similar family* of groups if for every $\omega \in \Omega$, for every $g \in G_\omega$ and for every $a \in X$, we have $g_a \in G_{\sigma(\omega)}$. ☙

Remark 2.6.37. If $\sigma(\omega) = \omega$, then G_ω is a self-similar group. ❧

We refer the reader to Section 2.8 for examples of self-similar groups and self-similar families of groups.

Boundary of rooted trees

Given a rooted tree T, one can consider its vertex set as a discrete topological space. The tree structure then gives us a natural way to compactify this space by adding a boundary. The action of a group on the tree T naturally extends to this boundary ∂T, a fact that will be useful in several instances. Let us first begin by defining this boundary.

Definition 2.6.38. Let $T = (V, E, v_0)$ be a rooted tree without leaves and where the degree of every vertex is finite, and let ∂T be the set of right-infinite paths in T starting at the root v_0 and without backtracking. For $\xi \in \partial T$ and $v \in V$, we will write $v \leq \xi$ if the path ξ passes through v. We will call the *tree compactification of* V the space $V \sqcup \partial T$ equipped with the topology defined by the basis
$$\{\{v\} \mid v \in V(T)\} \cup \{C_v \mid v \in V\},$$
where
$$C_v = \{w \in V(T) \sqcup \partial T \mid v \leq w\}.$$
We will call ∂T equipped with the subspace topology the *boundary of* T. ☙

What we called the tree compactification of the vertex set is in fact a compactification, and under some mild conditions, the boundary of a rooted tree is a Cantor space.

Proposition 2.6.39. *Let $T = (V, E, v_0)$ be a rooted tree without leaves and where every vertex has finite degree. Then, $V \sqcup \partial T$ is a compact metrisable space. Furthermore, if for every $v \in V$, there exists a vertex $w \geq v$ of degree greater than 2, then ∂T is a Cantor space.*

It will sometimes be convenient to have an explicit metric on $V \sqcup \partial T$ generating the topology on this set. We give such a metric in the next proposition.

Proposition 2.6.40. *Let $T = (V, E, v_0)$ be a rooted tree without leaves and where every vertex has finite degree. For $\xi, \zeta \in V \sqcup \partial T$ with $\xi \neq \zeta$, let $gcp(\xi, \zeta) \in \mathbb{N}$ be the length of the greatest common prefix of ξ and ζ. In other words,* *gcp*(ξ, ζ) *is the length of the maximal element* w *(with respect to the prefix relation* $\leq$*) satisfying both* $w \leq \xi$ *and* $w \leq \zeta$*. Then, the map*

$$\begin{aligned} dist\colon V \sqcup \partial V \times V \sqcup \partial V &\to \mathbb{R}_{\geq 0} \\ (\xi, \zeta) &\mapsto \begin{cases} 2^{-gcp(\xi,\zeta)} & \text{if } \xi \neq \zeta \\ 0 & \text{if } \xi = \zeta \end{cases} \end{aligned}$$

is a metric (in fact, even an ultrametric) inducing the topology of Definition 2.6.38 on $V \sqcup \partial T$*.*

This metric makes it easy to see that in this topology, a sequence $(w_n)_{n\in\mathbb{N}}$ of vertices of T converges to an element $\xi \in \partial T$ if and only if for every $m \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, the path from the root v_0 to w_n coincides with ξ on the first m edges.

Remark 2.6.41. Let X be a finite set and let X^* be the regular rooted tree over the alphabet X. Then, $\partial X^* = X^\infty$, with the topology given in Proposition 2.3.11. ❧

Any automorphism of a rooted tree can be naturally extended to a homeomorphism (in fact, even an isometry) on the boundary by passing to the limit.

Proposition 2.6.42. *Let* $T = (V, E, v_0)$ *be a rooted tree without leaves and where every vertex has finite degree, and let* $g \in \mathrm{Aut}(T)$ *be an automorphism of* T*. If we equip* $V \sqcup \partial V$ *with the metric of Proposition 2.6.40 then there exists a unique isometry* $\tilde{g}\colon V \sqcup \partial T \to V \sqcup \partial T$ *such that* $\tilde{g}(v) = g(v)$ *for all* $v \in V$*. In particular,* $\tilde{g}$ *restricts to an isometry of* ∂T*.*

Proof. It is clear from the definition of the metric on $V \sqcup \partial T$ that every $\xi \in \partial T$ is the limit of a sequence $(\xi_n)_{n\in\mathbb{N}}$ with $\xi_0 = v_0$, $\xi_n \leq \xi_{n+1}$ and $|\xi_{n+1}| = |\xi_n| + 1$, and that every such sequence defines a unique element of ∂T. As g is an automorphism of T, we have that $(g(\xi_n))_{n\in\mathbb{N}}$ is a sequence satisfying $g(\xi_0) = v_0$, $g(\xi_n) \leq g(\xi_{n+1})$ and $|g(\xi_{n+1})| = |g(\xi_n)| + 1$. Thus, this sequence converges to a unique element of ∂T that we will denote $\tilde{g}(\xi)$. As g preserves prefixes, it is easy to check that this gives us a well-defined isometry of $V \sqcup \partial T$. □

2.7 Branch and weakly branch groups

We now introduce an important class of groups acting on spherically homogeneous rooted trees, called *branch groups*.

Definition 2.7.1. Let T be a spherically homogeneous tree and let $G \leq \mathrm{Aut}(T)$ be a group of automorphisms of T. We say that G is a *weakly branch group* if

(i) for every $n \in \mathbb{N}$, G acts transitively on L_n, the n^{th} level of T,

(ii) $\mathrm{Rist}_G(v) \neq \{1\}$ for every $v \in T$.

If, furthermore, we have

(iii) $\mathrm{Rist}_G(n)$ is of finite index in G for all $n \in \mathbb{N}$, then we say that G is a *branch group.* ☙

Remark 2.7.2. Notice that if $\mathrm{Rist}_G(v) \neq \{1\}$ for every $v \in T$, then $|\mathrm{Rist}_G(v)| = \infty$, since $\mathrm{Rist}_G(w) \leq \mathrm{Rist}_G(v)$ for all $w \geq v$. ☙

In the case of self-similar group, one can introduce a stronger notion of branch groups by requiring the existence of a subgroup that contains a direct product of copies of itself.

Definition 2.7.3. Let X be a finite set and let $G \leq \mathrm{Aut}(X^*)$ be a self-similar group of automorphisms of the regular rooted tree X^* acting transitively on X^n for all $n \in \mathbb{N}$. If there exists a non-trivial subgroup $K \leq G$ such that $K^{|X|} \leq \psi_1(K)$, then we say that G is *regular weakly branch over* K. If, furthermore, K is of finite index in G, we say that G is *regular branch over* K. ☙

Notice that in particular, regular (weakly) branch groups are (weakly) branch.

The classical example of a branch group is the Grigorchuk group, which we will define in Example 2.8.3. More generally, we will see in Section 2.9 that every Šunić group except the infinite dihedral group is a regular branch group (see Proposition 2.9.18).

Branch groups were first introduced by Grigorchuk at the Group St-Andrews conference in 1997 (see [45]), but they appeared implicitly in prior work. In particular, they appear in Wilson's classification of just-infinite groups [85] as one of the three classes into which just-infinite groups naturally split, as was observed by Grigorchuk in [45].

Due to their large rigid stabilisers, branch and weakly branch groups possess a very particular subgroup structure. This has important algebraic consequences, as we will see below. We begin by a lemma regarding subnormal subgroups and rigid stabilisers. This is a generalised version of a very important lemma first proved by Grigorchuk in [45].

Lemma 2.7.4. *Let G be a group acting on a rooted tree T, and let $H \leq G$ be a k-subnormal subgroup of G, for some $k \in \mathbb{N}$. For all $v \in T$, if $H \not\leq \mathrm{St}_G(v)$, then $\mathrm{Rist}_G^{(k)}(v) \leq H$, where $\mathrm{Rist}_G^{(k)}(v)$ is the k^{th} derived subgroup of $\mathrm{Rist}_G(v)$.*

Proof. We will proceed by induction on k. If $k = 0$, then $H = G$ and the result is obvious. Let us now assume that it is true for some $k \in \mathbb{N}$ and let us prove it for $k + 1$.

If H is a $(k+1)$-subnormal subgroup of G, then by definition, there exists a k-subnormal subgroup $H_1 \leq G$ such that $H \trianglelefteq H_1$. If $v \in T$ is such that $H \not\leq \mathrm{St}_G(v)$, then as $H \leq H_1$, we also have that $H_1 \not\leq \mathrm{St}_G(v)$. Therefore, by the induction hypothesis, we have $\mathrm{Rist}_G^{(k)}(v) \leq H_1$.

Let us choose $h \in H$ such that $hv \neq v$, and let us consider two arbitrary elements $r, s \in \mathrm{Rist}_G^{(k)}(v)$. We have $hr^{-1}h^{-1} \in \mathrm{Rist}_G^{(k)}(hv) \leq H_1$. As $hv \neq v$, this implies that $hr^{-1}h^{-1}$ commutes with any element in $\mathrm{Rist}_G^{(k)}(v)$. Hence,

$$[hr^{-1}h^{-1}r, s] = [r, s].$$

On the other hand, since H is normal in H_1, we have $h(r^{-1}h^{-1}r) \in H$ and thus $[hr^{-1}h^{-1}r, s] \in H$. We conclude that $\mathrm{Rist}_G^{(k+1)}(v) \leq H$. □

As a corollary, we obtain the following lemma concerning rigid stabilisers and normal subgroups. This is essentially the same as Lemma 5.3 in [9], which itself was a slight generalisation of Grigorchuk's lemma in [45]. We give a proof here for completeness.

Lemma 2.7.5. *Let G be a group acting transitively on each level of a spherically homogeneous rooted tree T, and let $N \trianglelefteq G$ be a non-trivial normal subgroup of G. Then, there exists $n \in \mathbb{N}$ such that $\mathrm{Rist}'_G(n) \leq N$, where $\mathrm{Rist}'_G(n)$ is the derived subgroup of $\mathrm{Rist}_G(n)$.*

Proof. Since N is non-trivial, there exists $g \in N$ such that $g \neq 1$. In particular, there must exist $v \in T$ such that $gv \neq v$. By Lemma 2.7.4, we have that $\mathrm{Rist}'_G(v) \leq N$. Let $n \in \mathbb{N}$ be the level to which v belongs. Since G acts transitively on the n^{th} level L_n, we get by conjugating that

$$\mathrm{Rist}'_G(n) = \prod_{w \in L_n} \mathrm{Rist}'_G(w) \leq N.$$

□

Of course, the preceding lemma is only interesting if we know that $\mathrm{Rist}'_G(n)$ is non-trivial for every $n \in \mathbb{N}$. This turns out to be the case for weakly branch groups.

Lemma 2.7.6. *Let G be a weakly branch group acting on a spherically homogeneous rooted tree T. Then, for every $v \in T$ and $k \in \mathbb{N}$, the subgroup $\mathrm{Rist}_G^{(k)}(v)$ is non-trivial. In other words, $\mathrm{Rist}_G(v)$ is not solvable.*

Proof. We will proceed by induction on k. For $k = 0$, we need to prove that for all $v \in T$, we have that $\mathrm{Rist}_G(v)$ is non-trivial. This follows directly from the the fact that G is a weakly branch group. Let us now assume that for some $k \in \mathbb{N}$, we have that $\mathrm{Rist}_G^{(k)}(v)$ is non-trivial for all $v \in T$, and let us show that the same must thus be true for $k + 1$.

Let us fix $v \in T$. By assumption, we know that $\mathrm{Rist}_G^{(k)}(v)$ is non-trivial. In particular, there exists $g \in \mathrm{Rist}_G^{(k)}(v)$ and $w \in T$ such that $gw \neq w$. Given that $g \in \mathrm{Rist}_G^{(k)}(v) \leq \mathrm{Rist}_G(v)$, it is clear that we must have $w \geq v$. Therefore, we have $\mathrm{Rist}_G(w) \leq \mathrm{Rist}_G(v)$, and consequently $\mathrm{Rist}_G^{(k)}(w) \leq \mathrm{Rist}_G^{(k)}(v)$. Again using our assumption, let us choose a non-trivial $h \in \mathrm{Rist}_G^{(k)}(w)$. Since $gw \neq w$, we get that $\mathrm{Rist}_G^{(k)}(gw) \cap \mathrm{Rist}_G^{(k)}(w) = 1$. Thus, as $ghg^{-1} \in \mathrm{Rist}_G^{(k)}(gw)$, we conclude that

$$ghg^{-1}h^{-1} \neq 1.$$

Since $g, h \in \mathrm{Rist}_G^{(k)}(v)$, we have $ghg^{-1}h^{-1} \in \mathrm{Rist}_G^{(k+1)}(v)$, which concludes the proof. □

As an immediate consequence of the previous lemmas, we get that a non-trivial subnormal subgroup of a weakly branch group is always infinite.

Lemma 2.7.7. *Let G be a weakly branch group acting on a spherically homogeneous rooted tree T and let $H \leq G$ be a subnormal subgroup of G. If H is non-trivial, then H is infinite.*

Proof. By Lemma 2.7.4, there exists $k \in \mathbb{N}$ and $v \in T$ such that $\mathrm{Rist}_G^{(k)}(v) \leq H$. It follows from Lemma 2.7.6 that $\mathrm{Rist}_G^{(k)}(v)$ is infinite. Indeed, we have that $\mathrm{Rist}_G^{(k)}(w) \leq \mathrm{Rist}_G^{(k)}(v)$ for all $w \leq v$, and as there are infinitely many such w, all of which are non-trivial according to Lemma 2.7.6, it is easy to see that $\mathrm{Rist}_G^{(k)}(v)$ must be infinite. □

Let us now study some structural consequences of the previous lemmas. One immediate result is that two non-trivial normal subgroups of a weakly branch group always have a non-trivial intersection. In fact, we prove here a more general result.

Proposition 2.7.8. *Let G be a weakly branch group acting on a spherically homogeneous rooted tree T, let $H_1, H_2 \leq G$ be two subnormal subgroups of G acting spherically transitively on T, and let $L_1 \trianglelefteq H_1$, $L_2 \trianglelefteq H_2$ be non-trivial normal subgroups of H_1 and H_2, respectively. Then, $L_1 \cap L_2 \neq \{1\}$. In particular, if $N_1, N_2 \trianglelefteq G$ are two non-trivial normal subgroups, then $N_1 \cap N_2 \neq \{1\}$.*

Proof. As L_1 and L_2 are non-trivial subnormal subgroups of G, it follows from Lemma 2.7.4 that there exist $k_1, k_2 \in \mathbb{N}$ and $v_1, v_2 \in T$ such that $\mathrm{Rist}_G^{(k_i)}(v_i) \leq L_i$ for $i = 1, 2$. As L_i is normal in H_i, which acts spherically transitively on T, we get that $\mathrm{Rist}_G^{(k_i)}(n_i) \leq L_i$, where $n_i = |v_i|$. Let $n = \max\{n_1, n_2\}$ and let $k = \max\{k_1, k_2\}$. We then have $\mathrm{Rist}_G^{(k)}(n) \leq \mathrm{Rist}_G^{(k_i)}(n_i)$ for $i = 1, 2$. Therefore, $\mathrm{Rist}_G^{(k)}(n) \leq L_1 \cap L_2$. The conclusion then follows from Lemma 2.7.6. □

Another important consequence is that subnormal subgroups of weakly branch groups are never solvable.

Proposition 2.7.9. *Let G be a weakly branch group acting on a spherically homogeneous rooted tree T and let $H \leq G$ be a non-trivial subnormal subgroup of G. Then, H is not solvable.*

Proof. By Lemma 2.7.4, there exists $k \in \mathbb{N}$ and $v \in T$ such that $\mathrm{Rist}_G^{(k)}(v) \leq H$, and by Lemma 2.7.6, we have that $\mathrm{Rist}_G(v)$ is not solvable, which implies that $\mathrm{Rist}_G^{(k)}(v)$ is not solvable. Consequently, H is not solvable. □

As a corollary, we get that subnormal subgroups of branch groups are never of polynomial growth (see Section 2.5 for the definition of growth).

Corollary 2.7.10. *Let G be a weakly branch group and let $H \leq G$ be a non-trivial finitely generated subnormal subgroup. Then, H is not of polynomial growth. In particular, G is not of polynomial growth.*

Proof. Suppose that H is of polynomial growth. Then, by Gromov's theorem [53], there exists a nilpotent subgroup $L \leq H$ of finite index in H. Let $K \leq L$ be the normal core of L in H. As L is of finite index in H, we have that K is of finite index in H. In particular, K is non-trivial, since H must be infinite by Lemma 2.7.7. Therefore, K is a non-trivial subnormal subgroup of G. However, K is contained in L and must thus be nilpotent, a contradiction to Proposition 2.7.9. □

2.8 Spinal groups

In this section, we introduce a family of groups acting on rooted trees, called *spinal groups*, that will be very important in what follows. Indeed, many of the branch groups and groups of intermediate growth that we know belong to this family. Spinal groups were first formally introduced and studied by Bartholdi and Šunić in [13] in an effort to generalise the Grigorchuk groups. A more general version was later introduced by Bartholdi, Grigorchuk and Šunić in [9].

Spinal groups acting on regular rooted trees

To simplify matters, we will only consider spinal groups acting on regular rooted trees. Therefore, the definition we give here is not as general as the one given by Bartholdi, Grigorchuk and Šunić in [9]. However, it will be sufficient for our purposes.

Let X be a finite set of cardinality d. Let B be a finite group and let

$$\Omega = \left\{ \{\omega_{ij}\}_{i\in\mathbb{N}, 1\leq j\leq d-1} \;\middle|\; \omega_{ij} \in \mathrm{Hom}(B, \mathrm{Sym}(X)), \bigcap_{i\geq k}\bigcap_{j=1}^{d-1} \ker\omega_{ij} = \{1\} \forall k \in \mathbb{N} \right\}$$

be the set of sequences of $(d-1)$-tuples of homomorphisms from B to $\mathrm{Sym}(X)$ such that the intersection of the kernels is trivial no matter how far into the sequence we start. Let

$$\begin{aligned} \sigma\colon \Omega &\to \Omega \\ \omega = \{\omega_{ij}\}_{i\in\mathbb{N},1\leq j\leq d-1} &\mapsto \sigma(\omega) = \{\omega_{(i+1)j}\}_{i\in\mathbb{N},1\leq j\leq d-1} \end{aligned}$$

be the *shift* (with respect to the first index), which is well-defined thanks to the way the condition on the kernels was formulated.

For each $\omega = \{\omega_{ij}\}_{i\in\mathbb{N}, j\in\{1,2,\dots,d-1\}} \in \Omega$, we can recursively define a homomorphism

$$\begin{aligned} \iota_\omega\colon B &\to \mathrm{Aut}(X^*) \\ b &\mapsto (\omega_{01}(b), \omega_{02}(b), \dots, \omega_{0(d-1)}(b), \iota_{\sigma(\omega)}(b)) \end{aligned}$$

where we identify $\mathrm{Sym}(X)$ with rooted automorphisms of X^* thanks to Proposition 2.6.29 (see Figure 2.8 for a representation of this automorphism). The condition on the kernels of sequences in Ω ensures that this homomorphism is injective. Let us write $B_\omega = \iota_\omega(B) \leq \mathrm{Aut}(X^*)$.

For a fixed $\omega = \{\omega_{ij}\} \in \Omega$, let $A_\omega \leq \mathrm{Sym}(X)$ be any subgroup of $\mathrm{Sym}(X)$. For any $k \in \mathbb{N}^*$, we then define

$$A_{\sigma^k(\omega)} = \left\langle \bigcup_{j=1}^{d-1} \omega_{kj}(B) \right\rangle.$$

Definition 2.8.1. Using the notation above, the group $G_\omega = \langle A_\omega, B_\omega\rangle$ for some $\omega \in \Omega$ and $A_\omega \leq \mathrm{Sym}(X)$ is called a *spinal group* if $A_{\sigma^k(\omega)}$ acts transitively on X for all $k \in \mathbb{N}$. ☙

Remark 2.8.2. For $\omega \in \Omega$ and $A_\omega \leq \mathrm{Sym}(X)$, if $G_\omega = \langle A_\omega, B_\omega\rangle$ is a spinal group, then so is $G_{\sigma^k(\omega)} = \langle A_{\sigma^k(\omega)}, B_{\sigma^k(\omega)}\rangle$ for all $k \in \mathbb{N}$. ☙

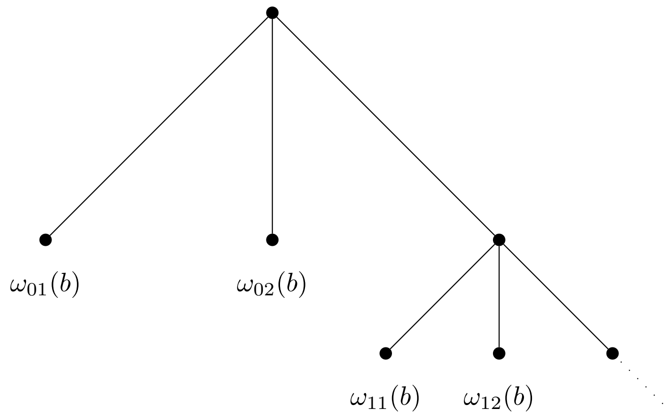


Figure 2.4: The automorphism $\iota_\omega(b)$.

Examples

We present here a few important examples of spinal groups. As we will see, many of the groups acting on rooted trees that were studied before belong to this family.

Example 2.8.3 (The first Grigorchuk group). Let[iii] $X = \{\mathbf{0}, \mathbf{1}\}$, $A = \mathrm{Sym}(X) \cong \mathbb{Z}/2\mathbb{Z}$ and $B = (\mathbb{Z}/2\mathbb{Z})^2$. Let a be the non-trivial element of A and b, c, d be the non-trivial elements of B. For $x \in \{b, c, d\}$, let $\omega_x \colon B \to A$ be the epimorphism that sends x to 1 and the other two non-trivial letters to a. For all $i \in \mathbb{N}$, let

$$\omega_i = \begin{cases} \omega_d & \text{if } i \equiv 0 \mod 3 \\ \omega_c & \text{if } i \equiv 1 \mod 3 \\ \omega_b & \text{if } i \equiv 2 \mod 3. \end{cases}$$

The group G_ω with $\omega = \{\omega_i\}_{i\in\mathbb{N}} = \omega_d\omega_c\omega_b\omega_d\omega_c\omega_b\dots$ (here, since $|X| - 1 = 1$, we write only one index) and $A_\omega = A$ is known as the first Grigorchuk group.

Although it is not readily apparent from the way we defined it, this group is self-similar. Indeed, the actions of b, c and d on X^* are given by

$$b = (a, c), \quad c = (a, d), \quad d = (1, b).$$

This group, which was first introduced in [41], is very important in the theory of groups acting on rooted trees. Indeed, it was the first example of a group of intermediate growth [42], and spinal groups were defined as a generalisation of it. ❧

[iii]Notice that the elements of the alphabet X are **bold** numbers. This is to avoid any possible confusion between elements of X and numbers in other contexts. We will follow the same convention throughout this thesis.

Example 2.8.4 (**Grigorchuk groups**)**.** Let $X = \{\mathbf{0}, \mathbf{1}, \dots, \mathbf{p-1}\}$, where p is a prime number, $A = \langle(\mathbf{0}\ \mathbf{1}\ \dots\ \mathbf{p-1})\rangle \cong \mathbb{Z}/p\mathbb{Z}$ and $B = (\mathbb{Z}/p\mathbb{Z})^2$. Let

$$\begin{aligned}\phi_k\colon (\mathbb{Z}/p\mathbb{Z})^2 &\to \mathbb{Z}/p\mathbb{Z}\\ (x, y) &\mapsto x + ky\end{aligned}$$

for $0 \le k \le p-1$ and let

$$\begin{aligned}\phi_p\colon (\mathbb{Z}/p\mathbb{Z})^2 &\to \mathbb{Z}/p\mathbb{Z}\\ (x, y) &\mapsto y.\end{aligned}$$

The groups G_ω with $A_\omega = A$ and $\omega = \{\omega_{ij}\}_{i\in\mathbb{N}, 1\le j\le p-1} \in \Omega$ such that, for all $i \in \mathbb{N}$, we have $\omega_{ij} = 1$ if $j \ne 1$ and $\omega_{i1} = \phi_{k_i}$ for some $0 \le k_i \le p-1$ are called Grigorchuk groups. They were introduced and studied by Grigorchuk in [44] as a generalisation of his first group of intermediate growth. ❧

Example 2.8.5 (**The Fabrykowski-Gupta group**)**.** Let $X = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$, $A = \langle(\mathbf{1}\ \mathbf{2}\ \mathbf{3})\rangle \cong \mathbb{Z}/3\mathbb{Z}$ and $B = \mathbb{Z}/3\mathbb{Z}$. Let $b \in B$ be a non-trivial element and let $a = (\mathbf{1}\ \mathbf{2}\ \mathbf{3}) \in A$. For all $i \in \mathbb{N}$, let $\omega_{i1}\colon B \to A$ be the unique homomorphism sending b to a and let $\omega_{i2}\colon B \to A$ be the trivial map. The group G_ω with $A_\omega = A$ and $\omega = \{\omega_{ij}\}_{i\in\mathbb{N}, j=1,2}$ is called the *Fabrykowski-Gupta group.*

This group is self-similar, since ω is a constant sequence. It is generated by a and b, with a cyclically permuting the first level and

$$b = (a, 1, b).$$

It was first introduced by Fabrykowksi and Gupta in [28] in order to produce a new example of a group of intermediate growth. Its growth was also studied by Bartholdi and Pochon in [10]. ❧

Example 2.8.6 (**GGS groups**)**.** Finally, let us introduce another important family of examples of spinal groups, known as GGS groups. These are a generalization of the second Grigorchuk group (introduced in [41]) and the groups introduced by Gupta and Sidki in [54], hence the name GGS. We present here the definition of GGS groups that was given in [9].

Let $X = \{\mathbf{1}, \mathbf{2}, \dots, \mathbf{d}\}$, $A = \langle(\mathbf{1}\ \mathbf{2}\ \dots\ \mathbf{d})\rangle \cong \mathbb{Z}/d\mathbb{Z}$, $B = \mathbb{Z}/d\mathbb{Z}$ and $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_{d-1}) \in (\mathbb{Z}/d\mathbb{Z})^d$ such that $\epsilon \ne 0$. Let $\omega = \{\omega_{ij}\} \in \Omega$, where

$$\omega_{ij}(x) = a^{x\epsilon_j}$$

for $x \in B$ and $a = (\mathbf{1}\ \mathbf{2}\ \dots\ \mathbf{d}) \in A$. Let $A_\omega = A$. If $\gcd(\epsilon_1, \epsilon_2, \dots, \epsilon_{d-1}, d) = 1$, then G_ω is called a GGS group. ❧

2.9 Šunić groups

In this section, we define yet another family of groups acting on rooted trees, that we will call *Šunić groups.* These groups were introduced by Šunić in [83] as close siblings to the first Grigorchuk group. They are a special class of self-similar spinal groups, but we devote an entire section to them as their study will form an important part of this thesis, in particular in Chapters 3, 4, 6 and 7.

Please note that most of the text this section was taken directly from Section 2 of the article [32] by the author and Alejandra Garrido, with a few minor modifications where necessary, and that, unless otherwise specified, the results are due to Šunić and were originally proved in [83].

Definition and examples

For the rest of this section, let p be a prime number and $m \geq 1$ be an integer. Let us set $X = \{\mathbf{0}, \mathbf{1}, \dots, \mathbf{p-1}\}$, $A = \langle (\mathbf{0}\ \mathbf{1}\ \dots\ \mathbf{p-1}) \rangle \cong \mathbb{Z}/p\mathbb{Z}$ and $B = (\mathbb{Z}/p\mathbb{Z})^m$. As p is prime, we can see A and B not only as groups, but also as vector spaces over the finite field $\mathbb{Z}/p\mathbb{Z}$. Let $a = (\mathbf{0}\ \mathbf{1}\ \dots\ \mathbf{p-1}) \in A$ and $b_0, \dots, b_{m-1} \in B$ be the vectors of the standard bases of A and B, respectively.

Let $\omega\colon B \to A$ be the epimorphism whose matrix in the bases $\{b_0, \dots, b_{m-1}\}$ and $\{a\}$ is

$$M_\omega = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

and let $\rho\colon B \to B$ be an automorphism whose matrix in the basis $\{b_0, \dots, b_{m-1}\}$ is

$$M_\rho = \begin{pmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_{m-1} \end{pmatrix}$$

for some $a_0, \dots, a_{m-1} \in \mathbb{Z}/p\mathbb{Z}$ with $a_0 \neq 0$. We see that M_ρ is the companion matrix to the polynomial $f(x) = x^m + a_{m-1}x^{m-1} + \dots + a_1 x + a_0$. Thus, ρ is uniquely determined by a polynomial of degree m whose constant term is non-zero.

Definition 2.9.1. Let $f(x) = x^m + a_{m-1}x^{m-1} + \dots + a_1 x + a_0$ be a monic polynomial with coefficients in $\mathbb{Z}/p\mathbb{Z}$ such that $a_0 \neq 0$. We will call the automorphism $\rho\colon B \to B$ whose matrix in the basis $\{b_0, \dots, b_{m-1}\}$ is the companion matrix to f the *automorphism associated to f*. ☙

The group A acts naturally on the rooted tree X^* as a group of rooted automorphisms. Thanks to ω and ρ, we can recursively define an action of B on X^* by

$$b = (\omega(b), 1, \dots, 1, \rho(b))$$

for all $b \in B$. It was shown by Šunić in [83] that this action is faithful, which allows us to see B as a group of automorphisms of the rooted tree X^*.

Proposition 2.9.2 (see Proposition 2 of [83])**.** *Using the notation above, we have*

(i) the action of B on X^* is faithful,

(ii) no non-trivial orbit of ρ is contained in $\ker(\omega)$*,*

From now on, when ρ is specified, we will thus frequently consider B as a group of automorphisms of X^*.

Definition 2.9.3 (Šunić group)**.** Let p be a prime number, $m \geq 1$ be an integer and $f(x) = x^m + a_{m-1}x^{m-1} + \dots + a_1 x + a_0$ be a polynomial with coefficients in $\mathbb{Z}/p\mathbb{Z}$ such that $a_0 \neq 0$. Using the notation above, let $\rho\colon B \to B$ be the automorphism associated to f. We will call the group $G_{p,f} = \langle A \cup B \rangle \leq \mathrm{Aut}(X^*)$ the *Šunić group* associated to m and f, where the action of B on X^* is given by ω and ρ as above. ☙

Remark 2.9.4. It is immediate from the definition that Šunić groups are self-similar. Furthermore, it is also easy to check that they are spinal groups. Unfortunately, our usage of ω as an epimorphism in this section (which comes from our desire to keep the same notation as in [83]) clashes with the usage of ω as a sequence of epimorphisms in Section 2.8. We will always keep those notations separate from each other, so there should hopefully be no confusion between the two.

Notation 2.9.5. As in [83], for all $i \in \mathbb{Z}$, we will set $B_i = \rho^i(\ker(\omega))$. Notice that, as we have chosen to denote elements of the alphabet X as **bold** numbers, there should be no confusion between this notation and the notation of Definition 2.6.25.

Remark 2.9.6. Notice that, from the definition of $\{b_0, \ldots, b_{m-1}\}$ and from the form of ω and ρ given above, we have

$$b_0 = (1, \ldots, 1, b_1), \ \ldots, \ b_{m-2} = (1, \ldots, 1, b_{m-1}),$$
$$b_{m-1} = (a, \ldots, 1, \rho(b_{m-1})).$$

In particular, we have $B_0 = \ker(\omega) = \langle b_0, \ldots, b_{m-2} \rangle$ and $B_1 = \langle b_1, \ldots, b_{m-1} \rangle$.

Let us now see a few important examples of Šunić group.

Example 2.9.7. The group $G_{2,x+1}$ is the infinite dihedral group. Indeed, in this case, we have $m = 1$, so $A \cong B \cong \mathbb{Z}/2\mathbb{Z}$ and the matrices for ω and ρ in the canonical bases are both the identity matrix of dimension 1. By the convention above, the non-trivial element of A is denoted by a, and since B also contains only one non-trivial element, we will denote it by b, so that we have $b = (a, b)$. We thus have that $G_{2,x+1} = \langle a, b \rangle$, with $a^2 = b^2 = 1$. To see that $G_{2,x+1}$ is indeed the infinite dihedral group, it thus remains to show that ab is an element of infinite order. This can easily be done by noticing that for all $n \in \mathbb{N}$, we have that $(ab)^{2n+1}$ is non-trivial (it acts non-trivially on X) and that $(ab)^{2n} = ((ba)^n, (ab)^n)$. We can then conclude by induction.

Example 2.9.8. The group G_{2,x^2+x+1} is the first Grigorchuk group (see Example 2.8.3).

Example 2.9.9. The group G_{2,x^2+1} is the only other self-similar group in the uncountable family defined in [43] (see Example 2.8.4). Its growth was studied by Erschler in ([27], Corollary 2'), which is why this group is sometimes referred to as the Grigorchuk-Erschler group. Unlike the Grigorchuk group, this group is not torsion.

It is generated by a, $b_0 = (1, b_1)$ and $b_1 = (a, b_0)$. Notice that

$$b_0 b_1 = (1, b_1)(a, b_0) = (a, b_1 b_0) = (a, b_0 b_1)$$

(using the fact that B is an abelian group). Therefore, $\langle a, b_0 b_1 \rangle$ is isomorphic to the infinite dihedral group, as we will see in Corollary 2.9.15.

Example 2.9.10. The group $G_{3,x-1}$ is the Fabrykowski–Gupta group (see Example 2.8.5), generated by a and $b = (a, 1, b)$.

Basic properties

For convenience, we will now collect here a few useful results about Šunić groups that were proved by Šunić in [83].

Notation 2.9.11. In what follows, we will denote by $\mathcal{G}_{p,m}$ the family of Šunić groups $G_{p,f}$ where f has degree m and by $\mathcal{G}$ the family of all Šunić groups. ♋

We begin by a simple but very useful contraction property of Šunić groups. This property is trivially established and was noticed by Šunić in equation (11) of [83].

Proposition 2.9.12. *Let $G \in \mathcal{G}$ be a Šunić group and let $|\cdot|: G \to \mathbb{N}$ be the word norm with respect to the generating set $A \cup B$ (see Definition 2.5.2). Then, for all $v \in X$ and for all $g \in G$, we have*

$$|\varphi_v(g)| \leq \frac{|g|+1}{2},$$

where $\varphi_v(g)$ is the projection of g to v (see Definition 2.6.22).

We will make extensive use of this property later on.

Next, let us mention that divisors of polynomials correspond to subgroups of Šunić groups.

Proposition 2.9.13 (Proposition 3 of [83])**.** *Let f be a monic polynomial with coefficients in $\mathbb{Z}/p\mathbb{Z}$ and non-zero constant coefficient. If $f = f_1 f_2$ for some non-constant monic polynomials f_1, f_2, then $G_{p,f_i} \leq G_{p,f}$ for $i = 1, 2$.*

The following proposition gives us a useful criterion to determine when a Šunić group is torsion.

Proposition 2.9.14 (Proposition 9 of [83])**.** *Let G be a group in $\mathcal{G}_{p,m}$ with $m \geq 2$. The following are equivalent:*

(i) G is torsion,

(ii) G is a p-group,

(iii) there exists r such that $B_0 \cup B_1 \cup \cdots \cup B_{r-1} = B$,

(iv) every non-trivial ρ-orbit intersects B_0.

Corollary 2.9.15. *Let f be a monic polynomial with coefficients in $\mathbb{Z}/2\mathbb{Z}$ and non-zero constant coefficient, and let us set $G = G_{2,f}$. The following are equivalent :*

(i) G contains an element of infinite order,

(ii) there exists $b \in B$ such that $b = (a, b)$,

(iii) f is divisible by $x + 1$,

(iv) G contains a copy of the infinite dihedral group.

Proof. (i) implies (ii): By Proposition 2.9.14, since G contains an element of infinite order, there exists an element $b \in B$ such that the ρ-orbit of b does not intersect $B_0 = \ker(\omega)$. Hence, $\omega(\rho^k(b\rho(b))) = \omega(\rho^k(b))\omega(\rho^{k+1}(b)) = a^2 = 1$ for all $k \in \mathbb{N}$. By (ii) of Proposition 2.9.2, we get that $\rho(b) = b$, so $b = (a, b)$.

(ii) implies (iii): Since $b = (a, b)$, we have $\rho(b) = b$, so b is an eigenvector of ρ with eigenvalue 1. It follows that f, the minimal polynomial of ρ, is divisible by $x + 1$.

(iii) implies (iv) is a direct consequence of Proposition 2.9.13 and Example 2.9.7 while (iv) trivially implies (i). □

The following proposition describes the abelianisation of a Šunić group. It is equivalent to Proposition 4 of [83], but we give a different proof here.

Proposition 2.9.16. *Let $G \in \mathcal{G}$, let us set $\Gamma := A * B$, and let $\pi\colon \Gamma \to G$ be the canonical projection map. Let $N \trianglelefteq \Gamma$ be the kernel of π. Then, N is a subgroup of Γ', the commutator subgroup of Γ. Therefore, the map π projects to an isomorphism $G/G' \cong \Gamma/\Gamma' \cong A \times B$.*

Proof. First note that $\pi^{-1}(G') = \Gamma' N$ and that

$$G/G' \cong (\Gamma/N)/(\Gamma' N/N) \cong \Gamma/(\Gamma' N),$$

so it suffices to show that $N \leq \Gamma'$ to prove that $G/G' \cong A \times B$.

Consider the subgroup S of index p in Γ generated by $\{a^i x a^{-i} \mid x \in B, i \in \mathbb{Z}/p\mathbb{Z}\}$. Since $\pi(a) \notin \mathrm{St}_G(1)$, the kernel N is contained in S and S is the preimage of $\mathrm{St}_G(1)$ in Γ. Let $\Psi\colon S \to \Gamma \times \cdots \times \Gamma$ (p factors) be the homomorphism defined by

$$\begin{aligned}
\Psi(x) &= (\omega(x), 1, \dots, 1, \rho(x))\\
\Psi(a^{-1}xa) &= (1, \dots, 1, \rho(x), \omega(x))\\
&\cdots\\
\Psi(axa^{-1}) &= (\rho(x), \omega(x), 1, \dots, 1)
\end{aligned}$$

for all $x \in B$. Defining $\pi_{G\times\cdots\times G} := \pi \times \cdots \times \pi$, we have $\pi_{G\times\cdots\times G} \circ \Psi = \psi_1 \circ \pi$, as the images of the generators of S by each of the maps coincide.

To show that $N \leq \Gamma'$, suppose that $\gamma \in N \backslash \Gamma'$, then $\gamma = a^i \beta z$ with $i \in \mathbb{Z}/p\mathbb{Z}$, $\beta \in B$, (not both trivial) and $z \in \Gamma'$. Since $\gamma \in N$ and $\beta z \in S$, we must have $i = 0$. Now,

$$\Psi(\gamma) = (\omega(\beta)z_0, z_1, \dots, \rho(\beta)z_{p-1}) \text{ where } \Psi(z) = (z_0, \dots, z_{p-1}).$$

Since $z \in \Gamma'$, it is easily seen (by considering $[a, x]$ for $x \in B$) that $z_0 z_1 \cdots z_{p-1} \in \Gamma'$. Thus the product of all entries in $\Psi(\gamma)$ is congruent to $\omega(\beta)\rho(\beta)$ modulo Γ'. Since $\pi_{G\times\cdots\times G} \circ \Psi = \psi_1 \circ \pi$, this product must be in N and so $\omega(\beta) = 1$. Repeating the above argument, we obtain that $\omega(\rho^n(\beta)) = 1$ for all $n \in \mathbb{N}$, which implies that $\beta = 1$. Thus, $\gamma = z \in \Gamma'$, as required. □

The next proposition implies that every Šunić group is a self-replicating group (see Definition 2.6.34).

Proposition 2.9.17 (Proposition 5 of [83])**.** *Let $G \in \mathcal{G}$. Then, the map ψ_1 induces a subdirect embedding of $\mathrm{St}_G(1)$ in $G \times \cdots \times G$. Consequently, G acts spherically transitively and is self-replicating.*

The following proposition tells us that all Šunić groups, with the exception of the infinite dihedral group, are regular branch groups.

Proposition 2.9.18 (Lemmas 1 and 6 of [83])**.**

(i) *Let $G \in \mathcal{G}_{p,m}$ be a Šunić group, where $p \geq 3$. Then, G is regular branch over its commutator subgroup G'.*

(ii) *Let $G \in \mathcal{G}_{2,m}$ be a Šunić group, where $m \geq 2$. Then, G is regular branch over the subgroup*

$$K = \langle [a, b] \mid b \in B_1 \rangle^G,$$

where $\langle [a, b] \mid b \in B_1 \rangle^G$ denotes the normal closure in G of the subgroup $\langle [a, b] \mid b \in B_1 \rangle$.

The next lemma shows that the subgroup over which a Šunić group is branch always contain a level stabiliser. This will be crucial in Chapter 4 when we discuss the congruence subgroup property.

Lemma 2.9.19 (Lemma 9 of [83])**.** *Let $G \in \mathcal{G}_{p,m}$ be a Šunić group. Then, $\mathrm{St}_G(m + 1) \leq G'$. Furthermore, if $p = 2$ and $m \geq 2$, then $\mathrm{St}_G(m + 1) \leq K$, where K is the subgroup from Proposition 2.9.18.*

Finally, the following lemma, which is a collection of a few lemmas from [83], tells us the structure of Šunić groups acting on the binary tree (with the exception of the infinite dihedral group). It will be useful at various points throughout this text.

Lemma 2.9.20 (See Lemmas 3, 5 and 7 of [83])**.** *Let $G \in \mathcal{G}_{2,m}$ be a Šunić group with $m \geq 2$ and let $\overline{B_1}$ be the normal closure of B_1 in G.*

(i) *For any element $d \in B_0 \setminus B_1$, we have*

$$G = \langle a, d \rangle \ltimes \overline{B_1} = \langle a, d \rangle \ltimes (B_1 \ltimes K) \tag{2.1}$$

(ii) *There exist elements $c \in B_{-1} \setminus B_0$ and $d \in B_0 \setminus B_1$ such that $c = (a, d)$ and*

$$\psi_1(\mathrm{St}_G(1)) = \hat{C} \ltimes (\overline{B_1} \times \overline{B_1}) \tag{2.2}$$

where $\hat{C} = \langle (a, d), (d, a) \rangle$ is a diagonal subgroup of $\langle a, d \rangle \times \langle a, d \rangle$, meaning that one component is trivial if and only if the other one is.

In particular, if $g \in \mathrm{St}_G(1)$ is such that $\psi(g) = (h, 1)$ or $\psi(g) = (1, h)$ with $h \in \langle a, d \rangle \leq G$ then, $g = 1$.

Chapter 3

Groups of intermediate growth

In this chapter, we investigate the growth of groups acting on rooted trees, with the aim of discovering new groups of intermediate growth.

The existence of groups of intermediate growth was first established by Grigorchuk in 1983 [42]. In that article, he used self-similarity and an argument of length contraction to show that the Grigorchuk group has intermediate growth. It was soon realised that this technique could be generalised and applied to many other groups. This was done by Grigorchuk in [44] for the family of p-groups known as Grigorchuk p-groups, and later by Bartholdi and Šunić for many of the torsion spinal groups [13].

However, not all groups of intermediate growth acting on a rooted tree possess this property of length contraction. For example, some elements of the non-torsion Grigorchuk groups acting on the binary rooted tree do not reduce in length, but Grigorchuk showed in [43] that such elements are rare enough that the growth of those groups is still intermediate. In 2006, Bux and Pérez generalised these ideas and showed that in general, as long as the proportion of elements reducing in length is large enough, a group acting on a rooted tree has subexponential growth [18]. They then used this result to prove that the iterated monodromy group[i] of the polynomial $z^2 + i$ has intermediate growth.

In a similar vein, Bartholdi and Pochon showed in 2009 that for a certain class of self-similar groups, if the set of elements whose length does not contract up to a fixed level is of subexponential growth, then the entire group is of subexponential growth. They also obtained a precise bound on the growth of the group when the growth of this set is linear. They then used their results to study the growth of the Fabrykowski-Gupta group, which was first studied by Fabrykowski and Gupta in [28, 29].

Although already quite powerful, Bartholdi and Pochon's criterion is somewhat limiting, since to apply it, one needs to understand not only elements with no length contraction, but also elements whose length contraction only appears after a significant number of steps. It is thus natural to ask if it would be sufficient to consider only the growth of elements with no length contraction at all. After some preliminaries in Section 3.1, we show that this is indeed the

[i]Although they are a very interesting class of groups acting on rooted trees, we will not discuss iterated monodromy groups in this thesis. We refer the interested reader to [65] for the definition and a survey of the topic.

case in Section 3.2 (Theorem 3.2.1). We then apply this criterion in Section 3.3, where we show in Theorem 3.3.6 that a large family of spinal groups (see Section 2.8) acting on the 3-regular rooted tree are of subexponential growth. Finally, in Section 3.4, we discuss a few natural questions that remain open.

The results of this chapter were already published in [31], and most of the text of this chapter comes from this article, with minor adjustments where necessary.

3.1 Preliminaries

Word pseudometrics and growth

In Section 2.5, we defined the word metric of a group (with respect to a given generating set) and the associated growth function. However, instead of working with a word metric, it is sometimes more convenient to work with what we will call a *word pseudometric*.

Definition 3.1.1. Let G be a finitely generated group and S be a symmetric finite generating set for G. A map $|\cdot|\colon S \to \{0,1\}$ that associates to every generator a length of 0 or 1 will be called a *pseudolength* on S. A pseudolength can be extended to a map

$$\begin{aligned} |\cdot|\colon G &\to \mathbb{N} \\ g &\mapsto \min\left\{ \sum_{i=1}^{k} |s_i| \;\middle|\; g = s_1 \dots s_k, s_i \in S \right\} \end{aligned}$$

called the *word pseudonorm* of G (associated to $(S, |\cdot|)$). We will call the corresponding pseudometric

$$\begin{aligned} d\colon G \times G &\to \mathbb{N} \\ (g,h) &\mapsto |g^{-1}h| \end{aligned}$$

the *word pseudometric* of G (associated to $(S, |\cdot|)$). ☙

Remark 3.1.2. If every generator is assigned a length of 1, then we obtain the word metric associated to S. ❧

If there is only a finite number of elements with length 0 in G, one can define a growth function for the group with respect to the given pseudometric. The growth function thus obtained is in fact equivalent to the usual growth function defined in Definition 2.5.4.

Proposition 3.1.3. *Let G be a group generated by a finite symmetric set S and $|\cdot|\colon S \to \{0,1\}$ be a pseudolength on S. Let*

$$G_0 = \langle \{ s \in S \mid |s| = 0 \} \rangle$$

be the subgroup of G generated by elements of S of length 0. If G_0 is finite, then the growth function

$$\begin{aligned} \gamma_{G,|\cdot|}\colon \mathbb{N} &\to \mathbb{N} \\ n &\mapsto \left| B_{G,|\cdot|}(n) \right| \end{aligned}$$

is well-defined, where $B_{G,|\cdot|}(n) = \{g \in G \mid |g| \leq n\}$*, and* $\gamma_{G,|\cdot|} \sim \gamma_{G,S}$*, where* $\gamma_{G,S}$ *is the usual growth function (see Definition 2.5.4).*

Proof. To show that $\gamma_{G,|\cdot|}$ is well-defined, we need to show that

$$\left|B_{G,|\cdot|}(n)\right| < \infty$$

for every $n \in \mathbb{N}$. Let $S_1 = \{s \in S \mid |s| = 1\}$ be the set of generators of length 1. For $g \in G$ with $|g| = n$, it follows from the definition of the word pseudonorm that there exist $s_1, \ldots, s_n \in S_1$, $g_0, \ldots, g_n \in G_0$ such that

$$g = g_0 s_1 g_1 \ldots s_n g_n.$$

Hence,

$$\left|B_{G,|\cdot|}(n)\right| \leq \sum_{k=0}^{n} |G_0|^{k+1} |S_1|^k < \infty.$$

We must now show that the growth function with respect to the word pseudometric is equivalent to the growth function with respect to the word metric. As in Section 2.5, we will denote by $|\cdot|_S$ the word norm in G with respect to S. Let

$$M = \max\{|g|_S \mid g \in G_0\}.$$

Then, the decomposition

$$g = g_0 s_1 g_1 \ldots s_n g_n$$

implies that

$$\begin{aligned} |g|_S &\leq (n+1)M + n \\ &= (|g|+1)M + |g| \\ &= |g|(M+1) + M \\ &\leq (2M+1)|g| \end{aligned}$$

if $|g| \geq 1$. Hence, if $n \geq 1$,

$$\gamma_{G,|\cdot|}(n) \leq \gamma_{G,S}((2M+1)n)$$

so $\gamma_{G,|\cdot|} \lesssim \gamma_{G,S}$. Since it is clear from the definition that $\gamma_{G,S} \lesssim \gamma_{G,|\cdot|}$, we have $\gamma_{G,|\cdot|} \sim \gamma_{G,S}$. □

Definition 3.1.4. Using the notation of Proposition 3.1.3, a word pseudometric such that G_0 is finite will be called a *proper word pseudometric.* ❧

Since the growth function coming from a proper word pseudometric is equivalent to the growth function coming from a word metric, we can study growth using whichever is more convenient. We will frequently make use of this fact in what follows.

In this chapter, we will be interested mainly in distinguishing between groups of exponential or subexponential growth. As we saw in Section 2.5, when we have a word metric, this can be done using the exponential growth rate of the group. It is straightforward to generalise this to proper word pseudometrics.

Proposition 3.1.5. *Let G be a finitely generated group, S be a finite symmetric generating set and $|\cdot|\colon G \to \mathbb{N}$ be a proper word pseudonorm. The limit*

$$\kappa_{G,|\cdot|} = \lim_{n\to\infty} \gamma_{G,|\cdot|}(n)^{\frac{1}{n}}$$

exists and is called the exponential growth rate *of the group G (with respect to the generating set S and the pseudonorm $|\cdot|$). Furthermore, if we denote by $\Omega_{G,|\cdot|}(n)$ the sphere of radius n in the word pseudometric $|\cdot|$, we have*

$$\kappa_{G,|\cdot|} = \lim_{n\to\infty} |\Omega_{G,|\cdot|}(n)|^{\frac{1}{n}}$$

as long as G is infinite.

The exponential growth rate depends on the pseudometric. However, as with the usual word metric, whether it is greater than 1 or not depends only on the growth type of the group.

Proposition 3.1.6. *Let G be a finitely generated group with a finite symmetric generating set S and a proper word pseudonorm $|\cdot|$. Then, $\kappa_{G,|\cdot|} > 1$ if and only if G is of exponential growth.*

Non-ℓ_1-expanding self-similar families of groups acting on rooted trees

A classical way of showing that a group acting on a rooted tree is of subexponential growth is to show that the projection of elements to some level induces a significant amount of length reduction. We introduce here a class of groups that seem well suited to this kind of argument, *non-ℓ_1-expanding self-similar families* of groups acting on rooted trees. This is a special case of self-similar families of groups, as defined in Definition 2.6.36.

Note that instead of considering self-similar families over a set Ω as in Definition 2.6.36, we will restrict ourselves to self-similar families over $\mathbb{N}$, where the map $\sigma\colon \mathbb{N} \to \mathbb{N}$ is simply the addition by 1. This has the advantage of simplifying the notation without causing any significant loss in generality, since if $(G_\omega)_{\omega\in\Omega}$ is a self-similar family, then for any $\omega \in \Omega$, $(G_\nu)_{\nu\in\mathbb{N}}$ is also a self-similar family, where $G_\nu = G_{\sigma^\nu(\omega)}$.

Definition 3.1.7. Let X be a finite alphabet of size d, and let $(G_\nu)_{\nu\in\mathbb{N}}$ be a self-similar family of groups of automorphisms of the rooted tree X^*. For each $\nu \in \mathbb{N}$, let S_ν be a finite symmetric generating set for G_ν and let $|\cdot|_\nu$ be a proper word pseudonorm on G_ν with respect to S_ν. The family $(G_\nu, S_\nu, |\cdot|_\nu)_{\nu\in\mathbb{N}}$ is said to be a *non-ℓ_1-expanding self-similar family* of groups of automorphisms of X^* if for all $\nu \in \mathbb{N}$ and all $g \in G_\nu$, we have

$$\sum_{i=1}^{d} |g_i|_{\nu+1} \leq |g|_\nu,$$

where (using the notation mentioned in Notation 2.6.33)

$$g = \tau(g_1, g_2, \dots, g_d)$$

with $g_1, g_2, \dots, g_d \in G_{\sigma(\omega)}$, $\tau \in \mathrm{Sym}(X)$.

If $G_v = G$, $S_\nu = S$ and $|\cdot|_\nu = |\cdot|$ for all $\nu \in \mathbb{N}$, we say that G is a *non-ℓ_1-expanding self-similar group* (with respect to S and $|\cdot|$). ❧

To show that a self-similar family of groups is non-ℓ_1-expanding, it is sufficient to look at the generators.

Proposition 3.1.8. *Let $(G_\nu, S_\nu, |\cdot|_\nu)_{\nu\in\mathbb{N}}$ be a self-similar family of groups. It is non-ℓ_1-expanding if and only if, for all $\nu \in \mathbb{N}$ and $s = \tau(s_1, s_2, \ldots, s_d) \in S_\nu$, we have*

$$\sum_{i=1}^{d} |s_i|_{\nu+1} \leq |s|_\nu.$$

Proof. This follows directly from the subadditivity of the word pseudonorm. □

In particular, we see that if $s = \tau(s_1, s_2, \ldots, s_d) \in S_\nu$, there is at most one s_i with positive length, and none if $|s|_\nu = 0$.

Notation 3.1.9. In order to keep the notation simple, if $(G_\nu, S_\nu, |\cdot|_\nu)_{\nu\in\mathbb{N}}$ is a non-ℓ_1-expanding self-similar family of groups, for $\nu \in \mathbb{N}$, we will write γ_ν for the growth function and κ_ν for the exponential growth rate of G_ν with respect to the pseudonorm $|\cdot|_\nu$. ♋

The exponential growth rates of a non-ℓ_1-expanding self-similar family of groups form a non-decreasing sequence, a fact that will be useful later.

Proposition 3.1.10. *Let $(G_\nu, S_\nu, |\cdot|_\nu)_{\nu\in\mathbb{N}}$ be a non-ℓ_1-expanding self-similar family of groups of automorphisms of X^*, where X is an alphabet of cardinality d. For any $\nu \in \mathbb{N}$,*

$$\kappa_\nu \leq \kappa_{\nu+1}.$$

Proof. Let $n \in \mathbb{N}$ be greater than d and let $g \in G_\nu$ be such that $|g|_\nu \leq n$. We have

$$g = \tau(g_1, g_2, \ldots, g_d)$$

with $g_1, g_2, \ldots, g_d \in G_{\nu+1}$, $\tau \in \mathrm{Sym}(X)$ and

$$\sum_{i=1}^{d} |g_i|_{\nu+1} \leq |g|_\nu \leq n.$$

Since g is determined by $g_1, g_2, \ldots, g_d$ and τ, we have

$$\gamma_\nu(n) \leq d! \sum_{r_1+r_2+\cdots+r_d\leq n} \gamma_{\nu+1}(r_1)\gamma_{\nu+1}(r_2)\ldots\gamma_{\nu+1}(r_d).$$

Let $C(k) = \frac{\gamma_{\nu+1}(k)}{\kappa_{\nu+1}^k}$ for any $k \in \mathbb{N}$.We have

$$\begin{aligned}\gamma_\nu(n) &\leq d! \sum_{r_1+r_2+\cdots+r_d\leq n} C(r_1)\kappa_{\nu+1}^{r_1} C(r_2)\kappa_{\nu+1}^{r_2} \ldots C(r_d)\kappa_{\nu+1}^{r_d} \\ &= d!\kappa_{\nu+1}^n \sum_{r_1+r_2+\cdots+r_d\leq n} C(r_1)C(r_2)\ldots C(r_d).\end{aligned}$$

Let $s(n) \in \{1, \dots, n\}$ be such that $C(s(n)) \geq C(r)$ for all $1 \leq r \leq n$. We then have

$$\begin{aligned}\gamma_\nu(n) &\leq d!\kappa_{\nu+1}^n \sum_{r_1+r_2+\cdots+r_d \leq n} C(s(n))^d \\ &\leq d!\kappa_{\nu+1}^n C(s(n))^d n^d\end{aligned}$$

It is clear from the definition that the sequence $s(n)$ is non-decreasing. Therefore, either it stabilises or it goes to infinity. Since $\lim_{k\to\infty} C(k)^{\frac{1}{k}} = 1$, in both cases we have $\lim_{n\to\infty} C(s(n))^{\frac{1}{n}} = 1$. Hence,

$$\begin{aligned}\kappa_\nu &= \lim_{n\to\infty} \gamma_\nu(n)^{\frac{1}{n}} \\ &\leq \kappa_{\nu+1} \lim_{n\to\infty} d!^{\frac{1}{n}} C(s(n))^{\frac{d}{n}} n^{\frac{d}{n}} \\ &= \kappa_{\nu+1}.\end{aligned}$$

□

Let us now see a few examples of non-ℓ_1-expanding self-similar families of groups. This list is by no means exhaustive, but serves to illustrate that many previously studied groups fall into this class.

Example 3.1.11 (Spinal groups)**.** Using the notation of Section 2.8, let $G_\omega = \langle A_\omega, B_\omega \rangle$ be a spinal group. For all $\nu \in \mathbb{N}$, let us set $G_\nu = G_{\sigma^\nu(\omega)}$, $A_\nu = A_{\sigma^\nu(\omega)}$, $B_\nu = B_{\sigma^\nu(\omega)}$ and $S_\nu = A_\nu \cup B_\nu$. Let $|\cdot|_\nu \colon G_\nu \to \mathbb{N}$ be the pseudonorm induced by setting $|a|_\nu = 0$ for all $a \in A_\nu$ and $|b|_\nu = 1$ for all $b \in B_\nu \setminus \{1\}$. Then, $(G_\nu, S_\nu, |\cdot|_\nu)_{\nu\in\mathbb{N}}$ is a non-ℓ_1-expanding self-similar family of groups. Indeed, for $a \in A_\nu$, we have $a = \tau(1, 1, \dots, 1)$, so $0 = |a|_\nu = \sum_{i=0}^{d} |1|_{\nu+1}$. For $b \in B_\nu \setminus \{1\}$, we have $b = \tau(a_1, \dots, a_{d-1}, c)$ with $a_1, \dots, a_{d-1} \in B_{\nu+1}$ and $c \in B_{\nu+1}$. Therefore,

$$1 = |a_1| + \cdots + |a_{d-1}| + |c| = |b|.$$

We conclude thanks to Proposition 3.1.8. ❧

Example 3.1.12 (Nekrashevych's family of groups $\mathcal{D}_\omega$)**.** Let $\{0,1\}^\infty$ be the set of right-infinite sequences of 0 and 1 and

$$\begin{aligned}\sigma \colon \{0,1\}^\infty &\to \{0,1\}^\infty \\ \omega_0\omega_1\omega_2\dots &\mapsto \omega_1\omega_2\omega_3\dots\end{aligned}$$

be the shift. Let X be an alphabet of two letters. For $\omega = \omega_0\omega_1\omega_2\dots \in \{0,1\}^\infty$, we can recursively define automorphisms $\beta_\omega, \gamma_\omega \in \mathrm{Aut}(X^*)$ by

$$\begin{aligned}\beta_\omega &= (\alpha, \gamma_{\sigma(\omega)}) \\ \gamma_\omega &= \begin{cases} (\beta_{\sigma(\omega)}, 1) & \text{if } \omega_0 = 0 \\ (1, \beta_{\sigma(\omega)}) & \text{if } \omega_0 = 1 \end{cases}\end{aligned}$$

where $\alpha \in \mathrm{Aut}(X^*)$ is the non-trivial rooted automorphism of X^*. We can then define the group $\mathcal{D}_\omega = \langle \alpha, \beta_\omega, \gamma_\omega \rangle$. This family of groups was first studied by Nekrashevych in [63].

It follows from the definition that $\alpha^2 = \beta_\omega^2 = \gamma_\omega^2 = 1$. Hence, the set $S_\omega = \{\alpha, \beta_\omega, \gamma_\omega\}$ is a finite symmetric generating set of $\mathcal{D}_\omega$. Let $|\cdot|_\omega \colon \mathcal{D}_\omega \to \mathbb{N}$ be the word pseudometric defined by $|\alpha|_\omega = 0$, $|\beta_\omega|_\omega = |\gamma_\omega|_\omega = 1$. Then, the family $(G_\nu, S_\nu, |\cdot|_\nu)_{\nu\in\mathbb{N}}$ is a non-ℓ_1-expanding self-similar family of automorphisms of T_2, where $G_\nu = \mathcal{D}_{\sigma^\nu(\omega)}$, $S_\nu = S_{\sigma^\nu(\omega)}$ and $|\cdot|_\nu = |\cdot|_{\sigma^\nu(\omega)}$. ❧

Example 3.1.13 (Peter Neumann's example)**.** We present here a group that first appeared as an example in Neumann's paper [68]. The description we use here is based on [9].

Let $X = \{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{6}\}$ and $A = \mathrm{Alt}(X)$. For every couple $(a, x) \in A \times X$ such that x is a fixed point of a, we can recursively define an automorphism of $\mathrm{Aut}(X^*)$ by

$$b_{(a,x)} = a(1, \ldots, b_{(a,x)}, \ldots, 1)$$

where the $b_{(a,x)}$ is in the x^{th} position. Let

$$S = \left\{b_{(a,x)} \in \mathrm{Aut}(X^*) \mid (a, x) \in A \times X \text{ and } ax = x\right\},$$

$G = \langle S\rangle$ and $|\cdot| \colon G \to \mathbb{N}$ be the word norm associated to S. Then, it is clear from the definition that $(G_\nu, S_\nu, |\cdot|_\nu)_{\nu\in\mathbb{N}}$ is a non-ℓ_1-expanding self-similar family of automorphisms of X^*, where $G_\nu = G$, $S_\nu = S$ and $|\cdot|_\nu = |\cdot|$ for all $\nu \in \mathbb{N}$. Hence, G is a non-ℓ_1-expanding self-similar group. ❧

Incompressible elements

In non-ℓ_1-expanding self-similar families of group, there are elements that never reduce in length no matter how many time we take their projections. We will call those *incompressible elements*.

Definition 3.1.14. Let X be a finite alphabet of size d and let $(G_\nu, S_\nu, |\cdot|_\nu)_{\nu\in\mathbb{N}}$ be a non-ℓ_1-expanding self-similar family of groups of automorphisms of X^*. For any $k \in \mathbb{N}^*$, we recursively define the sets $\mathcal{I}_k^\nu$ of elements of G_ν which have no length reduction up to level k as

$$\mathcal{I}_k^\nu = \left\{ g = \tau(g_1, g_2, \ldots, g_d) \in G_\nu \;\middle|\; g_1, g_2, \ldots, g_d \in \mathcal{I}_{k-1}^{\nu+1}, \sum_{i=1}^{d} |g_i|_{\nu+1} = |g|_\nu \right\}$$

where $\mathcal{I}_0^\nu = G_\nu$ for all $\nu \in \mathbb{N}$.

We will call the set

$$\mathcal{I}_\infty^\nu = \bigcap_{k=1}^{\infty} \mathcal{I}_k^\nu$$

the set of *incompressible elements* of G_ν. This is the set of elements which have no length reduction on any level. ☙

3.2 Incompressible elements and growth

The standard technique used to show that a self-similar family of groups acting on a rooted tree is of subexponential growth, pioneered by Grigorchuk [42], relies on the existence of significant length contraction when we look at the wreath decomposition of elements of the groups. Consequently, this method cannot be applied to groups containing incompressible elements. However, as we will see, as long as the incompressible elements do not grow too quickly, the groups still have subexponential growth.

More precisely, we will show that if every group in a non-ℓ_1-expanding self-similar family of groups of automorphisms of a regular rooted tree is generated by incompressible elements and the sets of incompressible elements grow uniformly subexponentially, then the groups themselves are also of subexponential growth. This result is a generalization of the first part of Proposition 5 proved by Bartholdi and Pochon in [10]. The main difference is that we show here that under our assumptions, it is sufficient to look at the growth of the set $\mathcal{I}_\infty^\nu$ of incompressible elements instead of the set $\mathcal{I}_k^\nu$ of elements that have no reduction up to level k for some $k \in \mathbb{N}$.

Theorem 3.2.1. *Let $A \in \mathbb{N}$ be an integer, $X = \{\mathbf{1}, \mathbf{2}, \dots, \mathbf{d}\}$ be an alphabet of size $d \geq 2$, $(G_\nu, S_\nu, |\cdot|_\nu)_{\nu \in \mathbb{N}}$ be a non-ℓ_1-expanding self-similar family of automorphisms of X^* such that $S_\nu \subseteq \mathcal{I}_\infty^\nu$ and $|S_\nu| \leq A$ for every $\nu \in \mathbb{N}$, and let $\Omega_\nu(n)$ be the sphere of radius $n \in \mathbb{N}$ in G_ν with respect to the pseudometric $|\cdot|_\nu$. If there exists a subexponential function $\delta \colon \mathbb{N} \to \mathbb{N}$ with $\ln(\delta)$ concave such that for infinitely many $\nu \in \mathbb{N}$, $\mathcal{I}_\infty^\nu \cap \Omega_\nu(n) \leq \delta(n)$ for all $n \in \mathbb{N}$, then the groups G_ν are of subexponential growth for every $\nu \in \mathbb{N}$.*

Proof. The proof is inspired by the one found in [10], with a few key modifications. The idea is to split the set $\Omega_\nu(n)$ in two, the set of elements which can be written as a product of a few incompressible elements and the set of elements which can only be written as a product of a large number of incompressible elements. The first set grows slowly because there are few incompressible elements, and the second set grows slowly because there is a significant amount of length reduction.

Let us fix $\nu \in \mathbb{N}$ such that $\mathcal{I}_\infty^\nu \cap \Omega_\nu(n) \leq \delta(n)$ for all $n \in \mathbb{N}$. In what follows, we will show that $\kappa_\nu = 1$. By Proposition 3.1.10, this will show that $\kappa_{\nu'} = 1$ for all $\nu' \leq \nu$.

Since $S_\nu \subseteq \mathcal{I}_\infty^\nu$, we have that for every $g \in G_\nu$, the set

$$\left\{ N \in \mathbb{N} \mid g = g_1 g_2 \dots g_N, g_i \in \mathcal{I}_\infty^\nu, \sum_{i=1}^{N} |g_i|_\nu = |g|_\nu \right\}$$

is not empty. Hence, we can define

$$N(g) = \min \left\{ N \in \mathbb{N} \mid g = g_1 g_2 \dots g_N, g_i \in \mathcal{I}_\infty^\nu, \sum_{i=1}^{N} |g_i|_\nu = |g|_\nu \right\}.$$

For any $0 < \epsilon < 1$ and any $n \in \mathbb{N}$, the sphere $\Omega_\nu(n)$ of radius n in G_ν can be partitioned in two by the subsets

$$\Omega_\nu^>(n, \epsilon) = \{g \in \Omega_\nu(n) \mid N(g) > \epsilon n\}$$
$$\Omega_\nu^<(n, \epsilon) = \{g \in \Omega_\nu(n) \mid N(g) \leq \epsilon n\}.$$

Let us assume that there are infinitely many values of n such that $\Omega_\nu^>(n, \epsilon) \neq \emptyset$ (otherwise, we can simply ignore this set and consider only $\Omega_\nu^<(n, \epsilon)$). For such an n, let $g \in \Omega_\nu^>(n, \epsilon)$. By definition of $N(g)$, there exists $g_1, g_2, \dots, g_{N(g)} \in$

$\mathcal{I}_\infty^\nu$ such that $g = g_1 g_2 \dots g_{N(g)}$ and $\sum_{i=1}^{N(g)} |g_i|_\nu = |g|_\nu$. Let $h_i = g_{2i-1}g_{2i}$ for $1 \leq i \leq \lfloor \frac{N(g)}{2} \rfloor$. Then,

$$g = \begin{cases} h_1 h_2 \dots h_{\frac{N(g)-1}{2}} g_{N(g)} & \text{if } N(g) \text{ is odd} \\ h_1 h_2 \dots h_{\frac{N(g)}{2}} & \text{if } N(g) \text{ is even.} \end{cases}$$

Notice that since

$$|g|_\nu = \sum_{i=1}^{N(g)} |g_i|_\nu,$$

we must have $|h_i|_\nu = |g_{2i-1}|_\nu + |g_{2i}|_\nu$. Hence, no h_i can be in $\mathcal{I}_\infty^\nu$ (otherwise, this would contradict the minimality of $N(g)$).

Let

$$S(g) = \left\{ i \;\middle|\; |h_i|_\nu \leq \frac{6}{\epsilon} \right\}$$

be the set of "small" factors of g and

$$L(g) = \left\{ i \;\middle|\; |h_i|_\nu > \frac{6}{\epsilon} \right\}$$

be the set of "large" factors. Clearly, $|S(g)| + |L(g)| = \lfloor \frac{N(g)}{2} \rfloor$. Since $g \in \Omega_\nu^>(n, \epsilon)$, $N(g)$ is not too small compared to n, which implies that as long as n is large enough, more than half of the factors of g must be small. More precisely, if $n > \frac{3}{\epsilon}$, then $|S(g)| \geq \frac{1}{2}\lfloor \frac{N(g)}{2} \rfloor$. Indeed, it that were not the case, then we would have $|L(g)| > \frac{1}{2}\lfloor \frac{N(g)}{2} \rfloor$, so

$$\begin{aligned} n &\geq \sum_{i=1}^{\lfloor \frac{N(g)}{2} \rfloor} |h_i|_\nu \geq \sum_{i \in L(g)} |h_i|_\nu \\ &> \frac{1}{2} \left\lfloor \frac{N(g)}{2} \right\rfloor \frac{6}{\epsilon} \geq \frac{(N(g)-1)}{4} \frac{6}{\epsilon} \\ &> \frac{3}{2} n - \frac{3}{2\epsilon} \\ &> n \end{aligned}$$

which is a contradiction. Therefore, if $n > \frac{3}{\epsilon}$,

$$\begin{aligned} |S(g)| &\geq \frac{1}{2} \left\lfloor \frac{N(g)}{2} \right\rfloor \geq \frac{N(g)-1}{4} \\ &> \frac{\epsilon}{4} n - \frac{1}{4} \\ &> \frac{\epsilon}{8} n. \end{aligned}$$

This means that the number of small factors is comparable with n. This is important because, as we will see, every small factor gives us some length reduction on a fixed level (fixed in the sense that it does not depend on n, but only on ϵ). Hence, on this level, we will see a large amount of length reduction.

For $r \in \mathbb{R}$, let

$$l_\nu(r) = \min\{k \in \mathbb{N} \mid (G_\nu \setminus \mathcal{I}_\infty^\nu) \cap B_\nu(r) \cap \mathcal{I}_k^\nu = \emptyset\}$$

where $B_\nu(r)$ is the ball of radius r in G_ν. Notice that since $(G_\nu \setminus \mathcal{I}_\infty^\nu) \cap B_\nu(r)$ is a finite set whose intersection with $\mathcal{I}_\infty^\nu = \bigcap_{k=1}^\infty \mathcal{I}_k^\nu$ is empty, $l_\nu(r)$ is well-defined. It is the first level on which every element of $(G_\nu \setminus \mathcal{I}_\infty^\nu) \cap B_\nu(r)$ has seen some length contraction.

Let us consider the $l_\nu(\frac{6}{\epsilon})^{\text{th}}$-level decomposition of g,

$$g = \tau(g_{\mathbf{11\ldots1}}, g_{\mathbf{11\ldots2}}, \ldots, g_{\mathbf{dd\ldots d}})$$

with $\tau \in \wr^{l_\nu(\frac{6}{\epsilon})} \operatorname{Sym}(X)$ and $g_{\mathbf{11\ldots1}}, \ldots, g_{\mathbf{dd\ldots d}} \in G_{\nu+l_\nu(\frac{6}{\epsilon})}$. Since

$$g = \begin{cases} h_1 h_2 \ldots h_{\frac{N(g)-1}{2}} g_{N(g)} & \text{if } N(g) \text{ is odd} \\ h_1 h_2 \ldots h_{\frac{N(g)}{2}} & \text{if } N(g) \text{ is even,} \end{cases}$$

we have

$$\sum_{j \in X^{l_\nu(\frac{6}{\epsilon})}} |g_j|_{\nu+l_\nu(\frac{6}{\epsilon})} \leq \left(\sum_{i=1}^{\frac{N(g)-1}{2}} \sum_{j \in X^{l_\nu(\frac{6}{\epsilon})}} |h_{i,j}|_{\nu+l_\nu(\frac{6}{\epsilon})} \right) + \sum_{j \in X^{l_\nu(\frac{6}{\epsilon})}} |g_{N(g),j}|_{\nu+l_\nu(\frac{6}{\epsilon})}$$

if $N(g)$ is odd and

$$\sum_{j \in X^{l_\nu(\frac{6}{\epsilon})}} |g_j|_{\nu+l_\nu(\frac{6}{\epsilon})} \leq \sum_{i=1}^{\frac{N(g)}{2}} \sum_{j \in X^{l_\nu(\frac{6}{\epsilon})}} |h_{i,j}|_{\nu+l_\nu(\frac{6}{\epsilon})}$$

if $N(g)$ is even, where $X^{l_\nu(\frac{6}{\epsilon})}$ is the set of words of length $l_\nu(\frac{6}{\epsilon})$ in the alphabet $X = \{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{d}\}$,

$$h_i = \tau_i(h_{i,\mathbf{11\ldots1}}, h_{i,\mathbf{11\ldots2}}, \ldots, h_{i,\mathbf{dd\ldots d}})$$

is the $l_\nu(\frac{6}{\epsilon})^{\text{th}}$-level decomposition of h_i and

$$g_{N(g)} = \tau_{N(g)}(g_{N(g),\mathbf{11\ldots1}}, g_{N(g),\mathbf{11\ldots2}}, \ldots, g_{N(g),\mathbf{dd\ldots d}})$$

is the $l_\nu(\frac{6}{\epsilon})^{\text{th}}$-level decomposition of $g_{N(g)}$.

It follows from the definition of $l_\nu(\frac{6}{\epsilon})$ that $h_i \notin \mathcal{I}_{l_\nu(\frac{6}{\epsilon})}^\nu$ for all $i \in S(g)$. Hence, for all $i \in S(g)$,

$$\sum_{j \in X^{l_\nu(\frac{6}{\epsilon})}} |h_{i,j}|_{\nu+l_\nu(\frac{6}{\epsilon})} \leq |h_i|_\nu - 1.$$

Therefore, as long as $n > \frac{3}{\epsilon}$,

$$\begin{aligned} \sum_{j \in X^{l_\nu(\frac{6}{\epsilon})}} |g_j|_{\nu+l_\nu(\frac{6}{\epsilon})} &\leq n - |S(g)| \\ &< n - \frac{\epsilon}{8} n \\ &= \frac{8-\epsilon}{8} n. \end{aligned}$$

Hence, by summing over every possible lengths of the factors in the $l_\nu(\frac{6}{\epsilon})^{\text{th}}$-level decomposition, we get that for $n > \frac{3}{\epsilon}$,

$$\begin{aligned}|\Omega_\nu^{>}(n,\epsilon)| &\leq \sum_{k_1+\cdots+k_{d^{l_\nu(\frac{6}{\epsilon})}} \leq \frac{8-\epsilon}{8}n} C|\Omega_{\nu+l_\nu(\frac{6}{\epsilon})}(k_1)|\dots|\Omega_{\nu+l_\nu(\frac{6}{\epsilon})}(k_{d^{l_\nu(\frac{6}{\epsilon})}})| \\ &\leq \left(\frac{8-\epsilon}{8}n\right)^{d^{l_\nu(\frac{6}{\epsilon})}} K(n)\kappa_{\nu+l_\nu(\frac{6}{\epsilon})}^{\frac{8-\epsilon}{8}n}\end{aligned}$$

where $C = \left[G_\nu : \mathrm{St}_{G\nu}\left(l_\nu\left(\frac{6}{\epsilon}\right)\right)\right]$ and $K(n)$ is a function such that

$$\lim_{n\to\infty} K(n)^{\frac{1}{n}} = 1.$$

We conclude that, for a fixed ϵ between 0 and 1,

$$\limsup_{n\to\infty} |\Omega_\nu^{>}(n,\epsilon)|^{\frac{1}{n}} \leq \kappa_{\nu+l_\nu(\frac{6}{\epsilon})}^{\frac{8-\epsilon}{8}}.$$

On the other hand,

$$\begin{aligned}|\Omega_\nu^{<}(n,\epsilon)| &\leq \sum_{i=1}^{\epsilon n}\sum_{k_1+\cdots+k_i=n}\prod_{j=1}^{i}\delta(k_j) \\ &\leq \sum_{i=1}^{\epsilon n}\sum_{k_1+\cdots+k_i=n}\delta\left(\frac{n}{i}\right)^i\end{aligned}$$

by Lemma 6 of [10], since $\ln(\delta)$ is concave. Hence, assuming that $\epsilon < \frac{1}{2}$, we have

$$\begin{aligned}|\Omega_\nu^{<}(n,\epsilon)| &\leq \sum_{i=1}^{\epsilon n}\binom{n}{i-1}\max_{1\leq i\leq \epsilon n}\left\{\delta\left(\frac{n}{i}\right)^i\right\} \\ &\leq \epsilon n\binom{n}{\epsilon n}\max_{1\leq i\leq \epsilon n}\left\{\delta\left(\frac{n}{i}\right)^i\right\}.\end{aligned}$$

Using the fact that $\binom{n}{\epsilon n} \leq \frac{n^{\epsilon n}}{(\epsilon n)!}$ and Stirling's approximation, we get

$$|\Omega_\nu^{<}(n,\epsilon)| \leq \epsilon n\left(\frac{e}{\epsilon}\right)^{\epsilon n}\frac{C(n)}{\sqrt{2\pi\epsilon n}}\max_{1\leq i\leq \epsilon n}\left\{\delta\left(\frac{n}{i}\right)^i\right\}$$

where $\lim_{n\to\infty} C(n) = 1$. Therefore,

$$\limsup_{n\to\infty}|\Omega_\nu^{<}(n,\epsilon)|^{\frac{1}{n}} \leq \left(\frac{e}{\epsilon}\right)^{\epsilon}\limsup_{n\to\infty}\delta\left(\frac{n}{i_n}\right)^{\frac{i_n}{n}}$$

where $1 \leq i_n \leq \epsilon n$ maximises $\delta\left(\frac{n}{i}\right)^i$. Let $k_n = \frac{n}{i_n}$. Then, $\frac{1}{\epsilon} \leq k_n \leq n$. Since $\lim_{k\to\infty}\delta(k)^{\frac{1}{k}} = 1$, there must exist $N \in \mathbb{N}$ such that $\sup_{\frac{1}{\epsilon}\leq k}\{\delta(k)^{\frac{1}{k}}\} = \sup_{\frac{1}{\epsilon}\leq k\leq N}\{\delta(k)^{\frac{1}{k}}\}$. Hence, there exists some $K_\epsilon \in \mathbb{N}$ such that $K_\epsilon \geq \frac{1}{\epsilon}$ and $\limsup_{n\to\infty}\delta\left(\frac{n}{i_n}\right)^{\frac{i_n}{n}} = \delta(K_\epsilon)^{\frac{1}{K_\epsilon}}$. We conclude that

$$\limsup_{n\to\infty}|\Omega_\nu^{<}(n,\epsilon)| \leq \left(\frac{e}{\epsilon}\right)^{\epsilon}\delta(K_\epsilon)^{\frac{1}{K_\epsilon}}$$

for some $K_\epsilon \geq \frac{1}{\epsilon}$.

Since, for any $0 < \epsilon < \frac{1}{2}$, we have $|\Omega_\nu(n)| = |\Omega_\nu^>(n,\epsilon)| + |\Omega_\nu^<(n,\epsilon)|$,

$$\begin{aligned}
\kappa_\nu &= \lim_{n\to\infty} |\Omega_\nu(n)|^{\frac{1}{n}} = \lim_{n\to\infty} \left(|\Omega_\nu^>(n,\epsilon)| + |\Omega_\nu^<(n,\epsilon)|\right)^{\frac{1}{n}} \\
&\leq \limsup_{n\to\infty} \left(2\max\left\{|\Omega_\nu^>(n,\epsilon)|, |\Omega_\nu^<(n,\epsilon)|\right\}\right)^{\frac{1}{n}} \\
&= \max\left\{\limsup_{n\to\infty} |\Omega_\nu^>(n,\epsilon)|^{\frac{1}{n}}, \limsup_{n\to\infty} |\Omega_\nu^<(n,\epsilon)|^{\frac{1}{n}}\right\} \\
&\leq \max\left\{\kappa_{\nu+l_\nu(\frac{6}{\epsilon})}^{\frac{8-\epsilon}{8}}, e^\epsilon \left(\frac{1}{\epsilon}\right)^\epsilon \delta\left(K_\epsilon\right)^{\frac{1}{K_\epsilon}}\right\}.
\end{aligned}$$

Let us now fix $0 < \epsilon < \frac{1}{2}$. There must exist a $k \in \mathbb{N}$ such that

$$\kappa_{\nu+k} \leq e^\epsilon \left(\frac{1}{\epsilon}\right)^\epsilon \delta\left(K_\epsilon\right)^{\frac{1}{K_\epsilon}}.$$

Indeed, otherwise we would have $\kappa_{\nu+i} > e^\epsilon \left(\frac{1}{\epsilon}\right)^\epsilon \delta\left(K_\epsilon\right)^{\frac{1}{K_\epsilon}}$ for all $i \in \mathbb{N}$. In particular, this would imply that

$$\kappa_\nu \leq \kappa_{\nu+l_\nu(\frac{6}{\epsilon})}^{\frac{8-\epsilon}{8}}.$$

Let $\nu' \in \mathbb{N}$ be such that $\nu' \geq \nu + l_\nu(\frac{6}{\epsilon})$ and $\mathcal{I}_\infty^{\nu'} \cap \Omega_{\nu'}(n) \leq \delta(n)$ for all $n \in \mathbb{N}$ (such a ν' exist by hypothesis). Then, we would also have

$$\kappa_{\nu'} \leq \kappa_{\nu'+l_{\nu'}(\frac{6}{\epsilon})}^{\frac{8-\epsilon}{8}},$$

and so, using the fact that by Proposition 3.1.10, $\kappa_{\nu+l_\nu(\frac{6}{\epsilon})} \leq \kappa_{\nu'}$, we would have

$$\kappa_\nu \leq \kappa_{\nu'+l_{\nu'}(\frac{6}{\epsilon})}^{\left(\frac{8-\epsilon}{8}\right)^2}.$$

By induction, we conclude that for any $m \in \mathbb{N}^*$, there exists $k_m \in \mathbb{N}$ such that

$$\kappa_\nu \leq \kappa_{\nu+k_m}^{\left(\frac{8-\epsilon}{8}\right)^m}.$$

Since $|S_i| \leq A$ for every $i \in \mathbb{N}$, we have that $\kappa_i \leq A$ for every $i \in \mathbb{N}$. Hence, we get that $\kappa_\nu \leq A^{\left(\frac{8-\epsilon}{8}\right)^m}$ for every $m \in \mathbb{N}^*$, which implies that $\kappa_\nu = 1$. This contradicts the hypothesis that $\kappa_\nu > e^\epsilon \left(\frac{1}{\epsilon}\right)^\epsilon \delta\left(K_\epsilon\right)^{\frac{1}{K_\epsilon}}$.

Therefore, there must exist some $i \in \mathbb{N}$ such that $\kappa_{\nu+i} \leq e^\epsilon \left(\frac{1}{\epsilon}\right)^\epsilon \delta\left(K_\epsilon\right)^{\frac{1}{K_\epsilon}}$. By Proposition 3.1.10, we must have

$$\kappa_\nu \leq e^\epsilon \left(\frac{1}{\epsilon}\right)^\epsilon \delta\left(K_\epsilon\right)^{\frac{1}{K_\epsilon}}.$$

As the above inequality is valid for any $0 < \epsilon < \frac{1}{2}$ and

$$\lim_{\epsilon\to 0} e^\epsilon \left(\frac{1}{\epsilon}\right)^\epsilon \delta\left(K_\epsilon\right)^{\frac{1}{K_\epsilon}} = 1$$

we must have $\kappa_\nu = 1$, and so G_ν is of subexponential growth. □

3.3 Growth of spinal groups

Using the techniques developed by Grigorchuk in [43], one can show that every spinal group acting on the binary rooted tree is of subexponential growth. The next natural step would then be to study the growth of spinal groups acting on the 3-regular rooted tree.

In this section, we will apply the criterion given in Theorem 3.2.1 to study the growth of some spinal groups acting on the 3-regular rooted tree. We will be able to prove that the growth is subexponential in several new cases. In particular, our results will imply that all the groups in Šunić's family acting on the ternary tree (see Section 2.9) are of subexponential growth. While this was already known for torsion groups, this was previously unknown for groups with elements of infinite order, except in the case of the Fabrykowski-Gupta group.

Unfortunately, we were unable for now to obtain similar results for spinal groups acting on rooted trees of higher degrees, as the methods used here do not seem to have obvious generalizations in those settings.

Growth of spinal groups acting on the 3-regular rooted tree

Let $X = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$, $m \in \mathbb{N}$, $\mathbb{Z}/3\mathbb{Z} \cong A = \langle(\mathbf{1}\ \mathbf{2}\ \mathbf{3})\rangle \leq \mathrm{Sym}(X)$ and $B = (\mathbb{Z}/3\mathbb{Z})^m$. Let

$$\Omega = \left\{ \{\omega_{ij}\}_{i\in\mathbb{N}, 1\leq j\leq 2} \;\middle|\; \omega_{i,1} \in \mathrm{Epi}(B, A), \omega_{i,2} = 1, \bigcap_{i\geq k} \ker(\omega_{i,1}) = 1 \forall k \in \mathbb{N} \right\}$$

be a set of sequences of pairs of homomorphisms of B into A and $\sigma\colon \Omega \to \Omega$ be the shift (see Section 2.8). For any $\omega \in \Omega$, let us define $A_\omega = A$. Using the notation of Section 2.8, we get spinal groups $G_\omega = \langle A, B_\omega\rangle$ acting on X^* which naturally come equipped with a word pseudonorm $|\cdot|_\omega$ assigning length 0 to elements of A and length 1 to non-trivial elements of B_ω.

Notation 3.3.1. In order to streamline the notation, we will drop the indices ω wherever it is convenient and rely on context to keep track of which group we are working in. We will also drop the second index in the sequences of Ω and write $\omega = \omega_0\omega_1 \cdots \in \Omega$, which is a minor abuse of notation.

The set of incompressible elements of G_ω will be denoted by $\mathcal{I}_\infty^\omega$, and we will write $\mathcal{I}_\infty^\omega(n)$ for the set of incompressible elements of length n.

We will write $a = (\mathbf{1}\ \mathbf{2}\ \mathbf{3}) \in A$, and for any $b \in B_\omega$, we will write $b^{a^i} = a^i b a^{-i}$ where $i \in \mathbb{Z}/3\mathbb{Z}$. ♫

Remark 3.3.2. We have that for every $\omega \in \Omega$, the group G_ω is a quotient of $A * B_\omega$. Hence, every element of G_ω can be written as an alternating product of elements of A and B_ω.

It follows that every $g \in G_\omega$ of length n can be written as

$$g = a^s \iota_\omega(\beta_1)^{a^{c_1}} \iota_\omega(\beta_2)^{a^{c_2}} \ldots \iota_\omega(\beta_n)^{a^{c_n}}$$

for some $s \in \mathbb{Z}/3\mathbb{Z}$, $\beta\colon \{1, 2, \ldots, n\} \to B$ and $c\colon \{1, 2, \ldots, n\} \to \mathbb{Z}/3\mathbb{Z}$, where $\iota_\omega\colon B \to B_\omega$ is the isomorphism described in Section 2.8, and where we use indices to denote the arguments of some functions. In order to further streamline the notation, we will sometimes drop the i_ω and write simply

$$g = a^s \beta_1^{a^{c_1}} \beta_2^{a^{c_2}} \dots \beta_n^{a^{c_n}}.$$

Although this is an abuse of notation, the group to which β_i belongs should always be clear from the context. ❧

Notation 3.3.3. For any $n \in \mathbb{N}$ at least 2 and $c\colon \{1, 2, \dots, n\} \to \mathbb{Z}/3\mathbb{Z}$, we will denote by $\partial c\colon \{1, 2, \dots n-1\} \to \mathbb{Z}/3\mathbb{Z}$ the *discrete derivative* of c, that is,

$$\partial c(k) = c_{k+1} - c_k.$$

☙

Lemma 3.3.4. *Let $\omega \in \Omega$ and $g \in G_\omega$ with $|g| = n \geq 2$. Writing*

$$g = a^s \beta_1^{a^{c_1}} \beta_2^{a^{c_2}} \dots \beta_n^{a^{c_n}}$$

for some $s \in \mathbb{Z}/3\mathbb{Z}$, $\beta\colon \{1, 2, \dots, n\} \to B$ and $c\colon \{1, 2, \dots, n\} \to \mathbb{Z}/3\mathbb{Z}$, if $g \in \mathcal{I}_\infty^\omega(n)$, then there exists $m_c \in \{1, 2, \dots, n\}$ such that

$$\partial c(k) = \begin{cases} 2 & \text{if } k < m_c \\ 1 & \text{if } k \geq m_c. \end{cases}$$

Proof. If there exists $k \in \{1, 2, \dots, n-1\}$ such that $\partial c(k) = 0$, then $c_k = c_{k+1}$, which means that

$$|g| = |a^s \beta_1^{a^{c_1}} \beta_2^{a^{c_2}} \dots (\beta_k \beta_{k+1})^{a^{c_k}} \dots \beta_n^{a^{c_n}}| \leq n - 1$$

a contradiction. Hence, $\partial c(k) \neq 0$ for all $k \in \{1, 2, \dots, n-1\}$.

Therefore, to conclude, we only need to show that if $\partial c(k) = 1$ for some $k \in \{1, 2, \dots, n-2\}$, then $\partial c(k+1) \neq 2$. For the sake of contradiction, let us assume that $\partial c(k) = 1$ and $\partial c(k+1) = 2$ for some $k \in \{1, 2, \dots, n-2\}$. Without loss of generality, we can assume that $c_k = 0$ (indeed, it suffices to conjugate by the appropriate power of a to recover the other cases). We have

$$\begin{aligned} \beta_k \beta_{k+1}^a \beta_{k+2} &= (\alpha_k, 1, \beta_k)(\beta_{k+1}, \alpha_{k+1}, 1)(\alpha_{k+2}, 1, \beta_{k+2}) \\ &= (\alpha_k \beta_{k+1} \alpha_{k+2}, \alpha_{k+1}, \beta_k \beta_{k+2}) \end{aligned}$$

for some $\alpha_k, \alpha_{k+1}, \alpha_{k+2} \in A$. Since

$$|\alpha_k \beta_{k+1} \alpha_{k+2}| + |\alpha_{k+1}| + |\beta_k \beta_{k+2}| = 2 < 3 = |\beta_k \beta_{k+1}^a \beta_{k+2}|,$$

there is some length reduction on the first level, so $g \notin \mathcal{I}_\infty^\omega$. □

It follows from Lemma 3.3.4 that an element

$$g = a^s \beta_1^{a^{c_1}} \beta_2^{a^{c_2}} \dots \beta_n^{a^{c_n}} \in \mathcal{I}_\infty^\omega$$

is uniquely determined by the data (β, s, c_1, m_c), where $\beta\colon \{1, 2, \dots, n\} \to B$, $s, c_1 \in \mathbb{Z}/3\mathbb{Z}$ and $m_c \in \{1, 2, \dots, n\}$. Of course, not every possible choice corresponds to an element of $\mathcal{I}_\infty^\omega$. In what follows, we will bound the number of good choices for (β, s, c_1, m_c).

Proposition 3.3.5. *Let $\omega = \omega_0\omega_1\omega_2\cdots \in \Omega$ and let $l \in \mathbb{N}$ be the smallest integer such that $\cap_{i=0}^{l} \ker(\omega_i) = 1$. Then, there exists a constant $C_l \in \mathbb{N}$ such that*

$$|\mathcal{I}_\infty^\omega(n)| \leq C_l n^{\frac{3^{l+2}-1}{2}}$$

for all $n \in \mathbb{N}$.

Proof. Let us fix $n \in \mathbb{N}$, $s, c_1 \in \mathbb{Z}/3\mathbb{Z}$ and $m_c \in \{1, 2, \ldots, n\}$, and let $c\colon \{1, 2, \ldots, n\} \to \mathbb{Z}/3\mathbb{Z}$ be the unique sequence such that $c(1) = c_1$ and

$$\partial c(k) = \begin{cases} 2 \text{ if } k < m_c \\ 1 \text{ if } k \geq m_c \end{cases}$$

for all $k \in \{1, 2, \ldots n-1\}$. For each map $\beta\colon \{1, 2, \ldots, n\} \to B \setminus \{1\}$, let us define

$$g^{(\beta)} = a^s \iota_\omega(\beta_1)^{a^{c_1}} \iota_\omega(\beta_2)^{a^{c_2}} \ldots \iota_\omega(\beta_n)^{a^{c_n}} \in G_\omega.$$

We will try to bound the number of maps $\beta\colon \{1, 2, \ldots, n\} \to B \setminus \{1\}$ such that $g^{(\beta)} \in \mathcal{I}_\infty^\omega(n)$.

Let us look at the first-level decomposition of $g^{(\beta)}$,

$$g^{(\beta)} = a^s(g_1^{(\beta)}, g_2^{(\beta)}, g_3^{(\beta)}).$$

If $g^{(\beta)} \in \mathcal{I}_\infty^\omega(n)$, we must have $n = |g_1^{(\beta)}| + |g_2^{(\beta)}| + |g_3^{(\beta)}|$. On the other hand, using the fact that

$$\iota_\omega(\beta_k)^{a^{c_k}} = \begin{cases} (\iota_{\sigma(\omega)}(\beta_k), \omega_0(\beta_k), 1) & \text{if } c_k = 1 \\ (1, \iota_{\sigma(\omega)}(\beta_k), \omega_0(\beta_k)) & \text{if } c_k = 2 \\ (\omega_0(\beta_k), 1, \iota_{\sigma(\omega)}(\beta_k)) & \text{if } c_k = 3 \end{cases} \qquad (\star)$$

for all $k \in \{1, 2, \ldots, n\}$, we see that $|g_i^{(\beta)}| \leq n_i$, where $n_i = |\{j \in \{1, 2, \ldots, n\} \mid c_j = i\}|$. Note that here and in what follows, we slightly abuse the notation and make no distinction between an integer and its equivalence class in $\mathbb{Z}/3\mathbb{Z}$. Since $n = n_1 + n_2 + n_3$, we must have $|g_i^{(\beta)}| = n_i$ for $i = 1, 2, 3$. In particular, the values of $|g_i^{(\beta)}|$ are independent of β.

For every $i \in \{1, 2, 3\}$ and every $k \in \{1, 2, \ldots, n_i\}$, let $\mu_k^{(i)}$ be the k^{th} element of the set $\{j \in \{1, 2, \ldots, n\} \mid c_j = i\}$. Using once again the decomposition $(\star)$, we see (dropping the ι_ω from the notation) that

$$g_i^{(\beta)} = \alpha_1^{(i,\beta)} \beta_1^{(i)} \alpha_2^{(i,\beta)} \beta_2^{(i)} \ldots \alpha_{n_i}^{(i,\beta)} \beta_{n_i}^{(i)} \alpha_{n_i+1}^{(i,\beta)}$$

where $\beta_k^{(i)} = \beta_{\mu_k^{(i)}}$ and

$$\alpha_k^{(i,\beta)} = \prod_{\substack{\mu_{k-1}^{(i)} < j < \mu_k^{(i)} \\ c_j = i-1}} \omega_0(\beta_j)$$

(where we set $\mu_0^{(i)} = \mu_1^{(i)} - 3$ and $\mu_{n_i+1}^{(i)} = \mu_{n_i}^{(i)} + 3$). Now, by hypothesis, the map $c\colon \{1, 2, \ldots, n\} \to \mathbb{Z}/3\mathbb{Z}$ satisfies

$$\partial c(k) = \begin{cases} 2 \text{ if } k < m_c \\ 1 \text{ if } k \geq m_c. \end{cases}$$

Therefore, c is a subsequence of one of the three sequences

$$\begin{aligned}&\dots 021021021\mathbf{0}12012012\dots\\&\dots 102102102\mathbf{1}20120120\dots\\&\dots 210210210\mathbf{2}01201201\dots,\end{aligned}$$

where the numbers in bold are the points where the discrete derivative of the sequence changes from 2 to 1 and thus correspond to m_c. Using these sequences, it is easy to see that if $\mu_k^{(i)} \neq m_c + 2$, we must have

$$\alpha_k^{(i,\beta)} = \begin{cases} \omega_0(\beta_{\mu_k^{(i)}-2}) & \text{if } \mu_k^{(i)} \leq m_c \\ \omega_0(\beta_{\mu_k^{(i)}-1}) & \text{if } \mu_k^{(i)} > m_c \end{cases}$$

(where we define $\beta_j = 1$ if $j \notin \{1, 2, \dots, n\}$). In the case where $\mu_k^{(i)} = m_c + 2$, we instead have $\alpha_k^{(i,\beta)} = \omega_0(\beta_{m_c-1})\omega_0(\beta_{m_c+1})$.

Now, since $g^{(\beta)} \in \mathcal{I}_\infty^\omega$, we must also have that $g_1^{(\beta)}, g_2^{(\beta)}, g_3^{(\beta)} \in \mathcal{I}_\infty^\omega$. Hence, by Lemma 3.3.4, for all $i \in \{1, 2, 3\}$, there must exist $s^{(i,\beta)}, c_1^{(i,\beta)} \in \mathbb{Z}/3\mathbb{Z}$ and $m_c^{(i,\beta)} \in \{1, 2, \dots, n_i\}$ such that the maps $\alpha^{(i,\beta)} \colon \{1, 2, \dots, n_i\} \to A$ are uniquely determined by $s^{(i,\beta)}, c_1^{(i,\beta)}$ and $m_c^{(i,\beta)}$. Consequently, we see that the values of $s^{(i,\beta)}, c_1^{(i,\beta)}$ and $m_c^{(i,\beta)}$ uniquely determine the values of $\omega_0(\beta_j)$ for all $j \in \{1, 2, \dots, n\}$, except for $j = m_c - 1$ and $j = m_c + 1$, where instead it is the product $\omega_0(\beta_{m_c-1})\omega_0(\beta_{m_c+1})$ that is determined. It follows that by specifying the values of $s^{(i,\beta)}, c_1^{(i,\beta)}$, $m_c^{(i,\beta)}$ and $\omega_0(\beta_{m_c-1})$, we completely determine $\omega_0(\beta_j)$ for all $j \in \{1, 2, \dots, n\}$.

Since every β_j appears in one of $g_1^{(\beta)}$, $g_2^{(\beta)}$ and $g_3^{(\beta)}$, by repeating this procedure on $g_1^{(\beta)}$, $g_2^{(\beta)}$ and $g_3^{(\beta)}$, we will determine the values of $\omega_1(\beta_j)$ for all $j \in \{1, 2, \dots, n\}$. By induction, the choice of $s^{(x,\beta)}, c_1^{(x,\beta)}, m_c^{(x,\beta)}$ for all words x of length at most $l+1$ in the alphabet X (that is, for all vertices of the tree up to level $l+1$) and of $\omega_{|y|}(\beta^{(y)}_{m_c^{(y,\beta)}-1})$ for all words y of length at most l in the alphabet X determines $\omega_i(\beta_j)$ for all $0 \leq i \leq l$ and all $j \in \{1, 2, \dots, n\}$. Since $\cap_{i=0}^l \ker \omega_i = \{1\}$, for each j, there is at most one $\beta_j \in B$ having the prescribed images $\omega_i(\beta_j)$ for all $0 \leq i \leq l$.

Since there are $\frac{3^{l+2}-1}{2}$ vertices in the tree up to level $l+1$, we have $3^{\frac{3^{l+2}-1}{2}}$ choices for $s^{(x,\beta)}$ and $3^{\frac{3^{l+2}-1}{2}}$ choices for $c_1^{(x,\beta)}$. Since $m_c^{(x,\beta)}$ satisfies $1 \leq m_c^{(x,\beta)} \leq n$, there are at most $n^{\frac{3^{l+2}-1}{2}}$ choices for $m_c^{(x,\beta)}$. Finally, there are at most $\frac{3^{l+1}-1}{2}$ instances where we must choose the value of $\omega_{|y|}(\beta^{(y)}_{m_c^{(y)}-1})$, which means that we must make at most $3^{\frac{3^{l+1}-1}{2}}$ choices. Once all these choices are made, there is at most one β satisfying all the required conditions. Hence, there are at most

$$C_l n^{\frac{3^{l+2}-1}{2}}$$

elements in $\mathcal{I}_\infty^\omega(n)$, where

$$C_l = 3^{\frac{7\cdot 3^{l+1}-3}{2}}.$$

□

With this, we can prove that many spinal groups are of subexponential growth.

Theorem 3.3.6. *Let $\omega \in \Omega$ and G_ω be the associated spinal group of automorphisms of the 3-regular rooted tree X^*. If there exists $l \in \mathbb{N}$ such that $\cap_{i=k}^{k+l} \ker(\omega_i) = 1$ for infinitely many $k \in \mathbb{N}$, then G_ω is of subexponential growth.*

Proof. According to Proposition 3.3.5, there exist infinitely many $k \in \mathbb{N}$ such that

$$|\mathcal{I}_\infty^{\sigma^k(\omega)}(n)| \leq C_l n^{\frac{3^{l+2}-1}{2}}$$

for some $C_l \in \mathbb{N}$. Since $\ln(C_l n^{\frac{3^{l+2}-1}{2}})$ is concave, the result follows from Theorem 3.2.1. □

Remark 3.3.7. It is easy to check that all of the groups considered in Theorem 3.3.6 are regular branch over their commutator subgroup. Therefore, according to Corollary 2.7.10, they are not of polynomial growth. Thus, they are in fact of intermediate growth. In particular, every Šunić group acting on the 3-regular rooted tree is of intermediate growth. ❧

3.4 Open questions

Theorem 3.3.6 gives us many new examples of groups of intermediate growth, but it still leaves many questions open. In this short section, we will discuss a few of those.

To begin with, one can wonder if the hypothesis of Theorem 3.3.6 on the sequence ω is necessary. Although we need this assumption for technical reasons, it seems likely that it could be relaxed.

Question 3.4.1. In Theorem 3.3.6, is the condition that there exists $l \in \mathbb{N}$ such that $\cap_{i=k}^{k+l} \ker(\omega_i) = 1$ for infinitely many $k \in \mathbb{N}$ necessary? What is the growth of the groups not satisfying this condition?

In particular, one might wonder what happens to the growth if one takes sequences converging to a constant sequence. A similar question was studied for a different family of groups by Nekrashevych in [64], where he found a nice example of a group of non-uniform exponential growth.

In a related question, one might wonder if the hypotheses of Theorem 3.2.1 could be relaxed.

Question 3.4.2. In Theorem 3.2.1, could the hypotheses on the growth of incompressible elements be relaxed? More precisely, could the condition that $\mathcal{I}_\infty^\nu \cap \Omega_\nu(n) \leq \delta(n)$ for all $n \in \mathbb{N}$ be replaced by the condition that there exists $N_\nu \in \mathbb{N}$ such that $\mathcal{I}_\infty^\nu \cap \Omega_\nu(n) \leq \delta(n)$ for all $n \geq N_\nu$?

A positive answer to this question would help to answer Question 3.4.1 and would make the criterion more powerful. However, it seems that a more careful analysis would be needed to achieve such a result.

Another very natural question that one might ask is if Theorem 3.3.6 generalises to groups acting on trees of higher degree.

Question 3.4.3. Do all Šunić groups (with the exception of the infinite dihedral group) have intermediate growth? More generally, do all spinal groups with a cyclic action on the first level have subexponential growth?

We would expect the answer to both questions to be positive. In principle, Theorem 3.2.1 could be used to study the growth of those groups. However, the difficulty is in counting the number of incompressible elements. Starting with trees of degree 4, due to commuting elements, there are exponentially many words representing the same element, which makes the combinatorics much harder to analyse.

So far, we have only considered spinal groups with a cyclic action on the first level. One might also ask if these conditions could be relaxed.

Question 3.4.4. Do all spinal groups have subexponential growth?

Once again, Theorem 3.2.1 could be used to attack this question, but the problem of estimating the number of incompressible elements seems out of reach of our current methods.

Chapter 4

Congruence subgroup property

In this chapter, we will study a property of groups acting on rooted trees known as the *congruence subgroup property*. Our interest in this property comes from the fact that in order to understand every finite quotient of a group possessing it, it is sufficient to understand quotients by level stabilisers. As the latter are often well-understood, this makes the study of finite quotients much easier, a fact that will prove invaluable in Chapter 6, where we will study maximal subgroups of some Šunić groups[i].

The congruence subgroup property for groups acting on rooted tree is an analogue of the property of the same name for linear groups. Briefly, a group acting on a rooted tree is said to have this property if every subgroup of finite index contains the stabiliser of some level of the tree. This property was shown to hold for many well-known examples of groups acting on rooted trees, such as the Grigorchuk group [8] and GGS groups with non-constant defining vectors [45, 75, 30]. However, not all groups have this property, not even among the more restricted class of finitely generated branch groups, as was shown by Pervova in [75].

A general investigation of the congruence subgroup property for branch groups was carried out by Bartholdi, Siegenthaler and Zalesskii in [11], where they obtained structural results about the congruence kernel of branch groups and computed this kernel for various examples, such as the Hanoi tower group. In the same paper, they asked if the congruence subgroup property for branch groups is a property of the group, or if it depends on the action. This was answered by Garrido, who showed in [35] that it is a property of the group.

In spite of all this research, there are still many branch groups for which it is not known whether they possess the congruence subgroup property or not. For instance, there is no general result concerning this property in the class of spinal groups[ii], or even in the smaller class of Šunić groups. In this chapter, we prove in Section 4.3 (Theorem 4.3.8) that every Šunić group, with the exception of the infinite dihedral group, possess the congruence subgroup property. Before we do this, however, we first give some definitions and basic properties in Section 4.1, and then study a link between the congruence subgroup property and what is known as the LERF property in Section 4.2. Finally, as an aside, we use in

[i]See Section 2.9 for the definition of Šunić groups.

[ii]See Section 2.8 for the definition of spinal groups.

Section 4.4 the results of Section 4.3 to prove that every Šunić group is just-infinite. We then show in Theorem 4.4.4 that this is not mere happenstance and that in fact, every finitely generated branch group with the congruence subgroup property must be just-infinite.

The results of Section 4.3 come from a joint work with Alejandra Garrido and were already published in [32]. Large parts of that section were taken from this article with some minor modifications.

4.1 Definitions and basic properties

In this section, we will give the definition of the congruence subgroup property for groups acting on rooted trees.

We begin by defining what is known as the profinite topology on a group.

Definition 4.1.1. Let G be a group. The *profinite topology* on G is the topology obtained by the basis of open sets

$$\{gH \mid g \in G,\ H \leq G \text{ of finite index}\}.$$

❧

Remark 4.1.2. We can also take the cosets of *normal* subgroups of finite index as a basis for the profinite topology, since every subgroup of finite index contains a normal subgroup of finite index. This is often more convenient. In both cases, it is easy to check that the conditions for being a basis are satisfied and that the topologies thus defined are the same. ❧

In groups acting on rooted trees, there is a very natural family of normal subgroups of finite index, namely the level stabilisers (see Definition 2.6.14 (ii)). We can thus define on those groups a coarser topology than the profinite topology by looking only at cosets of level stabilisers.

Definition 4.1.3. Let T be a rooted tree and let $G \leq \mathrm{Aut}(T)$ be a group of automorphisms of T. The *congruence topology* on G is the topology defined by the basis of open sets

$$\{g\,\mathrm{St}_G(n) \mid g \in G,\ n \in \mathbb{N}\}.$$

❧

Clearly, an open set in the congruence topology is also open in the profinite topology, but the converse is not necessarily true. Groups for which this holds are said to possess the *congruence subgroup property*.

Definition 4.1.4. Let T be a rooted tree and let $G \leq \mathrm{Aut}(T)$ be a group of automorphisms of T. We say that G possesses the *congruence subgroup property*, or CSP for short, if the profinite topology and the congruence topology on G are equal. In other words, G has the CSP if every open set in the profinite topology is also open in the congruence topology. ❧

It is easy to see that a group has the CSP if and only if every finite index subgroup contains a level stabiliser.

Proposition 4.1.5. *Let $G \leq \mathrm{Aut}(T)$ be a group of automorphisms of a rooted tree T. Then, G has the congruence subgroup property if and only if for every finite index subgroup H, there exists $n \in \mathbb{N}$ such that $\mathrm{St}_G(n) \leq H$.*

Proof. ($\Rightarrow$) If G has the CSP and H is a subgroup of G of finite index, then there exists $g \in G$ and $n \in \mathbb{N}$ such that $g\,\mathrm{St}_G(n) \subseteq H$. As g must belong to H and H is a subgroup, we have $\mathrm{St}_G(n) \leq H$.

($\Leftarrow$) If for $H \leq G$ of finite index, we have $n \in \mathbb{N}$ such that $\mathrm{St}_G(n) \leq H$, then for all $h \in H$, we have $h\,\mathrm{St}_G(n) \subseteq H$, so H is open in the congruence topology. As multiplication is continuous, every coset of H must also be open.

□

In general, it can be rather difficult to check that every finite index subgroup contains a level stabiliser. However, in the case of branch groups (see Definition 2.7.1), it can suffice to consider the derived subgroups of rigid stabilisers, as the next proposition shows.

Proposition 4.1.6. *Let G be a branch group acting on a rooted tree T. If for all $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $\mathrm{St}_G(m) \leq \mathrm{Rist}_G'(n)$, then G has the congruence subgroup property.*

Proof. Let $H \leq G$ be a subgroup of finite index. Then, by Lemma 2.7.5, there exists $n \in \mathbb{N}$ such that $\mathrm{Rist}_G'(n) \leq H$. Therefore, there exists $m \in \mathbb{N}$ such that $\mathrm{St}_G(m) \leq H$. The result then follows from Proposition 4.1.5. □

To apply this criterion, one must still look at an infinite number of subgroups. However, in the case of *regular* branch groups (see Definition 2.7.3), it suffices to look at one subgroup.

Proposition 4.1.7. *Let X be a finite alphabet of size d and let $G \leq \mathrm{Aut}(X^*)$ be a regular branch group over a subgroup $K \leq G$. If there exists $m \in \mathbb{N}$ such that $\mathrm{St}_G(m) \leq K'$, then for all $n \in \mathbb{N}$, we have $\mathrm{St}_G(m+n) \leq \mathrm{Rist}_G'(n)$. In particular, G has the congruence subgroup property.*

Proof. Let us fix $n \in \mathbb{N}$. We have that $K^{d^n} \leq \psi_n(\mathrm{Rist}_G(n))$. Therefore, we have $(K')^{d^n} \leq \psi_n(\mathrm{Rist}_G'(n))$. It follows that

$$\mathrm{St}_G(m)^{d^n} \leq \psi_n(\mathrm{Rist}_G'(n)).$$

Let us consider $H = \psi_n^{-1}(\mathrm{St}_G(m)^{d^n})$. As $H \leq \mathrm{St}_G(n)$ and $\psi_n(H) = (\mathrm{St}_G(m))^{d^n}$, we must have that $H \leq \mathrm{St}_G(m+n)$. On the other hand, $\psi_n(\mathrm{St}_G(m+n)) \leq (\mathrm{St}_G(m))^{d^n}$, which implies that $H = \mathrm{St}_G(m+n)$. Since the map ψ_n is injective, we conclude that $\mathrm{St}_G(m+n) \leq \mathrm{Rist}_G'(n)$. The fact that G has the congruence subgroup property then follows directly from Proposition 4.1.6. □

This idea was used to study the congruence subgroup property in many groups. For instance, it was used by Grigorchuk in [45] to prove that the Grigorchuk group has the congruence subgroup property, and was also used by Fernández-Alcober, Garrido and Uria-Albizuri [30] to show that all GGS groups with non-constant defining vectors (see Example 2.8.6) have the CSP.

4.2 CSP and the LERF property

As the congruence subgroup property allows us to reduce questions about finite index subgroups to questions about level stabilisers, it is very useful in the study of the LERF property. In this short section, we remark that a non-torsion group with the CSP acting on a tree of bounded degree cannot be LERF.

Let us first recall the definition of the LERF property.

Definition 4.2.1. A group G is said to be *locally extended residually finite*, or *LERF*, if every finitely generated subgroup of G is closed in the profinite topology. In other words, G is LERF if every finitely generated subgroup is the intersection of subgroups of finite index. ❧

In groups possessing the congruence subgroup property, being a closed subgroup in the profinite topology is the same as being a closed subgroup in the congruence topology, which is a very restrictive condition. Indeed, if such a group contains a subgroup isomorphic to $\mathbb{Z}$, then it must contain a subgroup that is never closed in this topology, which implies that non-torsion groups with the CSP are not LERF.

Proposition 4.2.2. *Let G be a group acting on a rooted tree T of bounded degree. If G has the congruence subgroup property and contains an element of infinite order, then G is not LERF.*

Proof. Let us suppose that G has the CSP and contains an element $g \in G$ of infinite order. Let $M \in \mathbb{N}$ be a bound on the degree of every vertex of T, and let $p \in \mathbb{N}$ be a number coprime with $M!$. We will show that $H = \langle g^p \rangle$ is not closed in the profinite topology on G.

Let us first notice that for all $n \in \mathbb{N}$, we have $H \operatorname{St}_G(n) = \langle g \rangle \operatorname{St}_G(n)$. Indeed, since every vertex of T has degree bounded by M, one can easily check that we must have

$$G/\operatorname{St}_G(n) \leq \wr^n \operatorname{Sym}(M).$$

As $\wr^n \operatorname{Sym}(M)$ is a group of order $(M!)^k$ by Proposition 2.2.5, where $k = \frac{M^n-1}{M-1}$, we get that the order of $G/\operatorname{St}_G(n)$ divides $(M!)^k$. As p is coprime with $M!$, it must also be coprime with $(M!)^k$, which means by Bézout's theorem that there exists $r, t \in \mathbb{Z}$ such that $rp + t(M!)^k = 1$. As the order of $G/\operatorname{St}_G(n)$ divides $(M!)^k$, we must have $g^{t(M!)^k} \in \operatorname{St}_G(n)$, which implies that

$$(g^p)^r \operatorname{St}_G(n) = g \operatorname{St}_G(n).$$

Consequently, we have $H \operatorname{St}_G(n) = \langle g \rangle \operatorname{St}_G(n)$.

This implies that H is not closed. Indeed, let us suppose for the sake of contradiction that H is closed in the profinite topology. Then, there exists a sequence of finite index subgroups $\{H_i\}_{i\in\mathbb{N}}$ such that

$$H = \bigcap_{i\in\mathbb{N}} H_i.$$

As G has the congruence subgroup property, there exists a sequence of numbers $\{n_i\}_{i\in\mathbb{N}}$ such that $\operatorname{St}_G(n_i) \leq H_i$. Since we must also have $H \leq H_i$, we get that $H \operatorname{St}_G(n_i) \leq H_i$. Since $H \operatorname{St}_G(n_i) = \langle g \rangle \operatorname{St}_G(n_i)$, this implies that $\langle g \rangle \leq H_i$ for all $i \in \mathbb{N}$. We conclude that $\langle g \rangle \leq H$, which is absurd, since g is of infinite order. Therefore, we conclude that H is not closed in the profinite topology, and so G is not LERF. $\square$

Notice that in the above proposition, the existence of an element of infinite order is crucial. Indeed, Grigorchuk and Wilson showed in [49] that the first Grigorchuk group (see Example 2.8.3), which has the congruence subgroup property, is LERF.

4.3 Congruence subgroup property for Šunić groups

In this section, we will show that every Šunić group, with the exception of the infinite dihedral group, possesses the congruence subgroup property. In fact, we could extend this result to a larger class of spinal groups with a cyclic action on the first level, since we do not really need the fact that the degree of the tree is a prime number, and we do not really need self-similarity either, as long as the groups belong to a well-behaved self-similar family. However, this would complicate the notation, and since we will not require this greater generality in what follows, we restrict ourselves to the setting of Šunić groups. In any case, the ideas involved are exactly the same.

Throughout this section, we will use the notation of Section 2.9. Recall that for a prime p and an integer $n \geq 1$, we set $X = \{\mathbf{0}, \mathbf{1}, \dots, \mathbf{p-1}\}$, $A = \mathbb{Z}/p\mathbb{Z}$ and $B = (\mathbb{Z}/p\mathbb{Z})^m$. Given a polynomial f with coefficients in $\mathbb{Z}/p\mathbb{Z}$ and non-zero constant coefficient, we can define faithful actions of A and B on X^*. The Šunić group $G_{p,f}$ is then the group of automorphisms of the regular rooted tree X^* generated by A and B.

As in Section 2.9, we will denote by $\mathcal{G}_{p,m}$ the set of all Šunić groups $G_{p,f}$ with f of degree m. We will need to consider two different cases: the case where p is an odd prime and the case where $p = 2$.

Notation 4.3.1. Although we are working with left actions, to stay consistent with standard group theoretic notation, we will write $g^h = h^{-1}gh$ and $[g,h] = g^{-1}h^{-1}gh$. ♠

The case where p is an odd prime

In this subsection, we will show that the second derived subgroup of a Šunić group acting on a rooted tree of odd prime degree must contain some level stabiliser. This will then imply that the group has the congruence subgroup property thanks to Proposition 4.1.7 and Proposition 2.9.18.

Lemma 4.3.2. *Let $G \in \mathcal{G}_{p,m}$ be a Šunić group, where p is an odd prime. Then,*

$$\psi_1(G'') \geq \gamma_3(G) \times \overset{p}{\dots} \times \gamma_3(G),$$

where $\gamma_3(G)$ is the third term in the lower central series of G and ψ_1 is the map defined in Proposition 2.6.31.

Proof. Let us first show that $\psi_1(G') \leq G \times \overset{p}{\dots} \times G$ is subdirect. For any $b \in B$, we have

$$\psi_1([a^{-1}, b]) = (\rho(b^{-1})\omega(b), \omega(b^{-1}), 1, \dots, 1, \rho(b)).$$

Since ρ is an automorphism of B, we can obtain all generators of B in the last coordinate of $\psi_1(G')$. Also, since there exists $b \in B$ such that $\omega(b) = a^{-1}$, we can obtain a in the last coordinate of $\psi_1(G')$ by taking $[a^{-1}, b]^{a^2}$. Thus $\psi_1(G') \leq G \times \overset{p}{\ldots} \times G$ maps onto G in the last coordinate. As G' is normal and the action of G on the first level is transitive, we conclude that $\psi_1(G')$ is subdirect in $G \times \overset{p}{\ldots} \times G$.

Now, by Proposition 2.9.18, we have that $\psi_1(G') \geq G' \times \overset{p}{\ldots} \times G'$. It follows that $\psi_1(G'') \geq \gamma_3(G) \times \overset{p}{\ldots} \times \gamma_3(G)$. □

Lemma 4.3.3. *Let $G \in \mathcal{G}_{p,m}$ be a Šunić group, where p is an odd prime. Then, the second derived subgroup G'' of G contains the stabiliser $\mathrm{St}_G(m+3)$.*

Proof. Let us first show that $\psi_1(\gamma_3(G)) \geq G' \times \overset{p}{\ldots} \times G'$. Let $b \in B$ be any element and let $c \in B$ be such that $\omega(c) = a$. Let us set $x = [[c, a], \rho^{-1}(b)] \in \gamma_3(G)$. Using classical commutator identities and the fact that B is an abelian group, we have

$$\begin{aligned} x &= [c^{-1}a^{-1}ca, \rho^{-1}(b)] \\ &= [c^{-1}, \rho^{-1}(b)]^{a^{-1}ca}[a^{-1}ca, \rho^{-1}(b)] \\ &= [a^{-1}ca, \rho^{-1}(b)]. \end{aligned}$$

Therefore,

$$\begin{aligned} \psi_1(x) &= \psi_1([a^{-1}ca, \rho^{-1}(b)]) \\ &= ([1, \omega(\rho^{-1}(b))], 1, \ldots, 1, [\rho(c), 1], [\omega(c), b]) \\ &= (1, \ldots, 1, 1, [a, b]). \end{aligned}$$

Using the fact that G' is normally generated by $\{[a, b] \mid b \in B\}$, that G is self-replicating (by Proposition 2.9.17) and that $\gamma_3(G)$ is a normal subgroup of G, we see that $\gamma_3(G) \geq 1 \times \cdots \times 1 \times G'$. As G acts transitively on the first level, using once again the normality of $\gamma_3(G)$, we conclude that $\psi_1(\gamma_3(G)) \geq G' \times \overset{p}{\ldots} \times G'$.

Now, by Lemma 2.9.19, we have that G' contains $\mathrm{St}_G(m+1)$. Therefore, by Lemma 4.3.2, we have

$$\begin{aligned} \psi_2(G'') &\geq \psi_1(\gamma_3(G)) \times \overset{p}{\ldots} \times \psi_1(\gamma_3(G)) \\ &\geq G' \times \overset{p^2}{\ldots} \times G' \\ &\geq \mathrm{St}_G(m+1) \times \overset{p^2}{\ldots} \times \mathrm{St}_G(m+1) \\ &= \psi_2(\mathrm{St}_G(m+3)). \end{aligned}$$

Since ψ_2 is injective, the claim follows. □

Theorem 4.3.4. *Let $G \in \mathcal{G}_{p,m}$ be a Šunić group, where p is an odd prime. Then, G has the congruence subgroup property.*

Proof. This follows directly from Proposition 4.1.7, Proposition 2.9.18 and Lemma 4.3.3. □

The case where $p = 2$

We will now turn our attention to Šunić groups acting on the binary rooted tree. If G is such a group, we see that for $n \in \mathbb{N}$,

$$G/\operatorname{St}_G(n) \leq \wr^n(\mathbb{Z}/2\mathbb{Z}).$$

Consequently, $G/\operatorname{St}_G(n)$ is a 2-group.

Using that fact, it is easy to see that the infinite dihedral group (which is a Šunić group, see Example 2.9.7) cannot possess the congruence subgroup property. Indeed, if that were the case, we would have that every finite quotient of the infinite dihedral group is a quotient of a 2-group, and thus a 2-group. This is of course absurd, since the infinite dihedral group admits every finite dihedral group as a quotient. Alternatively, one can also use Proposition 4.2.2 and the fact that the infinite dihedral group is LERF[iii].

However, we will show here that every other Šunić group acting on the binary rooted tree has the congruence subgroup property. The strategy will be the same as in the previous section. More precisely, we will show that for a group $G \in \mathcal{G}_{2,m}$ with $m \geq 2$, the subgroup K' contains some level stabiliser, where $K = \langle [a, b] \mid b \in B_1 \rangle^G$ is the subgroup defined in Proposition 2.9.18.

Recall that according to the notation established in Definition 2.6.25, for $v \in X^*$, we denote $K_v = \varphi_v(\operatorname{St}_K(v))$.

Lemma 4.3.5. *Let $G \in \mathcal{G}_{2,m}$ be a Šunić group, where $m \geq 2$. Then, there exists $n \in \mathbb{N}$ such that $K_v \geq \langle a, B_0 \rangle$, where $v = \mathbf{1}^n$.*

Proof. Recall from Section 2.9 that $\{b_0, \dots, b_{m-1}\}$ is the standard basis for $B = (\mathbb{Z}/2\mathbb{Z})^m$, that $B_1 = \langle b_1, \dots, b_{m-1} \rangle$ and that K is normally generated by $[a, b_1], \dots, [a, b_{m-1}]$. There are two cases to consider: the case where $m > 2$ and the case where $m = 2$.

Case $m > 2$: We have

$$\varphi_{\mathbf{1}}([a, b_i]) = \begin{cases} \rho(b_i) & \text{if } i = 1, \dots, m-2 \\ a\rho(b_{m-1}) & \text{if } i = m-1. \end{cases}$$

Hence, we have

$$\varphi_{\mathbf{11}}([a, b_i]) = \rho^2(b_i)$$

if $1 \leq i \leq m-2$. Furthermore, as

$$\varphi_{\mathbf{11}}([a, b_{m-1}]^2) = \omega(\rho(b_{m-1}))\rho^2(b_{m-1}),$$

we have that $K_{\mathbf{11}}$ contains $\rho^2(b_1), \dots, \rho^2(b_{m-2}), \omega(\rho(b_{m-1}))\rho^2(b_{m-1})$.

Let us set $x = [a, b_{m-2}]^{[a, b_{m-1}]}$. A direct computation shows that

$$\varphi_{\mathbf{11}}(x^a) = a.$$

Hence, $K_{\mathbf{11}}$ contains $a, \rho^2(b_1), \dots, \rho^2(b_{m-1})$ and therefore contains $\langle a, \rho^3(B_0) \rangle$. If $\rho^3(B_0) = B_0$, then we are done. If not, then as $\varphi_{\mathbf{1}}(\rho^3(B_0)) = \rho^4(B_0)$, we have

[iii] To see this, one can use the fact that the infinite dihedral group contains a subgroup of finite index isomorphic to $\mathbb{Z}$, which is easily seen to be LERF. One can then conclude using the fact that a group containing a LERF subgroup of finite index is also LERF.

$\rho^4(B_0) \in K_{\mathbf{111}}$. Now, since $\rho^3(B_0) \neq B_0$, there is some $y \in \rho^3(B_0) \setminus B_0$. By conjugating y by a, we obtain that $a \in K_{\mathbf{111}}$ and therefore $\langle a, \rho^4(B_0)\rangle \leq K_{\mathbf{111}}$. Since ρ is cyclic, we may repeat the above procedure until we reach the first n such that $\rho^n(B_0) = B_0$ at which point we have $\langle a, \rho^n(B_0)\rangle = \langle a, B_0\rangle \leq K_{\mathbf{1}^{n-1}}$.

Case $m = 2$**:** This case includes only the Grigorchuk–Erschler (with polynomial x^2+1) and Grigorchuk (with polynomial x^2+x+1) groups. First note that $K = \langle [a, b_1]\rangle^G$. For all $n \geq 1$ and for all $b \in B \setminus B_0$ (in other words, for $b = b_1$ or $b = b_0 b_1$), we have $\varphi_{\mathbf{1}}([a,b]^{2n}) = [a, \rho(b)]^n$. Let k be the smallest integer such that $\rho^k(b_1) = b_0$ (such an integer exists, since $b_1 = \rho(b_0)$ and ρ is of finite order). Then, by induction, we get

$$\varphi_{\mathbf{1}^k}([a, b_1]^{2^k}) = [a, b_0].$$

Hence,

$$\varphi_{\mathbf{1}^{k+1}}([a, b_1]^{2^k}) = \varphi_{\mathbf{1}}([a, b_0]) = b_1.$$

It follows that $b_0 = \varphi_v([a, b_1]^{2^k})$ where $v = \mathbf{1}^{2k+1}$.

Now, suppose there exists $g \in G$ which maps $u = \mathbf{1}^{2k}\mathbf{0}$ to v and such that $\varphi_v(g) = 1$. Then, since $a = \varphi_u([a, b_1]^{2^k})$, writing $h = [a, b_1]^{2^k}$, we have $\varphi_v(h^g) = \varphi_u(h)^{\varphi_v(g)} = a$. As K is normal in G, we conclude that $K_v \geq \langle a, b_0\rangle$.

It only remains to show that such a g exists. For the Grigorchuk–Erschler group, $k = 1$ and we may take $g = b_0^{b_1^a}$. For the Grigorchuk group, $k = 2$ and we take $g = b_1^{b_0^f}$, where $f = b_1^{b_1^a}$. □

Lemma 4.3.6. *Let $G \in \mathcal{G}_{2,m}$ be a Šunić group, with $m \geq 2$. Then, the derived subgroup K' of $K \leq G$ contains $\mathrm{St}_G(n+m+2)$, where n is as in Lemma 4.3.5.*

Proof. We begin by showing that $\psi_1([K, \langle a, B_0\rangle]) \geq K \times K$. Let $b \in B_1$ be an arbitrary element. Then, setting

$$x = [[b_{m-1}, a], \rho^{-1}(b)] \in [K, \langle a, B_0\rangle],$$

we have

$$\begin{aligned}
\psi_1(x) &= [\psi_1([b_{m-1}, a]), \psi_1(\rho^{-1}(b))]\\
&= ([a\rho(b_{m-1}), 1], [\rho(b_{m-1})a, b])\\
&= (1, a\rho(b_{m-1})b\rho(b_{m-1})ab)\\
&= (1, [a, b]).
\end{aligned}$$

Since G is self-replicating, we conclude that

$$\psi_1([K, \langle a, B_0\rangle]) \geq 1 \times K.$$

By the transitivity of the action of G on the first level, we then have $\psi_1([K, \langle a, B_0\rangle]) \geq K \times K$.

Now, by Lemma 4.3.5, there exists n such that $K_v \geq \langle a, B_0\rangle$, where $v = \mathbf{1}^n$. The fact that G is regular branch over K implies that

$$\psi_n(K \cap \mathrm{St}_G(n)) \geq 1 \times \cdots \times 1 \times K$$

where there are 2^n factors in the direct product. Thus,

$$\psi_n(K' \cap \mathrm{St}_G(n)) \geq 1 \times \cdots \times 1 \times [K, \langle a, B_0 \rangle].$$

This, together with the first claim, implies that

$$\psi_{n+1}(K' \cap \mathrm{St}_G(n+1)) \geq 1 \times \cdots \times 1 \times K \times K$$

where there are 2^{n+1} factors in the product. Since $K' \cap \mathrm{St}_G(n+1)$ is normal in G, which acts transitively on the $(n+1)$th level, we obtain that $\psi_{n+1}(K' \cap \mathrm{St}_G(n+1))$ contains a direct product of 2^{n+1} copies of K. By Lemma 2.9.19, we have that $\mathrm{St}_G(m+1) \leq K$, so

$$\begin{aligned}\psi_{n+1}(K' \cap \mathrm{St}_G(n+1)) &\geq \mathrm{St}_G(m+1) \times \cdots \times \mathrm{St}_G(m+1) \\ &= \psi_{n+1}(\mathrm{St}_G(n+m+2)).\end{aligned}$$

This yields that $K' \geq \mathrm{St}_G(n+m+2)$, since ψ_{n+1} is injective. □

As a direct consequence of the previous lemma and Propositions 4.1.7 and 2.9.18, we immediately get the following theorem.

Theorem 4.3.7. *Let $G \in \mathcal{G}_{2,m}$ be a Šunić group, with $m \geq 2$. Then, G has the congruence subgroup property.*

For convenience, let us now group the results of Theorems 4.3.4 and 4.3.7 into one theorem.

Theorem 4.3.8. *Let $G \in \mathcal{G} \setminus \mathcal{G}_{2,1}$. Then, G has the congruence subgroup property.*

4.4 Just-infiniteness and the congruence subgroup property

The results of Section 4.3 not only tell us that every Šunić group (with the exception of the infinite dihedral group) possesses the congruence subgroup property, but also allow us to conclude almost immediately that every Šunić group is just-infinite.

Let us first give a reminder of the definition of a just-infinite group, or more generally of a just-non-$\mathcal{P}$ group.

Definition 4.4.1. Let G be a group. We say that G is just-infinite if G is infinite, but every proper quotient of G is finite. More generally, if $\mathcal{P}$ is a property of groups, we say that G is just-non-$\mathcal{P}$ if G does not possess the property $\mathcal{P}$, but every proper quotient of G does. ☙

To prove that every Šunić group is just-infinite, we can simply use Lemma 4.3.3 and Lemma 4.3.6, along with Proposition 4.1.7, to conclude that there exists $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, we have $\mathrm{St}_G(m+n) \leq \mathrm{Rist}'_G(n)$. Now, by Lemma 2.7.5, every non-trivial subgroup $N \trianglelefteq G$ contains $\mathrm{Rist}'_G(n)$ for some $n \in \mathbb{N}$, which means that $\mathrm{St}_G(m+n) \leq N$. As $\mathrm{St}_G(n+m)$ is a subgroup of finite index, we conclude that every non-trivial normal subgroup of G must be of finite index.

However, although simple and direct, this proof does not make immediately obvious the fact that it sits in a more general framework for studying properties of quotients of regular branch groups. Therefore, before proving just-infiniteness of Šunić groups, we would first like to introduce the following result about quotients of regular weakly branch groups, which has the advantage of suggesting a general technique to attack this kind of questions. This will be useful in Chapter 8 to study quotients of the Basilica group. Although we would expect this result to be known, we do not know of any reference for it and thus give a complete proof.

Recall that for a property $\mathcal{P}$, a group is said to be *virtually* $\mathcal{P}$ if it contains a subgroup of finite index that has $\mathcal{P}$.

Theorem 4.4.2. *Let X be a finite alphabet of size d, let G be a regular weakly branch group over a normal subgroup K acting on the rooted tree X^*, and let $\mathcal{P}$ be a property of groups that is preserved under taking finite direct products, quotients and subgroups. Then, every proper quotient of G is virtually $\mathcal{P}$ if and only if G/K' is virtually $\mathcal{P}$.*

Proof. As K cannot be abelian by Proposition 2.7.9, G/K' is a proper quotient, so the necessity is obvious. Let us show that if G/K' is virtually $\mathcal{P}$, then so must every proper quotient of G.

Let $N \trianglelefteq G$ be a non-trivial normal subgroup of G. According to Lemma 2.7.5, there exists $n \in \mathbb{N}$ such that $\mathrm{Rist}'_G(n) \leq N$. Now, by definition of a regular weakly branch group over K, we have that there exists a subgroup $K_n \leq K$ such that $K^{d^n} = \psi_n(K_n)$. In particular, we see that $K_n \leq \mathrm{Rist}_G(n)$. Consequently, we have that $K'_n \leq \mathrm{Rist}'_G(n) \leq N$.

As K is normal in G and as G is self-similar, it follows from its definition that K_n must also be a normal subgroup of G. Consequently, as K'_n is a characteristic subgroup of K_n, we have that K'_n is a normal subgroup of G. Hence, we can take the quotient G/K'_n.

If we can prove that G/K'_n is virtually $\mathcal{P}$, then this will imply that G/N is also virtually $\mathcal{P}$. Indeed, as $K'_n \leq N \leq G$, we have that

$$G/N \cong (G/K'_n) \Big/ (N/K'_n).$$

If G/K'_n is virtually $\mathcal{P}$, then by the correspondence theorem, there exists $H \leq G$ of finite index such that $K'_n \leq H$ and such that H/K'_n has $\mathcal{P}$. Since $\mathcal{P}$ is preserved by taking quotients, we have that

$$HN/N \cong (H/K'_n) \Big/ (N/K'_n) \cap (H/K'_n)$$

has $\mathcal{P}$. It is also of finite index in G/N, since H is of finite index in G.

Thus, it suffices to prove that G/K'_n is virtually $\mathcal{P}$. In fact, since $K'_n \leq \mathrm{St}_G(n)$ and since $\mathrm{St}_G(n)$ is of finite index in G, it suffices to prove that $\mathrm{St}_G(n)/K'_n$ is virtually $\mathcal{P}$.

Now, since G is self-similar, we have $\psi_n(\mathrm{St}_G(n)) \leq (G)^{d^n}$. Hence,

$$\begin{aligned}\psi_n(\mathrm{St}_G(n))/\psi_n(K'_n) &\leq (G)^{d^n} / (K')^{d^n} \\ &= (G/K')^{d^n}.\end{aligned}$$

As G/K' is virtually $\mathcal{P}$, there exists a finite index subgroup $H \leq G$ containing K' such that H/K' has property $\mathcal{P}$. Since property $\mathcal{P}$ is preserved by finite direct products, $(H/K')^{d^n}$ is a finite index subgroup of $(G/K')^{d^n}$ with property $\mathcal{P}$. Let us set $L = \psi_n^{-1}(H^{d^n} \cap \psi_n(\mathrm{St}_G(n)))$. As $K' \leq H$, we clearly have that $K'_n \leq L$. We claim that L is a finite index subgroup of $\mathrm{St}_G(n)$ such that L/K'_n has $\mathcal{P}$.

To see that L is of finite index in $\mathrm{St}_G(n)$, it suffices to notice that since H^{d^n} is of finite index in G^{d^n}, we have that $H^{d^n} \cap \psi_n(\mathrm{St}_G(n))$ is of finite index in $\psi_n(\mathrm{St}_G(n))$. Since ψ_n restricted to $\mathrm{St}_G(n)$ is an isomorphism onto its image, we conclude that L is of finite index in $\mathrm{St}_G(n)$. To see that L/K'_n has $\mathcal{P}$, it suffices to notice that ψ_n gives us an isomorphism between L/K'_n and

$$(H^{d^n} \cap \psi_n(\mathrm{St}_G(n)))/(K')^{d^n} \leq (H/K')^{d^n}.$$

Since $(H/K')^{d^n}$ has $\mathcal{P}$ and since $\mathcal{P}$ is inherited by subgroups, we conclude that L/K' has $\mathcal{P}$. This proves that G/K'_n is virtually $\mathcal{P}$ and thus concludes the proof.

□

Thanks to this theorem, we can prove that every Šunić group is just-infinite.

Theorem 4.4.3. *Every Šunić group is just-infinite.*

Proof. Every Šunić group acts spherically transitively on a regular rooted tree and is thus infinite. Therefore, it remains to show that every proper quotient of a Šunić group is finite.

Let $G \in \mathcal{G}_{p,m}$ be a Šunić group. If $p > 2$, then by Lemma 4.3.3, we have $\mathrm{St}_G(m+3) \leq G''$. In particular, this implies that G'' is a subgroup of finite index of G. Therefore, G/G'' is finite. As G is regular weakly branch over G' by Proposition 2.9.18, we conclude by Theorem 4.4.2 that every proper quotient of G is finite.

Likewise, if $G \in \mathcal{G}_{2,m}$ is a Šunić group with $m \geq 2$, then by Lemma 4.3.6, we have that K' is of finite index, where K is the normal subgroup over which G is regular branch, as stated in Proposition 2.9.18. We once again conclude by Theorem 4.4.2.

The last remaining case is the infinite dihedral group, which is well-known to be just-infinite. □

As was made clear in this section, using Proposition 4.1.7 to prove that a regular branch group has the congruence subgroup property also immediately yields that the group is just-infinite. Since this proposition is the main method by which the CSP is proved, it is natural to ask if the congruence subgroup property implies just-infiniteness. In fact, this question was asked explicitly by Rachel Skipper in her thesis ([81], Question 5.3). In the following theorem, we show that this is indeed the case, thus yielding a different proof of the just-infiniteness of Šunić groups.

Theorem 4.4.4. *Let X be a finite alphabet, and let G be a finitely generated branch group acting on the rooted tree X^*. If G has the congruence subgroup property, then G is just-infinite.*

Proof. Suppose on the contrary that G is not a just-infinite group. Then, there exists a non-trivial normal subgroup $N \trianglelefteq G$ such that G/N is infinite. By Lemma 2.7.5, there exists some $n \in \mathbb{N}$ such that $\mathrm{Rist}'_G(n) \leq N$, where $\mathrm{Rist}'_G(n)$ is the derived subgroup of $\mathrm{Rist}_G(n)$. It follows that $G/\mathrm{Rist}'_G(n)$ is infinite. Since we have that $\mathrm{Rist}'_G(n)$ is non-trivial by Lemma 2.7.6, we can thus suppose without loss of generality that $N = \mathrm{Rist}'_G(n)$.

As G is a finitely generated branch group, we have that $\mathrm{Rist}_G(n)$ is a finitely generated subgroup of finite index of G, which implies that $\mathrm{Rist}_G(n)/\mathrm{Rist}'_G(n)$ is a finitely generated abelian subgroup of finite index of $G/\mathrm{Rist}'_G(n)$. As $G/\mathrm{Rist}'_G(n)$ is infinite, this implies that $\mathrm{Rist}_G(n)/\mathrm{Rist}'_G(n)$ is an infinite finitely generated abelian group.

Let $p \in \mathbb{N}$ be a natural number that is coprime with $|X|!$. As $\mathrm{Rist}_G(n)/\mathrm{Rist}'_G(n)$ is an infinite finitely generated abelian group, it is well-known that it must contain a subgroup of index p. By the correspondence theorem, there exists a subgroup $L \leq \mathrm{Rist}_G(n)$ such that $\mathrm{Rist}'_G(n) \leq L$ and $[\mathrm{Rist}_G(n) : L] = p$. Let $K \trianglelefteq G$ be the normal core of L in G. Since $\mathrm{Rist}_G(n)$ is of finite index in G, this implies that L is also of finite index in G. Therefore, K is also a subgroup of finite index of G. We have

$$K \leq L \leq \mathrm{Rist}_G(n) \leq G.$$

Therefore, we have

$$[G : K] = [G : \mathrm{Rist}_G(n)][\mathrm{Rist}_G(n) : L][L : K].$$

Since $[\mathrm{Rist}_G(n) : L] = p$, we see that p divides the order of G/K.

On the other hand, for any $m \in \mathbb{N}$, we have that $G/\mathrm{St}_G(m)$ is a subgroup of $\wr^m \mathrm{Sym}(|X|)$. It follows from Proposition 2.2.5 that the order of $G/\mathrm{St}_G(m)$ must divide $(|X|!)^k$ for $k = \frac{|X|^m-1}{|X|-1}$, and so the same must be true of any quotient of $G/\mathrm{St}_G(m)$. If G had the congruence subgroup property, this would then imply that the order of every finite quotient of G divides $(|X|!)^k$ for some $k \in \mathbb{N}$. However, we have constructed above a quotient of G whose order is divisible by p, with p coprime with $|X|!$. It follows that the order of this quotient cannot divide $(|X|!)^k$ for any $k \in \mathbb{N}$. This proves that G cannot have the congruence subgroup property. We have thus shown that a group that is not just-infinite cannot possess the CSP. □

Theorem 4.4.4 gives us a simple way to show that a finitely generated branch group does not possess the congruence subgroup property. For example, using the fact that the Hanoi tower group is not just-infinite (see [11]), we immediately obtain that it does not have the congruence subgroup property, a fact that was first proved by Bartholdi, Siegenthaler and Zalesskii in [11].

However, Theorem 4.4.4 does not allow us to conclude anything about weakly branch groups that are not branch. Thus, one might ask if there can exist a weakly branch group that is not branch but still possesses the CSP.

Question 4.4.5. Must a weakly branch group with the congruence subgroup property necessarily be branch?

Chapter 5

The class $\mathcal{MF}$ and dense subgroups

In this chapter, we study the class $\mathcal{MF}$ of groups such that every maximal subgroup is of finite index. More precisely, we study the link between this class, profinite topology and various notions of dense subgroups. The main result of this chapter is Theorem 5.4.3, which is a technical result concerning the projection proper dense subgroups of self-replicating weakly branch group. It is in fact a broad generalisation of a theorem first proved by Pervova in [73, 74] for Grigorchuk 2-groups, *GGS* and *EGS* groups, and later extended by the author and Alejandra Garrido in [32] to just-infinite branch groups. This is a crucial tool that will allow us, in the following chapters, to exploit length contraction and inductive arguments to study maximal subgroups of various branch and weakly branch groups.

We begin in Section 5.1 by giving a few important general facts about the class $\mathcal{MF}$. Then, in Section 5.2, we review the well-known link between this class and the existence of dense subgroups in the profinite topology. In Section 5.3, we investigate two other related notions of dense subgroups, namely pro-$\mathcal{MF}$-dense and prodense subgroups, and how they relate to maximal subgroups of infinite index. Finally, in Section 5.4, we prove Theorem 5.4.3 and its corollary, Corollary 5.4.4, which will form one of our main technical tools in Chapters 6, 7 and 8.

5.1 The class $\mathcal{MF}$

Let $\mathcal{MF}$ be the class of groups containing no maximal subgroups of infinite index. In this section, we collect a few facts about this class.

We first begin by observing that the class $\mathcal{MF}$ is well-behaved with respect to taking quotients, a fact that is well-known and easily established.

Proposition 5.1.1. *Let G be a group in $\mathcal{MF}$ and let $N \trianglelefteq G$. Then, G/N is in $\mathcal{MF}$.*

Proof. Let $\bar{M}$ be a maximal subgroup of G/N. By the correspondence theorem, there exists a maximal subgroup $M \leq G$ of the same index. As $G \in \mathcal{MF}$, the result follows. □

The class $\mathcal{MF}$ is also well-behaved with respect to taking extensions of finitely generated groups.

Theorem 5.1.2. *Let G be a finitely generated group and $N \trianglelefteq G$ be a finitely generated normal subgroup of G. If N and G/N are in $\mathcal{MF}$, then G is also in $\mathcal{MF}$.*

Proof. Let $M \leq G$ be a maximal subgroup of G. We will show that M is of finite index in G.

If $N \leq M$, then M/N is a maximal subgroup of G/N and therefore of finite index. It follows that M is of finite index in G.

If $N \nleq M$, let $H = N \cap M$. By assumption, H is a proper subgroup of N. As N is finitely generated, there exists a maximal subgroup $K \leq N$ such that $H \leq K$. Since N is in $\mathcal{MF}$, K is of finite index in N.

Let $L \leq N$ be the characteristic core of K in N. As K is a finite index subgroup of a finitely generated group, L is of finite index in N. Furthermore, as L is characteristic in a normal subgroup of G, it is normal in G.

Let $M' = ML$. Since $L \leq N$, we have

$$M' \cap N = (M \cap N)L = HL.$$

As both H and L are subgroups of K, we have $M' \cap N \leq K \neq N$, which implies that $M' \neq G$. By the maximality of M, this means that $M' = M$. Therefore, $L \leq M$.

As L is of finite index in N, we have that H is of finite index in N. Since $N \nleq M$, the maximality of M yields $MN = G$. This implies that M is of finite index in G, since

$$[MN : M] = [N : M \cap N].$$

□

As a corollary, we get that a group which is virtually in $\mathcal{MF}$ must be in $\mathcal{MF}$, thus recovering a result proved by Grigorchuk and Wilson in [49].

Corollary 5.1.3. *Let G be a finitely generated group. If there exists a finite index subgroup $H \leq G$ such that $H \in \mathcal{MF}$, then $G \in \mathcal{MF}$. In particular, any finitely generated virtually nilpotent group belongs to $\mathcal{MF}$.*

Proof. This follows immediately from Theorem 5.1.2 and from the fact that every maximal subgroup of a finitely generated nilpotent group is normal (see for instance [77], 12.1.5), since any finite group is obviously in $\mathcal{MF}$. □

5.2 The class $\mathcal{MF}$ and profinite topology

There is a nice and well-known characterisation of groups belonging to $\mathcal{MF}$ in terms of the profinite topology (see Definition 4.1.1), namely that a finitely generated group contains a maximal subgroup of infinite index if and only if it contains a dense subgroup in the profinite topology. In this section, we give a proof of this fact. To begin, we recall a useful characterisation of dense subgroups in the profinite topology.

Proposition 5.2.1. *Let G be a group. A subgroup $H \leq G$ is dense in the profinite topology if and only if, for all normal subgroups of finite index $N \trianglelefteq G$, we have $HN = G$.*

Proof. ($\Rightarrow$) If H is dense in the profinite topology, then for every normal subgroup of finite index $N \trianglelefteq G$ and every $g \in G$, we have $H \cap gN \neq \emptyset$. Therefore, $HN = G$.

($\Leftarrow$) If H is such that for all normal subgroups of finite index $N \trianglelefteq G$, we have $HN = G$, then for every $g \in G$, we have $H \cap gN \neq \emptyset$. As the cosets of normal subgroups of finite index of G form a basis for the profinite topology (see Remark 4.1.2), we conclude that H is dense.

□

We can now characterise the existence of maximal subgroups of infinite index in terms of profinite topology.

Proposition 5.2.2. *Let G be a finitely generated group. Then, G contains a maximal subgroup of infinite index if and only if it contains a proper subgroup that is dense in the profinite topology. Furthermore, any maximal subgroup of infinite index of G must be dense in the profinite topology.*

Proof. ($\Rightarrow$) Let $M < G$ be a maximal subgroup of infinite index. Then, for every normal subgroup of finite index $N \trianglelefteq G$, we must have $MN = G$. Indeed, otherwise, we would have $MN = M$ by the maximality of M, which would imply that M is of finite index in G. Therefore, by Proposition 5.2.1, M is dense.

($\Leftarrow$) Let $H < G$ be a proper dense subgroup in the profinite topology. As G is finitely generated, H must be contained in a maximal subgroup $M < G$. Since $H \leq M$, we have that M is also dense. We conclude by using the fact that a proper dense subgroup must be of infinite index, since every subgroup of finite index is closed in the profinite topology.

□

5.3 Pro-$\mathcal{MF}$-dense and prodense subgroups

We saw in Proposition 5.2.2 that the existence of maximal subgroups of finite index in finitely generated groups is equivalent to the existence of dense subgroups in the profinite topology. In this section, we introduce the notion of a *pro-$\mathcal{MF}$-dense subgroup*, which is often more convenient for the study of the index of maximal subgroups. We then show that being pro-$\mathcal{MF}$-dense is in fact equivalent to being dense in the profinite topology, even though the underlying topologies might be different. Finally, we show that if every proper quotient is in $\mathcal{MF}$, then being pro-$\mathcal{MF}$-dense is equivalent to being *prodense*, which is yet another notion of density.

Let us begin by introducing the notion of *prodense* and *pro-$\mathcal{MF}$-dense* subgroups. In Proposition 5.2.1, we saw that for a subgroup H of a group G, being dense in the profinite topology is the same as satisfying $HN = G$ for all normal subgroups $N \trianglelefteq G$ of finite index. Using this characterisation of a dense subgroup, one could naturally generalise this notion by changing the class of normal subgroups that are considered. For instance, by asking that $HN = G$ for all non-trivial normal subgroup $N \trianglelefteq G$, one arrives at the notion of a *prodense* subgroup, as is done in [36].

Definition 5.3.1. Let G be a group. A subgroup $H \leq G$ is said to be a *prodense subgroup* of G if we have $HN = G$ for all non-trivial normal subgroups $N \trianglelefteq G$. ❧

If one takes instead every normal subgroup such that the quotient by this normal subgroup belongs to $\mathcal{MF}$, one then arrives at the notion of what we will call a *pro-$\mathcal{MF}$-dense subgroup.*

Definition 5.3.2. Let G be a group. We will call a subgroup $H \leq G$ a *pro-$\mathcal{MF}$-dense subgroup* of G if we have $HN = G$ for all normal subgroups $N \trianglelefteq G$ such that $G/N \in \mathcal{MF}$. ❧

Remark 5.3.3. If the intersection of every pair of non-trivial normal subgroups is non-trivial (which is the case, for example, in the class of weakly branch groups by Proposition 2.7.8), one can define a topology on G whose basis of open set is

$$\{gN \mid g \in G,\ N \trianglelefteq G,\ N \neq \{1\}\}.$$

For more on this subject, see [36, 37], where this topology is called the *pro-normal* topology. Likewise, if for every normal subgroups $N_1, N_2 \trianglelefteq G$ such that $G/N_1, G/N_2 \in \mathcal{MF}$, we have $G/(N_1 \cap N_2) \in \mathcal{MF}$, then one can define a topology on G whose basis of open sets is

$$\{gN \mid g \in G,\ N \trianglelefteq G,\ G/N \in \mathcal{MF}\}.$$

We will call this topology, when it exists, the *pro-$\mathcal{MF}$ topology.*

Note that, in general, those topologies are not well-defined. Indeed, it is easy to construct, for example, groups where $N_1 \cap N_2 = \{1\}$ and G is not in $\mathcal{MF}$. However, when $G \in \mathcal{MF}$, the pro-$\mathcal{MF}$ topology is well-defined by Proposition 5.1.1 and is in fact equal to the discrete topology on G. ❧

In the cases where the pro-$\mathcal{MF}$ topology is defined, the pro-$\mathcal{MF}$-dense subgroups are exactly the dense subgroups in the pro-$\mathcal{MF}$ topology.

Proposition 5.3.4. *Let G be a group such that for every pair of normal subgroups $N_1, N_2 \trianglelefteq G$ with $G/N_1, G/N_2 \in \mathcal{MF}$, we also have $G/(N_1 \cap N_2) \in \mathcal{MF}$, so that the pro-$\mathcal{MF}$ topology on G is well-defined (see Remark 5.3.3). Then, a subgroup $H \leq G$ is dense in the pro-$\mathcal{MF}$ topology if and only if it is pro-$\mathcal{MF}$-dense.*

Proof. The proof is the same as the proof of Proposition 5.2.1, replacing normal subgroups of finite index with normal subgroups such that the quotient is in MF. □

Of course, a similar result holds for prodense subgroups and the pro-normal topology. However, for the moment, we will mainly focus on pro-$\mathcal{MF}$-dense subgroups and the pro-$\mathcal{MF}$ topology, since these are more directly related to the study of the index of maximal subgroups.

In general, when it exists, the pro-$\mathcal{MF}$ topology can be finer than the profinite topology, as the following example illustrates.

Example 5.3.5. Let

$$G = H_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, a, b, c \in \mathbb{Z} \right\}$$

be the discrete Heisenberg group. As G is a finitely generated nilpotent group, it is in $\mathcal{MF}$ by the theorem of Margulis-Soifer [59], and so are all of its quotients. It follows that the pro-$\mathcal{MF}$ topology on G is well-defined and that every normal subgroup of G is open in this topology. In particular, the trivial subgroup is open in the pro-$\mathcal{MF}$ topology. However, the trivial subgroup cannot be open in the profinite topology, since every open subgroup in the profinite topology must be of finite index, whereas G is infinite. ❧

As the pro-$\mathcal{MF}$ topology can be finer than the profinite topology, one might expect that the notion of a pro-$\mathcal{MF}$-dense subgroup could be stronger than the notion of a dense subgroup in the profinite topology. However, as the next proposition shows, the two are in fact equivalent for finitely generated groups.

Proposition 5.3.6. *Let G be a finitely generated group and let $H \leq G$ be a subgroup. Then, H is dense in the profinite topology on G if and only if H is pro-$\mathcal{MF}$-dense.*

Proof. ($\Rightarrow$) Let us suppose that H is dense in the profinite topology on G and let $N \trianglelefteq G$ be a normal subgroup of G such that $G/N \in \mathcal{MF}$. By Proposition 5.2.1, we have that $HF = G$ for all normal subgroup $F \trianglelefteq G$ of finite index. Therefore, we have that $(HN)(FN)/N = G/N$ for all normal subgroup $F \trianglelefteq G$ of finite index. It thus follows from the correspondence theorem and Proposition 5.2.1 that HN/N is a dense subgroup of G/N in the profinite topology. As G/N does not contain any maximal subgroup of infinite index by hypothesis, it follows from Proposition 5.2.2 that $HN/N = G/N$. Therefore, once again by the correspondence theorem, we have that $HN = G$, which shows that H is pro-$\mathcal{MF}$-dense.

($\Leftarrow$) Let us assume that H is pro-$\mathcal{MF}$-dense. Since every finite group is in $\mathcal{MF}$, we have $HN = G$ for all normal subgroup $N \trianglelefteq G$ of finite index. Therefore, H is dense in the profinite topology by Proposition 5.2.1.

□

As a corollary, we get that a finitely generated group is in $\mathcal{MF}$ if and only if it does not contain a proper pro-$\mathcal{MF}$-dense subgroup.

Corollary 5.3.7. *Let G be a finitely generated group. Then, G contains a maximal subgroup of infinite index if and only if it contains a proper pro-$\mathcal{MF}$-dense subgroup. Furthermore, any maximal subgroup of infinite index of G must be pro-$\mathcal{MF}$-dense.*

Proof. This follows directly from Propositions 5.3.6 and 5.2.1. □

Although we have just shown that they are equivalent for finitely generated groups, formulating everything in terms of pro-$\mathcal{MF}$-dense subgroups instead of dense subgroups in the profinite topology can often be more convenient for studying groups in $\mathcal{MF}$, since this formulation allows us to consider normal subgroups of infinite index (something that is not obvious when working with the profinite topology, as they are never open in this setting).

In fact, in what follows, this approach will allow us to consider every non-trivial normal subgroup. Indeed, we will primarily be interested in determining whether a given group G belongs to the class $\mathcal{MF}$ or not. If G admits a normal subgroup $N \trianglelefteq G$ such that G/N is not in $\mathcal{MF}$, then G itself cannot be in $\mathcal{MF}$ by Proposition 5.1.1. Therefore, the question of knowing if G is in $\mathcal{MF}$ is only interesting if G is infinite and if we know that every proper quotient of G is in $\mathcal{MF}$. In that case, the existence of a pro-$\mathcal{MF}$-dense subgroup is equivalent to the existence of a prodense subgroup.

Proposition 5.3.8. *Let G be an infinite finitely generated group such that for every non-trivial normal subgroup $N \trianglelefteq G$, we have $G/N \in \mathcal{MF}$. Then, a subgroup $H \leq G$ is prodense if and only if it is pro-$\mathcal{MF}$-dense.*

Proof. ($\Rightarrow$) Suppose that $H \leq G$ is prodense. Then, for every non-trivial normal subgroup $N \leq G$, we have $HN = G$. To show that H is pro-$\mathcal{MF}$-dense, it thus only remains to show that if $G \in \mathcal{MF}$, then $H = G$. Assume for the sake of contradiction that G is in $\mathcal{MF}$ but that H is a proper subgroup. Then, since G is finitely generated, H is contained in a maximal subgroup $M < G$ of finite index. Let $N \trianglelefteq G$ be the normal core of M. We have that N is of finite index in G, which means that it is non-trivial, since G is infinite. Therefore, using the fact that H is prodense, we find $HN = G$, which is absurd, since $HN \leq M \lneq G$.

($\Leftarrow$) Suppose that H is pro-$\mathcal{MF}$-dense. For every non-trivial normal subgroup $N \trianglelefteq G$, we have $G/N \in \mathcal{MF}$ by assumption, so that $HN = G$. Thus, H is prodense.

□

Thus, for groups satisfying the hypotheses of Proposition 5.3.8, studying pro-$\mathcal{MF}$-dense subgroups is equivalent to studying prodense subgroups.

It is interesting to note that, for finitely generated branch groups[i], the assumptions of Proposition 5.3.8 are always verified.

Proposition 5.3.9. *Let G be a finitely generated branch group. Then, the pro-$\mathcal{MF}$ topology is well-defined on G and for every non-trivial normal subgroup $N \trianglelefteq G$, N is open in this topology. In particular, every proper quotient of G is in $\mathcal{MF}$.*

Proof. Recall from Section 2.7 that a branch group is always infinite. If $G \in \mathcal{MF}$, then the result is true by Remark 5.3.3 and Proposition 5.1.1. Let us now suppose that G is not in $\mathcal{MF}$. By Proposition 2.7.8, the intersection of two non-trivial normal subgroups is non-trivial. Thus, to prove the result, it suffices to show that for every non-trivial normal subgroup $N \trianglelefteq G$, we have $G/N \in \mathcal{MF}$. It follows from Lemma 2.7.5 that G/N is a finitely generated virtually abelian group. As abelian and finite groups are in $\mathcal{MF}$, Theorem 5.1.2 implies that $G/N \in \mathcal{MF}$. □

[i]See Section 2.7 for the definition of branch groups.

Thus, a finitely generated branch group G is in $\mathcal{MF}$ if and only if it contains no prodense subgroups. This is a fact that we will rely on in the following chapters. This also has consequences for the Frattini[ii] subgroup, as the following simple proposition shows.

Proposition 5.3.10. *Let G be a finitely generated group that is just-non-$\mathcal{MF}$, meaning that G is not in $\mathcal{MF}$ but every proper quotient of G is in $\mathcal{MF}$. Then, the Frattini subgroup of G is trivial. In particular, any finitely generated branch group that is not in $\mathcal{MF}$ has a trivial Frattini subgroup.*

Proof. Let $F \trianglelefteq G$ be the Frattini subgroup of G (which is always characteristic, so in particular normal). Since G is a finitely generated group that is not in $\mathcal{MF}$, it must contain a maximal subgroup $M < G$ of infinite index. By definition, we have $F \leq M$. Thus, by the correspondence theorem, we have that M/F is maximal in G/F. If F were non-trivial, we would have that $G/F \in \mathcal{MF}$, which would imply that M is of finite index in G, a contradiction. Thus, F must be trivial.

The statement about branch groups follows directly from Proposition 5.3.9. □

5.4 Weakly branch groups and proper prodense subgroups

In this section, we study prodense subgroups of self-replicating weakly branch groups. As was mentioned in Section 5.3, under the assumption that every proper quotient is in $\mathcal{MF}$, the study of prodense subgroups is equivalent to the study of pro-$\mathcal{MF}$-dense subgroups, which itself is equivalent to the study of maximal subgroups of infinite index. The main result of this section is that under some technical and somewhat restrictive conditions, the projection to any vertex of a proper prodense subgroup is also a proper prodense subgroup (Theorem 5.4.3).

A restricted version of this result was first proved by Pervova in [73] and [74]. This was later generalised to a larger class of groups by Alejandra Garrido and the author in [32]. The version we present here is far more general and can also be applied to weakly branch groups, whereas the previous versions could only be applied to just-infinite branch groups. To our knowledge, this is the most general existing version of this criterion.

Although the hypotheses are somewhat restrictive, the main result of this section is still very useful in the study of maximal subgroups of weakly branch groups, as will become apparent in Chapters 6, 7 and 8. This is thanks to the fact that it allows us to exploit the phenomenon of length contraction that exists in many groups to simplify the analysis.

Let us first prove that under some suitable conditions, a prodense subgroup of a self-replicating weakly branch group projects to a prodense subgroup.

Proposition 5.4.1. *Let X be an alphabet of size d, let G be a self-replicating weakly branch group acting on the d-regular rooted tree X^*, let $H \leq G$ be a*

[ii] Recall that the Frattini subgroup of a group is the intersection of every maximal subgroup.

prodense subgroup and let $u \in X^*$ be any vertex of the tree. Then, H_u is a prodense subgroup of G, where $H_u = \varphi_u(H \cap \mathrm{St}_G(u))$.

Proof. It suffices to show that $H_u \,\mathrm{Rist}'_G(n) = G$ for all $n \in \mathbb{N}$. Indeed, since G is weakly branch, for every non-trivial normal subgroup $N \trianglelefteq G$, there exists $n \in \mathbb{N}$ such that $\mathrm{Rist}'_G(n) \leq N$ by Lemma 2.7.5. Furthermore, by Lemma 2.7.6, $\mathrm{Rist}'_G(n)$ is non-trivial for every $n \in \mathbb{N}$.

Let us suppose that u is on level m, and let $n \in \mathbb{N}$ be any natural number. Clearly, $\mathrm{Rist}_G(m+n) \leq \mathrm{St}_G(m) \leq \mathrm{St}_G(u)$, and

$$\varphi_u(\mathrm{Rist}_G(n+m)) \leq \mathrm{Rist}_G(n).$$

Therefore, $\varphi_u(\mathrm{Rist}'_G(n+m)) \leq \mathrm{Rist}'_G(n)$.

As $\mathrm{Rist}'_G(n+m)$ is a non-trivial normal subgroup of G, we have by hypothesis

$$H \,\mathrm{Rist}'_G(n+m) = G.$$

As G is self-replicating, for every $g \in G$, there exists $\tilde{g} \in \mathrm{St}_G(u)$ such that $\varphi_u(\tilde{g}) = g$. Since $H \,\mathrm{Rist}'_G(n+m) = G$, there exist $h \in H$ and $r \in \mathrm{Rist}'_G(n+m)$ such that $hr = \tilde{g}$. Since $\tilde{g}, r \in \mathrm{St}_G(u)$, we must have $h \in \mathrm{St}_G(u)$. Therefore, we get

$$\varphi_u(h)\varphi_u(r) = g,$$

with $\varphi_u(h) \in H_u$ and $\varphi_u(r) \in \mathrm{Rist}'_G(n)$. This shows that $H_u \,\mathrm{Rist}'_G(n) = G$. $\square$

As we have seen above, for self-replicating weakly branch group, the projection of any prodense subgroup to a vertex is still a prodense subgroup. However, to determine whether a group belongs to $\mathcal{MF}$ or not, we need to study *proper* prodense subgroups. The next theorem tells us that the projections of proper prodense subgroups stay proper if the action of the group on the first level is primitive[iii]. In order to prove it, though, we will first need a small lemma regarding the action of self-similar groups.

Lemma 5.4.2. *Let X be a finite alphabet and let G be a self-replicating group acting spherically transitively on the rooted tree X^*. If the action of G on X is primitive, then $\varphi_v(\mathrm{St}_G(1))$ acts spherically transitively on X^* for all $v \in X$.*

Proof. Let us fix a letter $x \in X$. We will begin by showing that $\mathrm{St}_G(n)$ acts transitively on the set x^nX for all $n \in \mathbb{N}$.

Suppose on the contrary that there exists $n \in \mathbb{N}$ such that the action of $\mathrm{St}_G(n)$ on x^nX is not transitive. We claim that it must therefore be trivial. Indeed, let us consider the subgroup $\varphi_{x^n}(\mathrm{St}_G(n)) \leq G$. Since G is self-replicating, we must have $\varphi_{x^n}(\mathrm{St}_G(x^n)) = G$. Therefore, as $\mathrm{St}_G(n)$ is normal in $\mathrm{St}_G(x^n)$, we obtain by applying φ_{x^n} that $\varphi_{x^n}(\mathrm{St}_G(n))$ is normal in G. Furthermore, since the action of $\mathrm{St}_G(n)$ on x^nX is not transitive, we see that the action of $\varphi_{x^n}(\mathrm{St}_G(n))$ on X is not transitive. It must therefore be trivial, since G acts primitively on X and $\varphi_{x^n}(\mathrm{St}_G(n))$ is normal in G. Indeed, otherwise, the orbits of this action would form a non-trivial partition of X preserved by the action of G.

We thus have that $\mathrm{St}_G(n)$ acts trivially on x^nX. As $\mathrm{St}_G(n)$ is normal in G, and since G acts spherically transitively on X^*, this implies that $\mathrm{St}_G(n)$ acts

[iii]See Definition 2.1.3 for the definition of a primitive action.

trivially on vX for all $v \in X^n$. In other words, $\mathrm{St}_G(n)$ acts trivially on X^{n+1}. Consequently, we have that $\mathrm{St}_G(n) \leq \mathrm{St}_G(n+1)$. As the other inclusion is trivially true, we conclude that $\mathrm{St}_G(n) = \mathrm{St}_G(n+1)$.

Let $g \in \mathrm{St}_G(n)$ be an arbitrary element. As we have seen above, we have also $g \in \mathrm{St}_G(n+1)$. Let $y \in X$ be an arbitrary letter, and let us consider $\varphi_y(g)$. Since G is self-similar, we have $\varphi_y(g) \in G$, and since $g \in \mathrm{St}_G(n+1)$, we must have $\varphi_y(g) \in \mathrm{St}_G(n)$. However, using once again the fact that $\mathrm{St}_G(n) = \mathrm{St}_G(n+1)$, we have that $\varphi_y(g) \in \mathrm{St}_G(n+1)$. This implies that $\mathrm{St}_G(n) = \mathrm{St}_G(n+2)$. By induction, we conclude that $\mathrm{St}_G(n) = \mathrm{St}_G(n+m)$ for all $m \in \mathbb{N}$, which implies that $\mathrm{St}_G(n) = \{1\}$. This is absurd, since $\mathrm{St}_G(n)$ is a subgroup of finite index of G, and G must be infinite, since it acts spherically transitively on X^*.

We have thus shown that $\mathrm{St}_G(n)$ acts transitively on x^nX for all $n \in \mathbb{N}$. As $\mathrm{St}_G(n+1) \leq \mathrm{St}_G(1)$ for all $n \in \mathbb{N}$, this shows in particular that $\mathrm{St}_G(1)$ acts transitively on $x^{n+1}X$ for all $n \in \mathbb{N}$. By induction, we see that this implies that $\mathrm{St}_G(1)$ acts transitively on xX^n for all $n \in \mathbb{N}$, and thus that $\varphi_x(\mathrm{St}_G(1))$ acts spherically transitively on X^*. □

We can now prove the announced theorem.

Theorem 5.4.3. *Let X be a finite alphabet of size d, let G be a finitely generated self-replicating weakly branch group acting on the d-regular rooted tree X^*, let $H \leq G$ be a prodense subgroup and let $u \in X^*$ be any vertex. If the action of G on X is primitive, then $H \neq G$ if and only if $H_u \neq G$, where $H_u = \varphi_u(\mathrm{St}_H(u))$.*

Proof. Since G is self-replicating, we have $G_u = G$, so if $H_u \neq G$, then clearly $H \neq G$.

Let us now assume that $H \neq G$ and let us show that $H_u \neq G$. It suffices to prove this fact for $u \in X$. Indeed, if this property holds on the first level of the rooted tree, we can then use induction to prove it for u on any level thanks to Proposition 5.4.1.

Therefore, let $u \in X$ be a vertex on the first level of the tree and let us assume for the sake of contradiction that $H \neq G$ but $H_u = G$.

The rigid stabiliser of the vertex u in H, $\mathrm{Rist}_H(u) = \mathrm{Rist}_G(u) \cap H$, is a normal subgroup of $\mathrm{St}_H(u)$. Since $H_u = G$, it is also a normal subgroup of $\mathrm{St}_G(u)$. Indeed, for any $g \in \mathrm{St}_G(u)$, there exists $h \in \mathrm{St}_H(u)$ such that $\varphi_u(g) = \varphi_u(h)$. Hence, since any $r \in \mathrm{Rist}_H(u)$ acts trivially outside of uX^*, the subtree of vertices prefixed by u, we have

$$grg^{-1} = hrh^{-1} \in \mathrm{Rist}_H(u).$$

Since $\mathrm{St}_G(1) \leq \mathrm{St}_G(u)$, we have that $\mathrm{Rist}_H(u) \trianglelefteq \mathrm{St}_G(1)$.

Now, since G acts transitively on X and since $H\,\mathrm{St}_G(1) = G$, we conclude that H must also act transitively on X. Therefore, for any $v \in X$, there exists $h \in H$ such that $\mathrm{St}_H(v) = h\,\mathrm{St}_H(u)h^{-1}$. Hence, $H_v = \varphi_v(\mathrm{St}_H(v)) = G$ for all $v \in X$. It follows that $\mathrm{Rist}_H(v) \trianglelefteq \mathrm{St}_G(1)$ for all $v \in X$. Therefore,

$$\mathrm{Rist}_H(1) = \prod_{v \in X} \mathrm{Rist}_H(v) \trianglelefteq \mathrm{St}_G(1).$$

Since $\mathrm{Rist}_H(1) \trianglelefteq H$ and $H \,\mathrm{St}_G(1) = G$, we conclude that

$$\mathrm{Rist}_H(1) \trianglelefteq G.$$

This implies that $\mathrm{Rist}_H(1) = \{1\}$. Indeed, otherwise, by hypothesis, we would have $H \,\mathrm{Rist}_H(1) = G$, which is absurd since $H \,\mathrm{Rist}_H(1) = H$ and $H \neq G$.

Let us show that, as a consequence of this fact, we must have

$$H \cap \prod_{v \in U} \mathrm{Rist}_{\mathrm{Aut}(X^*)}(v) = \{1\}$$

for all proper subset $U \subsetneq X$. We will proceed by induction on the cardinality of U. If $|U| = 1$, then $H \cap \mathrm{Rist}_{\mathrm{Aut}(X^*)}(v) = \mathrm{Rist}_H(v) = \{1\}$ by what was shown above. Let us now assume that the result holds if $|U| \leq n$ for some $n < d-1$, and let us show that it must then also hold for $|U| = n+1$.

Let $U \subset X$ be such that $|U| = n+1$ and let us assume for the sake of contradiction that

$$K = H \cap \prod_{v \in U} \mathrm{Rist}_{\mathrm{Aut}(X^*)}(v) \neq \{1\}.$$

In that case, there must exist some $v \in U$ such that $K_v = \varphi_v(K) \neq \{1\}$, and so, by the induction hypothesis, $K_v \neq \{1\}$ for all $v \in U$.

Clearly, K is normal in $\mathrm{St}_H(1)$. This implies that K_v is normal in $\varphi_v(\mathrm{St}_H(1))$ for all $v \in U$. We will prove that $\varphi_v(\mathrm{St}_H(1))$ is a normal subgroup of G acting spherically transitively on X^*.

Let us first begin by showing that $\varphi_v(\mathrm{St}_H(1))$ is a normal subgroup of G for all $v \in X$. Since $H_v = G$, for all $g \in G$, there exists $h \in \mathrm{St}_H(v)$ such that $\varphi_v(h) = g$. Now, since $\mathrm{St}_H(1)$ is normal in H, we have that $h \,\mathrm{St}_H(1) h^{-1} = \mathrm{St}_H(1)$. Therefore, by applying φ_v, we get $g\varphi_v(\mathrm{St}_H(1))g^{-1} = \varphi_v(\mathrm{St}_H(1))$, which proves that $\varphi_v(\mathrm{St}_H(1))$ is normal in G.

By the transitivity of the action of H on X and the normality of $\varphi_v(\mathrm{St}_H(1))$ in G, we see that $\varphi_v(\mathrm{St}_H(1)) = \varphi_w(\mathrm{St}_H(1))$ for all $v, w \in X$. Let us denote this subgroup by S_1.

Let us now show that S_1 acts spherically transitively on X^*. This is equivalent to showing that for one (and hence for all, by transitivity) $v \in X$ and for all $n \in \mathbb{N}$, the group $\mathrm{St}_H(1)$ acts transitively on the set vX^n. Let us fix $v \in X$ and $n \in \mathbb{N}$. It follows from Lemma 5.4.2 that $\mathrm{St}_G(1)$ acts transitively on vX^n. As $\mathrm{St}_G(n+1)$ is a normal subgroup of G, we have by hypothesis that $H \,\mathrm{St}_G(n+1) = G$. This implies that $\mathrm{St}_H(1)$ also acts transitively on vX^n. Indeed, let $w_1, w_2 \in X^n$ be two arbitrary elements. By transitivity, there exists $g \in \mathrm{St}_G(1)$ such that $g(vw_1) = vw_2$. Since $H \,\mathrm{St}_G(n+1) = G$, there exist $h \in H$ and $s \in \mathrm{St}_G(n+1)$ such that $hs = g$. We then have

$$h = gs^{-1} \in \mathrm{St}_G(1) \cap H = \mathrm{St}_H(1),$$

and since $s \in \mathrm{St}_G(n+1)$, we must have $h(vw_1) = vw_2$. We conclude that $S_1 = \varphi_v(\mathrm{St}_H(1))$ is a normal subgroup of G acting transitively on X^*.

Now, since the action of H on X is primitive, there exists $h \in H$ such that $0 < |U \cap hU| < |U|$. We have $hKh^{-1} = H \cap \prod_{v \in hU} \mathrm{Rist}_{\mathrm{Aut}(X^*)}(v)$. Hence,

$$[K, hKh^{-1}] \leq H \cap \prod_{v \in U \cap hU} \mathrm{Rist}_{\mathrm{Aut}(X^*)}(v) = \{1\}$$

by the induction hypothesis.

Let us choose $v \in U \cap hU$. Since K and hKh^{-1} commute, we must have that K_v and $\left(hKh^{-1}\right)_v$ commute. As hKh^{-1} is also normal in $\mathrm{St}_H(1)$, we see that K_v and $(hKh^{-1})_v$ are both normal in S_1. Therefore, $L = K_v \cap (hKh^{-1})_v$ is an abelian normal subgroup of S_1. As S_1 is normal in G, this means that L is an abelian subnormal subgroup of G. Therefore, by Proposition 2.7.9, we must have that $L = \{1\}$. On the other hand, since $S_1 \trianglelefteq G$ acts transitively on X^* and since K_v and $(hKh^{-1})_v$ are non-trivial by assumption, we have from Proposition 2.7.8 that L is non-trivial, which is a contradiction. Hence, K must be trivial and we have shown that

$$H \cap \prod_{v \in U} \mathrm{Rist}_{\mathrm{Aut}(X^*)}(v) = \{1\}$$

for all proper subset $U \subsetneq X$.

This implies that there exist isomorphisms $\alpha_2, \dots, \alpha_d \colon S_1 \to S_1$ such that for all $h \in \mathrm{St}_H(1)$,

$$\psi_1(h) = (h_1, \alpha_2(h_1), \dots, \alpha_d(h_1))$$

for some $h_1 \in S_1$. Indeed, let us write $X = \{\mathbf{1}, \mathbf{2}, \dots, \mathbf{d}\}$. We have shown that for all $v \in X$, the map φ_v is injective when restricted to $\mathrm{St}_H(1)$. It is thus an isomorphism onto its image, S_1. We can then define $\alpha_i = \varphi_{\mathbf{i}} \circ \varphi_{\mathbf{1}}^{-1}$.

Let $r \in \mathrm{Rist}_G(\mathbf{d})$ be any non-trivial element, and let us consider $\varphi_{\mathbf{d}}(r) \in G$. As G acts faithfully on a rooted tree of finite degree, we have that G is residually finite. Therefore, there exists a normal subgroup $L \trianglelefteq G$ of finite index such that $\varphi_{\mathbf{d}}(r) \notin L$.

Let us consider $L \cap S_1$, which is a normal subgroup of S_1 of finite index. Let $\pi_{L\cap S_1} \colon S_1 \to S_1/(L \cap S_1)$ be the standard projection onto the quotient, and let $K \trianglelefteq S_1$ be the kernel of the map $\pi_{L\cap S_1} \circ \alpha_d$. As $S_1/(L \cap S_1)$ is a finite group, we have that K is a normal subgroup of finite index of S_1. It follows that $K \cap L$ is a normal subgroup of finite index of S_1. In particular, it is non-trivial. Therefore, it follows from Lemma 2.7.4 that there exists $v \in X^*$ such that $\mathrm{Rist}''_G(v) \leq K \cap L$. Since S_1 acts spherically transitively on X^*, we conclude that $\mathrm{Rist}''_G(|v|) \leq K \cap L$, and as $\mathrm{Rist}''_G(|v|)$ is a normal subgroup of G, we use Lemma 2.7.5 to conclude that there exists $m \in \mathbb{N}$ such that $\mathrm{Rist}'_G(m) \leq K \cap L$.

Let us set $N = \mathrm{Rist}'_G(m+1)$. It is clear that N is a normal subgroup of G and that $\varphi_v(N) \leq \mathrm{Rist}'_G(m)$ for all $v \in X$. Since H is a prodense subgroup, we have $HN = G$. In particular, for the $r \in \mathrm{Rist}_G(\mathbf{d})$ chosen above, there must exist $h \in H$ and $n \in N$ such that $hn = r$. As both n and r belong to $\mathrm{St}_G(1)$, we must have $h \in \mathrm{St}_H(1)$. By our choice of r, we have

$$\psi_1(r) = (1, 1, \dots, 1, \varphi_{\mathbf{d}}(r)).$$

It follows from the fact that $N = \mathrm{Rist}'_G(m+1)$ that

$$\psi_1(n) = (n_1, n_2, \dots, n_d)$$

with $n_i \in \mathrm{Rist}'_G(m)$ for $i = 1, 2, \dots d$. Finally, since $h \in \mathrm{St}_H(1)$, we have

$$\psi_1(h) = (h_1, \alpha_2(h_1), \dots, \alpha_d(h_1)).$$

Consequently, we have

$$(1, 1, \dots, \varphi_{\mathbf{d}}(r)) = (h_1 n_1, \alpha_2(h_1) n_2, \dots, \alpha_d(h_1) n_d).$$

In particular, this means that $h_1 \in \mathrm{Rist}'_G(m) \leq K \cap L \leq K$. As K was defined as the kernel of the map $\pi_{L\cap S_1} \circ \alpha_d$, this implies that $\alpha_d(h_1) \in L$. Since we also have that $n_d \in \mathrm{Rist}'_G(m) \leq L$, we must have $\varphi_{\mathbf{d}}(r) \in L$. This is absurd, since we chose L such that $\varphi_{\mathbf{d}}(r) \notin L$.

Hence, our assumption that $H_u = G$ led us to a contradiction, so we must have $H_u \neq G$. □

Let us now state a direct corollary of this theorem and of a few other results in this chapter that will be useful in applications later on.

Corollary 5.4.4. *Let X be a finite alphabet and let G be a finitely generated self-replicating weakly branch group acting on X^*. Let us suppose that the action of G on X is primitive, and suppose that every proper quotient of G is in $\mathcal{MF}$ (this is the case, for instance, if G is branch, by Proposition 5.3.9). Let $H \leq G$ be a subgroup that is dense in the profinite topology (respectively, pro-$\mathcal{MF}$-dense) and let $v \in X^*$ be any vertex. Then, H_v is dense in the profinite topology (respectively, pro-$\mathcal{MF}$-dense), and $H \neq G$ if and only if $H_v \neq G$.*

Proof. The result follows immediately from Propositions 5.4.1, 5.3.8, 5.3.4 and Theorem 5.4.3. □

Corollary 5.4.4 states that when every proper quotient belongs to $\mathcal{MF}$, the projections of proper prodense subgroups are also proper prodense subgroups. In particular, by Corollary 5.3.7, the projections of maximal subgroups of infinite index (if they exist) are proper prodense subgroups. However, Corollary 5.4.4 tells us nothing about the maximality of these projections. It is thus natural to wonder if the projections of maximal subgroups of infinite index must necessarily be maximal. The next proposition shows that this is always the case, under the assumptions of Corollary 5.4.4.

Proposition 5.4.5. *Let X be a finite alphabet and let G be a self-replicating weakly branch group acting on X^* in such a way that the action of G on X is primitive. Suppose that every proper quotient of G is in $\mathcal{MF}$. If $M < G$ is a maximal subgroup of G of infinite index, then M_v is also a maximal subgroup of infinite index of G for any $v \in X^*$.*

Proof. By Corollary 5.4.4, we have that M_v is a proper prodense subgroup of G. Thus, to show that it is a maximal subgroup of infinite index, we only need to show that it is maximal.

Without loss of generality, we can assume that $v \in X$. Indeed, if the result is true for all $v \in X$, then we immediately get that it is true for all $v \in X^*$ by induction.

For the sake of contradiction, let us assume that M_v is not maximal in G. Then, there exists $g \in G$ such that

$$M_v \lneq \langle M_v, g\rangle \lneq G.$$

As G is self-replicating, there exists $\tilde{g} \in G$ such that $\tilde{g} \in \mathrm{St}_G(v)$ and $\varphi_v(\tilde{g}) = g$. Since M is prodense in G, we have that $M\,\mathrm{St}_G(1) = G$. Therefore, there exists $\tilde{m} \in M$ and $\tilde{s} \in \mathrm{St}_G(1)$ such that $\tilde{g} = \tilde{m}\tilde{s}$. Since both $\tilde{g}$ and $\tilde{s}$ belong to $\mathrm{St}_G(v)$, we must have that $\tilde{m} \in \mathrm{St}_G(v)$. Therefore, we have

$$\varphi_v(\tilde{s}) = \varphi_v(\tilde{m}^{-1}\tilde{g}) = \varphi_v(\tilde{m}^{-1})g.$$

Since $\tilde{m} \in M$, we have that $\varphi_v(\tilde{m}^{-1}) \in M_v$. We conclude that $\langle M_v, g\rangle = \langle M_v, \varphi_v(\tilde{m}^{-1})g\rangle$. Thus, replacing g by $\varphi_v(\tilde{m}^{-1})g$ and $\tilde{g}$ by $\tilde{s}$ if necessary, we can assume that $\tilde{g} \in \mathrm{St}_G(1)$.

Now, let $w \in X$ be any element of X different from v. Since G is self-replicating, and since $\mathrm{Rist}_G(w)$ is a normal subgroup of $\mathrm{St}_G(w)$, we have that $\varphi_w(\mathrm{Rist}_G(w))$ is a normal subgroup of G. By Corollary 5.4.4, we know that M_w is a proper prodense subgroup of G. Therefore, we have that $M_w\varphi_w(\mathrm{Rist}_G(w)) = G$. Consequently, there exists $m_w \in M_w$ and $r_w \in \mathrm{Rist}_G(w)$ such that $m_w\varphi_w(r_w) = \varphi_w(\tilde{g})$.

Let
$$\hat{g} = \tilde{g} \prod_{w \in X\setminus\{v\}} r_w^{-1} \in \mathrm{St}_G(1).$$

Then, for every $w \in X \setminus \{v\}$, we have

$$\begin{aligned}\varphi_w(\hat{g}) &= \varphi_w(\tilde{g}) \prod_{w' \in X\setminus\{v\}} \varphi_{w'}(r_{w'})^{-1} \\ &= \varphi_w(\tilde{g})\varphi_w(r_w)^{-1} \\ &= m_w \in M_w\end{aligned}$$

where the second equality comes from the fact that $r_{w'} \in \mathrm{Rist}_G(w')$, so $\varphi_w(r_{w'}) = 1$ if $w \neq w'$. Furthermore, by a similar computation, we have that $\varphi_v(\hat{g}) = g$.

Since $g \notin M_v$ by construction, we must have that $\hat{g} \notin M$. Let us write $H = \langle M, \hat{g}\rangle$. By the maximality of M, we must have that $H = G$. However, we will now prove that
$$H_v = \langle M_v, g\rangle \lneq G,$$

which will contradict the fact that $\langle M, \hat{g}\rangle = G$, since $G_v = G$. Thus, to finish the proof, we only need to prove the above claim.

Let $h \in \mathrm{St}_H(v)$ be an arbitrary element of H stabilising v. Since $h \in H = \langle M, \hat{g}\rangle$, there exist $n \in \mathbb{N}$, $i_1, \ldots, i_n \in \mathbb{Z}$ and $\mu_1, \ldots, \mu_{n+1} \in M$ such that

$$h = \mu_1\hat{g}^{i_1}\mu_1^{-1}\mu_2\hat{g}^{i_2}\mu_2^{-1} \ldots \mu_n\hat{g}^{i_n}\mu_n^{-1}\mu_{n+1}.$$

Notice that since $\hat{g} \in \mathrm{St}_G(1)$, we must have that $\mu_j g^{i_j}\mu_j^{-1} \in \mathrm{St}_G(1)$ for all $1 \leq j \leq n$, and since $h \in \mathrm{St}_G(v)$, this implies that $\mu_{n+1} \in \mathrm{St}_G(v)$.

We will now see that for all $1 \leq j \leq n$, we must have

$$\varphi_v(\mu_j\hat{g}^{i_j}\mu_j^{-1}) \in \langle M_v, g\rangle.$$

Indeed, if $\mu_j \in \mathrm{St}_G(v)$, then we have

$$\varphi_v(\mu_j\hat{g}^{i_j}\mu_j^{-1}) = \varphi_v(\mu_j)g^{i_j}\varphi_v(\mu_j)^{-1} \in \langle M_v, g\rangle.$$

If $\mu_j \notin \mathrm{St}_G(v)$, then we have $\mu_j^{-1} \cdot v \neq v$. Let us set $w = \mu_j^{-1} \cdot v$. We have that

$$\varphi_v(\mu_j\hat{g}^{i_j}\mu_j^{-1}) = \varphi_w(\mu_j)m_w^{i_j}\varphi_w(\mu_j)^{-1}.$$

Now, since $m_w^{i_j} \in M_w$, there exists some $\nu \in \mathrm{St}_M(w)$ such that $\varphi_w(\nu) = m_w^{i_j}$. It follows that
$$\varphi_v(\mu_j\hat{g}^{i_j}\mu_j^{-1}) = \varphi_v(\mu_j\nu\mu_j^{-1}).$$

Since $\nu \in \mathrm{St}_M(w)$ and since $\mu_j \in M$, we have that $\varphi_v(\mu_j \nu \mu_j^{-1}) \in \mathrm{St}_G(\mu_j \cdot w) = \mathrm{St}_M(v)$. Therefore, we conclude that $\varphi_v(\mu_j \hat{g}^{i_j} \mu_j^{-1}) \in M_v$.

Finally, since $\mu_{n+1} \in M_v$, we have that $\varphi_v(\mu_{n+1}) \in M_v$. Thus, we see that $\varphi_v(h) \in \langle M_v, g \rangle$. This concludes the proof. □

The hypotheses of Theorem 5.4.3 and its corollaries are rather strong. Indeed, requiring the group be self-replicating seems somewhat restrictive. One might wonder if the result remains true under weaker assumptions.

Question 5.4.6. Is it true that, under the assumption that every proper quotient is in MF, the projections of any proper prodense subgroup in a weakly branch group G is always a proper prodense subgroup of the projection of G? If not, is it true at least for weakly branch groups acting primitively on the first level of the rooted tree?

Chapter 6

Groups of intermediate growth not in $\mathcal{MF}$

In this chapter, we study the maximal subgroups of Šunić groups acting on the binary rooted tree. More precisely, we show that if such a Šunić group contains an element of infinite order, then it must admit exactly countably many maximal subgroups of infinite index. We then give a complete description of these maximal subgroups.

The existence of maximal subgroups of infinite index among Šunić groups acting on the binary rooted tree is interesting, since they are branch groups (Proposition 2.9.18) of intermediate growth (as mentioned in Section 3.3). Although these groups are neither the first examples of branch groups admitting maximal subgroups of infinite index (the first examples of such groups were given by Bondarenko in [14]) nor the first examples of groups of intermediate growth with the same property (the existence of such groups followed easily from Nekrashevych's discovery of simple groups of intermediate growth [66]), they are, to the best of our knowledge, the first examples of branch groups of intermediate growth that do not belong to the class $\mathcal{MF}$. This shows that unlike in linear groups, subexponential growth is not a sufficient condition for a branch group to belong to the class $\mathcal{MF}$.

It is also interesting to note that these groups are very close to the groups studied by Pervova in [73, 74]. In those articles, Pervova studied torsion Grigorchuk groups acting on the binary rooted tree. It follows from our work that the self-similar non-torsion Grigorchuk group sometimes called the Grigorchuk-Erschler group in the literature (due to the fact that the growth of this group was studied by Erschler in [27]) contains maximal subgroups of infinite index. Thus, our results show that, at least in this case, the assumption of periodicity was necessary.

In Section 6.1, we define for each Šunić group a family of dense subgroups in the profinite topology. We show in Section 6.2 that in the case of non-torsion Šunić groups acting on the binary rooted tree, these dense subgroups are proper, thus showing that these groups admit maximal subgroups of infinite index. Then, in Section 6.3, we show that some of the subgroups of this family (the ones corresponding to prime numbers) are in fact maximal. In Section 6.4, we show that these are in fact, up to conjugation, the only maximal subgroups of infinite index in those groups. To complete our investigation of maximal subgroups, we briefly describe in Section 6.5 the maximal subgroups of finite index of every Šunić group. We then conclude by discussing some open questions in Section 6.6.

The results in this chapter are part of a joint work with Alejandra Garrido and were already published in [32]. Most of text in Sections 6.1 to 6.4 was taken from this article, with minor modifications where required.

For this entire chapter, we will adopt the notation of Section 2.9. Recall that given a prime number p and a monic polynomial f with coefficients in $\mathbb{Z}/p\mathbb{Z}$ and non-zero constant coefficient, one can define the Šunić group $G_{p,f}$ acting on the regular rooted tree X^*, where $X = \{\mathbf{0}, \mathbf{1}, \dots, \mathbf{p-1}\}$. This group is generated by

$$A = \langle a = (\mathbf{0}\ \mathbf{1}\ \dots\ \mathbf{p-1})\rangle \cong \mathbb{Z}/p\mathbb{Z}$$

and $B = \langle b_0, b_1, \dots, b_{m-1}\rangle \cong (\mathbb{Z}/p\mathbb{Z})^m$, where m is the degree of the polynomial f. The action of A on X^* is by rooted automorphisms, and the action of B on X^* is given by the recurrence

$$b = (\omega(b), 1, \dots, 1, \rho(b))$$

where $\omega\colon B \to A$ is a particular epimorphism and $\rho\colon B \to B$ is an automorphism of B uniquely defined by f, as explained in Section 2.9.

Recall from Notation 2.9.11 that we denote by $\mathcal{G}$ the set of all Šunić groups and by $\mathcal{G}_{p,m}$ the set of Šunić groups with acting on the p-regular rooted tree and with defining polynomial of degree m. Throughout this chapter, we will be only interested in $\mathcal{G}_{2,m}$, the set of Šunić groups for acting on the binary rooted tree. We will also restrict our attention mainly to Šunić groups containing an element of infinite order. Recall that by Corollary 2.9.15, this happens if and only if the defining polynomial is divisible by $x + 1$.

Notice that the set $\mathcal{G}_{2,1}$ contains only one element, namely the infinite dihedral group. As this group is of a very different nature from the other Šunić groups (it is the only group in this family that is not a branch group), we will frequently omit it from our considerations.

6.1 Dense subgroups

In this section, we define a family of subgroups of Šunić groups acting on the binary rooted tree that are dense in the profinite topology. This is an important step towards the goal of finding maximal subgroups of infinite index, as was discussed in Chapter 5. We begin by giving a method to find dense subgroups in the congruence topology (see Chapter 4 for the definition of these topologies). The general version that we give here is due to Paul-Henry Leemann.

Proposition 6.1.1. *Let T be the d-regular rooted tree for some $d \geq 2$ and let $G \leq \operatorname{Aut} T$ be a group countably generated by $S = \{g_1, g_2, \dots\}$. If $m_1, m_2, \dots, \in \mathbb{N}$ are numbers coprime with $|G/\operatorname{St}_G(n)|$ for all $n \in \mathbb{N}$, then $H = \langle g_1^{m_1}, g_2^{m_2}, \dots\rangle$ is a dense subgroup of G with respect to the congruence topology.*

Proof. Recall from Definition 4.1.3 that the basic open sets of the congruence topology for G are the cosets of $\operatorname{St}_G(n)$ for $n \in \mathbb{N}$. Let us write $f_n = |G/St_G(n)|$. By assumption, we have that m_i and f_n are coprime for all i

and for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, the Euclidean algorithm yields $x, y \in \mathbb{Z}$ such that $xm_i + yf_n = 1$. We can thus write $g_i = (g_i^{m_i})^x (g_i^{f_n})^y$. Since $g_i^{f_n} \in \mathrm{St}_G(n)$, we have $g_i \,\mathrm{St}_G(n) = (g_i^{m_i})^x \,\mathrm{St}_G(n)$ and therefore $\langle g_i \rangle \,\mathrm{St}_G(n) = \langle g_i^{m_i} \rangle \,\mathrm{St}_G(n)$. This implies that

$$G = \langle g_1, g_2, \dots \rangle \,\mathrm{St}_G(n) = H \,\mathrm{St}_G(n)$$

for every n. In other words, H intersects every non-empty open set and is thus dense in G with respect to the congruence topology. □

With the above proposition, it is simple to find subgroups of Šunić groups acting on the regular rooted tree X^* that are dense in the profinite topology.

Corollary 6.1.2. *Let $G = \langle a, B \rangle \in \mathcal{G} \setminus \mathcal{G}_{2,1}$ be a Šunić group (different from the infinite dihedral group) acting on the p-regular rooted tree X^* for some prime p. Then, for any $q \in \mathbb{N}$ coprime to p and any $b \in B$, the subgroup $H(q) = \langle (ab)^q, B \rangle$ is dense in G with respect to the profinite topology.*

Proof. Since $b \in B$, we clearly have $G = \langle ab, B \rangle$. As $G/\mathrm{St}_G(n)$ is a p-group for all $n \in \mathbb{N}$ (this follows directly, for instance, from Proposition 2.2.5), it follows immediately from Proposition 6.1.1 that H is dense in G with respect to the congruence topology. The result then follows from the fact that, according to Theorem 4.3.8, G has the congruence subgroup property, so the congruence topology is the same as the profinite topology. □

Of course, the existence of the dense subgroups $H(q)$ does not tell us anything about the existence of maximal subgroups of infinite index, since those subgroups could very well not be proper. In fact, it is easy to see that if G is torsion, and thus a p-group (see Proposition 2.9.14), then $H(q) = G$.

However, we will show in the next section that in the case where $G \in \mathcal{G}_{2,m} \setminus \mathcal{G}_{2,1}$ is a non-torsion Šunić group acting on the binary rooted tree (and different from the infinite dihedral group), each $H(q)$ is in fact a proper subgroup of G when we take b to be the generator such that $b = (a, b)$, whose existence is guaranteed by Corollary 2.9.15. In particular, this will imply that such a G contains maximal subgroups of infinite index.

Before we go on to prove that they are proper, let us first notice an interesting fact about these subgroups $H(q) < G$. It turns out that they are in fact conjugate to the group G in the automorphism group of the tree.

Proposition 6.1.3. *Let $G \in \mathcal{G}_{2,m}$ be a non-torsion Šunić group acting on the binary rooted tree X^* and $q \geq 3$ be an odd number. Let $b \in B$ be the element such that $\psi_1(b) = (a, b)$. Then, the subgroup $H(q) = \langle (ab)^q, B \rangle$ is conjugate to G in $\mathrm{Aut}(X^*)$.*

Proof. Notice that the existence of b as in the statement is ensured by Corollary 2.9.15. Let $g \in \mathrm{Aut}(X^*)$ be defined by $\psi_1(g) = ((ba)^{\frac{q-1}{2}} g, g)$. We have

$$g^{-1}(ab)^q bg = g^{-1}(ab)^{q-1} agaa$$

and

$$\begin{aligned}
\psi_1(g^{-1}(ab)^{q-1}aga) &= \psi_1(g^{-1})\psi_1((abab)^{\frac{q-1}{2}})\psi_1(aga) \\
&= (g^{-1}(ab)^{\frac{q-1}{2}}, g^{-1})((ba)^{\frac{q-1}{2}}, (ab)^{\frac{q-1}{2}})(g, (ba)^{\frac{q-1}{2}} g) \\
&= (1, 1).
\end{aligned}$$

Since ψ_1 is injective, this means that $g^{-1}(ab)^q bg = a$.

Now, let $x \in B$ be any element. Then,

$$\begin{aligned}\psi_1(g^{-1}xg) &= (g^{-1}(ab)^{\frac{q-1}{2}}, g^{-1})(\omega(x), \rho(x))((ba)^{\frac{q-1}{2}} g, g) \\ &= (g^{-1}(ab)^{\frac{q-1}{2}}\omega(x)(ba)^{\frac{q-1}{2}} g, g^{-1}\rho(x)g) \\ &= (\omega(x), g^{-1}\rho(x)g).\end{aligned}$$

Indeed, if $\omega(x) = 1$, then $g^{-1}(ab)^{\frac{q-1}{2}}\omega(x)(ba)^{\frac{q-1}{2}} g = 1$, and if $\omega(x) = a$, then

$$g^{-1}(ab)^{\frac{q-1}{2}}\omega(x)(ba)^{\frac{q-1}{2}} g = g^{-1}(ab)^q bg = a = \omega(x).$$

This implies that $g^{-1}xg = x$. Since $H(q)$ is generated by $(ab)^q b$ and B, we conclude that $g^{-1}H(q)g = G$. □

As a corollary, we get that for all odd $q \geq 3$, the subgroup $H(q)$ defined in the above proposition is isomorphic to G.

In fact, we could also use this proposition to show that $H(q)$ is dense in G. Indeed, this proposition implies that $H(q)\operatorname{St}_G(n)/\operatorname{St}_G(n)$ is isomorphic to $G/\operatorname{St}_G(n)$ for all $n \in \mathbb{N}$. On the other hand, we have

$$H(q)\operatorname{St}_G(n)/\operatorname{St}_G(n) \leq G/\operatorname{St}_G(n).$$

Since $G/\operatorname{St}_G(n)$ is finite, it cannot contain a proper subgroup isomorphic to itself, so we must have $H(q)\operatorname{St}_G(n) = G$ for all $n \in \mathbb{N}$.

6.2 Proper dense subgroups

In this section, we will prove that for non-torsion Šunić groups acting on the binary rooted tree, the dense subgroups $H(q)$ defined in Proposition 6.1.3 are in fact proper for every odd number $q \geq 3$. Let us first introduce some notation that we will use throughout the rest of this chapter.

Notation 6.2.1. We will denote by $\widetilde{\mathcal{G}}_{2,m}$ the non-torsion groups in $\mathcal{G}_{2,m}$. For $G \in \widetilde{\mathcal{G}}_{2,m}$, we will denote by $b \in B$ the element such that $b = (a, b)$ whose existence is guaranteed by Corollary 2.9.15. Notice that $\langle a, b\rangle \leq G$ is isomorphic to the infinite dihedral group. For $q \in \mathbb{N}$, we will denote by $H(q) = \langle (ab)^q, B\rangle$ the subgroup of G generated by $(ab)^q$ and the elements of B. ♠

The main goal of this section will be to prove that for $G \in \widetilde{\mathcal{G}}_{2,m}$ and $q \geq 3$ odd, the subgroup $H(q)$ is a proper subgroup of G. Since we have already shown in the previous section that this subgroup is also dense, we get that G does not belong to the class $\mathcal{MF}$ of groups whose maximal subgroups are all of finite index (see Proposition 5.2.2).

In order to show that the subgroup $H(q)$ is proper, we will compare its action with the action of G on the boundary X^∞ of the rooted tree X^* (see Proposition 2.6.42). More precisely, we will show that the orbit of $\mathbf{1}^\infty \in X^\infty$ under the action of $H(q)$ is strictly smaller than the orbit of the same point under the action of G. This will immediately imply that $H(q)$ is a proper subgroup of G.

To do this, let us study the actions of G and $H(q)$ on X^∞. To describe the action of G, it is sufficient to look at the action of its generators, namely a and the elements of B. In order to more easily achieve this, let us first notice that we can define a natural operation of addition on the set $X = \{\mathbf{0}, \mathbf{1}\}$, namely the addition modulo 2. With this operation, for $\xi = \xi_0\xi_1\xi_2\cdots \in X^\infty$, the action of $a \in G$ on ξ is given by

$$a \cdot \xi_0\xi_1\xi_2\cdots = (\xi_0 + \mathbf{1})\xi_1\xi_2\ldots,$$

and the action of $x \in B$ on ξ is given recursively by

$$x \cdot \xi_0\xi_1\xi_2\cdots = \begin{cases} \mathbf{0}\xi_1\xi_2\ldots & \text{if } \xi_0 = \mathbf{0} \text{ and } \omega(x) = 1 \\ \mathbf{0}(\xi_1 + \mathbf{1})\xi_2\ldots & \text{if } \xi_0 = \mathbf{0} \text{ and } \omega(x) = a \\ \mathbf{1}\rho(x)\cdot(\xi_1\xi_2\ldots) & \text{if } \xi_0 = \mathbf{1} \end{cases}$$

where ω and ρ are the maps associated to G, as in Section 2.9.

Let us make a couple of simple yet very useful remarks regarding the action of G on X^∞.

Remark 6.2.2. Let $\xi = \xi_0\xi_1\xi_2\cdots \in X^\infty$ be any point on the boundary. Since we have $b = (a, b)$, it follows from the formulas for the action above that $b \cdot \xi$ is the sequence obtained by adding $\mathbf{1}$ (modulo 2) to the element immediately following the first $\mathbf{0}$ in the sequence. For example, we have $b \cdot \mathbf{10}^\infty = \mathbf{110}^\infty$. In particular, this implies that $\mathbf{1}^\infty$ is a fixed point of b.

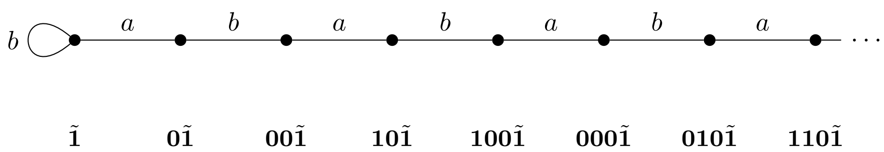
Remark 6.2.3. For $x \in B$ and $\xi \in X^\infty$, it follows again from the formulas for the action that either $x \cdot \xi = \xi$ or $x \cdot \xi$ is the sequence obtained by adding $\mathbf{1}$ (modulo 2) to the element immediately following the first $\mathbf{0}$ in the sequence. Thus, either ξ is fixed by x or $x \cdot \xi = b \cdot \xi$. It follows that for any $\xi \in X^\infty$, the orbit of ξ under the action of G is the same as the orbit of ξ under the action of $\langle a, b\rangle \leq G$.

Let us study the action of G on the orbit of $\tilde{\mathbf{1}} = \mathbf{1}^\infty \in X^\infty$. According to Remark 6.2.3, this orbit is the same as the orbit of $\tilde{\mathbf{1}}$ under $\langle a, b\rangle \cong D_\infty$. We will therefore restrict our attention to the action of this subgroup for the moment.

We will begin by describing the orbital graph of this action with respect to the generating set $\{a, b\}$. Let us first briefly recall the definition of an orbital graph.

Definition 6.2.4. Let G be a group with a symmetric generating set S acting on a space V. The orbital graph of the action of G on V (with respect to S) is the labelled graph whose vertex set is V and where two vertices $v, w \in V$ are connected by an edge labelled $s \in S$ whose origin is v and whose terminus is w if and only if $w = sv$.

Proposition 6.2.5. *The orbital graph of the action of $\langle a, b\rangle$ on the orbit of $\tilde{\mathbf{1}}$ is a half-line with a loop labeled by b at $\tilde{\mathbf{1}}$.*



Proof. Since $\langle a, b\rangle$ is generated by two elements of order 2, the orbital graph must be a connected graph where every vertex has degree 2 (where a loop adds only 1 to the degree of the vertex). Hence, there are only four possibilities : the graph is either a circle, a line, a segment or a half-line.

It follows from Remark 6.2.2 that $b\cdot\xi \neq \xi$ for all $\xi \in X^\infty$ containing at least one $\mathbf{0}$, and clearly, we have that $a \cdot \xi \neq \xi$ for all $\xi \in X^\infty$. Hence, b has exactly one fixed point, $\tilde{\mathbf{1}}$, and a has none. Therefore, by degree considerations, the only possible orbital graph of the action of $\langle a, b\rangle$ on the orbit of $\tilde{\mathbf{1}}$ must be a half-line with a loop at $\tilde{\mathbf{1}}$ labelled by b. □

Thanks to Proposition 6.2.5, we can define a bijection between the orbit of $\tilde{\mathbf{1}}$ and $\mathbb{Z}$.

Proposition 6.2.6. *The map $\zeta\colon \mathbb{Z} \to G \cdot \tilde{\mathbf{1}}$ given by $\zeta(n) = (ab)^n \cdot \tilde{\mathbf{1}}$ is a bijection between $\mathbb{Z}$ and the orbit $G \cdot \tilde{\mathbf{1}}$ of $\tilde{\mathbf{1}}$ under G.*

Proof. According to Remark 6.2.3, for every $\xi \in G \cdot \tilde{\mathbf{1}}$, there exists $g \in \langle a, b\rangle$ such that $\xi = g \cdot \tilde{\mathbf{1}}$. Since $\langle a, b\rangle$ is isomorphic to the infinite dihedral group, there exists $n \in \mathbb{Z}$, $m \in \{0, 1\}$ such that $g = (ab)^n b^m$. Since $b \cdot \tilde{\mathbf{1}} = \tilde{\mathbf{1}}$, we conclude that ζ is surjective.

Injectivity follows from Proposition 6.2.5. Indeed, it follows from the structure of the orbital graph (see the figure in that proposition) that $(ab)^n \cdot \tilde{\mathbf{1}}$ is the vertex at distance $2n - 1$ from $\tilde{\mathbf{1}}$ if $n > 0$ and at distance $2n$ if $n \leq 0$. □

The bijection ζ allows us to define an action of G on $\mathbb{Z}$. It turns out that the restriction of this action to $\langle a, b\rangle$ is the standard action of D_∞ on $\mathbb{Z}$. This will allow us to prove the main theorem of this section.

Theorem 6.2.7. *Let us consider $G \in \widetilde{\mathcal{G}}_{2,m}$ with $m \geq 2$. Then, for each odd number $q \geq 3$, the subgroup $H(q) = \langle (ab)^q, B\rangle$ is proper and dense in the profinite topology.*

Proof. It suffices to show that $H(q)$ is proper, since the density was shown in Corollary 6.1.2. Let $\zeta\colon \mathbb{Z} \to G\cdot\tilde{\mathbf{1}}$ be the map from Proposition 6.2.6. Then, G acts on $\mathbb{Z}$ by

$$g \cdot n = \zeta^{-1}(g \cdot \zeta(n))$$

for all $g \in G$ and $n \in \mathbb{N}$. In particular,

$$(ab)^q \cdot n = \zeta^{-1}((ab)^{q+n} \cdot \tilde{\mathbf{1}}) = q + n$$

and

$$b \cdot n = \zeta^{-1}(b(ab)^n \cdot \tilde{\mathbf{1}}) = \zeta^{-1}((ab)^{-n} b \cdot \tilde{\mathbf{1}}) = \zeta^{-1}((ab)^{-n} \cdot \tilde{\mathbf{1}}) = -n$$

for all $n \in \mathbb{Z}$. According to Remark 6.2.3, for $x \in B$, we also have $x \cdot n = \pm n$, depending on the value of n.

Since $H(q)$ is generated by $(ab)^q$ and B, it follows that $H(q) \cdot 0 = q\mathbb{Z} \subsetneq \mathbb{Z} = G \cdot 0$. Therefore, $H(q) \neq G$. □

Notice that if two odd numbers $q_1, q_2 \in \mathbb{N}$ are relatively prime, then $H(q_1)$ and $H(q_2)$ are contained in different maximal subgroups (necessarily of infinite index, since $H(q_1)$ and $H(q_2)$ are dense). Indeed, if they were contained in the same maximal subgroup M, this M would contain ab and B, so we would have $M = G$, a contradiction.

Thus, Theorem 6.2.7 immediately implies the following corollary.

Corollary 6.2.8. *Let $G \in \widetilde{\mathcal{G}}_{2,m}$ with $m \geq 2$. Then, G contains at least countably many maximal subgroups of infinite index.*

Another immediate corollary is that the Frattini subgroup of non-torsion Šunić groups is trivial.

Corollary 6.2.9. *Let $G \in \widetilde{\mathcal{G}}_{2,m}$ with $m \geq 2$. Then, the Frattini subgroup of G is trivial.*

Proof. This follows directly from Proposition 5.3.10. □

6.3 Maximal subgroups of infinite index

Let $G \in \widetilde{\mathcal{G}}_{2,m}$ be a non-torsion Šunić group acting on the binary rooted tree, with $m \geq 2$. In Section 6.2, we showed that the subgroups $H(q)$ defined in Proposition 6.1.3 are proper dense subgroups of G with respect to the profinite topology. It follows from the proof of Proposition 5.2.2 that $H(q)$ is contained in a maximal subgroup of infinite index of G. In this section, we will show that in fact, if q is prime, then $H(q)$ is already a maximal subgroup of infinite index of G.

The proof uses a similar strategy to the one developed by Pervova in [73] and [74] to study maximal subgroups of the Grigorchuk group, even though the conclusion we reach here is the opposite of the one she obtained. More precisely, the idea of the proof is to use an argument of length reduction to show that for any $g \notin H(q)$, there exists a vertex v such that the projection of $\langle H(q), g\rangle$ to v is not proper. This will then imply that $\langle H(q), g\rangle$ cannot be proper by Theorem 5.4.3.

Before we can begin the proof, we will need a few auxiliary results. we begin by constructing a homomorphism that is a lift of $\varphi_{\mathbf{1}}$, the projection onto the second coordinate.

A lift of $\varphi_{\mathbf{1}}$

If $G \in \mathcal{G}_{2,m} \setminus \mathcal{G}_{2,1}$ is a Šunić group acting on the binary rooted tree X^*, with $X = \{\mathbf{0}, \mathbf{1}\}$, then we have a projection map $\varphi_{\mathbf{1}}\colon G \to G$ (see Definition 2.6.22). As G is self-replicating (see Proposition 2.9.17), this map is surjective. In this subsection, we will define a homomorphism $\phi\colon G \to \mathrm{St}_G(1)$ that is a lift of the projection $\varphi_{\mathbf{1}}$, meaning that $\varphi_{\mathbf{1}} \circ \phi = id_G$. Of course, in a similar fashion, we could also define a lift for $\varphi_{\mathbf{0}}$, and the same could also be done for Šunić groups acting on trees of higher degree, but we shall not need such generality here.

Before we begin, let us set some notation.

Notation 6.3.1. Let $G \in \mathcal{G}_{2,m} \setminus \mathcal{G}_{2,1}$ be a Šunić group acting on the binary rooted tree. Recall from Notation 2.9.5 that for all $i \in \mathbb{Z}$, we write $B_i =$

$\rho^i(\ker(\omega))$. For the rest of this chapter, we will denote by $c \in B_{-1} \setminus B_0$ and $d \in B_0 \setminus B_1$ two fixed elements such that $c = (a, d)$. The existence of those elements is stated in Lemma 2.9.20.

We are now ready to define a lift for $\varphi_{\mathbf{1}}$.

Proposition 6.3.2. *Let $G \in \mathcal{G}_{2,m}$ be a Šunić group with $m \geq 2$ and let $c, d \in G$ be as in Lemma 2.9.20. Then, there exists a unique homomorphism $\phi\colon G \to \mathrm{St}_G(1)$ such that*

$$\phi(a) = aca$$
$$\phi(x) = \rho^{-1}(x)$$

for all $x \in B$.

Proof. If such a homomorphism exists, then it is clearly unique. Thus, it suffices to show that the above yields a well-defined homomorphism. Let us write $\Gamma = A * B$ and consider the homomorphisms $\pi, \Psi, \pi_{G\times G}$ defined in the proof of Proposition 2.9.16. As in the proof of Proposition 2.9.16, let S be the subgroup of index 2 of Γ generated by $\{x, axa \mid x \in B\}$.

Let $\Phi\colon A * B \to S$ be the unique homomorphism such that

$$\Phi(a) = aca$$
$$\Phi(x) = \rho^{-1}(x)$$

for all $x \in B$. Defining $\phi = \pi \circ \Phi \circ \pi^{-1}$, we obtain the following diagram, where the bottom square commutes.

$$\begin{array}{ccc} A * B & \xrightarrow{\ \pi\ } & G \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \phi} \\ S & \xrightarrow{\ \pi\ } & \mathrm{St}_G(1) \\ \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \psi_1} \\ (A * B) \times (A * B) & \xrightarrow{\ \pi_{G\times G}\ } & G \times G \end{array}$$

Let $N \leq A * B$ be the kernel of π. To show that ϕ is a well-defined homomorphism, it suffices to show that $\Phi(N) \leq N$. A direct computation shows that every element w of A or B in $\Gamma = A * B$ satisfies $\Psi(\Phi(w)) = (w', w)$ with $w' \in \langle a, d\rangle \leq \Gamma$. Therefore, the same is true for every element w of Γ. Consequently, if $w \in N$, then there exists $w' \in \langle a, d\rangle \leq \Gamma$ such that $\Psi(\Phi(w)) = (w', w)$. Hence,

$$\pi_{G\times G}(\Psi(\Phi(w))) = (\pi(w'), 1)$$

with $\pi(w') \in \langle a, d\rangle \leq G$. Since $\pi_{G\times G} \circ \Psi = \psi_1 \circ \pi$, we get

$$\psi_1(\pi(\Phi(w))) = (\pi(w'), 1)$$

with $\pi(w') \in \langle a, d\rangle \leq G$. It follows from Lemma 2.9.20 that $\pi(\Phi(w)) = 1$. □

Remark 6.3.3. With the notation of the previous proposition, for all $g \in G$, we have $\psi_1(\phi(g)) = (g', g)$ for some $g' \in \langle a, d\rangle \leq G$. In particular, ϕ is a right inverse of the projection $\varphi_{\mathbf{1}}$ on the second coordinate.

A nice property of this this is that it becomes an inverse of $\varphi_{\mathbf{1}}$ when we restrict ourselves to the rigid stabiliser of the vertex **1**, as the next proposition shows.

Proposition 6.3.4. *Using the notation of the previous proposition, let $g \in G$ be an arbitrary element. If there exists $h \in \mathrm{St}_G(1)$ such that $\psi_1(h) = (1, g)$, then $h = \phi(g)$.*

Proof. We have $\psi_1(\phi(g)) = (x, g)$, with $x \in \langle a, d\rangle$. Therefore, $\psi_1(\phi(g)h^{-1}) = (x, 1)$. It follows from Lemma 2.9.20 that $\phi(g) = h$. □

The structure of $H(q)$

Before we can show that they are maximal, we will use the homomorphism ϕ defined in Proposition 6.3.2 to prove a few auxiliary results about the structure of the subgroups $H(q)$. Let us first fix a few more notational conventions.

Notation 6.3.5. Since we will be mainly concerned with non-torsion Šunić groups, for the rest of this chapter, unless otherwise specified, we will denote by G a group in $\widetilde{\mathcal{G}}_{2,m} \setminus \mathcal{G}_{2,1}$. As before, we will denote by $b \in B$ the element such that $b = (a, b)$, and recall that we also have elements $c \in B_{-1} \setminus B_0$ and $d \in B_0 \setminus B_1$ such that $c = (a, d)$.

Also, unless stated otherwise, $q \geq 3$ will be a fixed odd integer (not necessarily prime), and when q is understood from the context we will denote $H(q) = \langle (ab)^q, B\rangle$ simply by H. ♋

Our main goal in this subsection will be to understand the projections of the subgroup H.

Lemma 6.3.6. *Consider the following subgroups of G:*

$$\Delta_b = \langle a, d^{(ab)^{\frac{q-1}{2}}}\rangle, \qquad \Delta_d = \langle a, d^{(ad)^{\frac{q-1}{2}}}\rangle.$$

There is a unique isomorphism $f\colon \Delta_b \to \Delta_d$ such that $f(a) = a$ and $f(d^{(ab)^{\frac{q-1}{2}}}) = d^{(ad)^{\frac{q-1}{2}}}$.

Proof. Let $D_4 = \langle s, t \mid s^2 = t^2 = (st)^4 = 1\rangle$ be the dihedral group of order 8. For all $g \in G$, we have

$$a^2 = (d^g)^2 = 1$$

and

$$\begin{aligned}\psi_1((ad^g)^4) &= \psi_1(ad^g ad^g)^2\\ &= (\rho(d)^{g'}, \rho(d)^{g'})^2\\ &= (1, 1)\end{aligned}$$

(for some $g' \in G$), which means that $(ad^g)^4 = 1$. It follows that there are unique homomorphisms $g_b \colon D_4 \to \Delta_b$ and $g_d \colon D_4 \to \Delta_d$ such that

$$\begin{aligned} g_b(s) &= a \\ g_b(t) &= d^{(ab)^{\frac{q-1}{2}}} \\ g_d(s) &= a \\ g_d(t) &= d^{(ad)^{\frac{q-1}{2}}}. \end{aligned}$$

These homomorphisms are clearly surjective, and a direct computation shows that they are injective. Therefore, we have the isomorphism $f = g_d \circ g_b^{-1}$. □

Lemma 6.3.7. *The stabiliser* $\mathrm{St}_H(1)$ *is generated by* $\{x, x^{a(ba)^{q-1}} \mid x \in B\}$.

Proof. Clearly, $\{x, x^{a(ba)^{q-1}} \mid x \in B\}$ generates a subgroup of $\mathrm{St}_H(1)$. On the other hand, if $h \in \mathrm{St}_H(1)$, then it can be written as a product $h = h_1 h_2 \dots h_n$, with $h_i \in \{a(ba)^{q-1}\} \cup B$ (since this set generates H). To act trivially on the first level, this product must contain an even number of a, and therefore an even number of $a(ba)^{q-1}$. Since $(a(ba)^{q-1})^{-1} = a(ba)^{q-1}$, this implies that h is indeed in the subgroup generated by $\{x, x^{a(ba)^{q-1}} \mid x \in B\}$. □

Proposition 6.3.8. *We have* $\psi_1(\mathrm{St}_H(1)) \leq H^{(ab)^{\frac{q-1}{2}}} \times H$. *Furthermore,* $\psi_1(\mathrm{St}_H(1))$ *is subdirect, or in other words, the projection of* $\psi_1(\mathrm{St}_H(1))$ *on each factor is surjective.*

Proof. For all $x \in B$, we have

$$\psi_1(x) = (\omega(x), \rho(x)).$$

If $\omega(x) = 1$, then $\omega(x)$ is clearly in $H^{(ab)^{\frac{q-1}{2}}}$. Otherwise,

$$\omega(x) = a = (ba)^{\frac{q-1}{2}}(ab)^q b (ab)^{\frac{q-1}{2}} \in H^{(ab)^{\frac{q-1}{2}}}.$$

Moreover, $\rho(x) \in B \subset H$, so $\psi_1(x) \in H^{(ab)^{\frac{q-1}{2}}} \times H$. Similarly,

$$\psi_1(x^{a(ba)^{q-1}}) = (\rho(x)^{(ab)^{\frac{q-1}{2}}}, \omega(x)^{(ba)^{\frac{q-1}{2}}}) \in H^{(ab)^{\frac{q-1}{2}}} \times H.$$

The first result then follows from the fact that by Lemma 6.3.7, $\mathrm{St}_H(1)$ is generated by the elements of B and their conjugates by $a(ba)^{q-1}$.

Now, for all $x \in B$, we have $\rho^{-1}(x) \in \mathrm{St}_H(1)$ with $\psi_1(\rho^{-1}(x)) = (\omega(\rho^{-1}(x)), x)$. Since we also have $(ab)^{2q} \in \mathrm{St}_H(1)$ with $\psi_1((ab)^{2q}) = ((ba)^q, (ab)^q)$, we see that the projection of $\psi_1(\mathrm{St}_H(1))$ on the second factor is surjective. To see that the projection on the first factor is also surjective, it suffices to notice that for all $h \in \mathrm{St}_H(1)$ with $\psi_1(h) = (h_1, h_2)$, we have $h^{a(ba)^{q-1}} \in \mathrm{St}_H(1)$ with $\psi_1(h^{a(ba)^{q-1}}) = (h_2^{(ab)^{\frac{q-1}{2}}}, h_1^{(ba)^{\frac{q-1}{2}}})$. □

The next proposition and its corollary roughly tell us that any element of G that projects to H must in fact belong to H. This will be crucial to prove the maximality of H.

Proposition 6.3.9. *If there exists* $g \in \mathrm{St}_G(1)$ *such that* $\psi_1(g) = (1, h)$ *for some* $h \in H$*, then* $g \in \mathrm{St}_H(1)$.

Proof. Since $h \in H$, there exist $h_1, h_2 \ldots, h_n \in \{a(ba)^{q-1}\} \cup B$ such that

$$h = h_1 h_2 \ldots h_n.$$

For $1 \leq i \leq n$, define

$$\tilde{h}_i := \begin{cases} \rho^{-1}(h_i) & \text{if } h_i \in B \\ c^{a(ba)^{q-1}} & \text{if } h_i = a(ba)^{q-1} \end{cases}$$

and $\tilde{h} = \tilde{h}_1 \tilde{h}_2 \ldots \tilde{h}_n$. Each of the terms in the product is in $\mathrm{St}_H(1)$, therefore so is $\tilde{h}$. Notice that $\psi_1(\tilde{h}_i) = (x_i, h_i)$, for $1 \leq i \leq n$, where $x_i \in \{1, a, d^{(ab)^{\frac{q-1}{2}}}\}$. Therefore, writing $x = x_1 x_2 \ldots x_n$, we have

$$\psi_1(\tilde{h}) = (x, h).$$

On the other hand, by direct computation, we see that $\psi_1(\phi(h_i)) = (f(x_i), h_i)$, where f is the isomorphism of Lemma 6.3.6 and ϕ is the homomorphism of Proposition 6.3.2. Since ϕ is a homomorphism, we have $\phi(h) = \phi(h_1)\phi(h_2) \ldots \phi(h_n)$. Hence,

$$\psi_1(\phi(h)) = (f(x), h).$$

However, by Proposition 6.3.4, since $\psi_1(g) = (1, h)$, we must have $g = \phi(h)$. Hence, $f(x) = 1$. Since f is an isomorphism, this means that $x = 1$. Therefore,

$$\psi_1(\tilde{h}) = (1, h) = \psi_1(g).$$

By the injectivity of ψ_1, we get that $g = \tilde{h} \in \mathrm{St}_H(1)$. □

Corollary 6.3.10. *If* $g \in \mathrm{St}_G(1)$ *is such that* $\psi_1(g) = (g_0, g_1)$ *with* $g_0 \in H^{(ab)^{\frac{q-1}{2}}}$ *and* $g_1 \in H$*, then* $g \in \mathrm{St}_H(1)$.

Proof. By Proposition 6.3.8, there exists $h \in \mathrm{St}_H(1)$ such that $\psi_1(h) = (g_0, h_1)$ for some $h_1 \in H$. Hence,

$$\psi_1(gh^{-1}) = (1, g_1 h_1^{-1}).$$

By Proposition 6.3.9, this means that $gh^{-1} \in \mathrm{St}_H(1)$, so $g \in \mathrm{St}_H(1)$. □

Maximality of $H(q)$

We are now almost ready to show that if q is an odd prime, then $H(q)$ is maximal in G. The argument will rely heavily on the length contraction property of Šunić groups (see Proposition 2.9.12).

To prove maximality, we will also need to study another quantity associated to elements of $\mathrm{St}_G(1)$. It is essentially the length of elements obtained by ignoring prefixes and suffixes in $\langle a, b\rangle$.

Notation 6.3.11. For any $g \in G$, we will write $l(g)$ for the word norm of g with respect to the generating set $A \cup B$. The reason for this somewhat unusual notation is that we will later need to consider the word pseudonorm on G defined in Example 3.1.11, so we will need to have clearly different notation for both norms.

Definition 6.3.12. For any $g \in \mathrm{St}_G(1)$, we define the *essential length* of g as

$$\lambda(g) = \min\left\{l(g') \mid g = \gamma g' \delta, \quad \gamma, \delta \in \mathrm{St}_{\langle a,b\rangle}(1), g' \in \mathrm{St}_G(1)\right\}.$$

Lemma 6.3.13. *If a subgroup $Q \leq G$ contains $H = H(q)$ properly, then there exist $n \in \mathbb{N}$ and $s \in \mathrm{St}_G(1) \setminus H$ with $\lambda(s) \leq 3$ such that $\langle s, H\rangle \leq Q_{\mathbf{1}^n}$ (recall that by Definition 2.6.25, $Q_{\mathbf{1}^n} = \varphi_{\mathbf{1}^n}(\mathrm{St}_Q(\mathbf{1}^n))$).*

Proof. By assumption, $Q \neq H$, so there exists some $g \in Q \setminus H$. Replacing g by $g(ab)^q$ if necessary, we may assume that $g \in \mathrm{St}_Q(1)$.

Let $\gamma, \delta \in \mathrm{St}_{\langle a,b\rangle}(1)$ and $g' \in \mathrm{St}_G(1)$ be such that $g = \gamma g' \delta$ and $\lambda(g) = l(g')$. We have

$$\psi_1(g) = (\gamma_0 g'_0 \delta_0, \gamma_1 g'_1 \delta_1)$$

where $\psi_1(\gamma) = (\gamma_0, \gamma_1), \psi_1(g') = (g'_0, g'_1), \psi_1(\delta) = (\delta_0, \delta_1)$. Note that $\gamma_0, \gamma_1, \delta_0, \delta_1 \in \langle a, b\rangle$ as $\delta, \gamma \in \mathrm{St}_{\langle a,b\rangle}(1)$.

If $\gamma_1 g'_1 \delta_1 \in H$ then, Corollary 6.3.10 implies that $(ab)^{\frac{q-1}{2}} \gamma_0 g'_0 \delta_0 (ba)^{\frac{q-1}{2}} \notin H$. Thus, replacing g by $g^{(ab)^q b}$ if necessary, we may assume that $\varphi_{\mathbf{1}}(g) = \gamma_1 g'_1 \delta_1 \notin H$.

If $\varphi_{\mathbf{1}}(g) \notin \mathrm{St}_G(1)$ then $\varphi_{\mathbf{1}}(g(ab)^{2q}) = \varphi_{\mathbf{1}}(g)(ab)^q \in \mathrm{St}_G(1)$ so, replacing g by $g(ab)^{2q}$ we can suppose that $\varphi_{\mathbf{1}}(g) = \gamma_1 g'_1 \delta_1 \in \mathrm{St}_G(1)$.

If $\gamma_1 \notin \mathrm{St}(1)$ then $\varphi_{\mathbf{1}}((ab)^{2q} g (ba)^{2q}) = (ab)^q \gamma_1 g'_1 \delta_1 (ba)^q$ with $(ab)^q \gamma_1 \in \mathrm{St}_{\langle a,b\rangle}(1)$. So, replacing g by $(ab)^{2q} g (ba)^{2q}$ if needed, we have

$$\varphi_{\mathbf{1}}(g) = \gamma_1 g'_1 \delta_1 \in \mathrm{St}_G(1) \setminus H \text{ with } \gamma_1, \delta_1 \in \langle a, b\rangle, \text{ and } \gamma_1 \in \mathrm{St}(1).$$

Now, if $\delta_1 \in \mathrm{St}(1)$ then $\lambda(\varphi_{\mathbf{1}}(g)) \leq l(g'_1)$. Otherwise, $g'_1 a \in \mathrm{St}_G(1)$ and $a\delta_1 \in \mathrm{St}_{\langle a,b\rangle}(1)$ so $\lambda(\varphi_{\mathbf{1}}(g)) \leq l(g'_1) + 1$. By Proposition 2.9.12, we have that $l(g'_1) \leq \frac{l(g')+1}{2}$, and $l(g') = \lambda(g)$ by construction. Hence

$$\lambda(\varphi_{\mathbf{1}}(g)) \leq \frac{\lambda(g) + 3}{2}.$$

By repeating this procedure (which we can do thanks to Proposition 6.3.8), we conclude by induction that there exist some $n \in \mathbb{N}$ and $y \in \mathrm{St}_Q(\mathbf{1}^n)$ such that $s = \varphi_{\mathbf{1}^n}(y) \notin H$ and $\lambda(s) \leq 3$. □

Lemma 6.3.14. *If a subgroup $Q \leq G$ contains $H = H(q)$ properly, then there exists $m \in \mathbb{N}$ such that $Q_{\mathbf{1}^m} \geq \langle (ab)^r, B\rangle = H(r)$ for some proper divisor r of q.*

Proof. By Lemma 6.3.13, there exist $n \in \mathbb{N}$ and $s \in \mathrm{St}_G(1) \setminus H$ with $\lambda(s) \leq 3$ such that $\langle s, H\rangle \leq Q_{\mathbf{1}^n}$. Thus, it suffices to show the result for $Q = \langle g, H\rangle$ for some $g \in \mathrm{St}_G(1)$ such that $\lambda(g) \leq 3$, which we do below in several cases.

The case $\lambda(g) = 0$. In this case, $g \in \langle a, b\rangle$. We can therefore assume that $g = (ab)^k$ for some $k \in \mathbb{Z}$ (multiplying on the left or on the right by $b \in H$ if necessary). Since k cannot be a multiple of q, there exist $i, j \in \mathbb{Z}$ such that $ik + jq = r$, where r is the greatest common divisor of k and q. Hence $(ab)^r = (ab)^{ik}(ab)^{jq} \in \langle g, H\rangle$ and so $Q = \langle g, H\rangle \geq H(r)$.

The case $g = (ab)^{-k}x(ab)^k$ for $x \in B\setminus\{1, b\}$ and $k \in \mathbb{Z}$. Conjugating by an appropriate power of $(ab)^q$ if necessary, we can assume that k is a positive odd number. Note that k cannot be a multiple of q as $g \notin H$.

Then

$$\psi_1(g) = ((ab)^{\frac{k-1}{2}} a\rho(x)a(ba)^{\frac{k-1}{2}}, (ba)^{\frac{k-1}{2}} b\omega(x)b(ab)^{\frac{k-1}{2}}).$$

If $\omega(x) = a$, then the second coordinate in the above expression is $(ba)^k b$. Since $H \leq Q_{\mathbf{1}}$ by Proposition 6.3.8, we have that $Q_{\mathbf{1}}$ contains $(ab)^k$ and therefore also $H(r)$, where $r < q$ is the greatest common divisor of q and k, by the same argument as in the previous case. If $\omega(x) = 1$, then consider $(ba)^q g(ab)^q$ instead of g. Its image under $\varphi_{\mathbf{1}}$ is $(ba)^{\frac{q+k}{2}}\rho(x)(ab)^{\frac{q+k}{2}}$ where $\frac{q+k}{2}$ cannot be a multiple of q (this is guaranteed by Corollary 6.3.10). Since $H_{\mathbf{1}} = H$ by Proposition 6.3.8, we may take $(ba)^{\frac{q+k}{2}}\rho(x)(ab)^{\frac{q+k}{2}}$ as our new g and repeat this case. By Proposition 2.9.2, there exists some minimal $m \in \mathbb{N}^*$ such that $\omega(\rho^{m-1}(x)) = a$. Therefore, by repeating the above procedure $m-1$ times, we get that $Q_{\mathbf{1}^m} \geq H(r)$ for some proper divisor r of q.

The case $\lambda(g) = 1$. The fact that $g = \gamma g'\delta$ with $\gamma, \delta \in \mathrm{St}_{\langle a,b\rangle}(1)$ and $g' \in \mathrm{St}_G(1)$ with $l(g') = 1$ immediately implies that $g' \in B$, $\gamma = [a](ba)^{k_1'}[b]$ and $\delta = [b](ab)^{k_2'}[a]$ for some $k_1', k_2' \in \mathbb{N}$ (where the square brackets mean that an element might not be present). Multiplying γ by b on the left and δ by b on the right if necessary and using the fact that $\gamma, \delta \in \mathrm{St}_G(1)$, we can assume that $\gamma = (ba)^{2k_1}[b]$ and $\delta = [b](ab)^{2k_2}$ for some $k_1, k_2 \in \mathbb{N}$. Therefore,

$$g = (ba)^{2k_1}[b]g'[b](ab)^{2k_2} = (ba)^{2k_1}x(ab)^{2k_2} \tag{6.1}$$

for some $x \in B \setminus \{1, b\}$.

If $\omega(x) = a$ then consider $(ba)^q g(ab)^q \in Q$:

$$\begin{aligned}\varphi_{\mathbf{1}}((ba)^q g(ab)^q) &= (ba)^{\frac{q-1}{2}+k_1} b\omega(x)b(ab)^{\frac{q-1}{2}+k_2} \\ &= (ba)^{\frac{q-1}{2}+k_1} bab(ab)^{\frac{q-1}{2}+k_2} \\ &= (ba)^{\frac{q+1}{2}+k_1}(ba)^{\frac{q-1}{2}+k_2} b = (ba)^{q+k_1+k_2} b.\end{aligned}$$

As $H \leq Q_{\mathbf{1}}$ by Proposition 6.3.8, we get that $Q_{\mathbf{1}}$ contains $(ba)^{k_1+k_2}$.

If $k_1 + k_2$ is not a multiple of q, then $Q_{\mathbf{1}} \geq H(r)$ by a previously considered case, where r is the greatest common divisor of q and $k_1 + k_2$.

If $k_1 + k_2 = nq$ for some $n \in \mathbb{N}$ then $g = (ba)^{2nq-2k_2}x(ab)^{2k_2}$. Hence

$$\begin{aligned}b(ba)^{-2nq}g &= b(ba)^{-2k_2}x(ab)^{2k_2} \\ &= b(ab)^{2k_2}x(ab)^{2k_2} \\ &= (ba)^{2k_2}(bx)(ab)^{2k_2}.\end{aligned}$$

As $bx \in B \setminus \{1, b\}$, this is a previously considered case, so we know that there exists $m \in \mathbb{N}$ such that $Q_{\mathbf{1}^m} \geq H(r)$ for some proper divisor r of q.

If $\omega(x) = 1$, then $\varphi_{\mathbf{1}}(g) = (ba)^{k_1}\rho(x)(ab)^{k_2} \notin H$ can be made of the same form as (6.1) by multiplying on the left and right by an appropriate power of $(ab)^q$. We can therefore repeat the argument explained above. This will terminate after a finite number of steps as there is some $j \in \mathbb{N}$ such that $\omega(\rho^j(x)) = a$. This concludes the case $\lambda(g) = 1$.

The case $\lambda(g) = 2$. Elements in G of length 2 are of the form ax or xa for some $x \in B$. None of these elements are in $\mathrm{St}_G(1)$, so this case does not arise.

The case $\lambda(g) = 3$. If $l(g') = 3$ in the minimal decomposition $g = \gamma g' \delta$, then $g' = axa$ or $g' = yaz$ for some $x, y, z \in B \setminus \{1, b\}$. However, $yaz \notin \mathrm{St}_G(1)$, so this case is impossible. Hence, $g' = axa$, so $g = \gamma axa\delta$ with $\gamma, \delta \in \mathrm{St}_{\langle a,b\rangle}(1)$. We have

$$\begin{aligned}(ab)^{-q} g (ab)^q &= (ab)^{-q}\gamma axa\delta(ab)^q \\ &= ((ab)^{-q}\gamma a)x(a\delta(ab)^q) \\ &= \gamma' x \delta'\end{aligned}$$

with $\gamma' = (ab)^{-q}\gamma a$ and $\delta' = a\delta(ab)^q \in \mathrm{St}_{\langle a,b\rangle}(1)$. This means that $\lambda((ab)^{-q}g(ab)^q) \leq 1$, so from what we have already shown, we conclude that there exist $m \in \mathbb{N}$ and a proper divisor r of q such that $Q_{\mathbf{1}^m} \geq H(r)$. □

Lemma 6.3.15. *If a subgroup $Q \leq G$ contains $H = H(q)$ properly then there exists $n \in \mathbb{N}$ such that $Q_{\mathbf{1}^n} = H(t)$ for some proper divisor t of q.*

Proof. By Lemma 6.3.14, there exist $m \in \mathbb{N}$ and a proper divisor r of q such that $Q_{\mathbf{1}^m} \geq H(r)$. If $Q_{\mathbf{1}^m} = H(r)$, then we are done. Otherwise, by applying Lemma 6.3.14 to $Q_{\mathbf{1}^m}$, we find $m' \in \mathbb{N}$ and a proper divisor s of r such that $Q_{\mathbf{1}^{m+m'}} \geq H(s)$.

As q only has a finite number of divisors, repeating this procedure as often as necessary, we will find some $n \in \mathbb{N}$ such that either $Q_{\mathbf{1}^n} = H(t)$ for some proper divisor $t \geq 3$ of q, or $Q_{\mathbf{1}^n} \geq H(1)$. Since $H(1) = G$ and $Q_{\mathbf{1}^n} \leq G$, the latter case yields $Q_{\mathbf{1}^n} = H(1)$. □

We are now ready to prove the main theorem of this section.

Theorem 6.3.16. *For every odd prime q, the subgroup $H(q) < G$ is maximal.*

Proof. We know that G is a self-replicating branch group (see Proposition 2.9.17 and 2.9.18), and its action on X is clearly primitive (see Example 2.9.18). Therefore, by Corollary 5.4.4, for any proper subgroup $M < G$ that is dense in the profinite topology and for any $v \in X^*$, we must have that M_v is a proper dense subgroup of G.

Let us fix an odd prime q and denote $H(q)$ by H. By Theorem 6.2.7, we have that H is proper and dense in the profinite topology. Since H is dense, the same must be true of $\langle g, H\rangle$ for any $g \in G \setminus H$.

Let us fix an arbitrary element $g \in G \setminus H$. By Lemma 6.3.15, there exists $n \in \mathbb{N}$ such that $(\langle g, H\rangle)_{\mathbf{1}^n} = H(t)$, where t is a proper divisor of q. Since

q is prime, we get $(\langle g, H\rangle)_{\mathbf{1}^n} = H(1) = G$. Hence, by Corollary 5.4.4, $\langle g, H\rangle$ cannot be proper. As this is valid for every $g \in G \setminus H$, we conclude that H is maximal. □

6.4 Finding all maximal subgroups of infinite index

In this section, we will show that for any $G \in \widetilde{\mathcal{G}}_{2,m}$ and any odd prime q, the subgroups $H(q) = \langle (ab)^q, B\rangle$, which were shown to be maximal in Section 6.3, are the only maximal subgroups of infinite index, up to conjugation. We will then briefly describe maximal subgroups of finite index and thus obtain a complete description of every maximal subgroup of G.

In order to prove that every maximal subgroup of infinite index is a conjugate of some $H(q)$, we will need many auxiliary results regarding maximal subgroups of infinite index of Šunić groups. Please note that we use here the same notation as in the previous section.

We begin by showing that maximal subgroups of infinite index are uniquely determined by their projections on some level.

Lemma 6.4.1. *Let $M < G$ be a maximal subgroup of infinite index. Then, for all $n \in \mathbb{N}$, we have*

$$\mathrm{St}_M(n) = \bigcap_{w \in X^n} \varphi_w^{-1}(M_w)$$

where we see φ_w as a map from $\mathrm{St}_G(w)$ to G, so that $\varphi_w^{-1}(M_w) \le \mathrm{St}_G(w)$.

Proof. Let us fix $n \in \mathbb{N}$ and write $J = \bigcap_{w \in X^n} \varphi_w^{-1}(M_w)$. Then, we clearly have $\mathrm{St}_M(n) \le J \le \mathrm{St}_G(n)$. Suppose for a contradiction that $J \neq \mathrm{St}_M(n)$. This means that there must exist $g \in J \setminus \mathrm{St}_M(n)$. Notice that we must have $g \notin M$, since $g \in \mathrm{St}_G(n)$. Therefore, by the maximality of M, we have $\langle M, g\rangle = G$. As G is self-replicating, this implies that $(\langle M, g\rangle)_w = G$ for all $w \in X^n$. We will obtain a contradiction by showing that $\varphi_w(\alpha) \in M_w$ for all $w \in X^n$ and all $\alpha \in \mathrm{St}(w) \cap \langle M, g\rangle$. Let us fix such w and α. As $\alpha \in \langle M, g\rangle$, we can write

$$\alpha = m_1 g^{\epsilon_1} m_2 \cdots m_k g^{\epsilon_k} = (\mu_1 g^{\epsilon_1} \mu_1^{-1})(\mu_2 g^{\epsilon_2} \mu_2^{-1}) \cdots (\mu_k g^{\epsilon_k} \mu_k^{-1})\mu_k$$

for some $k \in \mathbb{N}$, $m_i \in M$ and $\epsilon_i \in \{-1, 0, 1\}$, where $\mu_i = m_1 \cdots m_i$ for $i = 1, \dots, k$. Notice that since $\alpha \in \mathrm{St}(w)$ and $g \in \mathrm{St}(n)$, we must have $\mu_k \in \mathrm{St}_M(w)$. As $\mu_k \in M$, this implies that $\varphi_w(\mu_k) \in M_w$. If we can prove that $\varphi_w(m^{-1} g m) \in M_w$ for all $m \in M_w$, then we will get that $\varphi_w(\alpha)$ is a product of elements of M_w, as required.

To show this, let us fix $m \in M$ and let us write $u = m^{-1} \cdot w$. It follows from the definition of J that there exists $h \in \mathrm{St}_M(u)$ such that $\varphi_u(g) = \varphi_u(h)$. If we write $m = \tau\mu$ for some $\tau \in \wr^n \mathrm{Sym}(X)$ and $\mu \in \mathrm{St}(n)$, we see that $mgm^{-1} = \tau\mu g \mu^{-1}\tau^{-1}$. As $\tau^{-1} w = u$, we have

$$\varphi_w(mgm^{-1}) = \varphi_u(\mu)\varphi_u(g)\varphi_u(\mu)^{-1} = \varphi_u(\mu)\varphi_u(h)\varphi_u(\mu)^{-1} = \varphi_w(mhm^{-1}).$$

Notice that since $h \in \mathrm{St}_M(u)$ and $(\mathrm{St}_M(u))^m = \mathrm{St}_M(w)$, we get$\varphi_w(m^{-1} h m) \in M_w$.

We have thus proved that $(\langle M, g\rangle)_w = M_w$. As mentioned above, by the maximality of M, we have $\langle M, g\rangle = G$, which implies that $M_w = G_w = G$.

This is a contradiction, since M must be a proper subgroup of G that is dense in the profinite topology by Proposition 5.2.2, and it follows from Theorem 5.4.3 (or, more precisely, from Corollary 5.4.4) that M_w must then be a proper subgroup of G. □

Lemma 6.4.2. *Let $L, M < G$ be two maximal subgroups of infinite index. If there exists $n \in \mathbb{N}$ such that $L_w = M_w$ for all $w \in X^n$, then $L = M$.*

Proof. Let us set $S = \bigcap_{w\in X^n} \varphi_w^{-1}(M_w)$, where once again, we restrict φ_w to a map between $\mathrm{St}_G(w)$ and G, so that the $\varphi_w(M_w) \leq \mathrm{St}_G(w)$. By Lemma 6.4.1, we have that $S = \mathrm{St}_L(n) = \mathrm{St}_M(n)$. In particular, this implies that S is normal both in L and M. However, it cannot be normal in G, since G is just infinite and L, M are subgroups of infinite index. This means that the normaliser $N_G(S)$ of S in G must be a proper subgroup of G containing both L and M. As these subgroups are maximal, we must have $L = N_G(S) = M$. □

By Proposition 6.3.8, all projections of $H(q)$ are conjugates of $H(q)$. We will now show that conversely, given a collection of conjugates of $H(q)$, there is a conjugate of $H(q)$ whose projections are precisely this collection.

Lemma 6.4.3. *For any odd prime q and any $g \in G$, there exist $s \in \mathrm{St}_G(1)$ and $h_0, h_1 \in H(q)$ such that $s = (gh_0, h_1)$.*

Proof. For any $g \in G$, we can find $s_1 \in \mathrm{St}_G(1)$ such that $s_1 = (g, y)$ with $y \in \langle a, b\rangle$ by writing g as a word in $a \cup B \setminus \{1\}$ and replacing each instance of a by b and each $x \in B \setminus \{1\}$ by $a\rho^{-1}(x)a$.

Since $\langle a, b\rangle \cong D_\infty$, there exists $l \in \mathbb{Z}$ such that either $y = (ab)^l$ or $y = (ab)^l a$. Because q is coprime to 4, there exist $m, n \in \mathbb{Z}$ such that $4m + l = qn$. Let us set

$$s_2 = \begin{cases} (acab)^{4m} = ((da)^{4m}, (ab)^{4m}) = (1, (ab)^{4m}) & \text{if } y = (ab)^l \\ aba(acab)^{4m} = (b, a(ab)^{4m}) & \text{if } y = (ab)^l a \end{cases}.$$

We then have $s = s_1 s_2 = \begin{cases} (g, y(ab)^{4m}) = (g, (ab)^{qn}) & \text{if } y = (ab)^l \\ (gb, ya(ab)^{4m}) = (gb, (ab)^{qn}) & \text{if } y = (ab)^l a. \end{cases}$ □

Lemma 6.4.4. *Let q be an odd prime and let us write $H = H(q)$. Then, for any $g_0, g_1 \in G$, there exists $s \in G$ such that*

$$\varphi_{\mathbf{0}}(\mathrm{St}_{H^s}(1)) = H^{g_0} \text{ and } \varphi_{\mathbf{1}}(\mathrm{St}_{H^s}(1)) = H^{g_1}.$$

Proof. Since G is self-replicating, there exist $s_1 \in \mathrm{St}_G(1)$ and $y \in G$ such that $s_1 = (y, g_1)$. Proposition 6.3.8 states that $\varphi_{\mathbf{0}}(\mathrm{St}_H(1)) = H^{(ab)^{(q-1)/2}}$ and $\varphi_{\mathbf{1}}(\mathrm{St}_H(1)) = H$. Applying Lemma 6.4.3 to $(ba)^{(q-1)/2}$ and to yg_0^{-1}, we obtain $s_2, s_3 \in \mathrm{St}_G(1)$ and $h_0, h_1, k_0, k_1 \in H$ such that $s_2 = ((ba)^{(q-1)/2}h_0, h_1)$ and $s_3 = (yg_0^{-1}k_0, k_1)$. Let us set

$$s_4 = s_2 s_3^{-1} = ((ba)^{(q-1)/2} h_0 k_0^{-1} g_0 y^{-1}, h_1 k_1^{-1})$$

and

$$s = s_4 s_1 = ((ba)^{(q-1)/2} h_0 k_0^{-1} g_0, h_1 k_1^{-1} g_1).$$

We then have

$$\varphi_{\mathbf{0}}(\mathrm{St}_{H^s}(1)) = H^{h_0 k_0^{-1} g_0} = H^{g_0}$$

and

$$\varphi_{\mathbf{1}}(\mathrm{St}_{H^s}(1)) = H^{h_1 k_1^{-1} g_1} = H^{g_1}$$

since $h_0, h_1, k_0, k_1 \in H$, which concludes the proof. □

Lemma 6.4.5. *Let q be an odd prime and let us write $H = H(q)$. Then, for every $n \in \mathbb{N}$ and any set $\{g_v\}_{v \in X^n} \subset G$, there exists $s \in G$ such that $(H^s)_v = H^{g_v}$.*

Proof. We will proceed by induction on n. The case $n = 0$ is trivial, and the case $n = 1$ was proved in Lemma 6.4.4. Let us now suppose that the claim true for some $n \geq 1$ and let us show that it must then also hold for $n + 1$.

Let $\{g_v\}_{v \in X^{n+1}} \subset G$ be a subset of G, and let us write each $v \in X^{n+1}$ as $v = \boldsymbol{i}w$ with $\boldsymbol{i} \in X$ and $w \in X^n$. For each $\boldsymbol{i} \in X$, the inductive hypothesis implies that there exists $t_{\boldsymbol{i}} \in G$ such that for each $w \in X^n$, $(H^{t_{\boldsymbol{i}}})_w = H^{g_{\boldsymbol{i}w}}$. Lemma 6.4.4 then yields some $s \in G$ such that $(H^s)_{\boldsymbol{i}} = H^{t_{\boldsymbol{i}}}$ for each $\boldsymbol{i} \in X$. Thus, for each $v \in X^{n+1}$, we obtain, $(H^s)_v = ((H^s)_{\boldsymbol{i}})_w = (H^{t_{\boldsymbol{i}}})_w = H^{g_{\boldsymbol{i}w}} = H^{g_v}$. □

Now, the last remaining step in proving that every maximal subgroup of infinite index is a conjugate of some $H(q)$ is to show that such a subgroup must necessarily project to some $H(q)$. In order to do this, we will show in Proposition 6.4.15 that for any proper and dense subgroup $M < G$, there exists $v \in X^*$ and an odd $l > 1$ such that the projection M_v is equal to $H(l)$. The proof of this result uses similar techniques to the ones used by Pervova to study maximal subgroups of the Grigorchuk group in [74]. More precisely, we will study the length of $\varphi_{\mathbf{1}^n}((baz)^{2^n})$ for some $z \in G'$ and show that it generally reduces as n grows. However, unlike in the case of the Grigorchuk group, this sequence might eventually stabilise to something that is not a generator of G, for example if we started with $(ab)^k$ (see Lemma 6.4.13). However, we will show in Lemma 6.4.10 that this is essentially the only possibility. Thus, we will be able to show in Lemma 6.4.11 that any dense subgroup must project to $(ab)^k$ for some odd k. From there, once again using contraction properties of G, we will manage to show in Proposition 6.4.15 that any proper dense subgroup must project to some $H(l)$ for some odd l. We will then finally show in Lemma 6.4.16 that if the proper dense subgroup M is maximal, then this q is prime.

Lemma 6.4.6. *If $z \in G'$, then $\varphi_{\mathbf{0}}(z) \equiv \varphi_{\mathbf{1}}(z) \equiv y$ modulo G', where $y \in B_1 \cup abB_1$.*

Proof. As G' is generated by conjugates of $[a, x]$ with $x \in B$, and since

$$\psi_1([a, x]) = (\rho(x)\omega(x), \omega(x)\rho(x)),$$

we have $\varphi_{\mathbf{0}}([a, x]) \equiv \varphi_{\mathbf{1}}([a, x])$ modulo G'. If $x \in B_0$, then $\varphi_{\mathbf{1}}([a, x]) = \rho(x) \in B_1$. If $x \notin B_0$, then $x' = bx \in B_0$ and $x = bx'$, so $\varphi_{\mathbf{1}}([a, x]) = \varphi_{\mathbf{1}}([a, bx']) = ab\rho(x') \in abB_1$.

Since conjugating $[a, x]$ by an element of G conjugates and possibly permutes the projections $\varphi_{\mathbf{0}}([a, x])$ and $\varphi_{\mathbf{1}}([a, x])$, the result is true for the generators of G'. Hence, since $(B_1 \cup abB_1)\, G'$ is a subgroup of G/G', the result is also true for any $z \in G'$. □

Definition 6.4.7. We define $\Theta\colon G' \to G'$ by $\Theta(z) = a\varphi_{\mathbf{0}}(z)a\varphi_{\mathbf{1}}(z)$. This is well-defined by the previous lemma.

Notation 6.4.8. For $g \in G$, we will denote by $|g|$ the word pseudonorm on G defined in Example 3.1.11, which assigns length 0 to a and length 1 to every non-trivial element of B.

Remark 6.4.9. The advantage of considering this word pseudonorm is that with this pseudonorm, G is a non-ℓ_1-expanding self-similar group (see Definition 3.1.7). In other words, we have $|g| \geq |\varphi_{\mathbf{0}}(g)| + |\varphi_{\mathbf{1}}(g)|$ for all $g \in G$. In particular, for all $z \in G'$, we have $|\Theta(z)| \leq |z|$.

Lemma 6.4.10. *Let $z \in G'$ be such that $|\Theta^n(z)| = |z|$ for all $n \in \mathbb{N}$. If $|z| \geq 3$, then there exist $l \in \mathbb{N}$ and $x \in B$ such that*

$$z = axa(ba)^{2l}x.$$

Otherwise, either $z = 1$ or $|z| = 2$ and there exists $x \in B \setminus \{1\}$ such that $z = axax$ or $z = xaxa$.

Proof. If $|z| = 0$ then $z = 1$, because $z \in G'$. Proposition 2.9.16 implies that $|z| = 1$ cannot hold and that if $|z| = 2$ there must exist $x \in B \setminus \{1\}$ with $z = axax$ or $z = xaxa$.

Assume that $|z| = k$ with $k \geq 3$. Then, there exist $x_1, \dots x_k \in B \setminus \{1\}$ such that z can be written in one of four forms:

$$\begin{array}{lll} (1)\ x_1a\dots ax_k & \text{or} & (2)\ ax_1\dots x_ka \text{ if } k \text{ is odd,} \\ (3)\ ax_1\dots ax_k & \text{or} & (4)\ x_1a\dots x_ka \text{ if } k \text{ is even.} \end{array}$$

If z is of form (1) then

$$\psi_1(z) = (\omega(x_1)\rho(x_2)\dots\omega(x_k), \rho(x_1)\omega(x_2)\dots\rho(x_k))$$

so

$$\Theta(z) = a\omega(x_1)\rho(x_2)\dots\omega(x_k)a\rho(x_1)\omega(x_2)\dots\rho(x_k).$$

Since $|\Theta(z)| = |z|$, no cancellation occurs in the expression for $\Theta(z)$, except possibly the term $a\omega(x_1)$. This means that $\omega(x_k) = 1$ and $\omega(x_i) = a$ for $1 < i < k$. But then $a\omega(x_1) = 1$ because $\Theta(z) \in \mathrm{St}_G(1)$ and so a must occur an even number of times in the expression for $\Theta(z)$. Hence $\Theta(z)$ is of the same form as z, with last letter $\rho(x_k)$. Then, since $|\Theta^n(z)| = |z|$ for all n, by induction we obtain that $\rho^n(x_k) \in \ker\omega$ for all n, which implies that x_k is trivial. So form (1) cannot occur.

If z is of form (2), noticing that this is a conjugate of form (1) by a, we obtain that

$$\Theta(z) = a\rho(x_1)\omega(x_2)\dots\rho(x_k)a\omega(x_1)\rho(x_2)\dots\omega(x_k).$$

Similar arguments as in case (1) show that x_1 is trivial, another contradiction.

Form (3) yields

$$\Theta(z) = a\rho(x_1)\omega(x_2)\dots\omega(x_k)a\omega(x_1)\rho(x_2)\dots\rho(x_k).$$

Again, $|\Theta(z)| = |z|$ implies that there is no cancellation. Therefore, $\omega(x_k) = \omega(x_1)$ and $\omega(x_i) = a$ for all $1 < i < k$. Notice that $\Theta(z)$ is of the same form as z with first letter $\rho(x_1)$ and last letter $\rho(x_k)$. Since $|\Theta^n(z)| = |z|$ for all $n \in \mathbb{N}$, we obtain by induction that $\omega(\rho^n(x_i)) = a$ for $1 < i < k$ and $\omega(\rho^n(x_1)) = \omega(\rho^n(x_k))$ for all $n \in \mathbb{N}$. This means that $x_i = b$ for $1 < i < k$ and that $x_1 = x_k = x \in B \setminus \{1\}$. So $z = axa(ba)^{2l}x$ where $2l = k-2$, for some $x \in B \setminus \{1\}$.

If z is of form (4) then

$$\Theta(z) = a\omega(x_1)\rho(x_2)\ldots\rho(x_k)a\rho(x_1)\omega(x_2)\ldots\omega(x_k).$$

To avoid length reduction and cancellation we must have $\omega(x_i) = a$ for $1 < i < k$. This then implies, since $\Theta(z) \in \mathrm{St}_G(1)$, that we must have $\omega(x_1) = \omega(x_k)$. If $\omega(x_1) = \omega(x_k) = 1$ then $\Theta(z)$ is of form (3). The same argument as in the case (3) then implies that $\rho(x_1) = b$ and thus $\omega(x_1) = a$, a contradiction. So $\omega(x_1) = \omega(x_k) = a$ and therefore $\Theta(z)$ has the same form as z. Repeating the argument for $\Theta(z)$ instead of z, we obtain that $\omega(\rho(x_2)) = \omega(\rho(x_{k-1})) = a$ and $\omega(\rho(x_i)) = a$ also for all the other i. Hence, by induction, $\omega(\rho^n(x_i)) = a$ for $1 \leq i \leq k$ and all $n \in \mathbb{N}$. In other words, $z = (ba)^{2l}$ where $k = 2l$. □

Lemma 6.4.11. *Let $M \leq G$ be a dense subgroup in the profinite topology. Then, there exist an odd $k \in \mathbb{Z}$ and a vertex $v \in X^*$ such that the projection M_v contains $(ab)^k$.*

Proof. Since M is dense in the profinite topology and since G' is of finite index in G, we must have $MG' = G$. Therefore, there exists $z \in G'$ such that $baz \in M$. We have

$$\varphi_{\mathbf{1}}((baz)^2) = b\varphi_{\mathbf{0}}(z)a\varphi_{\mathbf{1}}(z) = ba\Theta(z),$$

so by induction $\varphi_{\mathbf{1}^n}((baz)^{2^n}) = ba\Theta^n(z)$ for all $n \in \mathbb{N}$. Since $\{|\Theta^n(z)|\}_{n\in\mathbb{N}}$ is a non-increasing sequence by Remark 6.4.9, it must eventually become constant. Hence, it follows from Lemma 6.4.10 that there exist $l \in \mathbb{N}$, $x \in B$ and a vertex $v \in X^*$ such that the projection M_v contains either $ba(axa(ba)^{2l}x)$ or $ba(xaxa)$. In the former case, we have

$$\begin{aligned}\varphi_{\mathbf{0}}((baaxa(ba)^{2l}x)^2) &= \varphi_{\mathbf{0}}(x(ba)^{4l+2}x)\\ &= \omega(x)(ab)^{2l+1}\omega(x) = \begin{cases}(ba)^{2l+1} & \text{if } \omega(x) = a\\ (ab)^{2l+1} & \text{if } \omega(x) = 1.\end{cases}\end{aligned}$$

If instead $ba(xaxa) \in M_v$, then either $x = b$, in which case we obtain the desired result, or $x \neq b$. In the latter case, we can assume without loss of generality that $\omega(x) = 1$. Indeed, if $\omega(x) = a$, then $\varphi_{\mathbf{1}}((baxaxa)^2) = ba\rho(x)a\rho(x)a$ so we can repeat this until $\rho^i(x) \in \ker\omega$, at which point $\varphi_{\mathbf{0}}((ba\rho^i(x)a\rho^i(x)a)^2) = ab$. □

Lemma 6.4.12. *Let $M \leq G$ be a dense subgroup in the profinite topology. Then, there exist $\beta \in B \setminus \{1\}$ and a vertex $v \in X^*$ such that $\beta \in M_v$.*

Proof. We will show that for all $x \in B \setminus \{1\}$ and all $z \in G'$, there exist some vertex $v \in X^*$ and some element $\beta \in B \setminus \{1\}$ such that $\beta \in \langle xz\rangle_v$. The result will then follow, since by the density of M, for any $x \in B \setminus \{1\}$, there exists some $z \in G'$ such that $xz \in M$. Recall from Proposition 2.9.12 that

$l(\varphi_v(g)) \leq (l(g)+1)/2$ for $g \in \mathrm{St}_G(1)$ and $v \in X$, where l is the standard word metric for the generating set $A \cup B$.

For $x \in B \setminus \{1\}$ and $z \in G'$, we have

$$\psi_1(xz) = (\omega(x)\varphi_{\mathbf{0}}(z), \rho(x)\varphi_{\mathbf{1}}(z)).$$

By Lemma 6.4.6, there exists $y \in B_1$ such that $\varphi_{\mathbf{0}}(z) \equiv_{G'} \varphi_{\mathbf{1}}(z)$ are both congruent modulo G' to y or aby. There are thus four cases:

(1)(i) $x \notin B_0$, $\varphi_{\mathbf{0}}(z) \equiv_{G'} y$	(1)(ii) $x \notin B_0$, $\varphi_{\mathbf{0}}(z) \equiv_{G'} aby$
(2)(i) $x \in B_0$, $\varphi_{\mathbf{0}}(z) \equiv_{G'} y$	(2)(ii) $x \in B_0$, $\varphi_{\mathbf{0}}(z) \equiv_{G'} aby$

Case (1)(i). Setting $x' = \rho(x)y$, we have $x' \neq 1$, since $\rho(x) \notin B_1$ and $y \in B_1$. Thus, there exists $z' \in G'$ such that $\varphi_{\mathbf{1}}(xz) = x'z'$ and $l(x'z') = l(\varphi_{\mathbf{1}}(xz)) \leq \frac{l(xz)+1}{2}$.

Case (1)(ii). We have $\omega(x) = a$ by assumption. Furthermore, note that $b \notin B_1$, since $b = (a, b)$, which means that $by \neq 1$. Consequently, there exists $z' \in G'$ such that $\varphi_{\mathbf{0}}(xz) = x'z'$, where $x' = \omega(x)aby = by \in B \setminus \{1\}$, and we have $l(x'z') = l(\varphi_{\mathbf{0}}(xz)) \leq \frac{l(xz)+1}{2}$.

Case (2)(i). We have $\varphi_{\mathbf{0}}(xz) = yz'$ and $\varphi_{\mathbf{1}}(xz) = \rho(x)yz''$ for some $z', z'' \in G'$. If $y \neq 1$, take the former, and take the latter otherwise. Either way, $l(\varphi_v(xz)) \leq \frac{l(xz)+1}{2}$ for $v \in X$.

Case (2)(ii). We have $\varphi_{\mathbf{0}}(xz) \equiv_{G'} aby$ and $\varphi_{\mathbf{1}}(xz) \equiv_{G'} aby\rho(x)$. If $y \neq 1$, take the former, and take the latter otherwise. In both cases, we have $\varphi_w(xz) = aby'z_2$ for some $w \in X$, $y' \in B_1 \setminus \{1\}$ and $z_2 \in G'$. If k is the smallest integer such that $\rho^k(y') \notin B_1$, then $\varphi_{\mathbf{0}^k}(\varphi_w(xz)^{2^k}) = b\rho^k(y')z'$ for some $z' \in G'$. Indeed, we have

$$\varphi_{\mathbf{0}}((aby'z_2)^2) = b\rho(y')\varphi_{\mathbf{1}}(z_2)a\omega(y')\varphi_{\mathbf{0}}(z_2) = a\omega(y')b\rho(y')z_3$$

for some $z_3 \in G'$ (noting that $\varphi_{\mathbf{1}}(z_2)\varphi_{\mathbf{0}}(z_2) \in G'$). If $\rho(y') \notin B_1$, then $y' \notin B_0$, which means that $\omega(y') = a$. Thus, $\varphi_{\mathbf{0}}((aby'z_2)^2) = b\rho(y')z_3$. Otherwise, we have $\varphi_{\mathbf{0}}((aby'z_2)^2) = ab\rho(y')z_3$ with $\rho(y') \in B_1 \setminus \{1\}$ and we can repeat this process. Thus, eventually, we will obtain $\varphi_{\mathbf{0}^k}(\varphi_w(xz)^{2^k}) = x'z'$, where $x' = b\rho^k(y') \in B \setminus \{1\}$.

Notice that $l(\varphi_{\mathbf{0}}(g^2)) \leq \frac{2l(g)+1}{2} \leq l(g)$ for all $g \in G$ (since l takes integer values). Thus, we get by induction that $l(\varphi_{\mathbf{0}^k}(g^{2^k})) \leq l(g)$. Therefore, $l(x'z') \leq l(\varphi_w(xz)) \leq \frac{l(xz)+1}{2}$.

Thus, in all cases, there is a vertex $v \in X^*$ and elements $x' \in B \setminus \{1\}$, $z' \in G'$ such that $x'z'$ belongs to $\langle xz \rangle_v$, with $l(x'z') \leq \frac{l(xz)+1}{2}$. By repeating this process as necessary, we can assume that $l(x'z') \leq 1$, which is equivalent to saying that $x'z' \in B$. Since $x' \neq 1$, we must have that $x'z' = x' \neq 1$ by Proposition 2.9.16, so x' is our desired β. □

Lemma 6.4.13. *Let $M \leq G$ be a subgroup that contains $(ab)^k$ for some $k \in \mathbb{Z}$. Then, $(ab)^k \in M_v$ for all $v \in X^*$.*

Proof. Because $\psi_1((ab)^{2k}) = ((ba)^k, (ab)^k)$, the result is true for the vertices of the first level, and hence for any vertex by induction. □

Lemma 6.4.14. *Let $M \leq G$ be a dense subgroup in the profinite topology. Then, there exist a vertex $v \in X^*$ and an odd $k \in \mathbb{N}$ such that M_v contains b and $(ab)^k$.*

Proof. It follows from Lemma 6.4.11 that there exist a vertex $w \in X^*$ and an odd $k \in \mathbb{N}$ such that $(ab)^k \in M_w$. By Proposition 5.4.1, since M is dense in G, we have that M_w is also dense in G. Lemma 6.4.12 then implies that there exist a vertex $w' \in X^*$ and an element $\beta \in B \setminus \{1\}$ such that $\beta \in (M_w)_{w'} = M_{ww'}$. Since $\varphi_{\mathbf{1}}(\beta) = \rho(\beta)$, we see by induction that there exist some $l \in \mathbb{N}$ and $x \in B \setminus B_0$ such that $x \in M_u$, where $u = ww'\mathbf{1}^l$.

As $(ab)^k \in M_w$, Lemma 6.4.13, guarantees that $(ab)^k \in M_u$. Then , we have

$$\varphi_{\mathbf{1}}((ab)^k x (ab)^k) = (ab)^{\frac{k-1}{2}} aab (ab)^{\frac{k-1}{2}} = (ab)^{\frac{k-1}{2}} (ba)^{\frac{k-1}{2}} b = b.$$

Hence, $b, (ab)^k \in M_v$, where $v = u\mathbf{1}$. □

Proposition 6.4.15. *Let $M < G$ be a dense subgroup in the profinite topology. Then, there exist a vertex $v \in X^*$ and an odd $l \in \mathbb{N}$ such that $M_v = \langle (ab)^l, B \rangle$.*

Proof. According to Lemma 6.4.14, there exist a vertex $w \in X^*$ and an odd $k \in \mathbb{N}$ such that $b, (ab)^k \in M_w$.

Let us write $B = \{1, \beta_1, \beta_2, \ldots \beta_{2^m-1}\}$. By Proposition 5.4.1, M_w is dense in G, since M is dense in G. Therefore, for every $\beta_i \in B$, there exists $z_i \in G'$ such that $\beta_i a z_i \in M_w$. This implies that $\rho(\beta_i) a \Theta(z_i) \in M_{w\mathbf{1}}$ for all $1 \leq i \leq 2^m - 1$. Indeed, we have

$$\rho(\beta_i) a \Theta(z_i) = \rho(\beta_i) \varphi_0(z_i) a \varphi_1(z_i) = \begin{cases} \varphi_{\mathbf{1}}((\beta_i a z_i)^2) & \text{if } \beta_i \notin \ker \omega \\ \varphi_{\mathbf{1}}(\beta_i a z_i b \beta_i a z_i) & \text{if } \beta_i \in \ker \omega. \end{cases}$$

Since $b \in M_{w\mathbf{1}}$, and $|\Theta(z)| \leq |z|$ for all $z \in G'$, we can repeat the above procedure until we reach some $N \in \mathbb{N}$ such that $|\Theta^n(z_i)| = |\Theta^N(z_i)|$ for all $n \geq N$ and all $i \in \{1, \ldots, 2^m - 1\}$. Lemma 6.4.10 then yields, for each $i \in \{1, \ldots, 2^m - 1\}$, elements $x_i \in B$ and $l_i \in \mathbb{N}$ such that $\beta_i a z_i' \in M_{w\mathbf{1}^N}$ where $z_i' = a x_i a (ba)^{2l_i} x_i$ or $x_i a x_i a$.

Since

$$\psi_1(z_i') = (\rho(x_i)(ab)^l \omega(x_i), \omega(x_i)(ba)^l \rho(x_i)) \text{ or } (\omega(x_i)\rho(x_i), \rho(x_i)\omega(x_i)),$$

we have

$$\begin{aligned} \rho(\beta_i) &= \rho(\beta_i)\rho(x_i)(ab)^l \omega(x_i)\omega(x_i)(ba)^l \rho(x_i) \text{ or } \rho(\beta_i)\omega(x_i)\rho(x_i)\rho(x_i)\omega(x_i) \\ &= \rho(\beta_i)\varphi_{\mathbf{0}}(z_i')\varphi_{\mathbf{1}}(z_i') = \begin{cases} \varphi_{\mathbf{1}}((\beta_i a z_i')^2) & \text{if } \beta_i \in \ker \omega \\ \varphi_{\mathbf{1}}(\beta_i a z_i' b \beta_i a z_i') & \text{if } \beta_i \notin \ker \omega. \end{cases} \end{aligned}$$

Thus $B \leq M_u$ where $u = w\mathbf{1}^{N+1}$. Moreover, $(ab)^k \in M_u$ by Lemma 6.4.13. Lemma 6.3.15 then implies that there exist a vertex $v \in X^*$ and an odd $l \in \mathbb{N}$ such that $M_v = \langle (ab)^l, B \rangle$. □

Lemma 6.4.16. *Let $M < G$ be a maximal subgroup of infinite index. Then, there exist $v \in X^*$ and an odd prime $q \in \mathbb{N}$ such that $M_v = \langle (ab)^q, B\rangle = H(q)$.*

Proof. Since a maximal subgroup of infinite index is dense in the profinite topology (see Proposition 5.2.2), by Proposition 6.4.15 there exist $v \in X^*$ and $l \in \mathbb{N}$ odd such that $M_v = \langle (ab)^l, B\rangle$. Furthermore, since M is maximal, M_v must also be maximal by Proposition 5.4.5. This implies that l is prime. Indeed, it is clear that if l' divides l, we have $H(l) \leq H(l')$, so we must have l prime (and we know that $H(l)$ is maximal if l is prime by Theorem 6.3.16). □

We are now finally ready to prove the main theorem of this section, which states that every maximal subgroup of infinite index is conjugate to some $H(q)$.

Theorem 6.4.17. *Let $G \in \widetilde{\mathcal{G}}_{2,m} \setminus \mathcal{G}_{2,1}$ and let $M < G$ be a maximal subgroup of infinite index. Then, there exist an odd prime q and an element $g \in G$ such that $M = H(q)^g$.*

Proof. By Lemma 6.4.16, there exist $n \in \mathbb{N}$, $v \in X^n$ and an odd prime q such that $M_v = H(q)$. To simplify the notation, let us write $H = H(q)$.

Since M is a maximal subgroup of infinite index, it is dense. As G acts spherically transitively on X^*, this implies that M also acts spherically transitively on X^*. In particular, M acts transitively on X^n. This means that, for each $w \in X^n$, there exists $m \in M$ taking w to v. Writing $m = \tau\mu$ with $\tau \in \wr^n \mathrm{Sym}(X)$ and $\mu \in \mathrm{St}(n)$, we have

$$M_w = \varphi_w(\mathrm{St}_M(w)) = \varphi_w(m^{-1}\,\mathrm{St}_M(v)m) = \mu_w^{-1} M_v \mu_w = H^{\mu_w}$$

where $\mu_w = \varphi_w(\mu) \in G$ by the self-similarity of G. By Lemma 6.4.5, there exists some $g \in G$ such that $(H^g)_w = H^{\mu_w}$ for each $w \in X^n$. Since H^g is also a maximal subgroup of infinite index of G, we must have $M = H^g$ by Lemma 6.4.2. □

6.5 Maximal subgroups of finite index

Thanks to Theorem 6.4.17, we now have a complete description of the maximal subgroups of infinite index of non-torsion Šunić groups acting on the binary rooted tree. In particular, thanks to Proposition 6.1.3, we see that they are all isomorphic to G. To complete our description of all maximal subgroups of G, it remains only to describe the maximal subgroups of finite index. This is the main purpose of this short section. Note that unlike maximal subgroups of infinite index, maximal subgroups of finite index are easy to describe for all Šunić groups. Therefore, in this section, we will once again consider general Šunić groups.

As the next proposition shows, except in the case of the infinite dihedral groups, maximal subgroups of Šunić groups are in bijection with subgroups of index p in $(\mathbb{Z}/p\mathbb{Z})^{m+1}$.

Proposition 6.5.1. *Let us fix a prime p and some $m \geq 1$, and let $G \in \mathcal{G}_{p,m}$ be a Šunić group different from the infinite dihedral group. Let $\pi_{ab}\colon A \times B \cong (\mathbb{Z}/p\mathbb{Z})^{m+1}$ the the canonical projection from G to its abelianisation (see Proposition 2.9.16). Then, a proper subgroup $M < G$ is a maximal subgroup of finite index of G if and only if $G' \leq M$ and $\pi_{ab}(M)$ is of index p in $A \times B$.*

In particular, there are exactly $\frac{p^{m+1}-1}{p-1}$ maximal subgroups of finite index in G.

Proof. If M is a proper subgroup containing G' and such that $\pi_{ab}(M)$ is of index p in $A \times B$, then it follows from the correspondence theorem that M is a maximal subgroup of finite index of G.

On the other hand, if M is a maximal subgroup of finite index, then it contains a normal subgroup $N \trianglelefteq G$ of finite index, which in turns contains $\mathrm{St}_G(n)$ for some $n \in \mathbb{N}$ by Proposition 4.1.7 and Lemma 2.7.5. Since $G/\mathrm{St}_G(n)$ is a finite p-group, every maximal subgroup of $G/\mathrm{St}_G(n)$ must be normal and thus of index p. Therefore, by the correspondence theorem, M is a normal subgroup of G of index p, and it follows that $G' \leq M$ and $\pi_{ab}(M)$ is of index p in $A \times B$.

To count the number of maximal subgroups of finite index of G, then, it suffices to count the number of subgroups of index p of $(\mathbb{Z}/p\mathbb{Z})^{m+1}$. Notice that we can see $(\mathbb{Z}/p\mathbb{Z})^{m+1}$ as a vector space over the field $\mathbb{Z}/p\mathbb{Z}$. A subgroup of index p then corresponds to a codimension 1 subspace. If we fix an inner product on this vector space, we can associate to each non-trivial $g \in (\mathbb{Z}/p\mathbb{Z})^{m+1}$ a unique subgroup of index p, namely the orthogonal complement to g. Notice that given two non-trivial elements $g_1, g_2 \in (\mathbb{Z}/p\mathbb{Z})^{m+1}$, the orthogonal complement of g_1 is the same as the orthogonal complement of g_2 if and only if g_1 and g_2 are colinear, or in other words, if and only if g_2 belongs to the subgroup generated by g_1. As there are $p^{m+1}-1$ non-trivial elements in this group and since each cyclic subgroup contains $p-1$ non-trivial elements, there must be $\frac{p^{m+1}-1}{p-1}$ different subgroups of index p. □

We have thus described every maximal subgroups of finite index of every Šunić group, with the exception of the infinite dihedral group, whose subgroups are very well-understood.

6.6 Open questions

In this section, we briefly describe a few open questions that follow naturally from what was discussed in this chapter.

In the proof of Theorem 6.2.7, we have shown that the subgroups $H(q)$ of a Šunić group $G \in \widetilde{\mathcal{G}}_{2,m}$ with $m \geq 2$ are proper for any odd $q \geq 3$ by showing that the orbit $H(q) \cdot \tilde{\mathbf{1}}$ of the point $\tilde{\mathbf{1}}$ under the action of $H(q)$ is a proper subset of the orbit of the same point under the action of G. In particular, this implies that $H(q)$ is contained in the stabiliser of the set $H(q) \cdot \tilde{\mathbf{1}}$. Now, when q is prime, we have shown in Theorem 6.3.16 that $H(q)$ is maximal. This immediately implies that $H(q)$ is the stabiliser of the set $H(q) \cdot \tilde{\mathbf{1}}$. Next, in Theorem 6.4.17, we showed that every maximal subgroup of infinite index of G is a conjugate of some $H(q)$, which means, in conjunction with our previous observation, that every maximal subgroup of infinite index of G is in fact the stabiliser of some subset of the boundary of the tree. One might wonder if this must necessarily be the case for any branch group.

Question 6.6.1. Is every maximal subgroup of infinite index of a branch group the stabiliser of some set on the boundary? If not in general, are there some classes of branch groups for which this holds?

A positive answer to this question would be very interesting, as it would suggest more geometric approaches to the study of maximal subgroups of infinite index in branch groups.

One might also wonder about the index of the maximal subgroups of the torsion Šunić groups. The only group in this class which was covered by Pervova's theorem is the Grigorchuk group. We expect every torsion Šunić group to belong to the class $\mathcal{MF}$, but so far, a proof remains out of reach.

Question 6.6.2. Does every periodic Šunić group belong to the class $\mathcal{MF}$?

More generally, one might ask if being a p-group is a sufficient condition for a branch group to belong to $\mathcal{MF}$.

Question 6.6.3. Does there exist a branch p-group with maximal subgroups of infinite index?

Chapter 7

Non-torsion branch groups in $\mathcal{MF}$

In Chapter 6, we showed that every non-torsion Šunić group acting on the binary rooted tree contains maximal subgroups of infinite index and therefore does not belong to the class $\mathcal{MF}$ (with the exception of the infinite dihedral group, which is not a branch group). It is thus natural to ask if the same is true of Šunić groups acting on trees of higher degree.

Our results in Chapter 6 also show that the assumption of periodicity was essential in Pervova's result on the index of maximal subgroups of torsion Grigorchuk 2-groups [73, 74]. One might naturally wonder if the same is true for the other families of groups studied in those papers, namely *GGS* and *EGS* groups, or for the generalisations given in [1] and [58]. In fact, as far as we are aware, every known examples of branch groups in $\mathcal{MF}$ are torsion. Thus, one could even wonder if a branch group belonging to the class $\mathcal{MF}$ must necessarily be periodic.

In this chapter, we provide a negative answer to those questions by showing that the generalised Fabrykowski-Gupta groups, a special family of non-torsion branch groups that are both Šunić groups and GGS groups, belong to the class $\mathcal{MF}$. To our knowledge, these are the first examples of non-torsion branch groups in $\mathcal{MF}$. They are also interesting as examples of branch groups that are not LERF (by Proposition 4.2.2) yet do not contain maximal subgroups of infinite index.

In Section 7.1, we give the definition of generalised Fabrykowski-Gupta groups and remark on some of their properties. Then, in Section 7.2, we prove that every maximal subgroup of these groups is of finite index. Finally, in Section 7.3, we discuss a few open questions.

Part of the results in this chapter were obtained in collaboration with Alejandra Garrido.

7.1 Generalised Fabrykowski-Gupta groups

The main goal of this chapter is to show that every Šunić group in a special family, that we will call *generalised Fabrykowski-Gupta groups*, belongs to the class $\mathcal{MF}$ of groups such that every maximal subgroup is of finite index. Before we go on to prove this result, let us first introduce generalised Fabrykowski-Gupta groups and the notation that we will use throughout this section.

Let $X = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$ be a finite alphabet with three letters. Recall from Section 2.8 that the Fabrykowski-Gupta group is the group $G = \langle a, b\rangle \leq \mathrm{Aut}(X^*)$ of automorphisms of the rooted tree X^* generated by two automorphisms a and b, where $a = (\mathbf{1}\ \mathbf{2}\ \mathbf{3})$ is the rooted automorphism cyclically permuting the first level, and where b is recursively defined by

$$b = (a, 1, b).$$

Let us set $A = \langle a\rangle$ and $B = \langle b\rangle$. It is easy to see that $A \cong B \cong \mathbb{Z}/3\mathbb{Z}$. Let $\omega\colon B \to B$ be the unique homomorphism sending b to a and let $\rho\colon B \to B$ be the identity map. We then have that $b = (\omega(b), 1, \rho(b))$, so G is the Šunić group associated with the maps ω and ρ. Under the notation of Section 2.9, the Fabrykowski-Gupta group is the group $G_{3,x-1}$.

This suggests a natural generalisation. We will call *generalised Fabrykowski-Gupta groups* the Šunić groups whose defining polynomial is $x - 1$. However, we will exclude the case of the binary rooted tree, since the group $G_{2,x-1}$ is the infinite dihedral group, which is not a branch group.

Definition 7.1.1. Let p be an odd prime and let $X = \{\mathbf{1}, \mathbf{2}, \dots, \mathbf{p}\}$ be an alphabet with p letters. We will call the Šunić group $G_{p,x-1}$ (see Definition 2.9.3) the *generalised Fabrykowski-Gupta group acting on* X^*, or the *generalised Fabrykowski-Gupta group of degree* p. More explicitly, the generalised Fabrykowski-Gupta group of degree p is the subgroup of $\mathrm{Aut}(X^*)$ generated by a and b, where $a = (\mathbf{1}\ \mathbf{2}\ \dots\ \mathbf{p})$ is the rooted automorphism acting cyclically on the first level, and where b is recursively defined as

$$b = (a, 1, \dots, 1, b).$$

☙

As generalised Fabrykowski-Gupta groups are a special class of Šunić groups, all the results of Section 2.9 hold for them. In particular, they are self-replicating regular branch groups over their commutator subgroup.

Let us set some notation for these groups.

Notation 7.1.2. For the rest of this section, unless otherwise specified, p will denote an odd prime, X will denote the alphabet $X = \{\mathbf{1}, \mathbf{2}, \dots, \mathbf{p}\}$ and $G = \langle a, b\rangle$ will denote the generalised Fabrykowski-Gupta group of degree p, where a and b are as in Definition 7.1.1.

Notice that unlike in Section 2.9, our alphabet goes from $\mathbf{1}$ to $\mathbf{p}$ instead of going from $\mathbf{0}$ to $\mathbf{p-1}$. The reason for this difference is that for $i \in \mathbb{Z}$, we will use the notation $b_i = a^i b a^{-i}$. Thanks to our choice, we then have $\varphi_{\mathbf{i}}(b_i) = b$. Please note that by convention, for $i \in \mathbb{Z}$, we will denote in bold the unique element $\mathbf{i} \in X$ of X that is equal to i modulo p.

We will denote by $|\cdot|\colon G \to \mathbb{N}$ the word pseudonorm of Example 3.1.11. More concretely, $|\cdot|$ is the pseudonorm induced by giving length 0 to a and length 1 to every non-trivial power of b.

If $N \trianglelefteq G$ is a normal subgroup of G and if $g_1, g_2 \in G$ are two elements belonging to the same coset by N, we will write $g_1 \equiv_N g_2$. ❧

Remark 7.1.3. For every $g \in G$ with $|g| = n$, there exist $i, i_1, i_2, \dots, i_n \in \{0, 1, \dots, p-1\}$ with $i_k \neq i_{k+1}$ and $j_1, \dots, j_n \in \{1, 2, \dots, p-1\}$ such that

$$g = a^i b_{i_1}^{j_1} b_{i_2}^{j_2} \dots b_{i_n}^{j_n}.$$

❧

Let us also make a very important remark about the contracting properties of the word pseudonorm.

Remark 7.1.4. Since G is non-ℓ_1-expanding, we know that if $g = \tau(g_1, \ldots, g_p)$, then $\sum_{i=1}^{p} |g_i| \leq |g|$. It is also follows easily from Remark 7.1.3 that we have

$$|g_k| \leq \frac{|g| + 1}{2}$$

for all $1 \leq k \leq p$. Indeed, if $|g| = n$, we can write $g = a^i b_{i_1}^{j_1} b_{i_2}^{j_2} \ldots b_{i_n}^{j_n}$ as in Remark 7.1.3, and the fact that $i_k \neq i_{k+1}$ for all k implies that each projection cannot be longer than $(n+1)/2$. Thus, the equivalent of Proposition 2.9.12 holds for this word pseudonorm. ❧

Please note that Remarks 7.1.3 and 7.1.4 hold more generally for any Šunić groups or even for any spinal groups. We just state them here in the context of generalised Fabrykowski-Gupta groups for convenience.

7.2 Maximal subgroups of generalised Fabrykowski-Gupta groups

Now that we have defined generalised Fabrykowski-Gupta groups and set the notation that we will use throughout this section, we can begin to prove that every maximal subgroup of a generalised Fabrykowski-Gupta group must be of finite index. The general strategy of the proof is similar to the one used by Pervova in [73] and [74] to study maximal subgroups of the Grigorchuk groups. More explicitly, we will show that no dense subgroup can be proper by exploiting Proposition 5.4.1, Theorem 5.4.3 and the length contraction properties stated in Remark 7.1.4.

In order to prove the result, we will first need several lemmas.

Lemma 7.2.1. *Let $g \in G'$ be an element of the derived subgroup of G and let $g_1, g_2, \ldots, g_p$ be such that $\psi_1(g) = (g_1, g_2, \ldots, g_p)$. Then, we have $g_1 g_2 \cdots g_n \in G'$.*

Proof. As $g \in G'$, it follows from Proposition 2.9.16 that we can write $g = b_{i_1}^{j_1} b_{i_2}^{j_2} \cdots b_{i_n}^{j_n}$ with $\sum_{k=1}^{n} j_k \equiv 0 \mod p$. For each $1 \leq k \leq n$, we have that $b_{i_k}^{j_k}$ adds b^{j_k} to g_{i_k} and a^{j_k} to b_{i_k+1} (where the indices are taken modulo p). Therefore, we have that

$$g_1 g_2 \cdots g_p \equiv_{G'} a^{\sum_{k=1}^{n} j_k} b^{\sum_{k=1}^{n} j_k} = 1.$$

□

Lemma 7.2.2. *Let $g \in G$ be such that $g \equiv_{G'} a^i b^j$, with $i, j \not\equiv 0 \mod p$, and let $g_1, g_2, \ldots, g_p \in G$ be such that $\psi_1(g) = a^i(g_1, g_2, \ldots, g_p)$. Then, for all $u \in X$, we have $\varphi_u(g^p) \equiv_{G'} a^j b^j$, with $|\varphi_u(g^p)| \leq \sum_{k=1}^{p} |g_k| \leq |g|$.*

Proof. By hypothesis, we can write $g = a^i b^j z$ for some $z \in G'$. Let us write $x = b^j z$, and let $z_1, z_2, \ldots, z_p \in G$ be such that $\psi_1(z) = (z_1, z_2, \ldots, z_p)$. Then, we have $\psi_1(x) = (g_1, \ldots, g_p) = (a^j z_1, z_2, \ldots, b^j z_p)$. Now, let us consider

$$g^p = (a^i x)^p = (a^i x a^{-i})(a^{2i} x a^{-2i}) \cdots (a^{pi} x a^{-pi}).$$

As i is different from 0 and p is prime, we have that every conjugate of x by a power of a appears exactly once in this product. Thus, for $u \in X$, we have that $\varphi_u(g^p) \equiv_{G'} g_1 g_2 \cdots g_p$. It follows from Lemma 7.2.1 that $\varphi_u(g^p) \equiv_{G'} a^j b^j$. Furthermore, since $\varphi_u(g^p)$ can be written as a product (in some specified order depending on i and u) of the elements g_i, we have

$$|\varphi_u(g^p)| \leq \sum_{k=1}^{n} |g_k| \leq |g|.$$

□

Lemma 7.2.3. *Let $g \in G$ be such that $g \equiv_{G'} b^{i_0}$ for some $i_0 \not\equiv 0 \mod p$. Then, exactly one of the two following cases is true.*

1. *There exists some $u \in X$ and some $j_0 \not\equiv 0 \mod p$ such that $\varphi_u(g) \equiv_{G'} b^{j_0}$.*
2. *For all $u \in X$, there exist $i_u, j_u \not\equiv 0 \mod p$ such that $\varphi_u(g) \equiv_{G'} a^{i_u} b^{j_u}$. Furthermore, there exists at least one $u \in X$ such that $i_u \not\equiv j_u \mod p$.*

Proof. As $g \equiv_{G'} b^{i_0}$, it follows from Proposition 2.9.16 that we can write

$$g = b_{l_1}^{j_1} b_{l_2}^{j_2} \cdots b_{l_n}^{j_n}$$

with $\sum_{k=1}^{n} j_k \equiv i_0 \mod p$. For every $l \in \{1, 2, \ldots, p\}$, we have

$$\varphi_u(b_l) = \begin{cases} b & \text{if } u = \mathbf{l} \\ a & \text{if } u = \mathbf{l+1} \\ 1 & \text{otherwise.} \end{cases}$$

Thus, we have that $\varphi_u(g) \equiv_{G'} a^{m_{u-1}} b^{m_u}$, where

$$m_u \equiv \sum_{k \text{ s.t. } \mathbf{l_k} = u} j_k \mod p$$

and all the indices are taken modulo p.

It is clear that cases 1 and 2 are mutually exclusive. Thus, we only need to show that if case 1 does not hold, then case 2 does. Let us now assume that case 1 does not hold.

This means that we have $\varphi_u(g) \not\equiv_{G'} b^{j_0}$ for every $j_0 \not\equiv 0 \mod p$ and for every $u \in X$. As $\varphi_u(g) \equiv_{G'} a^{m_{u-1}} b^{m_u}$, this is equivalent to saying that for every $u \in X$, if $m_{u-1} \equiv 0 \mod p$, then the same is true of m_u (where again, the indices are taken modulo p). This, in turn, implies that $m_u \not\equiv 0 \mod p$ for every $u \in X$. Indeed, if there existed one $u \in X$ such that $m_u \equiv 0 \mod p$, then we would have $m_{u+1} \equiv 0 \mod p$. By induction, this would imply that $m_u \equiv 0 \mod p$ for all $u \in X$. However, we have that

$$\sum_{u\in X} m_u \equiv \sum_{k=1}^{n} j_k \equiv i_0 \not\equiv 0 \mod p,$$

which means that we cannot have $m_u \equiv 0 \mod p$ for every $u \in X$.

Thus, we have shown that for all $u \in X$, we have $\varphi_u(g) \equiv_{G'} a^{m_{u-1}} b^{m_u}$ with $m_{u-1}, m_u \not\equiv 0 \mod p$. To show that we are in case 2, we now only need to show that there exists $u \in X$ such that $m_{u-1} \not\equiv m_u \mod p$. Let us assume that it is not the case. Then, we have $m_{u-1} \equiv m_u \mod p$ for all $u \in X$, which means that $m_u \equiv m_v \mod p$ for all $u, v \in X$. Thus, we have

$$\sum_{u\in X} m_u \equiv |X| m_u \equiv 0 \mod p,$$

a contradiction. This concludes the proof. □

The next proposition will show that for any dense subgroup $H \leq G$, we can find a vertex such that the projection of H to this vertex contains b.

Proposition 7.2.4. *Let $H \leq G$ be a dense subgroup with respect to the profinite topology on G. Then, there exists $v_0 \in X^*$ such that $b \in H_{v_0}$, where, as in Definition 2.6.25, $H_{v_0} = \varphi_{v_0}(\mathrm{St}_H(v_0))$.*

Proof. Let us consider the set

$$U = \left\{ g \in \bigsqcup_{v\in X^*} H_v \mid g \equiv_{G'} a^i b^i \text{ for some } i \not\equiv 0 \mod p \right\}.$$

Since H is a dense subgroup of G and G' is a normal subgroup of finite index of G, this set is not empty. Let $y \in U$ be an element of minimal length in U, meaning that for any $g \in U$, we have $|y| \leq |g|$ (as the length function takes values in $\mathbb{N}$, such an element exists). Let $w \in X^*$ be such that $y \in H_w$. By Proposition 5.4.1 (or, more precisely, by Corollary 5.4.4), H_w is a dense subgroup of G, and if we prove the result for H_w, we will also have proved it for H. Thus, without loss of generality, we can assume that w is the root of the rooted tree X^*, so that $H_w = H$, which we will do in order to simplify the notation.

Let us now consider the set

$$V = \left\{ g \in \bigsqcup_{v\in X^*} H_v \mid g \equiv_{G'} b^i \text{ for some } i \not\equiv 0 \mod p \right\}$$

and let $x \in V$ be an element of minimal length in V. Again, such an element exists as V is not empty by the density of H. Let $v \in X^*$ be such that $x \in H_v$. It follows from Lemma 7.2.2 and by the minimality of y that there exists $y' \in H_v$ such that $y' \equiv_{G'} y$ and $|y'| = |y|$. Thus, without loss of generality, we can assume that $x \in H$, replacing H by H_v and y by y' if necessary.

If $|x| = 1$, then $x = a^j b^i a^{-j}$ with $i \not\equiv 0 \mod p$. Thus, we have that $\varphi_{\mathtt{j}}(x) = b^i$. As i is invertible modulo p, we conclude that $b \in H_j$.

We will now show that $|x|$ must be equal to 1. The case $|x| = 0$ is of course impossible by Proposition 2.9.16, since that would imply that $x \in A = \langle a \rangle$, but we have $x \equiv_{G'} b^i$ for some $i \not\equiv 0 \mod p$. Let us now assume that $|x| > 1$. In this case, there can exist no $u \in X$ such that $\varphi_u(x) \equiv_{G'} b^j$ with $j \not\equiv 0 \mod p$. Indeed, if such a u existed, we would have $\varphi_u(x) \in V$, and by Remark 7.1.4, we would have

$$|\varphi_u(x)| \leq \frac{|x|+1}{2} < |x|$$

which would contradict the minimality of x. Thus, according to Lemma 7.2.3, for all $u \in X$, there exist $i_u, j_u \not\equiv 0 \mod p$ such that $\varphi_u(x) \equiv_{G'} a^{i_u} b^{j_u}$. Furthermore, there exists $u_0 \in X$ such that $i_{u_0} \not\equiv j_{u_0} \mod p$.

For $u \in X$, let us write $x_u = \varphi_u(x)$. By Lemma 7.2.2, we have $\varphi_v(x_u^p) \in U$ for all $v \in X$, with $|\varphi_v(x_u^p)| \leq |x_u|$. Thus, by the minimality of y, we must have $|x_u| \geq |y|$ for all $u \in X$.

Let us write $x' = x_{u_0}(\varphi_{u_0}(y^p))^{k_0}$, where $k_0 \in \mathbb{Z}$ is the unique integer such that $\frac{1-p}{2} \leq k_0 \leq \frac{p-1}{2}$ and $(\varphi_{u_0}(y^p))^{k_0} \equiv_{G'} a^{-i_u} b^{-i_u}$. We then have $x' \equiv_{G'} b^{j_u - i_u}$. As $i_u \not\equiv j_u \mod p$, this means that $x' \in V$. Thus, by the minimality of x, we must have $|x'| \geq |x|$.

On the other hand, since $x' = x_{u_0}(\varphi_{u_0}(y^p))^{k_0}$, we have

$$|x'| \leq |x_{u_0}| + |k_0||\varphi_{u_0}(y^p)| \leq |x_{u_0}| + \frac{p-1}{2}|y|.$$

Furthermore, as mentioned in Remark 7.1.4, we have $\sum_{u \in X} |x_u| \leq |x|$, so

$$|x_{u_0}| \leq |x| - \sum_{u \in X \setminus \{u_0\}} |x_u|.$$

As $|x_u| \geq |y|$ for all $u \in X$, we get $|x_{u_0}| \leq |x| - (p-1)|y|$. Therefore, we have

$$|x'| \leq |x| - (p-1)|y| + \frac{p-1}{2}|y| = |x| - \frac{p-1}{2}|y| < |x|$$

which is a contradiction to the minimality of x. We conclude that $|x|$ must be equal to 1 and the result is then proved. □

Now that we have shown that any dense subgroup admits a projection that contains b, we only need to show that we can also obtain a in the same projection to obtain G. This is what we will do in the next two lemmas.

Lemma 7.2.5. *Let $x \in G$ be an element of G. If there exist $u, v \in X$ such that $|\varphi_u(x)| + |\varphi_v(x)| = |x|$, then $|x| \leq 3$.*

Proof. We can write $x = a^{j_0} b_{i_1}^{j_1} \cdots b_{i_n}^{j_n}$, where $n = |x|$. Since $|\varphi_u(x)| + |\varphi_v(x)| = |x|$, we must have either $\mathbf{i_k} = u$ if k is odd and $\mathbf{i_k} = v$ if k is even, or $\mathbf{i_k} = v$ if k is odd and $\mathbf{i_k} = u$ if k is even. Indeed, otherwise, we could express u and v as products such that the number of b appearing in both of these products is less than n. Without loss of generality, we may assume that $\mathbf{i_k} = u$ if k is odd and $\mathbf{i_k} = v$ if k is even. For the rest of this proof, to simplify the notation, we will write b_u and b_v for b_{i_1} and b_{i_2}, respectively, so that we can write $x = a^{j_0} b_u^{j_1} b_v^{j_2} b_u^{j_3} \ldots b_{i_n}^{j_n}$.

If $n \geq 4$, then we have that $x = a^{j_0} x' x''$, where $x' = b_u^{j_1} b_v^{j_2} b_u^{j_3} b_v^{j_4}$ and $x'' = b_{i_5}^{j_5} \cdots b_{i_n}^{j_n}$. We have that $\varphi_u(x) = \varphi_u(x')\varphi_u(x'')$ and $\varphi_v(x) = \varphi_v(x')\varphi_v(x'')$.

Thus, if we can show that $|\varphi_u(x')|+|\varphi_v(x')| < |x'|$, this will imply that $|\varphi_u(x)|+|\varphi_v(x)| < |x|$. Indeed, we would then have

$$\begin{aligned}|\varphi_u(x)| + |\varphi_v(x)| &\leq |\varphi_u(x')| + |\varphi_u(x'')| + |\varphi_v(x')| + |\varphi_v(x'')| \\ &< |x'| + |x''| = n = |x|.\end{aligned}$$

Without loss of generality, let us assume that $u = \mathbf{1}$. If $v \neq \mathbf{p}$, then $\varphi_{\mathbf{1}}(b_v) = 1$ and thus $\varphi_u(x') = b^{j_1+j_3}$. Therefore, we have $|\varphi_u(x')| \leq 1$. As $|\varphi_v(x')| \leq 2$, we get that $|\varphi_u(x')| + |\varphi_v(x')| \leq 3 < 4 = |x'|$. If, on the other hand, we have $v = \mathbf{p}$, then we have $\varphi_{\mathbf{p}}(b_u) = 1$ and by the same argument, we find that $|\varphi_u(x')| + |\varphi_v(x')| \leq 3 < |x'|$. We conclude that if $|x| \geq 4$, it is impossible to have $|\varphi_u(x)| + |\varphi_v(x)| = |x|$. □

Lemma 7.2.6. *Let $H \leq G$ be a dense subgroup of G with respect to the profinite topology. Then, there exists $v \in X^*$ such that $H_v = G$.*

Proof. Let us set

$$U = \left\{ g \in \bigsqcup_{u \in X^*} H_u \mid g \equiv_{G'} a^i b^i \text{ for some } i \not\equiv 0 \mod p \right\}$$

and let $y \in U$ be an element of minimal length among the elements of U. Let $u \in X$ be such that $y \in H_u$. As H_u is dense in G by Proposition 5.4.1, we can assume without loss of generality (replacing H by H_u if necessary) that $y \in H$.

By Lemma 7.2.2 and the minimality of y, we conclude that for every u in T, there exists some $y_u \in H_u$ such that $y_u \in U$ and y_u is of minimal length among the elements of U.

By Proposition 7.2.4, there exists $v_0 \in T$ such that $b \in H_{v_0}$. Again, without loss of generality, we can assume that $b \in H$, replacing H by H_{v_0} if necessary.

Let us now consider $H_{\mathbf{1}}$. As $\psi_1(b) = (a, 1, \ldots, 1, b)$, we have that $a \in H_{\mathbf{1}}$. We also have $y_{\mathbf{1}} \in H_{\mathbf{1}}$ such that $y_{\mathbf{1}} \in U$ and is minimal. Let $x \in G$ and $i \in \mathbb{N}$ be such that $y_{\mathbf{1}} = a^i x$ with $x \equiv_{G'} b^i$. Notice that since $a \in H_{\mathbf{1}}$, we also have that $x \in H_{\mathbf{1}}$. If $|y_{\mathbf{1}}| = 1$, then we also have $|x| = 1$, and we see that we must therefore have $a, b \in H_{\mathbf{1}}$. In other words, we have $G = H_{\mathbf{1}}$ and the result holds in this case.

Let us now assume that $|y_{\mathbf{1}}| > 1$. As in the proof of Lemma 7.2.3, for each $u \in X$ there is $m_u \in \mathbb{N}$ such that $x_u := \varphi_u(x) \equiv a^{m_{u-\mathbf{1}}} b^{m_u}$ (where we define addition and subtraction on X modulo p).

There can exist no $u \in X$ such that $m_{u-\mathbf{1}}, m_u \not\equiv 0 \mod p$. Indeed, if such a u existed, we would have, by Lemma 7.2.2, $\varphi_{\mathbf{1}}(x_u^p) \in U$ with

$$|\varphi_{\mathbf{1}}(x_u^p)| \leq |x_u| \leq \frac{|x| + 1}{2} = \frac{|y_{\mathbf{1}}| + 1}{2} < |y_{\mathbf{1}}|,$$

a contradiction to the minimality of $y_{\mathbf{1}}$. However, by Lemma 7.2.3, there must exist some $u_0 \in X$ such that $m_{u_0} \not\equiv 0 \mod p$. We conclude that $x_{u_0} \equiv_{G'} b^{m_{u_0}}$ and $x_{u_0+\mathbf{1}} \equiv_{G'} a^{m_{u_0}}$.

As $a \in H_{\mathbf{1}}$, we have that $x_{u_0}, x_{u_0+\mathbf{1}} \in H_{\mathbf{11}}$. Thus, $x_{u_0} x_{u_0+\mathbf{1}} \in U$. As G is non-ℓ_1-expanding, we must have

$$|x_{u_0} x_{u_0+\mathbf{1}}| \leq |x_{u_0}| + |x_{u_0+\mathbf{1}}| \leq |x| = |y_{\mathbf{1}}|,$$

but by the minimality of $|y_{\mathbf{1}}|$, we must have $|x_{u_0}x_{u_0+\mathbf{1}}| \geq |y_{\mathbf{1}}|$, which implies that $|x_{u_0}| + |x_{u_0+\mathbf{1}}| = |x|$. By Lemma 7.2.5, this forces $|x| \leq 3$.

It thus follows from Remark 7.1.4 that we must have $|x_{u_0+\mathbf{1}}| \leq 2$. The case $|x_{u_0+\mathbf{1}}| = 0$ is impossible, since this would imply that $|x_{u_0}| = |x|$, which is only possible if $|x| = 1$. As $x_{u_0+\mathbf{1}} \equiv_{G'} a^{m_0}$, it follows from Proposition 2.9.16 that the case $|x_{u_0+\mathbf{1}}| = 1$ is also impossible, since $|x_{u_0+\mathbf{1}}|$ must be an even number.

Thus, the only remaining case is the case where $|x_{u_0+\mathbf{1}}| = 2$. In this case, we must also have $|x_{u_0}| = 1$ and $|x| = 3$. With these conditions, we must have that

$$x = b_{u_0+\mathbf{1}}^{j_1} b_{u_0}^{m_{u_0}} b_{u_0+\mathbf{1}}^{-j_1}$$

for some $j_1 \in \mathbb{N}$ such that $j_1 \not\equiv 0 \mod p$. In particular, we have $x_{u_0} = b^{m_{u_0}}$ and $x_{u_0+\mathbf{1}} = b^{j_1} a^{m_{u_0}} b^{-j_1}$. As $x_{u_0}, x_{u_0+\mathbf{1}} \in H_{11}$, we conclude that $a, b \in H_{11}$ and thus that $G = H_{11}$. □

Theorem 7.2.7. *Let G be a generalised Fabrykowski-Gupta group of degree p, where p is an odd prime. Then, every maximal subgroup of G is of finite index in G, so G belongs to the class $\mathcal{MF}$.*

Proof. It follows from Lemma 7.2.6 and Theorem 5.4.3 (or, more precisely, from Corollary 5.4.4) that there are no proper dense subgroups of G in the profinite topology. Thus, by Proposition 5.2.2, G contains no maximal subgroup of infinite index. □

7.3 Open questions

In this chapter, we have shown that, in contrast with the case of the binary rooted tree, there are Šunić groups that are branch, not torsion, but still belong to the class $\mathcal{MF}$. One might wonder if the same is true of every Šunić group.

Question 7.3.1. Is every non-torsion Šunić group acting on a p-regular rooted tree necessarily in $\mathcal{MF}$ if p is an odd prime?

One might notice that for the binary rooted tree, the Šunić group corresponding to the polynomial $x - 1$ is the infinite dihedral group, which does belong to the class $\mathcal{MF}$. Thus, one might think that groups corresponding to the polynomial $x - 1$ are somewhat exceptional in the family of Šunić groups and that their behaviour might not be representative of the other groups in the family. However, the infinite dihedral group is of a very different nature from the generalised Fabrykowski-Gupta groups, since it is not a branch group. Furthermore, it contains maximal subgroups of every prime index, whereas the generalised Fabrykowski-Gupta group acting on the p-regular rooted tree only contains maximal subgroups of index p. As it seems that these maximal subgroups of odd index in the infinite dihedral group are responsible for the existence of maximal subgroups of infinite index in Šunić groups acting on the binary rooted tree, we suspect that the same phenomenon will not happen for trees of higher degree.

As mentioned at the beginning of this chapter, generalised Fabrykowski-Gupta groups also belong to another important family of groups acting on rooted trees, namely *GGS* groups. It is thus natural to ask if every *GGS* group belongs to the class $\mathcal{MF}$.

Question 7.3.2. Does every GGS group belong to the class $\mathcal{MF}$?

This question was already mentioned by Alexoudas, Klopsch and Thillaisundaram in [1], where they wonder if every group in a larger family, that they call multi-edge spinal groups, belong to the class $\mathcal{MF}$. The results in this chapter form the first evidence supporting a positive answer to this question.

Lastly, notice that in the same way as we have done here, one could define a generalised Fabrykowski-Gupta group acting on a d-regular tree for any $d \geq 2$, not necessarily only for trees of prime degree. However, in that case, the action of the group on the first level would not be primitive (see Example 2.1.5). Therefore, in that case, one could not use Theorem 5.4.3, which was crucial in the proof. It would thus be interesting to see if the result still holds in this case, or if the primitivity of the action on the first level is essential for the absence of maximal subgroups of infinite index.

Question 7.3.3. Do generalised Fabrykowski-Gupta groups acting on a rooted tree of non-prime degree belong to the class $\mathcal{MF}$?

Chapter 8

Maximal subgroups of the Basilica group

In Chapter 7, we saw the first examples of non-torsion branch groups belonging to the class $\mathcal{MF}$ of groups containing no maximal subgroup of infinite index. This showed that torsion is not a necessary condition for branch groups to belong to this class.

However, the generalised Fabrykowski-Gupta groups studied in Chapter 7 still share many properties with the Grigorchuk group and other branch groups known to be in $\mathcal{MF}$. For instance, they are just-infinite, and every proper quotient is a finite p-group. One could wonder if these or other properties are necessary for a branch group to belong to $\mathcal{MF}$.

In this Chapter, we will make some partial progress towards an answer to these questions by showing that a group known as the Basilica group belongs to $\mathcal{MF}$. The Basilica group was first studied by Grigorchuk and Żuk as an interesting example of an automaton group on three states and two letters in [52]. In that article, they showed that this group is not subexponentially amenable, meaning that it cannot be built from groups of subexponential growth by taking subgroups, quotients, extensions and direct limits. The amenability of the Basilica group was later shown by Bartholdi and Virág [12], thus proving that the class of amenable groups is larger than the class of subexponentially amenable groups. The Basilica group is also important in the theory of iterated monodromy groups, as it is the iterated monodromy group of the polynomial $z^2 - 1$ (see [5]).

The Basilica group is quite different in many respect from other known examples of groups in $\mathcal{MF}$. For instance, it is torsion-free, and it admits quotients which are not p-groups, or not even nilpotent. It also contains a free subsemigroup, which means that it is of exponential growth. However, as we will show, this group is not a branch group, but only a weakly branch group. Nevertheless, it remains an interesting example to help us understand what kind of groups can belong to the class $\mathcal{MF}$.

In Section 8.1, we give the definition of the Basilica group and list some of its properties. In Section 8.2, we find a minimal system of generators for the commutator subgroup of the Basilica group. Although it is not strictly necessary, it makes our study of its quotients in Section 8.3 slightly easier.

The main result of that section is Proposition 8.3.6, which implies that every proper quotient of the Basilica group is in $\mathcal{MF}$, but we also take the time to investigate other properties of these quotients. Finally, in Section 8.4, we show (Theorem 8.4.12) that every maximal subgroup of the Basilica group is of finite index.

8.1 The Basilica group

In this section, we give the definition of the Basilica group and list a few of its basic properties.

For the rest of this chapter, we will write $X = \{\mathbf{0}, \mathbf{1}\}$. Let $\sigma \in \mathrm{Sym}(X)$ be the non-trivial permutation of X. We can recursively define two automorphisms $a, b \in \mathrm{Aut}(X^*)$ of the rooted tree X^* by the formulas

$$a = (1, b) \qquad b = \sigma(a, 1).$$

We can then define the Basilica group as the group generated by those two automorphisms.

Definition 8.1.1. The *Basilica group* is the group $G = \langle a, b\rangle \leq \mathrm{Aut}(X^*)$ of automorphisms of the binary rooted tree X^* generated by a and b. ❧

Remark 8.1.2. In their original article [52], Grigochuk and Zuk defined the Basilica group as the group generated by a and $b = (1, a)\sigma$ acting on the *right* on X^*, whereas we use a left action here. However, the resulting group is the same, since we can easily pass from the left to the right action with the formula $x \cdot g = g^{-1} \cdot x$. ❧

Let us now look at a few important properties of the Basilica group. The following theorem is a collection of results about this group that were shown in [52].

Theorem 8.1.3. *Let $G = \langle a, b\rangle$ be the Basilica group. Then,*

(i) G is a self-replicating,

(ii) G is regular weakly branch group over the commutator subgroup G',

(iii) G is torsion-free,

(iv) the semigroup generated by a and b is free (so in particular, G is of exponential growth),

(v) $G/G' \cong \langle a\rangle \times \langle b\rangle \cong \mathbb{Z}^2$,

(vi) G is just non-solvable.

Grigorchuk and Zuk also gave a presentation for the Basilica group, which can be useful in studying its quotients. The presentation we give here is the one given in Propostion 9 of [52].

Theorem 8.1.4. *The Basilica group G has the presentation*

$$G = \langle a, b \mid \lambda^k(\tau_m), m = 2l+1, k \in \mathbb{N}, l \in \mathbb{N}\rangle$$

where

$$\tau_m = [b^{-m}ab^m, a]$$

and

$$\lambda\colon \begin{cases} a \mapsto b^2 \\ b \mapsto a. \end{cases}$$

8.2 The commutator subgroup of the Basilica group

By Theorem 8.1.3, the Basilica group is regular weakly branch over its commutator subgroup. Therefore, having a good understanding of this subgroup is of great importance in the study of the Basilica group. In this section, we give a minimal generating set for the commutator subgroup and study its abelianisation. These results will be helpful to study the quotients of the Basilica group later on, and we believe that they may also be of independent interest to anyone working with this group.

Let us first introduce a convenient notation for the generators of the commutator subgroup, in order to keep the notation slightly more manageable.

Notation 8.2.1. Let $G = \langle a, b\rangle$ be the Basilica group, with a and b as in the previous section. In what follows, for all $s, t \in \mathbb{Z}$, we will write

$$\alpha_{s,t} = [a^s, b^t],$$

where $[a^s, b^t] = a^{-s}b^{-t}a^sb^t$.

Let us now establish some relations between these commutators.

Proposition 8.2.2. *For all $s, t \in \mathbb{Z}$, we have the following relations in the Basilica group G:*

$$\alpha_{s,2t+1} = \left(\alpha_{1,1}(\alpha_{1,-1}^{-1}\alpha_{1,1})^t\right)^s$$
$$\alpha_{s,2t} = \alpha_{1,1}^{s-1}\left(\alpha_{1,2}^t\alpha_{1,1}^{-1}\right)^{s-1}\alpha_{1,2}^t$$

Proof. The proof is mainly a direct computation, using the map ψ_1 defined in Proposition 2.6.31. We will make frequent use of the fact that

$$\psi_1(\alpha_{1,1}) = (a^{-1}ba, b^{-1})$$
$$\psi_1(\alpha_{1,-1}) = (b, b^{-1})$$
$$\psi_1(\alpha_{1,2}) = (1, b^{-1}a^{-1}ba) = (1, \alpha_{1,1}^{-1}).$$

By direct computation, we find

$$\psi_1(\alpha_{s,2t+1}) = (a^{-t-1}b^sa^{t+1}, b^{-s}).$$

Therefore, it follows from the injectivity of ψ_1 that we have

$$\alpha_{s,2t+1} = \alpha_{1,2t+1}^s.$$

Using this fact and a bit more direct computation, we see that

$$\begin{aligned}\psi_1((\alpha_{1,-1}^{-1}\alpha_{1,1})^t) &= (b^{-1}a^{-1}ba, 1)^t \\ &= (\alpha_{1,1}^{-1}, 1)^t \\ &= (\alpha_{t,1}^{-1}, 1).\end{aligned}$$

It follows, once again with the help of some more direct computations, that

$$\begin{aligned}\psi_1\left(\left(\alpha_{1,1}(\alpha_{1,-1}^{-1}\alpha_{1,1})^t\right)^s\right) &= (a^{-1}ba\alpha_{t,1}^{-1}, b^{-1})^s \\ &= (a^{-1}bab^{-1}a^{-t}ba^t, b^{-1})^s \\ &= (\alpha_{1,-1}a^{-t}ba^t, b^{-1})^s \\ &= (\alpha_{1,-1}a^{-t}\alpha_{1,-1}^{-1}a^{-1}ba^{t+1}, b^{-1})^s \\ &= ((\alpha_{1,-1}a^{-t}\alpha_{1,-1}^{-1}a^t)(a^{-t-1}ba^{t+1}), b^{-1})^s \\ &= ([\alpha_{1,-1}^{-1}, a^t](a^{-t-1}ba^{t+1}), b^{-1})^s.\end{aligned}$$

Now, using the fact that $\psi_1(\alpha_{1,-1}^{-1}) = (b^{-1}, b)$ and $\psi_1(a^t) = (1, b^t)$, we see that $\alpha_{1,-1}^{-1}$ and a^t commute. Therefore, we have

$$\begin{aligned}\psi_1\left(\left(\alpha_{1,1}(\alpha_{1,-1}^{-1}\alpha_{1,1})^t\right)^s\right) &= ((a^{-t-1}ba^{t+1})^s, b^{-s}) \\ &= \psi_1(\alpha_{s,2t+1}).\end{aligned}$$

The first relation then immediately follows from the injectivity of ψ_1.

To prove the second relation, let us notice that from direct computation, we immediately get

$$\psi_1(\alpha_{s,2t}) = (1, \alpha_{t,s}^{-1}).$$

Hence, we have

$$\begin{aligned}\psi_1((\alpha_{1,2}^t\alpha_{1,1}^{-1})^{s-1}) &= (a^{-1}b^{-1}a, \alpha_{1,1}^{-t}b)^{s-1} \\ &= (a^{-1}b^{1-s}a, (\alpha_{t,1}^{-1}b)^{s-1}),\end{aligned}$$

so we see that

$$\begin{aligned}\psi_1(\alpha_{1,1}^{s-1}\left(\alpha_{1,2}^t\alpha_{1,1}^{-1}\right)^{s-1}\alpha_{1,2}^t) &= (1, b^{1-s}(\alpha_{t,1}^{-1}b)^{s-1}\alpha_{t,1}^{-1}) \\ &= (1, b^{-s+1}(\alpha_{t,1}^{-1}b)^sb^{-1}) \\ &= (1, b^{-s+1}([b, a^t]b)^sb^{-1}) \\ &= (1, b^{-s+1}(b^{-1}a^{-t}ba^tb)^sb^{-1}) \\ &= (1, b^{-s}a^{-t}b^sa^t) \\ &= \psi_1(\alpha_{s,2t}).\end{aligned}$$

This proves the second relation. □

The previous proposition implies that the commutator subgroup of Basilica is generated by only three elements.

Proposition 8.2.3. *The commutator subgroup G' of the Basilica group G is generated by $\alpha_{1,1} = [a, b]$, $\alpha_{1,-1} = [a, b^{-1}]$ and $\alpha_{1,2} = [a, b^2]$.*

Proof. Since the Basilica group is generated by two elements, a and b, its commutator subgroup, G', is generated by the set

$$\{\alpha_{s,t} = [a^s, b^t] | s, t \in \mathbb{Z}\}.$$

Therefore, it follows from Proposition 8.2.2 that G' is generated by $\alpha_{1,1}$, $\alpha_{1,-1}$ and $\alpha_{1,2}$. □

In the next section, we will show that G'/G'' is isomorphic to $\mathbb{Z}^3$. This will imply that this system of generators is minimal.

8.3 Quotients of the Basilica group

As we are interested in the index of maximal subgroups of the Basilica group G, it is of great importance to understand its quotients. Indeed, from Proposition 5.1.1, we know that if G admits a quotient which is not in $\mathcal{MF}$, then G is not in $\mathcal{MF}$. On the other hand, if every quotient of G is in $\mathcal{MF}$, then we know that being prodense is equivalent to being pro-$\mathcal{MF}$-dense (Proposition 5.3.8), which means that we can use Theorem 5.4.3 to study maximal subgroups.

In this section, we will investigate the properties of proper quotients of the Basilica group. Although strictly speaking, we only require Proposition 8.3.6 in our study of maximal subgroups, we also investigate other properties of quotients, since they point out some important difference between the Basilica group and every other known examples of weakly branch groups belonging to $\mathcal{MF}$.

Virtually nilpotent quotients

We know from Theorem 8.1.3 that the Basilica group G is just non-solvable, meaning that every proper quotient of G is solvable. One could wonder if those quotients are in fact nilpotent, but it is not hard to show, using the presentation given in Theorem 8.1.4, that this is not the case.

Proposition 8.3.1. *The Baumslag-Solitar group $BS(1,-1)$ (or, in other words, the fundamental group of the Klein bottle) is a quotient of the Basilica group.*

Proof. Let G be the Basilica group and let $N = \langle b^{-1}aba \rangle^G$ be the smallest normal subgroup of G containing $b^{-1}aba$. If follows from Theorem 8.1.4 that

$$G/N = \langle a, b \mid b^{-1}aba, \lambda^k(\tau_m), m \in 2\mathbb{N}+1, k \in \mathbb{N} \rangle$$

where

$$\tau_m = [b^{-m}ab^m, a]$$

and

$$\lambda\colon \begin{cases} a \mapsto b^2 \\ b \mapsto a. \end{cases}$$

This implies that G/N has the presentation

$$G/N = \langle a, b \mid b^{-1}aba \rangle,$$

which is the presentation of the Baumslag-Solitar group $BS(1,-1)$. Indeed, it is easy to see that τ_m is a consequence of the relation $b^{-1}aba$ for any $m \in 2\mathbb{N}+1$. Furthermore, since $b^{-2}ab^2a^{-1}$ is also a consequence of $b^{-1}aba$, we see that $\lambda^k(\tau_m)$ must be a consequence of this relation for all $k \in \mathbb{N}$ and $m \in 2\mathbb{N}+1$. □

As an immediate corollary, we get that the Basilica group maps surjectively onto the infinite dihedral group.

Corollary 8.3.2. *The infinite dihedral group is a quotient of the Basilica group.*

Proof. The infinite dihedral group is a quotient of BS(1,-1). □

Those results immediately imply that the Basilica group admits quotients that are not nilpotent, in contrast with Šunić groups, where every proper quotient was a finite p-group.

Corollary 8.3.3. *The Basilica group is not just non-nilpotent.*

Proof. The fundamental group of the Klein bottle is not nilpotent (and neither is the infinite dihedral group). □

Although some quotients are not nilpotent, we will show that every proper quotient of the Basilica group is virtually nilpotent. To prove this, we will rely on the following lemma, which was proved by Grigorchuk and Zuk in ([52], Lemma 9). However, we present here a different proof.

Lemma 8.3.4 (Lemma 9 of [52])**.** *Let G be the Basilica group, let G' be its derived subgroup, let $G'' = [G', G']$ be its second derived subgroup and let $\gamma_3(G) = [G', G]$ be the third term in the lower central series of G. Then, we have*

$$\psi_1(G'') = \gamma_3(G) \times \gamma_3(G).$$

Proof. Let us first show that $\psi_1(G'') \leq \gamma_3(G) \times \gamma_3(G)$. It follows from Proposition 8.2.3 that G'' is generated by the conjugates in G' of $[\alpha_{1,1}, \alpha_{1,-1}]$, $[\alpha_{1,1}, \alpha_{1,2}]$ and $[\alpha_{1,-1}, \alpha_{1,2}]$. We find

$$\begin{aligned}
\psi_1([\alpha_{1,1}, \alpha_{1,-1}]) &= [(a^{-1}ba, b^{-1}), (b, b^{-1})] \\
&= (a^{-1}b^{-1}ab^{-1}a^{-1}bab, 1) \\
&= ([[b,a],b], 1) \in \gamma_3(G) \times \gamma_3(G),
\end{aligned}$$

$$\begin{aligned}
\psi_1([\alpha_{1,1}, \alpha_{1,2}]) &= [(a^{-1}ba, b^{-1}), (1, [b,a])] \\
&= (1, [b^{-1}, [b,a]]) \\
&= (1, [[b,a], b^{-1}]^{-1}) \in \gamma_3(G) \times \gamma_3(G),
\end{aligned}$$

$$\begin{aligned}
\psi_1([\alpha_{1,-1}, \alpha_{1,2}]) &= [(b, b^{-1}), (1, [b,a])] \\
&= (1, [b^{-1}, [b,a]]) \\
&= (1, [[b,a], b^{-1}]^{-1}) \in \gamma_3(G) \times \gamma_3(G).
\end{aligned}$$

Since $\gamma_3(G)$ is a normal subgroup of G, any conjugate of these elements will also belong to $\gamma_3(G) \times \gamma_3(G)$. Therefore, $G'' \leq \gamma_3(G) \times \gamma_3(G)$.

Now, let us show that $\gamma_3(G) \times \gamma_3(G) \leq \psi_1(G'')$. Since G is generated by a and b, we have that $\gamma_3(G)$ is normally generated in G by $[[b, a], a]$ and $[[b, a], b]$. Since

$$[[b, a], a] = [(a^{-1}b^{-1}a, b), (1, b)] = 1,$$

we conclude that $\gamma_3(G)$ is normally generated by $[[b, a], b]$. We have computed above that

$$\psi_1([\alpha_{1,1}, \alpha_{1,-1}]) = ([[b, a], b], 1) \in \gamma_3(B) \times 1.$$

Since G'' is normal in G and since G is self-replicating, we conclude that $\gamma_3(G) \times 1 \leq \psi_1(G'')$. Conjugating by b, we then get that $1 \times \gamma_3(G) \leq \psi_1(G'')$, from which we conclude that $\gamma_3(G) \times \gamma_3(G) \leq \psi_1(G'')$. □

Using this, we can show that G/G'' is virtually nilpotent.

Lemma 8.3.5. *The group G/G'' is virtually nilpotent.*

Proof. Thanks to Lemma 8.3.4, we have

$$\psi_1(\mathrm{St}_G(1))/\psi_1(G'') \leq (G/\gamma_3(G)) \times (G/\gamma_3(G)).$$

As the group $(G/\gamma_3(G)) \times (G/\gamma_3(G))$ is clearly nilpotent, $\psi_1(\mathrm{St}_G(1))/\psi_1(G'')$ is nilpotent. Since ψ_1 is injective, we have

$$\psi_1(\mathrm{St}_G(1))/\psi_1(G'') \cong \mathrm{St}_G(1)/G''.$$

As $\mathrm{St}_G(1)$ is of finite index in G, $\mathrm{St}_G(1)/G''$ is of finite index in G/G''. Hence, we found a nilpotent subgroup of finite index in G/G''. □

From this, we can see that every proper quotient of the Basilica group is virtually nilpotent.

Proposition 8.3.6. *The Basilica group G is just non-(virtually nilpotent). In particular, every proper quotient of G is of polynomial growth.*

Proof. The fact that every proper quotient of Basilica is virtually nilpotent follows from Theorem 4.4.2, Lemma 8.3.5 and the fact that G is regular weakly branch over G'. Thanks to Gromov's theorem (see [53]), this means that every proper quotient is of polynomial growth, and the same theorem tells us that G itself is not virtually nilpotent, since it is of exponential growth. Thus, G is just non-(virtually nilpotent). □

Non-virtually abelian quotients

Although every proper quotient of the Basilica group is virtually nilpotent, we will see that there exist some quotients which are not virtually abelian. In particular, this will imply that the Basilica group is not a branch group.

Proposition 8.3.7. *Let G be the Basilica group, $G' = [G, G]$ be its derived subgroup, $\gamma_3(G) = [G', G]$ be the third term in its lower central series, and $H_3(\mathbb{Z})$ be the discrete Heisenberg group. Then, $G/\gamma_3(G) \cong H_3(\mathbb{Z})$. In particular, $G/\gamma_3(G)$ is not virtually abelian.*

Proof. Let $F(x,y)$ be the free group on x and y. Then, for all $m,n \in \mathbb{Z}$, we have

$$\begin{aligned}[x^{-m}y^nx^m, y^n] &= y^{-n}y^nx^{-m}y^{-n}x^my^{-n}x^{-m}y^nx^my^{-n}y^ny^n \\ &= y^{-n}[y^{-n},x^m]y^{-n}[x^m,y^{-n}]y^ny^n \\ &= y^{-n}[[x^m,y^{-n}],y^n]y^n \\ &\in \gamma_3(F(x,y)),\end{aligned}$$

where $\gamma_3(F(x,y))$ is the third term in the lower central series of $F(x,y)$.

By Theorem 8.1.4 and the fact that $\gamma_3(G)$ is normally generated by $[[a,b],a]$ and $[a,b],b]$, we have

$$G/\gamma_3(G) = \langle a,b \mid [[a,b],a],[[a,b],b],\lambda^k(\tau_m), m \in 2\mathbb{N}+1, k \in \mathbb{N}\rangle$$

with

$$\tau_m = [b^{-m}ab^m, a]$$

and

$$\lambda\colon \begin{cases} a \mapsto b^2 \\ b \mapsto a. \end{cases}$$

However, since $\gamma_3(F(a,b))$ is normally generated by $[[a,b],a]$ and $[[a,b],b]$, we have by the above that $\lambda^k(\tau_m)$ is a consequence of these two relations for all $k \in \mathbb{N}$ and $m \in 2\mathbb{N}+1$. Therefore, we have

$$G/\gamma_3(G) = \langle a,b \mid [[a,b],a],[[a,b],b]\rangle$$

which is the presentation of the discrete Heisenberg group $H_3(\mathbb{Z})$. It is well-known that this group is not virtually abelian. □

A direct consequence of this fact is that the Basilica group is not a branch group.

Corollary 8.3.8. *The Basilica group is not a branch group.*

Proof. It follows from Lemma 2.7.5 that every proper quotient of a branch group is virtually abelian. □

We can also use this result to understand the quotient G'/G''.

Proposition 8.3.9. *Let G be the Basilica group, G' be its derived subgroup and G'' be its second derived subgroup. The map $\mathbb{Z}^3 \to G'/G''$ defined by sending the canonical generators of $\mathbb{Z}^3$ to $\alpha_{1,1}, \alpha_{1,-1}$ and $\alpha_{1,2}$ is an isomorphism. In particular, $\alpha_{1,1}$, $\alpha_{1,-1}$ and $\alpha_{1,2}$ form a minimal set of generators for G'.*

Proof. Since ψ_1 is an injective map, we have that

$$G'/G'' \cong \psi_1(G')/\psi_1(G'').$$

Now, by Lemma 8.3.4, we have that $\psi_1(G'') = \gamma_3(G) \times \gamma_3(G)$, where $\gamma_3(G) = [G',G]$. Thus, it follows from Proposition 8.3.7 that $\psi_1(G')/\psi_1(G'') \leq H_3(\mathbb{Z}) \times H_3(\mathbb{Z})$, where $H_3(\mathbb{Z})$ is the discrete Heisenberg group.

Let $f\colon \mathbb{Z}^3 \to G'/G''$ be the homomorphism sending $(1,0,0)$ to $\alpha_{1,1}$, $(0,1,0)$ to $\alpha_{1,-1}$ and $(0,0,1)$ to $\alpha_{1,2}$, and let $g\colon G'/G'' \to H_3(\mathbb{Z}) \times H_3(\mathbb{Z})$ be the homomorphism implied above. To prove the result, it suffices to show that the kernel of $g \circ f$ is trivial.

By direct computation, we see that

$$g(\alpha_{1,1}) = (bc^{-1}, b^{-1}), \quad g(\alpha_{1,-1}) = (b, b^{-1}), \quad g(\alpha_{1,2}) = (1, c^{-1})$$

where $H_3(\mathbb{Z}) = \langle a, b \mid [[a,b],a], [[a,b],b]\rangle$ and $c = [a,b]$. Now, let $(l,m,n) \in \ker(g \circ f)$ be an arbitrary element of the kernel of $g \circ f$. It follows from the above computations that

$$g \circ f(l,m,n) = (b^{l+m}c^{-l}, b^{-l-m}c^{-n})$$

and we quickly see that this is trivial if and only if $l = m = n = 0$. Thus, $g \circ f$ is injective, which implies that f is injective. By Proposition 8.2.3, it is also surjective and is thus an isomorphism. □

8.4 Maximal subgroups of the Basilica group

In this section, we will show that every maximal subgroup of the Basilica group is of finite index. The general strategy of the proof is similar to the one used by Pervova to study maximal subgroups of the Grigorchuk group, albeit with extensive modifications due to the fact that the Basilica group is very different. Before we delve into the proof, let us give an outline of the argument.

By Proposition 8.3.6, we know that every proper quotient of the Basilica group G is virtually nilpotent, and therefore in $\mathcal{MF}$. It follows from Proposition 5.3.8 and Corollary 5.3.7 that it admits maximal subgroups of infinite index if and only if it admits a proper subgroup $H < G$ such that $HN = G$ for all non-trivial normal subgroup $N \trianglelefteq G$. We will show that such subgroups do not exist by showing that any subgroup satisfying this property must project to G on some vertex, and therefore cannot be proper, according to Theorem 5.4.3.

To achieve this, we will require several intermediate steps. We will start with a few lemmas about the length of the projection of elements in the Basilica group, but before that, let us first fix some notation and terminology.

Notation 8.4.1. As in the rest of this chapter, unless otherwise specified, G will denote the Basilica group. We will denote by $|\cdot|\colon G \to \mathbb{N}$ the *word norm* (see Definition 2.5.2) with respect to the generating set $S = \{a, a^{-1}, b, b^{-1}\}$. In what follows, we will generally make no distinction in the notation between a word in the generating set S and the element it represents in the group G and rely on the context to distinguish between the two cases. In particular, if $w \in S^*$ is a word in the alphabet S, we will denote by $|w|$ the length of the corresponding element in G. A word $w = s_1 \dots s_n \in S^*$ will be called a *word of minimal length* or a *geodesic word* if $|s_1 \dots s_n| = n$. ♋

We begin by showing that with the standard word metric, the Basilica group is a non-ℓ_1-expanding self-similar group (see Definition 3.1.7).

Lemma 8.4.2. *Let $g = \sigma^\epsilon(g_1, g_2) \in G$, where $\epsilon \in \{0,1\}$, be an arbitrary element of G. Then, $|g_1| + |g_2| \leq |g|$. In other words, G is non-ℓ_1-expanding.*

Proof. As $a = (1, b)$, $b = (1, a)\sigma$, $a^{-1} = (1, b^{-1})$ and $b^{-1} = (a^{-1}, 1)\sigma$, we see that the given inequality is true for the generating set $S = \{a, b, a^{-1}, b^{-1}\}$. Therefore, by Proposition 3.1.8, G is non-ℓ_1-expanding. □

In the next lemmas, we will show a few technical length contraction properties.

Lemma 8.4.3. *Let $g = \sigma(g_1, g_2) \notin \mathrm{St}_G(1)$ be an element that does not belong to the stabiliser of the first level, and let $\alpha, \beta \in G$ be such that $g^2 = (\alpha, \beta)$. Then, $|\alpha|, |\beta| \leq |g|$.*

Proof. We have $g^2 = \sigma(g_1, g_2)\sigma(g_1, g_2) = (g_2g_1, g_1g_2)$. Hence, thanks to Lemma 8.4.2, we have $|\alpha| \leq |g_2| + |g_1| \leq |g|$, and likewise, $|\beta| \leq |g|$. □

Lemma 8.4.4. *Let $g = \sigma^\epsilon(g_1, g_2) \in G$ be an arbitrary element, where $\epsilon \in \{0, 1\}$, and let $x_1x_2 \dots x_n \in S^*$ be a word of minimal length representing g, where $S = \{a, b, a^{-1}, b^{-1}\}$. If there exist $1 \leq i < j \leq n$ such that $x_i = b$, $x_j = b^{-1}$, then $|g_1| + |g_2| < |g| = n$.*

Proof. As the word $x_1x_2 \dots x_n$ is reduced (otherwise, it would not be of minimal length), it follows from the hypothesis that it must contain a subword of the form ba^kb^{-1} for some $k \in \mathbb{Z}^*$. Seen as an element of G, we have that

$$ba^kb^{-1} = \sigma(a, 1)(1, b^k)(a^{-1}, 1)\sigma = (b^k, 1).$$

Since ba^kb^{-1} is a subword of a geodesic word, we must have $|ba^kb^{-1}| = |k| + 2$, since otherwise, we could replace it by a shorter word representing the same element. On the other hand, $|b^k| \leq |k|$. Thus, there is a difference of at least 2 between the length of ba^kb^{-1} and the sum of the length of its children. By using subadditivity and the fact that every subword of a geodesic must again be a geodesic, we can conclude that $|g_1| + |g_2| \leq |g| - 2 < |g|$. □

Lemma 8.4.5. *Let $g = \sigma^\epsilon(g_1, g_2) \in G$, be an arbitrary element, where $\epsilon \in \{0, 1\}$, and let $x_1x_2 \dots x_n \in S^*$ be a word in the alphabet $S = \{a, b, a^{-1}, b^{-1}\}$ of minimal length representing g. If $x_1x_2 \dots x_n$ contains a subword of the form $b^{-2}a^kb^2$, then $|g_1| + |g_2| < |g| = n$.*

Proof. In G, we have

$$b^{-2}a^kb^2 = (a^{-1}, a^{-1})(1, b^k)(a, a) = (1, a^{-1}b^ka).$$

As in the proof of Lemma 8.4.4, we observe that $|b^{-2}a^kb^2| = |k| + 4$ and $|a^{-1}b^ka| \leq k + 2$ and thus conclude that $|g_1| + |g_2| \leq n - 2 < n = |g|$. □

In addition to these facts regarding length contraction of elements of G, we will also need to know the equivalence classes of the projections of some elements modulo the commutator subgroup G'.

Notation 8.4.6. Let $g_1, g_2 \in G$ be two arbitrary elements. We will write $g_1 \equiv_{G'} g_2$ if $g_1G' = g_2G'$. ☙

Lemma 8.4.7. *Let $g \notin \mathrm{St}_G(1)$ and $g^2 = (g_1, g_2)$. Then,*

$$g \equiv_{G'} ab \Rightarrow g_1 \equiv_{G'} g_2 \equiv_{G'} ab$$
$$g \equiv_{G'} ab^{-1} \Rightarrow g_1 \equiv_{G'} g_2 \equiv_{G'} a^{-1}b$$
$$g \equiv_{G'} a^{-1}b \Rightarrow g_1 \equiv_{G'} g_2 \equiv_{G'} ab^{-1}.$$

Proof. If $g = abz$ for some $z \in G'$ with $z = (z_1, z_2)$, then

$$g^2 = (1, b)\sigma(a, 1)(z_1, z_2)(1, b)\sigma(a, 1)(z_1, z_2) = (z_2baz_1, baz_1z_2).$$

According to Lemma 5 of [52], we have $z_1 \equiv_{G'} z_2^{-1}$, so the result follows. Similarly, if $g = ab^{-1}z$, we have

$$g^2 = (1, b)(a^{-1}, 1)\sigma(z_1, z_2)(1, b)(a^{-1}, 1)\sigma(z_1, z_2) = (a^{-1}z_2bz_1, bz_1a^{-1}z_2),$$

and if $g = a^{-1}bz$, we have

$$g^2 = (1, b^{-1})\sigma(a, 1)(z_1, z_2)(1, b^{-1})\sigma(a, 1)(z_1, z_2) = (z_2b^{-1}az_1, b^{-1}az_1z_2).$$

□

We are now in position to prove, in the next few results, that any subgroup of G that is prodense must project to G on some vertex.

Proposition 8.4.8. *Let $g \in G$ be such that $g \equiv_{G'} ab$. Then, there exist a vertex $u \in X^*$ in the rooted tree X^* and an element $g' \in \mathrm{St}_G(u) \cap \langle g \rangle$ such that $\varphi_u(g') = ab$.*

Proof. Let us proceed by induction on the length of g.

By definition, the elements of length 1 of G are a, b, a^{-1}, b^{-1}, none of which are congruent to ab modulo G' by Theorem 8.1.3, so the case $|g| = 1$ is impossible. For $|g| = 2$, by the same theorem, the only possibilities are $g = ab$ or $g = ba$. The case $g = ab$ is trivial. If $g = ba$, we have $g^2 = baba = (ba, ab)$, and so $\varphi_{\mathbf{1}}(g^2) = ab$.

Now, let us assume that the result is true for any $h \in G$ such that $h \equiv_{G'} ab$ and $|h| < n$ for some $n \in \mathbb{N}$, and let $g \in G$ be such that $g \equiv_{G'} ab$ and $|g| = n$.

Since $g \equiv_{G'} ab$, we must have $g \notin \mathrm{St}_G(1)$, so $g = \sigma(g_1, g_2)$. Therefore, we have $g^2 = (g_2g_1, g_1g_2)$. By Lemma 8.4.7, $g_2g_1 \equiv_{G'} g_1g_2 \equiv_{G'} ab$, and by Lemma 8.4.3, $|g_2g_1|, |g_1g_2| \leq |g| = n$. If $|g_2g_1| < n$ or $|g_1g_2| < n$, we can then conclude by induction. Otherwise, we must have $|g_2g_1| = |g_1g_2| = n$. Therefore, the words representing g_1 and g_2 obtained from a geodesic of g by the substitution $a \mapsto (1, b)$ and $b \mapsto \sigma(a, 1)$ must be geodesics, and so must their concatenations g_2g_1 and g_1g_2 (since the sum of the length of the words for g_1 and g_2, before any reduction, is exactly n).

Let us write $g_1g_2 = \sigma(\alpha, \beta)$. If the geodesic word for g_1 discussed above contains b and the one for g_2 contains b^{-1}, then by Lemma 8.4.4, $|\alpha| + |\beta| < n$. Therefore, $(g_1g_2)^2 = (\beta\alpha, \alpha\beta)$ with $|\alpha\beta| < n$, $\alpha\beta \equiv_{G'} ab$. Hence, we can conclude by induction. Likewise, if g_2 contains b and g_1 contains b^{-1}, we can conclude by induction by using the projections of $(g_2g_1)^2$.

Since it follows from Theorem 8.1.3 that the sum of the exponents of a in any word representing g is 1, the exponents of b in g_1 and g_2 must sum up to 1. Hence, if g_1 and g_2 both contain some b, one of them must also contain

b^{-1}. Likewise, if both contain some b^{-1}, then one of them must contain b. Hence, the only remaining case is if $g_1 = a^k$ or $g_2 = a^k$ for some $k \in \mathbb{Z}$, with $|g_1 g_2| = |g_2 g_1| = |g|$. We will show that this can only occur if $g = ab$ or $g = ba$.

Let us notice that $a^{k_1} b^{2l+1} a^{k_2} = \sigma(b^{k_1} a^{l+1}, a^l b^{k_2})$. Hence, if g contains a subword of the form $a^{k_1} b^{2l+1} a^{k_2}$ with $k_1, k_2 \in \mathbb{Z}^*$ and $l \in \mathbb{Z}$, then both g_1 and g_2 contain some non-trivial power of b. Hence, if $g_1 = a^k$ or $g_2 = a^k$, then we must have

$$g = b^{2l_1+1} a^{k_1} b^{2l_2} a^{k_2} \dots b^{2l_i} a^{k_i}$$

or

$$g = a^{k_1} b^{2l_1} a^{k_2} b^{2l_2} \dots a^{k_i} b^{2l_i+1}$$

with $\sum_{j=1}^{i} l_j = 0$ and $\sum_{j=1}^{i} k_j = 1$. Indeed, we just saw that in a geodesic word representing g, odd powers of b cannot be sandwiched between non-zero powers of a. This means that odd powers of b must be either at the very beginning or at the very end of the word. Hence, there are only two possible positions, which implies that there are at most two odd powers of b. As the sum of the powers of b must be 1, we conclude that the word for g must contain exactly one b with an odd power, either at the beginning or at the end, thus obtaining the two possibilities above.

If $g = b^{2l_1+1} a^{k_1} b^{2l_2} a^{k_2} \dots b^{l_i} a^{k_i}$, it follows from Lemmas 8.4.4 and 8.4.5 that $g = ba$ or $g = b^{-1} a^{k_1} b^2 a^{k_2}$ with $k_1 + k_2 = 1$. Indeed, otherwise, g would contain a subword of the form $ba^k b^{-1}$ or $b^{-2} a^k b^2$, which contradicts the hypothesis that $|g_1 g_2| = |g_2 g_1| = n$. If $g = b^{-1} a^{k_1} b^2 a^{k_2}$, we have $g^2 = (a^{-1} b^{k_1} a b^{k_2} a, b^{k_1} a b^{k_2})$, and $|b^{k_1} a b^{k_2}| \leq |k_1| + |k_2| + 1 < |k_1| + |k_2| + 3 = |g|$, a contradiction. Hence, the only possible case is $g = ba$.

Similarly, if $g = a^{k_1} b^{l_1} a^{k_2} b^{l_2} \dots a^{k_i} b^{2l_i+1}$, then unless $g = ab$, g must contain a subword of the form $ba^k b^{-1}$ or $b^{-2} a^k b^2$, which is impossible according to Lemmas 8.4.4 and 8.4.5.

This concludes the proof. □

Proposition 8.4.9. *Let $g \in G$ be such that $g \equiv_{G'} ab^{-1}$. Then, there exist a vertex $u \in X^*$ and an element $g' \in \mathrm{St}_G(u) \cap \langle g \rangle$ such that $\varphi_u(g') = b^{-1}a$.*

Proof. We again proceed by induction on $|g|$.

The case $|g| = 1$ is impossible. If $|g| = 2$, we have $g = b^{-1}a$ or $g = ab^{-1}$. Since $(ab^{-1})^2 = (1, b)(a^{-1}, 1)\sigma(1, b)(a^{-1}, 1)\sigma = (a^{-1}b, ba^{-1})$ and $(a^{-1}b)^{-1} = b^{-1}a$, the result is true in those cases.

Let us now assume that the result is true for elements of length smaller than $n \in \mathbb{N}$ and let $g \in G$ be such that $g \equiv_{G'} ab^{-1}$ and $|g| = n$. Writing $g = \sigma(g_1, g_2)$, $g_1 g_2 = (\alpha, \beta)$ and $g_2 g_1 = (\alpha', \beta')$, if $|\alpha|, |\beta|, |\alpha'|$ or $|\beta'|$ is smaller than n, we find that the result is true by induction thanks to Lemma 8.4.7 and Lemma 8.4.3.

Notice that once again, unless $g_1 = a^k$ or $g_2 = a^k$ for some $k \in \mathbb{Z}$, then one of $|\alpha|, |\beta|, |\alpha'|$ or $|\beta'|$ must be smaller than n, thanks to Lemma 8.4.4 and the fact that the exponents of b in g_1 and g_2 must sum to 1.

As in the proof of Proposition 8.4.8, this means that g cannot contain a subword of the form $a^{k_1} b^{2l+1} a^{k_2}$ with $k_1, k_2 \in \mathbb{Z}^*$. Therefore, we must have

$$g = b^{2l_1-1} a^{k_1} b^{2l_2} a^{k_2} \dots b^{l_i} a^{k_i}$$

or

$$g = a^{k_1} b^{l_1} a^{k_2} b^{l_2} \dots a^{k_i} b^{2l_i - 1}$$

with $\sum_{j=1}^{i} l_j = 0$ and $\sum_{j=1}^{i} k_j = 1$.

If $g = b^{2l_1-1} a^{k_1} b^{2l_2} a^{k_2} \dots b^{l_i} a^{k_i}$, then unless $g = b^{-1}a$, g must contain a subword of the form $ba^k b^{-1}$ or $b^{-2} a^k b^2$, which is impossible according to Lemmas 8.4.4 and 8.4.5.

If $g = a^{k_1} b^{l_1} a^{k_2} b^{l_2} \dots a^{k_i} b^{2l_i-1}$, then for the same reasons, we must have $g = ab^{-1}$ or $g = a^{k_1} b^{-2} a^{k_2} b$ with $k_1 + k_2 = 1$. However, $(a^{k_1} b^{-2} a^{k_2} b)^2 = (a^{-1} b^{k_1} a^{-1} b^{k_2} a, b^{k_1} a^{-1} b^{k_2})$, and $|b^{k_1} a^{-1} b^{k_2}| \leq |k_1| + |k_2| + 1 < |g|$.

Hence, unless $g = ab^{-1}$ or $g = b^{-1}a$, we always have that one of $\alpha, \beta, \alpha', \beta'$ is of length smaller than $|g|$. We can therefore conclude by induction thanks to Lemma 8.4.7. □

Lemma 8.4.10. *Let u be a vertex of the rooted tree X^*. Then, there exists $g \in \langle ab \rangle \cap \mathrm{St}_G(u)$ such that $\varphi_u(g) = ab$ or $\varphi_u(g) = ba$.*

Proof. We have $(ab)^2 = (ba, ba)$ and $(ba)^2 = (ba, ab)$. The result follows by induction. □

Proposition 8.4.11. *Let $H \leq G$ be a subgroup such that $HN = G$ for all non-trivial normal subgroups $N \trianglelefteq G$ (in other words, H is prodense). Then, there exists a vertex $u \in X^*$ such that $H_u = G$, where, as in Definition 2.6.25, $H_u = \varphi_u(\mathrm{St}_H(u))$.*

Proof. Since $HG' = G$, there exists $g \in H$ such that $g \equiv_{G'} ab$. Hence, it follows from Proposition 8.4.8 that there exists $v \in X^*$ such that $ab \in H_v$. Now, it follows from Proposition 5.4.1 that $H_v G' = G$. Hence, there exists $h \in H_v$ such that $h \equiv_{G'} ab^{-1}$. Therefore, according to Proposition 8.4.9, there exists v' such that $b^{-1}a \in (H_v)_{v'} = H_{vv'}$. From Lemma 8.4.10, we also have that either $ab \in H_{vv'}$ or $ba \in H_{vv'}$.

If $ab, b^{-1}a \in H_{vv'}$, then $a^2 \in H_{vv'}$. Since $a^2 = (1, b^2)$ and $b^2 = (a, a)$, if we set $u = vv'\mathbf{11} \in X^*$, we have that a and either ab or ba are in H_u. Since G is generated by a and b, we get $H_u = G$.

Likewise, if $ba, b^{-1}a \in H_{vv'}$, then $b^2 \in H_{vv'}$, and since $b^2 = (a, a)$, by setting $u = vv'\mathbf{1} \in X^*$, we get that $a, b \in H_u$, so $H_u = G$. □

We can now finally prove that every maximal subgroup of the Basilica group is of finite index.

Theorem 8.4.12. *Every maximal subgroup of the Basilica group G is of finite index.*

Proof. Suppose that there exists a maximal subgroup $M < G$ of infinite index. Since Proposition 8.3.6 implies that every proper quotient of G is in $\mathcal{MF}$, by Corollary 5.3.7, we must have $MN = G$ for every non-trivial normal subgroup $N \trianglelefteq G$. According to Proposition 8.4.11, there exists $u \in X^*$ such that $M_u = G$. Theorem 5.4.3 then says that $M = G$, which is a contradiction. Hence, G admits no maximal subgroup of infinite index. □

We thus have a new example of a weakly branch (but not branch, as we have shown in Corollary 8.3.8) group belonging to $\mathcal{MF}$. This group is quite different from the other known examples of weakly branch groups in $\mathcal{MF}$, since it is torsion-free and admits non-nilpotent quotients. Also, unlike these other examples, the Basilica group possesses maximal subgroups which are not normal. This can easily be seen from the fact that it admits as a quotient (see Corollary 8.3.2) the infinite dihedral group, which contains non-normal maximal subgroups.

Chapter 9

Parabolic subgroups

As was explained in Section 2.6, a group acting on a regular rooted tree also acts naturally on the boundary of this tree, which is homeomorphic to a Cantor space. In this chapter, we depart from the study of maximal subgroups in groups acting on rooted trees to study another very important class of subgroups, namely the stabilisers of points in the boundary of the tree. Such subgroups are called *parabolic subgroups*.

Parabolic subgroups are of great importance in the theory of groups acting on rooted trees. Indeed, the Schreier graphs of parabolic subgroups correspond to the orbital graphs of the action on the boundary, which are the subject of intensive research. For instance, these graphs have been used to compute the spectrum of some operators on self-similar groups (for example, in [8, 26, 47]). In particular, these ideas were used by Grigorchuk and Żuk to compute the spectrum of the Markov operator of the Lamplighter group [51], which led to a counter-example to a strong version of the Atiyah conjecture on L^2-Betti numbers [48]. These graphs can also be used to find bounds on the growth of the group (see [66] and [4]).

Additionally, parabolic subgroups also appear naturally in the study of weakly maximal subgroups of groups acting on rooted trees. Recall that a subgroup is weakly maximal if it is a maximal element of the partially ordered set of subgroups of infinite index (ordered by inclusion). As maximal subgroups of infinite index are obviously weakly maximal, the study of weakly maximal subgroups can be seen as a natural generalisation of the study of the class $\mathcal{MF}$. Although they are never maximal, parabolic subgroups of branch groups are always weakly maximal (see [8], Proposition 3.16) and thus play an important role in the theory of weakly maximal subgroups (although they are far from the only weakly maximal subgroups, as is shown in [15]).

In this chapter, we will investigate various algebraic properties of parabolic subgroups of weakly branch groups. We begin in Section 9.1 by showing that under some suitable conditions, these subgroups are never finitely generated. Then, in Sections 9.2 and 9.3, we study the isomorphism classes of these parabolic subgroups. More precisely, in Section 9.2, we will show that, under suitable conditions, the stabilisers of points which are called *regular* are all isomorphic (Theorem 9.2.10 and Corollary 9.2.11). It is interesting to note that some of these ideas are not only valid for groups acting on rooted trees, but also for larger classes of groups acting on a topological space. Then, in Section 9.3, we show that under suitable conditions, points with different groups of germs cannot have isomorphic stabilisers (Theorem 9.3.3). In particular, the stabiliser of a singular point cannot be isomorphic to the stabiliser of a regular point. Finally, in Section 9.4, we study the index of parabolic subgroups and show that an infinite automaton group must necessarily admit a parabolic subgroup of infinite index (Theorem 9.4.8).

9.1 Infinitely generated parabolic subgroups

In this section, we will show that under suitable conditions, parabolic subgroups of weakly branch groups are never finitely generated. The proof is inspired by the proof of Mal'cev's theorem (see [25], III.19), which states that finitely generated residually finite groups are hopfian.

Theorem 9.1.1. *Let X be a finite alphabet of cardinality $d \geq 2$ and let G be a weakly branch group acting on the rooted tree X^*. If there exists $N \in \mathbb{N}$ such that for any $v \in X^*$, $\mathrm{Rist}_{G_v}(1)$ acts non-trivially on level N, then $\mathrm{St}_G(\xi)$ is not finitely generated for any $\xi \in X^\infty$.*

Proof. Let $\xi = v_0v_1v_2\cdots \in X^\infty$ be any point on the boundary of the tree and let

$$\begin{aligned}\sigma\colon X^\infty &\to X^\infty\\ v_0v_1v_2\ldots &\mapsto v_1v_2\ldots\end{aligned}$$

be the shift map.

For any $n \in \mathbb{N}$, we have $\mathrm{St}_G(\xi) \leq \mathrm{St}_G(v_0v_1\ldots v_n)$. The restriction of the map $\varphi_{v_0v_1\ldots v_n}$ (see Definition 2.6.22) to $\mathrm{St}_G(\xi)$ gives us a homomorphism

$$\varphi_{v_0v_1\ldots v_n}\colon \mathrm{St}_G(\xi) \to \mathrm{St}_{G_{v_0v_1\ldots v_n}}(\sigma^{n+1}(\xi)).$$

This homomorphism is clearly surjective.

For $n \in \mathbb{N}$, let

$$\pi_n\colon G_{v_0v_1\ldots v_n} \to G_{v_0v_1\ldots v_n}/\,\mathrm{St}_{G_{v_0v_1\ldots v_n}}(N)$$

be the quotient map by the stabiliser of level N. As $G_{v_0v_1\ldots v_n}/\,\mathrm{St}_{G_{v_0v_1\ldots v_n}}(N)$ is isomorphic to a subgroup of

$$\mathrm{Aut}(X^*)/\,\mathrm{St}(N) \cong \wr^N \mathrm{Sym}(X),$$

there exists for all $n \in \mathbb{N}$ a monomorphism $i_n\colon G_{v_0v_1\ldots v_n}/\,\mathrm{St}_{G_{v_0v_1\ldots v_n}}(N) \to \wr^N \mathrm{Sym}(X)$.

Hence, for all $n \in \mathbb{N}$, we have a homomorphism

$$\alpha_n\colon \mathrm{St}_G(\xi) \to \wr^N \mathrm{Sym}(X)$$

given by

$$\alpha_n = i_n \circ \pi_n \circ \varphi_{v_0v_1\ldots v_n}.$$

We will see that these maps are all different. For this, it suffices to show that for any $n \in \mathbb{N}$ and $m \in \mathbb{N}^*$, the maps α_n and α_{n+m} are different.

By hypothesis, for any $n \in \mathbb{N}$, there exists an element of $\mathrm{Rist}_{G_{v_0v_1\dots v_n}}(1)$ that acts non-trivially on level N. Therefore, by the spherical transitivity of the action of G on X^* (and thus of the action of $G_{v_0v_1\dots v_n}$ on X^*), there exists for any $v \in X$ an element of $\mathrm{Rist}_{G_{v_0v_1\dots v_n}}(v)$ acting non-trivially on level N. Let $v \in X$ be different from v_{n+1} and let $g \in \mathrm{Rist}_{G_{v_0v_1\dots v_n}}(v)$ be an element acting non-trivially on level N. Then, we have $g \notin \mathrm{St}_{G_{v_0v_1\dots v_n}}(N)$, so $\pi_n(g) \neq 1$. As $v \neq v_{n+1}$, we have that $\mathrm{Rist}_{G_{v_0v_1\dots v_n}}(v) \leq \mathrm{St}_{G_{v_0v_1\dots v_n}}(\sigma^n(\xi))$. Hence, by the surjectivity of $\varphi_{v_0v_1\dots v_n}$, there must exist some $h \in \mathrm{St}_G(\xi)$ such that $\varphi_{v_0v_1\dots v_n}(h) = g$. Therefore, $\alpha_n(h) \neq 1$.

Let $m \in \mathbb{N}^*$. Then,

$$\begin{aligned}\pi_{n+m}(\varphi_{v_0v_1\dots v_{n+m}}(h)) &= \pi_{n+m}(\varphi_{v_{n+2}\dots v_{n+m}}(\varphi_{v_{n+1}}(\varphi_{v_0v_1\dots v_n}(h))))\\ &= \pi_{n+m}(\varphi_{v_{n+2}\dots v_{n+m}}(\varphi_{v_{n+1}}(g)))\\ &= \pi_{n+m}(\varphi_{v_{n+2}\dots v_{n+m}}(1))\\ &= 1\end{aligned}$$

since $g \in \mathrm{Rist}_{G_{v_0v_1\dots v_n}}(v)$ with $v \neq v_{n+1}$. It follows that $\alpha_{n+m}(h) = 1$. Therefore, $\alpha_n \neq \alpha_{n+m}$.

We have thus found an infinite number of different homomorphisms

$$\alpha_n\colon \mathrm{St}_G(\xi) \to \wr^N \mathrm{Sym}(X).$$

Since $\wr^N \mathrm{Sym}(X)$ is a finite group, this implies that $\mathrm{St}_G(\xi)$ is not finitely generated (as a finitely generated group only admits a finite number of homomorphisms to a finite group). □

Notice that in particular, Theorem 9.1.1 applies to any self-replicating weakly branch group, which includes the vast majority of groups studied in this thesis.

9.2 Stabilisers of regular points

If a group G acts on a rooted tree T, the stabilisers of two points of the boundary ∂T in the same orbit under G are obviously isomorphic. However, if G is finitely generated, then its action on ∂T can never be transitive, since ∂T is uncountable. Therefore, it is natural to ask which parabolic subgroups are isomorphic and which ones are not. In this section, we provide a partial answer to this question. In order to do this, we will first study stabilisers in the general setting of groups acting by homeomorphisms on a topological space and then specialise our results to the case of groups acting on rooted trees.

Before we begin, let us discuss a few different notions of stabilisers for groups acting by homeomorphism on topological spaces.

Definition 9.2.1. Let $\mathcal{X}$ be a topological space and let G be a group acting on $\mathcal{X}$ by homeomorphisms. For $x \in \mathcal{X}$, the *neighbourhood stabiliser* of x is the subgroup

$$\mathrm{St}_G^0(x) = \{g \in G \mid \exists U \subseteq \mathcal{X} \text{ open such that } x \in U \text{ and } g|_U = \mathrm{id}|_U\}$$

of all elements of G that act trivially on some neighbourhood of x. ☙

It is clear that the neighbourhood stabiliser of a point is a subgroup of the stabiliser of the same point. In fact, it is even a normal subgroup.

Proposition 9.2.2. *Let $\mathcal{X}$ be a topological space and let G be a group acting on $\mathcal{X}$ by homeomorphisms. For every $x \in \mathcal{X}$, we have*

$$\mathrm{St}_G^0(x) \trianglelefteq \mathrm{St}_G(x).$$

Proof. Let $g \in \mathrm{St}_G^0(x)$ be an element fixing a neighbourhood U of x pointwise, and let $h \in \mathrm{St}_G(x)$ be an element fixing x. Then, $h(U)$ is also a neighbourhood of x and we have that hgh^{-1} fixes $h(U)$ pointwise. □

As the neighbourhood stabiliser is a normal subgroup of the stabiliser, one can consider the quotient group. This group is known as the *group of germs.*

Definition 9.2.3. Let $\mathcal{X}$ be a topological space, let G be a group acting on $\mathcal{X}$ by homeomorphisms and let $x \in \mathcal{X}$ be a point. The *group of germs* of G at x is the group

$$\mathrm{St}_G(x)/\,\mathrm{St}_G^0(x).$$

☙

One can then distinguish two classes of points, depending on whether this group is trivial or not.

Definition 9.2.4. Let $\mathcal{X}$ be a topological space and let G be a group acting on $\mathcal{X}$ by homeomorphisms. A point $x \in \mathcal{X}$ is said to be *regular* if its group of germs is trivial, and *singular* if it is non-trivial. In other words, x is a regular point if $\mathrm{St}_G^0(x) = \mathrm{St}_G(x)$ and singular if $\mathrm{St}_G^0(x) < \mathrm{St}_G(x)$. ☙

Just as in the case of groups acting on rooted tree, it is also possible to define a notion of rigid stabilisers for groups acting by homeomorphisms on a topological space.

Definition 9.2.5. Let $\mathcal{X}$ be a topological space, let G be a group acting on $\mathcal{X}$ by homeomorphisms, and let $U \subseteq \mathcal{X}$ be an open set. The *rigid stabiliser* of U in G is the subgroup

$$\mathrm{Rist}_G(U) = \{g \in G \mid g|_{\mathcal{X}\setminus U} = \mathrm{id}\,|_{\mathcal{X}\setminus U}\}$$

of elements acting trivially outside of U. ☙

Notice that in the case of a group acting on a rooted tree T, this definition coincides with Definition 2.6.14 if we set $\mathrm{Rist}_G(v) = \mathrm{Rist}_G(T_v)$, where T_v is the subtree of T rooted at the vertex v, which is an open set.

Notation 9.2.6. Let $\mathcal{X}$ be a topological space and $U \subseteq \mathcal{X}$ be an open set. We will denote by $\mathrm{Rist}(U)$ the rigid stabiliser of U in the group of all homeomorphisms of $\mathcal{X}$. ❧

We will now study conditions under which the neighbourhood stabilisers of any two points are isomorphic. In particular, this will imply that the stabilisers of regular points are all isomorphic. We begin by a simple but very general lemma.

Lemma 9.2.7. *Let $\mathcal{X}$ be a topological space, let G be a group acting on $\mathcal{X}$ by homeomorphisms, and let $x, y \in \mathcal{X}$ be two points. If there exists a homeomorphism $f\colon \mathcal{X} \to \mathcal{X}$ such that*

(i) $f(x) = y$,

(ii) for any neighbourhood U of x, there exists $g_U \in G$ such that $g_U^{-1} f \in \mathrm{Rist}(U)$,

then $f \operatorname{St}_G^0(x) f^{-1} = \operatorname{St}_G^0(y)$. In particular, $\operatorname{St}_G^0(x)$ and $\operatorname{St}_G^0(y)$ are isomorphic.

Proof. Let $g \in \operatorname{St}_G^0(x)$ be an arbitrary element. We need to show that $fgf^{-1} \in \operatorname{St}_G^0(y)$. Since g is in the neighbourhood stabiliser of x, there must exist a neighbourhood U of x such that g acts trivially on U. By condition (ii), we know that there exists $g_U \in G$ such that $g_U^{-1} f \in \mathrm{Rist}(U)$. Let us show that $g_U g g_U^{-1} = fgf^{-1}$.

Let us first notice that by definition, we have $g(U) = U$, and therefore $g(\mathcal{X} \setminus U) = \mathcal{X} \setminus U$. Now, let $z \in \mathcal{X} \setminus U$ be any point outside of U. Since $g_U^{-1} f \in \mathrm{Rist}(U)$, we have $g_U^{-1} f(z) = z$, so $f(z) = g_U(z)$. This immediately implies that $f(U) = g_U(U)$ and that $f^{-1}(z) = g_U^{-1}(z)$ for all $z \in \mathcal{X} \setminus f(U)$. Therefore, for $z \in \mathcal{X} \setminus f(U)$, we have

$$(g_U g g_U^{-1})(z) = g_U(g(f^{-1}(z))) = f(g(f^{-1}(z))) = (fgf^{-1})(z),$$

where the second equality comes from the fact that $g(f^{-1}(z)) \in \mathcal{X} \setminus U$. Now, for $z \in f(U)$, we have $f^{-1}(z) \in U$ and

$$g_U^{-1}(z) \in g_U^{-1} f(U) = U.$$

Since g acts trivially on U, we get that

$$g_U g g_U^{-1}(z) = z = fgf^{-1}(z).$$

We have thus proved that $g_U g g_U^{-1}(z) = fgf^{-1}(z)$ for all $z \in \mathcal{X}$, which means that $g_U g g_U^{-1} = fgf^{-1}$. As $g_U g g_U^{-1} \in G$, we thus get that $fgf^{-1} \in G$, and since g is in the neighbourhood stabiliser of x, we must have that fgf^{-1} is in the neighbourhood stabiliser of $f(x) = y$. Thus, we have shown that $f \operatorname{St}_G^0(x) f^{-1} \leq \operatorname{St}_G^0(y)$.

To show the other inclusion, we will simply use the same argument with f^{-1}. However, in order to do that, we first need to show that f^{-1} satisfies condition (ii).

Let V be a neighbourhood of y and let us set $U = f^{-1}(V)$. Then, U is a neighbourhood of x, Therefore, by condition (ii), there exists g_U such that $g_U^{-1} f \in \mathrm{Rist}(U)$. As $f(U) = V$, we have

$$f g_U^{-1} = f(g_U^{-1} f) f^{-1} \in f \,\mathrm{Rist}(U) f^{-1} = \mathrm{Rist}(f(U)) = \mathrm{Rist}(V).$$

By taking the inverse, we get $g_U f^{-1} \in \mathrm{Rist}(V)$. We conclude that f^{-1} also satisfies condition (ii). Therefore, by symmetry, we have $f^{-1} \operatorname{St}_G^0(y) f \leq \operatorname{St}_G^0(x)$. We conclude that $f \operatorname{St}_G^0(x) f^{-1} = \operatorname{St}_G^0(y)$. □

In order to use Lemma 9.2.7 to prove that neighbourhood stabilisers are isomorphic, we first need to construct a homeomorphism satisfying the hypotheses of that lemma. To this end, we will need the following lemma.

Lemma 9.2.8. *Let $\mathcal{X}$ be a Hausdorff topological space, let G be a group acting on $\mathcal{X}$ by homeomorphisms, and let $x, y \in \mathcal{X}$ be two points of $\mathcal{X}$. Suppose that there exist decreasing (with respect to inclusion) bases of neighbourhoods $\{U_i\}_{i\in\mathbb{N}}$ and $\{V_i\}_{i\in\mathbb{N}}$ of x and y, respectively, and a sequence $\{g_i\}_{i\in\mathbb{N}}$ of elements of G such that*

(i) $g_i(U_i) = V_i$

(ii) $g_{i+1}(z) = g_i(z)$ *for all* $z \in \mathcal{X} \setminus U_i$

for all $i \in \mathbb{N}$. Then, the sequence $\{g_i\}_{i\in\mathbb{N}}$ converges to a homeomorphism $f\colon \mathcal{X} \to \mathcal{X}$ such that $f(x) = y$. Furthermore, we have $f\,\mathrm{St}^0_G(x) f^{-1} = \mathrm{St}^0_G(y)$, so $\mathrm{St}^0_G(x)$ and $\mathrm{St}^0_G(y)$ are isomorphic.

Proof. Let us first show that the sequence $\{g_i\}_{i\in\mathbb{N}}$ converges. Notice that since $\mathcal{X}$ is Hausdorff, if the limit exists, it must be unique.

For all $i \in \mathbb{N}$, we have $g_i(x) \in V_i$, since $x \in U_i$ and $g_i(U_i) = V_i$ by hypothesis. As $\{V_i\}_{i\in\mathbb{N}}$ is a decreasing a basis of neighbourhoods of y, we conclude that for every neighbourhood V of y, there exists $N \in \mathbb{N}$ such that $g_i(x) \in V$ for all $i \geq N$. Thus, we have $\lim_{i\to\infty} g_i(x) = y$.

Now, for $z \in \mathcal{X}$ with $z \neq x$, since $\mathcal{X}$ is Hausdorff, we know that there must exist $N \in \mathbb{N}$ such that $z \notin U_N$, and thus $z \notin U_i$ for all $i \geq \mathbb{N}$, since the sequence is decreasing. Consequently, for every $m \in \mathbb{N}$, we have by hypothesis

$$g_{N+m+1}(z) = g_{N+m}(z).$$

Thus, by induction, we get that $g_i(z) = g_N(z)$ for all $i \geq N$. Therefore, we have $\lim_{i\to\infty} g_i(z) = g_N(z)$.

We have just shown that the sequence $\{g_i\}_{i\in\mathbb{N}}$ converges to a map $f\colon \mathcal{X} \to \mathcal{X}$ such that $f(x) = y$. It remains to show that this map is a homeomorphism.

To see this, let us first notice that the sequence $\{g_i^{-1}\}_{i\in\mathbb{N}}$ satisfies conditions (i) an (ii) if we exchange U_i and V_i. Indeed, we clearly have $g_i^{-1}(V_i) = U_i$. This implies that if $z \in \mathcal{X}\setminus V_i$, then we have that $g_i^{-1}(z) \in \mathcal{X}\setminus U_i$. Now, by condition (ii), we have that

$$g_{i+1}(g_i^{-1}(z)) = g_i(g_i^{-1}(z)) = z.$$

It follows that $g_{i+1}^{-1}(z) = g_i^{-1}(z)$ for all $z \in \mathcal{X} \setminus V_i$.

Therefore, the arguments above also apply to the sequence $\{g_i^{-1}\}_{i\in\mathbb{N}}$, which must then converge to a map $h\colon \mathcal{X} \to \mathcal{X}$ satisfying $h(y) = x$. We will show that h is the inverse of f. We already have $h(f(x)) = x$. Now, let $z \in \mathcal{X}$ be different from x. Then, as above, there exists $N \in \mathbb{N}$ such that $z \in \mathcal{X} \setminus U_N$, and therefore $f(z) = g_N(z)$. Since $z \notin U_N$, we must have that $g_N(z) \notin V_n$, since $g_N(U_N) = V_N$. Therefore, by a similar argument to the one above, we must have

$$h(g_N(z)) = g_N^{-1}(g_N(z)) = z.$$

This shows that $h \circ f$ is the identity map on $\mathcal{X}$. By a symmetric argument, we find that $f \circ h$ is also the identity, so $h = f^{-1}$.

To prove that f is a homeomorphism, we still need to prove that f and f^{-1} are both continuous. Let us prove that f is an open map. To see this, it suffices to show that for every open set $U \subseteq \mathcal{X}$ and for every $z \in U$, there exists some open subset $U' \subseteq U$ containing z and such that $f(U')$ is open. Now, if $z \neq x$, we can choose U' such that $U' \cap U_N = \emptyset$ for some $N \in \mathbb{N}$ large enough, since $\mathcal{X}$ is Hausdorff. In that case, by the above argument, we have $f(U') = g_N(U')$. As g_N is a homeomorphism, we find that $f(U')$ is open.

In the case where $z = x$, since $\{U_i\}_{i\in\mathbb{N}}$ is a basis of neighbourhoods, there exists $N \in \mathbb{N}$ such that $U_N \subseteq U$. We can thus choose $U' = U_N$. The result will then follow as soon as we show that $f(U_N) = V_N$. We have seen above that $f(w) = g_N(w)$ for all $w \in \mathcal{X} \setminus U_N$. As $g_N(U_N) = V_N$, this means that

$$f(\mathcal{X} \setminus U_N) = g_N(\mathcal{X} \setminus U_N) = \mathcal{X} \setminus V_N.$$

It follows from the fact that f is a bijection that $f(U_N) = V_N$.

We have thus shown that f is an open map. By symmetry, we must also have that f^{-1} is an open map, which means that f is a homeomorphism.

Finally, the fact that $f\,\mathrm{St}^0_G(x)f^{-1} = \mathrm{St}^0_G(y)$ follows directly from Lemma 9.2.7. Indeed, the map $f\colon \mathcal{X} \to \mathcal{X}$ is a homeomorphism that clearly satisfies condition (i) of Lemma 9.2.7. To see that it also satisfies condition (ii) of that same lemma, let $U \subset \mathcal{X}$ be a neighbourhood of x. Then, as $\{U_i\}_{i\in\mathbb{N}}$ is a basis of neighbourhoods of x, there exists $N \in \mathbb{N}$ such that $U_N \subseteq U$. By what we have seen above, we have $f(z) = g_N(z)$ for all $z \in \mathcal{X} \setminus U_N$. Therefore, we have $g_N^{-1}(f(z)) = z$ for all $z \in \mathcal{X} \setminus U_N$, which means that $g_N^{-1}f \in \mathrm{Rist}(U_N)$. As $U_N \subseteq U$, we have $\mathrm{Rist}(U_N) \leq \mathrm{Rist}(U)$, which concludes the proof. □

We can now formulate a condition under which the neighbourhood stabilisers of points are all isomorphic. This condition is slightly technical, but we hope that by stating it in this generality, it can be used to study not only groups acting on rooted trees, but also other interesting examples of groups acting on topological spaces.

Before we state this condition, let us first recall the definition of a minimal action.

Definition 9.2.9. Let G be a group acting on a topological space $\mathcal{X}$ by homeomorphisms. The action of G on $\mathcal{X}$ is said to be *minimal* if for every $x \in \mathcal{X}$, the orbit $G \cdot x$ is a dense subset of $\mathcal{X}$. ☙

We will now see that if a group acts by isometries in such a way that every rigid stabiliser acts minimally on some subset, then the stabilisers of regular points are all isomorphic.

Theorem 9.2.10. *Let $\mathcal{X}$ be a metric space and let G be a group acting on $\mathcal{X}$ by isometries. Suppose that the action of G on $\mathcal{X}$ is minimal and that for every open set $U \subseteq \mathcal{X}$ and every element $x \in U$, there exists an open subset $W \subseteq U$ containing x and such that the action of $\mathrm{Rist}_G(U)$ on W is minimal (in the sense that for every $y \in W$, the set $(\mathrm{Rist}_G(U) \cdot y) \cap W$ is a dense subset of W). Then, for all $x, y \in \mathcal{X}$, we have $\mathrm{St}^0_G(x) \cong \mathrm{St}^0_G(y)$. In particular, if x and y are both regular, then $\mathrm{St}_G(x) \cong \mathrm{St}_G(y)$.*

Proof. Let $x, y \in \mathcal{X}$ be two arbitrary elements of $\mathcal{X}$. Let us assume that for some $n \in \mathbb{N}$, we have two decreasing (with respect to inclusion) families $\{U_i\}_{0\leq i\leq n}$, $\{W_i\}_{0\leq i\leq n}$ of neighbourhoods of x and a family $\{g_i\}_{0\leq i\leq n}$ of elements of G with the following properties:

(i) $U_i \subseteq B(x, \frac{1}{i})$, where $B(x, \frac{1}{i})$ denotes the ball of radius $\frac{1}{i}$ centred at x, with the convention that $B(x, \frac{1}{0}) = \mathcal{X}$

(ii) $W_i \subseteq U_i$ and $\mathrm{Rist}_G(U_i)$ acts minimally on W_i

(iii) $y \in g_i(W_i)$

(iv) $g_i(z) = g_{i-1}(z)$ for all $z \in \mathcal{X} \setminus U_{i-1}$

for all $0 \leq i \leq n$ (where we define g_{-1} as the identity on $\mathcal{X}$ and $U_{-1} = \mathcal{X}$). We will now show that we can construct two neighbourhoods U_{n+1}, W_{n+1} of x and an element $g_{n+1} \in G$ such that the families $\{U_i\}_{0 \leq i \leq n+1}$, $\{W_i\}_{0 \leq i \leq n+1}$ and $\{g_i\}_{0 \leq i \leq n+1}$ satisfy conditions (i), (ii), (iii) and (iv).

As W_n is a neighbourhood of x, there exists some $m \in \mathbb{N}$ such that $B(x, \frac{1}{m}) \subseteq W_n$. Let us set $M = \max\{m, n+1\}$ and $U_{n+1} = B(x, M)$, so that we have $U_{n+1} \subseteq W_n$ and $U_{n+1} \subseteq B(x, \frac{1}{n+1})$. By hypothesis, there exists some open set $W_{n+1} \subseteq U_{n+1}$ containing x and such that $\mathrm{Rist}_G(U_{n+1})$ acts minimally on W_{n+1}. Therefore, U_{n+1} and W_{n+1} satisfy conditions (i) and (ii). We now only need to find $g_{n+1} \in G$ such that conditions (iii) and (iv) are satisfied.

By condition (iii), we have that $y \in g_n(W_n)$. Therefore, we have $g_n^{-1}(y) \in W_n$. As $\mathrm{Rist}_G(U_n)$ acts minimally on W_n by condition (ii), and since we have $W_{n+1} \subseteq U_{n+1} \subseteq W_n$, there must exist some $h \in \mathrm{Rist}_G(U_n)$ such that $h(g_n^{-1}(y)) \in W_{n+1}$. Let us set $g_{n+1} = g_n h^{-1}$. We then have that $y \in g_{n+1}(W_{n+1})$, so condition (iii) is satisfied. To see that condition (iv) is also satisfied, let us pick $z \in \mathcal{X} \setminus U_n$. We then have

$$g_{n+1}(z) = g_n(h^{-1}(z)) = g_n(z),$$

since $h \in \mathrm{Rist}_G(U_n)$.

Therefore, starting with $U_0 = W_0 = \mathcal{X}$ and g_0 the identity map on $\mathcal{X}$, we can construct by induction infinite families $\{U_i\}_{i \in \mathbb{N}}$, $\{W_i\}_{i \in \mathbb{N}}$ of neighbourhoods of x and a family $\{g_i\}_{i \in \mathbb{N}}$ of elements of G such that conditions (i), (ii), (iii) and (iv) are satisfied for all $i \in \mathbb{N}$.

It follows from condition (i) that $\{U_i\}_{i \in \mathbb{N}}$ is a basis of neighbourhoods of x. For all $i \in \mathbb{N}$, let us set $V_i = g_i(U_i)$. It follows from conditions (ii) and (iii) that V_i is a neighbourhood of y. Since g_i is an isometry, we have that $V_i \subseteq B(g_i(x), \frac{1}{i})$, and since $y \in V_i$, it follows that $V_i \subseteq B(y, \frac{2}{i})$. Therefore, $\{V_i\}_{i \in \mathbb{N}}$ is a basis of neighbourhoods of y.

To conclude the proof, we simply apply Lemma 9.2.8 with the families $\{U_i\}_{i \in \mathbb{N}}$, $\{V_i\}_{i \in \mathbb{N}}$ and $\{g_i\}_{i \in \mathbb{N}}$, which, as we have shown, satisfy the hypotheses of that lemma. □

Using this result, we can conclude that for many weakly branch groups, the stabilisers of regular points are isomorphic.

Corollary 9.2.11. *Let G be a weakly branch group acting on a spherically homogeneous rooted tree T (see Section 2.7 for definitions). If, for every $v \in T$, there exists $w \geq v$ such that $\mathrm{Rist}_G(v)$ acts spherically transitively on T_w (where T_w is the subtree rooted at w, see Definition 2.6.10), then the neighbourhood stabilisers of every points on the boundary ∂T are isomorphic. In particular, all parabolic subgroups of regular points on the boundary ∂T are isomorphic.*

Proof. As we have seen in Proposition 2.6.42, G acts by isometries on ∂T. Using the metric on ∂T, it is straightforward to show that if G acts spherically transitively on T, then its action on ∂T is minimal.

Now, let $U \subseteq \partial T$ be an open subset of the boundary and let $\xi \in U$ be an arbitrary element. It follows from the definition of the topology on ∂T (see Definition 2.6.38) that there exists some $v \in T$ such that $v \leq \xi$ and $\partial T_v \subseteq U$, where

$$\partial T_v = \{\zeta \in \partial T \mid v \leq \zeta\}.$$

By assumption, there exists $w \geq v$ such that $\mathrm{Rist}_G(v)(= \mathrm{Rist}_G(\partial T_v))$ acts spherically transitively on T_w. Without loss of generality, we can assume that $w \leq \xi$. Indeed, for every $g \in \mathrm{St}_G(v)$, we have that $\mathrm{Rist}_G(v)$ acts spherically transitively on $T_{g\cdot w}$, since

$$\mathrm{Rist}_G(v) = \mathrm{Rist}_G(g \cdot v) = g\,\mathrm{Rist}_G(v)g^{-1}.$$

As G acts spherically transitively on T, we thus have that $\mathrm{Rist}_G(v)$ acts spherically transitively on $T_{w'}$ for every $w' \in V$ such that $|w'| = |w|$ and $v \leq w'$. Thus, as $v \leq \xi$, we can replace w by the prefix of ξ of length $|w|$ if necessary, so that we can assume that $w \leq \xi$.

Using a similar argument to the one above, we see that the fact that $\mathrm{Rist}_G(v)$ acts spherically transitively on T_w implies that $\mathrm{Rist}_G(v)$ acts minimally on $\partial T_w \subset \partial T_v$. As $\partial T_v \subseteq U$, we have that $\mathrm{Rist}_G(v) \leq \mathrm{Rist}_G(U)$. Therefore, we have just shown that for every open set $U \subseteq \partial T$ and for every $\xi \in U$, there exists some $w \in V$ such that $\xi \in \partial T_w \subseteq U$ and the action of $\mathrm{Rist}_G(U)$ on ∂T_w is minimal. The result then follows immediately from Theorem 9.2.10. □

In particular, for branch groups, the parabolic subgroups of regular points are always isomorphic. To see this, we will need the following lemma.

Lemma 9.2.12. *Let G be a group acting spherically transitively on a spherically homogeneous rooted tree T and let $N \trianglelefteq G$ be a normal subgroup of finite index of G. Then, there exists $w \in T$ such that N acts spherically transitively on T_w.*

Proof. Let $\{w_i\}_{i\in\mathbb{N}}$ be a sequence of vertices of T such that $w_i < w_{i+1}$ and $|w_{i+1}| = |w_i| + 1$ for all $i \in \mathbb{N}$, and let us consider the sequence of subgroups $\{H_i = \mathrm{St}_G(w_i)N\}_{i\in\mathbb{N}}$. As $w_i < w_{i+1}$, we have that $\mathrm{St}_G(w_{i+1}) \leq \mathrm{St}_G(w_i)$. Therefore, we have

$$H_0 \geq H_1 \geq H_2 \geq \cdots \geq N.$$

As N is of finite index in G, this sequence must stabilise at some point, meaning that there exists $k \in \mathbb{N}$ such that $H_i = H_k$ for all $i \geq k$. We will prove that N acts transitively on T_{w_k}. For this, it suffices to prove that for all $v \in T_{w_k}$, there exists $i \in \mathbb{N}$ and $g \in N$ such that $g \cdot w_i = v$.

Let $v \in T_{w_k}$ be a vertex and let $i \in \mathbb{N}$ be the unique number such that $|w_i| = |v|$ (such a number exists and is unique thanks to our hypotheses on the sequence $\{w_j\}_{j\in\mathbb{N}}$). Since G acts spherically transitively on T, there must exist some $g \in G$ such that $g \cdot w_i = v$. Notice that since $w_k \leq w_i$ and $w_k \leq v$, we must have $g \cdot w_k = w_k$, which means that $g \in \mathrm{St}_G(w_k) \leq H_k$. Now, since $H_k = H_i$, this means that there exist $h \in \mathrm{St}_G(w_i)$ and $n \in N$ such that $g = nh$. Therefore, we get that $gh^{-1} \in N$. As $h \in \mathrm{St}_G(w_i)$, we have

$$gh^{-1} \cdot w_i = g \cdot w_i = v,$$

which concludes the proof. □

Corollary 9.2.13. *Let G be a branch group acting on a spherically homogeneous rooted tree T, and let $\xi, \zeta \in \partial T$ be two points on the boundary of T. Then, $\mathrm{St}_G^0(\xi) \cong \mathrm{St}_G^0(\zeta)$. In particular, if ξ and ζ are both regular, we have $\mathrm{St}_G(\xi) \cong \mathrm{St}_G(\zeta)$.*

Proof. By Corollary 9.2.11, it suffices to show that for all $v \in T$, there exists $w \geq v$ such that $\mathrm{Rist}_G(v)$ acts spherically transitively on T_w. Since G is a branch group, we have that $\mathrm{Rist}_G(|v|)$ is a normal subgroup of G of finite index. Therefore, by Lemma 9.2.12, there exists some $w \in T$ such that $\mathrm{Rist}_G(|v|)$ acts transitively on T_w. Let $v' \in T$ be the unique vertex such that $v' \leq w$ and $|v'| = |v|$. Without loss of generality, we can assume that $v' = v$. Indeed, it follows from spherical transitivity that if a normal subgroup acts spherically transitively on T_w, then it must act spherically transitively on $T_{w'}$ for all $w' \in T$ with $|w'| = |w|$.

Since $\mathrm{Rist}_G(|v|)$ acts transitively on T_w, and since $v \leq w$, we must have that $\mathrm{Rist}_G(v)$ acts transitively on T_w. Indeed,

$$\mathrm{Rist}_G(|v|) = \prod_{|v'|=|v|} \mathrm{Rist}_G(v'),$$

and $\mathrm{Rist}_G(v')$ acts trivially on T_w for all $v' \neq v$. This concludes the proof. □

The techniques employed in this section could also be used to study stabilisers in other kinds of groups than groups acting on rooted trees. In particular, Lemma 9.2.8 is quite general and could work in different settings. To illustrate this, we prove here a result regarding the stabilisers of irrational points in Thompson's group F. This result is not new, it was already proved by Golan and Sapir in [38]. However, we thought it might be interesting to include a proof of this fact using the techniques developed in this chapter to give an example of the different kinds of groups for which our method also applies.

Proposition 9.2.14. *Let F be Thompson's group and let $x, y \in [0,1]$ be two irrational points. Then, $\mathrm{St}_F(x) \cong \mathrm{St}_F(y)$.*

Proof. To prove the result, we will use Lemma 9.2.8. Before we begin, however, let us first prove that x and y are regular points.

Recall that Thompson's group F is the group of all piecewise linear homeomorphisms of $[0,1]$ with powers of 2 slopes and dyadic rational points of discontinuity for the derivative (see [19] for more information about this group). If $g \in F$ fixes the irrational point x, then we must have $2^k x + b = x$ for some $k \in \mathbb{Z}$ and some dyadic rational b. Therefore, we have $x(1-2^k) = b$. As x is irrational, this is only possible if $k = b = 0$. Therefore, we conclude that g must also fix some neighbourhood of x, which shows that x is a regular point. Likewise, y is also a regular point.

We now need to construct decreasing bases of neighbourhoods $\{U_i\}_{i\in\mathbb{N}}$ and $\{V_i\}_{i\in\mathbb{N}}$ of x and y, respectively, and a sequence $\{g_i\}_{i\in\mathbb{N}}$ of elements of the Thompson group F satisfying the hypotheses of Lemma 9.2.8.

Let $\{a_i\}_{i\in\mathbb{N}}$ be a strictly increasing sequence of dyadic rational numbers such that $a_0 > 0$ and $\lim_{i\to\infty} a_i = x$, and let $\{b_i\}_{i\in\mathbb{N}}$ be a strictly decreasing sequence of dyadic rational numbers such that $b_0 < 1$ and $\lim_{i\to\infty} b_i = x$ (such sequences obviously exist). Then, if we set $U_i = [a_i, b_i]$, it is clear that

$\{U_i\}_{i\in\mathbb{N}}$ is a decreasing basis of neighbourhoods of x. In a similar way, we construct sequences $\{c_i\}_{i\in\mathbb{N}}$ and $\{d_i\}_{i\in\mathbb{N}}$ of dyadic rational numbers such that $\{V_i = [c_i, d_i]\}_{i\in\mathbb{N}}$ is a decreasing sequence of neighbourhoods of y, with $c_0 > 0$ and $c_1 < 1$.

It is known that the Thompson group F acts transitively on the set of pairs $(a, b) \in [0, 1] \times [0, 1]$ where a and b are both dyadic rational number and $0 < a < b < 1$ (see for instance [19], Lemma 4.2). Therefore, there exists $g_0 \in F$ such that $g_0(U_0) = V_0$. Let us now assume that for some $n \in \mathbb{N}$, we have elements $\{g_i\}_{0\leq i\leq n}$ of the Thompson group F such that $g_i(U_i) = V_i$ for all $0 \leq i \leq n$ and $g_{i+1}(z) = g_i(z)$ for all $z \in [0, 1] \setminus U_i$ and for all $0 \leq i < n$. We will show that we can construct $g_{n+1} \in F$ with the same properties.

Recall that $U_n = [a_n, b_n]$ with a_n and b_n dyadic rational numbers. It is a well-known fact that $\mathrm{Rist}_F(U_n)$ acts transitively on pairs of dyadic rationals (a, b) such that $a_n < a < b < b_n$ (see again [19], Lemma 4.2). Now, since $V_{n+1} = [c_{n+1}, d_{n+1}]$ with c_{n+1}, d_{n+1} dyadic rationals with $c_n < c_{n+1} < d_{n+1} < d_n$, we have that

$$g_n^{-1}(V_{n+1}) = [g_n^{-1}(c_{n+1}), g_n^{-1}(d_{n+1})]$$

with $g_n^{-1}(c_{n+1})$, $g_n^{-1}(d_{n+1})$ dyadic rationals and

$$a_n < g_n^{-1}(c_{n+1}) < g_n^{-1}(d_{n+1}) < b_n$$

since $g_n(a_n) = c_n$ and $g_n(b_n) = d_n$, and since the Thompson group F preserves the order relation. It follows that there exists $h \in \mathrm{Rist}_F(U_n)$ such that $h(U_{n+1}) = g_n^{-1}(V_{n+1})$.

Let us set $g_{n+1} = g_n h$. We have that

$$g_{n+1}(U_{n+1}) = g_n(h(U_{n+1})) = g_n(g_n^{-1}(V_{n+1})) = V_{n+1}.$$

Furthermore, for $z \in [0, 1] \setminus U_n$, we have

$$g_{n+1}(z) = g_n(h(z)) = g_n(z)$$

since $h \in \mathrm{Rist}_F(U_n)$. Thus, by induction, we can construct a sequence $\{g_i\}_{i\in\mathbb{N}}$ of elements of F satisfying the hypotheses of Lemma 9.2.8, which concludes the proof.

□

It thus seem that this method is quite versatile. It would be interesting to better understand the class of groups to which it applies.

9.3 Isomorphism classes of parabolic subgroups

In Section 9.2, we saw that for weakly branch groups with sufficiently rich rigid stabilisers, the parabolic subgroups associated to regular points are all isomorphic. It is thus natural to ask if all parabolic subgroups are isomorphic for these groups. In this section, we will give a negative answer to this question by showing that under suitable conditions, points with non-isomorphic groups of germs cannot have isomorphic stabilisers. In particular, for those groups, the stabiliser of a singular point can never be isomorphic to the stabiliser of a regular point.

We begin by a simple observation regarding centralisers of elements in the neighbourhood stabiliser of a point for branch groups.

Lemma 9.3.1. *Let G be a finitely generated branch group acting on a spherically homogeneous rooted tree T and let $\xi \in \partial T$ be a point on the boundary of the tree. Let $g \in \mathrm{St}_G^0(\xi)$ be an element fixing some neighbourhood of ξ, let $C(g) \leq \mathrm{St}_G(\xi)$ be the centraliser of g in $\mathrm{St}_G(\xi)$, and let $N_g \trianglelefteq \mathrm{St}_G(\xi)$ be the largest normal subgroup of $\mathrm{St}_G(\xi)$ that is contained in $C(g)$. Then, the group $\mathrm{St}_G(\xi)/N_g$ is finitely generated.*

Proof. Let us first notice that N_g is well-defined, since the product of subgroups of $C(g)$ that are normal in $\mathrm{St}_G(\xi)$ is again a subgroup of $C(g)$ that is normal in $\mathrm{St}_G(\xi)$.

As g belongs to the neighbourhood stabiliser of ξ, there exists $v \in T$ with $v \leq \xi$ such that g acts trivially on T_v. Let us set $R = \mathrm{Rist}_G(v) \cap \mathrm{St}_G(\xi)$. We must have that $R \leq C(g)$, since the supports of g and of elements of $\mathrm{Rist}_G(v)$ are disjoint. Notice that R is a normal subgroup of $\mathrm{St}_G(\xi)$, since $\mathrm{St}_G(\xi) \leq \mathrm{St}_G(v)$. Therefore, we have that $R \leq N_g$, which implies that

$$\mathrm{St}_G(\xi)/N_g \cong (\mathrm{St}_G(\xi)/R)/(N_g/R).$$

As the quotient of a finitely generated group is finitely generated, it suffices to show that $\mathrm{St}_G(\xi)/R$ is finitely generated.

We have that

$$\begin{aligned}\mathrm{St}_G(\xi)/R &= \mathrm{St}_G(\xi)/(\mathrm{Rist}_G(v) \cap \mathrm{St}_G(\xi)) \\ &\cong (\mathrm{St}_G(\xi)\,\mathrm{Rist}_G(v))/\,\mathrm{Rist}_G(v).\end{aligned}$$

For any $v' \in T$ with $|v'| = |v|$ and $v' \neq v$, we have that $\mathrm{Rist}_G(v')$ acts trivially on T_v. Since $v \leq \xi$, we conclude that $\mathrm{Rist}_G(v') \leq \mathrm{St}_G(\xi)$. Therefore, we must have

$$\mathrm{Rist}_G(|v|) \leq \mathrm{St}_G(\xi)\,\mathrm{Rist}_G(v).$$

Since G is a branch group, $\mathrm{Rist}_G(|v|)$ is a subgroup of finite index in G, which implies that $\mathrm{St}_G(\xi)\,\mathrm{Rist}_G(v)$ must also be a subgroup of finite index. As G is finitely generated, we conclude that $\mathrm{St}_G(\xi)\,\mathrm{Rist}_G(v)$ must also be finitely generated, and therefore so must $\mathrm{St}_G(\xi)/R$. This concludes the proof. □

We will now show that the property of having a finitely generated quotient by centralising elements is a characterisation of elements belonging to the neighbourhood stabiliser. More precisely, we will show that if ξ is a singular point and $g \in \mathrm{St}_G(\xi) \setminus \mathrm{St}_G^0(\xi)$ is an element that does not belong to the neighbourhood stabiliser of ξ, then $\mathrm{St}_G(\xi)/N$ is not finitely generated for any normal subgroup N contained in the centraliser of g. The proof will be very similar to the proof of Theorem 9.1.1.

Lemma 9.3.2. *Let X be a finite alphabet and let G be a weakly branch group acting on the regular rooted tree X^*. Let $\xi \in X^\infty$ be a singular point, let $g \in \mathrm{St}_G(\xi) \setminus \mathrm{St}_G^0(\xi)$ be an element fixing ξ but not any neighbourhood of it, and let $C(g) \leq \mathrm{St}_G(\xi)$ be the centraliser of g in $\mathrm{St}_G(\xi)$. Let us suppose that there exists $M \in \mathbb{N}$ such that, for all $v \in X^*$, $\mathrm{Rist}_G(v)$ acts non-trivially on vX^M. Then, for every normal subgroup $N \trianglelefteq \mathrm{St}_G(\xi)$ such that $N \leq C(g)$, the group $\mathrm{St}_G(\xi)/N$ is not finitely generated.*

Proof. For any $i \in \mathbb{N}$, let us denote by $\xi_i \in X^i$ the prefix of length i of ξ. As $g \notin \mathrm{St}^0_G(\xi)$, we can find a strictly increasing subsequence $\{i_k\}_{k\in\mathbb{N}}$ and elements $x_k \in X$, $\zeta_k \in X^\infty$ such that $\xi_{i_k}x_k \neq \xi_{i_k+1}$ and $g \cdot (\xi_{i_k}x_k\zeta_k) \neq (\xi_{i_k}x_k\zeta_k)$. For all $k \in \mathbb{N}$, let us denote by $w_k \in X^M$ the prefix of length M of ζ_k.

As in the proof of Theorem 9.1.1, we will construct an infinite sequence of different maps from $\mathrm{St}_G(\xi)$ to a finite group. For every $k \in \mathbb{N}$, let us denote by $Y_k \subseteq X^{M+1}$ the orbit of x_kw_k under the action of $\varphi_{\xi_{i_k}}(\mathrm{St}_G(\xi))$. We can then define a map

$$\alpha_k\colon \mathrm{St}_G(\xi) \to \mathrm{Sym}(X^{M+1})$$

by

$$\alpha_k(h)(w) = \begin{cases} w & \text{if } w \notin Y_k \\ \varphi_{\xi_{i_k}}(h)\cdot w & \text{if } w \in Y_k. \end{cases}$$

In other words, α_k is the restriction of the action of $\varphi_{\xi_{i_k}}(\mathrm{St}_G(\xi))$ to Y_k. It is then clear that α_k is a homomorphism for all $k \in \mathbb{N}$.

As in the proof of Theorem 9.1.1, we will prove that these maps are all different by showing that for all $k, m \in \mathbb{N}$, we have $\alpha_k = \alpha_{k+m}$ if and only if $m = 0$.

Let us fix $k \in \mathbb{N}$, and let us consider $\mathrm{Rist}_G(\xi_{i_k}x_k)$. Notice that since $\xi_{i_k}x_k \neq \xi_{i_k+1}$, we have that $\mathrm{Rist}_G(\xi_{i_k}x_k)$ acts trivially on $\xi_{i_k+1}X^\infty$. In particular, this means that $\mathrm{Rist}_G(\xi_{i_k}x_k) \leq \mathrm{St}_G(\xi)$.

By assumption, there exists $r \in \mathrm{Rist}_G(\xi_{i_k}x_k)$ such that r acts non-trivially on $\xi_{i_k}x_kX^M$. In other words, there exists $w \in X^M$ such that $\varphi_{\xi_{i_k}x_k}(r)\cdot w \neq w$. Without loss of generality, we can assume that $w = w_k$. Indeed, by spherical transitivity of the action of G, we know that there must exist some $h \in G$ such that $h \cdot \xi_{i_k}x_kw = \xi_{i_k}x_kw_k$. We then have that $\varphi_{\xi_{i_k}x_k}(hrh^{-1}) \cdot w_k \neq w_k$. Since we have $h \in \mathrm{St}_G(\xi_{i_k}x_k)$, and since $\mathrm{Rist}_G(\xi_{i_k}x_k)$ is normal in $\mathrm{St}_G(\xi_{i_k}x_k)$, we get $hrh^{-1} \in \mathrm{Rist}_G(\xi_{i_k}x_k)$. Therefore, replacing r by some conjugate if necessary, we can assume that $\varphi_{\xi_{i_k}x_k}(r) \cdot w_k \neq w_k$. In particular, this implies that $\alpha_k(r) \neq 1$. However, for any $m \geq 1$, we have that $\varphi_{k+m}(r) = 1$, since $\varphi_{\xi_{i_{k+1}}}(r) = 1$. Thus, we conclude that all homomorphisms α_k are different.

To conclude the proof, we only need to show that if $N \trianglelefteq \mathrm{St}_G(\xi)$ is a normal subgroup such that $N \leq C(g)$, then $\alpha_k(N) = 1$ for all $k \in \mathbb{N}$. Indeed, this will imply that the homomorphisms $\alpha_k\colon \mathrm{St}_G(\xi) \to \mathrm{Sym}(X^M)$ project to homomorphisms $\tilde{\alpha}_k\colon \mathrm{St}_G(\xi)/N \to \mathrm{Sym}(X^M)$. We will then have infinitely many different homomorphisms between $\mathrm{St}_G(\xi)/N$ and the finite group $\mathrm{Sym}(X^M)$, which implies that $\mathrm{St}_G(\xi)/N$ cannot be finitely generated.

Let us now prove the claim. Let $N \trianglelefteq \mathrm{St}_G(\xi)$ be a normal subgroup contained in $C(g)$, and let $k \in \mathbb{N}$ be any number. We need to show that for any $h \in N$, we have $\alpha_k(h) = 1$. In other words, we need to show that $h\cdot\xi_{i_k}y = \xi_{i_k}y$ for all $y \in Y_k$. Let us suppose, for the sake of contradiction, that this is not the case. Then, using the fact that $\mathrm{St}_G(\xi)$ acts transitively on Y_k and that N is normal in $\mathrm{St}_G(\xi)$, we can assume without loss of generality that $h \cdot \xi_{i_k}x_kw_k \neq \xi_{i_k}x_kw_k$.

Using the fact that N is a normal subgroup of $\mathrm{St}_G(\xi)$ and that $N \not\leq \mathrm{St}_G(\xi_{i_k}x_kw_k)$, we get from Lemma 2.7.4 that $\mathrm{Rist}'_{\mathrm{St}_G(\xi)}(\xi_{i_k}x_kw_k) \leq N$. Now, since $\xi_{i_k}x_kw_k$ is not a prefix of ξ, we have that $\mathrm{Rist}_G(\xi_{i_k}x_kw_k)$ acts trivially on ξ, and thus $\mathrm{Rist}_G(\xi_{i_k}x_kw_k) \leq \mathrm{St}_G(\xi)$. Consequently, we have that $\mathrm{Rist}_{\mathrm{St}_G(\xi)}(\xi_{i_k}x_kw_k) = \mathrm{Rist}_G(\xi_{i_k}x_kw_k)$, and so $\mathrm{Rist}'_G(\xi_{i_k}x_kw_k) \leq N$. It follows that $\mathrm{Rist}'_G(v) \leq N$ for all $v \geq \xi_{i_k}x_kw_k$.

Now, by assumption, we have $g \cdot \xi_{i_k} x_k \zeta_k \neq \xi_{i_k} x_k \zeta_k$. Therefore, there must exist some $v_k \geq \xi_{i_k} x_k w_k$ such that $g \cdot v_k \neq v_k$. It follows that we have

$$g \operatorname{Rist}'_G(v_k) g^{-1} = \operatorname{Rist}'_G(g \cdot v_k)$$

with $g \cdot v_k \neq v_k$. On the other hand, since $\operatorname{Rist}'_G(v_k) \leq N \leq C(g)$, we must have

$$g \operatorname{Rist}'_G(v_k) g^{-1} = \operatorname{Rist}'_G(v_k).$$

As $\operatorname{Rist}_G(v_k) \cap \operatorname{Rist}_G(g \cdot v_k) = \{1\}$, we must have $\operatorname{Rist}'_G(v_k) = \{1\}$, which contradicts Lemma 2.7.6. We conclude that for all $h \in N$, we must have $\alpha_k(h) = 1$, which concludes the proof. □

Using these two lemmas, we can now conclude that for finitely generated branch groups with a sufficiently regular action, two parabolic subgroups with non-isomorphic groups of germs are not isomorphic.

Theorem 9.3.3. *Let X be a finite alphabet, let G be a finitely generated branch group acting on X^*, let $\xi, \zeta \in X^\infty$ be two points on the boundary with isomorphic stabilisers and let*

$$f \colon \operatorname{St}_G(\xi) \to \operatorname{St}_G(\zeta)$$

be an isomorphism. If there exists $M \in \mathbb{N}$ such that $\operatorname{Rist}_G(v)$ acts non-trivially on vX^M for all $v \in X^*$, then we must have

$$f(\operatorname{St}^0_G(\xi)) = \operatorname{St}^0_G(\zeta).$$

Proof. It suffices to show that for every $g \in \operatorname{St}^0_G(\xi)$, we have $f(g) \in \operatorname{St}^0_G(\zeta)$, since the other inclusion can be obtained by symmetry using f^{-1}.

Let $C(g) \leq \operatorname{St}_G(\xi)$ be the centraliser of g and let $N_g \trianglelefteq \operatorname{St}_G(\xi)$ be the largest normal subgroup of $\operatorname{St}_G(\xi)$ that is contained in $C(g)$. Then, by Lemma 9.3.1, the group $\operatorname{St}_G(\xi)/N_g$ is finitely generated. Therefore, by applying the isomorphism f, we find that $\operatorname{St}_G(\zeta)/f(N_g)$ is a finitely generated group. Now, since f is an isomorphism, we must have that $f(C(g)) = C(f(g))$, which means that $f(N_g)$ is a normal subgroup of $\operatorname{St}_G(\zeta)$ contained in the centraliser of $f(g)$. As $\operatorname{St}_G(\zeta)/f(N_g)$ is finitely generated, it follows from Lemma 9.3.2 that we must have $f(g) \in \operatorname{St}^0_G(\zeta)$. This concludes the proof. □

Corollary 9.3.4. *Let X be a finite alphabet, and let G be a finitely generated branch group acting on X^*. Suppose that there exists $M \in \mathbb{N}$ such that $\operatorname{Rist}_G(v)$ acts non-trivially on vX^M for all $v \in X^*$. Then, two parabolic subgroups with non-isomorphic groups of germs cannot be isomorphic. In particular, the stabiliser of a regular point cannot be isomorphic to the stabiliser of a singular point.*

Proof. Let $\xi, \zeta \in X^\infty$ be two points such that $\operatorname{St}_G(\xi)$ is isomorphic to $\operatorname{St}_G(\zeta)$, and let $f \colon \operatorname{St}_G(\xi) \to \operatorname{St}_G(\zeta)$ be such an isomorphism. Then, by Theorem 9.3.3, we must have $f(\operatorname{St}^0_G(\xi)) = \operatorname{St}^0_G(\zeta)$, which implies that f projects to an isomorphism between $\operatorname{St}_G(\xi)/\operatorname{St}^0_G(\xi)$ and $\operatorname{St}_G(\zeta)/\operatorname{St}^0_G(\zeta)$. □

The hypotheses of Theorem 9.3.3 are satisfied by any finitely generated self-similar regular branch group. We then get the following corollary.

Corollary 9.3.5. *Let X be a finite alphabet and let G be a finitely generated self-similar regular branch group acting on X^*. Then, the stabilisers of singular points on the boundary are never isomorphic to the stabilisers of regular points.*

In particular, using this result, we conclude that the Grigorchuk group has exactly two isomorphism classes of parabolic subgroups.

Corollary 9.3.4 tells us that parabolic subgroups with non-isomorphic groups of germs are not isomorphic. It would be interesting to know if the converse is true.

Question 9.3.6. Suppose that G is a finitely generated branch groups satisfying the hypotheses of Corollary 9.3.4. If two parabolic subgroups have isomorphic groups of germs, are they necessarily isomorphic?

We know from Corollary 9.2.13 that the answer to this question is yes in the special case where the groups of germs are trivial, but we do not know anything about the general case.

9.4 Index of parabolic subgroups

So far, in this chapter, we have mainly studied parabolic subgroups for groups acting spherically transitively on a rooted tree. For such groups, every parabolic subgroup must be of infinite index. However, in general, the index of a parabolic subgroup needs not be infinite. Obviously, this cannot be the case for finite groups, but it is not hard to construct examples of infinite groups acting on a rooted tree such that every parabolic subgroup is of finite index. It is thus natural to ask under which conditions an infinite group acting on a rooted tree must admit a parabolic subgroup of infinite index (or equivalently, a point on the boundary of the tree with an infinite orbit).

In this section, we provide an answer to this question in the case of automata groups, which are a special class of finitely generated self-similar groups. We will show that an automaton group is infinite if and only if it contains a parabolic subgroup of infinite index. In fact, using similar ideas, our result can be generalised to a special class of semigroups called automata semigroups. However, we will not prove it in such generality here, since this thesis only concerns itself with groups. We refer the interested reader to [21] for a more general statement.

Let us begin by giving a definition of automata groups.

Definition 9.4.1. Let X be a finite alphabet, and let $G \leq \mathrm{Aut}(X^*)$ be a self-similar group acting on the regular rooted tree X^*. The group G will be called an *automaton group* if it admits a finite symmetric generating set S such that $\varphi_x(s) \in S \cup \{1\}$ for all $s \in S$ and all $x \in X$. ❧

Notice that a self-similar groups is an automaton group if and only if it admits a generating set such that the projection maps are 1-Lipschitz maps in the word metric associated to this generating set.

Proposition 9.4.2. *Let X be a finite alphabet, and let $G \leq \mathrm{Aut}(X^*)$ be a self-similar group acting on the regular rooted tree X^*. Then, G is an automaton group if and only if there exists a finite symmetric generating set S such that*

$|\varphi_v(g)| \leq |g|$ *for all* $g \in G$ *and for all* $v \in X^*$*, where* $|\cdot|\colon G \to \mathbb{N}$ *is the word norm associated to* S *(see Definition 2.5.2).*

Proof. If G is an automaton group, we have $\varphi_x(s) \in S \cup \{1\}$ for all $x \in X$ and $s \in S$, so $|\varphi_x(s)| \leq |s|$. We conclude by subadditivity and induction.

On the other hand, if $|\varphi_v(g)| \leq |g|$ for all $g \in G$ and $v \in X^*$, then in particular, we have $|\varphi_x(s)| \leq |s| \leq 1$ for all $x \in X$ and $s \in S$, which means that $\varphi_x(s) \in S \cup \{1\}$. □

We have already seen many examples of automata groups. Indeed, it follows from Proposition 9.4.2 that every self-similar non-ℓ_1-expanding group (see Definition 3.1.7 is an automaton group. In particular, every Šunić group (see Section 2.9) is an automaton group. It is also easy to see that the Basilica group (see Section 8.1) is an automaton group. In fact, every finitely generated self-similar group mentioned in this thesis is an automaton group.

Remark 9.4.3. Automata groups are usually defined in terms of Mealy automata, which are deterministic finite-state transducers. However, for the purpose of this thesis, we shall not need the theory of automata. We thus preferred to define automata groups as in Definition 9.4.1, which is equivalent to the standard definition but is easier to state in our context. ❧

We will now prove that for every infinite automaton group G acting on a regular rooted tree X^*, there exists some $\xi \in X^\infty$ such that the orbit of ξ under G is infinite, which is equivalent to saying that $\mathrm{St}_G(\xi)$ has infinite index in G. In order to do this, we first define a map from X^* to $\mathbb{N} \cup \{\infty\}$ that we will call the *potential function*.

Definition 9.4.4. Let X be a finite alphabet and let $G \leq \mathrm{Aut}(X^*)$ be a group acting on X^*. We define the *potential function* $P_G\colon X^* \to \mathbb{N} \cup \{\infty\}$ on X^* by

$$P_G(v) = \sup_{\xi \in X^\infty} |G \cdot (v\xi)|$$

for $v \in X^*$. In other words, $P_G(v)$ is the supremum of the size of the orbits of points on the boundary below v. ❧

Remark 9.4.5. Notice that for any $v \in X^*$, we have

$$P_G(v) = \max_{x \in X} P_G(vx).$$

In particular, $P_G(v) = \infty$ if and only if there exists $x \in X$ such that $P_G(vx) = \infty$. ❧

As the next lemma shows, an automaton group is infinite if and only if the root of the rooted tree on which it acts has infinite potential.

Lemma 9.4.6. *Let* X *be a finite set and let* $G \leq \mathrm{Aut}(X^*)$ *be a finitely generated group acting on* X^**. Then,* G *is infinite if and only if* $P_G(\varepsilon) = \infty$*, where* $\varepsilon \in X^*$ *is the empty word.*

Proof. If $P_G(\varepsilon) = \infty$, then it is clear that G must be infinite. We now need to show the converse. Let us assume that $P_G(\varepsilon)$ is finite. Then, for any $\xi \in X^\infty$, the orbit of ξ is of size at most $P_G(\varepsilon)$. As G is finitely generated, there are only finitely many different homomorphisms from G to $\mathrm{Sym}(P_G(\varepsilon))$. This implies that

$$H = \bigcap_{\xi \in X^\infty} \mathrm{St}_G(\xi)$$

is of finite index in G. However, as the action of G on X^∞ is faithful, we must have $H = \{1\}$, which implies that G is finite. □

The potential function is well-behaved with respect to projections on some vertex, as the next lemma shows.

Lemma 9.4.7. *Let X be a finite alphabet, $G \leq \mathrm{Aut}(X^*)$ be a group acting on X^*, and $v \in X^*$ be a vertex. Then, for all $w \in X^*$, we have*

$$P_G(vw) = [G : \mathrm{St}_G(v)] P_{G_v}(w),$$

where, as in Definition 2.6.25, $G_v = \varphi_v(\mathrm{St}_G(v))$.

Proof. Let $\xi \in X^\infty$ be a point on the boundary. We have that

$$|G \cdot vw\xi| = [G : \mathrm{St}_G(v)] |\,\mathrm{St}_G(v) \cdot vw\xi|.$$

Now, it is clear that $|\,\mathrm{St}_G(v) \cdot vw\xi| = |G_v \cdot w\xi|$, and so

$$|G \cdot vw\xi| = [G : \mathrm{St}_G(v)] |G_v \cdot w\xi|.$$

Taking the supremum over all $\xi \in X^\infty$ on both sides yields the result. □

We are now ready to prove that an automaton group is infinite if and only if there is a point on the boundary with infinite orbit.

Theorem 9.4.8. *Let X be a finite alphabet and let G be an automaton group acting on X^*. Then, G is infinite if and only if there exists $\xi \in X^\infty$ such that $|G \cdot \xi| = \infty$ (or, equivalently, $[G : \mathrm{St}_G(\xi)] = \infty$).*

Proof. It is clear that if there exists some $\xi \in X^\omega$ such that $|G \cdot \xi| = \infty$, then G must be infinite. Let us prove the converse.

Suppose that G is infinite. For each $u \in X^*$, we define the set

$$B_u = \{v \in X^* \mid v \geq u \text{ and } P_G(v) = \infty\}.$$

It follows from Remark 9.4.5 that $B_u \neq \emptyset$ if and only if $P_G(u) = \infty$. By Lemma 9.4.6, we then get that $B_\varepsilon \neq \emptyset$.

We may assume that there exists some $w \in X^*$ with $P_G(w) = \infty$ and such that for all $v \in B_w$,

$$|G \cdot w| = |G \cdot v|.$$

Indeed, suppose that this is not the case. Then, for every $u \in X^*$ with $P_G(u) = \infty$, there exists some $v \in B_u$ such that $|G \cdot u| < |G \cdot v|$. Thus, starting from ε, which satisfies $P_G(\varepsilon) = \infty$, we can construct a strictly increasing sequence $\{u_i\}_{i \in \mathbb{N}}$ of elements of X^* such that $u_{i+1} \in B_{u_i}$ and $|G \cdot u_i| < |G \cdot u_{i+1}|$ for all $i \in \mathbb{N}$. Since the sequence $\{u_i\}_{i \in \mathbb{N}}$ is strictly increasing, there exists a unique element $\xi \in X^\infty$ such that u_i is a prefix of ξ for all $i \in \mathbb{N}$. It follows from the fact that $|G \cdot u_i| < |G \cdot u_{i+1}|$ that the size of the orbit of ξ is unbounded, which is what we were looking for. Thus, it suffices to consider the case where there exists some $w \in X^*$ as above.

In this case, for every $v \in B_w$, we have $\mathrm{St}_G(w) = \mathrm{St}_G(v)$, since $w \leq v$ and $|G \cdot w| = |G \cdot v|$. It follows that for every $v \in B_w$, we have

$$G_v = \varphi_v(\mathrm{St}_G(w)).$$

Since $\mathrm{St}_G(w)$ is a finite index subgroup of the finitely generated group G, we have that it is finitely generated. Let us fix a finite set D of generators for $\mathrm{St}_G(w)$, and let

$$L = \max\{|d| \mid d \in D\},$$

where $|\cdot|\colon G \to \mathbb{N}$ is the word norm with respect to a finite generating set S of G satisfying the property of Definition 9.4.1. Now, since for every $v \in B_w$, we have

$$|\varphi_v(d)| \leq |d| \leq L,$$

we see that there are only finitely many possibilities for the image by φ_v of generators of $\mathrm{St}_G(w)$. As φ_v is uniquely determined by the image of generators of $\mathrm{St}_G(w)$, we conclude that the set

$$\{G_v \mid v \in B_w\}$$

is finite. Therefore, the set

$$\mathcal{P} = \{P_{G_v}(a) \mid v \in B_w, a \in X\} \subset \mathbb{N} \cup \{\infty\}$$

is also finite. Let us denote by M the maximal element of the set $\mathcal{P} \setminus \{\infty\}$.

Let $\xi \in X^\infty$ be an element of the boundary. For every $i \in \mathbb{N}$, let us denote by $\xi_i \in X^*$ the prefix of ξ of length i. We are interested in studying the size of the orbit of $w\xi$ under G. If $w\xi_i \in B_w$ for all $i \in \mathbb{N}$, then we must have $|G \cdot w\xi| = |G \cdot w|$, since we have $|G \cdot w\xi_i| = |G \cdot w|$ for all $i \in \mathbb{N}$. If not, then let $i_0 \in \mathbb{N}$ be the largest element of $\mathbb{N}$ such that $w\xi_{i_0} \in B_w$. Notice that i_0 is well-defined, since if follows from Remark 9.4.5 that if $w\xi_j \notin B_w$, then $w\xi_{j+m} \notin B_w$ for all $m \in \mathbb{N}$. Let $a \in X$ be the unique element of X such that $\xi_{i_0}a = \xi_{i_0+1}$. By definition, we must have $|G \cdot w\xi| \leq P(w\xi_{i_0}a)$, and since $w\xi_{i_0}a \notin B_w$, we must have $P(w\xi_{i_0}a) < \infty$.

From Lemma 9.4.7, we have

$$\begin{aligned} P(w\xi_{i_0}a) &= [G : \mathrm{St}_G(w\xi_{i_0})] P_{G_{w\xi_{i_0}}}(a) \\ &= |G \cdot w| P_{G_{w\xi_{i_0}}}(a) \\ &\leq |G \cdot w| M \end{aligned}$$

where the last inequality comes from the fact that since $P(w\xi_{i_0}a) < \infty$, we must have $P_{G_{w\xi_{i_0}}}(a) < \infty$, and so $P_{G_{w\xi_{i_0}}}(a) < M$ by the definition of M.

We have thus shown that for every $\xi \in X^\infty$, we have $|G \cdot w\xi| \leq |G \cdot w|M$. Therefore, we must have $P_G(w) \leq |G \cdot w|M < \infty$, a contradiction to the fact that $P_G(w) = \infty$. We conclude that there must exist a point on the boundary with infinite orbit. □

Remark 9.4.9. It was remarked by Ivan Mitrofanov that the proof of Theorem 9.4.8 immediately generalises to finitely generated subgroups of automata groups, which are not necessarily themselves automata groups. ❧

Theorem 9.4.8 tells us that for an infinite automaton group G acting on X^*, there always exists a point $\xi \in X^\infty$ on the boundary such that the orbit of ξ under the action of G is infinite. However, we do not currently know anything about this points ξ. In particular, we do not know if it is possible to choose it in such a way that it is *periodic*, in the sense that there exists some $v \in X^*$ such that $\xi = v^\infty$.

Question 9.4.10. If G is an infinite automaton group acting on X^*, does there always exist $v \in X^*$ such that the orbit of v^∞ by the action of G is infinite?

This question is equivalent to asking if the dual semigroup of an automaton group can be infinite but contain only periodic elements, which is an analogue of the Burnside problem for automata semigroups. An answer to this question would be very interesting, since a positive answer would prove that the dual semigroup of an infinite automaton group always contain a non-abelian free semigroup and are thus of exponential growth, by a result of the author and Ivan Mitrofanov [33], whereas a negative answer would give us an example of an automaton group with different properties than any of the others we know so far. So far, this is only known in the very special case of groups generated by invertible and reversible automata of two states [56] and three states [57].

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