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C*-SIMPLE GROUPS: AMALGAMATED FREE PRODUCTS, HNN EXTENSIONS, AND FUNDAMENTAL GROUPS OF 3-MANIFOLDS

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Abstract. We establish sufficient conditions for the C*-simplicity of two classes of groups. The first class is that of groups acting on trees, such as amalgamated free products, HNN-extensions, and their normal subgroups; for example normal subgroups of Baumslag-Solitar groups. The second class is that of fundamental groups of compact 3-manifolds, related to the first class by their Kneser-Milnor and JSJ-decompositions.

Much of our analysis deals with conditions on an action of a group Γ on a tree T which imply the following three properties: abundance of hyperbolic elements, better called strong hyperbolicity, minimality, both on the tree T and on its boundary ∂T, and faithfulness in a strong sense. For this, we define in particular the notion of a *slender* automorphism of T, namely of an automorphism such that its set of fixed points on ∂T is nowhere dense with respect to the shadow topology.

1. Introduction

In the first part of this paper, we analyse actions of groups on trees, and we establish that the reduced C*-algebras of some of these groups are simple. In the second part, we apply this to fundamental groups of compact 3-manifolds and their subnormal subgroups.

Given a group Γ, recall that its *reduced C^*-algebra* $C^*_r(\Gamma)$ is the closure for the operator norm of the group algebra $\mathbf{C}[\Gamma]$ acting by the left-regular representation on the Hilbert space $\ell^2(\Gamma)$. For an introduction to group C*-algebras, see for example Chapter VII of [Davi–96].

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A group is *C^*-simple* if it is infinite and if its reduced C*-algebra has no non-trivial two-sided ideals.

Non-abelian free groups are C*-simple. The first proof, due to Powers [Powe–75], relies on a combinatorial property of free groups shared by many other groups, called for this reason *Powers groups*; the definition is recalled in Section 7 below. For now, we emphasize that

Powers groups are C^*-simple

(the converse does *not* hold). Whenever a group is known to be a Powers group, the proof involves Proposition 17 below, or a variant thereof. Let us agree that a

- *strongly Powers group* is a group Γ such that any subnormal subgroup $N \neq \{1\}$ of Γ is a Powers group.

(The definition of "subnormal" is recalled in the proof of Proposition 18.) More on these groups in Section 7 and in [Harp–07].

Part I, on groups acting on trees, begins with Section 2, where we collect essentially known facts which are useful for the proof of Proposition 1. We need first some terminology. A *pending ray* in a tree T is a ray with vertex set $(x_n)_{n\geq 1}$ such that x_n has degree 2 in T for all n large enough. Cofinal classes of rays are the elements of the *boundary* ∂T, which is a space with a natural *shadow topology* (more on this in Section 2). An action of a group Γ on a tree T is

- *minimal* if there does not exist any proper Γ-invariant subtree in T.

If Γ acts by automorphisms on T, it acts also by homeomorphisms on ∂T. It is important not to confuse the minimality of the action on the tree and the minimality of the action on the boundary (which means that the boundary does not have any non-trivial Γ-invariant closed subspace). An automorphism γ of T, with $\partial T \neq \emptyset$, is

- *slender* if its fixed point set in ∂T has empty interior,

and the action of Γ on T is *slender* if any $\gamma \in \Gamma$, $\gamma \neq 1$, is slender. Observe that a slender action is faithful. As our trees are assumed to be non-empty, observe also that a tree on which a group acts without fixed vertex and without fixed geometric edge has infinite diameter and non-empty boundary.

Proposition 1. *Let T be a tree without vertices of degree ≤ 1 and without pending rays. Let Γ be a countable group which acts on T. We assume that the action is minimal (in particular without fixed vertex), without fixed boundary point, and slender.*

Then Γ is a strongly Powers group.

Section 3 is a reminder on the fundamental group Γ of a graph of groups $\mathbf{G} = (G, Y)$, which acts on its Bass-Serre tree T. We recall known criteria for T to be "small" (a vertex, or a linear tree), for the action to be faithful, and for the action to be minimal. Most of this can be found in papers by H. Bass. In Section 4, we analyse further the two standard examples and obtain the following results; they rely on Proposition 18, which is a reformulation of Proposition 1.

Let A, B, C be three groups, let ι_A, ι_B be injections of C in A, B respectively, and let

$$\Gamma \,=\, A *_C B \,=\, \langle A, B \mid \iota_A(c) = \iota_B(c) \;\; \forall c \in C \rangle$$

be the corresponding amalgam. Below, we identify C to a subgroup of A and of B. We define inductively a decreasing sequence $C_0 \supset C_1 \supset \cdots$ of subgroups of C by $C_0 = C$ and

$$C_k \,=\, \big(\bigcap_{a \in A} a^{-1} C_{k-1} a \big) \cap \big(\bigcap_{b \in B} b^{-1} C_{k-1} b \big) \quad \text{for } k \geq 1.$$

Proposition 2. *Consider as above a countable amalgam $\Gamma = A *_C B$, with moreover $[A : C] \geq 3$ and $[B : C] \geq 2$. Assume that $C_k = \{1\}$ for some $k \geq 0$.*

Then Γ is a strongly Powers group.

The condition $C_k = \{1\}$ has the following geometrical interpretation: for $\gamma \in \Gamma$, if there exists $e \in E(T)$ such that the k-neighbourhood

$$\mathcal{V}_k(e) \,=\, \{x \in V(T) \mid \min\{d(x, s(e)),\ d(x, t(e))\} \leq k\}$$

is pointwise fixed by γ, then $\gamma = 1$. In particular, if $C_k = \{1\}$ for some $k \geq 0$, then the action of Γ on T is faithful.

Proposition 2 is an improvement on previous results in two ways. (i) It establishes a property of any subnormal subgroup $N \neq \{1\}$, and not only of $N = \Gamma$. (ii) Its hypothesis are stated in terms of A, B, and C only; on the contrary, with the same conclusion "Γ is a Powers group", Proposition 10 in [Harp–85] and Corollary 4.6 in [Ivan] have hypothesis stated in terms of the action of Γ on the edge set of its Bass-Serre tree (more precisely, this action should be strongly faithful, as defined in Section 2). Our conditions in Proposition 2 are also weaker than those of [Bedo–84].

A similar remark holds for Proposition 4 below and Proposition 11 in [Harp–85].

A particular case of Proposition 2 is well-known: the case of free products [PaSa–79]. A free product $A * B$ is *non-trivial* if neither A nor B is the group with one element. Recall that the *infinite dihedral* *group* is the free product of two groups of order two, and that it is the only non-trivial amenable free product.

Corollary 3. *A non-trivial free product $A * B$ is C^*-simple if and only if it is not isomorphic to the infinite dihedral group.*

At first sight, Corollary 3 follows from Proposition 2 for countable groups only. But a group Γ is C*-simple [respectively Powers, strongly Powers] as soon as, for any countable subgroup Γ_0 of Γ, there exists a countable C*-simple [respectively Powers, strongly Powers] subgroup of Γ containing Γ_0; see Proposition 10 in [BeHa–00] and Lemma 3.1 in [BoNi–88]. It follows that Corollary 3 holds as stated above [and also with "Powers group" or "strongly Powers group" instead of "C*-simple"].

Let G be a group, let θ be an isomorphism from a subgroup H of G to some subgroup of G, and let

$$\Gamma = HNN(G, H, \theta) = \langle G, \tau \mid \tau^{-1} h \tau = \theta(h) \ \forall h \in H \rangle$$

be the corresponding HNN-extension. We define inductively a decreasing sequence $H_0 \supset H_1 \supset \cdots$ of subgroups of H by $H_0 = H$ and

$$\begin{aligned} H'_k &= H_{k-1} \cap \tau^{-1} H_{k-1} \tau = H_{k-1} \cap \theta(H_{k-1}) \\ H_k &= \Big(\bigcap_{g \in G} g H'_{k-1} g^{-1} \Big) \cap \tau \Big(\bigcap_{g \in G} g H'_{k-1} g^{-1} \Big) \tau^{-1} \end{aligned}$$

for $k \geq 1$.

Proposition 4. *Consider as above a countable HNN-extension $\Gamma = HNN(G, H, \theta)$, with moreover $H \subsetneqq G$ and $\theta(H) \subsetneqq G$. Assume that $H_k = \{1\}$ for some $k \geq 0$.*

Then Γ is a strongly Powers group.

As for Proposition 2, the condition $H_k = \{1\}$ has the following geometrical interpretation: for $\gamma \in \Gamma$, if there exists $e \in E(T)$ such that the k-neighbourhood $\mathcal{V}_k(e)$ is pointwise fixed by γ, then $\gamma = 1$.

Though it does not quite follow from Proposition 4, similar arguments imply the next proposition. Its first part is due to Nikolay Ivanov, with a different proof (Theorem 4.9 in [Ivan]). In the second part, for $p \geq 0$, we denote abusively by $\theta^p(H)$ the image of the restriction of θ^p to $\{h \in H \mid \theta^j(h) \in H \text{ for } j = 1, \dots, p-1\}$, and similarly for $\theta^{-p}(H)$.

Proposition 5. *The Baumslag-Solitar group*

$$BS(m, n) = \langle \tau, b \mid \tau^{-1} b^m \tau = b^n \rangle = HNN(b^{\mathbf{Z}}, b^{m\mathbf{Z}}, \ b^{mk} \longmapsto b^{nk})$$

is a strongly Powers group if and only if it is C^*-simple, if and only if

$\min\{|m|,|n|\} \geq 2$ *and* $|m| \neq |n|$.

More generally, let Γ *be a countable group which is an HNN-extension* $HNN(G,H,\theta)$ *such that* $H \subsetneqq G$, $\theta(H) \subsetneqq G$, *and at least one of*

$$K_+ \doteqdot \bigcap_{p\geq 0} \theta^p(H) = \{1\}, \quad K_+ \doteqdot \bigcap_{p\geq 0} \theta^{-p}(H) = \{1\}$$

holds. Then Γ *is a strongly Powers group.*

Consider for example the homomorphism $\sigma : BS(m,n) \longrightarrow \mathbf{Z}$ defined by $\sigma(b) = 0$ and $\sigma(\tau) = 1$, and the *special Baumslag-Solitar group*

$$SBS(m,n) = \ker(\sigma).$$

We have:

Corollary 6. *If* $\min\{|m|,|n|\} \geq 2$ *and* $|m| \neq |n|$, *then* $SBS(m,n)$ *is a strongly Powers group.*

The subject of Part II is the fundamental group Γ of a connected compact 3-manifold M (with or without boundary). Let $\widehat{M}$ be the manifold obtained from M by filling all 2-spheres in ∂M with 3-balls (thus M is the connected sum of $\widehat{M}$ and as many 3-balls as the number of components of ∂M which are 2-spheres). Since $\pi_1(\widehat{M}) = \pi_1(M)$, it is often convenient to replace M by $\widehat{M}$. For background on 3-manifolds, we refer to [Hemp–76], [JaSh–79], [Scot–83], [Thur–97], and [Bona–02]; see also Section 5.

Let us first describe four classes of examples for which it is easy to see wether Γ is C*-simple or not.

(i) A *Seifert manifold* is a connected 3-manifold which can be foliated by circles, usually called *fibers*, such that each fiber has a neighbourhood, either a solid torus or a solid Klein bottle, which is a union of fibers in some standard way ("such that .." is automatic for a compact manifold, by a theorem of David Epstein). Let M be a Seifert manifold with infinite fundamental group Γ (equivalently: which is not covered by a 3-sphere, see Lemma 3.2 in [Scot–83]); if M is given a Seifert structure, a generic fiber generates a normal subgroup of Γ which is infinite cyclic. Hence Γ is not C*-simple, because a C*-simple group cannot have non-trivial amenable normal subgroups (Proposition 3 of [Harp–07]).

(ii) A *Sol-manifold* is a connected 3-manifold M of which the interior $\mathring{M}$ can be given a Riemannian structure such that the universal covering $\pi : Sol \longrightarrow \mathring{M}$ is a local isometry. Here, *Sol* denotes the 3-dimensional Lie group with underlying space $\mathbf{R}^3$, with product

$$(x,y,z)(x',y',z') = (x + e^{-z}x', y + e^{z}y', z + z'),$$

and with Riemannian structure

$$ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2.$$

The fundamental group of a Sol-manifold contains a subgroup of finite index Γ_0 which fits in a short exact sequence

$$\{1\} \longrightarrow K \longrightarrow \Gamma_0 \longrightarrow Q \longrightarrow \{1\}$$

where K [respectively Q] is a discrete subgroup of the group of isometries of a Euclidean plane [respectively of a line]; see Proposition 4.7.7 in [Thur–97]. In particular, Γ has a solvable subgroup of finite index, hence Γ is amenable, and therefore Γ is not C*-simple.

(iii) Let M be a connected 3-manifold which is hyperbolic and not elementary (for example of finite volume). If M is orientable, then Γ is isomorphic to a subgroup of $PSL_2(\mathbf{C})$ which is not almost solvable, and these are strongly Powers groups. (See Section 7; see also Proposition 6 in [Harp–85], which could be added to the list of Corollary 12 in [Harp–07].) If M is non-orientable, the orientation subgroup Γ' of Γ, which is of index two, is C*-simple, by the previous argument; since Γ is infinite and has infinite conjugacy classes (Lemma 9.1 in [HaPr–07]), Γ itself is C*-simple by [BeHa–00].

(iv) If the compact 3-manifold $\widehat{M}$ is not prime (= is a connected sum in a non-trivial way), and if we assume the Poincaré Conjecture, then Γ is a non-trivial free product. Suppose that, moreover, M is orientable. If Γ is an infinite dihedral group, then $\widehat{M}$ is Seifert; in all other cases, Γ is a strongly Powers group, by Proposition 2. (More on this in the discussion of Section 5.)

We sum up the previous discussion as follows:

Example 7. *Let M be a connected compact 3-manifold ! and let Γ denote its fundamental group.*

(i) *If $\widehat{M}$ is a Seifert manifold, Γ is not C^*-simple.*
(ii) *If $\widehat{M}$ is a Sol-manifold, Γ is not C^*-simple.*
(iii) *If $\widehat{M}$ is hyperbolic and non-elementary, Γ is C^*-simple.*

Suppose moreover that M is orientable, and that the Poincaré Conjecture is true.

(iv) *If $\widehat{M}$ is not prime and not Seifert, Γ is a strongly Powers group.*

In a sense, the existence of a Seifert structure and that of a Sol-structure on $\widehat{M}$ are the only obstructions to the C*-simplicity of Γ. More precisely, as a consequence of Propositions 2 and 4, and as we show in Section 6:

Proposition 8. *Let M be an orientable connected compact 3-manifold and let $\widehat{M}$ be as above. We assume that Thurston's Geometrisation Conjecture holds.*

If $\widehat{M}$ is not a Seifert manifold and is not a Sol manifold, then the fundamental group $\Gamma = \pi_1(M)$ is a strongly Powers group.

Corollary 9. *Let K be a knot in $\mathbf{S}^3$ and let Γ denote the fundamental group of its complement.*

Then Γ is a strongly Powers group if and only if K is not a torus knot.

For non-orientable manifolds, we will restrict ourselves to the following simple statement:

Proposition 10. *Let M be a non-orientable connected compact 3-manifold, Γ its fundamental group, and Γ' the fundamental group of the total space of the orientation cover of M; in particular, Γ' is a subgroup of index 2 in Γ.*

Then Γ is C^*-simple if and only if Γ' is C^*-simple.

Remark on groups of surfaces and groups of 3-manifolds. Let S be a connected compact surface. It follows from the classification of surfaces and from elementary arguments (compare with [HaPr–07]) that the following three properties are equivalent:

(i) S is not homeomorphic to a disc, a sphere, a projective plane, an annulus, a Möbius band, a 2-torus, or a Klein bottle;

(ii) Γ is icc (namely all its conjugacy classes other than $\{1\}$ are infinite); or equivalently the von Neumann algebra of Γ is a factor of type II_1;

(iii) Γ is C*-simple.

If Γ is now the fundamental group of a connected compact 3-manifold, the equivalence of (ii) and (iii) does not carry over. Indeed, the fundamental group of a Sol manifold can be icc, and is never C*-simple. In some sense, Propositions 8 and 10 provide the 3-dimensional analogue of the equivalence $(i) \Longleftrightarrow (iii)$.

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I. Fundamental groups of graphs of groups

2. On groups acting on trees

Let X be a *graph*, with vertex set $V(X)$ and edge set $E(X)$. We follow Bass [Bass–93] and Serre [Serr–77]; in particular, each geometric edge of X corresponds to a pair $\{e, \overline{e}\} \in E(X)$, with $\overline{e} \neq e$ and $\overline{\overline{e}} = e$. Each edge $e \in E(X)$ has a *source* $s(e) \in V(X)$ and a *terminus*, $t(e) \in V(X)$. We denote by $d(x, y)$ the *combinatorial distance* between two vertices $x, y \in V(X)$. In case of a *tree*, we write T rather than X, and we agree that $V(T) \neq \emptyset$.

We denote by ∂T the *boundary* of a tree T. Recall that a *ray* in T is a subtree with vertex set $(x_n)_{n\in\mathbf{N}}$, such that

$$d(x_m, x_n) \;=\; |m-n| \quad \text{for all } \; m, n \in \mathbf{N}.$$

The enumeration of the vertices of a ray will *always* be such that the above condition holds. Two rays with vertex sets $(x_n)_{n\in\mathbf{N}}$ and $(y_n)_{n\in\mathbf{N}}$ are *cofinal* if there exists some $k \in \mathbf{Z}$ such that $y_n = x_{n+k}$ for all n large enough, and ∂T is the set of cofinal classes of rays in T.

The set ∂T has a natural topology defined as follows. For any edge $e \in E(T)$, the *shadow* $(\partial T)_e$ of e in ∂T is the subset of ∂T represented by rays $(x_n)_{n\in\mathbf{N}}$ such that $d(t(e), x_n) < d(s(e), x_n)$ for all n large enough. The family of shadows $\{(\partial T)_e\}_{e\in E(T)}$ generates a topology on ∂T which is Hausdorff and totally disconnected; if T is countable, this topology is moreover metrisable. Given any $x_0 \in V(T)$, this topology coincides[1] with the inverse limit topology on the set ∂T, identified with the inverse limit of the discrete spaces $S(x_0, n) = \{x \in V(T) \mid d(x_0, x) = n\}$, namely of the *spheres* in T around x_0. It is known that ∂T is compact if and only if T is locally finite (and we do consider below trees *which are not* locally finite). See the last exercise in Section I.2.2 of [Serr–77], Section I.8.27 in [BrHa–99], and Section 4 in [MoSh–04].

Observe that, for ∂T to be non-empty, it suffices that T has more than one vertex and does not have any vertex of degree 1.

A ray cofinal with a pending ray is also pending. The classes of the pending rays are precisely the isolated points in ∂T; in particular, the

[1] Similarly, one can define two topologies on the disjoint union $\overline{T} = V(T) \sqcup \partial T$, one using appropriate shadows and one by identifying $\overline{T}$ to the inverse limit of the balls $\{x \in V(T) \mid d(x_0, x) \leq n\}$. These two topologies coincide if and only if the tree T is locally finite, but their restrictions on ∂T coincide in all cases. The shadow topology makes $\overline{T}$ a compact space, but ∂T need not be closed, and therefore need not be compact (unless T is locally finite). More on this in [MoSh–04].

topological space ∂T is *perfect* if the tree T does not have any pending ray. (Pending rays are defined before Proposition 1.)

Proposition 11. *For any countable tree T, the boundary ∂T is a Baire space.*

Proof. Choose a vertex $x_0 \in V(T)$. Since T is countable, each of the spheres $S(x_0, n)$ defined above is countable, and therefore is a polish space for the discrete topology. Since the inverse limit of a countable inverse system of polish spaces is polish ([Bou-TG9], § 6), ∂T is polish ("countable" is important, because a product of uncountably many polish spaces is not polish in general). Since polish spaces are Baire spaces ([Bou-TG9], § 5), this ends the proof. □

Recall that an action of a group Γ on a set Ω is

- *faithful* if, for any $\gamma \in \Gamma$, $\gamma \neq 1$, there exists $\omega \in \Omega$ such that $\gamma(\omega) \neq \omega$, and
- *strongly faithful* if, for any finite subset F of Γ not containing 1, there exists $\omega \in \Omega$ such that $\gamma(\omega) \neq \omega$ for all $\gamma \in F$.

An action of a group on a tree T is faithful, or strongly faithful, if it is the case for the action on the vertex set $V(T)$. Let Ω be a Hausdorff space and let the action of Γ be by homeomorphisms; if the action is strongly faithful and if F, ω are as above, observe that there exists a neighbourhood V of ω in Ω such that $\gamma(V) \cap V = \emptyset$ for all $\gamma \in F$. Recall also from the introduction that an action of a group Γ on a tree T is

- *slender* if any $\gamma \in \Gamma$, $\gamma \neq 1$, is slender, namely is such that the fixed point set $(\partial T)^\gamma$ has empty interior.

Corollary 12. *Let Γ be a countable group which acts by automorphisms on a tree T.*

If the action on T is slender, then the action on ∂T is strongly faithful.

Proof. Let F be a finite subset of $\Gamma \smallsetminus \{1\}$, as in the definition of "strongly faithful". Since the action is slender, the fixed point set $(\partial T)^\gamma$ has no interior for all $\gamma \in F$, so that there exists by Proposition 11 (Baire's property of ∂T) a point $\xi \in \partial T$ such that $\gamma(\xi) \neq \xi$ for all $\gamma \in F$.

[Observe that $\partial T \smallsetminus \bigcup_{\gamma \in \Gamma \smallsetminus \{1\}} (\partial T)^\gamma$ is dense, and therefore non-empty, so that there exists $\xi \in \partial T$ such that $\gamma(\xi) \neq \xi$ for all $\gamma \in \Gamma \smallsetminus \{1\}$. We shall not use this below.] □

Since [Tits–70], we know that an automorphism γ of a tree T can be of three different kinds:

- It is an *inversion* if there exists $e \in E(T)$ such that $\gamma(e) = \overline{e}$.
- It is *elliptic* if it has at least one fixed point in $V(T)$.
- It is *hyperbolic* if it is not an inversion and if

$$d(\gamma) \doteqdot \min\{d(y, \gamma(y)) \mid y \in V(T)\} \geq 1.$$

If γ is hyperbolic, it has an *axis*, which is a linear subtree T_γ, such that $x \in V(T_\gamma)$ if and only if $d(x, \gamma(x)) = d(\gamma)$, and $d(\gamma)$ is called the *translation length* of γ.

We translate here by

- *linear subtree*

what is a "chaîne" in [Tits–70] and a "droit chemin" in [Serr–77], namely a subtree of which the vertex set is of the form $(x_n)_{n \in \mathbf{Z}}$, with $d(x_m, x_n) = |m - n|$ for all $m, n \in \mathbf{Z}$. Linear trees are defined accordingly.

Remark 13. *(i) An automorphism γ of T which is hyperbolic has exactly two fixed points on ∂T which are its* source $\alpha(\gamma)$ *and its* sink $\omega(\gamma)$*, and the infinite cyclic group $\gamma^{\mathbf{Z}}$ acts freely on $\partial T \smallsetminus \{\alpha(\gamma), \omega(\gamma)\}$. If γ is an hyperbolic automorphism of a tree of which the boundary consists of more than two points (and therefore of which the boundary is infinite), then γ is always slender, since its fixed point set in ∂T consists of two non-isolated points.*

(ii) If T is a tree with non-empty boundary ∂T and if γ is an elliptic automorphism of T with fixed point set $V(T)^\gamma$ of finite diameter, then $(\partial T)^\gamma = \emptyset$, and in particular γ is slender.

*(iii) If T is the Bass-Serre tree corresponding to the Baumslag-Solitar group $\Gamma = BS(m, n)$ for some $m, n \in \mathbf{Z}$ (as in Subsection 4.2 below), there exist elliptic automorphisms $\gamma \in \Gamma$ of T with fixed-point sets of infinite diameter (*Exemple *4.2 of* [Stal–06]*).*

(iv) Let γ be an elliptic automorphism of a tree T such that $(\partial T)^\gamma \neq \emptyset$. Consider a ray with vertex set $(x_n)_{n \geq 0}$ of which the origin x_0 is fixed by γ and which represents a boundary point fixed by γ; then each of the vertices x_n is fixed by γ.

(v) Elliptic automorphisms need not be slender.

For *(v)*, consider for example the regular tree T of degree 3, a vertex x_0 of T, and the three isomorphic connected components T_1, T_2, T_3 obtained from T by deleting x_0 and the incident edges. An appropriate transposition of T_1 and T_2 fixing T_3 is not slender, since $(\partial T)^\sigma = \partial T_3$ is open non-empty, but a cyclic permutation γ of T_1, T_2, T_3 of order 3 is slender since $(\partial T)^\gamma = \emptyset$.

Let Γ be a group which acts by automorphisms on a tree T, and therefore by homeomorphisms on ∂T. Two elements in Γ which are hyperbolic (namely which act on T by hyperbolic automorphisms) are

- *transverse* if they do not have any common fixed point in ∂T, equivalently if the intersections of their axis is finite (possibly empty).

The action is

- *strongly hyperbolic* if Γ contains a pair of transverse hyperbolic elements.

If the action is strongly hyperbolic, the next lemma shows that there is an abundance of transverse pairs.

Lemma 14. *Let Γ be a group acting strongly hyperbolically on a tree T.*

For any hyperbolic element $\gamma_0 \in \Gamma$, there exist infinitely many hyperbolic elements γ_n, $n \geq 1$, such that the γ_n, $n \geq 0$, are pairwise transverse.

Proof. Let $\alpha, \beta \in \Gamma$ be hyperbolic and transverse. Upon replacing α with $\beta^k \alpha \beta^{-k}$ for an appropriate k, we can assume furthermore that γ_0 and α are transverse.

Set $\delta_m = \alpha^m \gamma_0 \alpha^{-m}$ for all $m \geq 1$. Then there exists a subsequence $(\gamma_n)_{n \geq 1}$ of $(\delta_m)_{m \geq 1}$ such that the γ_n, $n \geq 0$, are pairwise transverse. □

The next proposition is a restatement of well-known facts from the literature. See for example Proposition 2 of [PaVa–91] and Proposition 7.2 of [Bass–93].

Proposition 15. *Let T be an infinite tree which is not a linear tree. Let Γ be a group which acts on T, minimally and in such a way that its action on ∂T has no fixed points.*

Then the action of Γ on T is strongly hyperbolic.

Proof. By minimality, $\Gamma \neq \{1\}$ fixes no vertex and no pair of adjacent vertices. By Proposition 3.4 of [Tits–70], this implies that Γ contains at least one hyperbolic element, say γ_1, and that $\partial T \neq \emptyset$. Denote by $\alpha_1 \in \partial T$ the source of γ_1 and by ω_1 its sink.

By minimality and by the proof of Corollary 3.5 of [Tits–70], T is the union of the axis of the hyperbolic elements of Γ. As T is infinite and is not a linear tree, it follows that ∂T is infinite and that there exists an hyperbolic element $\gamma_2 \in \Gamma$, say with source α_2 and sink ω_2, such that $\{\alpha_2, \omega_2\} \neq \{\alpha_1, \omega_1\}$. Upon replacing γ_2 by γ_2^{-1}, we may assume that $\alpha_2 \notin \{\alpha_1, \omega_1\}$. If $\omega_2 \notin \{\alpha_1, \omega_1\}$, there is nothing left to prove; upon replacing γ_1 by γ_1^{-1} if necessary, we can assume from now that $\omega_2 = \omega_1$, and we denote this point by ω.

Since Γ does not fix any boundary point in ∂T, there exists $\delta \in \Gamma$ such that $\delta(\omega) \neq \omega$. Upon exchanging γ_1 and γ_2, we can assume that $\delta(\alpha_1) \neq \omega$.

We claim that all the orbits of Γ on ∂T are infinite. Since any orbit of $\gamma_1^{\mathbf{Z}}$ on $\partial T \smallsetminus \{\alpha_1, \omega\}$ is infinite, and similarly for $\gamma_2^{\mathbf{Z}}$ on $\partial T \smallsetminus \{\alpha_2, \omega\}$, the only point we have to check is that the orbit $\Gamma(\omega)$ is infinite. Observe that $\gamma_3 \doteqdot \delta\gamma_1\delta^{-1}$ is hyperbolic, that $\alpha(\gamma_3) = \delta(\alpha_1)$, and that $\omega(\gamma_3) = \delta(\omega)$. Since $\omega \notin \{\alpha(\gamma_3), \omega(\gamma_3)\}$, the $\gamma_3^{\mathbf{Z}}$-orbit of ω is infinite. *A fortiori*, the Γ-orbit of ω is infinite.

It is a general fact that, if a group Γ acts on a set Ω in such a way that all its orbits are infinite, and if F is any finite subset of Ω, there exists $\gamma \in \Gamma$ such that F and $\gamma(F)$ are disjoint; see[2] Lemma 2.3 in [NePM–76]. In our case, it implies that we can choose $\gamma \in \Gamma$ such that $\gamma(\{\alpha_1, \omega\}) \cap \{\alpha_1, \omega\} = \emptyset$, namely such that the hyperbolic elements γ_1 and $\gamma\gamma_1\gamma^{-1}$ are transverse. $\square$

Recall that an action of a group Γ on a topological space Ω is

- *minimal* if the only Γ-invariant closed subpaces of Ω are Ω itself and the empty subspace.

If a group Γ acts on a tree T in a minimal way (definition before Proposition 1), its action on ∂T *need not be minimal.* A first example is that of the standard action of $\mathbf{Z}$ on a linear tree; a second example is the action of a Baumslag-Solitar group $BS(1, n)$ on its Bass-Serre tree T, which is minimal, but with the corresponding action on the infinite boundary ∂T having a fixed point (see Subsection 4.2 below). However:

Proposition 16. *Let Γ be a group which acts on a tree T. Assume that the action is strongly hyperbolic and minimal.*

Then the action of Γ on ∂T is minimal.

Proof. Let T_0 be the subtree of T which is the union of the axis T_γ over all hyperbolic elements $\gamma \in \Gamma$. As already noted in the proof of Proposition 15, $T_0 = T$. Hence the set

$$L_\Gamma \doteqdot \{\eta \in \partial T \mid \eta = \omega(\gamma) \ \text{for some hyperbolic} \ \gamma \in \Gamma\}$$

is dense in ∂T.

Let C be a non-empty Γ-invariant closed subset of ∂T, and choose $\xi \in C$. Let $\eta \in L_\Gamma$; choose an hyperbolic element $\gamma \in \Gamma$ such that $\eta = \omega(\gamma)$. In case $\xi \neq \alpha(\gamma)$, we have $\eta = \lim_{n\to\infty} \gamma^n(\xi)$, so that $\eta \in C$.

[2] This is a straightforward consequence of the following lemma of B.H. Neumann [NeBH–54]: a group cannot be covered by finitely many cosets with respect to subgroups of infinite index. Other Proof: Theorem 6.6 in [BMMN–98].

Assume now that $\xi = \alpha(\gamma)$. By Lemma 14, there exists an hyperbolic element $\gamma' \in \Gamma$ transverse to γ. We have first $\omega(\gamma') \in C$ as in the previous case, and then $\eta = \lim_{n\to\infty} \gamma^n\big(\omega(\gamma')\big) \in C$.

Hence $C \supset L_\Gamma$, and $C = \overline{L_\Gamma} = \partial T$. In other terms, the action of Γ on ∂T is minimal. □

Remark. Proposition 16 shows that, under appropriate hypothesis, minimality on T implies minimality on ∂T. Here is a kind of converse:

Let T be an infinite tree without vertices of degree one. If a group Γ acts on T in such a way that its action on ∂T is minimal, and if Γ contains at least one hyperbolic element, then the action on T is also minimal.

Indeed, suppose that the action of Γ on T is not minimal, so that T_0 defined as above is a non-empty proper subtree of T. Since T does not have any vertex of degree 1, there exists a ray in T, disjoint from T_0, which defines a point $\xi \in \partial T$; denote by $(x_n)_{n\geq 0}$ the vertices of this ray, and by e the first edge of this ray, with $s(e) = x_0$ and $t(e) = x_1$. Then the shadow $(\partial T)_e$ is a neighbourhood of ξ in ∂T, disjoint from the Γ-invariant closed subset ∂T_0. Hence the action of Γ on ∂T is not minimal.

A homeomorphism γ of a Hausdorff space Ω is

- *hyperbolic* if it has the following property: there are two points $\alpha, \omega \in \Omega$ fixed by γ such that, for any neighbourhoods U of α and V of ω, we have $\gamma^n(\Omega \smallsetminus U) \subset V$ and $\gamma^{-n}(\Omega \smallsetminus V) \subset U$ for n large enough.

An action of a group Γ on Ω is

- *strongly hyperbolic* if Γ contains two hyperbolic homeomorphisms which are *transverse*, namely without common fixed point.

These generalise notions defined above in the following sense: an hyperbolic automorphism of a tree T induces an hyperbolic homeomorphism of the boundary ∂T, and a group acting on T strongly hyperbolically is strongly hyperbolic on ∂T. The following proposition is a reformulation of Proposition 11 and Theorem 13 in [Harp–07], indeed a reformulation of part of [Powe–75] (see also Proposition 32 below).

Proposition 17. *Let Γ be a group which acts by homeomorphisms on a Hausdorff topological space Ω. Assume that the action is strongly faithful, strongly hyperbolic, and minimal.*

Then Γ is a Powers group.

Proof. Here is first a general observation. Let Γ be a group which acts by homeomorphisms on a Hausdorff topological space Ω containing a

Γ-invariant dense subspace Ω_0. Then the action of Γ on Ω is strongly faithful if and only if the action of Γ on Ω_0 is strongly faithful.

Set $L_\Gamma \doteqdot \{\eta \in \Omega \mid \eta = \omega(\gamma) \text{ for some hyperbolic } \gamma \in \Gamma\}$, as in the proof of Proposition 16. If the action of Γ on Ω is strongly hyperbolic, L_Γ is non-empty, indeed infinite. If this action is minimal, L_Γ is dense in Ω. It follows that the action of Γ on L_Γ is strongly faithful, so that the hypothesis of Proposition 11 of [Harp–07] (= Proposition 32 here) are fulfilled. □

The particular case of $\Omega = \partial T$ provides the following variation on Proposition 1.

Proposition 18. *Let Γ be a countable group which acts on a tree T. Assume that the action is slender, strongly hyperbolic, and minimal.*

Then Γ is a strongly Powers group.

Proof. First step: Proposition 17 applies to Γ. The action of Γ on ∂T is strongly faithful by Corollary 12 and minimal by Proposition 16. It follows from Proposition 17 that Γ is a Powers group.

Second step: Proposition 17 applies to any normal subgroup $N \neq \{1\}$ of Γ. The subgroup N has no fixed vertex in T. Otherwise, since the set $V(T)^N$ is Γ-invariant, it would coincide with $V(T)$ by Γ-minimality, and Γ could not be faithful on T. The same argument shows that N has no fixed pair of adjacent vertices in T and (because Γ is faithful and minimal on ∂T) no fixed boundary point in ∂T. It follows that N contains hyperbolic elements, by Proposition 3.4 of [Tits–70].

The union of the axis of the hyperbolic elements in N is a Γ-invariant subtree, and coincides with T by Γ-minimality. It follows that the action of N on T is minimal, and also strongly hyperbolic (for example by Proposition 15). Proposition 17 again implies that N is a Powers group.

Coda. If $N \neq \{1\}$ is subnormal in Γ, namely if there exists a chain of subgroups $N_0 = N \leq N_1 \leq \cdots \leq N_k = \Gamma$ with N_{j-1} normal in N_j for $j = 1, \ldots, k$, an induction on j based on the second step shows that N is a Powers group. □

Proof of Proposition 1. Since there are in T neither vertices of degree one nor pending rays, ∂T is non-empty and perfect. The action of Γ on ∂T is strongly hyperbolic by Proposition 15, so that Proposition 18 applies. □

3. On the action of the fundamental group of a graph of groups on the corresponding Bass-Serre tree

Recall that a *graph of groups* $\mathbf{G} = (G, Y)$ consists of

- a non-empty connected graph Y,
- two families of groups $(G_y)_{y \in V(Y)}$ and $(G_e)_{e \in E(Y)}$, with $G_{\overline{e}} = G_e$ for all $e \in E(Y)$,
- a family of monomorphisms $\varphi_e : G_e \longrightarrow G_{t(e)}$, for $e \in E(Y)$.

An *orientation* of Y is a subset $E^+(Y)$ of $E(Y)$ containing exactly one of $e, \overline{e}$ for each $e \in E(Y)$; we denote by $E^-(Y)$ the complement of $E^+(Y)$ in $E(Y)$.

A graph of groups $\mathbf{G} = (G, Y)$ gives rise to the *fundamental group* $\Gamma = \pi_1(G, Y, M)$ of $\mathbf{G}$ and the *universal cover* $T = T(G, Y, M, E^+(Y))$, also called the *Bass-Serre tree of* $\mathbf{G}$, where M is a maximal tree in Y, and $E^+(Y)$ an orientation of Y; abusively, T is also called the *Bass-Serre tree of* Γ. Let us recall as follows part of the standard theory (§ I.5 in [Serr–77], and [Bass–93]).

- (BS-1) Γ has a presentation with generators the groups G_y, $y \in V(Y)$, and elements τ_e, $e \in E(Y)$, and relations

$$\begin{aligned} \tau_{\overline{e}} &= (\tau_e)^{-1} && \text{for all } e \in E(Y), \\ \tau_e^{-1} \varphi_{\overline{e}}(h) \tau_e &= \varphi_e(h) && \text{for all } e \in E(Y) \text{ and } h \in G_e, \\ \tau_e &= 1 && \text{for all } e \in E(M). \end{aligned}$$

  Moreover, the natural homomorphisms

$$G_y \longrightarrow \Gamma \quad \text{and} \quad \mathbf{Z} \longrightarrow \Gamma, \ k \longmapsto \tau_e^k$$

  are injective for all $y \in V(Y)$ and for all $e \in E(Y)$ with $e \notin E(M)$.

- (BS-2) T is a graph with

$$V(T) = \bigsqcup_{y \in V(Y)} \Gamma / G_y \quad \text{and} \quad E(T) = \bigsqcup_{e \in E(Y)} \Gamma / \varphi_e(G_e).$$

  The source map, the terminus map, and the inversion map, are given by

$$\begin{aligned} s(\gamma \varphi_e(G_e)) &= \begin{cases} \gamma G_{s(e)} & \text{if } e \in E^+(Y) \\ \gamma \tau_e^{-1} G_{s(e)} & \text{if } e \notin E^+(Y), \end{cases} \\ t(\gamma \varphi_e(G_e)) &= \begin{cases} \gamma \tau_e G_{t(e)} & \text{if } e \in E^+(Y) \\ \gamma G_{t(e)} & \text{if } e \notin E^+(Y), \end{cases} \\ \overline{\gamma \varphi_e(G_e)} &= \gamma \varphi_{\overline{e}}(G_{\overline{e}}) \end{aligned}$$

  for all $\gamma \in \Gamma$ and $e \in E(Y)$. The natural action of Γ is by automorphisms of graphs, and without inversions.

  Moreover, *T is a tree.*

- (BS-3) The natural mappings

$$V(T) \longrightarrow V(Y), \quad \gamma G_y \longmapsto y$$
$$E(T) \longrightarrow E(Y), \quad \gamma\varphi_e(G_e) \longmapsto e$$

  are the constituants of a morphism of graphs $p : T \longrightarrow Y$ which factors as an isomorphism $\Gamma\backslash T \approx Y$.

- (BS-4) The sections

$$V(Y) \longrightarrow V(T), \quad y \longmapsto \widetilde{y} \doteqdot 1G_y$$
$$E(Y) \longrightarrow E(T), \quad e \longmapsto \widetilde{e} \doteqdot 1\varphi_e(G_e)$$

  are such that the stabilizer of $\widetilde{y}$ in Γ is isomorphic to G_y for all $y \in V(Y)$, and similarly the stabilizer of $\widetilde{e}$ in Γ is isomorphic to G_e for all $e \in E(Y)$.

  The first section $V(M) = V(Y) \longrightarrow V(T)$ and the restriction $E(M) \longrightarrow E(T)$ of the second section are the constituents of an isomorphism of graph from M onto a subtree of T.

Moreover, up to isomorphisms, Γ and T do not depend on the choices of M and $E^+(Y)$.

For what we need below, it is important to observe that

the action of Γ *on* T *need be neither faithful nor minimal.*

We will often write G_e instead of $\varphi_e(G_e)$; this is abusive since, though $G_{\overline{e}} = G_e$, the cosets Γ/G_e and $\Gamma/G_{\overline{e}}$ are *different sets* of oriented edges, see the definitions of $E(T)$ and of the change of orientations in (BS-2).

3.1. **Small Bass-Serre trees, fixed points, and fixed boundary points.** Let $\mathbf{G} = (G, Y)$ be a graph of groups. An edge $e \in E(Y)$ is *trivial*[3] if $s(e) \neq t(e)$ and if $\varphi_{\overline{e}} : G_e \longrightarrow G_{s(e)}$ is an isomorphism onto. If $\mathbf{G} = (G, Y)$ has a trivial edge e, we can define a new graph Y/e obtained from Y by collapsing $\{e, \overline{e}\}$ to a vertex, and we can define naturally a new graph of groups $\mathbf{G}/e$, with fundamental group isomorphic to that of $\mathbf{G}$. Say that $\mathbf{G}$ is

- *reduced* if it does not contain any trivial edge.

[3]Terminological adjustment. "Trivial", here and on Page 193 of [ScWa–79], means the same as "directed" on Page 1096 of [Bass–76]. "Reduced" here means the same as "minimal" in [ScWa–79]. On page 42 of [Bass–93], a vertex $y \in V(Y)$ is *terminal* if there is a unique edge $e \in E(Y)$ with $s(e) = y$, and if $\varphi_{\overline{e}} : G_e \longrightarrow G_{s(e)}$ is an isomorphism onto; observe then that e is trivial. If Y is a segment of length 2 with two end vertices x, z, a middle vertex y, the vertex group G_y isomorphic to the two edge groups, and G_x, G_z large enough, then (G, Y) is not reduced but does not have any terminal vertex. The contraction process $\mathbf{G} \rightsquigarrow \mathbf{G}/e$ is a particular case of the process described with more details in Section 2 of [Bass–76].

Let X be a connected subgraph of Y. The corresponding *subgraph of groups* $\mathbf{F} = (F, X)$ is defined by $F_x = G_x$ for all $x \in V(X)$, and $F_e = G_e$ for all $e \in E(X)$, with the inclusion $F_e \longrightarrow F_{t(e)}$ being precisely φ_e for all $e \in E(X)$.

The next proposition collects observations from papers by Bass; we choose a maximal tree M of Y containing a maximal tree L of X, and an orientation $E^+(Y)$ of Y containing an orientation $E^+(X)$ of X.

Proposition 19. *Let $\mathbf{G} = (G, Y)$ be a graph of groups, and let the notation be as above.*

(i) *If $\mathbf{F} = (F, X)$ is a subgraph of groups of $\mathbf{G}$, the fundamental group $\pi_1(F, X, L)$ is isomorphic to a subgroup of $\pi_1(G, Y, M)$ and the universal cover $T(F, X, L, E^+(X))$ can be identified with a subtree of $T(G, Y, M, E^+(Y))$.*

In case $\mathbf{G}$ is, moreover, finite and reduced, and if X is a proper subgraph of Y, then $\pi_1(F, X, L)$ is a proper subgroup of $\pi_1(G, Y, M)$.

(ii) *If Y is finite, there exist a reduced graph of group $\mathbf{H} = (H, Z)$ and a* contraction *$\mathbf{G} = (G, Y) \longrightarrow \mathbf{H} = (H, Z)$ which induces an isomorphism from the fundamental group of $\mathbf{G}$ onto that of $\mathbf{H}$.*

Proof. For (i), see Items 1.14 and 2.15 in [Bass–93]. For (ii), see Proposition 2.4 in [Bass–76]. □

[Claim (ii) need not hold when Y is infinite; see the discussion in Section 7 of [ScWa–79].]

For the next proposition, compare for example with Theorem 6.1 in [Bass–76].

Proposition 20. *Let $\mathbf{G} = (G, Y)$ be a graph of groups, with universal cover T; we assume that $\mathbf{G}$ is reduced. Then:*

(i) *T is finite if and only if T is reduced to one vertex, if and only if Y is reduced to one vertex.*

(ii) *T has no vertex of degree 1.*

(iii) *If T is infinite, it does not have any vertex fixed by Γ.*

(iv) *T is a linear tree if and only if Y is*

– *either a segment with two vertices and one pair of edges $\{e, \overline{e}\}$, and with $[G_{s(e)} : \varphi_{\overline{e}}(G_e)] = [G_{t(e)} : \varphi_e(G_e)] = 2$ (case of a degenerate non-trivial amalgam, see Subsection 4.1),*

– *or a loop with one vertex y and one pair of edges $\{e, \overline{e}\}$, and with $\varphi_e(G_e) = G_y = \varphi_{\overline{e}}(G_e)$ (case of a semi-direct product $G_y \rtimes_\theta \mathbf{Z}$, see Subsection 4.2).*

(v) *T has a pending ray if and only if it is a linear tree.*

Proof. Let $\mathbf{G} = (G, Y)$ be a reduced graph of groups. If Y is not reduced to one vertex, it contains either a segment of length one which is not directed, or a loop. In both cases, T contains a linear subgraph X, by Proposition 19, and Γ contains an element γ which leaves X invariant and which induces on X an hyperbolic translation. Moreover, any edge of T is contained in a linear subgraph. Claims (i) to (iii) follow.

Suppose that Y contains an edge e with $s(e) \neq t(e)$; suppose moreover either that at least one of $[G_{s(e)} : \varphi_{\overline{e}}(G_e)]$, $[G_{t(e)} : \varphi_e(G_e)]$ is at least 3, or that Y contains at least one other edge than e and $\overline{e}$. Then T has an abundance of vertices of degrees at least 3, and therefore T does not have pending rays. Similarly, if Y contains a vertex y with at least two loops incident to y, then T has an abundance of vertices of degrees at least 4 and T does not have pending rays. This shows Claims (iv) and (v). □

3.2. **Faithful actions.** Let $\mathcal{N} = (N_y)_{y \in V(Y)}$ be the family of subgroups of the groups in the family $(G_y)_{y \in V(Y)}$ defined as follows:

- ∘ N_y is a normal subgroup of G_y for all $y \in Y$;
- ∘ the family is Y*-invariant*, which means that, for all $e \in E(Y)$, there exists a subgroup N_e of G_e, with $N_{\overline{e}} = N_e$, such that $N_{t(e)} = \varphi_e(N_e)$;
- ∘ and $\mathcal{N}$ is maximal with these properties (in the sense that N_y is maximal in G_y for all $y \in V(Y)$).

Note that, for all $y, y' \in V(Y)$, the groups N_y and $N_{y'}$ are isomorphic.

Our next proposition is Proposition 1.23 in [Bass–93].

Proposition 21. *Let* $\mathbf{G} = (G, Y)$ *be a graph of groups and let* $\Gamma, T, \mathcal{N} = (N_y)_{y \in V(Y)}$ *be as above. Choose a vertex* $y_0 \in V(Y)$*.*

Then the kernel of the action of Γ *on* T *is isomorphic to* N_{y_0}*. In particular, the action of* Γ *on* T *is faithful if and only if* $N_{y_0} = \{1\}$*.*

3.3. **Minimal actions.** The next proposition follows from the proof of Corollary 3.5 in [Tits–70], and is also part of Proposition 7.12 in [Bass–93].

Proposition 22. *Let* $\mathbf{G} = (G, Y)$ *be a graph of groups, and let* Γ, T *be as above; we assume for simplicity that the diameter of the underlying graph* Y *is finite.*

The action of Γ *on* T *is minimal if and only if* Y *is reduced.*

4. The two standard examples

4.1. **Amalgamated free products.** In this case, the underlying graph Y is a segment of length one, with two vertices x, y and one pair of edges

$\{e, \overline{e}\}$. The edge group G_e can be identified to a subgroup of both G_x and G_y. The fundamental group of $\mathbf{G} = (G_x, G_y, G_e, Y)$ is the free product with amalgamation, or, for short, the *amalgam* $\Gamma = G_x *_{G_e} G_y$. From now on in this subsection, we write A, B, C instead of G_x, G_y, G_e, so that in particular

$$\Gamma = A *_C B \quad \text{acts on its Bass-Serre tree } T.$$

The edge set of the universal cover T of $\mathbf{G}$ consists of two copies of Γ/C exchanged by the involution $e \longmapsto \overline{e}$, say $E(T) = E^+(T) \sqcup E^-(T)$ with $E^+(T) = \Gamma/C$ and $E^-(T) = \overline{\Gamma/C}$. The vertex set of T is the disjoint union $\Gamma/A \sqcup \Gamma/B$. The source and terminus mappings are defined to be the canonical projections

$$s : \Gamma/C \longrightarrow \Gamma/A, \quad \gamma C \longmapsto \gamma A$$
$$t : \Gamma/C \longrightarrow \Gamma/B, \quad \gamma C \longmapsto \gamma B$$

and

$$s : \overline{\Gamma/C} \longrightarrow \Gamma/B, \quad \overline{\gamma C} \longmapsto \gamma B$$
$$t : \overline{\Gamma/C} \longrightarrow \Gamma/A, \quad \overline{\gamma C} \longmapsto \gamma A.$$

In particular, the tree T is bipartite regular, with one class of vertices of degree $[A : C]$ and the other class of degree $[B : C]$. The action of Γ has two orbits on the vertex set; it is transitive on the orientation $E^+(T)$, or equivalently on the set of geometric edges of T.

The *kernel* of such an amalgam is the subgroup

$$\ker(A *_C B) = \bigcap_{\gamma \in \Gamma} \gamma^{-1} C \gamma \tag{1}$$

of C, namely the largest subgroup of C which is normal in both A and B. An amalgam is

- *faithful* if its kernel is reduced to $\{1\}$, equivalently if the action of Γ on T is faithful.

With the notation of Proposition 2, observe that

$$\ker(A *_C B) = \bigcap_{\ell \geq 0} C_\ell \subset C_k \quad \text{for all } k \geq 0.$$

The amalgam is

- *non-trivial* if $A \neq C \neq B$ (equivalently if $\mathbf{G}$ is reduced), and
- *non-degenerate* if moreover at least one of the indices $[A : C]$, $[B : C]$ is strictly larger than 2.

From the definition of the universal cover T of $\mathbf{G}$, the amalgam is trivial if and only if the diameter of T is finite (and this occurs if and only the diameter of T is at most 2). Also, the amalgam is non-trivial and degenerate if and only if T is a linear tree.

Remarks. (i) A non-trivial amalgam which is faithful is *a fortiori* non-degenerate, unless it is the infinite dihedral group.

(ii) The condition for Γ to act faithfully on T *implies* that Γ is icc (but the converse does not hold); see Corollary 2 in [Corn–09]. Recall that a group Γ is *infinite conjugacy class*, or shortly is *icc*, if it is infinite and if all its conjugacy classes except $\{1\}$ are infinite.

(iii) For example, if $m, n \geq 2$ are two coprime integers, the torus knot group $\langle a, b \mid a^m = b^n \rangle = \langle a \rangle *_{\langle a^m = b^n \rangle} \langle b \rangle$ *does not* act faithfully on its Bass-Serre tree; its kernel is infinite cyclic, generated by $a^m = b^n$. Any non-trivial free product $A * B$ acts faithfully on its Bass-Serre tree.

(iv) Let T be a regular tree of some degree $d \geq 3$. Let Γ denote the group of all automorphisms γ of T which are either elliptic or hyperbolic with an even translation length; it is a subgroup of index 2 in $\mathrm{Aut}(T)$, and simple [Tits–70]. Choose an edge $e \in V(T)$. Denote by A [respectively B, C] the pointwise stabiliser in Γ of $s(e)$ [respectively of $t(e)$, of $\{s(e), t(e)\}$], so that $\Gamma = A *_C B$. We leave it to the reader to check that (as already stated in the introduction), for any $k \geq 0$, the group C_k is the pointwise stabiliser of the neighbourhood $\mathcal{V}_k(e)$ defined in the introduction, and that

$$C \,=\, C_0 \supsetneqq C_1 \supsetneqq \cdots \supsetneqq C_k \supsetneqq C_{k+1} \supsetneqq \cdots \supsetneqq \bigcap_{k \geq 0} C_k \,=\, \ker(A *_C B) \,=\, \{1\},$$

is a *strictly* decreasing infinite sequence of subgroups of C. Observe that this group Γ is not countable, but that it contains dense countable subgroups giving rise similarly to strictly decreasing infinite sequences of C_k's. We are grateful to Laurent Bartholdi for suggesting these examples.

(v) For an integer $k \geq 1$, recall that $\Gamma = A *_C B$ is *k-acylindrical* if, whenever $\gamma \in \Gamma$ fixes pointwise a segment of length k in its Bass-Serre tree, then $\gamma = 1$ [Sela–97]. Thus the condition "$C_k = \{1\}$ for some $k \geq 0$" is substancially weaker than the condition "Γ is ℓ-acylindrical for some $\ell \geq 0$". The previous remark shows that the condition "$\ker(A *_C B) = \{1\}$" is the weakest of all.

Proposition 23. *Let $\Gamma = A *_C B$ be an amalgam acting on its Bass-Serre tree T as above.*

(i) *The amalgam is non-trivial if and only if there is no vertex in T fixed by Γ, if and only if the tree T is infinite; if these conditions hold, then the action of Γ on T is* minimal.

We assume from now on that the amalgam is non-trivial.

(ii) *The amalgam is non-degenerate if and only if T is not a linear tree, if and only if ∂T is perfect, if and only if the action of Γ on T is* strongly hyperbolic.

We assume from now on that the amalgam is non-degenerate.

(iii) *The action of Γ on ∂T is minimal.*
(iv) *The action of Γ on T is slender as soon as $C_k = \{1\}$ for some $k \geq 1$.*

Proof. For (i), see Propositions 20 and 22. [Alternatively, a direct argument is straightforward.]

(ii) Observe first that, if the amalgam is non-trivial and degenerate, then T is a linear tree, so that ∂T has two points, in particular is not perfect, and the action of Γ on ∂T is not strongly hyperbolic.

Suppose now that the amalgam is non-degenerate. Choose $q, r \in A$ such that the three cosets C, qC, rC in A/C are pairwise disjoint, and $s \in B$ with $s \notin C$ (so that $C \cap sC = \emptyset = C \cap s^{-1}C$ in B/C).

In T, there is a first segment of length 5, of which the 6 vertices and 5 oriented edges are, in "the" natural order,

$$qs^{-1}A,\ qs^{-1}C,\ qB,\ qC,\ \underline{A},\ \underline{C},\ \underline{B},\ \underline{sC},\ \underline{sA},\ sqC,\ sqB,$$

and similarly a second segment of length 5 with vertices and edges

$$rs^{-1}A,\ rs^{-1}C,\ rB,\ rC,\ \underline{A},\ \underline{C},\ \underline{B},\ \underline{sC},\ \underline{sA},\ srC,\ srB.$$

These two segments of length 5 have a common subsegment of length 2 with vertices and edges underlined above.

The element $sqsq^{-1} \in \Gamma$ maps the first edge of the first segment onto its last edge, and $srsr^{-1} \in \Gamma$ maps the first edge of the second segment onto its last edge. It follows that $sqsq^{-1}$ and $srsr^{-1}$ are hyperbolic elements, with axis sharing exactly two geometric edges (see if necessary Proposition 25 in Subsection I.6.4 of [Serr–77]). Thus these two hyperbolic elements are transverse.

If the amalgam is non-degenerate, at least every other vertex in T is of degree at least 3. In particular, T does not have any pending ray (Proposition 20), so that ∂T is perfect.

(iii) If the amalgam is non-degenerate, the action of Γ on T is minimal, by (i). Hence the action is also minimal on ∂T, by Proposition 16.

(iv) Let $\gamma \in \Gamma$ be such that the fixed point set $(\partial T)^\gamma$ of γ on the boundary has non-empty interior. To finish the proof, it suffices to show that, if the amalgam is non-degenerate and if $C_k = \{1\}$, then $\gamma = 1$.

Since ∂T is perfect, γ cannot be hyperbolic. We can therefore assume that γ has a fixed vertex $x_0 \in V(T)$. Let $(x_n)_{n \geq 0}$ be the vertices of a ray starting from x_0 and representing a boundary point ξ in the interior of $(\partial T)^\gamma$. By Remark 13.iv, the vertex x_n is fixed by γ for all $n \geq 0$. Since ξ is in the interior of $(\partial T)^\gamma$, there is an edge in the ray $(x_n)_{n \geq 0}$, say d from x_m to x_{m+1}, such that $(\partial T)_d \subset (\partial T)^\gamma$.

Let U be the subtree of T of which the vertices belong to rays with first two vertices x_m, x_{m+1}. Since rays in this tree represent boundary points in $(\partial T)^\gamma$, the same remark as above implies that γ fixes all vertices and all edges in U. Choose an edge $e \in E(U)$ such that all vertices at distance at most k from $s(e)$ or $t(e)$ are in $V(U)$; these vertices are fixed by γ. Choose moreover $\delta \in \Gamma$ such that $e = \delta C$; we can assume that $s(e) \in \Gamma/A$ and $t(e) \in \Gamma/B$.

Let us first assume for simplicity that $k = 1$. Choose transversals $R \subset A$ and $S \subset B$ such that $A = \bigsqcup_{r \in R} rC$ and $B = \bigsqcup_{s \in S} sC$ (disjoint unions). Since γ fixes the edges δrC and δsC, we have $\gamma \in \delta rC(\delta r)^{-1}$ and $\gamma \in \delta sC(\delta s)^{-1}$ for all $r \in R$ and $s \in S$, namely

$$\begin{aligned} \gamma \ &\in \ \big(\bigcap_{r \in R} \delta r C r^{-1} \delta^{-1}\big) \cap \big(\bigcap_{s \in R} \delta s C s^{-1} \delta^{-1}\big) \\ &= \ \delta \left(\big(\bigcap_{a \in A} aCa^{-1}\big) \cap \big(\bigcap_{b \in B} bCb^{-1}\big) \right) \delta^{-1} \\ &= \ \delta C_1 \delta^{-1} \ = \ \{1\}, \end{aligned}$$

hence $\gamma = 1$.

The argument in the general case, $k \geq 1$, is similar, and is left to the reader. Hence, in all cases, $\gamma = 1$. □

Proof of Proposition 2. We have now to assume that

A, B, and therefore also C, Γ, and T, are countable,

because our proof of Proposition 11 assumes countability. Moreover, we assume as in Proposition 2 that the amalgam is non-degenerate and that $C_k = \{1\}$.

Proposition 23 shows that the hypothesis of Proposition 18 are satisfied. Hence the proof of Proposition 2 is complete. □

4.2. **HNN extensions.** In this case, the underlying graph Y is a loop, with one vertex y, and one pair of edges $\{e, \overline{e}\}$. The edge group G_e can be identified (via $\varphi_{\overline{e}}$) to a subgroup of G_y, and we have a monomorphism $\varphi_e : G_e \longrightarrow G_y$. From now on in this subsection, we write G, H, θ

instead of G_y, G_e, φ_e. The fundamental group of $\mathbf{G} = (G, H, \theta(H), Y)$ is a *HNN-extension*, which has the presentation

$$\Gamma \ = \ HNN(G,H,\theta) \ = \ \langle G, \tau \mid \tau^{-1} h \tau = \theta(h) \ \ \forall h \in H \rangle$$

and which acts on its Bass-Serre tree T.

The edge set of T consists of two copies of Γ/H exchanged by the involution $e \longmapsto \overline{e}$, say $E(T) = E^+(T) \sqcup E^-(T)$ with $E^+(T) = \Gamma/H$ and $E^-(T) = \overline{\Gamma/H}$. The vertex set of T is Γ/G. The source and terminus mappings are

$$\begin{aligned} s : \Gamma/H &\longrightarrow \Gamma/G, \quad \gamma H \longmapsto \gamma G \\ t : \Gamma/H &\longrightarrow \Gamma/G, \quad \gamma H \longmapsto \gamma \tau G. \end{aligned}$$

and

$$\begin{aligned} s : \overline{\Gamma/H} &\longrightarrow \Gamma/G, \quad \overline{\gamma H} \longmapsto \gamma \tau G \\ t : \overline{\Gamma/H} &\longrightarrow \Gamma/G, \quad \overline{\gamma H} \longmapsto \gamma G. \end{aligned}$$

In particular, the tree T is regular, of degree $[G:H] + [G:\theta(H)]$. The action of Γ is transitive on the vertex set, as well as on the orientation $E^+(T)$, or equivalently on the set of geometric edges of T.

The *kernel* of an HNN-extension is the subgroup

$$\ker(HNN(G,H,\theta)) \ \doteqdot \ \bigcap_{\gamma \in \Gamma} \gamma^{-1} H \gamma \tag{2}$$

of $H \cap \theta(H)$, namely the largest subgroup of $H \cap \theta(H)$ which is both normal in G and invariant by θ. An HNN extension is

- *faithful* if its kernel is reduced to $\{1\}$, equivalently if the action of Γ on T is faithful.

With the notation of Proposition 4, observe that

$$\ker(HNN(G,H,\theta)) \ = \ \bigcap_{\ell \geq 0} H_\ell \ \subset \ H_k \quad \text{for all } \ k \geq 0.$$

The HNN-extension is

- *ascending* if at least one of $H, \theta(H)$ is the whole of G,
- *strictly ascending* if exactly one of $H, \theta(H)$ is the whole of G, and
- *non-degenerate* if at least one of $H, \theta(H)$ is a proper subgroup of G.

The HNN-extension is degenerate if and only if T is a linear tree, in which case θ is an automorphism of G and Γ is the corresponding semi-direct product $G \rtimes_\theta \mathbf{Z}$.

Remarks. (i) An HNN-extension with $H \neq \{1\}$ which is faithful is *a fortiori* non-degenerate.

(ii) The condition for Γ to act faithfully on T *implies* that Γ is icc (but the converse does not hold); see Example 2.9 in [Stal–06], and Corollary 4 in [Corn–09].

(iii) For example, if m, n are integers, the Baumslag-Solitar group

$$BS(m,n) \;=\; \langle \tau, b \mid \tau^{-1} b^m \tau = b^n \rangle \;=\; HNN(b^{\mathbf{Z}}, b^{m\mathbf{Z}}, b^{mk} \longmapsto b^{nk})$$

acts faithfully on the corresponding tree if and only if $|m| \neq |n|$. Indeed, on the one hand, if $n = \pm m$, then $H = \langle b^m \rangle$ is clearly a normal subgroup of Γ; and, on the other hand, it is a result of Moldavanskii that $BS(m,n)$ has an infinite cyclic normal subgroup if and only if $|m| = |n|$; see [Mold–91], or the exposition in the Appendix of [Souc–01].

Proposition 24. *Let $\Gamma = HNN(G, H, \theta)$ be an HNN-extension acting on its Bass-Serre tree T as above. Then:*

- (i) *There is no vertex in T fixed by Γ, the tree T is infinite and the action of Γ on T is minimal.*
- (ii') *The HNN-extension is non-degenerate if and only if T is not a linear tree, if and only if the space ∂T is perfect.*

We assume from now on that the extension is non-degenerate.

- (ii") *The HNN-extension is non-ascending if and only if the space ∂T is without Γ-fixed point, if and only if the action of Γ on T is strongly hyperbolic.*

We assume from now on that the extension is non-ascending.

- (iii) *The action of Γ on ∂T is minimal.*
- (iv) *The action of Γ on T is slender as soon as $H_k = \{1\}$ for some $k \geq 1$.*

Proof. For (i) and (iii), see the proof of Proposition 23.

(ii') If the extension is degenerate, T is a linear tree, so that ∂T has exactly two points. Otherwise, T is regular of degree has least 3, so that ∂T is perfect.

(ii") Suppose first that $H = G \supsetneqq \theta(H)$. The mapping $E^+(T) \longrightarrow V(T)$, $e \longmapsto s(e)$ is a bijection. Choose a geodesic in T with vertex set $(x_p)_{p \in \mathbf{Z}}$ such that $d(x_p, x_{p'}) = |p - p'|$ for all $p, p' \in \mathbf{Z}$, and such that the edge from x_p to x_{p+1} lies in $E^+(T)$ for all $p \in \mathbf{Z}$; consider the limit $\xi = \lim_{p \to \infty} x_p \in \partial T$. Consider also a geodesic ray with vertex set $(y_q)_{q \in \mathbf{N}}$ such that $d(y_q, y_{q'}) = |q - q'|$ for all $q, q' \in \mathbf{N}$, and such that the edge from y_q to y_{q+1} lies in $E^+(T)$ for all $q \in \mathbf{N}$; set $\eta = \lim_{q \to \infty} y_q \in \partial T$. We claim that $\eta = \xi$, from which it follows that ξ is fixed by Γ.

To prove the claim, it suffices to check that there exists $b \in \mathbf{Z}$ and $q \in \mathbf{N}$ such that $y_q = x_{q+b}$; inded, since $E^+(T) \longrightarrow V(T)$, $e \longmapsto s(e)$, is a bijection, this implies that $y_{q'} = x_{q'+b}$ for all $q' \geq q$, and therefore $\eta = \xi$.

If one could not find b, q such that $y_q = x_{q+b}$, there would exist a segment with vertex set $(z_r)_{0 \leq r \leq N}$, with $N \geq 1$, connecting the geodesic to the ray, namely with $z_0 = x_{p_0}$ for some $p_0 \in \mathbf{Z}$ and $z_N = y_{q_0}$ for some $q_0 \in \mathbf{N}$, moreover with $z_r \notin \{x_p\}_{p \in \mathbf{Z}}$ for $r \geq 1$ and $z_r \notin \{y_q\}_{q \in \mathbf{N}}$ for $r \leq N-1$. On the one hand, the edge with source z_{r+1} and terminus z_r would be in $E^+(T)$ for all r (as one checks inductively for $r = 0, 1, \ldots$), on the other hand, the edge with source z_{r-1} and terminus z_r would also be in $E^+(T)$ for all r (as one checks inductively for $r = N, N-1, \ldots$). But this is impossible, since $E^+(T)$ cannot contain both some edge e with source z_{r+1} and terminus z_r and the edge $\overline{e}$.

Suppose next that $H \subsetneqq G = \theta(H)$, so that the mapping $E^+(T) \longrightarrow V(T)$, $e \longmapsto t(e)$ is a bijection. An analogous argument shows that there exists a point in ∂T fixed by Γ.

It follows also that, if the HNN-extension is ascending, there cannot exist two transverse hyperbolic elements in Γ.

Suppose now that the extension is non-ascending. Choose $r, s \in G$ with $r \notin H$ and $s \notin \theta(H)$.

In T, there are two segments of length 2, with vertices and edges respectively

$$\begin{array}{ccccc} \tau^{-1}G & \tau^{-1}H & G & rH & r\tau G, \\ s^{-1}\tau^{-1}G & s^{-1}\tau^{-1}H & G & H & \tau G, \end{array}$$

sharing just one vertex, G. It follows that $r\tau$ and τs are two elements of Γ which are hyperbolic and transverse, with axis having in common the unique vertex G.

(iv) The proof of this claim is a somewhat tedious variation on that of Proposition 23 and is left to the reader. □

Proof of Proposition 4. We have now to assume that

$$G, \text{ and therefore also } H, \Gamma, \text{ and } T, \text{ are countable,}$$

because our proof of Proposition 11 assumes countability. Moreover, we assume as in Proposition 4 that the HNN-extension is non-ascending and that $H_k = \{1\}$.

Proposition 24 shows that the hypothesis of Proposition 18 are satisfied. Hence the proof of Proposition 4 is complete. □

Lemma 25. *Consider the Baumslag-Solitar group*

$$\Gamma = \langle \tau, b \mid \tau^{-1} b^m \tau = b^n \rangle$$

acting on its Bass-Serre tree T. Denote by G the cyclic subgroup of Γ generated by b, and by H that generated by b^m. Set

$$K_+ = \bigcap_{p \geq 0} \tau^{-p} H \tau^{p} \quad \text{and} \quad K_- = \bigcap_{p \geq 0} \tau^{p} H \tau^{-p}.$$

(i) We have

- *If $|m| = |n|$, then $K_+ = K_- = H$,*
- *if $n = \pm am$ for some $a \geq 2$, then $K_+ = \{1\}$ and $K_- = H$,*
- *if $m = \pm an$ for some $a \geq 2$, then $K_+ = H$ and $K_- = \{1\}$,*
- *In all other cases, $K_+ = K_- = \{1\}$.*

(ii) Assume that $|m| \neq |n|$. For any $k \in \mathbf{Z}$, $k \neq 0$, the automorphism $g = b^k$ of T is elliptic and slender.

(iii) If $|m| \neq |n|$, the action of Γ on T is slender.

Proof. (i) The normal form theorem for HNN extensions (Theorem 2.1 of Chapter IV in [LySc–77]) implies that

$$\begin{aligned} \tau^{-1} b^{\ell} \tau &\notin G \quad \text{if } \ell \notin m\mathbf{Z}, \\ \tau^{-1} b^{km} \tau &= b^{kn} \quad \text{for all } k \in \mathbf{Z}, \\ \tau b^{\ell} \tau^{-1} &\notin G \quad \text{if } \ell \notin n\mathbf{Z}, \\ \tau b^{kn} \tau^{-1} &= b^{km} \quad \text{for all } k \in \mathbf{Z}. \end{aligned}$$

Claim (i) follows.

For (ii) and (iii), we assume that $K_- = \{1\}$. The case with $K_+ = \{1\}$ will follow, since $HNN(G, H, \theta) \approx HNN(G, \theta(H), \theta^{-1})$. Let x_0 denote the vertex $G = 1G \in \Gamma/G = V(T)$. Observe that x_0 is fixed by g; indeed, the isotropy subgroup $\{\gamma \in \Gamma \mid \gamma x_0 = x_0\}$ coincides with G.

(ii) It suffices to show the following claim: $(\partial T)_e \not\subset (\partial T)^g$ for any $e \in E(T)$ with $d(x_0, s(e)) < d(x_0, t(e))$. If $g(t(e)) \neq t(e)$, then $g\left((\partial T)_e\right) \cap (\partial T)_e = \emptyset$, and the claim is obvious; we can therefore assume that $g(t(e)) = t(e)$. We distinguish two cases, depending on the Γ-orbit of e.

Suppose first that $e \in \Gamma/H$, namely that there exists $\gamma \in \Gamma$ such that $s(e) = \gamma\tau^{-1}(x_0)$ and $t(e) = \gamma(x_0)$. Then $(\gamma\tau^p G)_{p \in \mathbf{N}}$ are the vertices of a ray ρ in T starting at $t(e)$, with $d(\gamma\tau^p G, \gamma\tau^q G) = |p-q|$ for all $p, q \in \mathbf{N}$, and extending the segment from x_0 to $t(e)$. For $p \geq 1$, the vertex $\gamma\tau^p G$ is fixed by g if and only if $g \in \gamma G \gamma^{-1} \cap \gamma\tau^p G \tau^{-p} \gamma^{-1}$, namely if and only if $\gamma^{-1} g \gamma \in G \cap \tau^p G \tau^{-p}$. It follows from (i) that there exists an edge f in the ray ρ, with $d(x_0, s(f)) < d(x_0, t(f))$, such that $g(s(f)) = s(f)$ and $g(t(f)) \neq t(f)$, and consequently such that $g\left((\partial T)_f\right) \cap (\partial T)_f = \emptyset$. Since $(\partial T)_f \subset (\partial T)_e$, this implies $(\partial T)_e \not\subset (\partial T)^g$.

Suppose now that $e \in \overline{\Gamma/H}$, namely that there exists $\gamma \in \Gamma$ such that $s(e) = \gamma\tau(x_0)$ and $t(e) = \gamma(x_0)$. Observe that $H \neq G$, otherwise $K_- = G$ could not be $\{1\}$. Choose $u \in G$ with $u \notin H$, so that $\overline{\gamma u H}$ is an edge with source $\gamma u \tau G \neq s(e) = \gamma\tau G$ and with terminus $\gamma G = t(e)$. Then $(\gamma u \tau^p G)_{p \in \mathbf{N}}$ are the vertices of a ray in T starting at $\gamma G = \gamma u G = t(e)$, and the argument of the previous case carries over.

(iii) Let $\gamma \in \Gamma$, $\gamma \neq 1$, be an elliptic automorphism of T. Choose $\delta \in \Gamma$ such that the vertex δG is fixed by γ. Then $g \doteqdot \delta^{-1}\gamma\delta$ fixes x_0, so that $g = b^k$, as in Claim (ii). We have $(\partial T)^\gamma = \delta\left((\partial T)^g\right)$; in other words, the subspace $(\partial T)^\gamma$ of ∂T is the image by the homeomorphism δ of ∂T of the subspace $(\partial T)^g$, without interior points by (ii). Hence γ is slender. □

Proof of Proposition 5. If $\min\{|m|, |n|\} = 1$, the Baumslag-Solitar group $BS(m,n)$ is solvable, in particular amenable, and therefore it is not C*-simple. If $|m| = |n|$, the group $BS(m,n)$ contains an infinite cyclic normal subgroup (as already noted in Remark (iii) just before Proposition 24), so that $BS(m,n)$ is not C*-simple.

Let us assume that $\min\{|m|, |n|\} \geq 2$ and that $|m| \neq |n|$. Observe that, say in the case of m and n coprime to simplify the discussion, we have $G = \mathbf{Z}$ and

$$H_0 = m\mathbf{Z} \supsetneqq H_1 = (mn)^2\mathbf{Z} \supsetneqq \cdots \supsetneqq H_k = (mn)^{2k}\mathbf{Z} \supsetneqq \cdots$$
$$\supsetneqq \bigcap_{k\geq 0} H_k = \ker(BS(m,n)) = \{0\},$$

so that we cannot apply Proposition 4, even though the action of $BS(m,n)$ on T is faithful (by this same Remark (iii)).

However, the action of $BS(m,n)$ on its Bass-Serre tree T is slender, by Lemma 25, strongly hyperbolic and minimal, by Proposition 24. Hence, the hypothesis of Proposition 18 are satisfied, and the proof of Proposition 5 for $BS(m,n)$ is complete.

The proof of the more general case, which is similar, is left to the reader. □

4.3. **About the general case.** Let $\mathbf{G} = (G, Y)$ be a graph of groups. If Y is a tree, the analysis of the fundamental group of $\mathbf{G}$ can be done essentially as in Subsection 4.1. Assume now that Y is not a tree.

Choose a maximal tree M in Y. Let Z denote the graph obtained from Y by collapsing M to a vertex, namely the wedge of circles with a unique vertex, say z, and as many geometrical loops as there are geometrical edges in $Y \smallsetminus M$; and let $\mathbf{H} = (H, Z)$ be the corresponding graph of groups, with H_z the fundamental group of the subgraph of groups of $\mathbf{G}$ defined by M, and with H_e the group G_e for any edge in $E(Y) \smallsetminus E(M)$, appropriately identified to a subgroup of H_z. Observe that, if $\mathbf{G}$ is reduced, so is $\mathbf{H}$. Then the fundamental group of $\mathbf{H}$ is isomorphic to that of $\mathbf{G}$, and a multiple HNN extension of the form

$$\Gamma = \left\langle H, \tau_e, \ldots, \tau_f \;\middle|\; \begin{array}{c} \tau_e^{-1} h \tau_e = \theta_e(h) \;\; \forall h \in H_e \\ \cdots \\ \tau_f^{-1} h \tau_f = \theta_f(h) \;\; \forall h \in H_f \end{array} \right\rangle,$$

where $e, \ldots, f$ are the loops in some orientation $E^+(Z)$ of the wedge of circles Z. The Bass-Serre tree of $\mathbf{H}$ is regular of degree

$$[H_z : H_e] + [H_z : \theta_e(H_e)] + \cdots + [H_z : H_f] + [H_z : \theta_f(H_f)].$$

The analysis of Γ can be done essentially as in Subsection 4.2.

4.4. **Remark.** On several occasions in the past, including in [Harp–85] and [BeHa–86], the first author has been wrong in dealing with elliptic automorphisms of a tree T, concerning their fixed point sets in $V(T)$ and in ∂T. As a consequence of this, Claims (d) and (e) of Theorem 5 in [BeHa–86] are not correct as stated, as pointed out in [Stal–06] (many thanks to Yves Stalder). The notion of *slender* automorphism provides some way to fix at least part of the confusion.

II. Fundamental groups of 3-manifold

Manifolds which appear below are assumed to be *connected*, except explicit exceptions. In Items (5.1) to (5.3), M need not be orientable; from (5.4) onwards, M is assumed to be orientable.

5. A reminder on 3-manifolds and their groups

Let M be a compact[4] 3-manifold; set $\Gamma = \pi_1(M)$. We use $\approx$ to indicate both a homeomorphism of manifolds and an isomorphism of groups. We also use rather standard notation for particular manifolds, with the dimension in superscript: $\mathbf{S}^n$ for spheres, $\mathbf{P}^n$ for real projective spaces, $\mathbf{I}$, $\mathbf{D}^2$, and $\mathbf{B}^3$ for the interval, the 2-disc, and the 3-ball, $\mathbf{T}^2$ and $\mathbf{K}^2$ for the 2-torus and the Klein bottle.

(5.1) The manifold M is *prime* if it is not homeomorphic to the 3-sphere and if, for any connected sum $M_1 \sharp M_2$ homeomorphic to M, one of M_1, M_2 is a standard 3-sphere. Any compact 3-manifold has

[4] This can be slightly relaxed, because of the following theorem. *Let M be a 3-manifold with $\pi_1(M)$ finitely generated. Then there exists a compact 3-submanifold Q of M such that the induced homomorphism $\pi_1(Q) \longrightarrow \pi_1(M)$ is an isomorphism* (Theorem 8.6 in [Hemp–76]).

a decomposition in connected sum $M \approx M_1 \sharp \cdots \sharp M_k$, with the M_j prime, and with a small dose of non-uniqueness which can be precisely described; we have moreover a free product decomposition $\pi_1(M) = \pi_1(M_1) * \cdots * \pi_1(M_k)$, which is unique up to the order of the factors. This is the *Kneser-Milnor decomposition* of M (Chapter 3 of [Hemp–76]).

Note that we can have $\pi_1(M_j) = \{1\}$ for some j. Standard arguments of algebraic topology show that this can occur only if M_j is a 3-ball or a non-standard 3-sphere. Those M_j which are 3-balls correspond bijectively to spherical connected components of ∂M (Lemma 3.7 in [Hemp–76]). Denote by[5]

- $\mathcal{P}(M) = M_1 \sharp \cdots \sharp M_\ell$ the *Poincaré completion* of M

(as in Chapter 10 of [Hemp–76]), where $M_1, \dots, M_\ell$ stand now for the non simply connected prime factors of M; then $\pi_1(\mathcal{P}(M)) \approx \pi_1(M)$, and $\pi_1(M_j)$ is not the trivial group for $j = 1, \dots, \ell$.

To sum up this part of the discussion, there are three cases to distinguish:

(i) $\pi_1(M)$ is a non-degenerate free product,
(ii) $\pi_1(M)$ is an infinite dihedral group,
(*) $\mathcal{P}(M)$ is prime.

(Beware: these cases overlap, see Remark 27.)

(5.2) A 3-manifold M is *irreducible* if any 2-sphere in M bounds a 3-ball. A prime manifold M is either irreducible or a 2-sphere bundle over a circle (Lemma 3.13 in [Hemp–76]). Suppose M is a $\mathbf{S}^2$-bundle over $\mathbf{S}^1$; if M is orientable, this bundle is trivial, $M \approx \mathbf{S}^2 \times \mathbf{S}^1$; if M is non-orientable, this bundle is the non-trivial $\mathbf{S}^2$-bundle over $\mathbf{S}^1$, say $\mathbf{S}^2 \widetilde{\times} \mathbf{S}^1$; in both cases, M is a Seifert manifold.

Thus, we can modify the summing up of (5.1), and replace (*) by (iii) and (iv):

(iii) $\mathcal{P}(M)$ is a Seifert manifold,
(iv) $\mathcal{P}(M)$ is irreducible and is not a Seifert manifold.

(5.3) A manifold M is *∂-irreducible* if it is irreducible and if, moreover, all the connected components of its boundary are incompressible. (We refer to Chapter 6 of [Hemp–76] for the definition of "incompressible"; recall also that the fundamental group of any incompressible component of the boundary injects in the fundamental group of M.)

Lemma 26. *Let M be a 3-manifold such that $\mathcal{P}(M) = M$. Assume that M is irreducible and not ∂-irreducible.*

[5] Thus $\mathcal{P}(M) = \widehat{M}$ modulo the Poincaré Conjecture.

Then either $\pi_1(M)$ is a non-trivial free product, or M is a solid torus or a solid Klein bottle.

Proof. Since ∂M has a compressible component, say F, there exists a 2-disc D in M such that $\partial D = D \cap F$, with ∂D not contractible in F. Set $M_1 = M \setminus D$; more precisely, choose an open regular neighborhood $\mathcal{N}(D)$ of D and consider the complement M_1 of $\mathcal{N}(D)$ in M (thus M_1 is obtained by "splitting M along D", and our $M \setminus D$ is denoted by $\sigma_D(M)$ on Page 4 of [JaSh–79]). There are two cases to distinguish.

Suppose first that M_1 is connected. By the Seifert – Van Kampen theorem (for which the reference we like best is § 3 of [Rham–69]), we have $\pi_1(M) = \pi_1(M_1) * \mathbf{Z}$. If M_1 is simply connected, then M_1 is homeomorphic to a 3-ball (otherwise M would not be irreducible), and M is homeomorphic to either a solid torus or a solid Klein bottle, depending on the action of the gluing map on the orientation of D. If M_1 is not simply connected, then $\pi_1(M)$ is a non-trivial free product.

If M_1 has two connected components, say M' and M'', then $\pi_1(M) = \pi_1(M') * \pi_1(M'')$, again by the Seifert – Van Kampen Theorem. Observe that neither M' nor M'' is simply connected (otherwise one of M', M'' would be a 3-ball, and ∂D would be contractible in F), so that $\pi_1(M)$ is a non-trivial free product. □

Remark 27. *(i) Each case in the conclusion of Lemma 26 occurs.*

(ii) The fundamental group of a ∂-irreducible compact 3-manifold is not a non-trivial free product.

Proof. (i) Let M_1 be a compact 3-manifold which is irreducible and which has a non-empty boundary. Let D, D' be two disjoint 2-discs in ∂M_1, and let M be the result of attaching a handle $\mathbf{D}^2 \times \mathbf{I}$ to D and D'. Then $\pi_1(M) \approx \pi_1(M_1) * \mathbf{Z}$. The manifold M is irreducible and, if M_1 is not simply connected, $\pi_1(M)$ is a non-trivial free product; observe that M is not ∂-irreducible.

(ii) This is a result of Stallings (previously a conjecture of Kneser); see Theorem 7.1 in [Hemp–76]). □

Here is one way to sum up the discussion above.

Proposition 28. *Let M be a compact 3-manifold. Then at least one of the following statements is true:*

(i) *the group $\pi_1(M)$ is a non-degenerate free product, and is therefore a strongly Powers group.*
(ii) *$\pi_1(M)$ is an infinite dihedral group,*
(iii) *$\mathcal{P}(M)$ is a Seifert manifold,*
(iv) *$\mathcal{P}(M)$ is ∂-irreducible, is not a Seifert manifold, and $\pi_1(M)$ is not a non-trivial free product.*

(5.4) Suppose that $\pi_1(M)$ is an infinite dihedral group, and moreover that M is orientable; assume also that the Poincaré conjecture is true. Case (ii) in Proposition 28 is then contained in Case (iii); indeed, it follows from a theorem of [Tao–62] that $\widehat{M}$ is homeomorphic to $\mathbf{P}^3\sharp\mathbf{P}^3$. Recall that $\mathbf{P}^3\sharp\mathbf{P}^3$ is a circle bundle over $\mathbf{P}^2$, and *a fortiori* a Seifert manifold.

If $\pi_1(M)$ is a non-degenerate free product, $\mathcal{P}(M)$ cannot be a Seifert manifold, because the fundamental group of a Seifert manifold not covered by $\mathbf{S}^3$ has a normal infinite cyclic subgroup (Lemma 3.2 in [Scot–83]). Thus:

- *for orientable manifolds, exactly one of the statements (i), (iii), (iv) of Proposition 28 is true.*

(5.5) An irreducible manifold M is *atoroidal* if any incompressible torus in M is isotopic to a component of ∂M. Here is the basic *JSJ decomposition* theorem of Jaco-Shalen and Johansen (see Page 157 of [JaSh–79], Page 483 of [Scot–83], and Theorem 3.4 of [Bona–02]):

- *Let M be a compact 3-manifold which is ∂-irreducible and orientable. There exists a minimal finite family of disjoint essential tori $T_1, \dots, T_k$ such that each connected component of $M \smallsetminus \bigcup_{j=1}^k T_j$ is either atoroidal or a Seifert manifold, and this family is unique up to isotopy.*

Recall that an *essential* torus is a torus which is embedded in M, two-sided, incompressible, and not boundary-parallel. The components of $M \smallsetminus \bigcup_{j=1}^k T_j$ are called the *pieces*. As a corollary of this theorem, we have

- *If M and $T_1, \dots, T_k$ are as above, there is a graph of groups $\mathbf{G} = (G, Y)$, with one vertex for each piece and one geometric edge for each torus, and with $G_e = \pi_1(\mathbf{T}^2) \approx \mathbf{Z}^2$ for all $e \in E(Y)$, such that $\pi_1(M)$ is isomorphic to the fundamental group of $\mathbf{G}$.*

  Moreover, if $k > 0$, either M is a torus bundle over $\mathbf{S}^1$, and $k = 1$, or, for each edge $e \in E(Y)$, the images of G_e by φ_e and $\varphi_{\overline{e}}$ are proper subgroups of $G_{t(e)}$ and $G_{s(e)}$ respectively, and $k \geq 2$.

The last statement follows from Theorem 10.2 in [Hemp–76].

(5.6) An orientable 3-manifold M is *geometric* if the interior of M is homeomorphic to a Riemannian manifold $\Lambda\backslash X$, where X is in a standard list of eight 3-dimensional manifolds, traditionally denoted by

$$\mathbf{S}^3,\ \mathbf{E}^3,\ \mathbf{H}^3,\ \mathbf{S}^2 \times \mathbf{R},\ \mathbf{H}^2 \times \mathbf{R},\ \widetilde{PLS_2}(\mathbf{R}),\ Nil \text{ and } Sol,$$

and where Λ is a group of isometries of X acting freely on X. Equivalently (this is a theorem), a 3-manifold is geometric if its interior admits a complete Riemannian metric which is locally homogeneous, namely which is such that any pair of points in this interior has a pair of isometric neighbourhoods.

If X is one of these eight "models" which is neither $\mathbf{H}^3$ nor Sol then M is necessarily a Seifert manifold. It follows that a geometric 3-manifold is of one of three types:

- an hyperbolic manifold,
- a Seifert manifold,
- a Sol manifold.

There are many 3-manifolds which *are not* geometric: for example, a connected sum of two closed manifolds is geometric if and only if it is the connected sum of two copies of $\mathbf{P}^3$. For all this, see [Scot–83], in particular Page 403 for the last claim, as well as [Bona–02].

(5.7) An oriented 3-manifold is said to satisfy *Thurston's Geometrisation Conjecture* if the pieces of the decomposition (5.5) are geometric.

In three preprints made public in 2002 and 2003, Grisha Perelman, following the Hamilton program and using the Ricci flow, has sketched a proof of Thurston's Geometrisation Conjecture; particular cases have been known before. See [MoTi] and references there.

6. C*-simplicity of 3-groups: proofs of Propositions 8 to 10

We denote by $\mathbf{K}^2 \widetilde{\times} \mathbf{I}$ the non-trivial $\mathbf{I}$-bundle over the Klein bottle.

Lemma 29. *Let N be a compact orientable connected 3-manifold which is ∂-irreducible and with a connected component T of the boundary which is a 2-torus. Assume that the fundamental group $\pi_1(N)$ contains a normal subgroup $K \neq \{1\}$ which is peripheral, in the image of $\pi_1(T)$.*

Then one of the following conclusion holds:

(i) $K \approx \mathbf{Z}$ *and N is a Seifert manifold,*
(ii) $K \approx \mathbf{Z}^2$ *and* $N \approx \mathbf{T}^2 \times \mathbf{I}$,
(iii) $K \approx \mathbf{Z}^2$ *and* $N \approx \mathbf{K}^2 \widetilde{\times} \mathbf{I}$.

Proof. Observe that K, as a subgroup of $\pi_1(T)$, is isomorphic to one of $\mathbf{Z}$ or $\mathbf{Z}^2$. Consider two cases, depending on $Q \doteqdot \pi_1(N)/K$ is finite or not.

Case 1: Q is finite. Assume first that $K \approx \mathbf{Z}^2$. Theorem 10.6 in [Hemp–76] implies that N is a $\mathbf{I}$-bundle over a surface F; since F is necessarily homeomorphic to one of $\mathbf{T}^2$ or $\mathbf{K}^2$, one of (ii), (iii) holds.

Assume next that $K \approx \mathbf{Z}$; we will arrive at a contradiction. By a theorem of Epstein (9.8 in [Hemp–76]), $\pi_1(N)$ is torsion-free. Since $\pi_1(N)$ contains an infinite cyclic subgroup of finite index, $\pi_1(N)$ is itself infinite cyclic. But $\pi_1(N)$ contains $\pi_1(T)$ which is isomorphic to $\mathbf{Z}^2$, and this is impossible.

Case 2: Q is infinite. By Consequence (3) in Theorem 11.1 of [Hemp–76], we have $K \approx \mathbf{Z}$. By the theorem of Seifert fibration (see for our case Theorem II.6.4 in [JaSh–79]), N is a Seifert manifold. □

Let M be a orientable connected compact 3-manifold which is not simply connected; we assume that $\mathcal{P}(M) = M$ (or equivalently, modulo the Poincaré Conjecture, that $\widehat{M} = M$). We want to prove Proposition 8; by Proposition 28, we can assume that M is moreover ∂-irreducible.

If M is geometric, then M is either hyperbolic, or Seifert, or Sol, and Proposition 8 for these cases follows from Example 7. Thus, we can assume furthermore that

- *M is not geometric and M contains an essential 2-torus T;*

see (5.5). Proposition 8 follows now from the two next lemmas.

Lemma 30. *In the previous situation, if T is separating, then Γ is a strongly Powers group.*

Proof. Denote by M_1, M_2 the two components of the result of splitting M along T. Since M is ∂-irreducible and T is essential, M_1 and M_2 are also ∂-irreducible. Set $A = \pi_1(M_1)$, $B = \pi_1(M_2)$, and $C = \pi_1(T) \approx \mathbf{Z}^2$, so that $\Gamma = A *_C B$ by the Seifert – Van Kampen theorem; the amalgam is non-trivial, see (5.5). By Proposition 2, either Γ is a strongly Powers group, or $C_k \neq \{1\}$ for all $k \geq 1$. From now on, we will assume that $C_2 \neq \{1\}$ (so that $C_1 \neq \{1\}$), and we will obtain a contradiction.

Set $C_A = \bigcap_{a \in A} a^{-1}Ca$ and $C_B = \bigcap_{b \in B} b^{-1}Cb$, so that $C_1 = C_A \cap C_B$ is a non-trivial subgroup of $C \approx \mathbf{Z}^2$. Observe that C_A is isomorphic to either $\mathbf{Z}$ or $\mathbf{Z}^2$, and also that C_A is the largest subgroup of C which is normal in A.

Note that neither M_1 nor M_2 can be homeomorphic to $\mathbf{T}^2 \times \mathbf{I}$, otherwise T would be boundary parallel in M, and this is not the case by (5.5). Hence each of M_1, M_2 is either as in (i) or as in (iii) of Lemma 29, and we will end the proof by showing that each case is ruled out.

Suppose first that $M_1 \approx M_2 \approx \mathbf{K}^2 \widetilde{\times} \mathbf{I}$. Then $\partial M_1 \approx \partial M_2 \approx \mathbf{T}^2$, and we have two 2-coverings $N_j \longrightarrow M_j$ with $N_j \approx \mathbf{T}^2 \times \mathbf{I}$ $(j = 1, 2)$.

Thus we have a 2-covering $N \longrightarrow M$ where $N = N_1 \cup N_2$ is a $\mathbf{T}^2$-bundle over $\mathbf{S}^1$, so that N can be viewed as obtained from $\mathbf{T}^2 \times \mathbf{I}$ by identifying the two connected components of the boundary by a gluing homeomorphism of $\mathbf{T}^2$. A torus bundle over $\mathbf{S}^1$ is Sol if the gluing map is Anosov, and Seifert if the gluing map is periodic or reducible (Theorem 5.5 in [Scot–83] or Exercice 3.8.10 in [Thur–97]). If N is Sol, then M is also Sol (Theorem 5.3.i in [Scot–83]). If N is Seifert, then M is also Seifert (Theorem II.6.3 in [JaSh–79]). Thus, in all cases, M is geometric (see Page 483 of [Scot–83]), but this has been ruled out. Thus this case does not occur.

Suppose now that M_1 and M_2 are Seifert, with

$$A \rhd C_A \approx \mathbf{Z} \approx C_B \lhd B.$$

Since any subgroup of $\mathbf{Z}$ is characteristic, $C_1 = C_A \cap C_B$ is normal in both A and B, so that $C_1 \approx \mathbf{Z}$ is also normal in Γ. By the theorem of Seifert fibrations (already used in the proof of Lemma 29), it follows that M is Seifert, but this has been ruled out.

We can finally suppose that M_1 is Seifert with $Z \approx C_A \lhd A$, and $M_2 = \mathbf{K}^2\widetilde{\times}\mathbf{I}$ with $\partial M_2 \approx \mathbf{T}^2$ and $C = \pi_1(\partial M_2) \approx \mathbf{Z}^2$; since C_B is the maximal subgroup of C which is normal in B, we have $C_B = C$. It follows that $C_1 = C_A \cap C_B = C_A$. Since C_1 is normal in A,

$$\{1\} \neq C_2 = C_1 \cap \Big(\bigcap_{b \in B} b^{-1} C_1 b \Big) \lhd C_A \approx \mathbf{Z},$$

C_2 is normal in both A and B, and therefore also in Γ. (Though we will not use this, let us observe that $C_k = C_2$ for al $k \geq 2$.) It follows as above that M is Seifert, which has been ruled out. □

Lemma 31. *In the previous situation, if T is non-separating, then Γ is a strongly Powers group.*

Proof. To prove this claim, denote by M_1 the result of splitting M along T. Since M is ∂-irreducible and T is essential, M_1 is also ∂-irreducible. Set $G = \pi_1(M_1)$; the two boundary components of M_1 coming from the splitting correspond to two isomorphic subgroups of G, say H and $\theta(H)$ where θ is an isomorphism with domain H, and $H \approx \pi_1(T) \approx \mathbf{Z}^2$. If one had $H = G$ or $\theta(H) = G$, the manifold M would be a $\mathbf{T}^2$-bundle over $\mathbf{S}^1$ (Theorem 10.2 in [Hemp–76]), and this cannot be (as above). It follows that the HNN-extension $\Gamma = \pi_1(M) = HNN(G, H, \theta)$ is non-degenerate. By Proposition 4, either Γ is a strongly Powers group, or $H_k \neq \{1\}$ for all $k \geq 1$. From now on, we will assume that $H_2 \neq \{1\}$ (so that $H_1 \neq \{1\}$), we will obtain a contradiction.

Set $H_G = \bigcap_{g \in G} g^{-1}Hg$, so that $H_1 = H_G \cap \tau^{-1}H\tau \cap \tau H \tau^{-1}$ is a non-trivial subgroup of $H \approx \mathbf{Z}^2$. Observe that H_G is isomorphic to either $\mathbf{Z}$ or $\mathbf{Z}^2$, and also that H_G is the maximal subgroup of H which is normal in G.

The boundary of M_1 has at least two connected components which are 2-tori. Since $\partial(\mathbf{K}^2 \widetilde{\times} \mathbf{I}) \approx \mathbf{T}^2$ is connected, M_1 cannot fit in Case (iii) of Lemma 29. If one had $H_G \approx \mathbf{Z}^2$ and $M_1 \approx \mathbf{T}^2 \times \mathbf{I}$, the manifold M would be a $\mathbf{T}^2$-bundle over $\mathbf{S}^1$, consequently would be geometric, and this is ruled out (as above).

Therefore, because of Lemma 29, we can assume that M_1 is Seifert and that $H_G \approx \mathbf{Z}$. Choose a generator h of H_1, so that $H_1 = \langle h \rangle$. Since $\langle h \rangle$ is characteristic in H_G, the subgroup $\langle h \rangle$ is normal in G. Since

$$\{1\} \neq H_2 = \langle h \rangle \cap \tau^{-1}\langle h \rangle \tau \cap \tau \langle h \rangle \tau^{-1},$$

there exists a pair of non-zero integers p, q such that $\tau^{-1}h^p\tau = h^q$. By what we know about Baumslag-Solitar subgroups of 3-manifold groups (Theorem VI.2.1 in [JaSh–79]), this implies that $q = \pm p$. Hence Γ contains a normal subgroup $\langle h^p \rangle \approx \mathbf{Z}$. It follows as above that M is a Seifert manifold, in particular a geometric manifold, and this has been ruled out.

This ends the proof. □

Proof of Corollary 9. Let K be a knot in $\mathbf{S}^3$. Denote by M the complement of an open tubular neighbourhood of K, and let $\Gamma = \pi_1(M)$ the group of K. Then M is irreducible, by the Alexander-Schönflies Theorem. Assume moreover that K is not trivial, so that ∂M is an incompressible torus (Proposition 3.17 in [BuZi–85]), and M a ∂-irreducible manifold.

Suppose first that the JSJ decomposition of M is trivial, so that M is either Seifert or atoroidal (see 5.5). If M is a Seifert manifold, then Γ is not C*-simple beause K is a *torus knot*; see [Budn–06], Proposition 4, first case, with $n = 1$ (observe that, in the second case of the same proposition, $n \geq 1$ should be replaced by $n \geq 2$). If M is atoroidal, Thurston's Hyperbolisation Theorem implies that M is an hyperbolic manifold of finite volume (see e.g. [Bona–02], Section 6.1); in this case, K is a *hyperbolic knot* and Γ is a strongly Powers group, see Example 7.

Suppose now that the JSJ decomposition of M is non-trivial (i.e. involves at least one torus), so that K is a *satellite knot*. Geometric manifolds of this kind are classified; they are special kinds of Sol manifolds (see Theorem 2.11 in [Bona–02]) and cannot be knot complements. Hence M is not geometric, and Γ is a strongly Powers group by our Lemmas 30 and 31. □

Proof of Propositions 10. If Γ is of order two, neither Γ nor $\Gamma' = \{1\}$ is C*-simple, and the Proposition holds for a trivial reason. We assume from now on that Γ is not of order two, and it follows that Γ is infinite (a result of D. Epstein, Theorem 9.5 in [Hemp–76]).

Any subgroup of finite index in a C*-simple group is itself C*-simple [BeHa–00]. In particular, if Γ is C*-simple, so is Γ'.

Assume finally that Γ' is C*-simple. Then Γ' is icc (see Appendix J in [Harp–07]). It follows from Lemma 9.1 in [HaPr–07] that Γ is icc, and then from [BeHa–00] that Γ is C*-simple. □

7. A reminder on Powers groups and C*-simple groups

A group Γ is a *Powers groups* if it is not reduced to one element and if, for any finite subset F in $\Gamma \smallsetminus \{1\}$ and for any integer $N \geq 1$, there exists a partition $\Gamma = C \sqcup D$ and elements $\gamma_1, \dots, \gamma_N$ in Γ such that

$$fC \cap C = \emptyset \text{ for all } f \in F,$$
$$\gamma_j D \cap \gamma_k D = \emptyset \text{ for all } j, k \in \{1, \dots, N\},\ j \neq k.$$

Let us recall here the following facts:

- non-abelian free groups are Powers groups [Powe–75];
- Powers group have non-abelian free subgroups [BrPi];
- a Powers group is C*-simple, and moreover its reduced C*-algebra has a unique trace;
- Proposition 17 applies to many examples for showing them to be Powers groups;
- there are uncountably many countable groups Γ with pairwise non-isomorphic reduced C*-algebras (Corollary 9 in [AkLe–80], building up on [McDu–69]);
- if N is an amenable normal subgroup of a Powers group (or more generally of a C*-simple group), then $N = \{1\}$;
- there are C*-simple groups which are not Powers groups (examples: $PSL_n(\mathbf{Z})$ for large n, and direct products of non-abelian free groups);

see [Harp–07]. We do not know any example of a Powers group which is not a strongly Powers group, as defined in the introduction; indeed, the following question is open for us:

- *does there exist a pair* (Γ, N) *of a group and a non-trivial normal subgroup with* Γ *C*-simple and* N *not C*-simple?*

We are grateful to Bachir Bekka who has observed to us that, as a consequence of [Pozn], if such a pair (Γ, N) exists, then N cannot be linear (*a fortiori* Γ cannot be linear). Indeed, suppose that a C*-simple group Γ would contain a linear non-C*-simple non-trivial normal subgroup N; then, by Poznansky's results, the amenable radical R of N would be non-trivial; but amenable radicals are characteristic subgroups, so that N would be normal in Γ, and this is impossible since Γ is C*-simple. The question is closely related to that about the existence of a group Γ which would *not* be C*-simple and which would not contain any non-trivial amenable normal subgroup [BeHa–00].

Finally, let us state the following strenghtening of Proposition 11 of [Harp–07]. Given a group Γ of homeomorphisms of a topological space Ω, recall that the subset L_Γ of Ω has been defined in the proof of Proposition 17.

Proposition 32. *Let Γ be a group which acts by homeomorphisms on a Hausdorff topological space Ω. Assume the action of Γ is strongly hyperbolic on Ω and strongly faithful on L_Γ. Then*

(i) *any non-empty Γ-invariant closed subset of Ω contains $\overline{L_\Gamma}$.*

Let moreover N be a normal subgroup of Γ which contains an hyperbolic homeomorphism. Then

(ii) $\overline{L_N} = \overline{L_\Gamma}$;
(iii) *N is a Powers group.*

Proof. Observe that L_Γ is non-empty because Γ contains hyperbolic elements, and indeed infinite because the action of Γ is strongly hyperbolic. Claim (i) follows from the argument used for Proposition 16, and Claim (ii) is then straightforward. Thus the action of N on Ω satisfies the hypothesis of Proposition 11 in [Harp–07], so that Claim (iii) follows. □

Let M be a connected 3-manifold which is hyperbolic and not elementary, as in Example 7. Let $\Gamma = \pi_1(M)$ be acting in the canonical way on $\Omega = \mathbf{S}^2$, the boundary of the classical hyperbolic space $\mathbf{H}^3$ of dimension 3. Since Γ is discrete on $\mathbf{H}^3$ and not elementary, Γ is strongly hyperbolic on $\mathbf{S}^2$ and strongly faithful on L_Γ. It is then standard that

Corollary 33. *With the notation above, any normal subgroup $N \neq \{1\}$ of $\Gamma = \pi_1(M)$ contains transverse pairs of hyperbolic transformations.*

Proof. On the one hand, N cannot contain non-identity elliptic transformations, because Γ is torsion-free. On the other hand, if N contains parabolic transformations, then the closure of

$$\{\eta \in \mathbf{S}^2 \mid \gamma\eta = \eta \text{ for some parabolic } \gamma \in \Gamma\}.$$

coincides with $\overline{L_\Gamma}$ by (i) of the previous proposition, so that N contains pairs (p_1, p_2) of parabolic elements with distinct fixed points, and $p_1^k p_2^k$ is hyperbolic for k large enough.

Thus L_N is non-empty. Since L_N is Γ-invariant, $\overline{L_N} = \overline{L_\Gamma}$, and the conclusion follows. □

Similarly, in a Gromov-hyperbolic group, any infinite subnormal subgroup contains hyperbolic transformations.

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