# Planar networks and inequalities on eigenvalues

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UNIVERSITÉ DE GENÈVE FACULTÉ DES SCIENCES

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# Planar networks and inequalities on eigenvalues

THÉSE

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par

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# " Planar Networks and Inequalities on Eigenvalues "

La Faculté des sciences, sur le préavis de Messieurs A. ALEKSEEV, professeur ordinaire et directeur de thèse (Section de mathématiques), P. WITTWER, professeur titulaire (Département de physique théorique) et E. MEINRENKEN, professeur (Department of Mathematics, University of Toronto, Ontario, Canada), autorise l'impression de la présente thèse, sans exprimer d'opinion sur les propositions qui y sont énoncées.

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N.B. - La thèse doit porter la déclaration précédente et remplir les conditions énumérées dans les "Informations relatives aux thèses de doctorat à l'Université de Genève".

# Contents

# Résumé

Dans cette thèse, nous étudions la relation entre deux problèmes de l'algèbre linéaire concernant les valeurs propres des matrices hermitiennes et certains graphes planaires pondérés.

Le premier problème est appelé le problème de Horn. Considérons trois matrices hermitiennes $n \times n$ dont la somme est nulle. Considérons leurs $3n$ valeurs propres, où les valeurs propres de chaque matrice sont écrites dans l'ordre décroissant. Le problème est de décrire les relations entre ces valeurs propres. Il a été conjecturé par Horn et démontré par Klyachko et Knutson et Tao que la réponse est un cône polyédral, appelé le cône de Horn, qui est donné par une liste d'inégalités définie récursivement. Knutson et Tao ont reformulé ce problème en termes de "honeycombs" et "hives". Dans le modèle des hives, nous considérons la triangulation régulière d'ordre $n$ du triangle équilatéral. Le triangle est alors divisé en $n^2$ petits triangles, et leurs sommets forment $\frac{(n+1)(n+2)}{2}$ nœuds. Nous attribuons un nombre à chaque nœud. Chaque paire des triangles adjacents forme un rhombe. La condition de hive dit que sur chaque rhombe, la somme des nombres attribués aux bouts de la diagonale courte est plus grande ou égale à la somme des nombres attribués aux bouts de la diagonale longue. Knutson et Tao ont montré que le cône de Horn est donné par toutes les attributions de nombres aux nœuds de la triangulation qui vérifient la condition de hive et telles que des sommes de valeurs propres des trois matrices sont attribuées aux nœuds extérieurs.

Dans cette thèse, nous introduisons un cadre combinatoire où les inégalités des hives apparaissent naturellement. Nous considérons des réseaux planaires, i.e. certains graphes planaires orientés ayant $n$ sources et $n$ écoulements. Nous attribuons des poids aux arêtes du réseau. Dans ce cadre, certaines fonctions de ces poids, linéaires par morceaux, correspondent aux valeurs propres des matrices. Par exemple, nous définissons le poids d'un chemin comme la somme des poids de toutes ses arêtes, et alors la valeur propre maximale d'une matrice est remplacée par le poids maximal d'un chemin qui relie une source et un écoulement sur le réseau planaire. L'addition des matrices est remplacée par la concaténation des réseaux planaires, i.e. par la jonction des écoulements du premier réseau aux sources du deuxième réseau. Nous obtenons alors trois réseaux planaires: les deux réseaux et leur concaténation. Nous démontrons que leurs analogues des valeurs propres vérifient la condition de hive. Les nœuds internes de la triangulation ont aussi une interprétation naturelle en termes de poids de certains chemins sur ces réseaux planaires. Nous démontrons aussi que pour un choix particulier de réseaux planaires, leurs analogues des valeurs propres donnent l'entier du cône de Horn.

L'autre problème que nous étudions dans cette thèse est le précurseur du problème de Horn appelé le problème de Gelfand–Zeitlin. Considérons une matrice hermitienne $n \times n$ et ses valeurs propres ordonnées. Ces valeurs propres avec les valeurs propres de tous les sous-matrices principales de la matrice vérifient ce qu'on appelle les inégalités d'entrelacement, qui définissent le cône de Gelfand–Zeitlin. Ces inégalités ont un rôle important dans la description du système intégrable de Gelfand–Zeitlin. D'une façon similaire au problème de Horn, les inégalités d'entrelacement apparaissent naturellement dans le cadre des réseaux planaires pondérés: de nouveau, nous construisons des analogues des valeurs propres comme des fonctions linéaires par morceaux des poids des arêtes. Nous démontrons que ces analogues des valeurs propres vérifient les inégalités d'entrelacement et que pour un choix particulier du réseau planaire, ces analogues des valeurs propres donnent l'entier du cône de Gelfand–Zeitlin. Dans les annexes, nous démontrons que ces analogues des valeurs propres peuvent être interprétés comme des limites tropicales des vraies valeurs propres des matrices hermitiennes. Nous étudions aussi la relation entre les pré-images de l'application envoyant des matrices vers ses valeurs propres et son analogue dans le cadre des réseaux planaires. Enfin, nous généralisons ces analogues des valeurs propres pour le cas des semi-anneaux ordonnés arbitraires.

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# Chapter 1

# Horn problem

Let $A, B$, and $C$ be Hermitian $n \times n$ matrices, such that $A + B = C$ and let $\lambda_i$, $\mu_i$, and $\nu_i$, for $i = 1, \ldots, n$ be their ordered eigenvalues, i.e,

$$\begin{aligned} \lambda_1 \geq \lambda_2 \geq & \quad \ldots \quad \geq \lambda_n, \\ \mu_1 \geq \mu_2 \geq & \quad \ldots \quad \geq \mu_n, \\ \nu_1 \geq \nu_2 \geq & \quad \ldots \quad \geq \nu_n. \end{aligned}$$

A natural question of linear algebra is to describe the relations between these eigenvalues. One simple relation is the relation of traces. Indeed, we have $\mathrm{Tr}\, A + \mathrm{Tr}\, B = \mathrm{Tr}\, C$, and so

$$\sum_{i=1}^{n} \lambda_i + \sum_{i=1}^{n} \mu_i = \sum_{i=1}^{n} \nu_i. \tag{1.1}$$

This is the only relation which is an equality, the rest of them are inequalities. For example, the largest eigenvalue of the sum of two matrices is not greater than their individual maximal eigenvalues,

$$\lambda_1 + \mu_1 \geq \nu_1. \tag{1.2}$$

This follows from the following characterization of the maximal (and minimal) eigenvalues:

**Lemma 1** *For any Hermitian $n \times n$ matrix $A$, let $\lambda_1$ and $\lambda_n$ be its largest and smallest eigenvalues correspondingly. Then*

$$\lambda_1 = \max_{x^* x = 1} x^* A x$$

*and*

$$\lambda_n = \min_{x^* x = 1} x^* A x,$$

*i.e., for any $x$,*

$$\lambda_n x^* x \leq x^* A x \leq \lambda_1 x^* x.$$

**Proof** There exists a unitary matrix $U$ such that $A = UDU^*$, where $D = diag\{\lambda_1, \ldots, \lambda_n\}$. Thus we have

$$x^* A x = x^* U D U^* x = (U^* x)^* D (U^* x) = \sum_{i=1}^{n} \lambda_i |(U^* x)_i|^2,$$

and so

$$\lambda_n \sum_{i=1}^{n} |(U^*x)_i|^2 \leq x^*Ax \leq \lambda_1 \sum_{i=1}^{n} |(U^*x)_i|^2.$$

Since $U^* = U^{-1}$, we have

$$\sum_{i=1}^{n} |(U^*x)_i|^2 = (U^*x)^*(U^*x) = x^*UU^*x = x^*x,$$

which implies the desired inequalities.

□

There is a more general fact concerning maximal and minimal eigenvalues.

**Lemma 2 (Courant–Fischer minimax principle)**

$$\lambda_k = \max_{\substack{M \\ \dim M = k}} \min_{\substack{x \in M \\ x^*x=1}} x^*Ax = \min_{\substack{M \\ \dim M = n-k+1}} \max_{\substack{x \in M \\ x^*x=1}} x^*Ax$$

The lemma follows from the following inequalities:

**Lemma 3 (Poincaré inequalities)** *Let $M$ be of dimension $k$. Then there exist $x$ and $y$ unit vectors in $M$ such that*

$$x^*Ax \leq \lambda_k$$

*and*

$$y^*Ay \geq \lambda_{n-k+1}.$$

**Proof** Let $v_1, \ldots, v_n$ be the eigenvectors corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$. Consider the space $V = span\{v_k, \ldots, v_n\}$. We have $M \cap V \neq \{0\}$. Choose a unit vector $x \in M \cap V$. Then $x = \sum_{i=k}^{n} x_i v_i$, where $\sum_{i=k}^{n} |x_i|^2 = 1$. Therefore

$$x^*Ax = \sum_{i=k}^{n} |x_i|^2 \lambda_i \leq \sum_{i=k}^{n} |x_i|^2 \lambda_k = \lambda_k.$$

The second inequality is obtained by replacing $A$ by $-A$, then $-\lambda_i(A) = \lambda_{n-i+1}(-A)$.

□

**Proof of Lemma 2** The first Poincaré inequality implies that for any $M$ of dimension $k$ and for all unit vectors $x \in M$ we have

$$\min_x x^*Ax \leq \lambda_k.$$

If we take $M = span\{u_1, \ldots, u_k\}$, then we obtain an equality since $\lambda_k$ is the minimal eigenvector in $M$. Together these two facts imply the statement of the lemma.

□

More relations similar to (1.2) were added to this list over the years, proven mostly using the minimax principle and similar techniques. Here is the approximate history of the problem.

- Weyl, 1912 [10]: $\nu_{i+j-1} \leq \lambda_i + \mu_j$ for all $i$ and $j$ such that $i + j - 1 \leq n$.
- Fan [1], 1949: for all $k \leq n$,

$$\sum_{i=1}^{k} \nu_i \leq \sum_{i=1}^{k} \lambda_i + \sum_{i=1}^{k} \mu_i.$$

- Lidskii [9], 1950: for all $k \leq n$ and for all $I \subset \{1, \ldots, n\}$, such that $|I| = k$,

$$\sum_{i \in I} \nu_i \leq \sum_{i \in I} \lambda_i + \sum_{i=1}^{k} \mu_i.$$

- Finally, in 1962 Horn conjectured [5] the complete list of inequalities:

$$\sum_{i \in I} \nu_i \leq \sum_{j \in J} \lambda_j + \sum_{k \in K} \mu_k,$$

  for *admissible* triples of indices $I, J$, and $K$ of the same cardinality. The set $T_{r,n}$ of admissible triples of indices of cardinality $r$ is defined recursively. Let $I = \{i_1, \ldots, i_r\}$, $J = \{j_1, \ldots, j_r\}$, and $K = \{k_1, \ldots, k_r\}$ be subsets of $\{1, \ldots, n\}$ of cardinality $r < n$. The triple $(I, J, K)$ is in $T_{r,n}$ if

$$i_{a_1} + \cdots + i_{a_s} + j_{b_1} + \cdots + j_{b_s} = k_{c_1} + \cdots + k_{c_s} + \frac{s(s+1)}{2}$$

  for all $s < r$ and for all triples of indices $\{a_1, \ldots, a_s\}$, $\{b_1, \ldots, b_s\}$, and $\{c_1, \ldots, c_s\}$, subsets of $\{1, \ldots, r\}$, in $T_{s,r}$.

The Horn conjecture was proved more than 30 years later by Klyachko [6] using Schubert calculus and by Knutson and Tao [7] using a reformulation in terms of certain combinatorial objects called *honeycombs* or equivalent ones called *hives*.

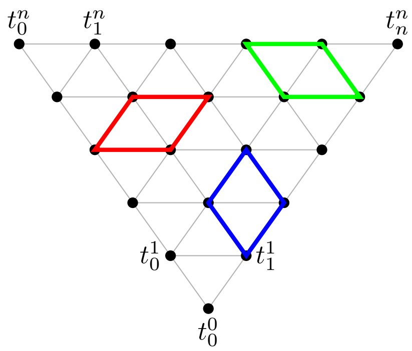


Figure 1.1: A hive triangle with three types of rhombi.

Consider the regular triangulation of order $n$ of an equilateral triangle (Figure 1.1). The triangle is divided into $n^2$ small triangles. Two adjacent triangles form a rhombus which can be of one of three types (shown in color in Figure 1.1).

Let us assign numbers $t_i^k$ to the nodes of the triangulation. Each rhombus gives rise to an inequality: the sum of the two numbers assigned to the endpoints of the short diagonal is greater than or equal to the sum of the two numbers assigned to the endpoints of the long diagonal. An assignment of numbers to the nodes is called a hive if it satisfies all the inequalities, i.e., if for $0 < i \leq k < n$,

$$\begin{array}{lcl} t_i^{k+1} + t_{i-1}^k & \geq & t_{i-1}^{k+1} + t_i^k, \\ t_i^{k+1} + t_i^k & \geq & t_{i+1}^{k+1} + t_{i-1}^k, \\ t_i^k + t_{i-1}^k & \geq & t_i^{k+1} + t_{i-1}^{k-1}. \end{array} \tag{1.3}$$

Now consider the boundary map:

$$\begin{array}{rcl} \partial & : & \{t_i^k,\ 0 \leq i \leq k \leq n\} \mapsto \\ & \mapsto & \{\lambda_i = t_0^i - t_0^{i-1},\ \mu_i = t_i^n - t_{i-1}^n,\ \nu_i = t_i^i - t_{i-1}^{i-1},\ 1 \leq i \leq n\}, \end{array}$$

i.e., we assign to the outer nodes of the triangulation sums of eigenvalues. Knutson and Tao showed that the Horn set of eigenvalues of triples of Hermitian matrices is given by the image of the cone defined by hives under the boundary map.

# Chapter 2

# Gelfand–Zeitlin problem

There is another, simpler problem concerning inequalities on eigenvalues of Hermitian matrices. Let $A$ be a Hermitian $n \times n$ matrix and for any $k$, let $A^{(k)}$ be its principle $k \times k$ submatrix, i.e., the $k \times k$ submatrix in the upper left corner of $A$. Let $\lambda_1^{(k)} \geq \cdots \geq \lambda_k^{(k)}$ be the ordered eigenvalues of $A^{(k)}$. An easy linear algebra problem is to describe the relations between these eigenvalues. It turns out that they satisfy the so-called *interlacing inequalities*:

**Lemma 4 (Cauchy interlacing theorem)** *For all* $0 < i \leq k < n$,

$$\lambda_i^{(k+1)} \geq \lambda_i^{(k)} \geq \lambda_{i+1}^{(k+1)}.$$

**Proof** Let us prove the lemma for $k+1 = n$. The other cases are proven the same way. Let $v_1, \ldots, v_i$ be the eigenvectors corresponding to the eigenvalues $\lambda_1^{(n-1)}, \ldots, \lambda_i^{(n-1)}$ and denote $M = span\{v_1, \ldots, v_i\}$. Then the minimax principle (Lemma 2) gives

$$\lambda_i^{(n-1)} = \min_{x \in M} x^* A^{(n-1)} x = \min_{x \in M} x^* A x \leq \lambda_i^{(n)}.$$

The second inequality is obtained by replacing $A$ by $-A$.

□

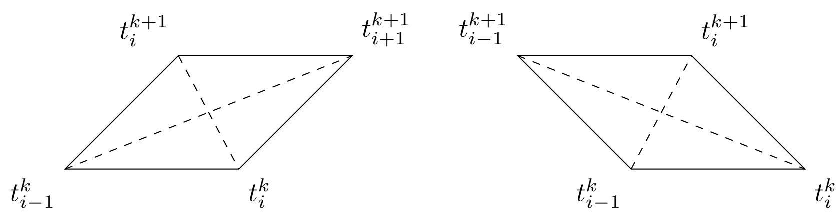


Figure 2.1: The two types of rhombi.

In fact, these inequalities can also be described in terms of hives: if we assign to the nodes of the triangulation (Figure 1.1) the sums of the eigenvalues, they satisfy the inequalities given by two of the three types of rhombi (see Figure 2.1), i.e.

$$\begin{array}{rcl} t_i^{k+1} + t_{i-1}^k & \geq & t_{i-1}^{k+1} + t_i^k, \\ t_i^{k+1} + t_i^k & \geq & t_{i+1}^{k+1} + t_{i-1}^k. \end{array} \tag{2.1}$$

These eigenvalues play an important role in the description of the Gelfand–Zeitlin integrable system [4] (for that reason we call the cone defined by the interlacing inequalities the Gelfand–Zeitlin cone). Indeed, the functions sending a matrix $A$ to the eigenvalue $\lambda_i^{(k)}(A)$ form a completely integrable system.

The interlacing inequalities also have applications in Schubert calculus [8].

# Chapter 3

# Planar networks

In this work we describe the inequalities that appear in the Horn problem and in the Gelfand–Zeitlin problem using certain planar directed graphs called *planar networks*.

**Definition 1** *A planar network is the following data:*

- *a finite graph* $\Gamma$ *with vertex set* $V\Gamma$ *and edge set* $E\Gamma$,
- *a pair of reals* $a < b$,
- *an embedding of* $\Gamma$ *into the strip* $\{a \leq x \leq b\} \subset \mathbb{R}^2$ *such that the image of each edge is a segment of a straight line that is not parallel to the* $y$*-axis.*

A planar network is naturally oriented from left to right. We will call the vertices on the line $x = a$ sources and the vertices on the line $x = b$ sinks of $\Gamma$; the other vertices will be called internal.

The simplest example of a planar network is a planar network with no edges. Another example is the planar network with exactly $n$ edges that are horizontal lines connecting the sources $\{1, \ldots, n\}$ with the sinks $\{1, \ldots, n\}$. Such a planar network $\Delta_0$ is shown in Figure 3.1.

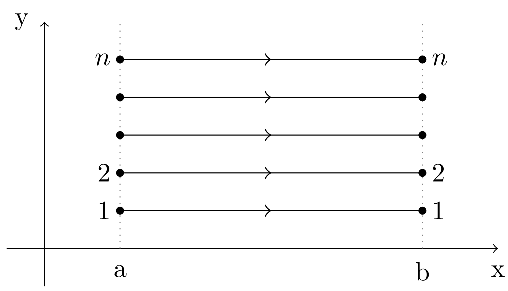


Figure 3.1: The planar network $\Delta_0$

In general, however, the graph can be rather complicated (for example, see Figure 3.2).

These graphs are an important tool in the theory of cluster algebras [3] and total positivity [2].

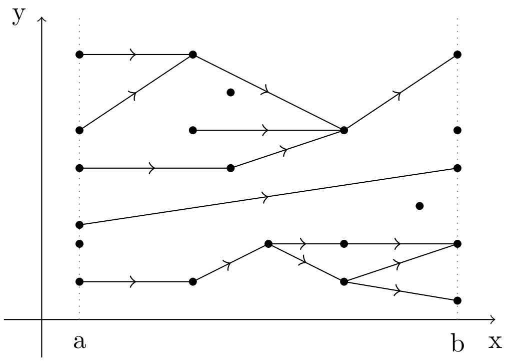


Figure 3.2: A planar network

## 3.1 Total positivity

**Definition 2** *A matrix is called totally positive if all its minors are positive.*

Totaly positive matrices arise in various areas of mathematics. It turns out that they are closely related to planar networks.

Consider the planar network $\Gamma^{(2)}$ shown in Figure 3.3 (from now on we will not draw the orientation of the edges since we always have the orientation left-to-right). Let us assign weights to its edges as shown in Figure 3.3, where $a, b, c$, and $d$ are real numbers.

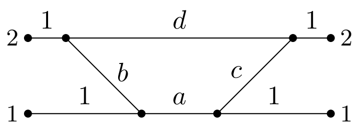


Figure 3.3: Planar network $\Gamma^{(2)}$.

We can associate to such planar network a $2 \times 2$ weight matrix $A$ by the following rule. Consider a path connecting the source $i \in \{1, 2\}$ with the sink $j \in \{1, 2\}$. The weight of a path we define as the product of weights of all edges along this path. For example, the weight of the only path in $\Gamma^{(2)}$ connecting the source 2 with the sink 1 is equal to $1 \cdot b \cdot a \cdot 1 = ab$. Now we take the sum of such weights over all possible paths connecting the given source with the given sink (for example, there are two paths from the source 2 to the sink 2), call this quantity $w_{i,j}$. Finally, we define the matrix entry $A_{i,j} = w_{i,j}$. We obtain

$$A = \begin{pmatrix} a & ac \\ ab & d + abc \end{pmatrix}.$$

Let us compute its determinant:

$$\det A = ad + a^2bc - a^2bc = ad.$$

Thus we note that if all the weights are positive, then the matrix $A$ is totally positive.

It turns out that all totally positive matrices can be obtained this way, i.e., as weight matrices of planar networks of the type shown in Figure 3.4.

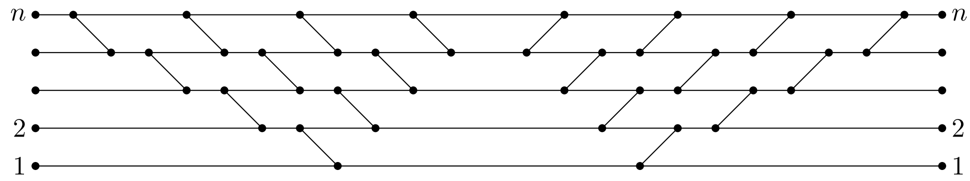


Figure 3.4: Planar network $\Gamma^{(n)}$.

Let us assign nonzero weights to all the slanted edges and to all $n$ horizontal edges in the middle of the graph. Let us assign weight 1 to all the other edges. Such a weighting is called essential. Consider the weight matrix $A$ of this weighting defined as in the 2 by 2 case, i.e., its entry $A_{i,j}$ is equal to the sum of weights of all paths connecting the source $i$ with the sink $j$.

**Theorem 1 (Fomin, Zelevinsky)** *The map sending a weighting of a planar network to its weight matrix restricts to a bijection between the positive essential weightings of* $\Gamma^{(n)}$ *and the totally positive* $n \times n$ *matrices.*

# Chapter 4

# Tropical weights and eigenvalues

In what follows the following example of a planar network motivated by the one discussed in Section 3.1 will play an important role. Consider the planar network $\Gamma_0$ shown in Figure 4.1. It has $n$ sources, $n$ sinks, and $\frac{n(n-1)}{2}$ slanted edges.

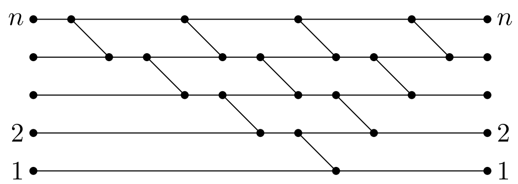


Figure 4.1: The planar network $\Gamma_0$.

## 4.1 Tropical weights and the Gelfand–Zeitlin problem

Let us consider tropical weightings of a planar network. That is, to every edge $e_i \in E\Gamma$, we assign the weight $w_i = w(e_i) \in \mathbb{T}$. Here $\mathbb{T}$ is the tropical semiring, $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$. We will study certain collections of paths on planar networks and their weights.

**Definition 3** *A $k$-path is a collection of $k$ vertex-disjoint paths connecting $k$ sources with $k$ sinks (see Figure 4.2).*

The *tropical weight* of a $k$-path is defined as the sum of weights of all its edges. We introduce a collection of functions sending a weighting $\omega = \{w_i\}_i$ of the planar network to the maximal tropical weight of its $k$-paths:

$$l_i\Gamma(\omega) = \max\{w(p_i),\ \text{for } p_i \text{ an } i-\text{path}\}.$$

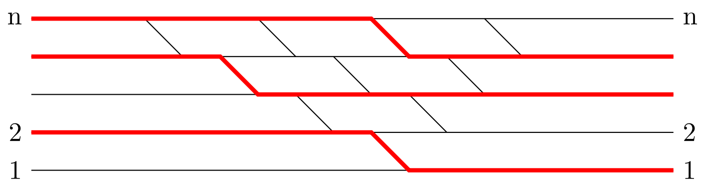


Figure 4.2: A 3-path in $\Gamma_0$.

For any $k \le n$, let us consider $k$-subnetworks of $\Gamma$: we define $\Gamma^{(k)}$ to be the planar network obtained from $\Gamma$ by discarding the sources and sinks above the line $y = k$. Consider the functions $l_i\Gamma^{(k)}$. We call $L\Gamma$ the map defined by $l_i\Gamma^{(k)}$ for all $i \le k \le n$.

These functions are the analogs of eigenvalues in this framework. We have the following results for the Gelfand–Zeitlin problem.

- In the article we show that the image of the map $L\Gamma$ is contained in the Gelfand–Zeitlin cone defined by the interlacing inequalities (4), i.e., for any weighting $\omega$ of $\Gamma$, the values $t_i^k = l_i\Gamma^{(k)}$ satisfy the inequalities defining the eigenvalues of Hermitian matrices.
- We also show in the article that for $\Gamma = \Gamma_0$ shown in Figure 4.1, the image of $L\Gamma_0$ coincides with the entire Gelfand–Zeitlin cone.
- In Appendix A we show that these tropical analogs of eigenvalues can be viewed as limits of true eigenvalues.
- In Appendix B we show the relation between the preimage of $L\Gamma$ and the preimage of the eigenvalue map $\{\lambda_i^k\}$.

We have similar results for the Horn problem.

## 4.2 Tropical weights and the Horn problem

Consider two planar networks $\Gamma$ and $\Delta$, each having $n$ sources and $n$ sinks. We can define their composition $\Gamma \circ \Delta$ by their concatenation. Then we can define multipaths on $\Gamma \circ \Delta$ in the following way. We say that a path $\gamma$ in $\Gamma$ and a path $\delta$ in $\Delta$ are *composable* if the set of sinks of $\gamma$ contains the set of sources of $\delta$.

**Definition 4** *A $(k,i)$-path is a collection $\alpha$ of paths in $\Gamma \circ \Delta$ such that $\alpha = \gamma \cup \delta$, where $\gamma$ is a $k$-path in $\Gamma$, $\delta$ is an $i$-path in $\Delta$, and the paths $\gamma$ and $\delta$ are composable.*

Similarly to the Gelfand–Zeitlin case we introduce the following functions. For a weighting $\omega = \{w_i\}_i$ of the planar network $\Gamma \circ \Delta$ we define

$$m_i^k\Gamma\Delta(\omega) = \max\{w(p_{k,i}),\ \text{for } p_{k,i} \text{ a } (k,i)\text{−path}\}.$$

Putting these functions together we obtain the map $M\Gamma\Delta$.

**Remark 1** *We have the following relations between these functions and the previously defined functions $l_i\Gamma^{(k)}$:*

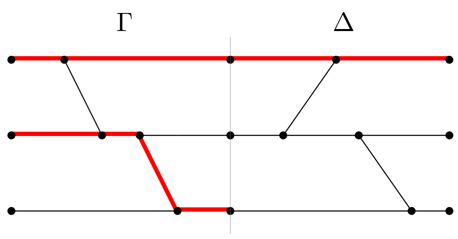


Figure 4.3: A $(2,1)$-path on $\Gamma \circ \Delta$

- *if* $i = 0$, *then* $m_i^k \Gamma\Delta = l_k \Gamma$,
- *if* $k = n$, *then* $m_i^k \Gamma\Delta = l_n \Gamma + l_i \Delta$,
- *if* $i = k$, *then* $m_i^k \Gamma\Delta = l_k(\Gamma \circ \Delta) = l_i(\Gamma \circ \Delta)$.

These functions are the analogs of the values assigned to the nodes of the hive triangle (Figure 1.1) in the description of the Horn cone. Namely, we have the following results for the Horn problem.

- In the article we show that the image of the map $M\Gamma\Delta$ is contained in the cone defined by the hive inequalities (1.3), i.e., for any weighting $\omega$ of $\Gamma \circ \Delta$, the values $t_i^k = m_i^k \Gamma\Delta$ satisfy the inequalities defining the eigenvalues of triples of Hermitian matrices.
- We also show in the article that for $\Gamma = \Gamma_0$ shown in Figure 4.1 and for $\Delta = \Delta_0$ shown in Figure 3.1, the image of $M\Gamma\Delta$ coincides with the entire cone.

# Chapter 5

# Semiring generalization

In the previous sections, we considered planar networks with weights in $\mathbb{R}$ and in $\mathbb{T}$. In fact, all the objects we defined can be generalized to an arbitrary semiring.

Let $\mathcal{R}$ be a semiring. Consider a planar network $\Gamma$ with $n$ sources and $n$ sinks and assign to each edge $e_i$ a weight $w_i \in \mathcal{R}$. We define the weight of a $k$-path as the product of weights of all its edges:

$$w(p) = \prod_{e_i \in p} w_i.$$

For any $i$ and $k$ such that $1 \leq i \leq k \leq n$, we define the functions $l_i\Gamma^{(k)}$:

$$l_i\Gamma^{(k)}(\omega) = \sum_{\substack{p_i \text{ an } i-\text{path} \\ \text{in } \Gamma^{(k)}}} w(p_i).$$

Note that these definitions are consistent with the ones given in Section 4.1 for the tropical semiring. Indeed, in the tropical semiring the multiplication is given by + and the addition is given by max.

If the semiring $\mathcal{R}$ is ordered, we can generalize the interlacing inequalities (written in the hive form in (2.1)): we say that the set $\{t_i^k\}_{1 \leq i \leq k \leq n}$ satisfies interlacing inequalities in $\mathcal{R}$ if for all $i$ and $k$,

$$\begin{array}{lcl} t_i^{k+1} \cdot t_{i-1}^k & \geq & t_{i-1}^{k+1} \cdot t_i^k, \\ t_i^{k+1} \cdot t_i^k & \geq & t_{i+1}^{k+1} \cdot t_{i-1}^k. \end{array} \tag{5.1}$$

In Appendix C we show that these inequalities hold for all positive weightings in $\mathcal{R}$.

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# THE HORN PROBLEM AND PLANAR NETWORKS

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Abstract. The problem of determining the set of possible eigenvalues of 3 Hermitian matrices that sum up to zero is known as the Horn problem. The answer is a polyhedral cone, which, following Knutson and Tao, can be described as the projection of a simpler cone in the space of triangular tableaux (or *hives*) to the boundary nodes of the tableau.

In this paper, we introduce a combinatorial problem defined in terms of certain weighted planar graphs giving rise to exactly the same polyhedral cone. In our framework, the values at the inner nodes of the triangular tableaux receive a natural interpretation. Other problems of linear algebra fit into the same scheme, among them the Gelfand–Zeitlin problem. Our approach is motivated by the works of Fomin and Zelevinsky on total positivity and by the ideas of tropicalization.

## 1. Introduction

This article is motivated by the following classical problem of linear algebra: under which conditions do three $n$-tuples of ordered real numbers $\lambda_1 \geq \cdots \geq \lambda_n$, $\mu_1 \geq \cdots \geq \mu_n$ and $\nu_1 \geq \cdots \geq \nu_n$ serve as the sets of eigenvalues of three Hermitian $n$-by-$n$ matrices, $A$, $B$, and $C$, related by the equality $C = A + B$?

This problem has a long history (see [8, 5] for background). The first nontrivial necessary condition,

$$\nu_1 \leq \lambda_1 + \mu_1,$$

was already known in the nineteenth century. Beginning with the work of Weyl in 1912 [18], different sets of inequalities of this type were found. Finally, in 1962, Horn put forward a very complex, recursively defined set of conditions of the form

$$\sum_{k\in K} \nu_k \leq \sum_{i\in I} \lambda_I + \sum_{j\in J} \mu_j, \text{ where } I, J, K \subset \{1, 2, \ldots, n\},$$

which, together with the obvious equality $\mathrm{Tr}(A) + \mathrm{Tr}(B) = \mathrm{Tr}(C)$, he conjectured to be necessary and sufficient.

Horn's conjecture was proved by Klyachko [14] and by Knutson and Tao [15], [16]. Since then several other proofs were given (e.g. [13]), and the results were extended to other problems of similar type (cf. [1], [3], [4]).

In this paper, we will use the "hive model" of Horn's polyhedral cone due to Knutson and Tao (see [15], [6]). Let $T_n$ be the $n$th order regular triangulation of the equilateral triangle. For real $n$-tuples $\lambda_1 \geq \cdots \geq \lambda_n$, $\mu_1 \geq \cdots \geq \mu_n$, and $\nu_1 \geq \cdots \geq \nu_n$ satisfying

$$\sum_i \lambda_i + \sum_j \mu_j = \sum_k \nu_k, \tag{1}$$

we associate real numbers to the boundary nodes of $T_n$ in the way demonstrated in Figure 1 for the case of $n = 3$ (see Section 2 for details).

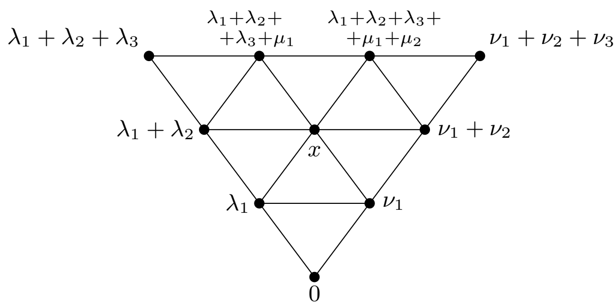


Figure 1. Triangulation $T_3$.

The Knutson–Tao theorem states that the three $n$-tuples may be realized as ordered sets of eigenvalues of Hermitian matrices $A, B$, and $C = A + B$ if and only if they satisfy the **hive condition**:

*There exists a concave function $f$ defined on the equilateral triangle, linear on each small triangle of the triangulation, and whose values at the boundary nodes of $T_n$ coincide with the values we ascribed to these nodes above.*

Naturally, such a function $f$ is uniquely determined by its values on the nodes of $T_n$. The condition of concavity translates into a set of inequalities parametrized by the internal edges, or, equivalently, by elementary rhombi of the triangulation. An example is the inequality

$$x \leq \lambda_1 + \nu_1, \tag{2}$$

where $x$ is the value of $f$ at the central node in Figure 1.

In the present paper motivated by constructions in the theory of cluster algebras and total positivity (see [7]), we introduce a combinatorial framework where the inequalities of the hive model arise in a natural way. Instead of Hermitian matrices, we consider certain oriented planar graphs, called *planar networks*, whose edges are weighted by real (or tropical) numbers. An example of a weighted planar network is shown in Figure 2.

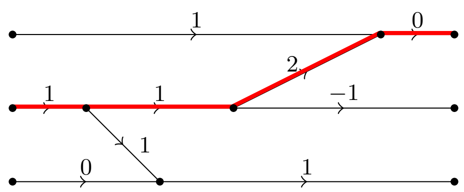


Figure 2. A planar network. The maximal path is shown in thick red.

In this setup, the eigenvalues of a matrix correspond to certain piecewise linear functions of the weights. The analogue of the top eigenvalue $\lambda_1$, for example, will be the maximum of the weights of oriented paths extending from the left to the right end of the network (where the weight of a path is the sum of weights of the edges contained in the path). For the weighted network in Figure 2, we obtain

$\lambda_1 = 1 + 1 + 2 + 0 = 4$. The definition of the other $\lambda$'s is given in Section 3; by analogy, we will call these quantities the *eigenvalues* of the weighted network.

The addition of matrices is replaced by concatenation of planar networks. In Figure 3, we give an example of a pair of networks $\Gamma$ and $\Delta$ concatenated to form the network $\Gamma \circ \Delta$. Providing $\Gamma$ and $\Delta$ with weights will then allow us to fill in the boundary values of the tableau in Figure 1: the value of $\nu_1$, for example, will be the maximum of the weights of paths extending from left to right, while $\lambda_1$ will be the maximum taken over paths extending from left to the middle line $L$.

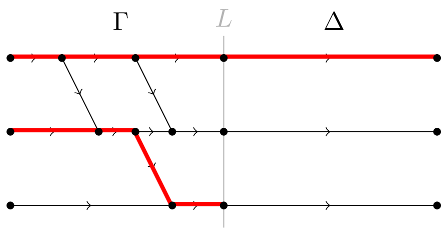


FIGURE 3. Concatenation of networks $\Gamma$ and $\Delta$. A path is shown in thick red.

Now we can formulate our first correspondence result, Theorem 4: for any pair of weighted planar networks $\Gamma$ and $\Delta$, the eigenvalues of $\Gamma$, $\Delta$, and $\Gamma \circ \Delta$ satisfy the hive condition and the trace condition (1).

A key feature of our construction is that there is a natural explicit definition of the values of the function $f$ at the internal nodes of $T_n$, and this makes the proofs rather straightforward. For example, the value $x$ of $f$ assigned to the middle node in Figure 1, is the maximal sum of the weights of two **disjoint** oriented paths in $\Gamma \circ \Delta$, one of which extends from left to right, while the other one from left to the middle line $L$. An example of such a pair is shown in Figure 3. Note that inequality (2) is now obvious: on the right-hand side of (2), the maximum is taken over all pairs of paths, one going from left to right and the other one from left to the middle, while on the left-hand side of (2), the maximum is taken only over a subset of such pairs, namely, over all pair of disjoint paths.

A natural question is to compare the set defined by these eigenvalue analogs with the set of eigenvalues of triples of Hermitian matrices. Theorem 5 states that any triple of ordered $n$-tuples satisfying the hive condition is the triple of the sets of eigenvalues of $\Gamma$, $\Delta$, and $\Gamma \circ \Delta$ for some weighted networks $\Gamma$ and $\Delta$. For $n = 3$, the graphs underlying $\Gamma$ and $\Delta$ can be chosen as shown in Figure 3.

We begin our paper with a planar network interpretation of the precursor of the Horn problem: the interlacing inequalities for eigenvalues of a Hermitian matrix and its principal submatrices (see [11]). These latter inequalities play a prominent role in the description of the Gelfand–Zeitlin integrable system (see [10]). Then we proceed to prove our main results, Theorems 4 and 5.

One can regard the eigenvalue problem for planar networks as a rather non-trivial tropicalization of the eigenvalue problem for Hermitian matrices. From this perspective, it is natural to expect but not very easy to prove that both problems are governed by the same set of inequalities. In the forthcoming paper [2], we will prove this "detropicalization" correspondence principle for the Horn problem, thus providing a new proof of the theorem of Knutson and Tao.

In the appendix, we generalize our construction of eigenvalues of weighted planar networks to arbitrary semirings. The interlacing inequalities can also be generalized in this way. We prove a stronger version of Theorem 2: we show that for a planar network with weights in a positive ordered semiring, its eigenvalues satisfy the generalized interlacing inequalities.

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## 2. Gelfand–Zeitlin and Horn problems

In this section, we recall two classical problems of linear algebra: the Gelfand–Zeitlin and Horn problems.

2.1. **The Gelfand–Zeitlin problem.** Let $\mathcal{H}_n$ be the set of $n$-by-$n$ Hermitian matrices. For $A \in \mathcal{H}_n$ and $1 \leq k \leq n$, denote by $A^{(k)}$ the principal submatrix of $A$ of size $k$, i.e., the $k$-by-$k$ submatrix sitting in the upper left corner of $A$. Let $(\lambda_1^{(k)} \geq \cdots \geq \lambda_k^{(k)})$ be the sequence of ordered eigenvalues of $A^{(k)}$; this way we obtain a set of functions

$$\lambda_i^{(k)} : \mathcal{H}_n \to \mathbb{R}, \quad 0 < i \leq k \leq n. \tag{3}$$

It is a classical result of linear algebra (see, e.g., [11]) that these functions satisfy the following *interlacing inequalities*:

$$\lambda_i^{(k+1)} \geq \lambda_i^{(k)} \geq \lambda_{i+1}^{(k+1)}, \; 0 < i \leq k < n. \tag{4}$$

The converse is also true: any set of numbers satisfying the interlacing inequalities appears as the eigenvalues of a Hermitian matrix and its principal submatrices.

One can recast inequalities (4) in the following form. Let $T = T_n$ be a regular triangulation of the equilateral triangle (Figure 4) with the set of nodes $VT$ parametrized by the indices $0 \leq i \leq k \leq n$; thus we have $|VT| = (n+1)(n+2)/2$. To each node of the triangulation, we associate a coordinate variable $t_i^k : \mathbb{R}^{VT} \to \mathbb{R}$.

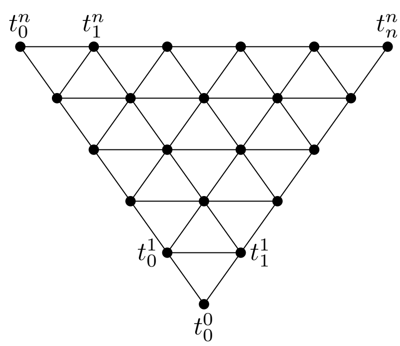


Figure 4. Triangulation $T_n$.

Denote by $\bar{E}T$ the set of horizontal edges of the triangulation $T$ parameterized by pairs $(i, k)$ satisfying $0 < i \leq k \leq n$; again, to each horizontal edge, we associate coordinate functions on $\mathbb{R}^{\bar{E}T}$, $h_i^{(k)} : \mathbb{R}^{\bar{E}T} \to \mathbb{R}$, $0 < i \leq k \leq n$. Note that this index set coincides with the index set of the eigenvalues of the principal submatrices of an Hermitian matrix (3).

**Definition 1.** *The cone $\mathcal{C}_2 \subset \mathbb{R}^{VT}$ is the polyhedral cone defined by the system of inequalities*

$$
\begin{array}{lcl}
t_i^{k+1} + t_{i-1}^k & \geq & t_{i-1}^{k+1} + t_i^k, \\
t_i^{k+1} + t_i^k & \geq & t_{i+1}^{k+1} + t_{i-1}^k
\end{array} \tag{5}
$$

*for $0 < i \leq k < n$.*

*The horizontal boundary map is the map*

$$\bar{\partial} : \mathbb{R}^{VT} \to \mathbb{R}^{\bar{E}T} : \{t_i^k\}_{0<i\leq k\leq n} \mapsto \{h_i^{(k)} = t_i^k - t_{i-1}^k\}_{0<i\leq k\leq n}.$$

Note that inequalities (5) are parametrized by the internal non-horizontal edges of the triangulation $T$, or, alternatively, by rhombi of two types having these edges as short diagonals (see Figure 5): for each rhombus, the corresponding inequality states that the sum of the two numbers assigned to the endpoints of the short diagonal is greater than or equal to the sum of the two numbers assigned to the endpoints of the long diagonal.

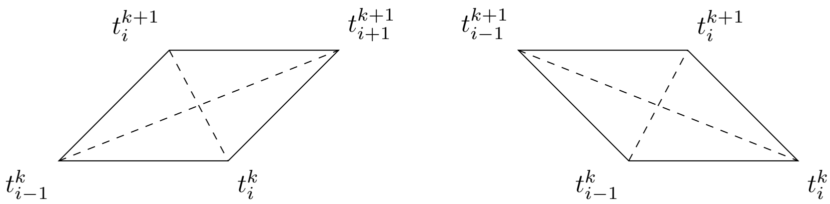


FIGURE 5. The two types of rhombi.

Observe that

- the set $\mathcal{C}_2$ and the linear map $\bar{\partial}$ are invariant under the transformations $t_i^k \mapsto t_i^k + c_k$ for $(c_k, \dots, c_0) \in \mathbb{R}^{k+1}$,
- the linear map $\bar{\partial}$ establishes a linear isomorphism between $\{t_0^k = 0 |\ k = 0, \dots, n\} \subset \mathbb{R}^{VT}$ and $\mathbb{R}^{\bar{E}T}$.

Then we have the following characterization of the interlacing inequalities.

**Proposition 1.** *Let $\mathcal{C}_{GZ} \subset \mathbb{R}^{\bar{E}T}$ be the cone defined by the interlacing inequalities* (4)*. Then*

$$\mathcal{C}_{GZ} = \bar{\partial}(\mathcal{C}_2).$$

*In fact, the linear map $\bar{\partial}$ establishes an isomorphism of polyhedral cones:*

$$\bar{\partial} : \mathcal{C}_2 \cap \{t_0^k = 0 |\ k = 0, \dots, n\} \to \mathcal{C}_{GZ}. \tag{6}$$

*Proof.* Let $t = \{t_i^k\}_{i,k}$ be in $\mathcal{C}_2$. Then we can rewrite inequalities (5) as

$$
\begin{array}{lcl}
t_i^{k+1} - t_{i-1}^{k+1} & \geq & t_i^k - t_{i-1}^k, \\
t_i^k - t_{i-1}^k & \geq & t_{i+1}^{k+1} - t_i^{k+1}.
\end{array}
$$

Therefore, the image $h = \{h_i^k\}_{i,k}$ of $t$ under the map $\bar{\partial}$ satisfies the condition

$$h_i^{k+1} \geq h_i^k \geq h_{i+1}^{k+1},$$

i.e., the interlacing inequalities.

Conversely, given a point $h = \{h_i^k\}_{i,k}$ in $\mathcal{C}_{GZ}$, define $t_i^k = h_1^k + \cdots + h_i^k$ for all $i$ and $k$. Then $h = \bar{\partial} t$, and the interlacing inequalities for $h_i^k$ imply precisely the inequalities (5) for $t_i^k$. □

2.2. **The Horn problem.** The Horn problem is related to the eigenvalues of triples of Hermitian matrices that add up to zero. More formally, consider the eigenvalue map

$$\Lambda^{\times 3} : \mathcal{H}_n \times \mathcal{H}_n \times \mathcal{H}_n \to \mathbb{R}^{3n}$$

listing the eigenvalues of a triple of Hermitian matrices of rank $n$, where the eigenvalues of each matrix are listed in decreasing order. The *Horn cone* is defined as the image

$$\mathcal{C}_{\mathrm{Horn}} = \Lambda^{\times 3}\left(\{(A,B,C) \in \mathcal{H}_n^{\times 3};\ A + B = C\}\right) \subset \mathbb{R}^{3n}.$$

Note that $A + B = C$ implies $\mathrm{Tr}(A) + \mathrm{Tr}(B) = \mathrm{Tr}(C)$ since the trace is a linear functional. For the Horn cone, this means that if

$$(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_n) \in \mathcal{C}_{\mathrm{Horn}},$$

then

$$\sum_{i=1}^n \lambda_i + \sum_{i=1}^n \mu_i = \sum_{i=1}^n \nu_i. \qquad (7)$$

In [12], Horn gave a rather complicated set of inequalities that would conjecturally define $\mathcal{C}_{\mathrm{Horn}}$; in particular, he suggested that $\mathcal{C}_{\mathrm{Horn}}$ is a polyhedral cone inside the hyperplane given by Equation (7). His conjecture was proved more than 30 years later [14, 15]. In the present paper, we will use the following elegant description of $\mathcal{C}_{\mathrm{Horn}}$ due to Knutson and Tao [15].

Consider again the triangulation $T = T_n$ and define the subcone $\mathcal{C}_3 \subset \mathcal{C}_2$ cut out from $\mathcal{C}_2$ by the inequalities corresponding to the third set of rhombi (see Figure 6).

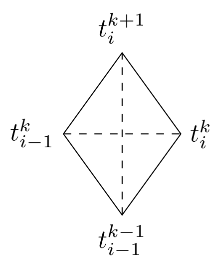


FIGURE 6. Third type of rhombi.

More precisely, we define the polyhedral cone $\mathcal{C}_3 \subset \mathbb{R}^{VT}$ by the following set of inequalities:

$$\begin{array}{lll} t_i^{k+1} + t_{i-1}^k & \geq & t_{i-1}^{k+1} + t_i^k, \\ t_i^{k+1} + t_i^k & \geq & t_{i+1}^{k+1} + t_{i-1}^k, \\ t_i^k + t_{i-1}^k & \geq & t_i^{k+1} + t_{i-1}^{k-1} \end{array} \qquad (8)$$

for $0 < i \leq k < n$.

**Remark 1.** We note that $\mathcal{C}_3$ is invariant under the translations $t_i^k \to t_i^k + c$ for $c \in \mathbb{R}$.

Now we define the map $\partial : \mathbb{R}^{VT} \to \mathbb{R}^{3n}$ by the formula

$$\partial : \{t_i^k,\ 0 \le i \le k \le n\} \mapsto \{\lambda_i = t_0^i - t_0^{i-1},\ \mu_i = t_i^n - t_{i-1}^n,\ \nu_i = t_i^i - t_{i-1}^{i-1},\ 1 \le i \le n\}.$$

In fact, $\partial$ is simply the cohomological boundary operator restricted to the outer edges of the triangulation $T_n$. Then Horn's conjecture may be formulated as follows.

**Theorem 1** (Knutson–Tao)**.** $\partial(\mathcal{C}_3) = \mathcal{C}_{\text{Horn}}$.

**Remark 2.** Since the operator $\partial$ is invariant under the translation mentioned in Remark 1, we can normalize $t_0^0 = 0$, and then the theorem may be equivalently stated as follows:

$$\partial(\mathcal{C}_3 \cap \{t_0^0 = 0\}) = \mathcal{C}_{\text{Horn}}. \tag{9}$$

## 3. Planar networks

In this section, we introduce the notion of planar networks, the key tool for the rest of the paper.

**Definition 2.** *A planar network is the following data:*

- *a finite graph $\Gamma$ with vertex set $V\Gamma$ and edge set $E\Gamma$,*
- *a pair of reals, $a < b$,*
- *an embedding of $\Gamma$ into the strip $\{a \le x \le b\} \subset \mathbb{R}^2$ such that the image of each edge is a segment of a straight line, which is not parallel to the $y$-axis.*

We will call the vertices on the line $\{x = a\}$ *sources* and the vertices on the line $\{x = b\}$ *sinks* of $\Gamma$; the other vertices will be called *internal.*

Observations:

- A *subnetwork* of a planar network $\Gamma$ is naturally defined as a subgraph of $\Gamma$, with the rest of the data unchanged. The sources and sinks of the subnetwork thus have to be subsets of the sources and sinks of $\Gamma$.
- A planar network $\Gamma$ is naturally oriented (from left to right), and we will use this orientation in what follows.
- Using this orientation, one can characterize the vertices of $\Gamma$ by the number of incoming and outgoing edges. A source, for example, is always a vertex of degree $(0, d)$ for some nonnegative integer $d$.

A crucial role in our analysis will be played by paths.

**Definition 3.** *A* multipath in $\Gamma$ is a subnetwork whose every vertex that is an internal vertex of $\Gamma$ is of degree $(1, 1)$.

A multipath has the same number of sources and sinks; a multipath with k sources is called a *$k$-path.* Each $k$-path is simply the union of $k$ disjoint paths of $\Gamma$ connecting a source with a sink. The set of $k$-paths in $\Gamma$ will be denoted by $P_k\Gamma$.

**Definition 4.** *Let $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ be the semifield of tropical numbers. A* weighting *of a planar network $\Gamma$ is an assignment of a tropical number to each edge of $\Gamma$. Identifying the set of weightings of $\Gamma$ with $\mathbb{T}^{E\Gamma}$, we can introduce*

- *the coordinate function $w_e : \mathbb{T}^{E\Gamma} \to \mathbb{T}$ for each edge $e \in E\Gamma$, and*
- *the weight functional*

$$w_\alpha : \mathbb{T}^{E\Gamma} \to \mathbb{T}, \quad w_\alpha = \sum_{e \in E\alpha} w_e$$

*for each subgraph $\alpha$ of $\Gamma$, in particular, for each multipath in $\Gamma$. When $\alpha$ has no edges, we set $w_\alpha = 0$.*

**Remark 3.** There is a more general definition of planar networks (cf. [9], [17]) on a disk or on an annulus, where the sources and sinks can come in an arbitrary order on the boundary, and the networks admit oriented cycles.

3.1. **The maximum functionals.** A certain collection of piecewise linear functions on the space $\mathbb{T}^{E\Gamma}$ of weightings of a planar network $\Gamma$ will play an important role in what follows. For a planar network $\Gamma$, $i > 0$, and a weighting $\epsilon \in \mathbb{T}^{E\Gamma}$, we define $l_i\Gamma : \mathbb{T}^{E\Gamma} \to \mathbb{T}$ by

$$l_i\Gamma(\epsilon) = \max\{w_\alpha(\epsilon) |\ \alpha \in P_i\Gamma\} \tag{10}$$

if the set $P_i\Gamma$ is nonempty; otherwise, we set $l_i\Gamma = -\infty$. By definition, we put $l_0\Gamma = 0$ and denote by $l\Gamma$ the $(n+1)$-tuple $(l_0\Gamma, \dots, l_n\Gamma)$.

**Remark 4.** If, for a weighting $\epsilon$, we have $l_i\Gamma(\epsilon) \in \mathbb{R}$ for all $i$, then we can define the eigenvalues associated to $\epsilon$ by the formula

$$\lambda_i = l_i\Gamma(\epsilon) - l_{i-1}\Gamma(\epsilon),$$

so as $l_i\Gamma(\epsilon) = \lambda_1 + \cdots + \lambda_i$.

**Example 2.** The simplest example is a planar network $\Gamma$ that contains no edges (Figure 7). Then $\mathbb{T}^{E(\Gamma)}$ is a single point, and the image of $l\Gamma$ is the point $(0, -\infty, \dots, -\infty)$.

**Example 3.** The next example is a planar network with exactly $n$ edges $e_i$ connecting the vertices $(a, i)$ and $(b, i)$ (Figure 8). Denote the corresponding weights by $w_i$ and let $(\varpi_1, \varpi_2, \dots, \varpi_n)$ be the permutation of the $n$-tuple $(w_1, w_2, \dots, w_n)$ such that $\varpi_i \geq \varpi_{i+1}$ for all $i = 1, \dots, n-1$. Then

$$l_i\Gamma = \varpi_1 + \cdots + \varpi_i.$$

The image of $l\Gamma$ is the closure in $\mathbb{T}^n$ of the polyhedral cone defined by the inequalities $\varpi_i \geq \varpi_{i+1}$.

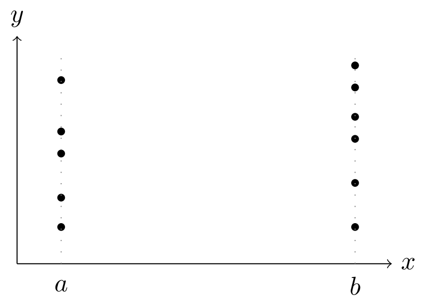


FIGURE 7

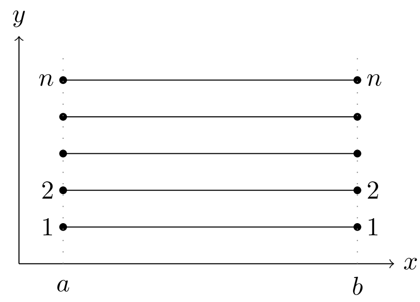


FIGURE 8

**Lemma 4.** (1) *If $\Gamma'$ is a subnetwork of $\Gamma$, then* $\mathrm{im}(l\Gamma') \subset \mathrm{im}(l\Gamma)$.
(2) *If the network $\Gamma'$ can be obtained from $\Gamma$ by insertion of a new vertex splitting an existing edge, then* $\mathrm{im}(l\Gamma') = \mathrm{im}(l\Gamma)$.

*Proof.* 1. Set the weights of all edges $e \in E\Gamma \backslash E\Gamma'$ equal to $-\infty$. The image of this subset of weightings under $l\Gamma$ coincides with the image of $l\Gamma'$.
2. Let $s : \mathbb{T}^{E\Gamma'} \to \mathbb{T}^{E\Gamma}$ be the map assigning to the split edge the sum of the weights of the two edges obtained by the insertion of the new vertex. Then $s$ preserves the functional $l$: $l\Gamma(s\epsilon) = l\Gamma'(\epsilon)$. This implies $\mathrm{im}(l\Gamma') = \mathrm{im}(l\Gamma)$. □

Our main goal is to study the image of the piecewise linear map $l\Gamma$ for a planar network $\Gamma$. We will need the following consequence of Lemma 4.

**Corollary 5.** *By allowing embedded edges that are unions of intervals, we can always replace a planar network* $\Gamma$ *by another planar network* $\Gamma'$ *having no vertices of degree (1,1) so that* $\mathrm{im}(l\Gamma) = \mathrm{im}(l\Gamma')$.

## 4. Main results

In this section, we state the main results of the paper. We establish a correspondence principle between the functionals $l\Gamma$ for planar networks $\Gamma$ and eigenvalues of Hermitian matrices. In particular, we show that the Gelfand–Zeitlin cone and the Horn cone appear as images of natural piecewise linear functions on the space of weightings of planar networks.

4.1. **Planar networks and Gelfand–Zeitlin.** In this section, we will assume that the planar network $\Gamma$ has precisely $n$ sources and sinks. Without loss of generality, we can assume that the set of $y$-coordinates of the sources and sinks is the set of the first $n$ integers $\{1, 2, \ldots, n\}$. We will say that such a network $\Gamma$ is a planar network *of rank n*.

For a planar network of rank $n$, we denote by $\Gamma^{(k)}$ the maximal subgraph of $\Gamma$ that does not contain the sinks or sources with $y$ coordinates above the line $\{y = k\}$; these are the vertices

$$(a, k+1), (b, k+1), \ldots, (a, n), (b, n).$$

Then $\Gamma^{(k)}$ is a planar network of rank $k$.

The collection of maps $l_i\Gamma^{(k)}$, $0 \leq i \leq k \leq n$, defines a map from the set of weightings of $\Gamma$ to the set of triangular tableaux (Figure 4) filled by tropical numbers:

$$L\Gamma : \mathbb{T}^{E\Gamma} \to \mathbb{T}^{VT}.$$

Note that each row of the tableau gives the map $l\Gamma^{(k)}$ for the appropriate $k$.

**Theorem 2.** *Let* $\Gamma$ *be a planar network of rank n. Then*

$$\mathrm{im}(L\Gamma) \subset \overline{\mathcal{C}_2} \cap \{t_0^k = 0 |\ k = 0, \ldots, n\}.$$

Here and below the closure is taken in $\mathbb{T}^N$.

The image of $L\Gamma$ depends on the planar network $\Gamma$. There are networks, however, for which this image is maximal. Let $\Gamma_0 = \Gamma_0[n]$ be the network in Figure 9.

**Remark 5.** Clearly, eliminating from a rank-$n$ planar network $\Gamma$ all vertices (with the adjacent edges) that cannot be reached from a source, we do not change the image of the functional $L\Gamma$. Combining this with Corollary 5, we see that we can replace the subnetwork $\Gamma_0^{(k)} \subset \Gamma_0$ with the network $\Gamma_0[k]$ (see Figure 10).

**Theorem 3.**

$$\mathrm{im}(L\Gamma_0) = \overline{\mathcal{C}_2} \cap \{t_0^k = 0 |\ k = 0, \ldots, n\}.$$

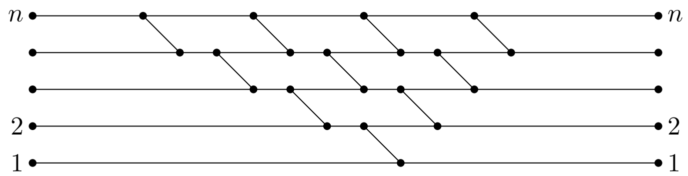


FIGURE 9. The planar network $\Gamma_0$.

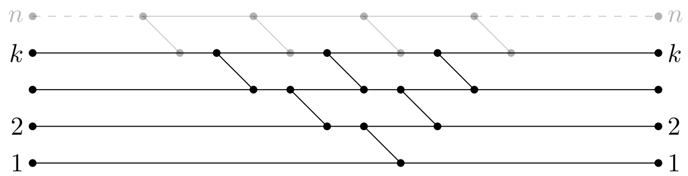


FIGURE 10. Subnetwork $\Gamma_0^{(k)}$.

Thus, the image of the map $L\Gamma_0$ coincides with the closure of the cone defined by the interlacing inequalities and describing the eigenvalues of Hermitian matrices.

4.2. **Planar networks and Horn.** We will call two planar networks $(\Gamma, [a, b])$ and $(\Delta, [a', b'])$ *composable* if $a' = b$ and the set of sources of $\Delta$ is a subset of the set of sinks of $\Gamma$. The network $(\Gamma \circ \Delta, [a, b'])$ is then defined in the obvious manner.

**Definition 5.** *We say that a subnetwork of $\Gamma \circ \Delta$ is a $\Gamma\Delta$-path if it is the union of the composable multipaths $\Gamma$ and $\Delta$. Such a subnetwork belongs to a set*

$$P_i^k \Gamma\Delta = \{\alpha = \gamma \cup \delta |\ \gamma \in P_k\Gamma,\ \delta \in P_i\Delta \quad \textit{a composable pair}\},$$

*for some $k \geq i \geq 0$.*

We have the following characterization of these subnetworks.

**Lemma 6.** *A subnetwork $\alpha$ of $\Gamma \circ \Delta$ is a $\Gamma\Delta$-path if it has only vertices of degrees $(0, 1)$, $(1, 0)$, and $(1, 1)$ and*

- *the vertices of degree $(0, 1)$ are sources of $\Gamma$,*
- *the vertices of degree $(1, 0)$ are sinks of $\Gamma$ or $\Delta$.*

Using this type of subnetworks, we can fill in the values at the nodes of the triangulation $T = T_n$ as follows. For $n \geq k \geq i \geq 0$ and a weighting $\epsilon : E(\Gamma \circ \Delta) \to \mathbb{T}$, we define the piecewise linear function

$$m_i^k \Gamma\Delta(\epsilon) = \max_{\alpha \in P_i^k \Gamma\Delta} w_\alpha(\epsilon).$$

Putting these together, we obtain a piecewise linear map $M\Gamma\Delta : \mathbb{T}^{E(\Gamma\circ\Delta)} \to \mathbb{T}^{VT}$ given by $t_i^k = m_i^k \Gamma\Delta(\epsilon)$ for $n \geq k \geq i \geq 0$.

**Theorem 4.** *Let $\Gamma$ and $\Delta$ be two composable networks of rank $n$. Then the piecewise linear map $M\Gamma\Delta$ satisfies*

$$\mathrm{im}(M\Gamma\Delta) \subset \overline{\mathcal{C}_3} \cap \{t_0^0 = 0\},$$

*where $\mathcal{C}_3$ is the polyhedral cone defined by inequalities (8).*

Similarly to the Gelfand–Zeitlin case, we have the following completeness result.

**Theorem 5.** *For planar networks $\Gamma_0$ shown in Figure 9 and $\Delta_0$ shown in Figure 8, we have*

$$\operatorname{im}(M\Gamma_0\Delta_0) = \overline{\mathcal{C}_3} \cap \{t_0^0 = 0\},$$

*and consequently*

$$\operatorname{im}(\partial \circ M\Gamma_0\Delta_0) = \overline{\mathcal{C}_{\mathrm{Horn}}}.$$

In other words, the image of the map $\partial \circ M\Gamma_0\Delta_0$ coincides with the closure of the Horn cone describing the eigenvalues of triples of Hermitian matrices.

## 5. Proofs

5.1. **Proof of Theorem 2.** We observe that a path in $P_1\Gamma$ can also be described as the graph of a continuous function $f : [a, b] \to \mathbb{R}$ such that $\mathrm{gr}(f) = \{(x, f(x)),\ x \in [a, b]\} \subset \Gamma$. In these terms, we can describe $P_k\Gamma$ as

$$P_k\Gamma = \{(f_1, \dots, f_k)|\ f_i : [a, b] \to \mathbb{R} \text{ continuous, } \mathrm{gr}(f_i) \subset \Gamma, f_i < f_{i+1},\ i = 1, \dots, k\}.$$

Here and below we use the notation $f < g$ as a shorthand for $f(x) < g(x)$, $x \in [a, b]$. To make our notation more readable, for a $k$-tuple $\mathbf{f} = (f_1, \dots, f_k) \in P_k\Gamma$, we will write

$$w[\mathbf{f}] \text{ for the functional } w_{\cup_{i=1}^k \mathrm{gr}(f_i)} : \mathbb{T}^{E\Gamma} \to \mathbb{T}.$$

Theorem 2 is equivalent to the inequalities

$$\begin{array}{lcl} l_i\Gamma^{(k)}(\epsilon) + l_{i-1}\Gamma^{(k-1)}(\epsilon) & \geq & l_{i-1}\Gamma^{(k)}(\epsilon) + l_i\Gamma^{(k-1)}(\epsilon), \\ l_i\Gamma^{(k)}(\epsilon) + l_i\Gamma^{(k-1)}(\epsilon) & \geq & l_{i+1}\Gamma^{(k)}(\epsilon) + l_{i-1}\Gamma^{(k-1)}(\epsilon), \end{array} \tag{11}$$

for $\epsilon \in \mathbb{T}^{E\Gamma}$.

Consider the first of the two inequalities. Our method of proof is to show that, for $\mathbf{f} \in P_{i-1}\Gamma^{(k)}$ and $\mathbf{g} \in P_i\Gamma^{(k-1)}$, there exist $\tilde{\mathbf{f}} \in P_{i-1}\Gamma^{(k-1)}$ and $\tilde{\mathbf{g}} \in P_i\Gamma^{(k)}$ such that

$$w[\tilde{\mathbf{f}}] + w[\tilde{\mathbf{g}}] = w[\mathbf{f}] + w[\mathbf{g}]. \tag{12}$$

For a positive integer $N$, we consider the map $\sigma = (\sigma_1, \dots, \sigma_N) : \mathbb{R}^N \to \mathbb{R}^N$ sending each vector to the vector with the same coordinates but listed in decreasing order: $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_N$. Thus $\sigma_1 : \mathbb{R}^N \to \mathbb{R}$ is simply the largest coordinate of a vector. Clearly, $\sigma$ is a piecewise linear, continuous map. Now, for a planar network $\Gamma$ and $2 \leq i \leq n$, we can define a map

$$P_{i-1}\Gamma \times P_i\Gamma \to P_{i-1}\Gamma \times P_i\Gamma$$

in the following way. For an $(i-1)$-path $\mathbf{f} = (f_1, \dots, f_{i-1})$ and an $i$-path $\mathbf{g} = (g_1, \dots, g_i)$, we set $\mathbf{f} \cup \mathbf{g} = (f_1, \dots, f_{i-1}, g_1, \dots, g_i)$. We observe that, for $x \in [a, b]$, no real number can occur more than twice in the sequence

$$(f_1(x), \dots, f_{i-1}(x), g_1(x), \dots, g_i(x))$$

of length $(2i-1)$. Define the function $\sigma_j(\mathbf{f} \cup \mathbf{g})$ by

$$\sigma_j(\mathbf{f} \cup \mathbf{g})(x) = \sigma_j(f_1(x), \dots, f_{i-1}(x), g_1(x), \dots, g_i(x)).$$

We have $\sigma_{j-1}(\mathbf{f} \cup \mathbf{g}) < \sigma_{j+1}(\mathbf{f} \cup \mathbf{g})$ for all $j$. Now we can define our map by sending the pair $(\mathbf{f}, \mathbf{g})$ to the pair $(\mathrm{even}(\mathbf{f}, \mathbf{g}), \mathrm{odd}(\mathbf{f}, \mathbf{g}))$, where $\mathrm{even}(\mathbf{f}, \mathbf{g}) = [(\sigma_2(\mathbf{f} \cup \mathbf{g}), \sigma_4(\mathbf{f} \cup \mathbf{g}), \dots, \sigma_{2i-2}(\mathbf{f} \cup \mathbf{g}))] \in P_{i-1}\Gamma$ and

$$\mathrm{odd}(\mathbf{f}, \mathbf{g}) = [(\sigma_1(\mathbf{f} \cup \mathbf{g}), \sigma_3(\mathbf{f} \cup \mathbf{g}), \dots, \sigma_{2i-1}\mathbf{f} \cup \mathbf{g}))] \in P_i\Gamma.$$

This map has the following two properties:

- if $\mathbf{f} \in P_{i-1}\Gamma$ and $\mathbf{g} \in P_i\Gamma$, then
$$w[\mathrm{even}(\mathbf{f},\mathbf{g})] + w[\mathrm{odd}(\mathbf{f},\mathbf{g})] = w[\mathbf{f}] + w[\mathbf{g}],$$
,
- if $\mathbf{f} \in P_{i-1}\Gamma^{(k)}$ and $\mathbf{g} \in P_i\Gamma^{(k-1)}$, then
$$\mathrm{even}(\mathbf{f},\mathbf{g}) \in P_{i-1}\Gamma^{(k-1)} \text{ and } \mathrm{odd}(\mathbf{f},\mathbf{g}) \in P_i\Gamma^{(k)}.$$

Thus, these two multipaths can be chosen as $\tilde{\mathbf{f}}$ and $\tilde{\mathbf{g}}$ in (12), which implies the first inequality in (11).

The second inequality is proved in a similar manner. We define a map
$$P_{i+1}\Gamma \times P_{i-1}\Gamma \to P_i\Gamma \times P_i\Gamma$$
by sending a pair of $i$-paths $(\mathbf{f},\mathbf{g})$ of length $2i$ to the pair $(\mathrm{even}(\mathbf{f},\mathbf{g}), \mathrm{odd}(\mathbf{f},\mathbf{g}))$, where
$$\mathrm{even}(\mathbf{f},\mathbf{g}) = [(\sigma_2(\mathbf{f}\cup\mathbf{g}), \sigma_4(\mathbf{f}\cup\mathbf{g}), \dots, \sigma_{2i}(\mathbf{f}\cup\mathbf{g}))] \in P_i\Gamma$$
and
$$\mathrm{odd}(\mathbf{f},\mathbf{g}) = [(\sigma_1(\mathbf{f}\cup\mathbf{g}), \sigma_3(\mathbf{f}\cup\mathbf{g}), \dots, \sigma_{2i-1}\mathbf{f}\cup\mathbf{g}))] \in P_i\Gamma.$$
As in the previous case, we observe that $w[\mathrm{even}(\mathbf{f},\mathbf{g})] + w[\mathrm{odd}(\mathbf{f},\mathbf{g})] = w[\mathbf{f}] + w[\mathbf{g}]$, and if $\mathbf{f} \in P_{i+1}\Gamma^{(k)}$ and $\mathbf{g} \in P_{i-1}\Gamma^{(k-1)}$, then $\mathrm{even}(\mathbf{f},\mathbf{g}) \in P_i\Gamma^{(k-1)}$ and $\mathrm{odd}(\mathbf{f},\mathbf{g}) \in P_i\Gamma^{(k)}$, which implies the second inequality in (11).

5.2. **Proof of Theorem 3.** We will prove a somewhat stronger statement: there exists a graph and a collection of multipaths for which the image of $L\Gamma$ is the full cone $\mathcal{C}_{GZ}$. In this section, we return to the functional $w_\alpha : \mathbb{T}^{E\Gamma} \to \mathbb{T}$, for $\alpha \in E\Gamma$, defined as $w_\alpha = \sum_{e\in\alpha} w_e$.

We will call a choice of multipaths $\alpha(k,i) \in P_i\Gamma^{(k)}$, $k = 1,\dots,n$, $i = 1,\dots,k$ a *collection*. Then to each collection $A = \{\alpha(k,i)\}_{i,k}$, we can associate the linear function $w_A : \mathbb{T}^{E\Gamma} \to \mathbb{T}^{VT} \cap \{t_0^k = 0,\ k = 0,\dots,n\}$ defined by $t_i^k = w_{\alpha(k,i)}$ for all $i$ and $k$. We say that the collection $A$ *non-degenerate* if $w_A$ is surjective.

Let $\Gamma$ be a planar network, and let $A = \{\alpha(k,i)\}$ be a non-degenerate collection of multipaths in $\Gamma$. Denote by $\mathcal{B}_{GZ}(A) \subset \mathbb{T}^{E\Gamma}$ the subset of those weightings of $\Gamma$ for which the collection $A$ satisfies the interlacing inequalities
$$\begin{aligned}\mathcal{B}_{GZ}(A) &= \{\epsilon \in \mathbb{T}^{E\Gamma} |\ (t_i^k = w_{\alpha(k,i)}(\epsilon), 0 < i \le k \le n) \in \overline{\mathcal{C}_2}\} = \\ &= w_A^{-1}(\overline{\mathcal{C}_2} \cap \{t_0^k = 0|\ k = 0,\dots,n\}),\end{aligned}$$
and let $\mathcal{B}_{\max}(A)$ be the subset of weightings for which each of the multipaths in $A$ is maximal:
$$\mathcal{B}_{\max}(A) = \{\epsilon \in \mathbb{T}^{E\Gamma} |\ l_i\Gamma^{(k)}(\epsilon) = w_{\alpha(k,i)}(\epsilon)\}.$$
In these terms, Theorem 2 is equivalent to the statement that, for any collection $A$, we have $\mathcal{B}_{\max}(A) \subset \mathcal{B}_{GZ}(A)$.

**Lemma 7.** *Suppose that a planar network $\Gamma$ has a non-degenerate collection $A$ such that $\mathcal{B}_{\max}(A) \supset \mathcal{B}_{GZ}(A)$. Then, in fact,*

$$L\Gamma(\mathcal{B}_{\max}(A)) = L\Gamma(\mathcal{B}_{GZ}(A)) = \overline{\mathcal{C}_2} \cap \{t_0^k = 0|\ k = 0,\dots,n\} \tag{13}$$

*Proof.* Indeed, we have $\mathcal{B}_{GZ}(A) = w_A^{-1}(\overline{\mathcal{C}_2} \cap \{t_0^k = 0|\ k = 0, \ldots, n\})$, and since $\Gamma$ is nondegenerate, we also have $w_A(\mathcal{B}_{GZ}(A)) = \overline{\mathcal{C}_2} \cap \{t_0^k = 0|\ k = 0, \ldots, n\}$. On the other hand, the restrictions $L\Gamma|\mathcal{B}_{\max}(A)$ and $w_A|\mathcal{B}_{\max}(A)$ coincide, and this, combined with $\mathcal{B}_{\max}(A) \supset \mathcal{B}_{GZ}(A)$ implies (13). □

Now we consider the planar network $\Gamma_0$ shown in Figure 9. Given a decreasing sequence of integers $\mathbf{a} = (a_1, \ldots, a_i)$ and an increasing sequence $\mathbf{b} = (b_1, \ldots, b_i)$, we will say that a multipath $\alpha \in P_i\Gamma_0$ is of type $[\mathbf{a}, \mathbf{b}]$ if its sources are given by the list $\mathbf{a}$ and its sinks are given by the list $\mathbf{b}$. It is easy to verify that there is a single multipath in $P_i\Gamma_0^{(k)}$ of type $[(k, \ldots, k-i+1), (1, \ldots, i)]$. Denote this multipath by $\alpha(k, i)$ and consider the collection $A = \{\alpha(k, i),\ 0 < i \leq k \leq n\}$ (see Figure 11).

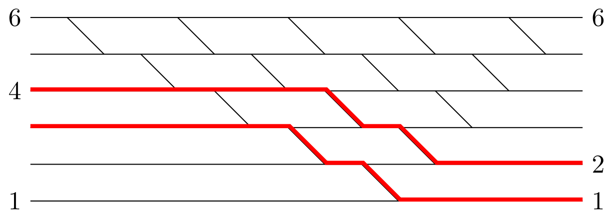


FIGURE 11. $\alpha(4, 2)$ for $n = 6$, shown in thick red.

**Lemma 8.** *The collection $A$ is non-degenerate.*

*Proof.* As shown in Figure 12, we introduce the following notation for the weights of the edges of $\Gamma$:

- we denote by $h_i$ the weights of the horizontal edges adjacent to the sinks,
- we denote by $a_{ij}$ the weights of the slanted edges,
- the weights of the rest of the edges we put equal to 0.

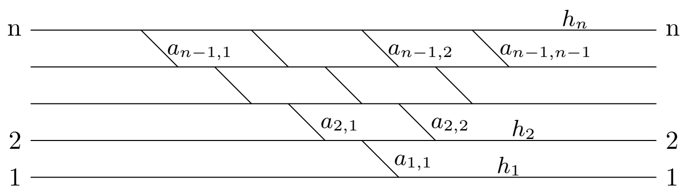


FIGURE 12. Weighting of the edges of $\Gamma_0$.

Then we can regard the map $w_A$ as a linear map from $\mathbb{T}^{\frac{n(n+1)}{2}}$ to itself, and in order to prove its surjectivity it is sufficient to show that the Jacobian of $w_A$ is nonzero.

We have $\dfrac{\partial w_{\alpha(k,i)}}{\partial h_k} = 0$ for $i < k$, since these multipaths $\alpha(k, i)$ do not contain the edge weighted $h_k$ and $\dfrac{\partial w_{\alpha(k,k)}}{\partial h_k} = 1$. Similarly, the first multipath of our collection $A$ containing the edge with weight $a_{r,s}$ is the multipath $\alpha(r+1, s)$, therefore we have $\dfrac{\partial w_{\alpha(k,i)}}{\partial a_{k-1,i}} = 1$ and $\dfrac{\partial w_{\alpha(k,i)}}{\partial a_{r,s}} = 0$ for $k \leq r$. Thus, with an appropriate ordering of the variables $h_i$ and $a_{i,j}$, the Jacobi matrix of $w_A$ is triangular with 1's on the diagonal. □

We prove Theorem 3 in two steps:

- we begin with a graphical description of the interlacing inequalities describing $\mathcal{B}_{GZ}(A)$ in terms of cells in the complement of $\Gamma_0$,
- then, again, using a graphical device, we show that should these inequalities hold, our collection $A$ will consist of maximal multipaths.

Thus, the theorem will be proved if we show that the conditions of Lemma 7 hold for our collection $A$.

5.2.1. *Graphical representation of the interlacing inequalities.* We enumerate the *cells* formed by the connected components of the complement of $\Gamma_0$ by the symbols $[k,i]$, $k=0,\dots,n-1$, $i=0,\dots,k$ as shown in Figure 13.

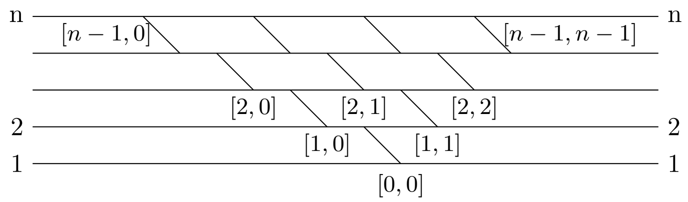


FIGURE 13. Enumeration of the cells of $\Gamma_0$.

To the cell $[k,i]$, we associate the functional $c_{[k,i]} : \mathbb{T}^{E\Gamma_0} \to \mathbb{T}$ that is the sum of the signed weights along the clockwise oriented boundary of the cell, where the sign depends on whether the orientation of the boundary coincides with the orientation of the edge or not (Figure 14). The same applies to the unbounded cells.

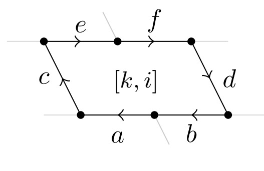


FIGURE 14. $c_{[k,i]} = -w_a - w_b - w_c + w_e + w_f + w_d$.

Next, we define the functionals

$$r^{\nearrow}_{[k,i]} = c_{[k,i]} + c_{[k-1,i-1]} + \cdots + c_{[k-i,0]}$$

and

$$r^{\searrow}_{[k,i]} = c_{[k,i]} + c_{[k-1,i]} + \cdots + c_{[i,i]},$$

which correspond to the boundary of the shaded regions shown in Figures 15 and 16 below.

**Lemma 9.** *The polyhedral cone $\mathcal{B}_{GZ}(A)$ is given by the weightings $\epsilon \in \mathbb{T}^{E\Gamma_0}$ satisfying*

$$\begin{aligned} r^{\nearrow}_{[k,i]}(\epsilon) &\geq 0, \quad 0 \leq i < k < n, \\ r^{\searrow}_{[k,i]}(\epsilon) &\leq 0, \quad 0 < i \leq k < n. \end{aligned} \tag{14}$$

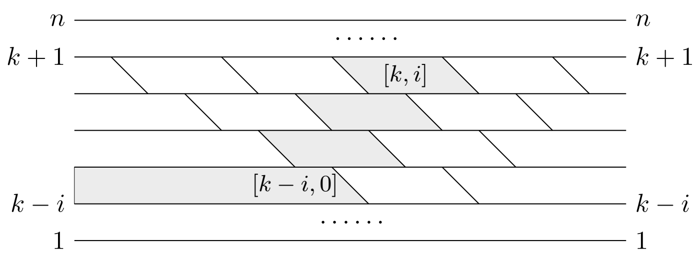


FIGURE 15. $r^{\nearrow}_{[k,i]}$.

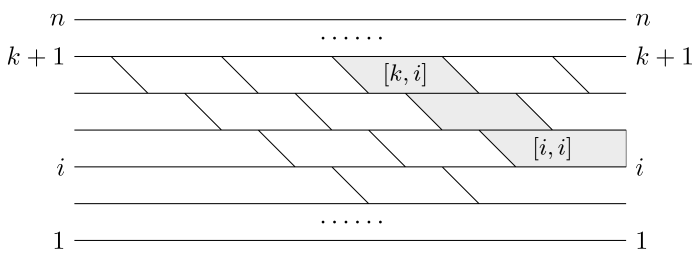


FIGURE 16. $r^{\searrow}_{[k,i]}$.

*Proof.* By Definition 1 that $\mathcal{B}_{GZ}(A)$ is defined by the inequalities

$$w_{\alpha(k+1,i)} + w_{\alpha(k,i-1)} - w_{\alpha(k+1,i-1)} - w_{\alpha(k,i)} \geq 0$$

and

$$w_{\alpha(k+1,i+1)} + w_{\alpha(k,i-1)} - w_{\alpha(k+1,i)} - w_{\alpha(k,i)} \leq 0,$$

for $0 < i \leq k < n$.

It is not difficult to identify these inequalities with inequalities (14). In Figures 17 and 18, we present a graphical proof of this equivalence for the case of the first inequality in (14) when $k + 1 = n = 5$ and $i = 2$:

- the thick (red) lines show $\alpha(k+1, i)$ and $\alpha(k, i)$,
- the dashed lines show $\alpha(k+1, i-1)$ and $\alpha(k, i-1)$,
- the sum of weights along the boundary of the shaded area represents the difference in the caption.

One then notes that the difference of the two shaded regions is the union of the cells $[k, i]$, …, $[k - i, 0]$, which, in view of the definition of $r^{\nearrow}_{[k,i]}$, completes the proof. The proof of the equivalence of the other pair of inequalities is analogous. □

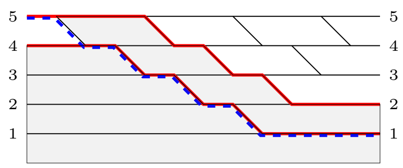


FIGURE 17. $w_{\alpha(k+1,i)} - w_{\alpha(k+1,i-1)}$.

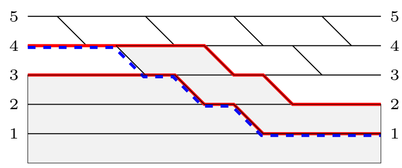


FIGURE 18. $w_{\alpha(k,i)} - w_{\alpha(k,i-1)}$.

5.2.2. *Maximality of the collection $A$.* Now we are ready to prove Theorem 3. We will call $\beta \in P_i\Gamma_0$ a *maximal multipath* for the weighting $\epsilon$ if $l_i\Gamma_0 = w_\beta(\epsilon)$. By Lemmas 7 and 9, we may assume that we are given a weighting $\epsilon$ of $\Gamma_0$ satisfying $r^{\nearrow}_{[k,i]}(\epsilon) \geq 0$ and $r^{\searrow}_{[k,i]}(\epsilon) \leq 0$. We must prove that for this weighting, the paths of the collection $A$ are maximal. We fix this $\epsilon$ for the rest of the section and will often drop it from our notation for brevity.

For any $k$-tuple $\mathbf{a} = (a_1, \ldots, a_k)$, we will denote the sum $a_1 + \cdots + a_k$ by $\sum \mathbf{a}$.

**Lemma 10.** *Let $\beta \in P_i\Gamma_0$ be a maximal multipath of type $[\mathbf{a}, \mathbf{b}]$ for the weighting $\epsilon \in \mathcal{B}_{GZ}$ such that the value of $\sum \mathbf{b}$ is minimal among all maximal multipaths in $P_i\Gamma_0$. Then $\mathbf{b} = (1, 2, \ldots, i)$.*

*Proof.* First consider the case $i = 1$. If $\beta$ ends at $b > 1$, then there is a cell $[b-1, b-1]$ whose upper edge is a part of $\beta$. Then $w_\beta \leq w_{\tilde\beta} = w_\beta - r^{\searrow}_{[k,b-1]}$ for some $k$, where $\tilde\beta$ is a path ending at $b-1$ (see Figure 19). This contradicts the assumption on $\beta$, and so the $i = 1$ case is proved.

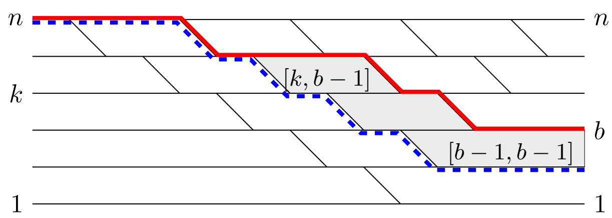


FIGURE 19. $w_\beta$ (solid) $= w_{\tilde\beta}$ (dashed) $- r^{\searrow}_{[k,b-1]}$ (shaded).

Now let $\beta \in P_i\Gamma_0$ be a maximal multipath for $\epsilon$ with the smallest possible value of $b_1 + \cdots + b_i$. Clearly, we can apply the above argument to the lowest path in $\beta$ and conclude that $b_1 = 1$. If $b_2 > 2$, then, as before, we can subtract the appropriate $r^{\searrow}_{[k,b_2-1]}$ from $w_\beta$, and obtain a multipath with a strictly lower sum of end-values. This contradicts the assumption on $\beta$, hence $b_2 = 2$. Repeating this argument, we can show that $b_j = j$ for $j = 1, \ldots, i$, which completes the proof. □

Now we consider the sources of maximal multipaths.

**Lemma 11.** *Let $\beta \in P_i\Gamma_0$ be a maximal multipath of type $[\mathbf{a}, (1, 2, \ldots, i)]$ for a weighting $\epsilon \in \mathcal{B}_{GZ}$ for which the value of $\sum \mathbf{a} = a_1 + \cdots + a_i$ is the largest among all the maximal multipaths in $P_i\Gamma_0$. Then $\mathbf{a} = (n, n-1, \ldots, n-i+1)$.*

*Proof.* We begin with the case $i = 1$. Suppose that a path $\beta \in P_1\Gamma_0$ connecting $a < n$ with the sink 1 satisfies the conditions of the lemma. Then $w_\beta \leq w_{\tilde\beta} = w_\beta + r^{\nearrow}_{[a,0]}$ for a path $\tilde\beta$ that starts at $a+1$. This contradicts our assumptions on $\beta$, and thus $a = n$.

If $i > 1$ and $\beta \in P_i\Gamma_0$ satisfies the conditions of the lemma, then we can still show as above that necessarily $a_1 = n$. Now assume that there is a gap between the sources of $\beta$, i.e. that, for some $j < n$, $j$ is a term of the sequence $\mathbf{a}$, but $j+1$ is not. If the first vertex $v_{j+1,1}$ of $\Gamma_0$ on the line $y = j+1$ is not in $\beta$, then we can again add $r^{\nearrow}_{[j,0]}$ to $w_\beta$ and obtain a new multipath with larger $\sum \mathbf{a}$.

If $v_{j+1,1} \in \beta$, then $\beta$ *has an elbow* at $v_{j+1,1}$, by which we mean that the two edges in $\beta$ containing $v_{j+1,1}$ are the downward incoming edge and the horizontal outgoing edge. The other two edges are excluded by our assumptions. Now consider the sequence of vertices $v_{j+1,1}, v_{j+2,2}, \ldots, v_{j+s,s} \in \beta$ such that $v_{j+s+1,s+1} \notin \beta$. It is easy to verify that $\beta$ has necessarily an elbow at all of the vertices of this sequence.

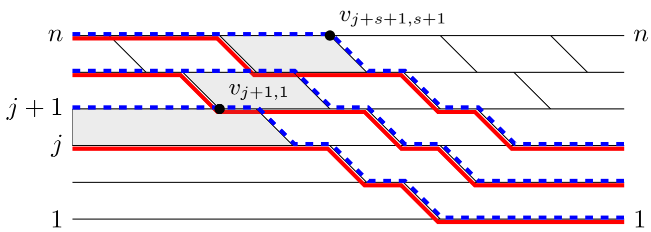


Figure 20. Graphical representation of $w_{\tilde{\beta}} = w_\beta + r^{\nearrow}_{[j+s-1,s-1]}$.

This implies that $w_\beta \leq w_{\tilde{\beta}} = w_\beta + r^{\nearrow}_{[j+s-1,s-1]}$ for a multipath $\tilde{\beta}$ with the source at height $j$ moved up to $j+1$. We give an example of this operation in Figure 20, where

- $i = 3$ and $s = 2$,
- $\beta$ is marked with thick (red) lines,
- $\tilde{\beta}$ is marked with dashed lines,
- the shaded area is the domain corresponding to $r^{\nearrow}_{[j+s-1,s-1]}$.

This contradicts our assumption that $\beta$ is a maximal multipath with the largest $\sum \mathbf{a}$, thus there can be no gaps among the sources of $\beta$, which completes the proof. □

Now we can quickly finish the proof of Theorem 3. We must show that $\mathrm{im}(L\Gamma_0) = \overline{\mathcal{C}_2} \cap \{t_0^k = 0 | \ k = 0, \ldots, n\}$. We consider the non-degenerate collection $A = \{\alpha(k,i)\}$. According to Lemma 7, it is sufficient to show that if the weights $w_{\alpha(k,i)}(\epsilon)$ satisfy the interlacing inequalities for a weighting $\epsilon$ of $\Gamma_0$, then they are also maximal. This obviously follows from Lemmas 10 and 11.

5.3. **Proof of Theorem 4.** Let $\Gamma$ and $\Delta$ be two planar networks of rank $n$. We must prove the inequalities

$$\begin{array}{rcl} m_i^k\Gamma\Delta + m_{i-1}^{k-1}\Gamma\Delta & \geq & m_{i-1}^k\Gamma\Delta + m_i^{k-1}\Gamma\Delta, \\ m_i^k\Gamma\Delta + m_i^{k-1}\Gamma\Delta & \geq & m_{i+1}^k\Gamma\Delta + m_{i-1}^{k-1}\Gamma\Delta, \\ m_i^k\Gamma\Delta + m_{i-1}^k\Gamma\Delta & \geq & m_i^{k+1}\Gamma\Delta + m_{i-1}^{k-1}\Gamma\Delta. \end{array} \tag{15}$$

We begin with the first inequality. As in the proof of Theorem 2, we will show that for $\alpha \in P_i^{k-1}\Gamma\Delta$ and $\beta \in P_{i-1}^k\Gamma\Delta$, one can find $\tilde{\alpha} \in P_{i-1}^{k-1}\Gamma\Delta$ and $\tilde{\beta} \in P_i^k\Gamma\Delta$ such that

$$w_\alpha(\epsilon) + w_\beta(\epsilon) = w_{\tilde{\alpha}}(\epsilon) + w_{\tilde{\beta}}(\epsilon), \text{ for any weighting } \epsilon \in \mathbb{T}^{E(\Gamma\Delta)}.$$

*We can informally describe this problem as follows. Imagine that we have a group of tourists with $k$ men and $k-1$ women; $i-1$ of the men and $i$ of the women are* *fit. The group is planning an excursion where the fit tourists go from one town to a neighboring town and the unfit tourists only go from the first town to a park halfway to the second town. The organizer of the excursion devises a route for each member of the group under the following special condition: the paths of the men should not intersect, and similarly, the paths of the women should not intersect. Just before the trip it turns out that there will be i fit men, and $i-1$ fit women, with the total numbers of men and women unchanged. Can the organizer redraw the routes under the same special condition, and so that if a certain segment was used only by one person in the original plan, then this segment will be used by only one person in the new plan as well?*

To prove the theorem it will be convenient to consider a slightly more general notion of a planar network in which we allow multiple edges (in fact, we will need only double edges). If we have a double edge emanating from a vertex, then this edge will contribute 2 to the out-degree of this vertex; similarly, this edges will contribute 2 to the in-degree of the vertex to which it points. We will also consider planar networks with a different number of sources and sinks. We will call such a planar network *of type* $[k_1, k_2]$ if the sum of out-degrees of all its sources is $k_1$ and the sum of all in-degrees of its sinks is $k_2$.

The sum of multipaths $\Theta = \alpha \cup \beta$ is naturally such a generalized planar network of type $[2k-1, 2i-1]$ with the edges $e \in E\alpha \cap E\beta$ having multiplicity 2. We will consider such a double edge as two separate edges. The end-line of $\Gamma$, which is also the start-line of $\Delta$, will be called the *middle line* of $\Theta$. According to our strategy, the inequality

$$m_i^k \Gamma\Delta + m_{i-1}^{k-1} \Gamma\Delta \geq m_{i-1}^k \Gamma\Delta + m_i^{k-1} \Gamma\Delta \tag{16}$$

will follow if we can decompose $\Theta$ as the union of multipaths. one from $P_{i-1}^{k-1}\Gamma\Delta$ and the other from $P_i^k \Gamma\Delta$. This will be shown in Proposition 14 below.

More generally, we consider decompositions of $\Theta$ into two multipaths from $P_q^p \Gamma\Delta$ and $P_{q'}^{p'} \Gamma\Delta$ for some integers $p, q, p'$ and $q'$. Clearly, we will have $q + q' = 2i - 1$ and $p + p' = 2k - 1$. To such a decomposition, we can associate a coloring of the edges of $\Theta$ in two colors, say, red and green. We will call such a coloring *valid*. Let us classify all valid colorings.

First we note that just as in the proof of Lemma 4, the operation of eliminating a vertex of degree $(1, 1)$ and replacing its two adjacent edges by a single edge will not influence the image of $M\Gamma\Delta$, so we will assume that $\Theta$ has no vertices of degree $(1, 1)$.

Recall that a closed path in an unoriented graph is a connected subgraph whose every vertex has degree 2; an open path has, in addition, two vertices of degree 1.

**Lemma 12.** *Assume that $\Theta$ has no vertices of degree $(1, 1)$ and consider the equivalence relation on the edges of $\Theta$ generated by the following relation: two edges are related if they are either both incoming edges of a vertex of $\Theta$ or both outgoing edges of a vertex of $\Theta$. Then the resulting equivalence classes are (possibly closed) unoriented paths in $\Theta$ with edges having alternating orientations.*

We will call this decomposition of $\Theta$ the *canonical path decomposition of* $\Theta$ (see Figure 21).

*Proof.* The fact that $\Theta$ is the union of two multipaths implies that

- the sources of $\Theta$ are of degree $(0, 1)$ or $(0, 2)$,

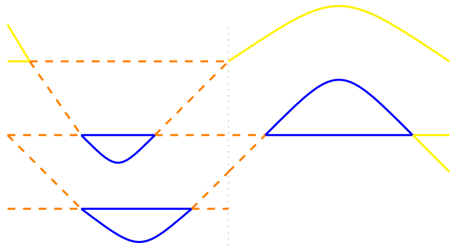

Figure 21. Canonical path decomposition of $\Theta$ of type $[5, 3]$ ($k = 3$ and $i = 2$).

- the degrees of all the other vertices have degrees from the following list:

$$(2, 2),\ (2, 1),\ (2, 0),\ (1, 1),\ (1, 0);$$

- moreover, the vertices that are neither sources nor sinks and that are not on the middle-line of $\Theta$ can have only degrees $(2, 2)$ or $(1, 1)$.

The statement of the lemma clearly follows from the fact that no vertex of $\Theta$ has in- or out-degree greater than two. □

**Remark 6.** Because of the alternating orientation of the edges, the following paths of the canonical path decomposition have an even number of edges:

- closed paths,
- open paths beginning and ending at a source,
- open paths beginning and ending at a sink,
- open paths beginning and ending at the middle line.

Now we define an *alternating coloring* of $\Theta$ as a coloring of the edges of $\Theta$ in two colors in such a way that the consecutive edges of each path of its canonical path decomposition are colored differently (see Figure 22). By Remark 6, such a coloring always exists.

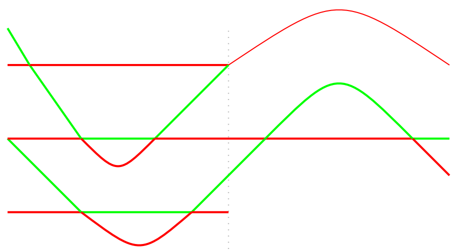

Figure 22. An alternating coloring of $\Theta$.

**Lemma 13.** *Assume that $\Theta$ has no vertices of degree $(1, 1)$, and denote the elements of its path decomposition by $Q$.*

(1) *Then there are precisely $2^{|Q|}$ alternating colorings of $\Theta$, corresponding to a coloring of each of the $|Q|$ paths chosen independently.*

(2) *The alternating colorings of* $\Theta$ *coincide with the valid colorings of* $\Theta$.

*Proof.* Clearly, every path has precisely two alternating colorings, and the colorings of different paths are independent of each other. This implies the first statement.

A coloring of the edges of $\Theta$ is valid if and only if

- at any vertex having in-degree 2, the two incoming edges are colored differently,
- at any vertex having out-degree 2, the two outgoing edges are colored differently. In particular, the two edges of a double edge have different colors.

These conditions coincide with the definition of an alternating coloring. □

Now we prove a somewhat strengthened version of the decomposition statement, which will imply our theorem:

**Proposition 14.** *Let* $\Theta$ *be a generalized planar network with the following properties:*

- $\Theta$ *is of type* $[2k-1, 2i-1]$,
- *the sources of* $\Theta$ *have degrees* $(0,1)$ *or* $(0,2)$,
- *all vertices of* $\Theta$, *apart from the sources, have degrees* $(d_1, d_2)$ *with* $2 \geq d_1 \geq d_2$.

*Then there is* $\tilde{\alpha} \in P_i^k\Theta$ *and* $\tilde{\beta} \in P_{i-1}^{k-1}\Theta$ *such that* $\tilde{\alpha} \cup \tilde{\beta} = \Theta$.

*Proof.* We can partition the set $Q$ of path components of $\Theta$ as follows:

$$Q = Q_{00} \cup Q_{L0} \cup Q_{0R} \cup Q_{LR} \cup Q_{LL} \cup Q_{RR} \cup Q_{cl},$$

where $Q_{cl}$ consists of all closed paths and the two indices of the rest of the $Q$'s indicate the beginning and the end of the path with the convention that

- $L$ stands for a source of $\Theta$,
- $R$ stands for a sink of $\Theta$,
- 0 stands for an internal vertex of $\Theta$.

We must show that among the $2^{|Q|}$ alternating colorings of the paths in $Q$, there is at least one for which precisely $k$ edges emanating from a source (source-edges), and precisely $i$ edges ending in a sink (sink-edges) are red. It is clear that in order to specify the coloring of a path, it is sufficient to color its source-edge or sink-edge whenever the path has one of these.

We note that the contribution of each of $Q_{00}$, $Q_{LL}$, $Q_{RR}$, and $Q_{cl}$ to the source-degree and the sink-degree is even, so their coloring is not essential. The total contribution of $Q_{L0}$ and $Q_{LR}$ to the source-degree and the total contribution of $Q_{0R}$ and $Q_{LR}$ to the sink-degree are odd. Therefore the coloring algorithm is as follows:

- if $|Q_{LR}|$ is even, then $|Q_{0R}|$ is odd and $|Q_{L0}|$ is odd. Then we color in red
  - the sink-edges of half of the paths in $Q_{LR}$ and of $(|Q_{0R}|+1)/2$ paths in $Q_{0R}$,
  - the source-edges of $(|Q_{L0}|+1)/2$ paths in $Q_{L0}$.
- if $|Q_{LR}|$ is odd, then $|Q_{0R}|$ and $|Q_{L0}|$ are even. Then we color in red
  - the sink-edges of $(|Q_{LR}|+1)/2$ of the paths in $Q_{LR}$ and of half of the paths in $Q_{0R}$,
  - the source-edges of half of paths in $Q_{L0}$.

This coloring algorithm ensures that the number of paths whose source-edges are colored in red is greater by 1 than the number of paths with source-edges colored in green. The same is true for the sink-edges. Thus we indeed obtain an element of $P_i^k\Theta$ colored in red and an element of $P_{i-1}^{k-1}\Theta$ colored in green. □

As we explained above, Proposition 14 implies Inequality (16).

The other two inequalities from (15) are proved in a similarly. Consider the second inequality. Let $\alpha \in P_{i-1}^{k-1}\Gamma\Delta$ and $\beta \in P_{i+1}^{k}\Gamma\Delta$. Then $\Theta = \alpha \cup \beta$ is a generalized planar network of type $[2k-1, 2i]$. As before, we obtain a path decomposition $Q = Q_{00} \cup Q_{L0} \cup Q_{0R} \cup Q_{LR} \cup Q_{LL} \cup Q_{RR} \cup Q_{cl}$ of $\Theta$ with $2^{|Q|}$ alternating colorings. Among these colorings, we must find one with precisely $k$ source-edges and precisely $i$ sink-edges colored in red. For this, we use the following algorithm:

- if $|Q_{LR}|$ is even, then $|Q_{0R}|$ is even while $|Q_{L0}|$ is odd. Then we color in red
  - the sink-edges of half of the paths in $Q_{LR}$ and in $Q_{0R}$,
  - the source-edges of $(|Q_{L0}|+1)/2$ paths in $Q_{L0}$.
- if $|Q_{LR}|$ is odd, then $|Q_{0R}|$ is odd and $|Q_{L0}|$ is even. Then we color in red
  - the sink-edges of $(|Q_{LR}|+1)/2$ of the paths in $Q_{LR}$ and of $(|Q_{0R}|-1)/2$ of the paths in $Q_{0R}$,
  - the source-edges of half of the paths in $Q_{L0}$.

This coloring algorithm ensures that the number of paths with red souce-edges is greater by 1 than the number of paths with green source-edges, whereas the number of paths with red sink-edges is equal to the number of green sink-edges, and so we indeed obtain an element of $P_i^k$ colored in red and an element of $P_i^{k-1}$ colored in green.

Finally, for the third inequality, we have a generalized planar network $\Theta$ of type $[2k, 2i-1]$. The coloring procedure is similar to the other two cases, and will be omitted.

We have thus proved inequalities (15), which completes the proof of the theorem.

**Remark 7.** A similar proof can be given for Theorem 2, which is the analog of Theorem 4 in the Gelfand–Zeitlin case.

5.4. **Proof of Theorem 5.** The strategy of the proof of Theorem 5 is analogous to that of Theorem 3. The steps of the proof are as follows:

1. We identify the special collection $B = \{\beta(k,i);\ 0 \le i \le k \le n\}$. Denote by $\tau(k)$ the $k$th horizontal line of $\Gamma_0 \circ \Delta_0$. Then

$$\beta(k,i) = \alpha(n-i, k-i) \cup \bigcup_{j=n-i+1}^{n} \tau(j).$$

The following counterpart of Lemma 7 is valid: the theorem follows from the inclusion

(17)

$$w_B^{-1}(\bar{\mathcal{C}}_3) \cap \{t_0^0 = 0\} \subset \{\epsilon \in \mathbb{T}^{E(\Gamma_0 \circ \Delta_0)} \mid m_i^k\Gamma_0\Delta_0(\epsilon) = w_{\beta(k,i)}(\epsilon),\, 0 \le i \le k \le n\};$$

in words, we must show that if, for some weighting, the weights of our special collection satisfy the inequalities defining $\mathcal{C}_3$ (where $t_0^0 = 0$), then the multipaths of our collection are maximal for this weighting.

2. We identify the regions corresponding to the inequalities (15) for $m_i^k = w_{\beta(k,i)}$. We denote the cells of $\Gamma_0$ as before and the cells of $\Delta_0$ by $[i, i+1]$ for

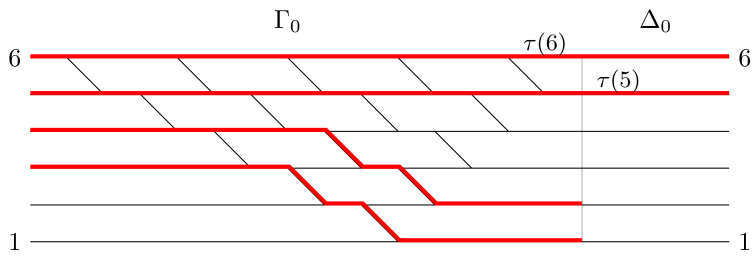


FIGURE 23. $\beta(4,2)$ for $n = 6$, shown in solid red.

$i = 0, \ldots, n$. The first inequality corresponds to $r^{\searrow}_{[k,i]}$, the third to $r^{\nearrow}_{[k,i]}$, while the second inequality corresponds to

$$r^{\rightarrow}_{[k,i]} = c_{[k,i]} + c_{[k,i+1]} + \cdots + c_{[k,k]} + c_{[k,k+1]}$$

(see Figure 24).

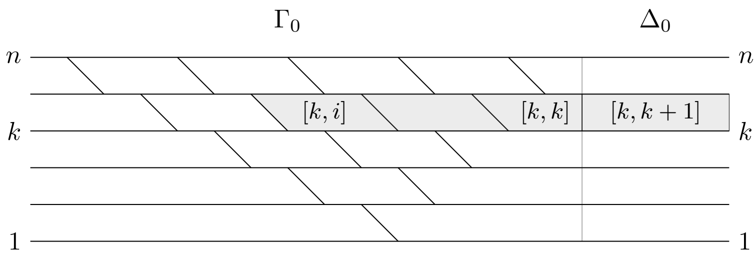


FIGURE 24. $r^{\rightarrow}_{[k,i]}$.

We thus obtain the graphical representation:

$$w_B(\epsilon) \in \mathcal{C}_3 \Leftrightarrow r^{\nearrow}_{[k,i]}(\epsilon) \geq 0,\ r^{\searrow}_{[k,i]}(\epsilon) \leq 0,\ r^{\rightarrow}_{[k,i]}(\epsilon) \geq 0. \tag{18}$$

3. Now we must show that if condition (18) holds for a weighting $\epsilon$ of $\Gamma_0 \circ \Delta_0$, i.e., if $r^{\nearrow}_{[k,i]}, -r^{\searrow}_{[k,i]}, r^{\rightarrow}_{[k,i]} \geq 0$ for all $i \leq k < n$, then the collection $B$ is maximal for $\epsilon$.

From now on, we assume that we have a fixed weighting $\epsilon$ of $\Gamma_0 \circ \Delta_0$ satisfying (18), and we will often drop $\epsilon$ from the notation. When we say that $\alpha \in P^k_i\Gamma_0\Delta_0$ is *maximal*, we mean that $w_\alpha(\epsilon) = m^k_i\Gamma_0\Delta_0(\epsilon)$.

Note that every $\alpha \in P^k_i\Gamma\Delta$ has two decompositions:

$$\alpha = \gamma \cup \delta = \alpha' \cup \alpha'',\ \gamma \in P_k\Gamma_0,\ \delta \in P_i\Delta_0,\ \alpha' \in P_i(\Gamma_0 \circ \Delta_0),\ \alpha'' \in P_{k-i}\Gamma_0.$$

We record the endpoints of these multipaths as follows (see Figure 25):

- denote by $\mathbf{a} = (a_1, \ldots, a_k)$ the decreasing sequence of sources of $\gamma$;
- denote by $\mathbf{b} = (b_1, \ldots, b_{k-i})$ the increasing sequence of sinks of $\alpha''$,
- denote by $\mathbf{b}' = (b'_1, \ldots, b'_k)$ the increasing sequence of sinks of $\gamma$. Note that $\mathbf{b}$ is a subsequence of $\mathbf{b}'$.
- denote by $\mathbf{c} = (c_1, \ldots, c_i)$ the decreasing sequence of sinks of $\alpha'$ (or $\delta$).

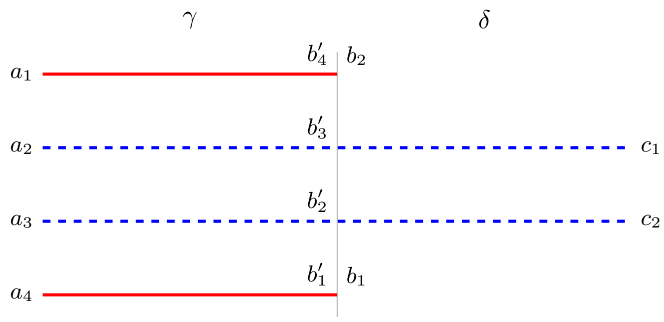


FIGURE 25. Path $\alpha'$ in dashed blue, path $\alpha''$ in solid red.

When necessary, we will indicate the index $\alpha$ explicitly by writing, for example, $\mathbf{b}_\alpha$ instead of $\mathbf{b}$.

Now, similarly to Lemmas 10 and 11, we can formulate a sequence of lemmas normalizing the form of a maximal path that eventually lead to the statement that $\beta(k,i)$ is maximal for the weighting $\epsilon$.

**Lemma 15.** *Let $\alpha \in P_i^k \Gamma_0 \Delta_0$ be a maximal path for which $\sum \mathbf{b}$ minimal. Then $b_{k-i} = b'_{k-i}$, which means that $\alpha''$ consists of the lowest $k-i$ paths of $\gamma$.*

*Proof.* To prove this, we observe that

$$c_{[j,j+1]}(\epsilon) \geq 0 \text{ for } 1 \leq j \leq n-1, \tag{19}$$

since $c_{[j,j+1]} = r^{\rightarrow}_{[j,j]} - r^{\searrow}_{[j,j]}$. Assume, contrary to the statement of the lemma, that for some $j \leq k-i$, we have $b_j > b'_j$ (see Figure 26). Then we obtain

$$w_{\tilde{\alpha}}(\epsilon) = w_\alpha(\epsilon) + c_{[b'_j, b'_j+1]}(\epsilon) + \cdots + c_{[b_j-1, b_j]}(\epsilon) \geq w_\alpha(\epsilon),$$

for a multipath $\tilde{\alpha}$. Clearly, $\tilde{\alpha}$ is maximal with $\sum \mathbf{b}_{\tilde{\alpha}} < \sum \mathbf{b}_\alpha$, which contradicts our assumption on $\alpha$. $\square$

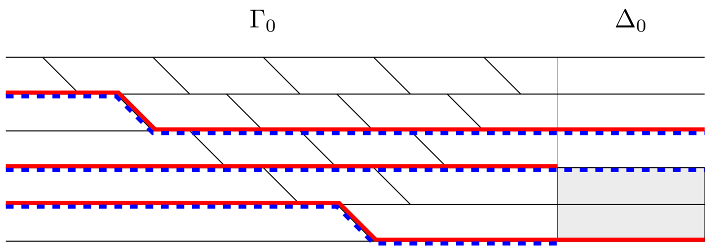


FIGURE 26. $w_\alpha$ (solid) $= w_{\tilde{\alpha}}$ (dashed) $- \sum c_{[l,l+1]}$ (shaded).

**Lemma 16.** *Let $\alpha \in P_i^k \Gamma_0 \Delta_0$ be a maximal path for which $b_{k-i} = b'_{k-i}$ and the value of $a_1$ is maximal. Then $a_1 = n$.*

The proof of this statement is identical to the first part of the proof of Lemma 11 (see Figure 27).

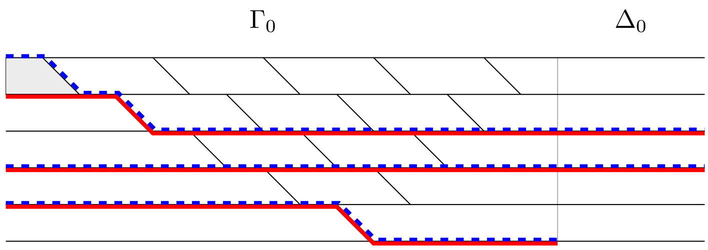


FIGURE 27. Raising the top source by adding $r^{\nearrow}_{[j,0]}$ (shaded).

**Lemma 17.** *Let $i > 0$ and let $\alpha \in P^k_i\Gamma_0\Delta_0$ be a maximal multipath for which $b_{k-i} = b'_{k-i}$, $a_1 = n$ and the value of $c_1$ is maximal. Then $c_1 = n$, and hence $\tau(n) \subset \alpha$.*

*Proof.* Since $b_{k-i} = b'_{k-i}$ for $i > 0$, we know that the highest path of $\alpha$ is a path in $\Gamma_0 \circ \Delta_0$. Assume that $c_1 \neq n$. Then, as shown in Figure 28, for an appropriate sequence $q_j$, $j = c_1, \ldots, n-1$, we have

$$w_{\tilde{\alpha}} = w_\alpha + \sum_{j=c_1}^{n-1} r^{\rightarrow}_{[j,q_j]} \geq 0,$$

where $\tilde{\alpha} \in P^k_i\Gamma_0\Delta_0$. Again, this contradicts the assumptions of the lemma, hence $c_1 = n$. □

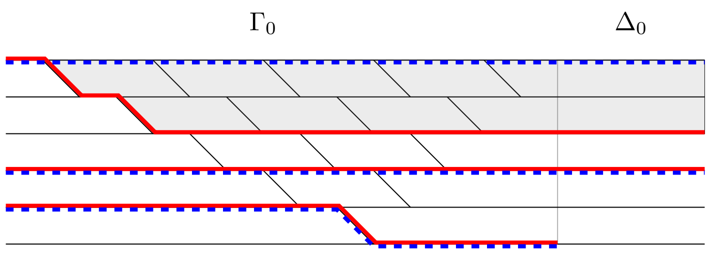


FIGURE 28. Raising the top sink by adding $\sum r^{\rightarrow}_{[j,l]}$ (shaded).

Now, using the proof of Theorem 3, we can quickly finish the proof of Theorem 5 by induction on $n$.

Indeed, Theorem 5 is trivial for $n = 1$. Assume that the theorem has already been proved for the case of rank $n-1$, and that $\epsilon$ is a weighting satisfying (18). Then the fact that $\beta(k, 0)$ is maximal for $k = 1, \ldots, n$ is a special case of Theorem 3. If $k \geq i > 0$, then the sequence of lemmas above shows that there is at least one maximal $\alpha \in P^k_i\Gamma_0\Delta_0$ containing $\tau(n)$. Proving that $\beta(k, i)$ is maximal is then equivalent to showing that $\beta(k-1, i-1)$ is maximal for the restriction of $\epsilon$ to the planar network obtained by removing $\tau(n)$ from $\Gamma_0 \circ \Delta_0$. This, however, follows from our inductive assumption, which completes the proof.

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# Appendix A

# Tropical limits

In this appendix, we show that the functions $l_i\Gamma^{(k)}$ defined in Section 4.1 can be viewed as tropical limits of eigenvalues of Hermitian matrices.

For any Hermitian matrix $A$, let us denote by $\Lambda$ the map sending $A$ to the collection $\{\lambda_i^{(k)}(A)\}$ defined in Chapter 2. On the subset of positive definite hermitian matrices, we define the family of maps $\Lambda_t = \frac{1}{t}\log\circ\Lambda$, where log is the componentwise logarithm map.

Let us recall the objects defined in Section 3.1. For clarity, we will use the word *real* when talking about them. Let $\Gamma$ be a planar network and $\omega$ its real weighting. The real weight of a path in $\Gamma$ is the product of weights of all its edges. To a weighted planar network, we assign the real weight matrix $\Omega$ by setting $\Omega_{i,j}$ to be the sum of weights of all paths connecting the source $i$ with the sink $j$.

We will now construct a family of matrices. Consider the planar network $\Gamma = \Gamma_0$ shown in Figure 4.1. Let $\omega$ be a tropical weighting of $\Gamma$, such that the weights of all horizontal edges, except the ones adjacent to the sinks, have weight 0. For every $i$ and $t \in \mathbb{R}$, we assign to the edge $e_i$ the real weight $\omega'(e_i) = e^{t\omega(e_i)}$. If $\omega(e_i) = -\infty$, we set $\omega'(e_i) = 0$. Note that by construction all the horizontal edges exept the ones adjacent to the sinks are of weight 1. Let $\Omega_t$ be the real weight matrix of $\Gamma$ with weights $\omega'$. Denote by $M_t$ the product $\Omega_t\Omega_t^*$.

It will be useful to describe $M_t$ differently. Consider the planar network $\Gamma^*$ which is the mirror image of $\Gamma$ (see Figure A.1) and denote by $e_i^*$ the mirror image of the edge $e_i$. We assign the weights $\omega''(e_i^*) = (\omega'(e_i))^* = e^{t\omega(e_i)}$ to the edges. Consider the concatenation $\Delta = \Gamma\circ\Gamma^*$ with the weighting $\omega'\cup\omega''$. Then $M_t$ is the real weight matrix of $\Delta$.

In the article we proved that there exists a subset $\mathcal{B}$ of weightings of $\Gamma = \Gamma_0$, such that its image under the map $L\Gamma$ is the entire Gelfand–Zeitlin cone. It is defined by the following condition: for all $i \le k \le n$, the maximal $i$-path in $\Gamma^{(k)}$ is the $i$-path connecting the sources $(k-i+1,\dots k)$ with the sinks $(1,\dots,k)$.

**Theorem 2** *Let $\omega$ be a tropical weighting and let $l = \{l_i^k\}$ be its image under the map $L\Gamma$. For every $t \in \mathbb{R}$, let $\mu(t) = \{\mu_i^k(t)\}$ be the image of $M_t$ under the*

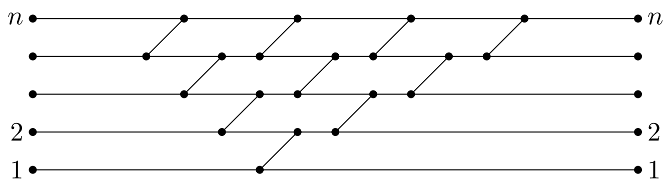


Figure A.1: Planar network $\Gamma^*$.

*map $\Lambda_t$. Then there exist constants $C \in \mathbb{R}$ and $t_0 \in \mathbb{R}$, such that*

$$|\sum_{j=1}^{i} \mu_j^k(t) - 2l_i^k| < \frac{C}{t}$$

*for all $t > t_0$.*

*Moreover, if $\omega$ is inside $\mathcal{B}$, then*

$$|\sum_{j=1}^{i} \mu_j^k(t) - 2l_i^k| < Ce^{-\alpha t}$$

*for some $\alpha \in \mathbb{R}_{>0}$.*

We will need the following characterization of minors of weight matrices. For any matrix $A$, denote by $\Delta_{I,J} = \Delta_{I,J}(A)$ its minor with row set $I$ and column set $J$.

**Lemma 5 (Lindström)** *Let $\Omega$ be the real weight matrix (of size $n \times n$) of a planar network $\Gamma$ with real weights $\omega$. Then $\Delta_{I,J}(\Omega)$ is equal to the sum of real weights of all multipaths connecting the sources labeled $I$ with the sinks labeled $J$.*

**Proof** Let us prove this for the case $|I| = |J| = n$. We have

$$\det \Omega = \sum_{\sigma \in S_n} sign(\sigma) \prod_{i=1}^{n} \Omega_{i,\sigma(i)}.$$

Each entry $\Omega_{i,\sigma(i)}$ is the sum of real weights of all paths connecting the source $i$ with the sink $j$. Expanding the product in the formula for the determinant we obtain

$$\det \Omega = \sum_{\sigma \in S_n} \sum_{P} sign(\sigma) w(P),$$

where $P$ are collections of paths $(p_1, \ldots, p_n)$, such that the path $p_i$ connects the source $i$ with the sink $\sigma(i)$ for all $i$. If the paths in $P$ are vertex-disjoint, then $\sigma$ is the identity permutation, and so $sign(\sigma) = 1$. Let us show that for every collection of paths which are not vertex-disjoint, there exists a collection of paths with the same real weight, but appearing in the determinant with an opposite sign, i.e., all non-vertex-disjoint collections of paths cancel out. Indeed, consider a collection of paths that are not vertex-disjoint. Take the rightmost intersection of paths in this collection and exchange the parts of the paths to the right of the intersection. Clearly, the weight of the new collection is the same, whereas the permutation is of opposite sign.

□

We will denote by $\Delta_I$ the principal minors, i.e. those, having the same row and column set. Let us denote by $\Delta_I^k$ the principal minors of the submatrix $\Omega^{(k)}$.

**Proof of Theorem 2** The eigenvalues $\{m_i^k(t) = e^{t\mu_i^k(t)}\}_{i,k}$ of $M_t$ are the solutions of the following system of equations

$$\sum_{\{i_1,...,i_s\}\subset\{1,..,k\}} e^{t\mu_{i_1}^k(t)} \cdot ... \cdot e^{t\mu_{i_s}^k(t)} = \sum_{I\subset\{1,...,k\},\, |I|=s} \Delta_I^k(t),$$

for all $k = 1, ..., n$ and $s = 1, ..., k$. By Lemma 5 and by construction of the real weighting, $\Delta_I^k(t) = \sum_q e^{t\delta_{s_q}^k}$, where the $\delta_{s_q}^k$ are the tropical weights of the $s-$paths of $\Delta = \Gamma \circ \Gamma^*$, connecting the sources labeled $I$ with the sinks labeled $I$. Let us denote by $Q_s^k$ the set of all indices for which $\delta_{s_q}^k$ is not maximal:

$$Q_s^k = \{s_q : \ \delta_{s_q}^k \neq 2l_s^k\}.$$

Let us assume that $\mu_i^k(t) \geq \mu_{i+1}^k(t)$ for all $i, k$, and $t$. We can rewrite the system of equations in the following form:

$$e^{t(\mu_1^k(t)+...+\mu_s^k(t))} + \sum_{\substack{\{i_1,...,i_s\}\subset\{1,..,k\}\\ \{i_1,...,i_s\}\neq\{1,...,s\}}} e^{t(\mu_{i_1}^k(t)+...+\mu_{i_s}^k(t))} = Ae^{2tl_s^k} + \sum_{s_q\in Q_s^k} e^{t\delta_{s_q}^k}$$

for some constant $A \in \mathbb{Z}$. We have

$$e^{t(\mu_1^k(t)+...+\mu_s^k(t)-2l_s^k)}\big(1 + \sum_{\substack{\{i_1,...,i_s\}\subset\{1,..,k\}\\ \{i_1,...,i_s\}\neq\{1,...,s\}}} e^{t(\mu_{i_1}^k(t)+...+\mu_{i_s}^k(t)-\mu_1^k(t)-...-\mu_s^k(t))}\big) =$$

$$= A + \sum_{s_q\in Q_s^k} e^{t(\delta_{s_q}^k - 2l_s^k)}.$$

Since $2l_s^k \geq \delta_{s_q}^k$ for every $k$, the right-hand side of this equation converges to a constant $C_1$ when $t \to \infty$. Since $\mu_{i_1}^k(t) + ... + \mu_{i_s}^k(t) - \mu_1^k(t) - ... - \mu_s^k(t) \leq 0$ for every $t$, we have

$$1 \leq 1 + \sum_{\substack{\{i_1,...,i_s\}\subset\{1,..,k\}\\ \{i_1,...,i_s\}\neq\{1,...,s\}}} e^{t(\mu_{i_1}^k(t)+...+\mu_{i_s}^k(t)-\mu_1^k(t)-...-\mu_s^k(t))} \leq C_2,$$

where $C_2 = \binom{k}{s}$. Thus, for every $\varepsilon > 0$, there exists a $t_0$, such that for $t > t_0$,

$$\frac{C_1 - \varepsilon}{C_2} \leq e^{t(\mu_1^k(t)+...+\mu_s^k(t)-2l_s^k)} \leq C_1 + \varepsilon,$$

and so

$$\frac{1}{t} \ln\left(\frac{C_1 - \varepsilon}{C_2}\right) \leq \mu_1^k(t) + ... + \mu_s^k(t) - 2l_s^k \leq \frac{1}{t} \ln(C_1 + \varepsilon).$$

We therefore have $\lim_{t\to\infty}\left(\mu_1^k(t)+...+\mu_s^k(t)\right)=2l_s^k$ for all $s$ and $k$.

If the weighting $\omega$ is inside $\mathcal{B}$, then the inequalities on all $l_i^j$'s (inequalities (11) in the article) are strict, which, together with the limit above implies

$$\lim_{t\to\infty}(\mu_{i_1}^k(t)+...+\mu_{i_s}^k(t)-\mu_1^k(t)-...-\mu_s^k(t)) =: \alpha_s^k<0,$$

i.e., for every $\varepsilon>0$, there exists a $t_1$, such that for all $t>t_1$

$$\alpha_s^k-\varepsilon<(\mu_{i_1}^k(t)+...+\mu_{i_s}^k(t)-\mu_1^k(t)-...-\mu_s^k(t))<\alpha_s^k+\varepsilon,$$

and we obtain the following bounds:

$$\frac{1+\sum e^{t(\delta_{s_q}^k-2l_s^k)}}{1+\sum e^{t(\alpha_s+\varepsilon)}}<e^{t(\mu_1^k(t)+...+\mu_s^k(t)-2l_s^k)}<\frac{1+\sum e^{t(\delta_{s_q}^k-2l_s^k)}}{1+\sum e^{t(\alpha_s-\varepsilon)}},$$

or

$$\frac{1}{t}\ln(1+\sum e^{t(\delta_{s_q}^k-2l_s^k)})-\frac{1}{t}\ln(1+\sum e^{t(\alpha_s^k+\varepsilon)})<\mu_1^k(t)+...+\mu_s^k(t)-2l_s^k \quad < \\ <\frac{1}{t}\ln(1+\sum e^{t(\delta_{s_q}^k-2l_s^k)})-\frac{1}{t}\ln(1+\sum e^{t(\alpha_s^k-\varepsilon)}) \quad .$$

Since $2l_s^k>\delta_{s_q}^k$ and since for small enough $\varepsilon$, both $\alpha_s^k-\varepsilon$ and $\alpha_s^k+\varepsilon$ are negative, the Taylor expansions for the logarithms give the desired bound.

□

A similar result can be obtained for complex weights. For any planar network we define the complex weight of a path and the complex weight matrix by the same formulas as in the real case.

Similarly to the real case we construct a family of matrices. Given the planar network $\Gamma=\Gamma_0$ with a tropical weighting $\omega$, for every $i$ and $t\in\mathbb{R}$, and for $\varphi_i\in[0,2\pi]$, we assign the complex weight $\omega'(e_i)=e^{t\omega(e_i)+i\varphi_i}$ to the edge $e_i$ of $\Gamma$. Consider the mirror image $\Gamma^*$ of $\Gamma$ and denote by $e_i^*$ the mirror image of the edge $e_i$. We assign the weights $\omega''(e_i^*)=(\omega'(e_i))^*=e^{t\omega(e_i)-i\varphi_i}$ to the edges. Consider the concatenation $\Delta=\Gamma\circ\Gamma^*$ with the weighting $\omega'\cup\omega''$. Let $M_t$ be its complex weight matrix.

**Theorem 3** *Let $\omega$ be a tropical weighting inside $\mathcal{B}$ and let $l=\{l_i^k\}$ be its image under the map $L\Gamma$. For every $t\in\mathbb{R}$, let $\mu(t)=\{\mu_i^k(t)\}$ be the image of $M_t$ under the map $\Lambda_t$. Then for every $i$ and $k$,*

$$\sum_{j=1}^{i}\mu_j^k(t)\xrightarrow[t\to\infty]{} l_i^k.$$

**Proof** There are two types of multipaths in $\Delta$: the ones that are symmetric with respect to the middle line and the ones that are not symmetric. Inside $\mathcal{B}$ the maximal multipaths are symmetric and so the phases $\varphi_i$ along these paths cancel. The asymmetric paths come in pairs: for every path with the set of edges $\{e_i,\ i\in I\}\cup\{e_j^*,\ j\in J\}$ there is the path with the edge set $\{e_j,\ j\in J\}\cup\{e_i^*,\ i\in I\}$ (see Figure A.2). The complex weights of these paths have equal absolute values, but opposite phases.

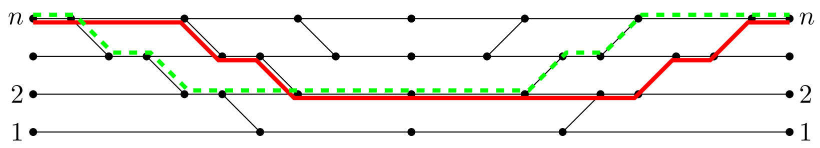


Figure A.2: A pair of paths with complex weights of equal absolute value and of opposite phase.

Similarly to Theorem 2, the eigenvalues $\{m_i^k(t) = e^{t\mu_i^k(t)}\}_{i,k}$ of $M_t$ are the solutions of the following system of equations

$$\begin{aligned} e^{t(\mu_1^k(t)+\cdots+\mu_s^k(t))} \quad &+ \sum_{\substack{\{i_1,\ldots,i_s\}\subset\{1,\ldots,k\}\\ \{i_1,\ldots,i_s\}\neq\{1,\ldots,s\}}} e^{t(\mu_{i_1}^k(t)+\cdots+\mu_{i_s}^k(t))} = \\ &= \; e^{2tl_s^k} + 2\sum_{s_q\in\tilde{Q}_s^k} e^{t\delta_{s_q}^k}\cos\phi_{s_q}^k + \sum_{s_q\in\tilde{\tilde{Q}}_s^k} e^{t\delta_{s_q}^k}, \end{aligned}$$

where the $\phi_{s_q}^k$ are the sums of the phases $\varphi_i$ along the path with weight $\delta_{s_q}^k$, $\tilde{Q}_s^k$ are the sets of indices of the asymmetric paths, and $\tilde{\tilde{Q}}_s^k$ are the sets of indices of the symmetric non-maximal paths. The rest of the proof is similar to that of Theorem 2.

□

# Appendix B

# Preimages

In this appendix, we show the relation between the preimages of the map $L\Gamma$ and the map $\Lambda$.

Consider the planar network $\Gamma = \Gamma_0$ shown in Figure 4.1. Choose a point $\lambda = \{\lambda_i^k\}$ inside the Gelfand–Zeitlin cone defined by the interlacing inequalities (4). Let us denote by $a_{ij}^\infty$ the edge weights of the preimage of $\lambda$ under the map $L\Gamma$ (see Figure B.1; the unlabeled edges are of weight 0).

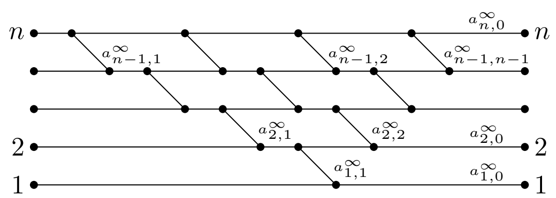


Figure B.1: $L\Gamma^{-1}(\lambda)$

Let $f$ be the map sending an upper-triangular matrix $A$ to the product $AA^*$.

**Theorem 4** *There exists a $T_0$, such that for $t \geq T_0$, the preimage of $\lambda$ under the map $f \circ \Lambda_t$ is given by the complex weight matrices of the planar network $\Gamma = \Gamma_0$ with edge weights $e^{ta_{ij}(t)+i\varphi_{ij}}$, where $\varphi_{ij} \in [0, 2\pi]$ and*

$$a_{ij}(t) \to a_{ij}^\infty.$$

**Proof** Let $l_s^{k,(\infty)}$ be the tropical weight of the maximal (with respect to the weighting $\{a_{ij}^\infty\}$) $s$-path $p_s^k$ of $\Gamma^{(k)}$. We have

$$l_s^{k,(\infty)} = \lambda_1^k + \cdots + \lambda_s^k.$$

The system of equations for $\Lambda_t^{-1}(\lambda)$ has the form

$$e^{t(\lambda_1^k+\cdots+\lambda_s^k)} + \sum_{\substack{\{i_1,\ldots,i_s\}\subset\{1,\ldots,k\}\\ \{i_1,\ldots,i_s\}\neq\{1,\ldots,s\}}} e^{t(\lambda_{i_1}^k+\cdots+\lambda_{i_s}^k)} =$$
$$= e^{2tl_s^k(t)} + 2\sum_{s_q\in\tilde{Q}_s^k} e^{t\delta_{s_q}^k(t)}\cos\phi_{s_q}^k + \sum_{s_q\in\tilde{\tilde{Q}}_s^k} e^{t\delta_{s_q}^k(t)}$$

for $k = 1, \ldots, n$ and $s = 1, \ldots, k$. Here

- $e^{2l_s^k(t)}$ is the complex weight (with respect to the weighting $\{e^{ta_{ij}(t)+i\varphi_{ij}}\} \cup \{e^{ta_{ij}(t)-i\varphi_{ij}}\}$) of the path $p_s^k \cup (p_s^k)^*$, where $(p_s^k)^*$ is the mirror image of $p_s^k$ in $\Gamma \circ \Gamma^*$;

- the $e^{t\delta_{s_q}^k(t)}$ are the absolute values of the complex weights of the rest of the paths $p_{s_q}^k$;

- the sets $\tilde{Q}_s^k$ and $\tilde{\tilde{Q}}_s^k$ are the sets of indices of asymmetric and symmetric paths in $\Gamma \circ \Gamma^*$, correspondingly.

Since $l_s^k$ are linear functions of $a_{ij}$, proving that $a_{ij}(t) \to a_{ij}^\infty$ is equivalent to proving that $l_s^k(t) \to l_s^{k,(\infty)}$.

Note that all $\delta_{s_q}^k(t)$ are actually linear functions of $l_i^j(t)$, for $i = 1, \ldots, j$ and $j = 1, \ldots, k$. Therefore, the system can be solved by induction on $k$.

The case $k = 1$ is clear since the only equation is

$$e^{2tl_1^1(t)} = e^{t\lambda_1^1},$$

which gives $l_1^1(t) = \lambda_1^1$. Assume that the system of equations is solved for all $k < K$, and the solution is as stated in the theorem (i.e., such that $a_{ij}(t) \to a_{ij}^\infty$, for $j < K$). Let us solve the system for $k = K$ iteratively. The functions $\delta_{s_q}^K(t)$, being linear functions of $l_j^k$, can be described more specifically. In fact, there are three possibilities. If the path with the corresponding weight contains the source (and, therefore, the sink) number $K$, then corresponding weight $\delta_{s_q}^K$ can be expressed as either

$$\delta_{s_q}^K = 2l_i^K + \sum_{k<K} \sum_{m \leq k} \alpha_{s_q,m}^k l_m^k$$

for some $i$ and some coefficients $\alpha_{s_q,m}^k$, or

$$\delta_{s_q}^K = l_i^K + l_j^K + \sum_{k<K} \sum_{m \leq k} \beta_{s_q,m}^k l_m^k$$

for some $i$ and $j$ and coefficients $\beta_{s_q,m}^k$. If the path does not contain the source and sink $K$, then its weight can be written as

$$\delta_{s_q}^K = \sum_{k<K} \sum_{m \leq k} \gamma_{s_q,m}^k l_m^k$$

for some coefficients $\gamma_{s_q,m}^k$.

Denote $e^{2l_i^K(t)} = x_i(t)$. Then the system can be rewritten as

$$x_s(t) = c^s(t) - \sum_{i=1}^{K} c_i^s(t) x_i(t) - \sum_{\substack{i,j=1 \\ i<j}}^{K} c_{ij}^s(t) \sqrt{x_i(t) x_j(t)}, \qquad \text{(B.1)}$$

where $c^s$, $c_i^s$, and $c_{ij}^s$ depend only on $l_m^k$, for $m \leq k < K$, and thus have been determined by the induction hypothesis.

Now we construct the iteration procedure. Let $x_i^{(n)}$ denote the $n$-th iteration of $x_i$. We put $x_i^{(0)} = e^{2l_i^{K,(\infty)}}$. For the $(n+1)$-st iteration we have

$$x_s^{(n+1)}(t) = c^s(t) - \sum_{i=1}^{K} c_i^s(t) x_i^{(n)}(t) - \sum_{\substack{i,j=1\\ i<j}}^{K} c_{ij}^s(t)\sqrt{x_i^{(n)}(t) x_j^{(n)}(t)}.$$

In order for this iterative procedure to be defined, we need to show that $x_i^{(n)} > 0$, for all $i$.

**Lemma 6** *We have $x_i^{(n)} > 0$ for all $i$.*

**Proof** Let us show this by induction on $n$. The induction base is obvious. Let $x_i^{(n)} > 0$ for all $i$, and let us prove that $x_i^{(n+1)} > 0$. We have

$$x_s^{(n)} = c^s(t) - \sum_{i=1}^{K} c_i^s(t) x_i^{(n-1)}(t) - \sum_{\substack{i,j=1\\ i<j}}^{K} c_{ij}^s(t)\sqrt{x_i^{(n-1)}(t) x_j^{(n-1)}(t)}. \tag{B.2}$$

Replacing all the $c^s, c_i^s, c_{i,j}^s$, and $x_i^{(n-1)}$ on the right-hand side by their expressions in terms of $l$'s and $\delta$'s and bounding all the cosines in the obtained expression by $-1$ we obtain

$$x_s^{(n)} \leq \sum_{\{i_1,\dots,i_s\}\subset\{1,\dots,K\}} e^{t(\lambda_{i_1}+\dots+\lambda_{i_s})} + 2\sum_{s_q\in\tilde{Q}_s^K} e^{t\delta_{s_q}^{K,(n-1)}(t)} + \sum_{s_q\in\tilde{\tilde{Q}}_s^K} e^{t\delta_{s_q}^{K,(n-1)}(t)}.$$

We have $\delta_{s_q}^{K,(n-1)}(t) \leq l_s^{K,(n-1)}(t)$ and $l_s^{K,(n-1)}(t) \to (\lambda_1^K + \dots + \lambda_s^K)$. Therefore, for any $\epsilon_1 > 0$ there exists a $t_1$, such that for all $t > t_1$,

$$x_s^{(n)} \leq (1+\epsilon_1) e^{t(\lambda_1^K+\dots+\lambda_s^K)}. \tag{B.3}$$

Thus

$$c_i^s(t) x_i^{(n)}(t) \leq c_i^s(t)(1+\epsilon_1) e^{t(\lambda_1^K+\dots+\lambda_i^K)} = (1+\epsilon_1) e^{t\delta_{s_q}^{K,(n)}(t)} e^{(\lambda_1^K+\dots+\lambda_i^K - l_i^K(t))}.$$

Denote by $d$ the distance $dist(\lambda, \partial\mathfrak{C}^{gz})$. Since $\delta_{s_q}^{K,(n)} < l_s^{K,(n)}$ and $l_i^{K,(n)}(t) \to (\lambda_1^K + \dots + \lambda_i^K)$, we have that for any $\epsilon_2 > 0$, there exists a $t_2$ such that for all $t > t_2$,

$$c_i^s(t) x_i^{(n)}(t) \leq (1+\epsilon_2) e^{t(\lambda_1^K+\dots+\lambda_s^K - d)}.$$

Similarly we have for all $t > t_3$,

$$c_{ij}^s(t)\sqrt{x_i^{(n)}(t) x_j^{(n)}(t)} \leq (1+\epsilon_2) e^{t(\lambda_1^K+\dots+\lambda_s^K - d)}.$$

Bounding all the cosines in the expression for $c^s(t)$ by 1, we obtain that for any $\epsilon_3 > 0$, there exists a $t_4$ such that for $t > t_4$ we have

$$c^s(t) \geq e^{t(\lambda_1^K+\dots+\lambda_s^K)} - \sum e^{t\delta_{s_q}^{K,(n)}(t)} > (1-\epsilon_3) e^{t(\lambda_1^K+\dots+\lambda_s^K)}.$$

This leads to a lower bound for $x_s^{(n+1)}$: bounding the cosines in expression (B.2) by 1 we obtain

$$x_s^{(n+1)} > e^{t(\lambda_1^K+\dots+\lambda_s^K)} \left((1-\epsilon_3) + \sum (1+\epsilon_2) e^{-td}\right).$$

$d$ is positive, so for any $\epsilon > 0$ there exists a $T$ such that for all $t > T$,

$$x_s^{(n+1)} > (1-\epsilon)e^{t(\lambda_1^K+\cdots+\lambda_s^K)} > 0.$$

$\square$

Thus the iterative procedure is defined. Now we fix a $t > T$. Using the bounds obtained in the proof of Lemma 6, we obtain

$$0 < x_s^{(n)} \leq (1+\epsilon_1)e^{t(\lambda_1^K+\cdots+\lambda_s^K)} < 2e^{t(\lambda_1^K+\cdots+\lambda_s^K)},$$

i.e., $x_s^{(n)}$ is a bounded sequence. Therefore it contains a convergent subsequence $x_s^{(n_i)} \xrightarrow[n_i\to\infty]{} x_s$. By construction, the limit $x_s$ is a solution of equation (B.1). Moreover, for any $\epsilon > 0$ there exists a $T$ (which does not depend on $n$) such that for all $t > T$,

$$1-\epsilon < x_s^{(n)}e^{-t(\lambda_1^K+\cdots+\lambda_s^K)} < 1+\epsilon.$$

Passing to the limit (with respect to $n$) we obtain that $x_s \xrightarrow[t\to\infty]{} e^{t(\lambda_1^K+\cdots+\lambda_s^K)}$, i.e., the solution satisfies the desired condition.

$\square$

# Appendix C

# Interlacing inequalities in ordered semirings

Let $\mathcal{R}$ be an ordered semiring and $\mathcal{P}$ its positive subsemiring (i.e., if $x \in \mathcal{P}$, then $x \geq 0_{\mathcal{R}}$). Consider a planar network $\Gamma$ with $n$ sources and $n$ sinks and assign a weighting $\omega = \{w_i \in \mathcal{P}\}$ to its edges. Recall, that in Chapter 5 we defined the functions $l_i\Gamma^{(k)}$:

$$l_i\Gamma^{(k)}(\omega) = \sum_{p_i \in P_i\Gamma^{(k)}} w(p_i)$$

for all $i$ and $k$ such that $1 \leq i \leq k \leq n$, where the weight $w$ of an $i$-path $p_i$ is defined as the product of all its edges.

**Theorem 5** *The collection* $\{t_i^k = l_i\Gamma^{(k)}(\omega)\}_{i,k}$ *satisfies the interlacing inequalities in* $\mathcal{P}$ (5.1).

**Proof** We must show that

$$\begin{array}{lcl} l_i\Gamma^{(k)}(\omega) \cdot l_{i-1}\Gamma^{(k-1)}(\omega) & \geq & l_{i-1}\Gamma^{(k)}(\omega) \cdot l_i\Gamma^{(k-1)}(\omega), \\ l_i\Gamma^{(k)}(\omega) \cdot l_i\Gamma^{(k-1)}(\omega) & \geq & l_{i+1}\Gamma^{(k)}(\omega) \cdot l_{i-1}\Gamma^{(k-1)}(\omega). \end{array}$$

The proof is similar to the proof of Theorem 4 of the article.

Let us first prove the first equation. By definition of $l_i\Gamma^{(k)}$ we must show that

$$\sum_{\substack{p \in P_i\Gamma^{(k)} \\ q \in P_{i-1}\Gamma^{(k-1)}}} w(p)w(q) \ \geq \sum_{\substack{p \in P_{i-1}\Gamma^{(k)} \\ q \in P_i\Gamma^{(k-1)}}} w(p)w(q). \tag{C.1}$$

Let $p$ be in $P_{i-1}\Gamma^{(k)}$ and let $q$ be in $P_i\Gamma^{(k-1)}$. Consider the generalized planar network $\Theta = p \cup q$. There can be other multipaths in $P_{i-1}\Gamma^{(k)}$ and in $P_i\Gamma^{(k-1)}$ that give the same $\Theta$ as their union. Let $\Theta_R$ be the set of all decompositions of $\Theta$ into a pair of multipaths, one from $P_{i-1}\Gamma^{(k)}$ and the other from $P_i\Gamma^{(k-1)}$. Let $\Theta_L$ be the set of all decompositions of $\Theta$ into a pair of multipaths, one from $P_i\Gamma^{(k)}$ and the other from $P_{i-1}\Gamma^{(k-1)}$. We will show that $|\Theta_L| \geq |\Theta_R|$, which would imply (C.1).

Consider the canonical path decomposition of $\Theta$. Similarly to the proof of Theorem 4 of the article, the following paths of the canonical path decomposition have an even number of edges:

- closed paths $Q_{cl}$,
- open paths beginning and ending at a source $Q_{LL}$,
- open paths beginning and ending at a sink $Q_{RR}$.

We consider alternate colorings of $\Theta$ in red and green. To be specific, if an alternate coloring results in an element of $\Theta_R$, let the color corresponding to the element of $P_{i-1}\Gamma^{(k)}$ be red, and if an alternate coloring results in an element of $\Theta_L$, let the color corresponding to the element of $P_i\Gamma^{(k)}$ be red. We must show that the number of such colorings giving $\Theta_R$ is not greater than the number of such colorings giving $\Theta_L$. The colorings of $Q_{LL}, Q_{RR}$, and $Q_{cl}$ resulting in elements of $\Theta_L$ are the same as those resulting in elements of $\Theta_R$. Therefore it remains to color the paths in the set $Q_{LR}$, i.e., starting at a source and ending at a sink. The coloring of each path is uniquely determined by the coloring of the source-edge it starts with.

We have $|Q_{LR}| = 2s-1$ for some $s$. There are two possibilities. One possibility is if there are no paths in $Q_{LR}$ containing the source-edge or the sink-edge labeled $k$. Then to obtain an element of $\Theta_R$, we must color in red $s-1$ of the source-edges, so the remaining $s$ source-edges will be colored in green. The number of such colorings is $\binom{2s-1}{s-1} = \binom{2s-1}{s}$. In order to obtain an element of $\Theta_L$, we must color $s$ source-edges in red and then $s-1$ remaining source-edges will be colored in green. Obviously we also have $\binom{2s-1}{s}$ such colorings.

In the case when there is a path in $Q_{LR}$ containing the source or sink-edge labeled $k$, its coloring is fixed: in order to obtain an element of $\Theta_R$ or $\Theta_L$, this source or sink-edge must be colored in red. Therefore it remains to color $2s-2$ edges. To obtain an element of $\Theta_R$, we must color in red $s-2$ source-edges, and so there are $\binom{2s-2}{s-2}$ such colorings. To obtain an element of $\Theta_L$, we must color in red $s-1$ source-edges, this gives $\binom{2s-2}{s-1}$ colorings. We have $\binom{2s-2}{s-2} < \binom{2s-2}{s-1}$.

The second equation is proved in the same way. For a generalized planar network $\Theta$, we call $\Theta_R$ the set of all decompositions of $\Theta$ into a pair of multipaths, one from $P_{i+1}\Gamma^{(k)}$ and the other from $P_{i-1}\Gamma^{(k-1)}$, and we call $\Theta_L$ the set of all decompositions of $\Theta$ into a pair of multipaths, one from $P_i\Gamma^{(k)}$ and the other from $P_i\Gamma^{(k-1)}$. Again, we consider alternate colorings of the canonical path decomposition of $\Theta$. To be specific, if an alternate coloring results in an element of $\Theta_R$, let the color corresponding to the element of $P_{i+1}\Gamma^{(k)}$ be red, and if an alternate coloring results in an element of $\Theta_L$, let the color corresponding to the element of $P_i\Gamma^{(k)}$ be red. As in the previous case, it is sufficient to count the number of colorings of $Q_{LR}$.

We have $|Q_{LR}| = 2s$ for some $s$. In the case when $Q_{LR}$ does not contain the source or sink-edge labeled $k$, in order to obtain an element of $\Theta_R$, we must color in red $s+1$ of the source-edges. There are $\binom{2s}{s+1}$ such colorings. To obtain an element of $\Theta_L$, we must color in red $s$ of the source-edges, and there are $\binom{2s}{s}$ such colorings. We have $\binom{2s}{s} > \binom{2s}{s+1}$.

In the case when $Q_{LR}$ contains the source or sink-edge labeled $k$, in order to obtain an element of $\Theta_R$ or $\Theta_L$, this edge must be colored in red. This leaves $2s-1$ source-edges. To obtain an element of $\Theta_R$, we must color in red $s$ of the source edges, which gives $\binom{2s-1}{s}$ colorings. To obtain an element of $\Theta_L$, we must color in red $s-1$ of the source-edges, which gives $\binom{2s-1}{s-1}$ colorings. We have $\binom{2s-1}{s} = \binom{2s-1}{s-1}$. □