Euler's Exponential Formula for Semigroups

Cachia, Vincent

How to cite

CACHIA, Vincent. Euler's Exponential Formula for Semigroups. In: Semigroup Forum, 2004, vol. 68, n° 1, p. 1–24. doi: 10.1007/s00233-002-0025-1

This publication URL: https://archive-ouverte.unige.ch/unige:121488

Publication DOI: 10.1007/s00233-002-0025-1

Semigroup Forum Vol. 68 (2004) 1–24

DOI: 10.1007/s00233-002-0025-1

RESEARCH ARTICLE

Euler's Exponential Formula for Semigroups

Vincent Cachia

Communicated by Rainer Nagel

Abstract

The aim of this paper is to show that Euler's exponential formula $\lim_{n\to\infty} (I-tA/n)^{-n}x = e^{tA}x$, well known for C_0 semigroups in a Banach space $X \ni x$, can be used for semigroups not of class C_0, the sense of the convergence being related to the regularity of the semigroup for $t > 0$. Although the strong convergence does not hold in general for not strongly continuous semigroups, an integrated version is stated for once integrated semigroups. Furthermore by replacing the initial topology on X by some (coarser) locally convex topology τ, the strong τ-convergence takes place provided the semigroup is strongly τ-continuous; in particular this applies to the class of bi-continuous semigroups [9]. On the other hand, for bounded holomorphic semigroups not necessarily of class C_0, Euler's formula is shown to hold in operator norm, with the error bound estimate $\mathcal{O}(\ln n/n)$, uniformly in $t > 0$. All these results also concern degenerate semigroups.

The Euler exponential formula $\lim_{n\to\infty}(I - tA/n)^{-n} = e^{tA}$ (a special case of the Post-Widder inversion formula) is well known as a useful method to construct or to approximate C_0 semigroups in the strong operator topology. In particular, Euler's approximation $(I - tA/n)^{-n}$ coincides with the solution of the difference equation associated to the differential equation $u' = Au$ [13, Remark 1.8.5].

It is known that this formula converges in the operator norm topology for bounded holomorphic C_0 semigroups (see [4] in a Hilbert space, and [3] in a Banach space). This suggests that its convergence is related to the regularity of the semigroup. On the other hand the class of C_0 semigroups is not always sufficient for the applications to evolution equations and this fact has motivated various other theories: semigroups generated by multi-valued linear operators, integrated semigroups, bi-continuous semigroups, etc.

In the first section we present some examples of Hille-Yosida operators for which Euler's approximation is, or is not, strongly convergent. The second section deals with generalizations of Euler's formula for integrated semigroups: on the one hand an integrated version converges strongly to the integrated semigroup, on the other hand if a k-times integrated semigroup is k-times strongly τ-differentiable, then one obtains Euler's formula in the sense of the strong τ-convergence. This is a method to study τ-continuous semigroups, which are not easy to characterize. In the last part we consider bounded holomorphic semigroups (not necessarily of class C_0) and prove that Euler's formula holds in operator norm with an error bound estimate $\mathcal{O}(\ln n/n)$, uniformly in $t > 0$.

Throughout this paper X denotes a Banach space with norm $\|\cdot\|$, $\mathcal{L}(X)$ denotes the space of all bounded linear operators on X with the operator norm also denoted by $\|\cdot\|$, and $R(\lambda, A) = (\lambda - A)^{-1}$ is the resolvent of an operator A at $\lambda \in \mathbb{C}$. A sequence of bounded linear operators A_n is said to converge to A in the strong operator topology if $\lim_{n\to\infty} \|A_n x - Ax\| = 0$ for all $x \in X$. The notions of strong continuity and strong differentiability without other indication refer to the Banach space norm on X.

1. Euler's formula in the strong operator topology

1.1. Hille-Yosida operators

Definition 1.1. A linear operator A in a Banach space X is called a Hille-Yosida operator if its resolvent set $\rho(A)$ contains a half-line $(\omega, +\infty)$ and the Hille-Yosida condition holds:

$$\sup_{\lambda>\omega, n\in\mathbb{N}} \|(\lambda-\omega)^n(\lambda-A)^{-n}\| < \infty. \tag{1.1}$$

In this definition we also admit multi-valued linear operators. These are in one-to-one correspondence with the linear subspaces of $X \times X$ viewed as graphs. The resolvent set is defined as the set of all complex numbers λ such that $(\lambda-A)^{-1}$ (as a graph) is the graph of an everywhere defined bounded linear operator in X [14]. In place of a multi-valued Hille-Yosida operator A, one can equivalently consider a pseudo-resolvent family $\{R(\lambda), \lambda \in \Omega \subset \mathbb{C}\} \subset \mathcal{L}(X)$, with a possibly non trivial kernel $\ker R(\lambda) = A0$, satisfying the condition (1.1).

In general, the Hille-Yosida condition (1.1) is not sufficient for A to generate a C_0 semigroup on X. However the part $A_|$ of A in $\overline{\mathrm{dom}(A)}$ generates a C_0 semigroup on this closed subspace [14, Theorem 3.2]. Moreover $X_0 = \overline{\mathrm{dom}(A)} + A0$ is a topological direct sum in X [14, Theorem 2.4]. Thus Euler's formula holds in X_0:

Lemma 1.2. *If A is a Hille-Yosida operator in X, then Euler's approximation converges strongly in $X_0 = \overline{\mathrm{dom}(A)} + A0$, to the C_0 semigroup generated by $A_|$ in $\overline{\mathrm{dom}(A)}$ and to 0 in $A0$:*

$$\lim_{n\to\infty} (I - tA/n)^{-n} x = e^{tA_|} P_0 x, \; x \in X_0, t \geq 0, \tag{1.2}$$

where P_0 is the projection from $X_0 = \overline{\mathrm{dom}(A)} + A0$ on $\overline{\mathrm{dom}(A)}$.

Proof. Since $A0 = \ker(\lambda - A)^{-1}$ for any $\lambda \in \rho(A)$, $(I - tA/n)^{-n}x = 0$ for any $x \in A0$. Since the convergence is well known for C_0 semigroups, it remains to observe that $(I - tA/n)^{-1}x = (I - tA_|/n)^{-1}x \in X_0$ for any $x \in X_0$. ∎

Following [1, Theorem 6.2], a Hille-Yosida operator generates a semigroup on the full space X if X has the Radon-Nikodym property (the proof extends clearly to the multi-valued case, with degenerate semigroups). For example, reflexive spaces and separable dual spaces have this property. This leads to:

Theorem 1.3. *Let X be a Banach space with the Radon-Nikodym property. If A is a Hille-Yosida operator in X with $\omega \leq 0$, then Euler's approximation converges strongly to a semigroup T_t which is strongly continuous for $t > 0$.*

Proof. By [1, Theorem 6.2] there exists a semigroup of bounded linear operators $(T(t))_{t>0}$, strongly continuous for $t > 0$, such that $\|T(t)\| \leq M$ and

$$R(\lambda, A) = \int_0^\infty e^{-\lambda s} T(s) ds, \ \lambda > \omega. \tag{1.3}$$

By the resolvent equation one obtains that

$$R(\lambda, A)^n = \frac{(-1)^{n-1}}{(n-1)!} R(\lambda, A)^{(n-1)} = \frac{(-1)^{n-1}}{(n-1)!} \int_0^\infty (-s)^{n-1} e^{-\lambda s} T(s) ds, \tag{1.4}$$

and then, for $n/t > \omega$,

$$\left[\frac{n}{t} R(n/t, A)\right]^n = \frac{1}{(n-1)!} \int_0^\infty \sigma^{n-1} e^{-\sigma} T(\sigma t/n) d\sigma. \tag{1.5}$$

Since $\sigma^{n-1} e^{-\sigma} d\sigma/(n-1)!$ is a probability measure on $(0, \infty)$ with associated mean n and variance n, one has for all $x \in X$ and $t > 0$:

$$\begin{aligned}
&\left\| \left[\frac{n}{t} R(n/t, A)\right]^n x - T(t)x \right\| \\
&\quad \leq \frac{1}{(n-1)!} \int_{|\sigma - n| > \delta} \sigma^{n-1} e^{-\sigma} \|(T(\sigma t/n) - T(t))x\| d\sigma \\
&\qquad + \frac{1}{(n-1)!} \int_{|\sigma - n| \leq \delta} \sigma^{n-1} e^{-\sigma} \|(T(\sigma t/n) - T(t))x\| d\sigma \\
&\quad \leq \frac{2Mn}{\delta^2} + \sup_{|s-t| \leq \delta t/n} \|T(s)x - T(t)x\|.
\end{aligned}$$

Let us set $\delta_n = n^\alpha$ for some $\alpha \in (1/2, 1)$: the first term goes to 0 as $2n^{1-2\alpha}$ and the last term tends to 0 by the continuity of $T(t)x$, when $n \to \infty$. ■

1.2. Counterexample

We will show by an example that Euler's formula does not necessarily hold (in the sense of the Banach space norm) for vectors $x \notin X_0$ (when X does not have the Radon-Nikodym property). Let $X = \mathcal{C}_b(\mathbb{R})$ the Banach space of bounded continuous functions $\mathbb{R} \to \mathbb{C}$ with the sup-norm, and let $T(t)f(\cdot) = f(\cdot + t)$ be the left translation semigroup. By the integral (for $\operatorname{Re}\lambda > 0$)

$$[R(\lambda)f](u) = \int_0^{+\infty} e^{-\lambda s} f(u+s)ds, \tag{1.6}$$

this semigroup is associated to a resolvent which clearly satisfies the condition (1.1) with $\omega = 0$. $R(\lambda)$ can be identified as the resolvent of the generator by considering the once integrated semigroup $S(t) = \int_0^t T(\tau)d\tau$ (see section 2.1 below). The subspace of strong continuity is $X_0 = \mathcal{C}_{ub}(\mathbb{R})$, the set of uniformly continuous and bounded functions. Let us consider a function $f \in X \backslash X_0$, defined by $f(u) = \min\{1, p|u-p| : p \in \mathbb{N}\}$, and prove that Euler's formula does not hold on f for the uniform convergence on $\mathbb{R}$, for any $t > 0$. We denote by $F_n(u)$ the difference (for some fixed $t > 0$)

$$\left(\frac{n}{t} R(n/t)\right)^n f(u) - T(t)f(u) = \int_0^{+\infty} \frac{n^n}{(n-1)!} s^{n-1} e^{-ns}(f(u+st) - f(u+t))ds$$

and consider the sequence $u_n = n - t$. It will be shown that $F_n(u_n)$ does not converge to 0 as $n \to \infty$, which means that $\sup_{u\in\mathbb{R}} |F_n(u)|$ does not converge to 0. Let $r_n(s) = \frac{n^n}{(n-1)!} s^{n-1} e^{-ns}$, so that

$$F_n(u_n) = \int_0^{\infty} r_n(s) f(n + (s-1)t)ds, \tag{1.7}$$

and $d\rho_n = r_n(s)ds$ is the associated probability measure on $(0, \infty)$. Then we observe that

$$\sup_{s\geq 0} r_n(s) = r_n\left(1 - \frac{1}{n}\right) = n\frac{(n-1)^{n-1}}{(n-1)!} e^{-(n-1)} = \frac{n}{\sqrt{2\pi(n-1)}}(1 - o(1)), \tag{1.8}$$

and thus

$$\rho_n([1 - 1/nt, 1 + 1/nt]) \leq \frac{2}{nt} \sup_{s\geq 0} r_n(s) \leq \frac{2/t}{\sqrt{2\pi(n-1)}}(1 - o(1)). \tag{1.9}$$

On the other hand, by the Chebychev inequality, we have for sufficiently large n:

$$\rho_n([1 - 1/2t, 1 + 1/2t]) \geq 1 - \frac{4t^2}{n}, \tag{1.10}$$

which leads to

$$\rho_n([1 - 1/2t, 1 - 1/nt] \cup [1 + 1/nt, 1 + 1/2t]) \geq 1 - \frac{4t^2}{n} - \frac{2/t}{\sqrt{2\pi(n-1)}}(1 - o(1)). \tag{1.11}$$

Since $f(n+(s-1)t)=1$ on these intervals for any $n \geq 3$, one concludes that $F_n(u_n) \geq 1-o(1)$, hence $F_n(u_n)$ does not converge to 0 as $n \to \infty$.

However it can be shown that $\lim_{n\to\infty} F_n(u)=0$ for each $u \in \mathbb{R}$ (in fact, ρ_n converges (weak-$*$) to the Dirac measure at 1, see the proof of Theorem 1.3), or even uniformly on the compact subsets of $\mathbb{R}$. This suggests that when Euler's exponential formula does not hold in the strong topology, one could find some weaker topology on X ensuring the convergence.

2. A generalization of Euler's formula

In order to study semigroups such as the translation semigroup in the previous example, which are not strongly continuous for the Banach space norm (but perhaps for some coarser topology on X), we shall use integrated semigroups. By this way we clarify the sense in which the generator is understood.

2.1. Integrated semigroups

The semigroup property $T(t+s) = T(t)T(s)$ corresponds to the resolvent equation for the Laplace transform: similarly in [1] one considers operator families whose Laplace transform are of the form $R(\lambda)/\lambda^k$, where $R(\lambda)$ is a pseudo-resolvent. This leads to another functional equation which characterizes integrated semigroups. The following definition [2, Def 3.2.1] states the relation between the integrated semigroup and its generator.

Definition 2.1. Let A be an operator on a Banach space X and $k \in \mathbb{N} \cup \{0\}$. A is called the generator of a k-times integrated semigroup if there exist $\omega \geq 0$ and a strongly continuous function S: $\mathbb{R}_+ \to \mathcal{L}(X)$ having a Laplace transform for $\lambda > \omega$, such that $(\omega,\infty) \subset \rho(A)$ and

$$R(\lambda, A) = \lambda^k \int_0^\infty e^{-\lambda t} S(t)dt \quad (\lambda > \omega). \tag{2.1}$$

In this case S is called the k-times integrated semigroup generated by A.

The particular case of a 0-times integrated semigroup coincides with the notion of C_0 semigroup. The generators of integrated semigroups are characterized as follows [1, Theorem 4.1]:

Proposition 2.2. *Let $k \in \mathbb{N} \cup \{0\}$, $\omega \in \mathbb{R}$, $M \geq 0$. A linear operator A is the generator of an $(k+1)$-times integrated semigroup $(S(t))_{t\geq 0}$ satisfying*

$$\limsup_{h\downarrow 0} \frac{1}{h}\|S(t+h)-S(t)\| \leq Me^{\omega t} \ (t \geq 0) \tag{2.2}$$

if and only if there exists $a \geq \max\{\omega, 0\}$ such that $(a,\infty) \subset \rho(A)$ and

$$\|(\lambda-\omega)^{n+1}[R(\lambda,A)/\lambda^k]^{(n)}/n!\| \leq M \tag{2.3}$$

for all $\lambda > a$, $n = 0, 1, 2, \dots$

Here are some preliminaries in order to generalize Euler's formula for integrated semigroups.

Lemma 2.3. *Let $S(t)$ be an exponentially bounded k-times integrated semigroup on X: $\|S(t)\| \leq Me^{\omega t}$. Let $R(\lambda)/\lambda^k = \int_0^\infty e^{-\lambda t} S(t)dt$ be its Laplace transform for $\lambda > \max\{\omega, 0\}$, where $R(\lambda)$ is a pseudo-resolvent. Then Euler's approximation for n sufficiently large can be written as*

$$[(n/t_0)R(n/t_0)]^n = \frac{(n/t_0)^{k+1}}{(n-1)!} \int_0^\infty \left(\frac{nt}{t_0}\right)^{n-k-1} P_{n,k}(nt/t_0)e^{-nt/t_0} S(t)dt, \tag{2.4}$$

where $t_0 > 0$ and $P_{n,k}$ denotes the polynomial (for $n > k$)

$$P_{n,k}(\lambda) = \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \lambda^{k-\ell} \frac{(n-1)!}{(n-\ell-1)!}. \tag{2.5}$$

Remark. The polynomials $P_{n,k}$ are related to the Laguerre polynomials by

$$P_{n,k}(\lambda) = (-1)^k k! L_k^{n-k-1}(\lambda). \tag{2.6}$$

Proof. For a given $t_0 > 0$, let n be such that $n/t_0 > \omega$ and $n > k$. By the resolvent equation one obtains

$$[(n/t_0)R(n/t_0)]^n = \left(\frac{n}{t_0}\right)^n \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{d\lambda^{n-1}} R(\lambda)\bigg|_{\lambda=n/t_0},$$

and thus

$$[(n/t_0)R(n/t_0)]^n = \left(\frac{n}{t_0}\right)^n \frac{(-1)^{n-1}}{(n-1)!} \int_0^\infty \frac{d^{n-1}}{d\lambda^{n-1}} (\lambda^k e^{-\lambda t})\bigg|_{\lambda=n/t_0} S(t)dt.$$

Then by the Leibniz formula,

$$\begin{aligned}
&[(n/t_0)R(n/t_0)]^n \\
&\quad = \frac{(-1)^{n-1}}{(n-1)!} \sum_{\ell=0}^{k} \binom{n-1}{\ell} \frac{k!}{(k-\ell)!} \left(\frac{n}{t_0}\right)^{n+k-\ell} \int_0^\infty (-t)^{n-\ell-1} e^{-nt/t_0} S(t)dt \\
&\quad = \frac{(n/t_0)^{k+1}}{(n-1)!} \int_0^\infty \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^\ell \left(\frac{nt}{t_0}\right)^{n-\ell-1} \\
&\qquad \times \frac{(n-1)!}{(n-\ell-1)!} e^{-nt/t_0} S(t)dt,
\end{aligned} \tag{2.7}$$

which leads to (2.4). ∎

Lemma 2.4. *The polynomials $P_{n,k}$ obey the following relations for $n > k > 0$:*

$$P_{n,k+1}(\lambda) = (\lambda - n + k + 1)P_{n,k}(\lambda) - \lambda k P_{n,k-1}(\lambda) \tag{2.8}$$

$$P'_{n,k}(\lambda) = kP_{n,k-1}(\lambda) \tag{2.9}$$

$$P_{n,k}(\lambda t)\lambda^{n-k-1}e^{-\lambda t} = \frac{d^k}{d\lambda^k}(\lambda^{n-1}e^{-\lambda t}). \tag{2.10}$$

Proof. From the identity (for any numbers $a_0, \ldots, a_{k+1}$):

$$\begin{aligned}\sum_{\ell=0}^{k+1} \binom{k+1}{\ell} (-1)^\ell a_\ell &= a_0 + \sum_{\ell=1}^{k} \left(\binom{k}{\ell-1} + \binom{k}{\ell} \right) (-1)^\ell a_\ell + (-1)^{k+1} a_{k+1} \\ &= \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} (a_\ell - a_{\ell+1}),\end{aligned}$$

one deduces

$$\begin{aligned}P_{n,k+1}(\lambda) &= \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \lambda^{k-\ell} \frac{(n-1)!}{(n-\ell-1)!} (\lambda - n + k + 1 - (k-\ell)) \\ &= (\lambda - n + k + 1)P_{n,k}(\lambda) - \lambda \sum_{\ell=0}^{k-1} \frac{(-1)^\ell k!}{\ell!(k-1-\ell)!} \lambda^{k-1-\ell} \frac{(n-1)!}{(n-\ell-1)!} \\ &= (\lambda - n + k + 1)P_{n,k}(\lambda) - \lambda k P_{n,k-1}(\lambda)\end{aligned}$$

which gives (2.8). By (2.5) one finds directly (2.9):

$$P'_{n,k}(\lambda) = \sum_{\ell=0}^{k-1} \frac{k!(-1)^\ell}{\ell!(k-1-\ell)!} \lambda^{k-1-\ell} \frac{(n-1)!}{(n-\ell-1)!} = kP_{n,k-1}(\lambda).$$

Relation (2.10) is clearly verified for $k = 1$. Suppose that it holds for some integer k: then by

$$\begin{aligned}&[P_{n,k}(\lambda t)\lambda^{n-k-1}e^{-\lambda t}]' \\ &\quad = \{\lambda t[P'_{n,k}(\lambda t) - P_{n,k}(\lambda t)] + (n-k-1)P_{n,k}(\lambda t)\}\lambda^{n-k-2}e^{-\lambda t} \\ &\quad = P_{n,k+1}(\lambda t)\lambda^{n-k-2}e^{\lambda t}\end{aligned}$$

(where we use (2.8)) one deduces that (2.10) holds for $k+1$, and thus for any $k < n$ by induction. ∎

2.2. Integrated Euler's formula

It will be shown that a once integrated semigroup can be approximated in the strong operator topology by integration of Euler's approximation.

Theorem 2.5. *Let $(S(t))_{t\geq 0}$ be an exponentially bounded, once integrated semigroup on X: $\|S(t)\| \leq Me^{\omega t}$. The associated pseudo-resolvent family is $R(\lambda) = \lambda\int_0^\infty e^{-\lambda t}S(t)dt$ for $\lambda > \omega$, and one defines $F(t) = t^{-1}R(t^{-1})$, $0 < t < \omega^{-1}$. Then for any $t_0 > 0$*

$$s-\lim_{n\to\infty} s-\lim_{\epsilon\to 0}\int_\epsilon^{t_0} F(\tau/n)^n d\tau = S(t_0). \tag{2.11}$$

Proof. Since the pseudo-resolvent is holomorphic in the half-plane $\operatorname{Re} z > \omega$, the function $t \mapsto F(t) = t^{-1}R(t^{-1})$ is holomorphic in the open disc of diameter $[0,\omega^{-1}]$. In particular $F(t/n)^n$ is strongly continuous and bounded on $[\epsilon, t_0]$ if n is sufficiently large. This implies that the integral in (2.11) is well defined. By the representation (2.4) one has

$$\int_\epsilon^{t_0} F(\tau/n)^n d\tau = \int_\epsilon^{t_0}\frac{(n/\tau)^2}{(n-1)!}\int_0^\infty\left(\frac{nt}{\tau}\right)^{n-2} e^{-nt/\tau}(nt/\tau - n + 1)S(t)d\tau dt. \tag{2.12}$$

Then by setting $\lambda = n/\tau$, and by using Fubini's theorem:

$$\begin{aligned}\int_\epsilon^{t_0} F(\tau/n)^n d\tau &= \int_{nt_0^{-1}}^{n\epsilon^{-1}}\frac{n}{(n-1)!}\int_0^\infty(\lambda t)^{n-2}e^{-\lambda t}(\lambda t - n + 1)S(t)d\lambda dt\\ &= \int_0^\infty\frac{nt^{n-2}}{(n-1)!}S(t)\int_{nt_0^{-1}}^{n\epsilon^{-1}}\lambda^{n-2}e^{-\lambda t}(\lambda t - n + 1)dtd\lambda\\ &= \int_0^\infty\frac{nt^{n-2}}{(n-1)!}S(t)\big[-\lambda^{n-1}e^{-\lambda t}\big]_{nt_0^{-1}}^{n\epsilon^{-1}}dt.\end{aligned}$$

Since $s-\lim_{t\to 0} S(t) = 0$, one has

$$\begin{aligned}&s-\lim_{\epsilon\to 0}\frac{n^n}{(n-1)!}\int_0^\infty S(t)(t/\epsilon)^{n-2}e^{-nt/\epsilon}\frac{dt}{\epsilon}\\ &\quad= s-\lim_{\epsilon\to 0}\frac{n^n}{(n-1)!}\int_0^\infty S(\epsilon u)u^{n-2}e^{-nu}du = 0\end{aligned}$$

by Lebesgue's theorem, for each n sufficiently large. It remains to show that

$$s-\lim_{n\to\infty}\frac{n^n}{(n-1)!}\int_0^\infty S(t)(t/t_0)^{n-2}e^{-nt/t_0}\frac{dt}{t_0} = S(t_0).$$

This follows from the argument in the proof of Theorem 1.3 with $n' = n - k$, and the observation that $\frac{n^n}{(n-1)!} \sim \frac{n'^{n'}}{(n'-1)!}e^k$ when $n \to \infty$. ∎

Remark. It follows clearly from the last argument that the limits can be inverted in (2.11):

$$s-\lim_{\epsilon\to 0} s-\lim_{n\to\infty}\int_\epsilon^{t_0} F(\tau/n)^n d\tau = s-\lim_{\epsilon\to 0}[S(t_0) - S(\epsilon)] = S(t_0). \tag{2.13}$$

Remark. The natural generalization to k-times integrated semigroups is not straightforward. We do not know whether the integral corresponding to (2.11) converges for arbitrary k-times integrated semigroups ($k > 1$).

2.3. Euler's formula for differentiable integrated semigroups

Let the Banach space X be endowed with a topology τ, coarser than the norm topology, such that (X, τ) is a locally convex topological vector space, which is sequentially complete on norm-bounded sets. This topology is given by a family of seminorms $\{p_i\}_{i \in I}$, and we can assume that $p_i(x) \leq \|x\|$ for all $x \in X$ and $i \in I$. The completeness ensures that the τ-Riemann integral is well defined for τ-continuous functions $[a, b] \to (X, \tau)$ which are norm-bounded.

Theorem 2.6. *Let x: $(0, \infty) \to X$ be an exponentially norm-bounded, τ-continuous function, $\|x(t)\| \leq Me^{\omega t}$ ($M > 0$, $\omega \in \mathbb{R}$). Suppose that, for some $k \in \mathbb{N}$ and some $t_0 > 0$ with $\omega t_0 \leq k$, there exist $x_0, \ldots, x_k \in X$ and ϵ: $(0, \infty) \to X$ such that:*

$$x(t) = \sum_{m=0}^{k} \frac{(t - t_0)^m}{m!} x_m + (t - t_0)^k \epsilon(t), \ \forall t > 0, \tag{2.14}$$

with $\tau - \lim_{t \to t_0} \epsilon(t) = 0$ and $\sup_{t > 0} \|\epsilon(t)\| \leq Me^{\omega t}$. Then

$$\tau - \lim_{n \to \infty} \left(\frac{n}{t_0}\right)^k \frac{1}{(n-1)!} \int_0^\infty \sigma^{n-k-1} P_{n,k}(\sigma) e^{-\sigma} x(\sigma t_0 / n) d\sigma = x_k. \tag{2.15}$$

The following lemmata are useful for the proof of the theorem:

Lemma 2.7. *Let $k \in \mathbb{N}$. For any integer $n \geq k + 1$:*

$$\frac{1}{(n-1)!} \int_0^\infty \sigma^{n-k-1+m} P_{n,k}(\sigma) e^{-\sigma} d\sigma = \begin{cases} 0 \text{ for } m = 0, 1, \ldots, k-1 \\ k! \text{ for } m = k \end{cases} \tag{2.16}$$

Proof. Since $\int_0^\infty \sigma^p e^{-\sigma} d\sigma = p!$, the left hand side of (2.16) equals

$$\sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \frac{(n-1-\ell+m)!}{(n-1-\ell)!}. \tag{2.17}$$

Since $n \geq k+1$, the product $\frac{(n-1-\ell+m)!}{(n-1-\ell)!} = (n-\ell)(n+1-\ell)\cdots(n+m-1-\ell)$ is a polynomial of degree m in the variable ℓ. We first show that

$$\sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} P_m(\ell) = 0 \tag{2.18}$$

for each polynomial P_m of degree $m = 0, 1, \ldots, k-1$, which implies (2.16) for these values of m. To check (2.18) we use the basis: $P_0(\ell) = 1$,

$P_1(\ell) = \ell, \ldots, P_m(\ell) = \ell(\ell-1)\cdots(\ell-m+1)$. By observing that $P_m(\ell) = 0$ for $\ell = 0, 1, \ldots, m-1$ and $P_m(\ell) = \ell!/(\ell-m)!$ for $\ell = m, m+1, \ldots$, the left hand side of (2.18) becomes

$$\begin{aligned}\sum_{\ell=m}^{k}(-1)^\ell \binom{k}{\ell} \frac{\ell!}{(\ell-m)!} &= \sum_{\ell=m}^{k}(-1)^\ell \frac{k!}{(k-\ell)!(\ell-m)!} \\ &= \frac{(-1)^m k!}{(k-m)!}\sum_{\ell'=0}^{k-m}(-1)^{\ell'}\frac{(k-m)!}{\ell'!(k-m-\ell')!} = 0,\end{aligned}$$

where $\ell' = \ell - m$ and (2.18) is proved. Similarly, one has for $P_k(\ell) = \ell(\ell - 1)\cdots(\ell-k+1)$

$$\sum_{\ell=0}^{k}(-1)^\ell \binom{k}{\ell} P_k(\ell) = (-1)^k k! \tag{2.19}$$

Since $(n-\ell)(n+1-\ell)\cdots(n+k-1-\ell) = (-1)^k P_k(\ell) + Q(\ell)$ where the degree of Q is $k-1$, one obtains the result for $m = k$. ∎

Lemma 2.8. *The integral $\int_0^\infty \frac{\sigma^{n-k-1}}{(n-1)!}e^{-\sigma}|(\sigma-n)^k P_{n,k}(\sigma)|d\sigma$ is bounded independently of $n > k$, for each fixed $k \geq 0$.*

Proof. Let $\phi_n(\sigma) = \frac{\sigma^{n-k-1}}{(n-1)!}e^{-\sigma}(\sigma-n)^k P_{n,k}(\sigma)$ for a given integer k. By Lemma 2.7 the integral $\int_0^\infty \phi_n(\sigma)d\sigma = k!$ is bounded independently of $n > k$. Since the function $\phi_n(\sigma)$ has no constant sign, we consider its positive part ϕ_n^+ and its negative part ϕ_n^- such that $|\phi_n| = \phi_n^+ + \phi_n^- = \phi_n + 2\phi_n^-$. Therefore one has to estimate the negative part of the function, in order to prove that its contribution to the integral is also bounded independently of $n > k$. Since $(\sigma-n)^k P_{n,k}(\sigma)$ goes to $+\infty$ as $\sigma \to \pm\infty$, the negative part ϕ_n^- has support in the interval between the smallest and the largest root of $(\sigma-n)P_{n,k}(\sigma)$ (except for the trivial case $k=0$).

Let us observe that the degree of $P_{n,k}(n)$ as a polynomial in n is $[k/2] = \sup\{p \in \mathbb{N}, p \leq k/2\}$. Indeed $P_{n,0}(n) = 1$, $P_{n,1}(n) = 1$, and by (2.8) one has $P_{n,k+1}(n) = (k+1)P_{n,k}(n) - nkP_{n,k-1}(n)$. Thus the observation follows by induction on k. By Taylor's formula and relation (2.9) one has for any $a \in \mathbb{R}$

$$P_{n,k}(\sigma) = \sum_{\ell=0}^{k}\binom{k}{\ell}(\sigma-a)^\ell P_{n,k-\ell}(a). \tag{2.20}$$

Let $\{\alpha_n\}_{n>k>0}$ be a sequence of roots for a given k: $P_{n,k}(\alpha_n) = 0$, and let $\beta_n = (\alpha_n - n)n^{-1/2}$. Then by (2.20) for $a = n$, $P_{n,k}(\alpha_n) = n^{k/2}\tilde{P}_{n,k}(\beta_n)$, where $\tilde{P}_{n,k}$ is a polynomial of degree k with coefficient of order ℓ:

$$\binom{k}{\ell} n^{-\frac{k-\ell}{2}} P_{n,k-\ell}(n) = \mathcal{O}(1)$$

when $n \to \infty$. Thus β_n as root of $\tilde{P}_{n,k}$ is bounded independently of $n > k > 0$. This leads to $|\alpha_n - n| \le \mathcal{O}(n^{1/2})$.

Let a_n, b_n be the smallest and the largest root of $(\sigma - n)P_{n,k}(\sigma)$, then for any $\sigma \in [a_n, b_n]$ one has $|\sigma - n| \le \max\{|a_n - n|, |b_n - n|\} \le \mathcal{O}(n^{1/2})$. Thus one has $|P_{n,k}(\sigma)| \le \mathcal{O}(n^{k/2})$, $\sigma \in [a_n, b_n]$, and then $\sup_{\sigma \in [a_n, b_n]} |(\sigma - n)^k P_{n,k}(\sigma)| \le \mathcal{O}(n^k)$. This leads to the estimate for ϕ_n^-:

$$\phi_n^-(\sigma) \le \mathcal{O}(n^k) \frac{(n-k-1)!}{(n-1)!} \frac{\sigma^{n-k-1}}{(n-k-1)!} e^{-\sigma}. \tag{2.21}$$

Finally one obtains

$$\begin{aligned}&\int_0^\infty \frac{\sigma^{n-k-1}}{(n-1)!} e^{-\sigma} \left|(\sigma - n)^k P_{n,k}(\sigma)\right| d\sigma \\ &\qquad \le k! + \mathcal{O}(n^k) \frac{(n-k-1)!}{(n-1)!} \int_0^\infty \frac{\sigma^{n-k-1}}{(n-k-1)!} e^{-\sigma} d\sigma,\end{aligned}$$

which shows that the integral on the left hand side is bounded independently of $n > k$. ■

Proof of Theorem 2.6. Inserting the expression (2.14) into the integral, one obtains for the left hand side of (2.15)

$$\sum_{m=0}^{k} \frac{x_m}{m!} \left(\frac{n}{t_0}\right)^{k-m} \frac{1}{(n-1)!} \int_0^\infty \sigma^{n-k-1} P_{n,k}(\sigma) e^{-\sigma} (\sigma - n)^m d\sigma \tag{2.22}$$

$$+ \frac{1}{(n-1)!} \int_0^\infty \sigma^{n-k-1} P_{n,k}(\sigma) e^{-\sigma} (\sigma - n)^k \epsilon(\sigma t_0/n) d\sigma. \tag{2.23}$$

By Lemma 2.7 the terms with power in σ less than k vanish in (2.22), thus only x_k remains for this line. Line (2.23) can be rewritten as $\int_0^\infty r_{n,k}(s)\epsilon(st_0)ds$ where the function

$$r_{n,k}(s) = \frac{n^n e^{-ns}}{(n-k-1)!} s^{n-k-1} (s-1)^k \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^\ell n^{k-\ell} s^{k-\ell} \frac{(n-k-1)!}{(n-\ell-1)!}, \tag{2.24}$$

defined for $s \ge 0$ and $n > k$, converges pointwise to 0: $\lim_{n\to\infty} r_{n,k}(s) = 0$ for any $s \ge 0$. This can be seen as follows:

$$n^{k-\ell} \frac{(n-k-1)!}{(n-\ell-1)!} \le \left(\frac{n}{n-k}\right)^{k-\ell} \underset{n\to\infty}{\longrightarrow} 1, \text{ hence is bounded.} \tag{2.25}$$

Setting $n' = n - k - 1$ one has:

$$\frac{n^n}{n'!} = n^{k+1} \left(\frac{n}{n'}\right)^{n'} \frac{n'^{n'}}{n'!} \underset{n\to\infty}{\sim} \frac{(n'e)^{k+1} e^{n'}}{\sqrt{2\pi n'}} \tag{2.26}$$

thus

$$|r_{n,k}(s)| \leq C|s^2-1|^k e^{-(k+1)s} {n'}^{k+1/2} e^{n'(1-s+\log s)} \tag{2.27}$$

where the constant C depends only on k. Since $1-s+\log s<0$ for any $s \in (0,\infty)\backslash\{1\}$, one obtains the convergence for these values of s, and finally $r_{n,k}(0)=r_{n,k}(1)=0$. Moreover let $\delta \in (0,1)$: for $|s-1|>\delta$, one has $s-1-\log s>\epsilon>0$ which shows that the convergence is uniform on $(0,\infty)\backslash(1-\delta,1+\delta)$, and dominated by the integrable function $\tilde{C}|s^2-1|^k e^{-(k+1)s}$.

To complete the proof of the theorem, let p_i be any continuous seminorm on (X,τ). Since $\tau-\lim_{t\to t_0}\epsilon(t)=0$, for any $\varepsilon>0$ there exists $\delta>0$ such that $\sup_{|s-1|<\delta} p_i(\epsilon(st_0))<\varepsilon$. One has

$$\begin{aligned} p_i\left(\int_0^\infty r_{n,k}(s)\epsilon(st_0)ds\right) &\leq \int_0^\infty |r_{n,k}(s)|p_i(\epsilon(st_0))ds \\ &\leq \sup_{t>0} e^{-kt/t_0}p_i(\epsilon(t))\int_{|s-1|>\delta} e^{ks}|r_{n,k}(s)|ds \\ &\quad + \varepsilon\int_0^\infty |r_{n,k}(s)|ds, \end{aligned}$$

where $\sup_{t>0} e^{-kt/t_0}p_i(\epsilon(t)) \leq \sup_{t>0} e^{-\omega t}\|\epsilon(t)\|$ (since $\omega t_0 \leq k$) is bounded, $\int_{|s-1|>\delta} e^{ks}|r_{n,k}(s)|ds$ tends to 0 when $n\to\infty$ by Lebesgue's theorem, and $\int_0^\infty |r_{n,k}(s)|ds$ is bounded uniformly in $n>k$ by Lemma 2.8. Hence the proof is complete. ■

Theorem 2.9. *Let A be a Hille-Yosida (multi-valued) operator on a Banach space X. Let τ be another topology on X, coarser than the norm topology, such that (X,τ) is a locally convex topological vector space, which is sequentially complete on norm bounded sets. Let $S(t)$ denote the associated once integrated semigroup on X, and $x\in X$. If $t\mapsto S(t)x$ is τ-differentiable at $t_0>0$ then:*

$$\tau-\lim_{n\to\infty}(I-t_0A/n)^{-n}x=S'(t_0)x. \tag{2.28}$$

Moreover if $S(\cdot)x \in \mathcal{C}^1((0,\infty),(X,\tau))$ for all $x\in X$, then $(S'(t))_{t>0}$ is a τ-continuous semigroup, $(\lambda-A)^{-1}x=\int_0^\infty e^{-\lambda t}S'(t)xdt$ as a generalized τ-Riemann integral, and the Euler type formula (2.28) holds for the strong τ-convergence.

More generally, if A is the generator of a k-times integrated semigroup $S(t)$ in X, $\|S(t)\|\leq Me^{\omega t}$, and if $S(t)x$ is k-times τ-differentiable at $t_0>0$ for some $x\in X$, then:

$$\tau-\lim_{n\to\infty}(I-t_0A/n)^{-n}x=S^{(k)}(t_0)x. \tag{2.29}$$

Proof. A k-times integrated semigroup $S(t)$ is related to its generator A by the Laplace transform

$$(\lambda-A)^{-1}=\lambda^k\int_0^\infty e^{-\lambda t}S(t)dt. \tag{2.30}$$

Thus by Lemma 2.3 one has

$$(I - t_0A/n)^{-n}x = \frac{(n/t_0)^{k+1}}{(n-1)!}\int_0^\infty \left(\frac{nt}{t_0}\right)^{n-k-1} P_{n,k}(nt/t_0)e^{-nt/t_0}S(t)xdt.$$

Then the limit (2.29) for an exponentially bounded k-times integrated semigroup follows from Theorem 2.6, after some verifications. A k-times integrated semigroup is strongly $\|\cdot\|$-continuous, thus also strongly τ-continuous, thus one takes $x(t) = S(t)x$. If $\omega t_0 > k$, one considers the k'-times integrated semigroup $\tilde{S}(t)$, with $k' \geq \omega t_0$, obtained by integration of $S(t)$, and which is clearly k'-times τ-differentiable at x in t_0:

$$\tilde{S}(t) = \int_0^t \frac{(t-s)^{k'-k-1}}{(k'-k-1)!} S(s)ds. \tag{2.31}$$

If the Hille-Yosida operator A generates a once integrated semigroup $S(t)$ such that $S(\cdot)x \in \mathcal{C}^1((0,\infty),(X,\tau))$ for all $x \in X$, then by Proposition 2.2 $\limsup_{h\downarrow 0} \|S(t+h) - S(t)\|/h \leq Me^{\omega t}$ and the linear operators $S'(t)$ are bounded by $Me^{\omega t}$. Hence the strong τ-Riemann integral $\int_a^b e^{-\lambda s}S'(s)ds$ is well defined, and converges as $a \downarrow 0$ and $b \to \infty$ in operator norm. Then by integration by parts

$$(\lambda - A)^{-1}x = \lambda\int_0^\infty e^{-\lambda s}S(s)xds = [-e^{-\lambda s}S(s)x]_0^\infty + \int_0^\infty e^{-\lambda s}S'(s)xds \tag{2.32}$$

one obtains that the resolvent of A is the Laplace transform of $S'(t)$, and thus $S'(t)$ satisfies the semigroup equation $S'(t)S'(s) = S'(t+s)$ by [1, Proposition 2.2]. ∎

2.4. Example: bi-continuous semigroups

Let A be a Hille-Yosida operator in a Banach space X. Assuming certain properties of the topological vector space (X,τ), F. Kühnemund was able to state sufficient conditions for the τ-differentiability of the associated once integrated semigroup $F(t)$, and then construct the class of bi-continuous semigroups. The following proposition [9, Theorem 16] summarizes this theory:

Proposition 2.10. *Let A be a Hille-Yosida operator in X, and let $F(t)$ be the associated once integrated semigroup. In order that $F(\cdot)x \in \mathcal{C}^1((0,\infty),(X,\tau))$ for all $x \in X$ and $(F'(t))_{t>0}$ is locally bi-equicontinuous, it is necessary and sufficient that* $\mathrm{dom}(A)$ *is bi-dense and the family $\{(s-\alpha)^k R(s,A)^k : k \in \mathbb{N}, s \geq \alpha\}$ is bi-equicontinuous for every $\alpha > \omega$. When these conditions are satified, the operator A is said to generate a bi-continuous semigroup $(F'(t))_{t>0}$ on X.*

Other results about bi-continuous semigroups can be found in [7, 9, 10]. Theorem 2.9 applies clearly to this class of semigroups and one obtains:

Corollary 2.11. *Let $T(t)$ be a bi-continuous semigroup in X, then Euler's formula holds for the strong τ-convergence:*

$$\tau - \lim_{n\to\infty}(I - tA/n)^{-n}x = T(t)x,\ x \in X.$$

For the example given in [9, Theorem 20], one obtains Euler's formula for a jointly continuous flow in the compact open topology (a direct proof of this result can be found in [5]).

Corollary 2.12. *Let ϕ: $\mathbb{R}_+ \times \Omega \to \Omega$ be a jointly continuous flow on the topological space Ω, and let the operator A in $\mathcal{C}_b(\Omega)$ be its Lie generator. Then for each $f \in \mathcal{C}_b(\Omega)$*

$$(I + tA/n)^{-n} f \underset{n\to\infty}{\longrightarrow} f \circ \phi_t$$

uniformly on the compact subsets of Ω.

3. Holomorphic semigroups

3.1. Real characterization

This section concerns holomorphic semigroups in the extended sense (cf [2, definition 3.7.3], and [8] for the degenerate case):

Definition 3.1. A family $\{T(t)\}_{t>0} \subset \mathcal{L}(X)$ is a bounded holomorphic semigroup of semi-angle θ if $T(t)T(s) = T(t+s)$ for any $t, s > 0$, and the function $t \mapsto T(t)$ has a holomorphic extension in $S_\theta = \{z \in \mathbb{C}\backslash\{0\}, |\arg z| < \theta\}$ which is bounded in each sector S_δ, for $0 < \delta < \theta$, by a constant $M_\delta > 0$.

Such a semigroup $T(t)$ has a Laplace transform $\hat{T}(\lambda) = \int_0^\infty e^{-\lambda t}T(t)dt$ for $\operatorname{Re}\lambda > 0$, and $\hat{T}$ is a pseudo-resolvent by [1, Prop 2.2]. Thus a multivalued linear operator A is said to generate the bounded holomorphic semigroup T if $\hat{T}(\lambda) = (\lambda - A)^{-1}$, $\operatorname{Re}\lambda > 0$. Here is a real characterization of the generators of such semigroups, in fact a generalization of [13, Theorem 2.5.5] to semigroups which are not in the class C_0.

Theorem 3.2. *Let A be a multi-valued linear operator in a Banach space X, such that $(0, +\infty) \subset \rho(A)$. There exists $\theta > 0$ such that A generates a bounded holomorphic semigroup of semi-angle θ if and only if the family $\{\lambda R(\lambda, A) : \lambda > 0\}$ is uniformly power bounded and uniformly analytic, i.e.*

$$\sup_{\lambda>0, n\in\mathbb{N}} \|[\lambda R(\lambda, A)]^n\| < \infty \text{ and } \sup_{\lambda>0, n\in\mathbb{N}} n\|[\lambda R(\lambda, A)]^n(I - \lambda R(\lambda, A))\| < \infty. \tag{3.1}$$

Moreover one has the representations, for $t > 0$ and $\operatorname{Re}\lambda > 0$,

$$e^{tA} = \frac{1}{2\pi i}\int_\gamma e^{\mu t}(\mu - A)^{-1}d\mu \text{ and } (\lambda - A)^{-1} = \int_0^\infty e^{-\lambda t}e^{tA}dt, \tag{3.2}$$

where γ is a smooth curve in $S_{\pi/2+\theta}$ running from $\infty e^{-i(\pi/2+\delta)}$ to $\infty e^{i(\pi/2+\delta)}$ for some $\theta > \delta > 0$.

The following lemma is not a new result (cf e.g. [11] and the references therein), but we need a more precise statement with explicit bounds. In fact the converse of this result is also known.

Lemma 3.3. *Let $T \in \mathcal{L}(X)$ be power bounded and analytic:* $\sup_{n\in\mathbb{N}} \|T^n\| = M < \infty$, $\sup_{n\in\mathbb{N}} n\|T^n(T-I)\| = N < \infty$. *Then for* $\theta = \arcsin(\min\{1/4eN, 1\})$ *and any $\delta < \theta$, one has*

$$\sup_{z-1\in S_{\pi/2+\delta}} |z-1|\|(z-T)^{-1}\| \leq \frac{M}{\sin\epsilon} + \frac{1}{(1-4eN\sin(\delta+\epsilon))\sin\epsilon} \tag{3.3}$$

where $\epsilon \in (0, \theta-\delta)$.

Proof. First we estimate for each $t > 0$:

$$\|e^{t(T-I)}\| \leq e^{-t}\sum_{n=0}^{\infty}\frac{t^n}{n!}\|T^n\| \leq M, \tag{3.4}$$

$$\|t(T-I)e^{t(T-I)}\| \leq te^{-t}\sum_{n=0}^{\infty}\frac{t^n}{n!}\|T^n(T-I)\| \leq (1-e^{-t})2N \leq 4N. \tag{3.5}$$

Then we show that for each $\delta < \theta$, $\sup_{z\in S_\delta}\|e^{z(T-I)}\| < \infty$. By the analyticity of $z \mapsto e^{z(T-I)}$ one has (in particular for each $t > 0$):

$$\begin{aligned} e^{z(T-I)} &= \sum_{n=0}^{\infty}\frac{(z-t)^n}{n!}\frac{d^n}{dt^n}e^{t(T-I)} \\ &= e^{t(T-I)} + \sum_{n=1}^{\infty}\frac{(z-t)^n}{t^n}\frac{n^n}{n!}\left[\frac{t}{n}(T-I)e^{\frac{t}{n}(T-I)}\right]^n. \end{aligned}$$

By using (3.5) and $n^n e^{-n}/n! \leq 1$, the series is dominated by a geometric series if $4eN|z-t|/t < 1$. If $|\arg z| \leq \delta < \theta$, we set $t = |z|^2/\operatorname{Re} z$, such that $|z-t|/t \leq \sin|\arg z| \leq \sin\delta < 1/4eN$, and find the estimate

$$\|e^{z(T-I)}\| \leq M + \frac{1}{1-4eN\sin\delta} = m_\delta. \tag{3.6}$$

Then by the Laplace transform one has $\|(\lambda - e^{i\delta}(T-I))^{-1}\| \leq m_\delta/\operatorname{Re}\lambda$ for $\operatorname{Re}\lambda > 0$ and $|\delta| < \theta$, and thus for $\mu = \lambda e^{-i\delta} \in S_{\pi/2+\theta}$:

$$\|(\mu + I - T)^{-1}\| \leq \frac{m_\delta}{\operatorname{Re}(\mu e^{i\delta})}. \tag{3.7}$$

Let $\arg\mu \in (0, \pi/2+\delta)$ for some $\delta \in (0,\theta)$, one has for $\epsilon \in (0, \theta-\delta)$: $|\mu|/\operatorname{Re}(\mu e^{-i(\delta+\epsilon)}) < (\sin\epsilon)^{-1}$ and thus

$$\|(\mu + I - T)^{-1}\| \leq \frac{m_{\delta+\epsilon}}{|\mu|\sin\epsilon}. \tag{3.8}$$

This gives the announced inequality (3.3). ∎

Proof of Theorem 3.2. By Lemma 3.3, the conditions of power boundedness and analyticity of $\{\lambda R(\lambda)\}$ (uniformly in $\lambda > 0$) imply that there exists $\theta > 0$ such that for each $\delta \in (0, \theta)$:

$$\sup_{z \in 1+S_{\pi/2+\delta}} |z-1| \|(z - \lambda R(\lambda, A))^{-1}\| \leq \tilde{m}_\delta < \infty, \tag{3.9}$$

uniformly for $\lambda > 0$. The Yosida approximation of A

$$A_n = \frac{I - nR(n, A)}{1/n} \tag{3.10}$$

is known to converge in the norm resolvent sense to A (for the multi-valued case, see e.g. [14, Lemma 3.1])

$$\lim_{n\to\infty} \|(\lambda - A_n)^{-1} - (\lambda - A)^{-1}\| = 0 \tag{3.11}$$

for any $\lambda \in \rho(A)$. Let us now estimate $\|(\zeta - A_n)^{-1}\|$:

$$\|(\zeta - A_n)^{-1}\| = \|n^{-1}(\zeta/n + I - nR(n, A))^{-1}\| \leq \frac{\tilde{m}_\delta}{|\zeta|} \tag{3.12}$$

for any $\zeta \in S_{\pi/2+\delta}$ by (3.9). Since this estimate is uniform in n, one obtains $\|(\zeta - A)^{-1}\| \leq \tilde{m}_\delta/|\zeta|$ for any $\zeta \in S_{\pi/2+\delta}$. Then one can define the semigroup generated by A for $t \in S_\theta$, as

$$e^{tA} = \frac{1}{2\pi i} \int_\gamma e^{\lambda t} (\lambda - A)^{-1} d\lambda \tag{3.13}$$

where γ is a smooth curve in $S_{\pi/2+\theta}$ running from $\infty e^{-i(\pi/2+\delta)}$ to $\infty e^{i(\pi/2+\delta)}$ for some $\delta < \theta$. The boundedness, analyticity and semigroup property are verified by standard arguments (see e.g. [6, Proposition II.4.3]). For the representation of the resolvent as Laplace transform of the semigroup: let $\operatorname{Re}\lambda > 0$ and consider a curve γ such that λ lies on the right of γ, then one has

$$\begin{aligned}
\int_0^\infty e^{-\lambda t} e^{tA} dt &= \frac{1}{2\pi i} \int_0^\infty \int_\gamma e^{(z-\lambda)t} (z - A)^{-1} dt dz \\
&= \frac{1}{2\pi i} \int_\gamma (z - A)^{-1} \int_0^\infty e^{(z-\lambda)t} dz dt \\
&= \frac{1}{2\pi i} \int_\gamma (z - A)^{-1} \frac{dz}{z - \lambda} = (\lambda - A)^{-1}
\end{aligned}$$

by Fubini's theorem and by Cauchy's integral theorem, closing the curve γ by circles with increasing diameter on the right.

In order to prove the necessity of the conditions of power boundedness and analyticity, let e^{tA} be some bounded holomorphic semigroup of semi-angle θ.

By [2, Theorem 2.6.1], the Laplace transform $R(\lambda, A)$ of e^{tA} has a holomorphic extension to $S_{\pi/2+\theta}$ and satisfies the estimate $\sup_{z\in S_{\pi/2+\delta}} \|zR(z,A)\| < \infty$ for each $\delta \in (0,\theta)$. Furthermore, one has the representation (3.13). By the resolvent equation, one finds for $\lambda > 0$

$$\frac{d^n}{d\lambda^n} R(\lambda, A) = (-1)^n \int_0^\infty t^n e^{-\lambda t} e^{tA} dt = (-1)^n n! R(\lambda, A)^{n+1}. \tag{3.14}$$

Thus, for $\lambda > 0$,

$$\|[\lambda R(\lambda, A)]^n\| \le \frac{\lambda^n}{(n-1)!} \int_0^\infty t^{n-1} e^{-\lambda t} \|e^{tA}\| dt \le \sup_{t>0} \|e^{tA}\| \tag{3.15}$$

which shows that $\{\lambda R(\lambda, A)\}_{\lambda>0}$ is uniformly power bounded. Furthermore,

$$\begin{aligned} [\lambda R(\lambda, A)]^n (I - \lambda R(\lambda, A)) &= \frac{\lambda^n}{(n-1)!} \int_0^\infty \left(1 - \frac{\lambda t}{n}\right) t^{n-1} e^{-\lambda t} e^{tA} dt \\ &= \frac{\lambda^n}{n!} \int_0^\infty t^n e^{-\lambda t} \left(\frac{d}{dt} e^{tA}\right) dt \end{aligned}$$

by integration by parts, using $(t^n e^{-\lambda t})' = (nt^{n-1} - \lambda t^n) e^{-\lambda t}$. Since

$$\begin{aligned} t \left\| \frac{d}{dt} e^{tA} \right\| &\le \frac{1}{2\pi} \int_\gamma |e^{zt}| \|z(z-A)^{-1}\| |tdz| \\ &\le \frac{1}{2\pi} \sup_{z\in S_{\pi/2+\delta}} \|z(z-A)^{-1}\| \int_{t\gamma} e^{\operatorname{Re} u} |du| \end{aligned} \tag{3.16}$$

for any curve $\gamma \subset S_{\pi/2+\theta}$ running from $\infty e^{-i(\pi/2+\delta)}$ to $\infty e^{i(\pi/2+\delta)}$, one can replace γ by γ/t and conclude that $\sup_{t>0} t \|\frac{d}{dt} e^{tA}\|$ is finite. Thus

$$\|[\lambda R(\lambda, A)]^n (I - \lambda R(\lambda, A))\| \le \frac{1}{n} \sup_{t>0} t \|\frac{d}{dt} e^{tA}\| \tag{3.17}$$

is bounded, which means that $\{\lambda R(\lambda, A)\}_{\lambda>0}$ is uniformly analytic. ∎

Corollary 3.4. *Let A be a multi-valued linear operator in a Banach space X, such that $(0,+\infty) \subset \rho(A)$. Consider for each $\delta > 0$*

$$\sup_{\lambda\in S_\delta, n\in\mathbb{N}} \|[\lambda R(\lambda, A)]^n\| = \tilde{M}_\delta \tag{3.18}$$

$$\sup_{\lambda\in S_\delta, n\in\mathbb{N}} n\|[\lambda R(\lambda, A)]^n (I - \lambda R(\lambda, A))\| = \tilde{N}_\delta. \tag{3.19}$$

The operator A generates a bounded holomorphic semigroup of semi-angle θ if and only if $\tilde{M}_\delta < \infty$ and $\tilde{N}_\delta < \infty$ for each $\delta \in (0,\theta)$. Furthermore for each

$\delta \in (0,\theta)$ one has

$$\tilde{M}_\delta \;\leq\; M_\delta = \sup_{t\in S_\delta} \|e^{tA}\| < \infty \tag{3.20}$$

$$\tilde{N}_\delta \;\leq\; N_\delta = \sup_{t\in S_\delta} t \left\| \frac{d}{dt} e^{tA} \right\| < \infty. \tag{3.21}$$

Remark. These inequalities extend to $\delta = 0$ by replacing S_δ by $(0,\infty)$ in the suprema. Since $\|nR(n,A)x - x\| \to 0$ for each $x \in \overline{\mathrm{dom}(A)}$ [14, Lemma 2.5], one has also $\tilde{M}_\delta \geq \tilde{M}_0 \geq 1$, and thus $M_\delta \geq M_0 \geq 1$.

3.2. Euler's Formula

The operator-norm convergence of Euler's formula has been established in [4, Theorem 5.1] (with the error bound $\mathcal{O}(\ln n/n)$) for m-sectorial generators in a Hilbert space. Then this convergence has been extended in [3, Corollary 1.3] to any bounded holomorphic C_0 semigroup on a Banach space (without error bound). It will be shown here that the operator-norm convergence holds for any bounded holomorphic semigroup (in the sense of definition 3.1) with the error bound estimate $\mathcal{O}(\ln n/n)$.

Theorem 3.5. *Let $\{R(\lambda)\}_{\lambda\in\Omega}$ be a pseudo-resolvent family in $\Omega \supset (0,\infty)$, such that $\{\lambda R(\lambda)\}_{\lambda>0}$ is uniformly power bounded and uniformly analytic in $(0,\infty)$. Then the sequence $[(n/t)R(n/t)]^n$ converges in operator norm to a bounded holomorphic semigroup, with the error bound estimate $\mathcal{O}(\ln n/n)$ uniformly in $t > 0$.*

Corollary 3.6. *Let A generate a bounded holomorphic semigroup in X, then Euler's formula holds in operator norm with the estimate:*

$$\|(I - tA/n)^{-n} - e^{tA}\| \leq \mathcal{O}(\ln n/n).$$

uniformly for $t > 0$.

The main estimate is similar to [4], but the case of a multi-valued operator A requires a careful treatment, and A is not necessarily boundedly invertible. For that reason, the preliminary estimates must be improved by the following lemma.

Lemma 3.7. *Let A generate a bounded holomorphic semigroup of semi-angle θ in X. Then one has, uniformly for $\mu > 0$*

$$\|(\mu - A)^{-1}(e^{tA} - (I - tA)^{-1})\| \;\leq\; L_1 t \quad (t > 0), \tag{3.22}$$

$$\|(\mu - A)^{-1}(e^{tA} - (I - tA)^{-1})(\mu - A)^{-1}\| \;\leq\; L_2 t^2 \quad (t > 0). \tag{3.23}$$

Proof. One sets for simplicity $R(\lambda) = (\lambda - A)^{-1}$ for any $\lambda \in \rho(A)$ and first proves for $\mu > 0$ that $\lim_{t\downarrow 0} \|R(\mu)(e^{tA} - I)\| = 0$. Let γ be a smooth curve as in (3.13), such that μ lies on the left of γ. By Cauchy's theorem, closing the curve γ on the left by increasing circles,

$$\frac{1}{2\pi i}\int_\gamma \frac{e^{(\lambda-\mu)t}}{\lambda-\mu}d\lambda = 1.$$

Hence

$$\begin{aligned} R(\mu)(e^{tA}-I) &= \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\left(R(\lambda)R(\mu) - \frac{e^{-\mu t}}{\lambda-\mu}R(\mu)\right)d\lambda \\ &= \frac{1}{2\pi i}\int_\gamma \frac{e^{\lambda t}}{\lambda-\mu}\left((1-e^{-\mu t})R(\mu) - R(\lambda)\right)d\lambda \\ &= (e^{\mu t}-1)R(\mu) - \frac{1}{2\pi i}\int_\gamma \frac{e^{\lambda t}}{\lambda-\mu}R(\lambda)d\lambda. \end{aligned}$$

The first term tends clearly to 0 as $t\downarrow 0$ and the last term also by Lebesgue's theorem, because the integrand is dominated by $c|(\lambda-\mu)\lambda|^{-1}\sup_{z\in S_{\pi/2+\delta}}$ $\|zR(z)\|$ (for some $c>0$) and

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{\lambda-\mu}R(\lambda)d\lambda = 0$$

by Cauchy's theorem, closing the curve γ on the right by increasing circles. Then in the operator norm topology,

$$\lim_{\varepsilon\downarrow 0}\int_\varepsilon^t \frac{d}{ds}R(\mu)e^{sA}ds = \lim_{\varepsilon\downarrow 0}R(\mu)(e^{tA}-e^{\varepsilon A}) = R(\mu)(e^{tA}-I). \tag{3.24}$$

Let us now find another expression for $R(\mu)de^{tA}/dt$:

$$\begin{aligned} R(\mu)\left(\mu e^{tA} - \frac{d}{dt}e^{tA}\right) &= \frac{1}{2\pi i}\int_\gamma e^{\lambda t}(\mu-\lambda)R(\mu)R(\lambda)d\lambda \\ &= \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\left(R(\lambda)-R(\mu)\right)d\lambda = e^{tA}, \end{aligned} \tag{3.25}$$

where one uses the resolvent equation and the fact that $\int_\gamma e^{\lambda t}d\lambda = 0$ by Cauchy's theorem (closing the curve γ on the left). It follows that

$$\|R(\mu)(e^{tA}-I)\| = \left\|\int_0^t (\mu R(\mu)-I)e^{sA}ds\right\| \le (1+M_0)M_0 t. \tag{3.26}$$

By the resolvent equation (for $t\neq \mu^{-1}$):

$$R(\mu)[(I-tA)^{-1}-I] = \frac{t}{\mu t-1}[t^{-1}R(t^{-1}) - \mu R(\mu)]. \tag{3.27}$$

Let us consider the holomorphic function $F(x) = (I - xA)^{-1} = x^{-1}R(x^{-1})$ for $\mathrm{Re}\, x > 0$. One finds $\sup_{x>0} \|F(x)\| \le M_0$, $F'(x) = x^{-2}[x^{-1}R(x^{-1}) - I]R(x^{-1})$, and then $\sup_{x>0} \|xF'(x)\| \le M_0(1 + M_0)$. For any $\mu > 0$, this gives the estimates:

$$\begin{aligned}
\sup_{t>0, |\mu t-1|>1/2} \|(\mu t - 1)^{-1}[F(t) - F(\mu^{-1})]\| &\le 4M_0 && (3.28)\\
\sup_{t>0, |\mu t-1|<1/2} \|(\mu t - 1)^{-1}[F(t) - F(\mu^{-1})]\| &\le \sup_{\frac{1}{2\mu}<x<\frac{3}{2\mu}} \mu^{-1}\|F'(x)\| && (3.29)\\
&\le \sup_{\frac{1}{2\mu}<x<\frac{3}{2\mu}} (x\mu)^{-1}\|xF'(x)\| \\
&\le 2\sup_{x>0} \|xF'(x)\| \le 2M_0(1 + M_0).
\end{aligned}$$

Thus one obtains $\|R(\mu)[(I - tA)^{-1} - I]\| \le 2M_0(1 + M_0)t$ and finally (3.22), where $L_1 = 3(1 + M_0)M_0$ is independent of μ.

In order to verify the estimate (3.23), one considers two parts: $D_1 = R(\mu)\left[R(\mu)(e^{tA} - I) - t(\mu R(\mu) - I)\right]$ and $D_2 = R(\mu)\left[R(\mu)((I - tA)^{-1} - I) - t(\mu R(\mu) - I)\right]$. For the first part, one has by using (3.26) twice:

$$D_1 = R(\mu)\int_0^t (\mu R(\mu) - I)(e^{sA} - I)ds = (\mu R(\mu) - I)^2 \int_0^t \int_0^s e^{-s_1 A} ds ds_1,$$

which leads to $\|D_1\| \le (1+M_0)^2 M_0 t^2/2$. For the second part, by the resolvent equation (for $t \ne \mu^{-1}$):

$$\begin{aligned}
D_2 &= \frac{tR(\mu)}{\mu t - 1}\left[t^{-1}R(t^{-1}) - I - \mu t(\mu R(\mu) - I)\right]\\
&= \frac{t^2}{(\mu t - 1)^2}\left[t^{-1}R(t^{-1}) - \mu R(\mu) - (t - \mu^{-1})\mu^2 R(\mu)(\mu R(\mu) - I)\right]\\
&= \frac{t^2}{(\mu t - 1)^2}\left[F(t) - F(\mu^{-1}) - (t - \mu^{-1})F'(\mu^{-1})\right].
\end{aligned}$$

One has

$$\sup_{|\mu t-1|>1/2} t^{-2}\|D_2\| \le 8M_0 + 2M_0(1 + M_0), \tag{3.30}$$

and with $F''(x) = 2x^{-3}R(x^{-1})(I - x^{-1}R(x^{-1}))^2$:

$$\begin{aligned}
\sup_{|\mu t-1|<1/2} t^{-2}\|D_2\| &\le \sup_{\frac{1}{2\mu}<x<\frac{3}{2\mu}} \mu^{-2}\|F''(x)\| \le \sup_{\frac{1}{2\mu}<x<\frac{3}{2\mu}} (x\mu)^{-2}\|x^2F''(x)\|\\
&\le 4\sup_{x>0}\|x^2F''(x)\| \le 8M_0(1 + M_0)^2. && (3.31)
\end{aligned}$$

Thus $\|D_1 - D_2\| \le \mathcal{O}(t^2)$, which gives (3.23) with $L_2 = 9M_0(1 + M_0)^2$. ∎

Proof of Theorem 3.5. By Theorem 3.2 the (multi-valued) operator $A = \lambda - R(\lambda)^{-1}$ generates a bounded holomorphic semigroup e^{tA}. Then

$$\begin{aligned}
[(n/t)R(n/t)]^n - e^{tA} &= \sum_{k=0}^{n-1}[(n/t)R(n/t)]^{n-k-1}[(n/t)R(n/t) - e^{tA/n}]e^{ktA/n} \\
&= [(n/t)R(n/t)]^{n-1}[(n/t)R(n/t) - e^{tA/n}] \\
&\quad + \sum_{k=1}^{n-2}[(n/t)R(n/t)]^{n-k-1}[(n/t)R(n/t) - e^{tA/n}]e^{ktA/n} \\
&\quad +[(n/t)R(n/t) - e^{tA/n}]e^{(n-1)tA/n}.
\end{aligned}$$

By (3.25) one has

$$e^{ktA/n} = R(\mu)\left(-\frac{n}{k}\frac{d}{dt}e^{ktA/n} + \mu e^{ktA/n}\right), \tag{3.32}$$

and by (3.27)

$$\begin{aligned}
[(n/t)R(n/t)]^{n-k-1} &= \frac{n}{t}(\mu t/n - 1)[(n/t)R(n/t)]^{n-k-2}[(n/t)R(n/t) - I]R(\mu) \\
&\quad -\mu[(n/t)R(n/t)]^{n-k-2}R(\mu).
\end{aligned} \tag{3.33}$$

Thus by Lemma 3.7, one obtains the estimates

$$\begin{aligned}
&\|[(n/t)R(n/t)]^{n-1}[(n/t)R(n/t) - e^{tA/n}]\| \\
&\leq \|\frac{n}{t}(\mu t/n - 1)[(n/t)R(n/t)]^{n-2}[(n/t)R(n/t) - I]\| \\
&\quad \times \|R(\mu)[e^{tA/n} - (n/t)R(n/t)]\| \\
&\quad + \mu\|[(n/t)R(n/t)]^{n-2}\|\|R(\mu)[e^{tA/n} - (n/t)R(n/t)]\| \\
&\leq |\mu t/n - 1|\frac{N_0 L_1}{n-2} + \frac{\mu t}{n}M_0 L_1,
\end{aligned} \tag{3.34}$$

$$\begin{aligned}
&\|[(n/t)R(n/t) - e^{tA/n}]e^{(n-1)tA/n}\| \\
&\leq \|[(n/t)R(n/t) - e^{tA/n}]R(\mu)\|\|\frac{n}{n-1}\frac{d}{dt}e^{(n-1)tA/n} - \mu e^{(n-1)tA/n}\| \\
&\leq \frac{L_1 N_0}{n-1} + \frac{\mu t}{n}L_1 M_0,
\end{aligned} \tag{3.35}$$

$$\begin{aligned}
&\|[(n/t)R(n/t)]^{n-k-1}\left[(n/t)R(n/t) - e^{tA/n}\right]e^{ktA/n}\| \\
&\leq \left\|\Big(\frac{n}{t}(\mu t/n - 1)[(n/t)R(n/t)]^{n-k-2}[(n/t)R(n/t) - I] \right. \\
&\qquad \left. - \mu[(n/t)R(n/t)]^{n-k-2}\Big)\right\|
\end{aligned}$$

$$
\begin{aligned}
&\times \|R(\mu)[(n/t)R(n/t) - e^{tA/n}]R(\mu)\| \left\| \left(-\frac{n}{k}\frac{d}{dt}e^{ktA/n} + \mu e^{ktA/n} \right) \right\| \\
\leq{}& |\mu t/n - 1| \frac{2N_0^2 L_2}{(n-k-1)k} + L_2 M_0 N_0 \frac{\mu t}{n} \left(2\frac{|\mu t/n - 1|}{n-k-1} + \frac{1}{k} \right) \\
&+ L_2 M_0^2 \frac{(\mu t)^2}{n^2},
\end{aligned}
\tag{3.36}
$$

for any $\mu > 0$. Since the left hand side of these inequalities does not depend on μ, one can take $\mu \downarrow 0$ in the right hand side. Then one finds the announced estimate

$$
\|[(n/t)R(n,t)]^n - e^{tA}\| \leq \frac{L_1 N_0}{n-2} + \frac{L_1 N_0}{n-1} + \sum_{k=1}^{n-2} \frac{2N_0^2 L_2}{(n-k-1)k} \leq \mathcal{O}(\ln n/n), \tag{3.37}
$$

by observing that

$$
\sum_{k=1}^{n-2} \frac{1}{(n-k-1)k} \leq 4\frac{\ln n}{n}. \tag{3.38}
$$

■

Corollary 3.8. *Let A generate a bounded holomorphic semigroup of semi-angle θ, or equivalently $\tilde{M}_\delta < \infty$ and $\tilde{N}_\delta < \infty$ for each $\delta \in (0,\theta)$. Then the sequence of bounded holomorphic functions $[(n/t)R(n/t,A)]^n$ converges in operator norm to e^{tA} for any $t \in S_\theta$ with the error bound estimate $\mathcal{O}(\ln n/n)$ uniform on each sector S_δ, $\delta \in (0,\theta)$. Moreover the inequalities (3.20) and (3.21) are in fact equalities: for each $\delta \in (0,\theta)$,*

$$
\sup_{\lambda \in S_\delta, n \in \mathbb{N}} \|[\lambda R(\lambda)]^n\| = \sup_{t \in S_\delta} \|e^{tA}\| \tag{3.39}
$$

$$
\sup_{\lambda \in S_\delta, n \in \mathbb{N}} n\|[\lambda R(\lambda)]^n (I - \lambda R(\lambda))\| = \sup_{t \in S_\delta} t\|\frac{d}{dt}e^{tA}\|. \tag{3.40}
$$

Proof. The convergence (with the error bound estimate) of $F(t/n)^n = [(n/t)R(n/t,A)]^n$ on the half-line $\{t = re^{i\delta} : r > 0\}$ follows from Theorem 3.5 for $e^{-i\delta}A$, i.e. for the pseudo-resolvent family $R(\lambda e^{-i\delta})$, where the constants M_0, N_0 are just replaced by M_δ, N_δ. Since $e^{tA} = \lim_{n\to\infty}[(n/t)R(n/t)]^n$ in operator norm for any $t \in S_\delta$, one has

$$
\|e^{tA}\| = \lim_{n\to\infty} \|[(n/t)R(n/t)]^n\| \leq \sup_{\lambda \in S_\delta, n \in \mathbb{N}} \|[\lambda R(\lambda)]^n\| \tag{3.41}
$$

and one obtains (3.39). By a classical result on holomorphic functions, one has the convergence of all derivatives of $F(t/n)^n$ to the corresponding derivatives of e^{tA}, uniformly on the compact subsets of S_θ. In particular

$$
\frac{d}{dt}e^{tA} = \lim_{n\to\infty} \frac{d}{dt}[F(t/n)^n] = \lim_{n\to\infty} \frac{n}{t}(F(t/n) - I)F(t/n)^n \tag{3.42}
$$

which leads to $N_\delta \leq \tilde{N}_\delta$ and thus to (3.40). ■

3.3. Concluding remark

One would naturally expect the error bound estimate $\mathcal{O}(1/n)$ instead of $\mathcal{O}(\ln n/n)$ in Theorem 3.5. This is actually true, by an induction argument of V. Paulauskas [12]. From the same preliminary estimates, this new method allows to skip the factor $\ln n$ in the final estimate, and thus to obtain the optimal one. The author is grateful to V. Paulauskas for the early communication of his work.

References

[1] Arendt, W., *Vector-valued Laplace transforms and Cauchy problems*, Israel J. Math. **59** (1987), 327–352.

[2] Arendt, W., C. Batty, M. Hieber, and F. Neubrander, "Vector-valued Laplace transforms and Cauchy problems", Monographs in Mathematics, Vol. 96, Birkhäuser Verlag, Basel, 2001.

[3] Blunck, S., *Some remarks on operator-norm convergence of the Trotter product formula on Banach spaces*, J. Funct. Anal. **195**(2) (2002), 350–370.

[4] Cachia, V. and V. Zagrebnov, *Operator-norm approximation of semigroups by quasi-sectorial contractions*, J. Funct. Anal. **180** (2001), 176–194.

[5] Dorroh, J. R. and J. W. Neuberger, *A theory of strongly continuous semigroups in terms of Lie generators*, J. Funct. Anal. **136**(1) (1996), 114–126.

[6] Engel, K.-J. and R. Nagel, "One-Parameter Semigroups for Linear Evolution Equations", Graduate Texts in Math., Vol. 194, Springer Verlag, New-York, 2000.

[7] Farkas, B., *Perturbations of bi-continuous semigroups*, Tübinger Berichte zur Funktionalanalysis **11** (2001/2002), 97–113.

[8] Favini, A. and M. Fuhrman, *Approximation results for semigroups generated by multivalued linear operators and applications*, Differential and Integral Equations **11**(5) (1998), 781–805.

[9] Kühnemund, F., *A Hille-Yosida theorem for bi-continuous semigroups*, Semigroup Forum **67** (2003), 205–225.

[10] Kühnemund, F., *Approximation of bi-continuous semigroups*, Tübinger Berichte zur Funktionalanalysis **10** (2000/2001), 100–117.

[11] Nagy, B. and J. Zemánek, *A resolvent condition implying powerboundedness*, Studia Math. **134**(2) (1999), 143–151.

[12] Paulauskas, V., *An operator-norm approximation of some semigroups by quasi-sectorial operators*, to appear in J. Funct. Anal., 2003.

[13] Pazy, A., "Semigroups of Linear Operators and Applications to Partial Differential Equations", Applied Mathematical Sciences, Vol. 44, Springer Verlag, New York, 1983.

[14] Yagi, A., *Generation theorem of semigroup for multivalued linear operators*, Osaka J. Math. **28** (1991), 385–410.

Department of Theoretical Physics
quai Ernest-Ansermet 24
CH-1211 Geneva 4, Switzerland
Vincent.Cachia@physics.unige.ch

Received November 14, 2002
and in final form March 14, 2003
Online publication August 15, 2003