A mould-theoretic perspective on Kashiwara-Vergne theory

Raphael, Elise

This publication URL: https://archive-ouverte.unige.ch/unige:127774
Publication DOI: 10.13097/archive-ouverte/unige:127774

UNIVERSITÉ DE GENÈVE
Section de mathématiques

FACULTÉ DES SCIENCES
Anton Alekseev

A Mould-Theoretic Perspective on Kashiwara-Vergne Theory.

THÈSE

présentée à la Faculté des Sciences de l'Université de Genève
pour obtenir le grade de Docteur ès Sciences, mention Mathématiques

par
Elise Raphael
de
Nancy (France)

Thèse N°5391

Genève
Atelier d'impression ReproMail
2019

UNIVERSITÉ DE GENÈVE

FACULTÉ DES SCIENCES

DOCTORAT ÈS SCIENCES, MENTION MATHEMATIQUES

Thèse de Madame Elise RAPHAEL

intitulée :

«A Mould-Theoretic Perspective on Kashiwara-Vergne Theory»

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N.B. - La thèse doit porter la déclaration précédente et remplir les conditions énumérées dans les "Informations relatives aux thèses de doctorat à l'Université de Genève".

Résumé

Cette thèse étudie les applications de techniques provenant des moules à la théorie de Kashiwara-Vergne. Développée par Jean Ecalle, la théorie des moules s'est avérée particulièrement adaptée à l'étude des Valeurs Zeta Multiples. L'isomorphisme conjectural entre les algèbres de Lie de double shuffle et de Kashiwara-Vergne a naturellement amené à transférer les techniques mouliennes utilisées d'un espace à l'autre.

• Le premier chapitre est consacré à la présentation des trois espaces jouant un rôle dans cette thèse : l'algèbre de Lie de double shuffle $\mathfrak{ds}$ provenant de la théorie des nombres, l'algèbre de Lie de Grothendieck-Teichmüller $\mathfrak{grt}_1$ liée à la topologie et l'algèbre de Lie de Kashiwara-Vergne .

• Le second chapitre constitue une introduction aux moules, avec des examples provenant de $\mathfrak{ds}$ and $\mathfrak{krv}$. Il y est ensuite expliqué comment traduire les propriétés définissant ces espaces dans ce nouveau language. Nous suivons ici les travaux de Leila Schneps. Le premier résultat de cette thèse est la définition d'une version linéarisée de l'algèbre de Lie de Kashiwara-Vergne $\mathfrak{lkv}$, analogue à l'algèbre de Lie $\mathfrak{ls}$, ainsi qu'une injection $\mathfrak{ls} \hookrightarrow \mathfrak{lkv}$.
Cela nous permet de montrer que les parties de profondeurs $d = 1, 2, 3$ de ces espaces sont isomorphes pour tous degrés n, ce qui fournit les dimensions des parties bigraduées de $\mathfrak{lkv}$ et $gr\mathfrak{krv}$ en profondeurs $1, 2, 3$ pour tout degré, ces dimensions étant connues dans le cas de $\mathfrak{ls}$.

• Le troisième et dernier chapitre est dédié à la version elliptique de $\mathfrak{krv}$ et contient les principaux résultats de cette thèse.
Nous définissons la version elliptique $\mathfrak{krv}_{ell}$ comme un sous-espace des dérivations de l'algèbre de Lie libre à deux générateurs, et nous prouvons qu'il est fermé pour le crochet de Lie des dérivations. On définit également un morphisme de Lie injectif $\mathfrak{krv} \hookrightarrow \mathfrak{krv}_{ell}$ analogue à celui $\mathfrak{grt} \hookrightarrow \mathfrak{grt}_{ell}$ et à l'application définie dans les moules $\mathfrak{ds} \hookrightarrow \mathfrak{ds}_{ell}$, ainsi qu'une application injective $\mathfrak{ds}_{ell} \hookrightarrow \mathfrak{krv}_{ell}$.

Enfin, nous démontrons que l'algèbre de Lie $\mathfrak{krv}^{1,1}$ définie indépendamment pat Alekseev-Kawazumi-Kuno-Naef, est égale à $\mathfrak{krv}_{ell}$.

Summary

This thesis studies the applications of mould techniques to Kashiwara-Vergne theory. Developed by Jean Ecalle, originally to the purpose of resurgence theory, mould theory proved to be particularly well suited to the study of Multiple Zeta Values. The conjectural isomorphism between the double shuffle and Kashiwara-Vergne Lie algebras naturally led to the transfer of mould techniques from one space to the other.

• The first chapter is dedicated to presenting the three main spaces: the double shuffle Lie algebra $\mathfrak{ds}$ from number theory, the Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}_1$ related to topology and the Kashiwara-Vergne Lie algebra from Lie theory.

• The second chapter introduces mould theory, using examples linked to $\mathfrak{ds}$ and $\mathfrak{krv}$. It then explains how to translate the defining properties of these spaces into mouls, following Schneps' works. The first result of this thesis is the definition of a linearized version of the Kashiwara-Vergne Lie algebra $\mathfrak{lkv}$ analoguous to the existing $\mathfrak{ls}$, together with an injection $\mathfrak{lds} \hookrightarrow \mathfrak{lkv}$.
It allows us to show that the parts of these spaces of depths $d = 1, 2, 3$ are isomorphic for all weights n, which yields the dimensions of the bigraded parts of $\mathfrak{lkv}$ and $gr\mathfrak{krv}$ of depths $1, 2, 3$ in all weights, since these dimensions are well-known for $\mathfrak{ls}$.

• The third and last chapter is dedicated to the elliptic versions of $\mathfrak{ds}$ and $\mathfrak{krv}$ and contains the main results of this thesis.
We define the elliptic version $\mathfrak{krv}_{ell}$ as a subspace of derivations of the free Lie algebra on two generators, and prove that it is closed under the Lie bracket of derivations.
We also define an injective Lie morphism $\mathfrak{krv} \hookrightarrow \mathfrak{krv}_{ell}$ in analogy with the section map $\mathfrak{grt} \hookrightarrow \mathfrak{grt}_{ell}$ and the mould-theoretic double shuffle map $\mathfrak{ds} \hookrightarrow \mathfrak{ds}_{ell}$, as well as an injective map $\mathfrak{ds}_{ell} \hookrightarrow \mathfrak{krv}_{ell}$.
Finally, we show that the Lie algebra $\mathfrak{krv}^{1,1}$ independently defined by Alekseev-Kawazumi-Kuno-Naef is equal to $\mathfrak{krv}_{ell}$.

Remerciements

Je voudrais remercier mon directeur de thèse, Anton Alekseev, tout d'abord d'avoir cru en ma capacité à entreprendre cette thèse, mais surtout pour son infinie patience, ses encouragements, la qualité de ses conseils et de son suivi. Les rencontres effectuées ces dernières années (souvent grâce à son intermédiaire) m'ont permis de réaliser que peu de doctorants ont la chance d'être encadrés d'une telle manière.

Merci infiniment à Leila de m'avoir accueillie à Paris puis à Cambridge et fait découvrir le monde complexe et fascinant des moules. Cette collaboration m'a énormément apporté, et ton enthousiasme est contagieux. Merci également d'avoir accepté de faire partie du jury.

Thank you Pavol for accepting to be part of my jury, but above all for sharing with me your oh-so precious sense of humour. I keep some of your emails dearly (and not only for blackmail purposes, I promise).

To Flo and Matteo, my closest academic brothers: I feel incredibly lucky to have shared a research topic, an office and amazing trips with you guys. Thanks for the shared laughs and pre-talk anxieties, but also for the non-judgemental answers to the most stupid math questions I ever dared to voice. You both showed me that being a brilliant researcher doesn't always come with self-confidence (did I mention you guys are brilliant?).

These five years would not have been the same without the Battelle crew. What our big office lacked in calm and quietness, it made up for in laughter and kindness.

Merci à mes cobureaux de ces dernières années, Alan et Solenn, pour les discussions sans queue ni tête que nous avons pu avoir au numéro 22.

Sans Mounir et Anthony, je n'aurais sans doute jamais commencé de thèse : merci de m'avoir encouragée à me lancer dans l'aventure, et puis pour tout le reste.

A ma femme qui se reconnaîtra : la fin n'est que le début, non? Mais je suis heureuse que nous ayions partagé les joies et les déboires de la thèse ensemble. Merci.

Au début de cette thèse, je pouvais encore faire des groupes d'amis avec intersection vide : ce temps est révolu, peu d'entre vous n'ont pas encore fini sur un mur d'escalade (mais vous ne perdez rien pour attendre)... Un grand merci pêle-mêle aux matheux et/ou grimpeurs Ben, Charlotte, Marie, Maxime, Seb, Thibault.Sans vous, j'aurais coulé maintes fois.

Merci aussi à Zoé d'avoir été aux petits soins pour le zombie que j'ai été ces derniers temps!

Durant cette thèse, j'ai également commencé à travailler pour SwissMAP et continuerai avec grand plaisir à le faire : merci à Ana, Isabelle, Mayra, Patricia d'avoir supporté mon intermittence, à Joana et Shaula, ainsi qu'à Paul.

A mes parents et ma soeurette : bien sûr, sans vous rien de possible. Merci de m'avoir accompagnée, supportée et soutenue dans les moments difficiles, sans jamais ne me mettre de pression.

Anne, merci mille fois pour ton indéfectible soutien à distance.

Matt, this could never have be done without you. Thank you to the moon and back. What I owe you can't be repaid, but I'll make sure I get plenty of time to try.

A mon grand-père Alain.

Contents

Chapter I

Introduction to the different Lie algebras

This chapter is devoted to the introduction of the three main objects studied in this thesis.
In the first section, we state the main definitions, results and conjectures leading to the construction of the double shuffle Lie algebra and its linearized version. In the second section, we give some background on the origin of the Grothendieck-Teichmüller groups, then focus on the prounipotent framework to introduce $\mathfrak{grt}_1$. The third section is devoted to the Kashiwara-Vergne conjecture. After detailing the original analytic story, we state the main definitions and results from [AT], culminating in the definition of the Kashiwara-Vergne Lie algebra. Finally, we quickly recall the existing ties between the main protagonists.

I.1 The double shuffle Lie algebra $\mathfrak{ds}$

Much of this introduction to MZVs is inspired by the excellent *Multiple zeta values : from numbers to motives* by José Ignacio Burgos Gil and Javier Fresán ([BF]), which was my own introduction to the topic.
The aim of this section is to be able to define the Lie algebra of Doubles mélanges et régularisations $\mathfrak{dmr}_0$, also called formal double shuffle Lie algebra $\mathfrak{ds}$. Therefore, some important definitions, results, proofs or conjectures might be missing if we did not feel they were essential for the understanding.

I.1.1 Riemann Zeta values

Definition I.1.1. The **Riemann Zeta values** are the values taken by the Riemann Zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ on positive integers equal or greater than 2.

Theorem I.1.2. *(Euler, 1735.) The values of the zeta function at even positive integers are given by*

$$\zeta(2k) = (-1)^{k-1} \frac{2\pi^{2k}}{2(2k)!} B_{2k},$$

where B_{2k} are the Bernouilli numbers defined by their generating function

$$\sum_{k\geq 0}^{\infty} B_k \frac{x^k}{k!} = \frac{x}{e^x - 1}, \quad |x| < 2\pi.$$

Quid of odd positive integers ? No such closed formulas are known. Actually, not much is known:

- In 1979, Apéry ([Ap]) proved the irrationality of $\zeta(3)$. Other proofs were found later, but none of them seem to generalize to other odd numbers.
- In 2001, Rivoal and Ball ([BR]) showed that infinitely many of the $\zeta(2n+1)$ are irrational and even linearly independent over $\mathbb{Q}$.
- Zudilin showed in 2001 ([Zu]) that amongst $\zeta(3), \zeta(5), \zeta(7)$ and $\zeta(9)$ at least one is irrational.
- One of the latest result (2018) by Rivoal and Zudilin ([RZu]) : there exist at least two irrational numbers amongst the 33 odd zeta values $\zeta(5), \zeta(7), ..., \zeta(69)$

The most important conjecture about zeta values is known as the transcendence conjecture :

Conjecture I.1.3. *The numbers $\pi, \zeta(3), .., \zeta(2k+1)$ are algebraically independent i.e.*

$$\forall k \geq 0,\ \forall P \in Z[x_0, ..., x_k] \quad P(\pi, \zeta(3), ..., \zeta(2k+1)) \neq 0.$$

As the aforementioned results infer, the actual research is far from proving this conjecture.

It is rather natural to ask ourselves what happens if we multiply two such objects. Let us have a look:

$$\begin{aligned}
\zeta(s_1) \cdot \zeta(s_2) &= (\sum_{n_1=1}^{\infty} \frac{1}{n_1^{s_1}})(\sum_{n_2=1}^{\infty} \frac{1}{n_2^{s_2}}) \\
&= \sum_{n_1,n_2=1}^{\infty} \frac{1}{n_1^{s_1} n_2^{s_2}} \\
&= \sum_{n_2>n_1\geq 1}^{\infty} \frac{1}{n_1^{s_1} n_2^{s_2}} + \sum_{n_1>n_2\geq 1}^{\infty} \frac{1}{n_1^{s_1} n_2^{s_2}} + \sum_{n=n_1=n_2\geq 1}^{\infty} \frac{1}{n^{s_1+s_2}}.
\end{aligned}$$

The last term is simply $\zeta(s_1 + s_2)$. The two other terms are a generalization of Riemann zeta values : double zeta values. This leads us to defining multiple zeta values.

I.1.2 Multiple zeta values

Definition I.1.4. A multi-index $\mathbf{s} = (s_1, .., s_l) \in \mathbb{Z}^l$ is said to be positive if $s_i \geq 1$ for all i and admissible if it is positive and $s_1 \geq 2$.

Proposition I.1.5. *If s is admissible, then*

$$\zeta(\mathbf{s}) = \sum_{n_1 > ... > n_l \geq 1} \frac{1}{n_1^{s_1} ... n_l^{s_l}},$$

$n_i \in \mathbb{N}$ is absolutely convergent.

Proof. We have

$$\zeta(\mathbf{s}) \leq \zeta(2, 1, ...1) = \sum_{n_1 > ... > n_l \geq 1} \frac{1}{n_1^2 n_2 ... n_l}.$$

Therefore,

$$\zeta(\mathbf{s}) \leq \sum_{n_1 \geq 1} \frac{1}{n_1^2} \Big(\sum_{k=1}^{n_1} \frac{1}{k} \Big)^{l-1} \leq \sum_{n \geq 1} \frac{(1 + \log(n))^{l-1}}{n^2},$$

and the latter converges since $(1 + \log(n))^{l-1} n^{-2} < n^{-3/2}$ for n sufficiently big. □

Definition I.1.6. The **multiple zeta value** associated to an admissible multi-index s is defined as

$$\zeta(\mathbf{s}) = \sum_{n_1 > ... > n_l \geq 1} \frac{1}{n_1^{s_1} ... n_l^{s_l}}.$$

The integer $s_1 + ... + s_l := wt(\mathbf{s})$ is called the **weight** of **s** and l its **length** (also called depth, a term we will use again in different contexts).
By convention, $\zeta(\emptyset) = 1$.

Here is a table of the first MZVs with respect to weight and length :

weight ╲ length	1	2	3	4
2	$\zeta(2)$			
3	$\zeta(3)$	$\zeta(2,1)$		
4	$\zeta(4)$	$\zeta(3,1), \zeta(2,2)$	$\zeta(2,1,1)$	
5	$\zeta(5)$	$\zeta(4,1), \zeta(3,2), \zeta(2,3)$	$\zeta(3,1,1), \zeta(2,2,1), \zeta(2,1,2)$	$\zeta(2,1,1,1)$

Definition I.1.7. We denote by $\mathcal{Z}$ the $\mathbb{Q}$-vector space generated by all multiple zeta values :

$$\mathcal{Z} = \langle 1, \zeta(2), \zeta(3), \zeta(2,1), \dots \rangle_{\mathbb{Q}}.$$

We also consider different subspaces associated to the weight and length of MZVS for given $k, l \geq 0$:

$$\begin{aligned} \mathcal{Z}_k &:= \langle \zeta(\mathbf{s}) |\ wt(\mathbf{s}) = k \rangle_{\mathbb{Q}} \\ \mathcal{F}_l \mathcal{Z} &:= \langle \zeta(\mathbf{s}) |\ l(\mathbf{s}) \leq l = k \rangle_{\mathbb{Q}} \\ \mathcal{F}_l \mathcal{Z}_k &:= \langle \zeta(\mathbf{s}) |\ wt(\mathbf{s}) = k,\ wt(\mathbf{s}) = k \rangle_{\mathbb{Q}} \end{aligned}$$

Note that $\mathcal{Z}_0 = \mathbb{Q}$ and $\mathcal{Z}_1 = \{0\}$.

Remark I.1.8. There is an increasing filtration on $\mathcal{Z}$ defined by the length :

$$\mathbb{Q} = \mathcal{F}_0 \mathcal{Z} \subseteq \mathcal{F}_1 \mathcal{Z} \subseteq \mathcal{F}_2 \mathcal{Z} \subseteq \dots$$

We are naturally interested in the dimensions of these spaces.
Let us first count the number of admissible multi indices : for a given weight k and length l, there are

$$\binom{k-2}{k-l-1} = \binom{k-2}{l-1}$$

possibilities, since $l+1$ elements are already fixed.
We then have

$$\sum_{l=1}^{k-1} \binom{k-2}{k-l-1} = 2^{k-2}$$

multi indices of weight k. This provides us with an upper bound for the dimension of $\mathcal{Z}_k$.
But some relations between MZVs of a given weight have been known for a long time.

Theorem I.1.9. *(Sum theorem). For any depth $l \in \mathbb{N}$ and weight $s > 1$,*

$$\sum_{\substack{s_1+\cdots+s_l=s \\ s_1>1}} \zeta(s_1, \ldots, s_l) = \zeta(s).$$

Example I.1.10. The case $s = 3$ was already shown by Euler :

$$\zeta(2,1) = \zeta(3).$$

Remark I.1.11. All known relations are between MZVs of the same weight.

Here is the main conjecture regarding the dimension of $\mathcal{Z}_k$:

Conjecture I.1.12. *(Zagier)*
Let the integer sequence $(d_k)_{k\in\mathbb{N}}$ be defined by the recursion

$$d_0 = 1,\ d_1 = 0,\ d_2 = 1, \quad d_k = d_k - 2 + d_k - 3 \quad (k \geq 3).$$

Then

$$dim_{\mathbb{Q}}(\mathcal{Z}_k) = d_k.$$

The upper bound has been achieved by Goncharov and Terasoma. There is no non trivial lower bound known.

Theorem I.1.13. *(Terasoma([Ter]), Deligne-Goncharov([DG]).*

$$dim_{\mathbb{Q}}(\mathcal{Z}_k) \leq d_k.$$

The table below shows the conjectural bound for each weight up to 12, as well as the number of admissible indices corresponding :

k	0	1	2	3	4	5	6	7	8	9	10	11	12
d_k	1	0	1	1	1	2	2	3	4	5	7	9	12
2^{k-2}			1	2	4	8	16	32	64	128	256	512	1024

The magnitude of d_k is indeed far lower than 2^{k-2}, hence there should be a very large number of relations amongst MZVs. The example given clearly does not give rise to enough relations, so one of the main goals of the MZVs theory is to find a generating set of relations.
In order to do this, remember our first naive multiplication of two zeta values

$$\zeta(s_1) \cdot \zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2),$$

which first hinted at an additional algebra structure on $\mathcal{Z}$.

Theorem I.1.14. *The multiplication of real numbers induces an algebra structure on $\mathcal{Z}$ compatible with length and weight filtrations :*

$$\mathcal{F}_{l_1}\mathcal{Z}_{k_1} \cdot \mathcal{F}_{l_2}\mathcal{Z}_{k_2} \subseteq \mathcal{F}_{l_1+l_2}\mathcal{Z}_{k_1+k_2}.$$

As mentioned earlier, there are no observed relations between MZVs of different weight i.e known relations are homogeneous, which leads to the following conjecture :

Conjecture I.1.15. *The weight defines a grading on $\mathcal{Z}$:*

$$\mathcal{Z} = \bigoplus_{k\geq 0} \mathcal{Z}_k.$$

The algebra structure defined on $\mathcal{Z}$ raises some new questions, for example on the number of algebra generators of $\mathcal{Z}$.

I.1.3 The integral representation and double shuffle relations

So far, we have defined multiple zeta values as infinite series, and seen that $\mathcal{Z}$ can be equipped with a product (called harmonic or stuffle product), turning it into an algebra.
Here, we will define the integral representation of MZVs. The idea behind it is as follows :

$$\sum_{k=1}^{\infty} \frac{z^k}{k} = -log(1-z) = \int_0^z \frac{dt}{1-t}$$

for $z \in]0,1[$. Although both the series and the integral diverge when z goes to 1, the equality suggests something similar might exist for convergent series.
Here is an example :

Example I.1.16.

$$\begin{aligned}
\int_{1\geq t_1\geq t_2\geq 0} \frac{dt_1}{t_1}\frac{dt_2}{1-t_2} &= \int_0^1 (\frac{dt_1}{t_1}\int_0^{t_1} \frac{dt_2}{1-t_2}) \\
&= \int_0^1 (\frac{dt_1}{t_1}\int_0^{t_1} \sum_{n\geq 1} t_2^{n-1} dt_2) \\
&= \int_0^1 \frac{dt_1}{t_1} \sum_{n\geq 1} \frac{t_1^n}{n} \\
&= \sum_{n\geq 1} \frac{1}{n} \int_0^1 t_1^{n-1} dt_1 \\
&= \sum_{n\geq 1} \frac{1}{n^2} = \zeta(2).
\end{aligned}$$

In order to describe the general integral representation of multiple zeta values, we need a few preliminary definitions.

Definition I.1.17. For $0 \leq t \leq 1$, we define :

$$\Delta^p(t) = \{(t_1, ..., t_p) \in \mathbb{R}^p |\ t \geq t_1 \geq t_2 \geq \cdots \geq t_p \geq 0\}$$

and the two differential forms

$$\omega_0(t) = \frac{dt}{t} \qquad \omega_1(t) = \frac{dt}{1-t}.$$

Now let $\mathbf{s}$ be a positive multi-index, and $r_i = \sum_{k=1}^{i} s_k$. We define a differential form of degree $r_l = \mathrm{wt}(\mathbf{s})$ by :

$$\begin{aligned}\omega_{\mathbf{s}} &= \omega_0(t_1) \wedge \cdots \wedge \omega_0(t_{r_1-1}) \wedge \omega_1(t_{r_1}) \\ &\wedge \omega_0(t_{r_1+1}) \cdots \wedge \omega_0(t_{r_2-1}) \wedge \omega_1(t_{r_2}) \\ &\wedge \dots \\ &\wedge \omega_0(t_{r_{l-1}+1}) \wedge \cdots \wedge \omega_0(t_{r_l-1}) \wedge \omega_1(t_{r_l}).\end{aligned}$$

Here are two examples of such forms :

Example I.1.18.

$$\omega_{(2)} = \omega_0(t_1) \wedge \omega_1(t_2) = \frac{dt_1}{t_1} \wedge \frac{dt_2}{1-t_2}$$

and

$$\omega_{(2,1)} = \omega_0(t_1) \wedge \omega_1(t_2) \wedge \omega_1(t_3) = \frac{dt_1}{t_1} \wedge \frac{dt_2}{1-t_2} \wedge \frac{dt_3}{1-t_3}.$$

Theorem I.1.19. *(Kontsevich)Let* $\mathbf{s} = (s_1, ..., s_l)$ *be an admissible multi-index. The multiple zeta value* $\zeta(\mathbf{s})$ *can be obtained by a convergent improper integral:*

$$\zeta(\mathbf{s}) = \int_{\Delta^{wt(\mathbf{s})}(1)} \omega_{\mathbf{s}}.$$

This new way of representing multiple zeta values allows the theory of MZVs to be linked to other mathematical objects, such as period polynomials . In this thesis, we will simply use the fact that multiplying two such integrals yields a different MZV than the product of the two corresponding series, providing us with a set of linear relations among MZVs.

Example I.1.20. Let us have a look at the simplest example of such a product :

$$\begin{aligned}\zeta(2) \cdot \zeta(2) &= \int_{1 \geq t_1 \geq t_2 \geq 0} \frac{dt_1 dt_2}{t_1(1-t_2)} \cdot \int_{1 \geq u_1 \geq u_2 \geq 0} \frac{du_1 du_2}{u_1(1-u_2)} \\ &= \int_{\substack{1 \geq t_1 \geq t_2 \geq 0 \\ 1 \geq u_1 \geq u_2 \geq 0}} \frac{dt_1 dt_2 du_1 du_2}{t_1(1-t_2)u_1(1-u_2)} \\ &= \sum_{i=1}^{6} \int_{U_i} \frac{dt_1 dt_2 du_1 du_2}{t_1(1-t_2)u_1(1-u_2)}\end{aligned}$$

where the sets U_i are defined as follows :

$$\begin{aligned}U_1 &= 1 \geq t_1 \geq u_1 \geq t_2 \geq u_2 \geq 0; \\ U_2 &= 1 \geq t_1 \geq u_1 \geq u_2 \geq t_2 \geq 0; \\ U_3 &= 1 \geq t_1 \geq t_2 \geq u_1 \geq u_2 \geq 0; \\ U_4 &= 1 \geq u_1 \geq t_1 \geq u_2 \geq t_2 \geq 0; \\ U_5 &= 1 \geq u_1 \geq t_1 \geq t_2 \geq u_2 \geq 0; \\ U_6 &= 1 \geq u_1 \geq u_2 \geq t_1 \geq t_2 \geq 0.\end{aligned}$$

Using the theorem, we obtain :

$$\zeta(2) \cdot \zeta(2) = 4 \cdot \zeta(3,1) + 2 \cdot \zeta(2,2).$$

When computing the product of the defining infinite series $\zeta(2)^2$, one gets :

$$\zeta(2)\cdot\zeta(2) = 2\zeta(2,2) + \zeta(4).$$

The example above thus yields the following equality :

$$\zeta(4) = 4\zeta(3,1).$$

The two representations of multiple zeta values provide us with some new relations when comparing products. In order to be able to refer to one or the other, we will name the product of infinite series the **stuffle** or harmonic product and denote it by $*$.
The product of integrals will be called **shuffle** product and denoted by $\sqcup\!\sqcup$.
Due to linear relations among MZVs, only the weight and length of multi indices are well defined : the products therefore have to be defined at the level of indices.
The algebraic setup and the exact formulas describing both products will be described in the next subsection, but it is already possible to state the **double shuffle relations** : for $\mathbf{s_1}, \mathbf{s_2}$ admissible :

$$\zeta(\mathbf{s_1} * \mathbf{s_2}) = \zeta(\mathbf{s_1} \sqcup\!\sqcup \mathbf{s_2}).$$

Remark I.1.21. The double shuffle relations provide us with a whole set of linear relations, but the smallest weight in which such relations exist is 4. Therefore the relation $\zeta(2,1) = \zeta(3)$ cannot be obtained this way, and double shuffle relations cannot be conjectured to describe all relations among MZVs.

Still, let us persist in this direction by disregarding convergence conditions and compute $\zeta(1)\cdot\zeta(2)$. The $*$ product yields $\zeta(1)\cdot\zeta(2) = \zeta(1,2)+\zeta(2,1)+\zeta(3)$, where the $\sqcup\!\sqcup$ product gives us $\zeta(1)\cdot\zeta(2) = 2\zeta(2,1)+\zeta(1,2)$.
Equating these expressions, the divergent terms cancel out and we get back Euler's relation.
It therefore seems that considering not only admissible multi-indices, but positive ones could provide us with more relations. This theory of assigning real values to divergent MZV-like series and integrals corresponding to mutli indices starting with 1 is called regularization.
We will now describe the algebraic setting defined by Hoffman in order to develop this idea of regularization more rigorously.

I.1.4 Hoffman's notations and regularization theory

Let $\mathfrak{h} := \mathbb{Q}\langle x, y\rangle$ be the non-commutative polynomial algebra over the rationals in two indeterminates x and y, and $\mathfrak{h}_1$ and $\mathfrak{h}_0$ its subalgebras $\mathbb{Q} + \mathfrak{h}y$ and $\mathbb{Q} + x\mathfrak{h}y$ respectively.
Let $Z : \mathfrak{h}_0 \longrightarrow \mathbb{R}$ the evaluation map taking a monomial $u_1u_2\dots u_l$ to the multiple integral:

$$\int\cdots\int_{1\geq t_1\geq\cdots\geq t_l>0} \omega_{u_1}(t_1)\dots\omega_{u_l}(t_l)$$

where $\omega_x(t) = \frac{dt}{t}$ and $\omega_y(t) = \frac{dt}{1-t}$.
Since the word $u_1\dots u_l$ belongs to $\mathfrak{h}_0$, it starts in x and ends in y and the corresponding integral under Z converges.
The theorem of integral representation of MZVs seen previously then yields :

$$Z(x^{s_1-1}yx^{s_2-1}y\dots x^{s_l-1}y) = \zeta(s_1, s_2, \dots, s_l).$$

The weight of the index $\mathbf{s}$ now corresponds to the total degree of the monomial $x^{s_1-1}yx^{s_2-1}y\dots x^{s_l-1}y$ and its depth to the y-degree.

Let us write $y_k := x^{k-1}y$. Then $\mathfrak{h}_1$ is freely generated by the y_k with $k \geq 1$, corresponding to positive multi indices, whereas words in $\mathfrak{h}_0$ are in bijection with admissible multi indices.

We will now equip $\mathfrak{h}_1$ with two different algebraic structures corresponding to the two products on MZVs.

Definition I.1.22. The **harmonic** or **stuffle** product $*$ on $\mathfrak{h}_1$ is defined inductively by

$$1 * w = w * 1 = w,$$

$$y_k w_1 * y_l w_2 = y_k(w_1 * y_l w_2) + y_l(y_k w_1 * w_2) + y_{k+l}(w_1 * w_2),$$

for all $k, l \geq 1$ and any words $w, w_1, w_2 \in \mathfrak{h}_1$, and then extended by $\mathbb{Q}$-bilinearity.
This product is associative and commutative.

Theorem I.1.23. *(First law of multiplication of MZVs.)*
The evaluation map $Z : \mathfrak{h}_0 \longrightarrow \mathbb{R}$ is an algebra homomorphism with respect to the multiplication $*$:

$$Z(w_1 * w_2) = Z(w_1)Z(w_2).$$

Example I.1.24. The stuffle product

$$y_k * y_l = y_k y_l + y_l y_k + y_{k+l}$$

corresponds to the identity

$$\zeta(k)\zeta(l) = \zeta(k, l) + \zeta(l, k) + \zeta(k + l).$$

Definition I.1.25. We define a second commutative product called **shuffle** product on $\mathfrak{h}$. It corresponds to the product of two integral representations of MZVs and is defined inductively by setting:

$$w ⧢ 1 = 1 ⧢ w = w,$$

$$uw_1 ⧢ vw_2 = u(w_1 ⧢ vw_2) + v(uw_1 ⧢ w_2),$$

for any words $w, w_1, w_2 \in \mathfrak{h}$ and $u, v \in \{x, y\}$ and again extended by $\mathbb{Q}$-bilinearity.

Theorem I.1.26. *(Second law of multiplication of MZVs.)*
The evaluation map $Z : \mathfrak{h}_0 \longrightarrow \mathbb{R}$ is an algebra homomorphism with respect to the multiplication ⧢:

$$Z(w_1 ⧢ w_2) = Z(w_1)Z(w_2).$$

By equating these two evaluation maps, we obtain the **finite double shuffle relation** :

$$\zeta(w_1 * w_2) = \zeta(w_1 ⧢ w_2)$$

for all $w_1, w_2 \in \mathfrak{h}_0$.

As mentioned before, the finite double shuffle relations do not generate all relations between MZVs. Extending such relations to non convergent indices seemed to be a solution. Ihara, Kaneko and Zagier found extensions to words in $\mathfrak{h}_1$.

Proposition I.1.27. *We have two algebra homomorphisms*

$$Z^* : \mathfrak{h}_1 \longrightarrow \mathbb{R}[T] \quad \text{and} \quad \text{and}\, Z^{⧢} : \mathfrak{h}_1 \longrightarrow \mathbb{R}[T]$$

that are uniquely characterized by the properties that they both extend the evaluation map $Z : \mathfrak{h}_0 \to \mathbb{R}$ and send y to T.

For a multi-index $\mathbf{s}$, we write $Z^*_{\mathbf{s}}(T)$ and $Z^{⧢}_{\mathbf{s}}(T)$ the images of the corresponding word $z_{s_1} \dots z_{s_l}$ by the maps Z^* and $Z^{⧢}$ respectively.

Remark I.1.28. If $\mathbf{s}$ is admissible, then $Z^*_{\mathbf{s}}(T) = Z^{⧢}_{\mathbf{s}}(T) = \zeta(\mathbf{s})$.
Below are a few example on non-admissible indices:

$\mathbf{s}$	(1)	(1,1)	(1,2)
$Z^*_{\mathbf{s}}(T)$	T	$\frac{1}{2}T^2 - \frac{1}{2}\zeta(2)$	$\zeta(2)T - \zeta(2,1) - \zeta(3)$
$Z^{⧢}_{\mathbf{s}}(T)$	T	$\frac{1}{2}T^2$	$\zeta(2)T - 2\zeta(2,1)$

Theorem I.1.29. *Regularization theorem (Ihara, Kaneko, Zagier, [IKZ]).*
For any multi index $\mathbf{s}$*, we have :*

$$Z^{⧢}_{\mathbf{s}}(T) = \rho(Z^*_{\mathbf{s}}(T)).$$

The map $\rho : \mathbb{R}[T] \to \mathbb{R}[T]$ *is given by :*

$$\rho(e^{Tu}) = A(u)e^{Tu}$$

with

$$A(u) = exp\left(\sum_{n=2}^{\infty} \frac{(-1)^n}{n}\zeta(n)u^n\right).$$

Example I.1.30. Let us come back to the previous table and compare some values of $Z^*_{\mathbf{s}}(T)$ and $Z^{⧢}_{\mathbf{s}}(T)$. Note that $\rho(T) = T$.
Comparing the two entries for $\mathbf{s} = (1,2)$ we get :

$$\zeta(2)T - \zeta(2,1) - \zeta(3) = \zeta(2)T - 2\zeta(2,1)$$

which gives us back the Euler relation $\zeta(2,1) = \zeta(3)$ we could not obtain via double shuffle relations. Let us look at more interesting case, i.e. when the role of ρ is not restricted to the identity. For $\mathbf{s} = (1,1,2)$ we have :

$$Z^{⧢}_{\mathbf{s}}(T) = \frac{1}{2}\zeta(2)T^2 - 2\zeta(2,1)T + 3\zeta(2,1,1)$$

and

$$Z^*_{\mathbf{s}}(T) = \frac{1}{2}\zeta(2)T^2 - (\zeta(3) + \zeta(2,1))T + \frac{1}{2}\zeta(4) + \zeta(3,1) + \zeta(2,1,1).$$

Now $\rho(T^2) = T^2 + \zeta(2)$, so from the regularization theorem we obtain 2 relations this time :

$$\zeta(2,1) = \zeta(3)$$

again, and

$$2\zeta(2,1,1) = \zeta(2)^2 + \frac{1}{2}\zeta(4) + \zeta(3,1).$$

We call these relations the regularized (or extended) double shuffle relations. They are conjectured to generate all algebraic relations over $\mathbb{Q}$ among MZVs.

I.1.5 Formal MZVs and $\mathfrak{ds}$

We will now define one of the three main objects of this thesis : the double shuffle Lie algebra $\mathfrak{ds}$, originally called $\mathfrak{dmr}_0$ par Racinet ([R]) for Doubles Mélanges et Régularisation.

Definition I.1.31. Let $\mathcal{FZ}$ denote the $\mathbb{Q}$-algebra generated by formal symbols $Z(w)$ for all words w subject to the extended double shuffle relations.We call it the algebra of **formal MZVs**.
For any admissible word w, we again write $w = z_{s_1} \dots z_{s_l}$ and $Z(s_1, \dots, s_l) = Z(w)$.

Conjecture I.1.32. *There is an algebra isomorphism between the algebra of formal MZVs and the algebra generated by real MZVs :*

$$\mathcal{FZ} \simeq \mathcal{Z}.$$

Definition I.1.33. Following Furusho, we define the space of "new formal zetas"

$$\mathfrak{nfz} := (\mathcal{FZ})_{>2}/(\mathcal{FZ}_{>0})^2,$$

where $(\mathcal{FZ}_{>0})^2$ is the ideal generated by products of elements in $(\mathcal{FZ})$ of weight at least 1. This space is composed of algebraic generators of $\mathcal{Z}$, called "new" by Furusho since they do not arise as the product of two other MZVs.

We are indeed interested in the structure of $\mathfrak{nfz}$ as a filtered, graded vector space. We can consider it as an algebra, but its multiplication law is of course trivial.
We will in fact study the graded dual of $\mathfrak{nfz}$: called $\mathfrak{ds}$, it is obtained by dualizing the defining equations of $\mathfrak{nfz}$, which are simply linearized versions of the double shuffle equations. It can be endowed with a Lie algebra structure: the original proof is very complex and due to Racinet ([R]), who named it $\mathfrak{dmr}_0$ for Double mélange et régularisation. Its structure can be given more economically in terms of Lie algebra generators than $\mathfrak{nfz}$ and the dimension of the graded pieces are indeed the same.

Definition I.1.34. The **double shuffle** Lie algebra $\mathfrak{ds}$ is composed of elements $f \in \mathfrak{h}$ of degree at least 3 such that:

$$c_{u \sqcup\!\sqcup v}(f) = 0 \qquad \forall u, v \in \mathfrak{h}$$

and

$$c_{u*v}(f) = 0 \qquad \forall u, v \in \mathfrak{h}_1 \text{ s.}t.\ (u, v) \neq (y^m, y^n)$$

where $c_w(f)$ denotes the coefficient of the word w in the polynomial f (sometimes written $(f|w)$).

We will see later that the first condition (shuffle) on $f \in \mathfrak{h}$ is equivalent to f being a Lie polynomial. Therefore the definition of $\mathfrak{ds}$ is sometimes given as:

$$\mathfrak{ds} := \{f \in (\mathfrak{lie}(x, y))_{\geq 3} |\ c_{u*v}(f) = 0\}$$

for all $u, v \in \mathfrak{h}_1$ such that u, v are not both powers of y.

Remark I.1.35. Several equivalent definitions of $\mathfrak{ds}$ or $\mathfrak{dmr}_0$ can be found in the literature. Some of them translate more easily in mould language than others. The original definition by Racinet ([R]) necessitates more background that we are willing to give, but here is the closest form of it used by Schneps:

Definition I.1.36. The Lie algebra $\mathfrak{ds}$ is the dual of the Lie coalgebra $\mathfrak{nfz}$ of new formal multizeta values. It can be defined directly as the set of polynomials $f \in \mathfrak{h}$ having the two following properties:

(1) The coefficients of f satisfy the shuffle relations :

$$\sum_{w \in sh(u,v)} (f|w) = 0$$

where u, v are words in x, y and $sh(u, v)$ is the set of words obtained by shuffling them. This condition is equivalent to the assertion that $f \in \mathfrak{lie}_2$.

(2) Let $f_* = \pi_y(f) + f_{corr}$, where $\pi_y(f)$ denotes the projection of f onto words ending in y and

$$f_{corr} = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (f|x^{n-1}y)y^n.$$

(When f is homogeneous of degree n, which we usually assume, then f_{corr} is just the monomial $\frac{(-1)^n}{n}(f|x^{n-1}y)y^n$.) The coefficients of f_* satisfy the stuffle relations:

$$\sum_{w \in st(u,v)} (f_*|w) = 0$$

where now u, v and $w \in \mathfrak{h}_1$, considered as rewritten in the variables $y_i = x^{i-1}y$, and $st(u, v)$ is the set of words obtained by stuffling them.

Example I.1.37. The first known non-trivial element of $\mathfrak{ds}$ is

$$f_3 = [x, [x, y]] + [[x, y], y] = x^2y - 2xyx + yx^2 + y^2x - 2yxy + xy^2.$$

Let us first have a look at the shuffle condition : we have to consider all pairs of words u, v such that some element of their shuffle has non zero coefficient in f.
Let $u = x, v = xy$. Then $sh(u, v) = [x^2y, x^2y, xyx]$ and $(f|x^2y)+(f|x^2y)+(f|xyx) = 1+1-2 = 0$. Similarly, let $u = x^2, v = y$. Then $sh(u, v) = [x^2y, xyx, yx^2]$ and $(f|x^2y) + (f|yx^2) + (f|xyx) = 1 + 1 - 2 = 0$. The remaining possibilities give the same result and f verifies the first condition.

To check the stuffle condition, let us first give the explicit expression of f^* in this case:

$$f_* = \pi_y(f) + f_{corr} = x^2y - 2yxy + xy^2 + \frac{-y^3}{3}.$$

Rewriting it in the variables y_i, it yields

$$f^* = y_3 - 2y_1y_2 + y_2y_1 + \frac{1}{3}y_1^3.$$

Now here are the two possibles couples u, v and their associated stuffle :

$$st(y_1, y_2) = [y_1y_2, y_2y_1, y_3]$$

gives

$$(f^*|y_1y_2) + (f^*|y_2y_1) + (f^*|y_3) = -2 + 1 + 1 = 0,$$

and

$$st(y_1, y_1^2) = [3y_1^3, y_1y_2, y_2y_1]$$

yields

$$(f^*|3y_1^3) + (f^*|y_1y_2) + (f^*|y_2y_1) = \frac{1}{3} \cdot 3 - 2 + 1 = 0.$$

Definition I.1.38. We define the Poisson or Ihara Lie bracket on the underlying vector space of $\mathfrak{lie}_2$ by :

$$\{f,g\} = [f,g] + D_f(g) - D_g(f)$$

where to each element $f \in \mathfrak{lie}_2$ one associates the derivation D_f such that $D_f(x) = 0$ and $D_f(y) = [y,f]$.
This bracket corresponds to the Lie bracket on the derivations of $\mathfrak{lie}_2$ in the sense that $D_{\{f,g\}} = [D_f, D_g]$.

Theorem I.1.39. *(Racinet, [R]) The double shuffle space $\mathfrak{ds}$ is a Lie algebra under the Poisson bracket.*

The proof contained in Racinet's thesis is extremely complicated. Salerno and Schneps ([SS]) gave in 2015 an elegant mould theoretic version of this theorem, which we will mention in Chapter 2 .
Note that $\mathfrak{ds}$ inherits from $\mathfrak{lie}_2$ a grading by **weight** or degree n given by the number of letters x and y in the Lie word, and a filtration by **depth**, the minimal number of ys in a Lie polynomial f.

Conjecture I.1.40. *The Lie algebra $\mathfrak{ds}$ is freely generated by one generator of weight n for each odd $n \geq 3$.*

The existence of the depth filtration on $\mathfrak{ds}$ brings the necessity to study the associated graded algebra $gr(\mathfrak{ds})$, and to see what becomes of the defining property of $\mathfrak{ds}$ when truncated to their lowest depth part.
The shuffle is an operation that respects depth, therefore the first property remains untouched. However, the stuffle produces terms of the same depth (given by the shuffle) and additional terms of higher depths which disappear when truncating to the lowest depth part (see II.2.8 for an example). This yields the definition of the linearized double shuffle space $\mathfrak{ls}$.

Definition I.1.41. The linearized double shuffle space $\mathfrak{ls}$ is defined to be the set of polynomials f in x, y of degree ≥ 3 satisfying the shuffle relations (as mentioned earlier, this is equivalent to f being in $\mathfrak{lie}_2$) and :

$$\sum_{w \in sh(u,v)} (\pi_y(f)|w) = 0$$

where as above, $\pi_y(f)$ is the projection of f onto the words ending in y, rewritten in the variables $y_i = x_{i-1}y$, u, v are words in the y_i and w belongs to their shuffle in the alphabet y_i.
However, we exclude from $\mathfrak{ls}$ all (linear combinations of) the depth 1 even degree polynomials, namely $ad(x)^{2n+1}(y)$, $n \geq 1$.

The space $\mathfrak{ls}$ is bigraded by weight and depth, since the shuffle relations respect the depth. We will give a simple mould theoretic proof by Salerno and Schneps of the following result in Chapter 2 :

Proposition I.1.42. *The space $\mathfrak{ls}_n^d$ of weight n and depth d is zero if $n \neq d$ mod 2.*
In particular, the graded quotient $\mathfrak{ds}_n^d/\mathfrak{ds}_n^{d+1}$ which lies inside it is zero if $n \neq d$ mod 2.

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I.2 The Grothendieck-Teichmüller Lie algebra

In this section, we give some background on how the profinite version of the Grothendieck-Teichmüller group arises from the study of the absolute Galois group of $\mathbb{Q}$, then define its prounipotent version and the associated Lie algebra $\mathfrak{grt}_1$.

I.2.1 The absolute Galois group $Gal(\mathbb{Q})$

Let $\mathbb{K}$ be a perfect field, i.e. every irreducible polynomial over $\mathbb{K}$ has distinct roots.
Let $\bar{\mathbb{K}}$ be the algebraic closure of $\mathbb{K}$. The **absolute Galois group** of $\mathbb{K}$ is :

$$\begin{aligned} Gal(\mathbb{K}) &:= Gal(\bar{\mathbb{K}}/K) \\ &= Aut(\bar{\mathbb{K}}/K) \\ &= \{\varphi|\varphi \in Aut(\bar{\mathbb{K}}),\ \varphi(x) = x\ \forall x \in \mathbb{K}\} \end{aligned}$$

Example I.2.1. The absolute Galois group of $\mathbb{R}$ is

$$Gal(\mathbb{R}) = Gal(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z},$$

where the non-trivial element is complex conjugation.
The absolute Galois group of $\mathbb{C}$ is indeed trivial.

What about $Gal(\mathbb{Q})$? This is actually a very difficult and obscure problem to tackle : as of today, the only explicitly defined elements are the identity and complex conjugation.
The idea of Alexander Grothendieck ([Gr]) was to look at this group via its action on simpler objects, i.e. understand $Aut(O)$ for some type of objects O then find a map from $Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \to Aut(O)$ (ideally an isomorphism).

More precisely, Grothendieck suggested to consider the outer action of $Gal(\mathbb{Q})$ on the algebraic fundamental groups of the moduli spaces of closed genus g curves defined over $\mathbb{Q}$ with n marked points $\mathcal{M}_{g,n}$.
In particular, taking the variety $\mathcal{M}_{0,4} = \mathbb{P}^1 - \{0, 1, \infty\}$ we obtain a morphism

$$Gal(\mathbb{Q}) \to Out(\hat{\pi}_1(\mathcal{M}_{0,4})) = Out(\widehat{Free}(x, y))$$

where $\hat{\pi}_1(\mathcal{M}_{0,4}) = \widehat{Free}(x, y)$ is the profinite completion of the free group on two generators and $Out()$ denotes the group of outer automorphisms.

Theorem I.2.2. *(Belyi [B]). The morphism*

$$Gal(\mathbb{Q}) \to Out(\widehat{Free}(x, y))$$

is injective.

Therefore, in order to understand $Gal(\mathbb{Q})$, we should then study its image in $Out(\widehat{Free}(x, y))$, which is a subgroup characterized by some equations explicited by Drinfel'd in [Dr].

I.2.2 The profinite Grothendieck-Teichmüller group

From these equations, we get the following definition:

Definition I.2.3. (Drinfel'd, 1991) The profinite Grothendieck Teichmüller group, denoted $\widehat{GT}$, is the subgroup of $Aut(Free(x, y))$ consisting of automorphisms ϕ such that :

$$\phi(x) = x\lambda \qquad\qquad \phi(y) = f^{-1}y\lambda f$$

where $\lambda \in \mathbb{Z}^{\times}$ and $f \in \widehat{Free}(x, y)$ satisfies

$$f(y, x) = f(x, y)^{-1}$$

$$f(z, x)z^m f(y, z)y^m f(x, y)x^m = 1 \qquad \text{for } xyz = 1$$

$$f(x_{12}, x_{23}x_{24})f(x_{13}x_{23}, x_{34}) = f(x_{23}, x_{34})f(x_{12}x_{13}, x_{24}x_{34})f(x_{12}, x_{23})$$

where the last equation takes values in the pro-finite completion of the pure braid group $\widehat{PB}_4$ with generators x_{ij}.

Theorem I.2.4. *(Drinfeld [Dr]). The outer action of $Gal(\mathbb{Q})$ on $\widehat{Free}(x, y)$ factors through $\widehat{GT}$, i.e. there is an injective morphism $Gal(\mathbb{Q}) \hookrightarrow \widehat{GT}$ making this diagram commute :*

$$\begin{tikzcd}
Gal(\mathbb{Q}) \arrow[r] \arrow[d] & \widehat{Free}(x, y) \\
\widehat{GT} \arrow[ur] &
\end{tikzcd}$$

Conjecture I.2.5. *The map $Gal(\mathbb{Q}) \hookrightarrow \widehat{GT}$ is an isomorphism.*

I.2.3 The pro-unipotent setting and GT/GRT

However, in the same seminal article [Dr] Drinfel'd also defines a version of the Grothendieck Teichmüller group in the prounipotent setting. While its ties with the course of action defined above are looser, it is a very interesting object in itself, linked to a variety of important mathematical objects, including MZVs and kashiwara-Vergne theory as we will see in section 4.

Definition I.2.6. The (prounipotent) Grothendieck Teichmüller group $GT(\mathbb{K})$ is defined as the set of pairs (λ, f) with $\lambda \in \mathbb{K}^*$ and $f \in \mathbb{K}\langle x, y\rangle$ such that:

$$f(y, x) = f(x, y)^{-1}, \tag{I.2.1}$$

$$f(z, x)e^{\frac{\lambda-1}{2}z} f(y, z)e^{\frac{\lambda-1}{2}y} f(x, y)e^{\frac{\lambda-1}{2}x} = 1 \tag{I.2.2}$$

where $e^x e^y e^z = 1$, and

$$f(x_{12}, x_{23}x_{24})f(x_{13}x_{23}, x_{34}) = f(x_{23}, x_{34})f(x_{12}x_{13}, x_{24}x_{34})f(x_{12}, x_{23}) \tag{I.2.3}$$

Here the last equation takes place in the pro-unipotent completion of the pure braid group, $xyz = 1$ and $m = (\lambda - 1)/2$. The group structure on $GT(\mathbb{K})$ is defined by considering the pairs (λ, f) as an automorphism F of the pro-unipotent completion of the free group in two generators x, y by setting $x \to x^\lambda$ and $y \to f^{-1}y^\lambda f$. Concretely, it is given by the equation :

$$(\lambda, f) \cdot (\lambda', f') = (\lambda\lambda', fF(f')).$$

Definition I.2.7. We denote by GT_1 the kernel of the group homomorphism :

$$GT(\mathbb{K}) \to \mathbb{K}$$
$$(\lambda, f) \mapsto \lambda.$$

Definition I.2.8. The **Kohno-Drinfeld Lie algebra** $\mathfrak{t}_n$ is generated by $n(n-1)/2$ elements $t_{ij} = t_{ji}$, where $1 \leq i \neq j \leq n$ and relations

$$[t_{ij}, t_{kl}] = 0$$
$$[t_{ij} + t_{ik}, t_{jk}] = 0$$

for all i, j, k, l all distinct.

Definition I.2.9. Consider group-like elements $\Phi \in \mathbb{K}\langle\langle X, Y\rangle\rangle$ satisfying the following equations :

$$\Phi(Y, X) = \Phi(X, Y)^{-1} \tag{I.2.4}$$

$$e^{\frac{\mu}{2}Z}\Phi(X, Y)e^{\frac{\mu}{2}X}\Phi(Y, Z)e^{\frac{\mu}{2}Y}\Phi(Z, X) = 1 \tag{I.2.5}$$

$$\Phi(t_{12}, t_{23} + t_{24})\Phi(t_{13} + t_{23}, t_{34}) = \Phi(t_{23}, t_{34})\Phi(t_{12} + t_{13}, t_{24} + t_{34})\Phi(t_{12}, t_{23}) \tag{I.2.6}$$

where for equation (I.2.5) $X + Y + Z = 0$ and the last equation takes values in the $\mathfrak{t}_4$.
The set of pairs (μ, Φ) solving these equations with $\mu \neq 0$ is called the set of **Drinfeld associators** $DAss$.
The set of solutions Φ of these equations for $\mu = 0$ is called the **graded Grothendieck-Teichmüller group** GRT_1 .

Theorem I.2.10. *(Drinfel'd, [Dr]) The set of associators $DAss$ is non-empty.*

I.2.4 The Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}_1$

We are now able to introduce the corresponding graded Lie algebra $\mathfrak{grt}_1$. Let $\mathbb{K}$ be a field of characteristic zero and let $\mathfrak{lie}_2$ be the **free Lie algebra** over $\mathbb{K}$ on x, y.

Definition I.2.11. The **Grothendieck-Teichmüller Lie algebra** $\mathfrak{grt}_1$ is spanned by elements ψ in the degree completion of $\mathfrak{lie}_2$ satisfying the following relations :

$$\psi(x, y) = -\psi(y, x) \tag{I.2.7}$$

$$\psi(x, y) + \psi(y, z) + \psi(z, x) = 0 \quad if \quad x + y + z = 0 \tag{I.2.8}$$

$$\psi(t_{12}, t_{2,34}) + \psi(t_{12,3}, t_{34}) = \psi(t_{23}, t_{34}) + \psi(t_{1,23}, t_{23,4}) + \psi(t_{12}, t_{23}) \tag{I.2.9}$$

where the last equation takes values in $\mathfrak{t}_4$ and $t_{ij,k} := t_{ik} + t_{jk}$.
The Lie bracket on $\mathfrak{grt}_1$ is the Ihara bracket given by :

$$\{\psi_1, \psi_2\} = D_{\psi_1}(\psi_2) - D_{\psi_2}(\psi_1) + [\psi_1, \psi_2]$$

where $D_{\psi_1}(x) = 0, D_{\psi_1}(y) = [y, \psi]$.
The Lie algebra $\mathfrak{grt}_1$ inherits a grading from $\mathfrak{lie}_2$ given by the **degree** or weight (i.e. total number of x and y of any Lie monomial).

Remark I.2.12.

1. The defining equations of $\mathfrak{grt}_1$ admit no solutions in degree 1 and 2.
2. The third equation is often referred to as the pentagon equation. In 2010, Furusho ([F1]) proved that the pentagon equation actually implies the two others.

Conjecture I.2.13. *(Deligne-Drinfeld-Ihara)*
The Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}_1$ *is isomorphic (up to completion) to the free Lie algebra*

$$\mathfrak{lie}(\sigma_3, \sigma_5, \sigma_7, \cdots)$$

generated by elements σ_{2n+1} *of odd degree given by :*

$$\sigma_{2n+1} = \sum_{k=1}^{2n} \frac{(2n+1)!}{k!(2n+1-k)!} ad_x^{k-1} ad_y^{n-k}[x, y].$$

This conjecture has been verified up to degree 16, and a further step in that direction is given by the following theorem.

Theorem I.2.14. *(Brown, [Br]) There is an injection*

$$\mathfrak{lie}(\sigma_3, \sigma_5, \sigma_7, \cdots) \hookrightarrow \mathfrak{grt}_1.$$

Example I.2.15. [Ih2] The two following examples are the degree 3 and 5 potential generators, normalized to have integer coefficients.

$$\begin{aligned} \sigma_3 = \quad & [x, [x, y]] - [y, [y, x]]. \\ \sigma_5 = \quad & 2[x, [x, [x, [x, y]]]] - 2[y, [y, [y, [y, x]]]] + 4[x, [x, [y, [x, y]]]] - 4[y, [y, [x, [y, x]]]] - \\ & -3[[x, y], [x, [x, y]]] + 3[[y, x], [y, [y, x]]]. \end{aligned}$$

Remark I.2.16. The Lie algebra corresponding to $GRT = \mathbb{K}^\times \ltimes GRT_1$ may be written as the semi-direct product $\mathfrak{grt} := \mathfrak{t}_2 \ltimes \mathfrak{grt}_1$.

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I.3 The Kashiwara-Vergne Lie algebra

In 1978, Kashiwara and Vergne conjectured in [KV] a universal property on the Baker-Campbell-Hausdorff formula of a real finite dimensional Lie algebra $\mathfrak{g}$.
As a corollary, this conjecture gives a simple proof of the Duflo isomorphism and extends it to germs of invariant distributions.

- 1978: Kashiwara and Vergne give a proof of their conjecture for solvable Lie algebras.
- 1981 : Rouvière shows that the conjecture holds for $sl_2(\mathbb{R})$.
- 1999 : Michèle Vergne proves it for quadratic Lie algebras.
- 2005 : the conjecture is settled positively by Alekseev and Meinrenken using deformation quantization techniques.
- 2008 : Alekseev and Torossian link the KV problem to Drinfeld's theory of associators, giving a new proof of the conjecture.

In the next pages, we introduce the basic notions necessary to the understanding of the conjecture, explain its different forms then describe the formalism needed to define the Kashiwara-Vergne Lie algebra.

I.3.1 The Baker-Campbell Hausdorff formula

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{R}$.
Lie's third theorem states the existence of a simply connected real Lie group G of Lie algebra $\mathfrak{g}$, together with an exponential mapping

$$exp : \mathfrak{g} \to G.$$

The exponential map defines a diffeomorphism from a neighbourhood of 0 in $\mathfrak{g}$ to a neighbourhood of the identity in G.

One can then read the multiplication on law on G in exponential coordinates, i.e. there exists a infinite series $ch(x, y)$ in $\mathfrak{g}$ such that :

$$\exp_{\mathfrak{g}}(x) \cdot_G \exp_{\mathfrak{g}}(y) = \exp_{\mathfrak{g}}(ch(x, y)).$$

The series $ch(x, y)$ is called the **Baker-Campbell-Hausdorff formula** (also denoted $V(x, y)$ in [Rou] and $Z(x, y)$ in [T]).
Its first few terms are given by :

$$ch(x, y) = \log(e^x e^y) = x + y + \frac{1}{2}[x, y] + \frac{1}{12}([x, [x, y]] + [y, [y, x]]) + \dots$$

where … indicate higher order brackets of x and y.

I.3.2 A motivation to the Kashiwara-Vergne conjecture : the Duflo isomorphism

Definition I.3.1. First, we construct the **tensor algebra** of the Lie algebra $\mathfrak{g}$:

$$T_{\mathfrak{g}} := \bigoplus_{k=0}^{\infty} T_{\mathfrak{g}}^k = \bigoplus_{k=0}^{\infty} \underbrace{\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}}_{k \text{ times}}.$$

The **universal enveloping algebra** of $\mathfrak{g}$ is the quotient

$$U_{\mathfrak{g}} := T_{\mathfrak{g}}/\mathcal{I}$$

where $\mathcal{I}$ is the two-sided ideal generated by $g \otimes h - h \otimes g - [g, h]$ with $g, h \in T_{\mathfrak{g}}$.
The **symmetric algebra** of $\mathfrak{g}$ is the quotient

$$S_{\mathfrak{g}} := T_{\mathfrak{g}}/\mathcal{J}$$

where $\mathcal{J}$ is the two-sided ideal generated by $g \otimes h - h \otimes g$ with $g, h \in \mathfrak{g}$.

Both spaces are filtered by the number of generators. Note that $S_{\mathfrak{g}}$ is indeed commutative, whereas $U_{\mathfrak{g}}$ is not. Still, one can relate the two spaces via the PBW map as follows.

Theorem I.3.2. *(Poincaré-Birkhoff-Witt).*
The **symmetrization map**

$$\begin{aligned} I_{PBW} : S_{\mathfrak{g}} &\longrightarrow U_{\mathfrak{g}} \\ x_1 \dots x_n &\longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)} \end{aligned}$$

is an isomorphism of filtered vector spaces.

Due to the non commutativity of $U_{\mathfrak{g}}$, this is not an isomorphism of algebras unless $\mathfrak{g}$ is abelian. We denote by ad the adjoint representation from $\mathfrak{g}$ to $End(\mathfrak{g})$. Now, one can extend the adjoint action ad of $\mathfrak{g}$ on itself to $S_{\mathfrak{g}}$: for any $x, y \in \mathfrak{g}$ and $n \in \mathbb{N}^*$,

$$ad_x(y^n) = n[x, y]y^{n-1}.$$

There is also an adjoint action of g on $U_{\mathfrak{g}}$: for any $x \in \mathfrak{g}$ and $u \in U_{\mathfrak{g}}$,

$$ad_x(u) = xu - ux.$$

Let us denote by $U_{\mathfrak{g}}^{\mathfrak{g}} = Z(U_{\mathfrak{g}})$ the center of the universal enveloping algebra of $\mathfrak{g}$, i.e the set of elements commuting with all other elements of $U\mathfrak{g}$.
Similarly, we consider the elements of $S_{\mathfrak{g}}$ that are invariant by the adjoint action :

$$S_{\mathfrak{g}}^{\mathfrak{g}} := \{f \in S_{\mathfrak{g}} \mid ad_x(f) = 0 \ \forall x \in \mathfrak{g}\}.$$

One can easily see that $ad_x \circ I_{PBW} = I_{PBW} \circ ad_x$ for all $x \in \mathfrak{g}$. Therefore I_{PBW} restricts to an isomorphism (of vector spaces still) from $S(\mathfrak{g})^{\mathfrak{g}}$ to the center $Z(U(\mathfrak{g}))$.We are now dealing with commutative algebras on both sides, but the I_{PBW} fails to respect the product.
Duflo's theorem provides us with the solution:

Theorem I.3.3. *Duflo [Duf],77. There is an isomorphism of algebras :*

$$Z(U_{\mathfrak{g}}) \cong (S_{\mathfrak{g}})^{\mathfrak{g}}$$

given by the composition :

$$\gamma := I_{PBW} \circ \partial_{J^{1/2}} : (S_{\mathfrak{g}})^{\mathfrak{g}} \to Z(U_{\mathfrak{g}}),$$

where $J := det\left(\frac{1-e^{-ad}}{ad}\right)$ *is the Duflo element .*

The Duflo element $J^{\frac{1}{2}}$ is a serie in $S_{\mathfrak{g}^*}$ of infinite order. It can therefore be seen as a differential operator of infinite order on $\mathfrak{g}^*$ with constant coefficients, denoted $\partial_{J^{1/2}}$.

We refer the reader to the lectures notes by Calaque and Rossi ([CR]) for a proof based on deformation theory and (co)homological algebra and of course to Duflo ([Duf]) for the original proof.

The Kashiwara-Vergne method provides another proof of the Duflo isomorphism uniquely based on the exponential map and BCH formula properties.

In order to understand how, we shall shed a different light on the Duflo isomorphism. Let us start with U_g:

Theorem I.3.4. *(Schwartz) The universal enveloping algebra $U_{\mathfrak{g}}$ is isomorphic to :*
- the algebra of differential operators on G invariant by right translations
- to the algebra of distributions on G supported at the identity e $D'_e(G)$, with the convolution product as multiplication.

To each element $x \in \mathfrak{g}$, associate the right-invariant vector field L_x on G, which can be considered as a first order differential operator. This gives us a linear map from g into differential operators on G invariant by right translations. Then any such differential operator D can be written in the form $Df = T * f$ with $T \in D'_e(G)$.

One can identify the symmetric algebra $S_{\mathfrak{g}}$ to the algebra of differential operators with constant coefficients on $\mathfrak{g}$: for an element P in $S_{\mathfrak{g}}$, we denote the associated differential operator by $P(_{\partial_x})$. Similarly, each of these operators defines a distribution supported at 0 on $\mathfrak{g}$.

In terms of differential operators, the Duflo map then reads :

$$\gamma(P)(\varphi(g)) = P_{\partial_x}(j^{\frac{1}{2}}(x))\varphi(ge^x)\Big|_{x=0}$$

where $P \in S_{\mathfrak{g}}$, $g \in G$ and φ is a function on G.
The Duflo isomorphism then transfer the distribution given by the convolution product of two distributions at 0 on $\mathfrak{g}$ to the convolution product of two distributions supported at e on G.
This is a particular case of the original problem defined in the next part.

I.3.3 The original problem

We follow here the exposition of the first chapter of [Rou].

Let G be a real, finite dimensional Lie group and $\mathfrak{g}$ its Lie algebra.

Definition I.3.5. Convolution products.

The convolution of two distributions u, v on the vector space $\mathfrak{g}$ is the distribution $u *_{\mathfrak{g}} v$ defined by:

$$\langle u *_{\mathfrak{g}} v, f \rangle := \langle u(x) \otimes v(y), f(x+y) \rangle$$

where x, y are elements in $\mathfrak{g}$, $\langle , \rangle$ denotes the duality between distributions and functions and f is an arbitrary test function (smooth and compactly supported) on $\mathfrak{g}$.

Similarly, the convolution of two distributions U, V on G is defined by:

$$\langle U *_G V, \varphi \rangle := \langle U(g) \otimes V(h), \varphi(gh) \rangle$$

with g, h in G and φ an arbitrary test function on G.

The product $*_{\mathfrak{g}}$ is commutative whereas in general $*_G$ is not.
We would now like to relate both convolution products via the exponential map.

Definition I.3.6. We first define the **transfer** $f \to \tilde{f}$ of a function f from G to $\mathfrak{g}$ by the following formula :

$$j(x)^{1/2}\tilde{f}(e^x) = f(x)$$

where $j(x) := det(\frac{1-e^{-ad_x}}{ad_x})$ is the Jacobian of exp.
By duality the transfer $u \to \tilde{u}$ of a distribution u from G to $\mathfrak{g}$ is defined by :

$$\langle \tilde{u}, \tilde{f} \rangle = \langle u, f \rangle.$$

Definition I.3.7. A distribution u on $\mathfrak{g}$ is said to G**-invariant** if for all $f, g \in G$ and $x \in \mathfrak{g}$:

$$\langle u(x), f(Ad_g(x)) \rangle = \langle u(x), f(x) \rangle$$

where Ad denotes the adjoint action of G on $\mathfrak{g}$.

We are now able to state the original problem that led to the Kashiwara-Vergne conjecture :

Problem. Prove that :

$$(u *_{\mathfrak{g}} v)\tilde{} = \tilde{u} *_G \tilde{v} \tag{I.3.1}$$

for any G-invariant distributions u, v on $\mathfrak{g}$ (with suitable supports).

Requiring the distributions u, v to be G-invariant is necessary to ensure the commutativity of the convolution product on G.
Applying the LHS of equation (I.3.1) to a test function $\tilde{f}$ and using the BCH formula, the equation becomes :

$$\langle u(x) \otimes v(y), f(x+y) \rangle = \left\langle u(x) \otimes v(y), \left(\frac{j(x)j(y)}{j(Z(x,y))} \right)^{\frac{1}{2}} f(ch(x,y)) \right\rangle. \tag{I.3.2}$$

The idea behind the Kashiwara-Vergne method is to prove this equality by **deformation**: we endow $\mathfrak{g}$ with the bracket $[\ ,\ ]_t$ defined by $[x, y]_t := t[x, y]$ with $t \in [0, 1]$ and write $\mathfrak{g}_t$ for $(\mathfrak{g}, [,]_t)$. Therefore $\mathfrak{g}_0$ is abelian, and $\mathfrak{g}_1 = \mathfrak{g}$.
This deformation yields a "scaled" version of the Baker-Campbell-Hausdorff formula, given by:

$$ch_t(x, y) = t^{-1}ch(tx, ty) = x + y + \frac{t}{2}[x, y] + \frac{t^2}{12}([x, [x, y]] + [y, [y, x]]) + \dots$$

for $t \neq 0$, and $Z_0(x, y) = x + y$.
It will therefore suffice to prove that for any G-invariant distributions u, v on $\mathfrak{g}$ and any test function f:

$$\frac{\partial}{\partial t}\left\langle u(x) \otimes v(y), \left(\frac{j(tx)j(ty)}{j(tZ_t(x,y))} \right)^{\frac{1}{2}} f(ch_t(x,y)) \right\rangle = 0 \tag{I.3.3}$$

since the LHS and RHS of (I.3.2) correspond respectively to the cases $t = 0$ and $t = 1$ above.

This leads to the formulation of the two equations known as the **combinatorial conjecture**, which imply (I.3.3).

Theorem I.3.8. *(KV Conjecture, '78)(Alekseev-Meinrenken [AM] '05)*
For any finite dimensional Lie algebra $\mathfrak{g}$, there exist series $A(x, y)$ and $B(x, y)$ in the free lie algebra $\mathfrak{lie}_2$ such that A, B give convergent power series at the neighbourhood of $(0, 0) \in \mathfrak{g}^2$ and :

$$x + y - log(exp_{\mathfrak{g}}(y) \cdot exp_{\mathfrak{g}}(x)) = (1 - e^{-ad_x})A(x, y) + (e^{ad_y} - 1)B(x, y)) \tag{I.3.4}$$

$$tr_{\mathfrak{g}}(ad_x \circ \partial_x A + (ad_y \circ \partial_y B) = \frac{1}{2} tr_{\mathfrak{g}} \left(\frac{ad_x}{e^{ad_x} - 1} + \frac{ad_y}{e^{ad_y} - 1} - \frac{ad_{ch(x,y)}}{e^{ad_{ch(x,y)}} - 1} - 1 \right) \tag{I.3.5}$$

where tr denotes the trace in the adjoint representation and $\partial_x A(x, y) \in End(\mathfrak{g})$ is the linear map :

$$u \to \left. \frac{\partial}{\partial t} \right|_{t=0} A(x + ut, y)$$

This conjecture was solved positively in 2006 in by Alekseev and Meinrenken, using Kontsevitch's quantization deformation theory and results. Details can be found in the orginal article [AM] and in [Rou].
Equation (I.3.3) was also solved independently from the conjecture in [ADS].

I.3.4 Alekseev-Torossian approach and the Kashiwara-Vergne Lie algebra

We are mainly concerned with the algebraic approach explored by Alekseev and Torossian in [AT] , '08. In this paper, the authors establish a relation between the KV conjecture and Drinfel'd's theory of associators, briefly mentioned in the previous section. While studying the uniqueness issue for the KV problem, they re-prove the conjecture using the existence of associators.
Along the way, they define the Kashiwara-Vergne Lie algebra, which is our main subject of study.
This approach relies on a reformulation of the Kashiwara-Vergne problem in the algebraic setting defined below.

Definitions and notations.

Let $\mathbb{K}$ be a field of characteristic zero. Let $\mathfrak{lie}_2$ be the **free Lie algebra** over $\mathbb{K}$ on x, y. By an abuse of notation, we denote similarly its degree completion (where the generators x and y have degree one)

$$\mathfrak{lie}_2 = \prod_{k=1}^{\infty} \mathfrak{lie}^k(x, y).$$

Example I.3.9. We write $\mathfrak{lie}_2^k$ for the k-th graded piece, spanned by words in k letters : $\mathfrak{lie}_{\not{2}}{}^1(x, y)$ is spanned by x and y, $\mathfrak{lie}_2^2(x, y)$ by $[x, y]$, $\mathfrak{lie}_2^3(x, y)$ by $[x, [x, y]]$ and $[x, y], y]$.

Recall that if a Lie algebra $\mathfrak{g}$ is finite dimensional, then one can associate to it via the exponential map a connected and simply connected Lie group G.

Now $\mathfrak{lie}_2$ is not, but it is positively graded with all grade components of finite dimension : we can associate to $\mathfrak{lie}_2$ a group coinciding with $\mathfrak{lie}_2$ as a set and whose group multiplication is given by the CBH formula. We keep denoting exp the map from $\mathfrak{lie}_2$ to its associated group $exp(\mathfrak{lie}_2.)$

Definition I.3.10. An automorphism of $\mathfrak{lie}_2$ is called **inner automorphism** if it is of the form Ad_g, where g is an element of $exp(\mathfrak{lie}_2)$ and Ad the adjoint action.
An automorphism F of $\mathfrak{lie}_2$ is said to be **tangential** if there exists F_1, F_2 inner automorphisms of $\mathfrak{lie}_2$ such that $F(x) = F_1(x)$ and $F(y) = F_2(y)$. We denote $TAut_2$ the group of tangential automorphisms.

Definition I.3.11. A **derivation** on an Lie algebra $\mathfrak{g}$ is a linear operator $u : \mathfrak{g} \to \mathfrak{g}$ such that

$$u([g_1, g_2]) = [g_1, u(g_2)] + [u(g_1), g_2] \quad \forall g_1, g_2 \in \mathfrak{g}$$

Let $\mathfrak{der}_2$ be the algebra of derivations on $\mathfrak{lie}_2$. It is a Lie algebra under the Lie bracket given by the commutator of maps.
An element $u \in \mathfrak{der}_2$ is entirely determined by its values on the generators x, y.

Definition I.3.12. A derivation u is said to be **tangential** if there exists $a, b \in \mathfrak{lie}_2$ such that $u(x) = [x, a]$ and $u(y) = [y, b]$.

In what follows, we will denote a tangential derivation u by (a, b), since tangential derivations are in one-to-one correspondence with pairs of elements of $\mathfrak{lie}_2$ (a, b) such that a has no linear term in x and b has no linear term in y.

Proposition I.3.13. *The tangential derivations $\mathfrak{tder}_2$ form a Lie subalgebra of $\mathfrak{der}_2$.*

The universal enveloping algebra of $\mathfrak{lie}_2$ is the free associative algebra on two non-commuting variables $\mathbb{K}\langle x, y\rangle$, sometimes denoted by Ass or Ass_2 in the literature.
Every element $a \in \mathbb{Q}\langle x, y\rangle$ admits a unique decomposition

$$a = a_0 + (\partial_x a)x + (\partial_y a)y$$

where $a_0 \in \mathbb{K}$ and ∂_x is a linear operator $\mathbb{Q}\langle x, y\rangle \to \mathbb{Q}\langle x, y\rangle$ such that

$$\partial_x(u_1 \dots u_{k-1}u_k) = \begin{cases} u_1 \dots u_{k-1} \text{ if } u_k = x \\ 0 \text{ otherwise} \end{cases}$$

The operator ∂_y is similarly defined. In the next chapters, we will favour the notation

$$a = a_0 + a_x x + a_y y$$

as it the one used in most of L.Schneps papers.

Example I.3.14. Let $a \in \mathbb{Q}\langle x, y\rangle$, $a = xy - yx$. Then $\partial_x(a) = -y$ and $\partial_y(a) = x$.

Definition I.3.15. The vector space $\mathfrak{tr}_2$ is defined in [AT] as the following quotient

$$\mathfrak{tr}_2 = \mathbb{Q}\langle x, y\rangle^+ / \langle (ab - ba)\rangle$$

with $a, b \in \mathbb{Q}\langle x, y\rangle$, where $\mathbb{Q}\langle x, y\rangle^+$ and $\langle (ab - ba)\rangle$ is the $\mathbb{K}$-linear subspace of $\mathbb{Q}\langle x, y\rangle$ spanned by commutators.

We denote by

$$tr : \mathbb{Q}\langle x, y\rangle \to \mathfrak{tr}_2$$

the natural projection.
Following the definition of $\mathfrak{tr}_2$, we have $tr(ab) = tr(ba)$ for all $a, b \in \mathbb{Q}\langle x, y\rangle$.
Not that he vector space $\mathfrak{tr}_2$ is not an algebra, but comes with an action of $\mathfrak{tder}_2$ (which extends from $\mathfrak{lie}_2$ to $\mathbb{Q}\langle x, y\rangle$ and descends to $\mathfrak{tr}_2$.

Example I.3.16. Graded components $\mathfrak{tr}_2^k$ of $\mathfrak{tr}_2$ are spanned by words of length k modulo cyclic permutations : $\mathfrak{tr}_2^1$ is spanned by $tr(x)$, $tr(y)$ and $\mathfrak{tr}_2^2$ by $tr(x^2)$, $tr(xy)$ and $tr(y^2)$.

Definition I.3.17. We define a "divergence" map by :

$$\begin{aligned} \text{div} \ : \mathfrak{tder}_2 &\longrightarrow \mathfrak{tr}_2 \\ u = (a, b) &\longmapsto tr(x\partial_x a + y\partial_y b) \end{aligned}$$

Example I.3.18. Let $u = ([x, y], [x, [x, y]])$ be a tangential derivation, meaning that $u(x) = [x, [x, y]]$ and $u(y) = [y, [x, [x, y]]]$.
To compute its divergence, one must first inject a and b into $\mathbb{Q}\langle x, y\rangle$ and obtain $a = xy - yx$, $b = x^2y - 2xyx + yx^2$.
The divergence of u is then :

$$div(u) = tr(x\partial_x(xy - yx) + y\partial_y(x^2y - 2xyx + yx^2)) = tr(-xy + yx^2).$$

Proposition I.3.19. *The divergence is a 1-cocycle, i.e*

$$div([u, v]) = u \cdot div(v) - v \cdot div(u).$$

Definition I.3.20. The Lie algebra cocycle div gives rise to a **"Jacobian" group cocycle** $j : TAut_2 \to \mathfrak{tr}_2$ uniquely defined by

$$j(id) = 0 \tag{I.3.6}$$

and

$$\frac{d}{dt}(j(exp(tu)))|_{t=0} = div(u) \tag{I.3.7}$$

verifying the cocycle condition : $j(gh) = j(g) + g \cdot j(h)$.

The generalized KV problem

We can now enunciate the **generalized KV problem**:
Find a tangential automorphism F of $\mathfrak{lie}_2$ such that :

$$F(x + y) = ch(x, y) \tag{I.3.8}$$

$$j(F) = tr(f(x) - f(ch(x, y)) + f(y)) \tag{I.3.9}$$

for some f in $x^2\mathbb{K}[x]$.
It is linked to the original formulation of the KV problem by the following theorem:

Theorem I.3.21. *([AT], 2008) An element $F \in TAut_2$ is a solution of the generalized KV problem if and only if $u = \kappa(F) = (A(x, y), B(x, y))$ satisfies :*

$$x + y - ch(y, x) = (1 - exp(-ad_x))A(x, y) + (exp(ad_y) - 1)B(x, y) \tag{I.3.10}$$

and

$$div(u) = tr\,(-f(x + y) + f(x) + f(y)) \tag{I.3.11}$$

with $f \in tr_1$, where $\kappa(u) = \mathrm{id} - \mathrm{gidg}^{-1}$.

Remark I.3.22. As its name infers, the generalized KV problem actually implies the original KV problem (I.3.8), in which the function f had to be even.

We denote the set of solutions to the generalized KV problem by $SolKV$ ($Sol\widehat{KV}$ in [AT]).
As seen previously, one can deform or rescale these equations by introducing a parameter t : one then ask for $F(x + y) = ch_t(x, y)$ and $j(F) = tr(f(x) - f(ch_s(x, y) + f(y))$.
For $t = 0$, Sol_0KV is the Kashiwara Vergne group KRV :

Definition I.3.23. The Kashiwara Vergne group KRV is composed of tangential automorphism F of $\mathfrak{lie}(x, y)$ such that :

$$F(x + y) = x + y \tag{I.3.12}$$

$$j(F) = tr(f(x) - f(x + y) + f(y)) \tag{I.3.13}$$

for some f in $x^2\mathbb{K}[x]$.

Theorem I.3.24. *([AT], 2008) The group KRV acts on $SolKV$ by multiplication on the right. This action is free and transitive.*

The Kashiwara-Vergne Lie algebra is the Lie algebra of the symmetry group KRV of the KV problem.

Definition I.3.25. The corresponding Lie algebra is the Kashiwara-Vergne Lie algebra $\mathfrak{krv}'$ given by:

$$\mathfrak{krv}' := \{u = (a, b) \in \mathfrak{tder}_2 \ :$$

$$[x, a] + [y, b] = 0, \tag{I.3.14}$$

$$\text{div } (u) = \text{tr } (-f(x + y) + f(x) + f(y))\} \tag{I.3.15}$$

for some element $f \in tr_1$.

Example I.3.26. 1. The element $t = (y, x)$ belongs to $\mathfrak{krv}'$ with $f = 0$.

2. Let $u = ([y, [y, x]], [x, [x, y]]) \in \mathfrak{tder}_2$. Let us check that u is in $\mathfrak{krv}'$. We have :

$$u(x) + u(y) = [x, [y, [y, x]]] + [y, [x[x, y]]] = 0$$

by using Jacobi's identity on the first term of the sum.
One then computes

$$\partial_x a = \partial_x(y^2x - 2yxy + xy^2) = y^2$$

and

$$\partial_y b = \partial_y(x^2y - 2xyx + yx^2) = x^2.$$

Therefore

$$\text{div}(u) = tr(xy^2 + yx^2) = \frac{1}{3}tr(x^3 + y^3 - (x + y)^3)$$

and u is indeed an element of $\mathfrak{krv}'$.

Remark I.3.27. The original notation in [AT] for the Kashiwara-Vergne Lie algebra was $\hat{\mathfrak{kv}}_2$, changed into $\mathfrak{krv}$ in [AET].
For us, it will be more convenient to let $\mathfrak{krv}$ denote the elements of degree at least 3 of $\mathfrak{krv}'$:

$$\mathfrak{krv}' = \mathbb{K}t \oplus \mathfrak{krv}.$$

From now on, we will work with $\mathfrak{krv}$ instead of $\mathfrak{krv}'$.

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I.4 Relating the Grothendieck-Teichmüller, Double shuffle and Kashiwara-Vergne Lie algebras

The following diagram summarizes the existing maps between the double shuffle, Grothendieck-Teichmüller and Kashiwara-Vergne different spaces.

$$\begin{tikzcd}
\text{Torsor} & Assoc(k) \arrow[rr, hook, bend left] & Form.MZVs & SolKV \\
 & \circlearrowright & \circlearrowright & \circlearrowright \\
\text{Group} & GRT_1(k) \arrow[r, hook, "Furusho"] & DS \arrow[r, hook, "Schneps"] & KRV \\
\text{Lie algebra} & \mathfrak{grt}_1(k) \arrow[r, hook] & \mathfrak{ds} \arrow[r, hook] & \mathfrak{krv}
\end{tikzcd}$$

There are natural injections from $\mathfrak{grt}_1$ and $\mathfrak{ds}$ to $\mathfrak{tder}_2$, simply given by :

$$\begin{aligned} \mathfrak{grt}_1 &\longrightarrow \mathfrak{tder}_2 \\ \psi &\longmapsto D_\psi = (0, \psi) \end{aligned}$$

where $D_\psi(x) = 0$ and $D_\psi(y) = [y, \psi]$, and similarly

$$\begin{aligned} \mathfrak{ds} &\longrightarrow \mathfrak{tder}_2 \\ f &\longmapsto D_f = (0, f). \end{aligned}$$

All three Lie algebras can then be seen as subalgebras of derivations of $\mathfrak{lie}_2$, and come equipped with the same grading by the weight (total number of x and y's). They also inherit from $\mathfrak{lie}_2$ a filtration by depth.
Following the notation from the third section, the three Lie algebras have no elements of degree 1 or 2.
The three Lie morphisms defined below respect both the grading and filtration.

Theorem I.4.1. *(Furusho, 2008) The map*

$$f(x, y) \mapsto f(x, -y)$$

gives an injective Lie algebra homomorphism from $\mathfrak{grt}$ *to* $\mathfrak{ds}$.

Theorem I.4.2. *(Alekseev-Torossian [AT], 2008) The map*

$$\nu : \psi \mapsto (\psi(-x-y, x), \psi(-x-y, y))$$

is an injective Lie algebra homomorphism mapping $\mathfrak{grt}_1$ *to* $\mathfrak{krv}$.

Theorem I.4.3. *([S2], 2013) The map*

$$f(x, y) \mapsto (f(-x-y, -x), f(-x-y, -y))$$

is an injective Lie homomorphism from $\mathfrak{ds}$ *to* $\mathfrak{krv}$.

Conjecture I.4.4. *The three spaces are isomorphic :*

$$\mathfrak{grt}_1 \cong \mathfrak{ds} \cong \mathfrak{krv}.$$

Chapter II

Moulds

They rejoice in the mellifluous names of ARI//GARI, ALI//GALI, ALA//GALA, ILI//GILI, AWI//GAWI, AWA//GAWA, IWI//GIWI.

Jean Ecalle, *Combinatorial tidbits from resurgence theory and mould calculus.*

The "elaborate toolbox" of moulds, as Jean Ecalle refers to it in some papers, first emerged from his work on resurgence theory.
The mould setting then appeared extremely suited to what Ecalle calls **dimorphy** : the existence of two encodings and two multiplication rules for one object. Indeed, the main example of dimorphy is the theory of MZV introduced earlier.
Ecalle's plan was to replace multizetas by suitable generating functions, one for each of the two different encodings, such that the multiplications rules and the correspondence between both encodings should be given by simple operations on the generating functions.
He found the framework of the algebra ARI and its group GARI together with the involution *swap* connecting the two encodings to be extremely suitable.
Over the years, Ecalle has written different survey-type articles on moulds, some of them focusing on the multizeta dimorphy. Notations can vary from one article to the other : we try to mention them all, even though he mentioned having fixed them in the latest survey [**?**].

We will here only be using but a very small part of the toolbox, and we refer the reader to all of Ecalle's work (but especially [E1],[E2]) to get a better picture of the complexity and diversity of the system he discovered and is still exploring.
We restrict to presenting the material useful to the study of the Lie algebras $\mathfrak{ds}$ and $\mathfrak{krv}$ defined previously. For most proofs, I will refer to the very detailed text by Schneps [S2]. J. Cresson's text *Calcul Moulien*[C], more directed to the study of analysis, will be of help to us as well. Finally, it is important to keep in mind throughout the rest of this thesis that:
(i) all these operators given in mould-theoretic terms can be applied to a much wider class of moulds than merely polynomial-valued moulds, which permits a number of proofs of results on polynomial-valued moulds (and thus polynomials in x, y) that are not accessible otherwise
(ii) there are some very important mould operators that are not translations of anything that can be phrased in the polynomial situation; this is where the real richness of mould theory comes into play. We do not use any of these in this chapter, but some of them will play a key role in the next .

II.1 Definitions and dictionary with the non-commutative framework

II.1.1 Definition

Ecalle describes moulds as "functions of a variable number of variables", depending on sequences $\boldsymbol{\omega} := (w_1, \dots, w_r)$ of arbitrary length r.

Example II.1.1. We define the mould $T^\bullet$ by :

$$T(\emptyset) = 0, \qquad T(w) = 0 \quad \forall w \in \Omega.$$
$$T(w_1, \dots, w_r) = \frac{1}{(w_2 - w_1)} \cdots \frac{1}{(w_{r-1} - w_r)}.$$

Remark II.1.2. The notation $M^{\boldsymbol{\omega}}$ is the one favored by Ecalle, due to the existence of objects named co-moulds $N_{\boldsymbol{\omega}}$ that can be contracted with moulds. We will prefer to use $M(\boldsymbol{\omega})$, since we are not concerned with comoulds and believe the notation to be more explicit. We therefore generally write M for $M^\bullet$.

The definition and notations we will use most frequently are the ones given by Schneps in ([S1], [S2], [SS]):

Definition II.1.3. Let $(u_1, u_2, \dots), (v_1, v_2, \dots)$ be two infinite sequences of indeterminates. A **bimould** M is a collection of functions

$$M_r\left(\begin{array}{cccc} u_1 & u_2 & \dots & u_r \\ v_1 & v_2 & \dots & v_r \end{array}\right)$$

for each $r \geq 0$, with each M_r a function on the $2r$ variables u_i and v_i. We call M_r the **depth** r part of the mould M. Note that $M(\emptyset) := M_0$ is a constant.
A **mould** is a bimould that is actually only a function of the u_i, and a v**-mould** is a function only of the v_i .
The space of all bimoulds is denoted $BIMU$.

Remark II.1.4. While we will mostly work with moulds, the definition of bimoulds is important to understand some of the operations defined by Ecalle which mix up the two sequences in u_i and v_i. Most moulds encountered in this thesis will be either polynomial or rational with at most a very precise denominator. In addition to these already severe restrictions, we will very often work with moulds **concentrated** in a certain depth r, i.e $M_r(u_1, \dots, u_r)$ is non zero only for a given r.
One should keep in mind that the properties discovered by Ecalle apply to a larger set of objects than what we often restrict them to.

Example II.1.5. 1. The moulds Log and Exp are given by

$$Log(\emptyset) = Exp(\emptyset) = 0$$
$$Log_r(\boldsymbol{\omega}) = Log(w_1, \dots, w_r) = \frac{(-1)^{r+1}}{r}$$
$$Exp_r(\boldsymbol{\omega}) = Exp(w_1, \dots, w_r) = \frac{1}{r!}$$

2. The two v-moulds pic and poc will be useful later :

$$pic(\emptyset) = poc(\emptyset) = 1$$
$$pic(v_1, \dots, v_r) = \frac{1}{v_1 \dots v_r}$$
$$poc(v_1, \dots, v_r) = \frac{1}{v_1(v_1 - v_2) \dots (v_{r-1} - v_r)}$$

II.1.2 Correspondence maps

We now need to understand the connection between moulds and the non-commutative world in which the Lie algebras we are interested in are defined.

Let us consider formal power series $\mathbb{Q}\langle\langle x, y\rangle\rangle$ in non-commutative variables x, y. There is a decreasing filtration on $\mathbb{Q}\langle\langle x, y\rangle\rangle$ induced by the y-degree, called the **depth filtration**. Then every power series $P \in \mathbb{Q}\langle\langle x, y\rangle\rangle$ can be uniquely decomposed as :

$$P = \sum_{r\geq 0} P^{(r)}.$$

Definition/Proposition II.1.6. *There is a $\mathbb{Q}$-linear isomorphism from the space of non-commutative power series in words y-degree r (denoted by $\mathbb{Q}^r\langle\langle x, y\rangle\rangle$) to the complete R-module of power series in $r+1$ commuting variables:*

$$\begin{aligned} \iota_X \ : \quad \mathbb{Q}^r\langle\langle x, y\rangle\rangle &\longrightarrow \mathbb{Q}[[z_0, \dots, z_r]] \\ x^{i_0}y\dots yx^{i_r} &\longrightarrow z_0^{i_0}z_1^{i_1}\dots z_r^{i_r} \end{aligned}$$

By applying this isomorphism to each $P^{(r)}$, we can uniquely represent a power series in $\mathbb{Q}^r\langle\langle x, y\rangle\rangle$ by an infinite sequence of power series $\iota_X(P^{(r)}) \in \mathbb{Q}[[z_0, \dots, z_r]]$ for all $r \geq 0$.

Definition II.1.7. Let $f \in \mathbb{Q}\langle x, y\rangle$ be a polynomial in the non commutative variables x, y. We write f^r for the depth r part of f, i.e the monomials containing exactly r ys, and

$$f = \sum_{\boldsymbol{a}=(a_0,\dots,a_r),r\geq 1} f_{\boldsymbol{a}} x^{a_0}y\dots yx^{a_r}.$$

The mould $vimo_f$ is uniquely defined by

$$vimo_f^r = \iota_X(f^r) = \sum_{\boldsymbol{a}=(a_0,\dots,a_r)} f_{\boldsymbol{a}} z_0^{a_0}\dots z_r^{a_r}$$

Example II.1.8. Let $f = x^2y + yxy + 2y^2x$. Then

$$\begin{aligned} &vimo_f^0(z_0) = 0 \\ &vimo_f^1(z_0, z_1) = z_0^2 \\ &vimo_f^2(z_0, z_1, z_2) = z_1 + 2z_2. \end{aligned}$$

All other $vimo_f^r$ are 0.

This is the simplest way to go from the non-commutative world to the commutative one, but it is not the most useful one for us. From now on, we will often work with non-commutative variables other than x and y.

Definition II.1.9. Let $C_i = ad(x)^{i-1}(y)$, $\mathbb{Q}\langle C\rangle$ be the subring of $\mathbb{Q}\langle x, y\rangle$ generated by the C_i, and let $\mathbb{Q}\langle C\rangle_n$ denote the vector subspace of polynomials in $\mathbb{Q}\langle C\rangle$ of homogeneous degree n in x and y
We denote $\mathbb{Q}\langle C\rangle^r$ the subspace of polynomials of homogeneous degree r (i.e. linear combinations of monomials of the form $C_{a_1}\dots C_{a_r}$), and $\mathbb{Q}\langle C\rangle_n^r$ the intersection.
The space $\mathbb{Q}\langle C\rangle$ is bigraded. If $f \in \mathbb{Q}\langle C\rangle$, we write f_n for its weight n part and f^r for its depth r part.

The following lemma provides us with a way to distinguish elements of $f \in \mathbb{Q}\langle C\rangle$ by their image in moulds.

Lemma II.1.10. *If $f \in \mathbb{Q}\langle x, y\rangle$, then $f \in \mathbb{Q}\langle C\rangle$ if and only if*

$$vimo_f(z_0, ..., z_r) = vimo_f(0, z_1 - z_0, z_2 - z_0, ..., z_r - z_0) \tag{II.1.1}$$

for $r \geq 1$.

Example II.1.11. Let us consider two polynomials in $\mathbb{Q}\langle x, y\rangle$, $f_1 = yx^2 + xyx$ and $f_2 = x^2y - 2xyx + yx^2$. Both are of degree 3 and depth 1, and their images under $vimo$ are given by :

$$vimo_{f_1}(z_0, z_1)) = z_1^2 + z_0 z_1,$$

$$vimo_{f_2}(z_0, z_1) = z_0^2 - 2z_0 z_1 + z_1^2.$$

We see that

$$vimo_{f_1}(0, z_1 - z_0) = z_1^2 - 2z_0 z_1 + z_0^2 \neq vimo_{f_1}(z_0, z_1)$$

whereas

$$vimo_{f_2}(0, z_1 - z_0) = z_1^2 - 2z_0 z_1 + z_0^2 = vimo_{f_2}(z_0, z_1).$$

Indeed, f_1 cannot be rewritten in the C_i, whereas $f_2 = C_3$.

Definition II.1.12. We denote f_C the polynomial f rewritten in the variables Ci, and f_Y the polynomial $\beta(\pi_Y(f))$, where β is the "backward writing" operator and $\pi_Y(f)$ the projection of f onto words starting in y and written in the variables y_i.
We then define the two maps :

$$\iota_Y \,:\, y_{a_1} \dots y_{a_r} \longrightarrow v_1^{a_1-1} \dots v_r^{a_r-1}$$
$$\iota_C \,:\, C_{a_1} \dots C_{a_r} \longrightarrow u_1^{a_1-1} \dots u_r^{a_r-1}$$

from monomials in $\mathbb{Q}^r\langle\langle x, y\rangle\rangle$ to monomials in commutative variables in v and u. From them, we construct the moulds ma_f and v-mould mi_f as follows:

$$ma_f(u_1, \dots, u_r) = (-1)^{r+n}\iota_C(f_C^r)$$
$$mi_f(v_1, \dots, v_r) = \iota_Y(f_Y^r)$$

The maps ma and mi are related to $vimo$ as follows :

Lemma II.1.13. *(Schneps, [S2], Lemma 3.2.1). The moulds ma_f and mi_f are obtained from $vimo_f$ by the formulas*

$$ma_f(u_1, \dots, u_r) = vimo_f(0, u_1, u_1 + u_2, \dots, u_1 + \dots + u_r) \tag{II.1.2}$$

$$mi_f(v_1, ..., v_r) = vimo_f(0, v_r, v_{r-1}, ..., v_1) \tag{II.1.3}$$

Therefore, if $f \in \mathbb{Q}\langle C\rangle$, $vimo_f$, ma_f and mi_f are simply different encodings of the same information.
Here is an example we will follow through the different sections :

Example II.1.14. Let

$$f = [x, [x, y]] + [[x, y], y] = x^2y - 2xyx + yx^2 + xy^2 - 2yxy + y^2x.$$

We have $f_Y = y_3 + y_2 y_1 - 2y_1 y_2$ and $f_C = C_3 - C_1 C_2 + C_2 C_1$.
The Lie polynomial f is of weight 3 and has monomials of depth up to 2.

$$\begin{aligned}
&vimo_f(z_0) = 0\\
&vimo_f(z_0, z_1) = z_0^2 - 2z_0 z_1 + z_1^2\\
&vimo_f(z_0, z_1, z_2) = z_0 - 2z_1 + z_2\\
&vimo_f(z_0, z_1, z_2, z_3) = 0
\end{aligned}$$

$$\begin{aligned}
&ma_f(\emptyset) = 0\\
&ma_f(u_1) = u_1^2\\
&ma_f(u_1, u_2) = -u_1 + u_2\\
&ma_f(u_1, u_2, u_3) = 0
\end{aligned}$$

$$\begin{aligned}
&mi_f(\emptyset) = 0\\
&mi_f(v_1) = v_1^2\\
&mi_f(v_1, v_2) = v_1 - 2v_2\\
&mi_f(v_1, v_2, v_3) = 0
\end{aligned}$$

Remark that the degree d of polynomials ma_f are given by $d = n - r$ with n the weight of f and r its depth.
Of these three maps, *ma* will be the most useful one to go back and forth between the commutative and non-commutative worlds: it will directly provide us with different Lie algebra isomorphisms between moulds and the non-commutative setting.

II.1.3 Operations on moulds

We first introduce basic binary and unary operations on moulds, including the fundamental $swap$ operator, then describe the Lie bracket ari and its ties to the Poisson bracket. Finally, we give the definitions of alternality and its corresponding description for non-commutative words.

Definition II.1.15. Basic binary moulds operations.
Let $M, N \in BIMU$. The addition of moulds is defined by :

$$(M + N)(\boldsymbol{\omega}) = M(\boldsymbol{\omega}) + N(\boldsymbol{\omega}).$$

The multiplication (*mu* or $\times$) of moulds is associative, but non-commutative :

$$(M \times N)(\boldsymbol{\omega}) = mu(M, N)(\boldsymbol{\omega}) = \sum_{\substack{(\boldsymbol{u},\boldsymbol{v}) \text{ s.t.} \\ \boldsymbol{u}\boldsymbol{v}=\boldsymbol{\omega}}} M(\boldsymbol{u}) \cdot N(\boldsymbol{v}) = \sum_{0 \le i \le r} M(w_1, \dots, w_i) N(w_{i+1}, \dots, w_r).$$

The identity is the mould with value 1 on $\emptyset$ and 0 in all other depths.
We denote by $invmu(M)$ the inverse of M for the *mu* multiplication.

Example II.1.16. Let M, N be two moulds on Ω and $w_1, w_2 \in \Omega$. Then

$$mu(M, N)(w_1, w_2) = M(\emptyset)N(w_1, w_2) + M(w_1)N(w_2) + M(w_1, w_2)N(\emptyset)$$

Proposition II.1.17. $(BIMU, +, \times)$ *is an associative, non-commutative algebra.*

Definition II.1.18. Let $BARI$ (resp. ARI, $\overline{ARI}$) be the space of bimoulds M (resp. moulds, resp v-moulds) such that $M_0 := M(\emptyset) = 0$.
We define a Lie bracket on these vector spaces by :

$$limu(M, N) = mu(M, N) - mu(N, M).$$

The bracket $limu$ is sometimes denoted by lu and $BARI$ sometimes called $BIMU^*$.

Proposition II.1.19. *The vector space $BARI$ (resp. ARI, $\overline{ARI}$) equipped with the operation $limu$ forms a Lie algebra.*

We write ARI^{pol} for the susbspace of ARI of polynomial moulds.

Theorem II.1.20. *The map :*

$$ma : \mathbb{Q}\langle C\rangle \to ARI^{pol}$$

is a ring isomorphism, where $\mathbb{Q}\langle C\rangle$ is equipped with the ordinary (concatenation) multiplication of polynomials, and ARI^{pol} with the multiplication mu.

Proof. From the definition of ι_c and ma, one sees that ma gives an isomorphism of vector spaces from $\mathbb{Q}\langle C\rangle$ to ARI^{pol}.
Let $f = C_{a_1} \dots C_{a_r}$ and $g = f = C_{b_1} \dots C_{b_s}$ in $\mathbb{Q}\langle C\rangle$ (by additivity, it is enough to assume that f and g are monomials in the C_i) :

$$ma_{fg} = (-1)^{r+s} u_1^{a_1-1} \dots u_{a_r-1}^{r} u_{r+1}^{b_1-1} \dots u_{r+s}^{b_s-1} = mu(ma_f, ma_g)$$

as both moulds are concentrated in depth r and s respectively. □

Definition II.1.21. Frequently used unary operations. The following linear operations on $BIMU$ are involutions :

$$\begin{aligned} neg(M)(w_1, \dots, w_r) &= M(-w_1, \dots, -w_r) \\ anti(M)(w_1, \dots, w_r) &= M(w_r, w_{r-1}, \dots, w_1) \\ mantar(M) = (w_1, \dots, w_r) &= (-1)^{r-1} M(w_r, w_{r-1}, \dots, w_1) \end{aligned}$$

The $push$ operator is a cyclic permutation of order $r + 1$ on each $BIMU_r$:

$$push(M)\begin{pmatrix} u_1 & \dots & u_r \\ v_1 & \dots & v_r \end{pmatrix} = M\begin{pmatrix} -u_1 \dots - u_r & u_1 & \dots & u_{r-1} \\ -v_r & v_1 - v_r & \dots & v_{r-1} - v_r \end{pmatrix}.$$

A (bi)mould M is said to be $push$ (resp neg, $anti$, $mantar$)-invariant if $push(M) = M$ (resp neg, $anti$, $mantar$).

These first four operators can be considered as operators on ARI or $\overline{ARI}$ by forgetting about the variables u_i or v_i.

Definition II.1.22. The $swap$ operator, though, exchanges the variables u_i and v_i.

$$swap(M)\begin{pmatrix} u_1 & \dots & u_r \\ v_1 & \dots & v_r \end{pmatrix} = M\begin{pmatrix} v_r & v_{r-1} - v_r & \dots & v_1 - v_2 \\ u_1 + \dots + u_r & u_1 + \dots + u_{r-1} & \dots & u_1 \end{pmatrix}$$

It can still be seen as an operator from ARI to $\overline{ARI}$ where

$$M(u_1, \dots, u_r) \mapsto (swapM)(v_1, \dots, v_r) = M(v_r, v_{r-1} - v_r, \dots, v_1 - v_2)$$

or indeed as an operator from $\overline{ARI}$ to ARI with :

$$M(v_1, \dots, v_r) \mapsto (swapM)(u_1, \dots, u_r) = M(u_1 + \dots + u_r, u_1 + \dots + u_{r-1}, \dots, u_1).$$

Ecalle defines a (bi)mould M to be **dimorphic** if both M and $swap(M)$ belong to a certain *symmetry type* (not necessarily the same).

Remark II.1.23. From the definitions of *ma* and *mi* and equations II.1.2 and II.1.3, one easily obtains the relation :

$$swap(ma_f) = mi_f.$$

Example II.1.24. Coming back to example II.1.14, in depth 2 we had $ma_f(u_1, u_2) = -u_1 + u_2$. Then

$$swap(ma_f)(v_1, v_2) = ma_f(v_2, v_1 - v_2) = -v_2 + v_1 - v_2 = v_1 - 2v_2 = mi_f(v_1, v_2).$$

This relation between ma_f and mi_f gives us a way to translate properties of a non-commutative polynomial f into (potentially) a dimorphic property of the associated mould.

II.1.4 The Lie algebra ARI

The ari-bracket

The $limu$ bracket defined previously is the first and easiest Lie algebra structure one can define on moulds, but Ecalle introduced a different Lie bracket called *ari* which turns out to be extremely important for us.
In order to construct the *ari* bracket, we will introduce the essential building blocks of operations on bimoulds, called **flections**, or flexions, or **sequence contractions** (all Ecalle's terms).

Definition/Example II.1.25. *The four* ***flectors*** *are denoted by the symbols* $\rceil$, $\lceil$, $\rfloor$, $\lfloor$ *and depend on the factorisation of a given sequence*

$$\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r) = \left(\begin{smallmatrix} u_1 & \ldots & u_r \\ v_1 & \ldots & v_r \end{smallmatrix} \right).$$

For the formal definition, we refer you to [S1], 2.2, Flexions.
We prefer to provide the reader with some enlightening examples :

Let $\boldsymbol{\omega} = \boldsymbol{ab}$ *with* $\boldsymbol{a} = \left(\begin{smallmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{smallmatrix} \right)$ *and* $\boldsymbol{b} = \left(\begin{smallmatrix} u_4 & u_5 & u_6 \\ v_4 & v_5 & v_6 \end{smallmatrix} \right)$. *Then*

$$\boldsymbol{a}\rfloor = \left(\begin{matrix} u_1 & u_2 & u_3 \\ v_1 - v_4 & v_2 - v_4 & v_3 - v_4 \end{matrix} \right)$$

$$\lceil \boldsymbol{b} = \left(\begin{matrix} u_1 + u_2 + u_3 + u_4 & u_5 & u_6 \\ v_4 & v_5 & v_6 \end{matrix} \right)$$

$$\boldsymbol{a}\rceil = \left(\begin{matrix} u_1 & u_2 & u_3 + u_4 + u_5 + u_6 \\ v_1 & v_2 & v_3 \end{matrix} \right)$$

$$\lfloor \mathbf{b} = \left(\begin{matrix} u_4 & u_5 & u_6 \\ v_4 - v_3 & v_5 - v_3 & v_6 - v_3 \end{matrix} \right)$$

.

Remark II.1.26. The definitions of flexions on ARI (resp. $\overline{ARI}$) are restricted to modifications of the u_i (resp. v_i).

We first give an explicit definition of the *ari* product, then give another presentation using the derivation $arit$.

Definition II.1.27. The product *ari* on $BARI$ is explicitely defined by :

$$\begin{aligned} ari(M,N)(\boldsymbol{\omega}) :=& limu(M,N)(\boldsymbol{\omega}) + \\ &\sum_{\boldsymbol{\omega}=\boldsymbol{bcd}} (M(\lfloor \boldsymbol{c})N(\boldsymbol{b}\rceil \boldsymbol{d}) - (N(\lfloor \boldsymbol{c})M(\boldsymbol{b}\rceil \boldsymbol{d})) + \\ &\sum_{\boldsymbol{\omega}=\boldsymbol{abc}} (M(\boldsymbol{a}\rceil \boldsymbol{c})N(\boldsymbol{b}\rfloor) - N(\boldsymbol{a}\rceil \boldsymbol{c})M(\boldsymbol{b}\rfloor)) \end{aligned}$$

with $\boldsymbol{b}, \boldsymbol{c} \neq \emptyset$.

Example II.1.28. Let $M, N \in BARI$ and $\boldsymbol{\omega} = (w_1, w_2)$. Then

$$\begin{aligned} ari(M,N)(\boldsymbol{\omega}) =& M(w_1)N(w_2)) - N(w_1)M(w_2) \\ &+ M\binom{u_2}{v_2 - v_1} N\binom{u_1+u_2}{v_1} - N\binom{u_2}{v_2-v_1} M\binom{u_1+u_2}{v_1} \\ &+ M\binom{u_1+u_2}{v_2} N\binom{u_1}{v_1-v_2} - N\binom{u_1+u_2}{v_2} M\binom{u_1}{v_1-v_2}. \end{aligned}$$

Now let $M, N \in ARI$ and $\boldsymbol{\omega} = (u_1, u_2)$. We simply get :

$$\begin{aligned} ari(M,N)(\boldsymbol{\omega}) =& M(u_1)N(u_2)) - N(u_1)M(u_2) \\ &+ M(u_2)N(u_1+u_2) - N(u_2)M(u_1+u_2) \\ &+ M(u_1+u_2)N(u_1) - N(u_1+u_2)M(u_1). \end{aligned}$$

Remark II.1.29. The definition on ARI (resp. $\overline{ARI}$) is given by the same formula with no lower flectors (resp. upper).

The Lie bracket *ari* is traditionally constructed from derivations (*anit*, *amit*, *arit*) built from the above flexions themselves.

Definition/Proposition II.1.30. *For any bimould $B \in BIMU$, the operators $anit(B)$ and $amit(B)$ are defined by :*

$$anit(B) \cdot A = \sum_{\substack{\boldsymbol{abc} \\ \boldsymbol{a},\boldsymbol{b}\neq\emptyset}} A(a\lceil c)B(\rfloor b), \tag{II.1.4}$$

$$amit(B) \cdot A = \sum_{\substack{\boldsymbol{abc} \\ \boldsymbol{b},\boldsymbol{c}\neq\emptyset}} A(a\rceil c)B(b\lfloor). \tag{II.1.5}$$

Finally, the operator $arit(B)$ is given by :

$$arit(B) \cdot A = amit(B) \cdot A - anit(B) \cdot A \tag{II.1.6}$$

All three operators are derivations for the bracket $limu$ ([S2], prop 2.2.1).
The ari- bracket can then be written as :

$$ari(A,B) = arit(B) \cdot A - arit(A) \cdot B + limu(A,B). \tag{II.1.7}$$

Proposition II.1.31. *The ari bracket is a Lie bracket, therefore ARI, $\overline{ARI}$ and $BARI$ are Lie algebras under ari.*

From now on, we will write ARI for ARI_{ari}, i.e equipped with the Lie bracket ari (similarly for $\overline{ARI}$). If we want to work with a different Lie bracket (such as $limu$ or the yet to be defined $Dari$), it will be indicated as a subscript.

Let us now relate the ari-bracket to the non-commutative world, and see why it will have a central place in the study of mould theoretic $\mathfrak{ds}$ and $\mathfrak{krv}$ spaces.

Proposition II.1.32. *Let $f \in \mathbb{Q}\langle C\rangle_n$ be of homogeneous depth r and $g \in \mathbb{Q}\langle C\rangle_m$ of homogeneous depth s. Let D_f be as usual the derivation of $\mathbb{Q}\langle C\rangle$ defined by $D_f(x) = 0, D_f(y) = [y, f]$. Then*

$$ma_{D_f(g)} = -(arit(ma_f)) \cdot (ma_g). \tag{II.1.8}$$

This allows us to state the following result linking the Poisson bracket to the ari-bracket.

Corollary II.1.33. *As above, let $f \in \mathbb{Q}\langle C\rangle_n$ be of homogeneous depth r and $g \in \mathbb{Q}\langle C\rangle_m$ of homogeneous depth s. Then :*

$$ma_{\{f,g\}} = ari(ma_f, ma_g). \tag{II.1.9}$$

Remark II.1.34. Note that this only applies to polynomials in the C_i, whereas the ari-bracket is defined on a much bigger space. One should then be very careful and precise in which setting the ari and Poissn bracket are "the same".

The group GARI

Since we have turned the space ARI into a Lie algebra, we now construct the associated Lie group.

Definition II.1.35. We denote $GARI$ (resp. $\overline{GARI}$) the set of moulds (resp. v-moulds) with constant term 1, and $GBARI$ the set of bimoulds with constant term 1.

For each mould B in $GBARI$ we can associate an automorphism of $GBARI$ denoted $garit_B$ by the formula:

$$garit_B \cdot A = \sum_{\substack{w=a_1b_1c_1\ldots a_sb_sc_s \\ b_i\neq\emptyset, a_ic_{i+1}\neq\emptyset}} A(\lceil b_1\rceil \ldots \lceil b_s\rceil)B(a_1\rfloor) \ldots B(a_s\rfloor)invmu(B)(\lfloor c_1) \ldots invmu(B)(\lfloor c_s) \tag{II.1.10}$$

with $s \geq 1$. As for ari, the expression for $garit_B$ on $GARI$ resp. $\overline{GARI}$ are obtained by ignoring the lower resp. upper flexions.
We denote by $inv_{gari}(B)$ the inverse of a mould B for the group law $gari$.

Definition II.1.36. The group law on $GARI$, denoted $gari$, is given by :

$$gari(A, B) = mu(garit_B \cdot A, B). \tag{II.1.11}$$

We call $expari$ the standard exponential map on the Lie algebra ARI, and $logari$ its inverse map. One retrieves the group law $gari$ on $GARI$ via the BCH formula :

$$gari(expari(M_1), expari(M_2)) = expari(ch(M_1, M_2)).$$

$GARI$ acts on its Lie algebra ARI by the standard adjoint action, denoted Ad_{ari} (sometimes $adari$ in Ecalle's work or $Adari$ in [S3]), by :

$$\begin{aligned} Ad_{ari}(A) \cdot B =& \frac{d}{dt}|_{t=0}(gari(A, expari(tB), invgari(A)) \\ =& B + ari(logari(A), B) + 2ari(logari(A), ari(logari(A), B)) + \cdots \end{aligned}$$

II.1.5 Alternality

We present here one of fundamental mould symmetry, alternality, and give its relation to the non-commutative framework.

Definition II.1.37. Let $\boldsymbol{u}, \boldsymbol{v}$ be two sequences. The **shuffle** $sh(\boldsymbol{u}, \boldsymbol{v})$ is given by the set of sequences $\boldsymbol{\omega}$ obtained by shuffling the elements of the two sequences $\boldsymbol{u}$ and $\boldsymbol{v}$ while preserving the internal order of the elements in these sequences.

Example II.1.38. Let $\boldsymbol{u} = (w_1, w_2)$ and $\boldsymbol{v} = (w_3)$.Then

$$sh(\boldsymbol{u}, \boldsymbol{v}) = \{(w_1, w_2, w_3), (w_1, w_3, w_2), (w_3, w_1, w_2)\}.$$

Definition II.1.39. A mould $M \in ARI$ or $\overline{ARI}$ is said to be **alternal** if for all pairs of sequences u, v in the u_i :

$$\sum_{\boldsymbol{\omega} \in sh(u,v)} M(\boldsymbol{\omega}) = 0. \tag{II.1.12}$$

A mould $M \in GARI$ or $\overline{GARI}$ is **symmetral** if for all pairs of sequences u, v in the u_i, we have :

$$\sum_{\boldsymbol{\omega} \in sh(u,v)} M(\boldsymbol{\omega}) = M(u)M(v).$$

Note that alternality is a property preserved by depth, so to prove that a mould is alternal one "only" needs to check II.1.12 for each depth.

Remark II.1.40. In depth 2, the definition yields the following equation:

$$M(w_1, w_2) + M(w_2, w_1) = 0,$$

In depth 3, there are two possible shuffles to consider : $sh((w_1, w_2), w_3)$ and $sh(w_1, (w_2, w_3))$, yielding the two equations :

$$M(w_1, w_2, w_3) + M(w_1, w_3, w_2) + M(w_3, w_1, w_2) = 0$$

$$M(w_1, w_2, w_3) + M(w_2, w_1, w_3) + M(w_2, w_3, w_1) = 0$$

The second equation is in fact automatically satisfied if the first is, by using the change of variables $w_1 \mapsto w_3$, $w_2 \mapsto w_1$ and $w_3 \mapsto w_2$.
It is actually enough to check the relation for the pairs $(\boldsymbol{\omega}^1, \boldsymbol{\omega}^2) = (w_1, \dots, w_s), (w_{s+1}, \dots, w_r)$ for $1 \leq s \leq [\frac{r}{2}]$ since all shuffle relations can be deduced from these by a change of variables.

Example II.1.41.

1. Take A to be the mould concentrated in depth 3 given by

   $$A(u_1, u_2, u_3) = \frac{1}{u_1 u_2} - \frac{2}{u_1 u_3} + \frac{1}{u_2 u_3}.$$

   Then

   $$A(u_1, u_2, u_3) + A(u_1, u_3, u_2) + M(u_3, u_1, u_2) = 0.$$

2. The mould B given by $B(\emptyset) = 1$, $B(u_1, \dots, u_r) = \frac{1}{u_1(u_1+u_2)\dots(u_1+\dots+u_r)}$ is symmetral. In depth 2, one sees that:

   $$\frac{1}{u_1(u_1 + u_2)} + \frac{1}{u_2(u_1 + u_2)} = \frac{u_2 + u_1}{u_1 u_2 (u_1 + u_2)} = \frac{1}{u_1 u_2} = B(u_1) \cdot B(u_2).$$

   One proves the result for all depths by a simple recurrence.

3. The mould T defined in example II.1.2 is alternal, see [C], 6.1 for the full proof.

4. The mould $lopil$ in $\overline{ARI}_{\overline{ari}}$ defined by :

$$lopil(v_1, \dots, v_r) = c_r \frac{v_1 + \cdots + v_r}{v_1(v_1 - v_2) \cdots (v_{r-1} - v_r)v_r}$$

   is alternal.

Proposition II.1.42. *(see [S2], prop 2.6.1) We have*

$$expari(ARI_{al}) = GARI_{as}.$$

Now, let us see why the alternality is such an important property for our purposes.

Definition/Proposition II.1.43. *Let $\mathfrak{lie}_C$ denote the degree completion of the Lie algebra $L[[C_1, C_2, \dots]]$ on the C_i. By Lazard elimination, $\mathfrak{lie}_C$ is free on the C_i and*

$$\mathfrak{lie}_2 \simeq \mathbb{Q}[x] \oplus \mathfrak{lie}_C. \tag{II.1.13}$$

Thus, Lazard elimination shows that every polynomial $b \in \mathfrak{lie}_2$ having no linear term in x can be written uniquely as a Lie polynomial in the C_i.

Now, let Δ denote the standard cobracket on $\mathbb{Q}\langle C\rangle$, defined by

$$\Delta(C_i) = C_i \otimes 1 + 1 \otimes C_i.$$

Then $\mathfrak{lie}_C$, seen as a subspace of $\mathbb{Q}\langle C\rangle$, is the space of primitive elements for Δ, i.e. elements satisfying

$$\Delta(f) = f \otimes 1 + 1 \otimes f.$$

This condition on f is given explicitly on the coefficients of f by the family of shuffle relations :

$$\sum_{u \in sh(C_{a_1} \dots C_{a_r}, C_{b_1} \dots C_{b_s})} (f|u) = 0,$$

But these conditions are exactly equivalent to the alternality relations

$$\sum_{u \in sh((a_1, \dots, a_r), (b_1 \dots b_s))} (f|u) = 0.$$

Together with Theorem II.1.20 and Corollary II.1.33, this yields :

Proposition II.1.44. *The map*

$$ma : \mathfrak{lie}_C \to ARI_{al}^{pol} \tag{II.1.14}$$

is an isomorphism of Lie algebras, where $\mathfrak{lie}_C$ is equipped with the Poisson bracket and ARI with the bracket ari.

Remark II.1.45. The Lie algebra $\mathfrak{lie}_C$ equipped with the Poisson bracket is often denoted $\mathfrak{mt}$ for the Twisted Magnus Lie algebra.

II.1.6 Subalgebras of ARI

Notations

We use the notation $ARI_{a/b}$ for the space of moulds having property a and whose swaps have property b; for example, $ARI_{al/al}$ denotes the space of alternal moulds with alternal swap.
The slightly more general notation ARI_{a*b} denotes the space of moulds having property a and whose swap has property b up to adding on a constant-valued mould; thus, we write ARI_{al*al} for the space of alternal moulds whose swaps are alternal up to adding on a constant-valued mould.
Finally, the notation $ARI_{\underline{a}/\underline{b}}$ denotes the subspace of $ARI_{a/b}$ of moulds that are even functions of u_1 in depth 1.

Example II.1.46. An example of a mould in ARI_{al*al} is the mould $\Delta^{-1}(A)$, where A is the polynomial mould concentrated in depth 3 given by

$$A(u_1,u_2,u_3) = -\frac{1}{4}u_1^3u_2 + \frac{1}{4}u_1^3u_3 - \frac{1}{4}u_1^2u_2^2 + \frac{1}{2}u_1^2u_3^2 + \frac{1}{4}u_1u_3^3 - \frac{1}{4}u_2^2u_3^2 - \frac{1}{4}u_2u_3^3$$
$$-\frac{1}{12}u_1^2u_2u_3 + \frac{1}{6}u_1u_2^2u_3 - \frac{1}{12}u_1u_2u_3^2$$

and $\Delta^{-1}(A)(u_1,u_2,u_3) = \frac{1}{u_1u_2u_3(u_1+u_2+u_3)}A(u_1,u_2,u_3)$. It is easy to check that $\Delta^{-1}(A)$ is alternal, but its swap is not alternal unless one adds on the constant $1/3$.

Proposition II.1.47. *The subspace ARI_{pol} of polynomial-valued moulds in ARI forms a Lie algebra under the ari bracket.*

This follows immediately from the definition of *ari*, since the flexion operations are polynomial.

Theorem II.1.48. *(Ecalle, Schneps) The subspaces ARI_{al} and $\overline{ARI}_{al}$ are Lie algebras under the Lie bracket ari.*

We refer the reader to [S2], Annex A5 for the proof. The following theorem will prove extremely important for the next section.

Theorem II.1.49. *([SS], Theorem 3.3) The subspaces $ARI_{\underline{al}/\underline{al}}$ and $ARI_{\underline{al}*\underline{al}}$ form Lie algebras under the ari bracket.*

The proof of this theorem, not too complicated but technical, is based on two following properties:

Proposition II.1.50. *([SS], 3.4) If $A \in ARI_{\underline{al}*\underline{al}}$, then A is neg-invariant and push-invariant.*

Proposition II.1.51. *([SS], 3.5)*
If A, B are push-invariant moulds in ARI, then

$$swap(ari(swap(A), swap(B))) = ari(A, B).$$

We are now equipped to tackle the study of $\mathfrak{ls}$ and $\mathfrak{ds}$ in the mould framework.

II.2 The mould double shuffle Lie algebra

II.2.1 Introduction

In [SS], Salerno and Schneps reproved Racinet's theorem stating that $\mathfrak{ds}$ is a Lie algebra under the Poisson bracket using the moulds machinery. In this section, we give the mould theoretic versions of $\mathfrak{ls}$ and $\mathfrak{ds}$ and state some important results leading to Salerno and Schneps' proof.
The mould *pal* defined by equation (II.2.2) is key to this proof : it is instrumental in proving the existence of a Lie algebra isomorphism between dimorphic moulds $\underline{al} * \underline{al}$ and moulds with the $\underline{al} * \underline{il}$ dimorphy, the latter containing $\mathfrak{ds}$.

II.2.2 The linearized double shuffle space $\mathfrak{ls}$

Recall the definition of $\mathfrak{ls}$:

Definition II.2.1. The linearized double shuffle space $\mathfrak{ls}$ is defined to be the set of polynomials f in x, y of degree ≥ 3 satisfying the shuffle relations and :

$$\sum_{w \in sh(u,v)} (\pi_y(f)|w) = 0$$

where $\pi_y(f)$ is the projection of f onto the words ending in y, rewritten in the variables $y_i = x_{i-1}y$, u, v are words in the y_i and w belongs to their shuffle in the alphabet y_i.

The previous subsection on alternality provided us with the tools to state the following theorem :

Theorem II.2.2. *([S2], Theorem 3.4.3)*
The map $f \to ma_f$ yields a Lie algebra isomorphism

$$\mathfrak{ls} \to ARI^{pol}_{\underline{al} * \underline{al}}.$$

As promised in the introduction, this description of $\mathfrak{ls}$ in moulds allows for a very simple proof of the property below.

Proposition II.2.3. *The space $\mathfrak{ls}^d_n$ of weight n and depth d is zero if $n \neq d$ mod 2.*

Proof. We first translate this statement into the mould language : it means that if $M \in ARI^{pol}_{\underline{al}/\underline{al}}$ is a homogeneous mould $M(u_1, \dots, u_d)$ of odd degree $n - d$ then M must be zero. But we saw previously that elements of $ARI_{\underline{al}/\underline{al}}$ are *neg*-invariant, i.e

$$M(-u_1, \dots, -u_d) = M(u_1, \dots, u_d).$$

If M is homogeneous of odd degree and respects the equation above then M is zero. □

II.2.3 The second defining condition of $\mathfrak{ds}$ and alternility

Recall the definition of $\mathfrak{ds}$ given in Chapter 1 :

Definition II.2.4. The Lie algebra $\mathfrak{ds}$ is the set of polynomials $f \in \mathbb{Q}\langle C\rangle$ having the two following properties :

(1) The coefficients of f satisfy the shuffle relations :

$$\sum_{w \in sh(u,v)} (f|w) = 0$$

where u, v are words in x, y and $sh(u, v)$ is the set of words obtained by shuffling them. This condition is equivalent to the assertion that $f \in \mathfrak{lie}_2$.

(2) Let $f_* = \pi_y(f) + f_{corr}$, where $\pi_y(f)$ denotes the projection of f onto words ending in y and

$$f_{corr} = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n}(f|x^{n-1}y)y^n.$$

(When f is homogeneous of degree n, which we usually assume, then f_{corr} is just the monomial $\frac{(-1)^n}{n}(f|x^{n-1}y)y^n$.) The coefficients of f_* satisfy the stuffle relations:

$$\sum_{w\in st(u,v)} (f_*|w) = 0$$

where now u, v and $w \in \mathfrak{h}_1$, considered as rewritten in the variables $y_i = x^{i-1}y$, and $st(u, v)$ is the stuffle of two such words.

In order to translate these defining properties in the moulds language, we rephrase the definition of $\mathfrak{ds}$ slightly:

Lemma II.2.5. *([S2], 3.4.5) The Lie algebra $\mathfrak{ds}$ is equal to the set of $f \in \mathfrak{lie}_2$ of degree ≥ 3 such that f_Y , rewritten in the variables y_i, satisfies all the stuffle relations except for those where both words in the pair (u, v) are powers of y.*

Indeed, we have already translated the first defining condition of $\mathfrak{ds}$: for f to be a Lie word, ma_f must be alternal (see proposition II.1.44).
The second condition will translate as a condition on $swap(ma_f)$ called alternility.

The alternility property relates, of course, to the stuffle product : to each stuffle sum, it associates a alternility sum. Let us start with an example, which is the only alternility sum associated to the stuffle of two letters.

Example II.2.6. Remember that the stuffle of two elements u, v is given by the set $st(u, v) = \{(u, v), (v, u), (u+v)\}$. The associated alternility sum is

$$M(v_1, v_2) + M(v_2, v_1) + \frac{M(v_1) - M(v_2)}{v_1 - v_2} = 0$$

The LHS above is called the alternility sum $M_{1,1}$. Alternility sums do not depend on the sequences that are "stuffled" but entirely on their length, hence the notation.

Definition II.2.7. A mould M in ARI is said to be alternil if it satisfies the alternility relation

$$M_{r,s} = 0$$

for all pairs of integers $1 \leq r \leq s$.
To define $M_{r,s}$, let $\boldsymbol{Y} = (y_1, \dots, y_r)$ and $\boldsymbol{Y'} = (y_{r+1}, \dots, y_{r+s})$ be a couple of sequences of length r and s respectively.
To each term $(y_{i_1}, \dots, y_{i_{r+s}})$ of length $r + s$ in the stuffle we associate the term $M(v_{i_1}, \dots, v_{i_{r+s}})$.
Other terms containing contractions y_{j+k} are associated to terms $\frac{1}{v_j - v_k}(M(\dots, v_j, \dots) - M(\dots, v_k, \dots))$.
The expression $M_{r,s}$ is then given by the sum of all these terms (one for each in the stuffle sum).

Note that if M is a polynomial-valued mould, then the alternility sums are polynomials.

Example II.2.8. Let us detail one of the alternility sums given by the stuffle of four terms, $M_{2,2}$. The stuffle $st((y_i, y_j)(y_k, y_l))$ is given by the sum of the following 13 terms :

$$\{(y_i, y_j, y_k, y_l), (y_i, y_k, y_j, y_l), (y_i, y_k, y_l, y_j),\\ (y_k, y_i, y_j, y_l), (y_k, y_i, y_l, y_j), (y_k, y_l, y_i, y_j),\\ (y_i, y_{j+k}, y_l), (y_{i+k}, y_j, y_l), (y_i, y_k, y_{j+l}),\\ (y_{i+k}, y_l, y_j), (y_k, y_i, y_{j+l}), (y_k, y_{i+l}, y_j), (y_{i+k}, y_{j+l})\}$$

The associated alternility sum for a mould M will therefore be composed of 6 terms involving the depth 4 part of M M_4, corresponding to the 6 first terms (which are similar to the shuffle terms for the same elements), 6 terms involving M_3 corresponding to the 6 terms with one contraction, and finally one term with M_2.

$$\begin{aligned}
&M_{2,2} = M(v_1, v_2, v_3, v_4) + M(v_1, v_3, v_2, v_4) + M(v_1, v_3, v_4, v_2)\\
&+M(v_3, v_1, v_2, v_4) + M(v_3, v_1, v_4, v_2) + M(v_3, v_4, v_1, v_2)\\
&+\frac{1}{v_2 - v_3}(M(v_1, v_2, v_4) - M(v_1, v_3, v_4)) + \frac{1}{v_1 - v_3}(M(v_1, v_2, v_4) - M(v_3, v_2, v_4))\\
&+\frac{1}{v_2 - v_4}(M(v_1, v_3, v_2) - M(v_1, v_3, v_4)) + \frac{1}{v_1 - v_3}(M(v_1, v_4, v_2) - M(v_3, v_4, v_2))\\
&+\frac{1}{v_2 - v_4}(M(v_3, v_1, v_2) - M(v_3, v_1, v_4)) + \frac{1}{v_1 - v_4}(M(v_3, v_1, v_2) - M(v_3, v_4, v_2))\\
&+\frac{1}{(v_1 - v_3)(v_2 - v_4)}(M(v_1, v_2) - M(v_1, v_4) + M(v_3, v_2) - M(v_3, v_4)).
\end{aligned}$$

The definition of alternility now allows us to state the main theorem of this section :

Theorem II.2.9. *([SS], Theorem 4.3) The map ma restricts to a Lie algebra isomorphism between the spaces :*

$$ma : \mathfrak{ds} \to ARI^{pol}_{\underline{al}*\underline{il}}. \tag{II.2.1}$$

Whereas it is relatively easy to prove the existence of a vector space isomorphism between $\mathfrak{ds}$ and $ARI^{pol}_{\underline{al}*\underline{il}}$, proving that $ARI^{pol}_{\underline{al}*\underline{il}}$ is indeed a Lie algebra requires some additional work, and the use of more advanced results and tools from Ecalle. It provides a proof that $\mathfrak{ds}$ is a Lie algebra under the Poisson bracket, different from Racinet's and Furusho's. Before presenting the necessary material, let us come back to example II.1.14 and see that it belongs to $ARI^{pol}_{\underline{al}*\underline{il}}$.

Example II.2.10. Let us once more consider the polynomial $f = x^2y-2xyx+yx^2+xy^2-2yxy+y^2x$. We previously showed that it belongs to $\mathfrak{ds}$, see I.1.37.
We have seen in II.1.14 that ma_f and mi_f are only non zero in depth 1 and 2 :

$$ma_f(u_1) = u_1^2 \qquad ma_f(u_1, u_2) = u_2 - u_1,$$

$$mi_f(v_1) = v_1^2 \qquad mi_f(v_1, v_2) = v_1 - 2v_2.$$

First observe that in depth one, both ma_f and mi_f are given by even functions.
The alternality condition in depth 2 is simply antisymmetry, which is verified by ma_f.

Recall that the first alternility sum is given in depth 2 by :

$$M(v_1, v_2) + M(v_2, v_1) + \frac{M(v_1) - M(v_2)}{v_1 - v_2}.$$

In this case, it yields

$$v_1 - 2v_2 + v_2 - 2v_1 + \frac{v_1^2 - v_2^2}{v_1 - v_2} = 0,$$

which shows that $mi_f = swap(ma_f)$ is alternil.
The element f therefore belongs to $ARI^{pol}_{\underline{al}*\underline{il}}$ with constant 0.

II.2.4 The mould pal and its adjoint action

We now introduce some necessary theory in order to state the theorem allowing us to prove that $ARI^{pol}_{\underline{al}*\underline{il}}$ is a Lie algebra.

Definition II.2.11. Let $dupal \in ARI$ be the mould defined explicitly as follows: $dupal(\emptyset) = 0$ and for each $r \geq 1$,

$$dupal(u_1, \dots, u_r) = \frac{B_r}{r!} \frac{1}{u_1 \dots u_r} \left(\sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} u_{i+1} \right).$$

Remark II.2.12. Note that $dar \cdot dupal$ is a polynomial-valued mould, given explicitely as the image of a power series under ma by :

$$dar \cdot dupal = ma\left(x - \frac{ad_{-y}}{e^{ad_{-y}} - 1}(x) \right).$$

Definition II.2.13. The mould $pal \in GARI$ is then defined recursively by $pal(\emptyset) = 1$ and :

$$dur \cdot pal = mu(pal, dupal). \tag{II.2.2}$$

The first terms are given by

$$\begin{aligned}
pal(\emptyset) &= 0 \\
pal(u_1) &= \frac{1}{2u_1} \\
pal(u_1, u_2) &= \frac{u_1 + 2u_2}{12u_1u_2(u_1 + u_2)} \\
pal(u_1, u_2, u_3) &= \frac{-1}{24u_1u_3(u_1 + u_2)}.
\end{aligned}$$

We denote $invpal$ its inverse $inv_{gari}(pal)$ in $GARI$ for the group law $gari$.

Theorem II.2.14. *(Ecalle, Schneps, [SS] thm 7.2)*
The adjoint map $Ad_{ari}(pal)$ induces a Lie isomorphism of Lie subalgebras of ARI_{ari}:

$$Ad_{ari}(pal) : ARI_{\underline{al}*\underline{al}} \to ARI_{\underline{al}*\underline{il}}.$$

Since we know ARI^{pol} and $ARI_{\underline{al}*\underline{al}}$ to be Lie algebras (see proposition II.1.47 and theorem II.1.49), this powerful theorem yields as a corollary :

Corollary II.2.15. *$ARI^{pol}_{\underline{al}*\underline{il}}$ forms a Lie algebra under the ari-bracket.*

The map $Ad_{ari}(pal)$ and the isomorphism it induces will appear again in Chapter 3, as we will use it in more details in the construction of the injection $\mathfrak{krv} \hookrightarrow \mathfrak{krv}_{ell}$.

II.3 Reformulation of $\mathfrak{krv}$ and definition of the linearized Lie algebra $\mathfrak{lkv}$

In this section, we first describe the defining properties of the Kashiwara Vergne Lie algebra in terms of combinatoric properties of non-commutating words in x, y. All results in this first parts are due to Leila Schneps ([S1]). We then define the linearized Kashiwara-Vergne space and give its mould theoretic version. This finally allows us to compare it to $\mathfrak{ds}$ and $\mathfrak{ls}$, stating isomorphisms results and conjectures.

II.3.1 Special derivations and the push-invariance property

The concept of push-invariance described in this subsection is linked to special derivations, i.e. derivations $u = (a, b) \in \mathfrak{tder}_2$ such that

$$u(x + y) = [x, a] + [y, b] = 0.$$

We would like to consider properties of a or b and therefore be able to work with a certain set of Lie polynomials and the associated depth filtration. In [S1], Leila Schneps defines the "specialness" of a Lie polynomial as follows :

Definition II.3.1. Let $k \geq 3$ and $f \in \mathfrak{lie}_2^k$. Set $b = f(-x - y, y)$. Then f is said to be **special** if there exists a unique $a \in \mathfrak{lie}_2$ such that

$$[x, a] + [y, b] = 0.$$

Example II.3.2. Let us give a simple example by working backwards : we know the derivation $u = ([[x, y], y], [x, [x, y]])$ to be special, since

$$[x, [[x, y], y]] + [[x, [x, y]]] = 0$$

by Jacobi's identity.
Now the associated special f is given by $f = b(-(x + y), y)$ i.e

$$f = [x, [x, y]] + [y, [x, y]].$$

Following Schneps ([S1]), we define some combinatorial properties of monomials in $\mathbb{Q}\langle x, y\rangle$.

Definition II.3.3. Let $w = x^{a_0}yx^{a_1}y...yx^{a_r}$ a monomial of depth r in $\mathbb{Q}\langle x, y\rangle$. We define

$$anti(x^{a_0}yx^{a_1}y...yx^{a_r}) = x^{a_r}yx^{a_{r-1}}y...yx^{a_0}$$

to be the palindrome or backwards writing operator, and the *push* operator by :

$$push(x^{a_0}yx^{a_1}y...yx^{a_r}) = x^{a_r}yx^{a_0}y...yx^{a_{r-1}}.$$

Extending these operators to Lie polynomials by linearity, we then say that a polynomial $f \in \mathfrak{lie}_2$ of degree $k \geq 3$ is **push-invariant** if

$$push(f) = f.$$

It is said to be palindromic if $f = (-1)^{k-1}anti(f)$, and **antipalindromic** if

$$f = (-1)^k anti(f).$$

Example II.3.4. 1. Let $f = [x, y] = xy - yx = x^1yx^0 - x^0yx^1$. Then

$$push(f) = yx - xy = -f$$

and we see that f is not push-invariant.

2. All Lie words are antipalindromic.

Proposition II.3.5. [Schneps, [S1]] *Let $n \geq 3$ and let $b \in \mathfrak{lie}_C$; write $b = b_x x + b_y y = xb^x + yb^y$. Then the following are equivalent:*

(i) There exists a unique element $a \in \mathfrak{lie}_C$ such that $[x, a] + [y, b] = 0$;

(ii) b is push-invariant;

(iii) $b_y = b^y$.

Proof. Let ∂_x denote the derivation of $\mathbb{Q}\langle x, y\rangle$ defined by $\partial_x(x) = 1, \partial_x(y) = 0$. For any polynomial h in x and y, set

$$s(h) = \sum_{i\geq 0} \frac{(-1)^i}{i!}\partial_x^i(h)yx^i \text{ and } s'(h) = \sum_{i\geq 0} \frac{(-1)^i}{i!}x^i y\partial_x^i(h).$$

It is shown by Racinet in [R] that for $f \in \mathfrak{lie}_2$ with usual decomposition we have $f = s(f_y)$ and in a similar way we have $f = s'(f^y)$.

(i)⇒ (iii) If f is special, there exists a unique $a \in \mathfrak{lie}_2$ such that $[x, a] + [b, y] = 0$. Setting $T = yb - by = ax - xa$ and using the decompositions a and b , we get :

$$T = yb_y y + yb_x x - yb^y y - xb^x y = xa^x x + ya^y x - xa_x x - xa_y y.$$

Comparing the terms starting with x and ending with y, we find that $-xb^x y = -xa_y y$, so $b^x = a_y$. Now since b is a Lie word, we must have $b = s(a_y) = s(b^x) = s'(b_x)$.

(iii)⇒ (i) Since $b_y = b^y$, we get

$$b = yb_y y + yb_x x - yb^y y - xb^x y = yb_x x - xb^x y$$

Therefore b has no terms starting and ending in y. By proposition 2.2 in [S1], there exists $a \in (\mathfrak{lie}_2)_{n-1}$ such that $T = ax - xa$. But then the derivation $D_{a,b}$ is special, so b is special.

(ii)⇒ (iii) Let us assume that b is push invariant, and show that $b_y = b^y$. By assumption ,we have :

$$(b|x^{a_0}y...yx^{a_r}) = (b|x^{a_r}yx^{a_0}y...yx^{a_{r-1}}).$$

In particular, for all words with $a_r = 0$, we have $(b|x^{a_0}y...y) = (b|yx^{a_0}y...yx^{a_{r-1}})$, i.e :

$$(b_y y|x^{a_0}y...y) = (yb^y|yx^{a_0}y...yx^{a_{r-1}}),$$

so

$$(b_y|x^{a_0}y...x^{a_{r-1}}) = (b^y|x^{a_0}y...yx^{a_{r-1}}).$$

Thus $b_y = b^y$.

(iii)⇒ (ii) Since b is a Lie polynomial such that $b_y = b^y$, we have $b = s(b_y) = s'(b^y) = s'(b_y)$, i.e.

$$b = \sum_{i\geq 0} \frac{(-1)^i}{i!}\partial_x^i(b_y)yx^i = \sum_{i\geq 0} \frac{(-1)^i}{i!}x^i y\partial_x^i(b^y) = \sum_{i\geq 0} \frac{(-1)^i}{i!}x^i y\partial_x^i(b_y).$$

Using the second and fourth term of the previous equation, we compute the coefficient of a word in b as

$$\begin{aligned}(b|x^{a_0}y...yx^{a_r}) &= \frac{(-1)^{a_r}}{(a_r)!}(\partial_x^{a_r}(b_y)yx^{a_r}|x^{a_0}y...yx^{a_r})\\ &= \frac{(-1)^{a_r}}{(a_r)!}(\partial_x^{a_r}(b_y)|x^{a_0}y...yx^{a_{r-1}})\\ &= \frac{(-1)^{a_r}}{(a_r)!}(x^{a_r}y\partial_x^{a_r}(b_y)|x^{a_r}yx^{a_0}y...yx^{a_{r-1}})\\ &= (b|x^{a_r}yx^{a_0}y...yx^{a_{r-1}}).\end{aligned}$$

Thus b is push-invariant. □

Thanks to this proposition, we can now reformulate the first defining condition of $\mathfrak{krv}$ as follows: the pair of polynomials $a, b \in \mathfrak{lie}_C$ satisfies $[x, a] + [y, b] = 0$ if and only if b is push-invariant and a is its partner.

II.3.2 Push-constance and the trace property.

Definition II.3.6. A polynomial $f \in \mathbb{Q}\langle x, y\rangle$ is said to be **push-constant** if there exists a constant A such that

$$\sum_{v\in Push(w)} (f|v) = A$$

for all $w \neq y^n$, and $(f|y^n) = 0$. The list $Push(w)$ contains the $r+1$ words obtained from w by iterating the $push$ operation (some words can occur more than once in this list).

Example II.3.7. The simplest example of a push-constant polynomial is the sum of all monomials of a given depth, for example

$$b = x^ayx^byx^c + x^cyx^ayx^b + x^byx^cyx^a + x^ayx^cyx^b + x^byx^ayx^c + x^cyx^byx^a.$$

Indeed $Push(x^ayx^byx^c) = [x^ayx^byx^c, x^cyx^ayx^b, x^byx^cyx^a]$ and one easily sees that b is push-constant for the value 3. More interesting push-constant polynomials can be obtained from elements $\psi \in \mathfrak{grt}$ by taking the projection of ψ onto the words ending in y and writing this as by. In this way we obtain for example:

$$b = 2x^2y^2 - \frac{11}{2}xyxy + \frac{9}{2}xy^2x - \frac{1}{2}yx^2y + 2yxyx - \frac{1}{2}y^2x^2.$$

Take $u = x^2y^2$, then $Push(u) = [x^2y^2, yx^2y, y^2x^2]$ and

$$\sum_{v\in Push(u)} (f|v) = 2 + \frac{-1}{2} + \frac{-1}{2} = 1$$

Similarly, for $u = xyxy$, $Push(u) = [xyxy, yxyx, xy^2x]$ and one obtains

$$\sum_{v\in Push(u)} (f|v) = \frac{-11}{2} + 2 + \frac{9}{2} = 1.$$

We see that b is push-constant for the constant 1.

Remark II.3.8. Let us denote by $\overline{C}(w)$ the list of the n words obtained by cyclic permutations of w, and let $v = uy$ be a word ending in y.
We define $\overline{C}_y(v)$ to be $\overline{C}(v) - \{\text{words ending in } x\}$. Writing $\overline{C}_y(v) = \{u_1y, \ldots, u_ry\}$, one obtains $\{u_1, \ldots, u_r\} = Push(u)$.

Example II.3.9. Let $v = x^2yxy$, $v = uy$ with $u = x^2yx$. Then $\overline{C} = \{x^2yxy, xyxyx, yxyx^2, xyx^2y, yx^2yx\}$ and $\overline{C}_y = \{x^2yxy, xyx^2y\}$. Indeed, $Push(u) = \{xyx^2, x^2yx\}$.

Theorem II.3.10. *(Schneps,[S1]) A special derivation $u = (a, b)$ satisfies the trace equation of $\mathfrak{krv}$ if and only if $b^y - b^x$ is push-constant for the constant $(b|x^{n-1}y)$.*

Proof. Recall the second defining equation of $\mathfrak{krv}$ given by:

$$tr(xa_x + b_yy) = \mathrm{tr}\,(-f(x + y) + f(x) + f(y))\} \tag{II.3.1}$$

for some $f \in x^2\mathbb{K}[[x]]$

First, we observe that $tr(f) = 0$ for any Lie polynomial, and therefore $tr(a) = tr(a_xx + a_yy) = 0$, yielding $tr(a_xx) = -tr(a_yy)$. Secondly, we use the specialness of u to obtain $a_y = b^x$ and $b^y = b_y$ by antipalindromy (see proof of thm 1, first implication). The LHS then reads :

$$tr(xa_x + b_yy) = tr(b_yy - a_yy) = tr(b_yy - b^xy) = tr((b_y - b^x)y) = tr((b^y - b^x)y).$$

With only a slight rewriting of the RHS, we will now work with the trace equation

$$tr((b^y - b^x)y) = Atr((x + y)^n) - x^n - y^n)$$

for some constant A and $n \geq 2$.
To show that the equality of both sides is equivalent to the push-constance of $b^y - b^x$, we need to look at it in terms of coefficients of each equivalent class C of cyclic words :

$$\sum_{v\in C}((b^y - b^x)y|v) = |C|A.$$

Since the higher degree terms cancel out in the RHS, the coefficient of y^n in the LHS has to be zero, and we get $(b^y - b^x|y^{n-1}) = 0$. Now for any word $v = uy$, $v \neq y^n$, we have :

$$|C|A = \frac{|C|}{n}\sum_{v\in\overline{C}}((b^y - b^x)y|v) = \frac{|C|}{n}\sum_{v\in\overline{C}_y}((b^y - b^x)y|v) = \frac{|C|}{n}\sum_{u'\in Push(u)}(b^y - b^x|u').$$

This is equivalent to

$$\sum_{u'\in Push(u)}(b^y - b^x|u') = nA$$

for all $u' \neq y^{n-1}$. Together with the fact that $(b^y - b^x|y^{n-1}) = 0$, this is exactly the definition of $b^y - b^x$ being push-constant (for the constant nA).
It remains to prove that $nA = (b|x^{n-1}y)$. If $r = 1$, then w is of depth 1, $|c_w| = n$ and $x^{n-1}y$ is the only word in c_w ending in y. Thus it comes down to

$$\big((b^y - b^x)y\,|\,x^{n-1}y\big) = nA.$$

But since b is a Lie polynomial, we have $(b|x^n) = (b^x|x^{n-1}) = 0$, so using $b^y = b_y$, we also have

$$\big((b^y - b^x)y\,|\,x^{n-1}y\big) = (b^y - b^x\,|\,x^{n-1}) = (b^y|x^{n-1})$$

$$= (b_y|x^{n-1}) = (b_yy|x^{n-1}y) = (b|x^{n-1}y),$$

which proves that $nc = (b|x^{n-1}y)$ as desired. Note that this condition means that if b has no depth 1 part, then $b^y - b^x$ is push-neutral.

□

Definition II.3.11. Let $V_{\mathfrak{krv}}$ be the vector space spanned by polynomials $b \in \mathfrak{lie}_C$ of homogeneous degree $n \geq 3$ such that

(i) b is push-invariant, and

(ii) $b^y - b^x$ is push-constant for the value $(b \mid x^{n-1}y)\}$,

equipped with the Lie bracket

$$\{b, b'\} = [b, b'] + u(b') - u'(b)$$

where $u = (a, b)$, $u' = (a', b')$ and a, a' are the (unique) partners of b and b' respectively.

The aforementioned properties show that the map

$$\begin{array}{rcl} \mathfrak{krv} & \overset{\sim}{\to} & V_{\mathfrak{krv}} \\ u = (a, b) & \mapsto & b \end{array} \tag{II.3.2}$$

is an isomorphism of vector spaces. Since $\mathfrak{krv}$ is known to be a Lie subalgebra of $\mathfrak{sder}_2$, the bracket on $V_{\mathfrak{krv}}$ is inherited directly from this and makes $V_{\mathfrak{krv}}$ into a Lie algebra.

Example II.3.12. Recall the example (2) from chapter 1 : the derivation $u = ([y, [y, x]], [x, [x, y]])$ is indeed an element of $\mathfrak{krv}$.
Now let us check that u satisfies the properties defined above. The special Lie polynomial associated to u is given by

$$f = b(-x - y, y) = [x, [x, y]] + [y, [x, y]].$$

The Lie word $b = [x, [x, y]] = x^2y - 2xyx + yx^2$ is push-invariant:

$$push(x^2y - 2xyx + yx^2) = yx^2 - 2xyx + x^2y.$$

Let us check the push-constance of $b^y - b^x = x^2 - xy + 2yx$. We only have two words to consider:
- for $u = x^2$ the list $Push(u)$ is given by x^2 alone and $(b^y - b^x|x^2) = 1$
- for $u = xy$, $Push(u) = [xy, yx]$ and $(b^y - b^x|xy) + (b^y - b^x|yx) = -1 + 2 = 1$,
which confirms the push-constance of $b^y - b^x$ for the constant $(b|x^2y) = 1$.

Remark II.3.13. We will not translate these properties straight away in terms of moulds. As one will see in the next paragraph, the push-invariance is easily translated, but the push-constance is a bit less tractable.

This description of $\mathfrak{krv}$ is the one used by Schneps to show the existence of an injection form $\mathfrak{ds}$ to $\mathfrak{krv}$, see [S1].

II.3.3 The linearized Kashiwara-Vergne Lie algebra $\mathfrak{lkv}$

This part of the thesis follows the article by Leila Schneps and the author ([RS1]).

Definition

We now consider the depth-graded versions of the defining conditions of $V_{\mathfrak{krv}}$, i.e. determine what these conditions say about the lowest-depth parts of elements $b \in V_{\mathfrak{krv}}$.
The push-invariance being a depth-graded condition, the lowest depth part of b is still push-invariant; in particular, by Proposition II.3.5 it admits of a unique partner $a \in \mathfrak{lie}_C$ such that $[x, a] + [y, b] = 0$, i.e. such that the associated derivation $u = (a, b)$ lies in $\mathfrak{sder}_2$.

For the second condition, let us consider two possibilities.
If b is of degree n and depth $r = 1$ and b^1 denotes the lowest-depth part of b, then $(b^1)^y = x^{n-1}$, so the push-constance condition on b^1 is empty since $(b^1)^y = (b|x^{n-1}y)x^{n-1}$.
If $r > 1$, however, then $(b|x^{n-1}y) = 0$ and so the push-constance condition on $b^y - b^x$ is actually push-neutrality (i.e $A = 0$). In $b^y - b^x$, the part of minimal depth $r-1$ is given by $(b^r)^y$, so the condition is the push-neutrality of $(b^r)^y$.

The property of push constance relate to the non-commutative word $b_y - b_x$, which is not ideal. In the following definition, we reformulate this as a property of b.

Definition II.3.14. A polynomial $b \in \mathbb{Q}\langle C\rangle$ of homogeneous degree n is said to be **circ-constant** if, setting $c = (b|x^{n-1}y)$, we have $b = b_0 + \frac{c}{n}y^n$ where b_0^y is push-constant for the value c .
It is **circ-neutral** if $c = 0$.

Example II.3.15. Consider the polynomial $b = x^2y - xy^2 + 2yxy + \frac{1}{3}y^3$. Using Example (II.3.7), we see that b is circ-constant for the constant $c = 1$.

This leads to the following definition :

Definition II.3.16. The *linearized Kashiwara-Vergne Lie algebra* is defined by

$$\mathfrak{lkv} = \Big\{ b \in \mathfrak{lie}_2 \,\Big|\, \begin{array}{l} (i)\ b \text{ is push-invariant} \\ (ii)\ b \text{ is circ-neutral if } b \text{ is of depth } > 1, \end{array} \Big\},$$

Theorem II.3.17. *([RS1]) The space $\mathfrak{lkv}$ is bigraded by weight and depth, and forms a Lie algebra under the Poisson bracket.*

The proof of Proposition II.3.17 above, that $\mathfrak{lkv}$ is closed under the proposed Lie bracket, is deferred to the next section : it comes as a byproduct in the construction of $\mathfrak{krv}_{ell}$.

Proposition II.3.18. *([RS1]) There is an injective Lie algebra morphism*

$$gr\,\mathfrak{krv} \hookrightarrow \mathfrak{lkv}.$$

This is proven by the very fact that the defining properties of $\mathfrak{lkv}$ are properties held by the lowest-depth parts of elements of $\mathfrak{krv}$, since this means precisely that there is an injective linear map

$$gr\,\mathfrak{krv} \hookrightarrow \mathfrak{lkv},$$

which is a Lie morphism as both spaces are equipped with the Lie bracket coming from $\mathfrak{sder}_2$. It is, however, an open question as to whether these two spaces are equal, since it is not clear that an element satisfying the defining conditions of $\mathfrak{lkv}$ necessarily lifts to an element of $\mathfrak{krv}$. It would be interesting to try to prove equality by starting with a polynomial $\mathfrak{lkv}$ of depth $r > 1$ and finding a way to construct a depth by depth lifting to an element of $\mathfrak{krv}$.

Translating the defining properties of $\mathfrak{lkv}$ into mould language.

Recall the push-operator on moulds in ARI defined by :

$$(push\, M)(u_1, \dots, u_r) = M(u_0, u_1, ..., u_{r-1})$$

where $u_0 = -u_1 - u_2 - \cdots - u_r$.
A mould M is push-invariant if $push(M) = M$.

The following proposition shows that this definition is precisely the translation into mould terms of the property of push-invariance for a Lie polynomial given previously.

Proposition II.3.19. *Let $b \in \mathfrak{lie}_C$. Then b is a push-invariant polynomial if and only if $ma(b)$ is a push-invariant mould.*

Proof. If $b = y$, then $ma(b)$ is concentrated in depth 1 with value $ma(b)(u_1) = 1$, so these are both clearly push-invariant.

Now let $b \in (\mathfrak{lie}_C)_n^{r-1}$ with $n \geq r \geq 2$. We write

$$b = \sum_{\underline{a}=(a_1,\dots,a_r)} k_{\underline{a}}\, x^{a_1} y \cdots y x^{a_r}.$$

Let $f = yb$, so that $b = f^y$. Recalling that $y = C_1$, the associated moulds are related by the formula

$$ma(f)(u_1, ..., u_r) = ma(C_1 b) = u_1^0 ma(b)(u_2, ..., u_r) = ma(b)(u_2, ..., u_r). \tag{II.3.3}$$

Since $b \in (\mathfrak{lie}_C)_n$, we have $\beta(b) = (-1)^{n-1} b$. Set

$$g = \beta(y f^y) = \beta(yb) = (-1)^{n-1} by = (-1)^{n-1} \sum_{\underline{a}} k_{\underline{a}}\, x^{a_1} y ... y x^{a_r} y.$$

As described in , we have

$$swap\big(ma(f)\big)(v_1, \dots, v_r) = (-1)^{n-1} \sum_{\underline{a}} k_{\underline{a}}\, v_1^{a_1} ... v_r^{a_r}.$$

Looking at

$$push(b)y = \sum_{\underline{a}} k_{\underline{a}}\, x^{a_r} y x^{a_1} y \cdots x^{a_{r-1}} y,$$

we see that $push(b)y$ is obtained from by by cyclically permuting the groups $x^{a_i} y$.
Since $b = push(b)$ if and only if $k_{(a_1,\dots,a_r)} = k_{(a_r,a_1,\dots,a_{r-1})}$ for each $\underline{a}$, this is equivalent to

$$swap\big(ma(f)\big)(v_1, ..., v_r) = swap\big(ma(f)\big)(v_r, v_1, \dots, v_{r-1}). \tag{II.3.4}$$

Using the definition of the $swap$, we rewrite (II.3.4) in terms of $ma(f)$ as

$$ma(f)(v_r, v_{r-1} - v_r, \dots, v_1 - v_2) = ma(f)(v_{r-1}, v_{r-2} - v_{r-1}, \dots, v_r - v_1) \tag{II.3.5}$$

We now make the change of variables $v_r = u_1 + ... + u_r$, $v_r - v_1 = u_r$, $v_1 - v_2 = u_{r-1}, \dots, v_{r-2} - v_{r-1} = u_2$, $v_{r-1} = u_1$ in this equation, obtaining

$$ma(f)(u_1 + \cdots + u_r, -u_2 - \cdots - u_r, u_2, \dots, u_{r-1}) = ma(f)(u_1, u_2, \dots, u_r).$$

Finally, using relation (II.3.3), we write this in terms of $ma(b)$ as

$$ma(b)(-u_2 - \cdots - u_r, u_2, \dots, u_{r-1}) = ma(b)(u_2, \dots, u_r).$$

Making the variable change $u_i \mapsto u_{i-1}$ changes this to

$$ma(b)(-u_1 - \cdots - u_{r-1}, u_1, \dots, u_{r-2}) = ma(b)(u_1, \dots, u_{r-1}),$$

which is just the condition of mould push-invariance $ma(b)$ in depth $r - 1$. □

Let us now show how to reformulate the second defining property of elements of $\mathfrak{lkv}$ in terms of moulds.

Definition II.3.20. Let $circ$ be the mould operator defined on moulds in $\overline{ARI}$ by

$$circ(B)(v_1, \dots, v_r) = B(v_2, \dots, v_r, v_1).$$

A mould $B \in \overline{ARI}$ is said to be **circ-neutral** if for $r > 1$ we have

$$\sum_{i=1}^{r} circ^i(B)(v_1, \dots, v_r) = 0.$$

If B is a polynomial-valued mould of homogeneous degree n (i.e. the polynomial $B(v_1, \dots, v_r)$ is of homogeneous degree $n - r$ for $1 \le r \le n$), we say that B is **circ-constant** if

$$\sum_{i=1}^{r} circ^i(B)(v_1, \dots, v_r) = c\Big(\sum_{\substack{a_1+\cdots+a_r=d \\ a_i \ge 0}} v_1^{a_1} \cdots v_r^{a_r} \Big)$$

for all $1 < r \le n$, where $B(v_1) = cv_1^{n-1}$. (If $c = 0$, then a circ-constant mould is circ-neutral.)

Example II.3.21. Let $\psi \in \mathfrak{grt}$ be homogeneous of degree n. Then as we saw in example (II.3.7), the polynomial ψ^y is push-constant, so $\psi^y y$ is circ-constant. For example if $n = 5$, then $\psi^y y$ is given by

$$\psi^y y = x^4y - 2x^3y^2 + \frac{11}{2}x^2yxy - \frac{9}{2}xyx^2y + 3yx^3y + 2x^2y^3 - \frac{11}{2}xyxy^2 + \frac{9}{2}xy^2xy$$
$$-\frac{1}{2}yx^2y^2 + 2yxyxy - \frac{1}{2}y^2x^2y - xy^4 + 4yxy^3 - 6y^2xy^2 + 4y^3xy$$

which is easily seen to be circ-constant.

For an example of a circ-constant mould, we take $B = swap\big(ma(\psi)\big)$, which has the same coefficients as $\psi^y y$: it is given by

$$\begin{aligned} B(v_1) &= v_1^4 \\ B(v_1, v_2) &= -2v_1^3 + \frac{11}{2}v_1^2v_2 - \frac{9}{2}v_1v_2^2 + 3v_2^3 \\ B(v_1, v_2, v_3) &= 2v_1^2 - \frac{11}{2}v_1v_2 - \frac{1}{2}v_2^2 + \frac{9}{2}v_1v_3 + 2v_2v_3 - \frac{1}{2}v_3^2 \\ B(v_1, v_2, v_3, v_4) &= -v_1 + 4v_2 - 6v_3 + 4v_4. \end{aligned}$$

One easily sees that

$$\begin{aligned} &B(v_1, v_2) + B(v_2, v_1) = v_1^3 + v_2^3 + v_1^2v_2 + v_1v_2^2 \\ &B(v_1, v_2, v_3) + B(v_2, v_3, v_1) + B(v_3, v_1, v_2) = v_1^2 + v_2^2 + v_3^2 + v_1v_2 + v_2v_3 + v_1v_3 \\ &\sum_{i=1}^{4} circ^i(B)(v_1, v_2, v_3, v_4) = v_1 + v_2 + v_3 + v_4. \end{aligned}$$

The following result proves that the circ-constance of a polynomial b and that of the associated mould $ma(b)$ are always connected as in the example II.3.21 By additivity, it suffices to prove the result for b a homogeneous polynomial of degree n, so that the circ-constance of b is relative to just one constant $c_n = c = (b|x^{n-1}y)$.

Proposition II.3.22. *Let $b \in \mathbb{Q}\langle C\rangle$ be of homogeneous weight $n \ge 3$. Then b is a circ-constant polynomial if and only if $swap\big(ma(b)\big)$ is a circ-constant mould, and b is circ-neutral if and only if $swap\big(ma(b)\big)$ is circ-neutral.*

Proof. Let β be the backwards-writing operator on $\mathbb{Q}\langle C\rangle$ (or *anti*). Write $b = xb^x + yb^y$, and let $g = \beta(yb^y) = \beta(b^y)y$. For $r \geq 1$, let g^r denote the depth r part of g. If we write the polynomial g^r as

$$g^r = \beta\big((b^y)^{r-1}\big)y = \sum_{\underline{a}=(a_1,\ldots,a_r)} k_{\underline{a}}\, x^{a_1}y\cdots yx^{a_r}y, \tag{II.3.6}$$

then we saw in that

$$swap\big(ma(b)\big)(v_1,\ldots,v_r) = \sum_{\underline{a}=(a_1,\ldots,a_r)} k_{\underline{a}}\, v_1^{a_1}\cdots v_r^{a_r}. \tag{II.3.7}$$

Observe that a polynomial is push-constant if and only it is also push-constant written backwards, so in particular, b^y is push-constant if and only if $\beta(b^y)$ is. Suppose that b is circ-constant, i.e. that b^y and thus $\beta(b^y)$ are push-constant for the value $c = (b|x^{n-1}y)$. In view of (II.3.6), this means that $\sum_{\underline{a}'} k_{\underline{a}'} = c$ when $\underline{a}'$ runs through the cyclic permutations of $\underline{a} = (a_1,\ldots,a_r)$ for every tuple $\underline{a}$, and this in turns means precisely that the mould $swap\big(ma(b)\big)$ is circ-constant. As for the circ-neutrality equivalence, it follows from the circ-constance, since circ-neutrality is nothing but circ-constance for the constant 0. □

The notion of circ-constance will play a role again later, but in this section we only need circ-neutrality. Indeed, we showed that a polynomial b lies in $\mathfrak{lkd}$, i.e. b is a Lie polynomial that is push-invariant and circ-neutral, if and only if the associated mould $ma(b)$ is alternal, push-invariant (by Proposition II.3.19) and its swap is circ-neutral (by Proposition II.3.22). In other words, we have shown that

Proposition II.3.23. *The map ma gives a vector space isomorphism :*

$$ma : \mathfrak{lkd} \xrightarrow{\sim} ARI^{pol}_{al+push/circneut}, \tag{II.3.8}$$

where the right-hand space is the subspace of ARI of polynomial-valued moulds in ARI that are alternal and push-neutral with circ-neutral swap. In fact this map is an isomorphism

$$\mathfrak{lkd}^r_n \simeq ARI^r_{n-r} \cap ARI^{pol}_{al+push/circneut}, \tag{II.3.9}$$

of each bigraded piece, where in general we write ARI^r_d for the subspace of polynomial-valued moulds of homogeneous degree d concentrated in depth r.

We will show in Chapter 3 below that $ARI^{pol}_{al+push/circneut}$ is a Lie algebra under the *ari*-bracket, and thus by the compatibility (II.1.33) of the *ari*-bracket with the Poisson bracket, we will then be able to conclude that $\mathfrak{lkd}$ is also a Lie algebra, proving Proposition II.3.17 of this paper.

II.3.4 Relation to $\mathfrak{ls}$ and $\mathfrak{ds}$

The injective Lie algebra morphism from $\mathfrak{ds}$ to $\mathfrak{krv}$ yields a corresponding bigraded injective map:

$$gr\,\mathfrak{ds} \hookrightarrow gr\,\mathfrak{krv}. \tag{II.3.10}$$

Our next result extends this map to the more general linearized spaces $\mathfrak{ls}$ and $\mathfrak{lkd}$.

Theorem II.3.24. *([RS1]) The Lie injection (II.3.10) extends to a bigraded Lie injection on the associated linearized spaces, giving the following commutative diagram:*

$$\begin{array}{ccc} gr\,\mathfrak{ds} & \hookrightarrow & gr\,\mathfrak{krv}. \\ \downarrow & & \downarrow \\ \mathfrak{ls} & \hookrightarrow & \mathfrak{lkd}. \end{array}$$

For all $n \geq 3$ and $r = 1, 2, 3$, the map is an isomorphism of the bigraded parts

$$\mathfrak{ls}^r_n \simeq \mathfrak{lkd}^r_n.$$

Theorem II.3.24 can be stated very simply in terms of moulds as

$$ARI^{pol}_{\underline{al}/\underline{al}} \subset ARI^{pol}_{al+push/circneut}.$$

We will actually prove the more general result without the polynomial hypothesis.

Theorem II.3.25. *([RS1]) There is an inclusion of mould subspaces*

$$ARI_{\underline{al}/\underline{al}} \subset ARI_{al+push/circneut},$$

Moreover in depths $r \leq 3$, we have

$$ARI^r \cap ARI_{\underline{al}/\underline{al}} = ARI^r \cap ARI_{al+push/circneut}.$$

Proof. It is well-known that every alternal mould satisfies

$$A(u_1, \dots, u_r) = (-1)^{r-1} A(u_r, \dots, u_1)$$

(cf. [S2], Lemma 2.5.3) and that a mould that is al/al and even in depth 1 is also push-invariant (cf. [S2], Lemma 2.5.5). Thus in particular $ARI_{\underline{al}/\underline{al}} \subset ARI_{al+push}$. It remains only to show that a mould in $ARI_{\underline{al}/\underline{al}}$ is necessarily circ-neutral. In fact, since the circ-neutrality condition is void in depth 1, we will show that even a mould in $ARI_{al/al}$ is circ-neutral; the condition of evenness in depth 1 is there to ensure the push-invariance, but not needed for the circ-neutrality.

The first alternality relation is given by

$$A(u_1, \dots, u_r) + A(u_2, u_1, \dots, u_r) + \cdots + A(u_2, \dots, u_r, u_1) = 0.$$

Since A is push-invariant, this is equal to

$$push^r A(u_1, \dots, u_r) + push^{r-1} A(u_2, u_1, \dots, u_r) + \cdots + push\, A(u_2, \dots, u_r, u_1) = 0.$$

But explicitly considering the action of the push operator on each term shows that

$$\begin{aligned} push^i A(u_2, \dots, u_{r-i}, u_1, u_{r-i+1}, \dots, u_r) &= A(u_{i+1}, \dots, u_r, u_0, u_2, \dots, u_i) \\ &= circ^{r-i} A(u_0, u_2, \dots, u_r), \end{aligned}$$

where $u_0 = -u_1 - \cdots - u_r$, so this sum is equal to

$$\sum_{i=0}^{r-1} circ^i A(u_0, u_2, \dots, u_r) = 0,$$

which proves that A is circ-neutral. This gives the inclusion.

Let us now prove the isomorphism in the cases $r = 1, 2, 3$. The case $r = 1$ is trivial since the alternality conditions are void in depth 1. A polynomial-valued mould concentrated in depth 1 is a scalar multiple of u_1^d, which is automatically in $ARI_{al/al}$, and lies in $ARI_{\underline{al}/\underline{al}}$ if and only if d is even. Such a mould is automatically alternal and the circ-neutral condition is void; it is push-invariant thanks to the evenness of d. This shows that in depth 1, both spaces are generated by moulds u_1^d for even d, and are thus isomorphic.

Now consider the case $r = 2$. Let $A \in ARI^{pol}_{al+push/circneut}$ be concentrated in depth 2. The circ-neutral property of the swap is explicitly given in depth 2 by $swap(A)(v_1, v_2) + swap(A)(v_2, v_1) = 0$. But this is also the alternality condition on $swap(A)$, so $A \in ARI_{al/al}$. The isomorphism in depth 2 is thus trivial.

Finally, we consider the case $r = 3$. Let $A \in ARI^{pol}_{al+push/circneut}$ be concentrated in depth 3, and let $B = swap(A)$. Again, we only need to show that B is alternal, which in depth 3 means that B must satisfy the single equation

$$B(v_1, v_2, v_3) + B(v_2, v_1, v_3) + B(v_2, v_3, v_1) = 0.$$

The circ-neutrality condition on B is given by

$$B(v_1, v_2, v_3) + B(v_3, v_1, v_2) + B(v_2, v_3, v_1) = 0.$$

It is enough to show that B satisfies the equality

$$B(v_1, v_2, v_3) = B(v_3, v_2, v_1),$$

since applying this to the middle term of (II.3.4) immediately yields the alternality property (II.3.4) in depth 3. So let us show how to prove (II.3.4).

We rewrite the push-invariance condition in the v_i, which gives

$$\begin{aligned} B(v_1, v_2, v_3) &= B(v_2 - v_1, v_3 - v_1, -v_1) && \text{(II.3.11)} \\ &= B(v_3 - v_2, -v_2, v_1 - v_2) && \text{(II.3.12)} \\ &= B(-v_3, v_1 - v_3, v_2 - v_3). && \text{(II.3.13)} \end{aligned}$$

Making the variable change exchanging v_1 and v_3, this gives

$$\begin{aligned} B(v_3, v_2, v_1) &= B(v_2 - v_3, v_1 - v_3, -v_3) && \text{(II.3.14)} \\ &= B(v_1 - v_2, -v_2, v_3 - v_2) && \text{(II.3.15)} \\ &= B(-v_1, v_3 - v_1, v_2 - v_1). && \text{(II.3.16)} \end{aligned}$$

By (II.3.11), the term $B(v_2 - v_1, v_3 - v_1, -v_1)$ is circ-neutral with respect to the cyclic permutation of v_1, v_2, v_3, so we have

$$B(v_2 - v_1, v_3 - v_1, -v_1) = -B(v_3 - v_2, v_1 - v_2, -v_2) - B(v_1 - v_3, v_2 - v_3, -v_3). \quad \text{(II.3.17)}$$

But the circ-neutrality of B also lets us cyclically permute the three arguments of B, so we also have

$$-B(v_3 - v_2, v_1 - v_2, -v_2) = B(-v_2, v_3 - v_2, v_1 - v_2) + B(v_1 - v_2, -v_2, v_3 - v_2).$$

Using (II.3.11) and substituting this into the right-hand side of (II.3.17) yields

$$\begin{aligned} B(v_1, v_2, v_3) = {} & B(-v_2, v_3 - v_2, v_1 - v_2) \\ & + B(v_1 - v_2, -v_2, v_3 - v_2) - B(v_1 - v_3, v_2 - v_3, -v_3). \end{aligned} \quad \text{(II.3.18)}$$

Now, exchanging v_1 and v_2 in (II.3.16) gives

$$B(v_3, v_1, v_2) = B(-v_2, v_3 - v_2, v_1 - v_2),$$

and doing the same with (II.3.14) gives

$$B(v_3, v_1, v_2) = B(v_1 - v_3, v_2 - v_3, -v_3).$$

Substituting these two expressions as well as (II.3.15) into the right-hand side of (II.3.18), we obtain the desired equality (II.3.4). This concludes the proof of Theorem II.3.24. $\square$

Remark. We conjecture that the inclusion of Theorem II.3.25 is an isomorphism. But even the proof of the simple equality (II.3.4) is surprisingly complicated in depth 3, let alone in higher depth. Computer calculation does lead to the general conjecture:

Conjecture. *If $A \in ARI_{al+push/circneut}$ and $B = swap(A)$, then B is mantar-invariant, i.e. for all $r > 1$, we have*

$$B(v_1, \ldots, v_r) = (-1)^{r-1} B(v_r, \ldots, v_1). \tag{II.3.19}$$

The above *mantar* invariance of B would also yield the following useful partial result, which is the mould analog for $\mathfrak{lkv}$ of the well-known result for $\mathfrak{ls}$, namely that the bigraded part $\mathfrak{ls}^r_n = 0$ when $n \not\equiv r \bmod 2$.

Lemma II.3.26. *Fix $1 \leq r \leq n$. Let $A \in ARI^r_{n-r} \cap ARI^{pol}_{al+push/circneut}$ and let $B = swap(A)$. Assume that B satisfies (II.3.19). Then if $n - r$ is odd, $A = 0$.*

*Proof.*Recall the operator on moulds in ARI (resp. $\overline{ARI}$) defined by

$$mantar(A)(u_1, \ldots, u_r) = (-1)^{r-1} A(u_r, \ldots, u_1) \tag{II.3.20}$$

(resp. the same expression with v_i instead of u_i) and the identity

$$neg{\circ}push = mantar{\circ}swap{\circ}mantar{\circ}swap,$$

where

$$neg(A)(u_1, \ldots, u_r) = A(-u_1, \ldots, -u_r).$$

Let $A \in ARI_{al+push/circneut}$; then A is push-invariant, so applying the left-hand operator to A gives $neg(A)$. Assuming (II.3.19) for $B = swap(A)$, i.e. assuming that $B = mantar(B)$, we see that applying the right-hand operator to A fixes A since on the one hand $swap{\circ}swap = id$ and on the other, $mantar(A) = A$ for all alternal moulds (cf. [S2], Lemma 2.5.3). Thus A must satisfy $neg(A) = A$, i.e. if $A \neq 0$ then the degree $d = n - r$ of A must be even. □

This implies the following result, which is the analogy for $\mathfrak{lkv}$ of the similar well-known result on $\mathfrak{ls}$.

Corollary II.3.27. *If the swaps of all elements of $ARI^{pol}_{al+push/circneut}$ are mantar-invariant, then $ARI^r_d \cap ARI^{pol}_{al+push/circneut} = 0$ whenever d is odd, i.e. by (II.3.9),*

$$\mathfrak{lkv}^r_n = 0 \quad \text{when } n \not\equiv r \bmod 2.$$

Chapter III

The Elliptic story

III.1 Introduction

As mentioned in Chapter 1, Grothendieck-Teichmüller theory was intended to study the automorphism groups of the profinite mapping class groups – the fundamental groups of moduli spaces $\mathcal{M}_{g,n}$ with the goal of discovering new properties of the absolute Galois group $Gal(\mathbb{Q})$.
However, due to the ease of study of the genus zero mapping class groups, which are essentially braid groups, the genus zero case garnered most of the attention, starting from the definition of the Grothendieck-Teichmüller group $\widehat{GT}$ by V.G. Drinfel'd ([Dr]) and of the Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}$ ([I]) in 1991.
The extension of the definition to a Grothendieck-Teichmüller group acting on the profinite mapping class groups in all genera was subsequently given in 2000 by A. Hatcher, P. Lochak, L. Schneps and H. Nakamura (cf. [HLS], [NS]). The higher genus profinite Grothendieck-Teichmüller group satisfies the two-level principle articulated by Grothendieck, which states that the subgroup of $\widehat{GT}$ consisting of automorphisms that extend to the genus one mapping class groups with one or two marked points will automatically extend to automorphisms of the higher mapping class groups.

It has proved much more difficult to extend the Lie algebra Grothendieck-Teichmüller construction to higher genus. Indeed, while the genus zero mapping class groups have a natural Lie algebra analog in the form of the braid Lie algebras, there is no good Lie algebra analog of the higher genus mapping class groups. The only possible approach for the moment seems to be to replace the higher genus mapping class groups by their higher genus braid subgroups, which do have good Lie algebra analogs. An early piece of work due to H. Tsunogai ([Ts]) in 2003 computed the relations that must be satisfied by a derivation acting on the genus one 1-strand braid Lie algebra (which is free on two generators) to ensure that it extends to a derivation on the genus one 2-strand braid Lie algebra, in analogy with the derivations in $\mathfrak{grt}$, shown by Ihara to be exactly those that act on the genus zero 4-strand braid Lie algebra and extend to derivations of the 5-strand braid Lie algebra.

After this, the next real breakthrough in the higher genus Lie algebra situation came with the work of B. Enriquez ([E], 2014).
In particular, using the same approach as Tsunogai of replacing the higher genus mapping class groups with their higher genus braid subgroups, Enriquez was able to extend the definition of $\mathfrak{grt}$ to an elliptic version $\mathfrak{grt}_{ell}$, which he identified with an explicit Lie subalgebra of the algebra of derivations of the algebra of the genus one 1-strand braid Lie algebra that extend to derivations of the 2-strand genus one braid Lie algebra. He showed in particular that there is a canonical surjection $\mathfrak{grt}_{ell} \to \mathfrak{grt}$, and a canonical section of this surjection, $\gamma : \mathfrak{grt} \to \mathfrak{grt}_{ell}$.

An answer was proposed for the double shuffle Lie algebra in [S3], which proposes a definition of an elliptic double shuffle Lie algebra $\mathfrak{ds}_{ell}$ based on mould theory and an elliptic interpretation of a strong theorem due to Ecalle.
The elliptic double shuffle Lie algebra is compatible with Enriquez's construction; in particular there is an injective Lie algebra morphism $\gamma : \mathfrak{ds} \hookrightarrow \mathfrak{ds}_{ell}$ which extends Enriquez's section in the sense that the image of the Lie subalgebra $\mathfrak{grt} \subset \mathfrak{ds}$ is equal to Enriquez's image $\gamma(\mathfrak{grt})$.

One interesting aspect of the mould theoretic approach is that it reveals a close relationship between the elliptic double shuffle Lie algebra and the associated graded of the usual double shuffle Lie algebra for the depth filtration.
In the second section of this chapter, we show that an analogous approach works to construct an elliptic version of $\mathfrak{krv}$, denoted $\mathfrak{krv}_{ell}$, which is given by two defining mould theoretic properties, and again has the key features of:

• being naturally identified with a Lie subalgebra of the derivation algebra of the free Lie algebra on two generators;

• being equipped with an injective Lie algebra morphism $\gamma : \mathfrak{krv} \hookrightarrow \mathfrak{krv}_{ell}$ which extends the Grothendieck-Teichmüller and double shuffle maps;

• having a structure closely related to that of the associated graded of $\mathfrak{krv}$ for the depth filtration.

In independent work, A. Alekseev et al. ([AKKN2]) took a different approach to the construction of higher genus Kashiwara-Vergne Lie algebras $\mathfrak{krv}^{(g,n)}$ for all $g, n \geq 1$, following the classical approach to the Kashiwara-Vergne problem which focuses on determining graded formality isomorphisms between prounipotent fundamental groups of surfaces and their graded counterparts (i.e. the exponentials of the associated gradeds of their associated Lie algebras).

More precisely, if Σ denotes a compact oriented surface of genus g with $n+1$ boundary components, the space $g(\Sigma)$ spanned by free homotopy classes of loops in Σ carries the structure of a Lie bialgebra equipped with the Goldman bracket and the Turaev cobracket. The Goldman-Turaev formality problem is the construction a Lie bialgebra homomorphism θ from $g(\Sigma)$ to its associated graded $\mathrm{gr}\, g(\Sigma)$ such that $\mathrm{gr}\,\theta = \mathrm{id}$. In order to solve this problem, Alekseev et al. defined a family $KV(g, n+1)$ of Kashiwara-Vergne problems. In the particular situation where $(g, n) = (1, 0)$, the surface Σ is of genus 1 with one boundary component, and its fundamental group is free on two generators X, Y, with the boundary loop being given by $C = (X, Y)$. The prounipotent fundamental group is then free on two generators e^x, e^y with a boundary element e^c satisfying $e^c = (e^x, e^y)$. The associated Lie algebra is free on generators x, y and the logarithm of the boundary loop is given by

$$c = ch\Big(ch\big(ch(x, y), -x\big), -y\Big) = [x, y] + \text{higher order terms},$$

where ch denotes the Campbell-Hausdorff law on $\mathfrak{lie}_2$. To define the genus one Kashiwara-Vergne Lie algebra $\mathfrak{krv}^{(1,1)}$, Alekseev et al. first defined the space of derivations u of $\mathfrak{lie}_2$ that annihilate the boundary element c and further satisfy a certain non-commutative divergence condition, and then took $\mathfrak{krv}^{(1,1)}$ to be the associated graded of the above space, which comes down to using the same defining conditions with c replaced by $[x, y]$. They showed that the resulting space is a Lie algebra under the bracket of derivations, and also that, like $\mathfrak{krv}_{ell}$, it is equipped with an injective Lie algebra morphism $\mathfrak{krv} \hookrightarrow \mathfrak{krv}^{(1,1)}$ that extends the morphism $\mathfrak{grt} \hookrightarrow \mathfrak{grt}_{ell}$ constructed by Enriquez.

The last part of this thesis is dedicated to the comparison of these two elliptic Kashiwara-Vergne Lie algebras.

III.2 The elliptic double shuffle Lie algebra

An elliptic version $\mathfrak{ds}_{ell}$ of the double shuffle Lie algebra $\mathfrak{ds}$ was constructed in [S3] using mould theory, and it is shown there that like $\mathfrak{grt}_{ell}$, $\mathfrak{ds}_{ell}$ is a Lie subalgebra of $\mathfrak{oder}_2$, and that there is an injective Lie morphism $\tilde{\gamma} : \mathfrak{ds} \to \mathfrak{ds}_{ell}$ that makes the diagram

$$\begin{array}{ccc} \mathfrak{grt} & \hookrightarrow & \mathfrak{ds} \\ \gamma \downarrow & & \downarrow \tilde{\gamma} \\ \mathfrak{grt}_{ell} & & \mathfrak{ds}_{ell} \\ & \setminus \; / & \\ & \mathfrak{oder}_2 & \end{array}$$

commute.

III.2.1 The Δ operator and the *Dari* bracket

The construction of $\mathfrak{krv}_{ell}$ is modelled on $\mathfrak{ds}_{ell}$. We present here some definitions and facts about moulds that will be used in both this section and the next.
In this chapter, we will not be restricted to polynomial moulds any more, but to rational moulds with a "controlled" denominator.

Definition III.2.1. The operators dar, dur and Δ. We define the following operators on moulds :

$$dar(M)(u_1, \dots, u_r) = u_1 \cdots u_r M(u_1, \dots, u_r) \tag{III.2.1}$$

$$dur(M)(u_1, \dots, u_r) = (u_1 + \cdots + u_r) M(u_1, \dots, u_r) \tag{III.2.2}$$

$$\Delta(M)(u_1, \dots, u_r) = u_1 \cdots u_r (u_1 + \cdots + u_r) M(u_1, \dots, u_r) \tag{III.2.3}$$

for $r \geq 1$, and let ARI^{Δ} denote the space of rational-function moulds A such that $\Delta(A)$ is a polynomial mould (i.e. the denominator of the rational function A is "at worst" $u_1 \cdots u_r(u_1 + \cdots + u_r)$). We write ARI^{Δ}_{*} for the space of moulds in $ARI^{\Delta} \cap ARI_{*}$, where $*$ may represent any (or no) properties on moulds in ARI.

The next lemma describes the action of the operators dar, dur and Δ on non-commutative polynomials.

Lemma III.2.2. *[R]*
Let $f \in \mathbb{Q}\langle C \rangle_n$. Then for $0 \leq r \leq n$, we have :

$$ma_{[x,f]}(u_1, \dots, u_r) = -(u_1 + \cdots + u_r) ma_f(u_1, \dots, u_r). \tag{III.2.4}$$

i.e

$$ma_{[f,x]} = dur(ma_f).$$

Similarly, for $f \in (\mathfrak{lie}_C)_n$:

$$ma_{f(x,[y,x])}(u_1, \dots, u_r) = u_1 \dots u_r ma_f(u_1, \dots, u_r) \tag{III.2.5}$$

i.e.

$$ma_{f(x,[y,x])} = dar(ma_f).$$

Finally,

$$ma_{([f(x,[y,x]),x])} = \Delta(ma_f). \tag{III.2.6}$$

Proof. Let us denote f^r the depth r part of f, and write $f^r = \sum_{\boldsymbol{a}} c_{\boldsymbol{a}} C_{a_1} \dots C_{a_r}$ with $\boldsymbol{a} = (a_1, \dots, a_r)$. Then

$$\begin{aligned}
[x, f^r] &= \sum_{\boldsymbol{a}} c_{\boldsymbol{a}} [x, C_{a_1} \dots C_{a_r}] \\
&= \sum_{\boldsymbol{a}} c_{\boldsymbol{a}} \sum_{i=1}^{r} C_{a_1} \dots C_{a_{i-1}} [x, C_{a_i}] C_{a_{i+1}} \dots C_{a_r} \\
&= \sum_{\boldsymbol{a}} c_{\boldsymbol{a}} \sum_{i=1}^{r} C_{a_1} \dots C_{a_{i-1}} C_{a_i+1} C_{a_{i+1}} \dots C_{a_r}
\end{aligned}$$

It yields :

$$\begin{aligned}
ma_{[x,f]}(u_1, \dots, u_r) &= (-1)^{r+n+1} \sum_{\boldsymbol{a}} c_{\boldsymbol{a}} \sum_{i=1}^{r} u_1^{a_1-1} \dots u_i^{a_i} \dots u_r^{a_r-1} \\
&= -(u_1 + \dots u_r)(-1)^{r+n} \sum_{\boldsymbol{a}} c_{\boldsymbol{a}} u_1^{a_1-1} \dots u_r^{a_r-1} \\
&= -(u_1 + \dots u_r) ma_f(u_1, \dots, u_r).
\end{aligned}$$

This concludes the proof of equation (III.2.4).

For equation (III.2.5), note that replacing y with $[y, x]$ in C_k yields $-C_{k+1}$. Therefore the substitution o making the substitution in a monomial $C_{k_1} \dots C_{k_r}$ gives $(-1)^r C_{k_1+1} \dots C_{k_r+1}$ and we get:

$$ma((-1)^r C_{k_1+1} \dots C_{k_r+1}) = (-1)^r (-1)^n u_1 \dots u_r ma(C_{k_1} \dots C_{k_r}).$$

Equation (III.2.6) is easily derived from (III.2.4) and (III.2.5). □

Example III.2.3. Let $f = [x, y] \in \mathfrak{lie}_2, f = C_2$. The mould ma_f is concentrated in depth 1 and $ma_f(u_1) = u_1$. Now

$$dur(ma_f)(u_1) = dar(ma_f)(u_1) = u_1^2$$

and

$$[f, x] = [[x, y], x] = -C_3,$$

so $ma_{[f,x]}(u_1) = u_1^2$. Similarly $f(x, [y, x]) = [x, [y, x]] = -C_3$. Finally,

$$[f(x, [y, x]), x] = [x, [x, [x, y]]] = C_4$$

which corresponds under ma to the mould

$$\Delta(ma_f)(u_1) = u_1^3.$$

III.2.2 Definition of the elliptic double shuffle Lie algebra

Definition III.2.4. The elliptic double shuffle Lie algebra $\mathfrak{ds}_{ell}$ is the set of Lie polynomials which map under ma to polynomial-valued Δ-bialternal moulds that are even in depth 1, i.e.

$$\mathfrak{ds}_{ell} = ma^{-1}\big(\Delta(ARI^{\Delta}_{\underline{al}*\underline{al}})\big).$$

As in the mould-theoretic proof that $\mathfrak{ds}$ is a Lie algebra, the mould pal and its inverse for $gari$ play a major role, through theorem (II.2.14). However, to prove that $\mathfrak{ds}_{ell}$ is a Lie algebra, one also need the following theorem by Baumard.

Theorem III.2.5. *([B], Thms. 3.3, 4.35] The space ARI^{Δ} forms a Lie algebra under the ari-bracket, and we have an injective Lie algebra morphism :*

$$Ad_{ari}(invpal) \, : \, ARI^{pol}_{\underline{al}*\underline{il}} \hookrightarrow ARI^{\Delta}$$

with both spaces equipped with the ari bracket.

The moulds *pal* and *invpal* are again key to the proof. We will see in the next section how they come of use in the case of the elliptic Kashiwara-Vergne Lie algebra.

III.3 The elliptic Kashiwara-Vergne Lie algebras

In this section we follow the procedure of [S3] for the double shuffle Lie algebra to define a natural candidate for the elliptic Kashiwara-Vergne Lie algebra, closely related to the linearized Kashiwara-Vergne Lie algebra, and give some of its properties.

III.3.1 Special types of derivations of $\mathfrak{lie}_2$

As in the previous chapters, let $\mathfrak{lie}_2$ denote the degree completion of the free Lie algebra over $\mathbb{Q}$ on non-commutative variables x and y. We write $(\mathfrak{lie}_2)_n$ for the graded part of weight n, $(\mathfrak{lie}_2)^r$ for the graded part of depth r, and $(\mathfrak{lie}_2)^r_n$ for the intersection, which is finite-dimensional.

For $a, b \in \mathfrak{lie}_2$, we write $D_{b,a}$ for the derivation defined by $x \mapsto b$ and $y \mapsto a$. The bracket is explicitly given by

$$[D_{b,a}, D_{b',a'}] = D_{\tilde{b},\tilde{a}} \tag{III.3.1}$$

with

$$\tilde{b} = D_{b,a}(b') - D_{b',a'}(b), \quad \tilde{a} = D_{b,a}(a') - D_{b',a'}(a). \tag{III.3.2}$$

• Let $\mathfrak{oder}_2$ denote the Lie subalgebra of $\mathfrak{der}_2$ of derivations $D = D_{b,a}$ that annihilate the bracket $[x, y]$ and such that neither $D(x)$ nor $D(y)$ have a linear term in x. The map $\mathfrak{oder}_2 \to \mathfrak{lie}_2$ given by $D \mapsto D(x)$ is injective (see Corollary III.3.3).

• Let $\mathfrak{tder}_2$ denote the Lie subalgebra of $\mathfrak{der}_2$ of *tangential derivations*, previously mentioned in the definition of the Kashiwara-Vergne Lia algebra. In this chapter, we denote such derivations $E_{a,b}$ with elements $a, b \in \mathfrak{lie}_2$ such that a has no linear term in x and b has no linear term in y, such that there exists $c \in \mathfrak{lie}_2$ such that setting $z = -x - y$,

$$E_{a,b}(x) = [x, a], \quad E_{a,b}(y) = [y, b] \text{ and } E_{a,b}(z) = [z, c].$$

The Lie bracket is explicitly given by

$$[E_{a,b}, E_{a',b]}] = E_{\tilde{a},\tilde{b}} \tag{III.3.3}$$

where

$$\tilde{a} = [a, a'] + E_{a,b}(a') - E_{a',b'}(a), \quad \tilde{b} = [b, b'] + E_{a,b}(b') - E_{a',b'}(b). \tag{III.3.4}$$

• Let $\mathfrak{sder}_2$ denote the Lie subalgebra of $\mathfrak{tder}_2$ of *special tangential derivations*, i.e. derivations such that $E_{a,b}(z) = [x, a] + [y, b] = 0$.

• Let $\mathfrak{ider}_2$ be the Lie subalgebra of $\mathfrak{tder}_2$ of *Ihara derivations*, which are those that annihilate x, i.e. those of the form $d_b = E_{0,b}$. The derivation d_b is defined by its values on x and y

$$d_b(x) = 0, \quad d_b(y) = [y, b]. \tag{III.3.5}$$

The Lie bracket on $\mathfrak{ider}_2$ is given by $[d_b, d_{b'}] = d_{\{b,b'\}}$, where $\{b, b'\}$ is the *Poisson* (or Ihara) bracket given by

$$\{b, b'\} = [b, b'] + d_b(b') - d_{b'}(b), \tag{III.3.6}$$

i.e. the second term of (III.3.4).

We have the following diagram showing the connections between these subspaces:

$$\begin{array}{c} \mathfrak{oder}_2 \hookrightarrow \mathfrak{der}_2 \\ \uparrow \\ \mathfrak{tder}_2 \\ / \quad \backslash \\ \mathfrak{sder}_2 \xrightarrow{\sim} \mathfrak{ider}_2 \end{array} \tag{III.3.7}$$

The isomorphism between $\mathfrak{sder}_2$ and $\mathfrak{ider}_2$ is given in Lemma III.3.14.

III.3.2 Definition of the elliptic Kashiwara-Vergne Lie algebra

The Kashiwara-Vergne Lie algebra

Definition III.3.1. The mould elliptic Kashiwara-Vergne vector space is the subspace of polynomial-valued moulds

$$\Delta\big(ARI^{\Delta}_{al+push*circneut}\big).$$

The elliptic Kashiwara-Vergne vector space is the subspace $\mathfrak{krv}_{ell} \subset \mathfrak{lie}_C$ such that

$$ma\big(\mathfrak{krv}_{ell}\big) = \Delta\big(ARI^{\Delta}_{al+push*circneut}\big). \tag{III.3.8}$$

The operator Δ trivially respects push-invariance of moulds, so the space $\mathfrak{krv}_{ell}$ lies in the space $\mathfrak{lie}_C^{push}$ of push-invariant elements of $\mathfrak{lie}_C$. We will now show that the subspace $\mathfrak{krv}_{ell}$ is actually a Lie subalgebra of $\mathfrak{lie}_C^{push}$, which is itself a Lie algebra thanks to the following lemma, of which a more explicit version (with a formula for the partner) is proved in [S3] (Lemma 2.1.1).

Lemma III.3.2. *Let $b \in \mathfrak{lie}_C$. Then $b \in \mathfrak{lie}_C^{push}$ if and only if there exists a unique element $a \in \mathfrak{lie}_C$ (the partner of b), such that if $D_{b,a}$ is the derivation of $\mathfrak{lie}_2$ defined by $x \mapsto b$, $y \mapsto a$, then $D_{b,a}$ annihilates $[x, y]$.*

By identifying $\mathfrak{lie}_C^{push}$ with the space of derivations that annihilate $[x, y]$, this lemma shows that $\mathfrak{lie}_C^{push}$ is a Lie algebra under the bracket of derivations. We state this as a corollary.

Corollary III.3.3. *The map $b \mapsto D_{b,a}$ gives an isomorphism*

$$\partial : \mathfrak{lie}_C^{push} \to \mathfrak{oder}_2 \tag{III.3.9}$$

whose inverse is $D_{b,a} \mapsto D_{b,a}(x) = b$, and this becomes a Lie isomorphism when $\mathfrak{lie}_C^{push}$ is equipped with the Lie bracket

$$\langle b, b'\rangle = [D_{b,a}, D_{b',a'}](x) = D_{b,a}(b') - D_{b',a'}(b). \tag{III.3.10}$$

Thus we know that $\mathfrak{lie}_C^{push}$ is a Lie algebra and it contains the elliptic Kashiwara-Vergne space $\mathfrak{krv}_{ell}$ as a subspace. This leads to our first main result on $\mathfrak{krv}_{ell}$.

Theorem III.3.4. *([RS1]) The subspace $\mathfrak{krv}_{ell} \subset \mathfrak{lie}_C^{push}$ is a Lie subalgebra.*

A new bracket on ARI : $Dari$.

In order to prove Theorem III.3.4, we need to define a new Lie bracket on ARI.

Definition III.3.5. Let M be a mould. We define the operator $Darit(M)$ as:

$$Darit(M) = -dar(arit(\Delta^{-1}(M)) - ad(\Delta^{-1}(M)))\circ dar^{-1}.$$

It is a derivation of ARI_{lu} for all M, since $arit(M)$ and $ad(M)$ are derivations of ARI_{lu} and dar is an automorphism.

Remark III.3.6. The derivation $Darit(M)$ preserves ARI^{pol} if M is polynomial-valued.

Definition III.3.7. Using the definition of $Darit$, we define the Lie bracket $Dari$ on ARI by :

$$Dari(A, B) = Darit(A) \cdot B - Darit(B) \cdot A$$

We denote ARI_{Dari} the Lie algebra with underlying space ARI and bracket $Dari$.

Proposition III.3.8. *(Schneps,[S3], Proposition 3.2.1.) The operator Δ is a Lie algebra isomorphism from ARI_{ari} to ARI_{Dari} :*

$$Dari(A, B) = \Delta(ari(\Delta^{-1}(A), \Delta^{-1}(B))). \tag{III.3.11}$$

Finally, we compare the $Dari$-bracket to the bracket $\langle\, ,\, \rangle$ on $\mathfrak{lie}_C^{push}$ given in Corollary III.3.3.

Proposition III.3.9. *The map*

$$ma : \mathfrak{lie}_C^{push} \to ARI_{Dari},$$

is a Lie algebra morphism, i.e.

$$ma\big(\langle b, b'\rangle\big) = Dari\big(ma(b), ma(b')\big).$$

Proof. The main point is the following result [BS] (see Theorem 3.5): if D_1 and D_2 lie in $\mathfrak{oder}_2$, then the map

$$\begin{aligned} \mathfrak{oder}_2 &\to ARI_{ari} \\ D &\mapsto \Delta^{-1}\big(ma\big(D(x)\big)\big), \end{aligned}$$

is an injective Lie morphism, i.e.

$$\Delta^{-1}\Big(ma\big([D_1, D_2](x)\big)\Big) = ari\Big(\Delta^{-1}\big(ma(D_1(x))\big), \Delta^{-1}\big(ma(D_2(x))\big)\Big).$$

Applying Δ to both sides of this and using (III.3.11), this is equivalent to

$$ma\big([D_1, D_2](x)\big) = Dari\Big(ma\big(D_1(x)\big), ma\big(D_2(x)\big)\Big),$$

which in turn means that

$$ma : \mathfrak{oder}_2 \to ARI_{Dari} \tag{III.3.12}$$

is a Lie algebra morphism. By composition with the Lie isomorphism $\mathfrak{lie}_C^{push} \xrightarrow{\sim} \mathfrak{oder}_2$ given in Corollary III.3.3,

$$b \mapsto D_{b,a} \overset{\Psi}{\mapsto} \Delta^{-1}\big(ma(D_{b,a}(x))\big) \overset{\Delta}{\mapsto} ma(b)$$

is an injective Lie morphism $\mathfrak{lie}_C^{push} \to ARI_{Dari}$, which proves the result. $\square$

A well-needed lemma

The last result we need to prove Theorem III.3.4 and obtained the long-delayed proof that $\mathfrak{lkv}$ is a Lie algebra is the following.

Lemma III.3.10. *The space $\overline{ARI}_{circneut}$ of circ-neutral moulds $A \in \overline{ARI}$ forms a Lie algebra under the $\overline{ari}$-bracket.*

Proof. Let $A, B \in \overline{ARI}_{circneut}$. We need to show that

$$\sum_{i=1}^{r} \overline{ari}(A,B)(v_i, \dots, v_r, v_1, \dots, v_{i-1}) = 0,$$

where the formula for the $\overline{ari}$-bracket on $\overline{ARI}$ is given by the expression

$$\begin{aligned}\overline{ari}(A,B) &= lu(A,B) + \overline{arit}(B)\cdot A - \overline{arit}(A)\cdot B\\ &= lu(A,B) + \overline{amit}(B)\cdot A - \overline{anit}(B)\cdot A - \overline{amit}(A)\cdot B + \overline{anit}(A)\cdot B.\end{aligned}$$

We will show that this expression is circ-neutral because in fact, each of the five terms in the sum is individually circ-neutral. Let us start by showing this for the first term, $lu(A,B)$.

Let σ denote the cyclic permutation of $\{1, \dots, r\}$ defined by

$$\sigma(i) = i+1 \ \text{ for } \ 1 \le i \le r-1, \ \ \sigma(r) = 1.$$

By additivity, since the circ-neutrality property is depth-by-depth, we may assume that A is concentrated in depth s and B in depth t, with $s \le t$, $s+t=r$. In this simplifed situation, we have

$$lu(A,B)(v_1, \dots, v_r) = A(v_1, \dots, v_s)B(v_{s+1}, \dots, v_r) - B(v_1, \dots, v_t)A(v_{t+1}, \dots, v_r).$$

We have

$$\begin{aligned}&\sum_{i=0}^{r-1} lu(A,B)(v_{\sigma^i(1)}, \dots, v_{\sigma^i(r)})\\ =&\sum_{i=0}^{r-1} A(v_{\sigma^i(1)}, \dots, v_{\sigma^i(s)})B(v_{\sigma^i(s+1)}, \dots, v_{\sigma^i(i)})\\ &- B(v_{\sigma^i(1)}, \dots, v_{\sigma^i(t)})A(v_{\sigma^i(t+1)}, \dots, v_{\sigma^i(i-1)}),\\ =&\sum_{i=0}^{r-1} A(v_{\sigma^i(1)}, \dots, v_{\sigma^i(s)})B(v_{\sigma^i(s+1)}, \dots, v_{\sigma^i(i)})\\ &- A(v_{\sigma^{i+t}(1)}, \dots, v_{\sigma^{i+t}(s)})B(v_{\sigma^{i+t}(s+1)}, \dots, v_{\sigma^{i+t}(r)})\\ =&0\end{aligned}$$

as the terms cancel out pairwise.

We now prove that the second term

$$\big(\overline{amit}(B)\cdot A\big)(v_1, \dots, v_r) = \sum_{i=1}^{s} A(v_1, \dots, v_{i-1}, v_{i+t}, \dots, v_r)B(v_i - v_{i+t}, \dots, v_{i+t-1} - v_{i+t})$$

is circ-neutral. Fix $j \in \{1, \dots, s\}$ and consider the term

$$A(v_1, \dots, v_{j-1}, v_{j+t}, \dots, v_r)B(v_j - v_{j+t}, \dots, v_{j+t-1} - v_{j+t}).$$

Thus for each of the other terms

$$A(v_1, \dots, v_{i-1}, v_{i+t}, \dots, v_r)B(v_i - v_{i+t}, \dots, v_{i+t-1} - v_{i+t})$$

in the sum, with $i \in \{1, \dots, s\}$, there is exactly one cyclic permutation, namely σ^{j-i}, that maps this term to

$$A(v_{\sigma^{j-i}(1)}, \dots, v_{\sigma^{j-i}(i-1)}, v_{\sigma^{j-i}(i+t}, \dots, v_{\sigma^{j-i}(r)})B(v_j - v_{j+t}, \dots, v_{j+t-1} - v_{j+t}).$$

For fixed $j \in \{1, \dots, s\}$, the values of $k = j - i \bmod s$ as i runs through $\{1, \dots, s\}$ are exactly $\{0, \dots, s-1\}$. Therefore, the coefficient of the term $B(v_j - v_{j+t}, \dots, v_{j+t-1} - v_{j+t})$ in the sum of the cyclic permutations of $\overline{amit}(B) \cdot A$ is equal to

$$\sum_{k=0}^{s-1} A(v_{\sigma^k(1)}, \dots, v_{\sigma^k(i-1)}, v_{\sigma^k(i+t)}, \dots, v_{\sigma^k(r)}),$$

which is zero due to the circ-neutrality of A. Thus the coefficient of the term $B(v_j - v_{j+t}, \dots, v_{j+t-1} - v_{j+t})$ in the sum of the cyclic permutations of $\overline{amit}(B) \cdot A$ is zero, and this holds for $1 \leq j \leq s$, so the entire sum is 0, i.e. $\overline{amit}(B) \cdot A$ is circ-neutral. The proof of the circ-neutrality of the term $\overline{anit}(B) \cdot A$ is analogous. By exchanging A and B, this also shows that $\overline{amit}(A) \cdot B$ and $\overline{anit}(A) \cdot B$ are circ-neutral, which concludes the proof of the lemma. $\square$

Proof that $\mathfrak{krv}_{ell}$ is a Lie algebra

This subsection is devoted to the proof of Theorem III.3.4, i.e. that the subspace $\mathfrak{krv}_{ell} \subset \mathfrak{lie}_C^{push}$ is closed under the bracket $\langle\ ,\ \rangle$.

From Proposition III.3.9, *ma* gives an injective Lie algebra morphism

$$\mathfrak{lie}_C^{push} \to ARI_{Dari}.$$

Thus it is equivalent to prove that the image of the subspace $\mathfrak{krv}_{ell} \subset \mathfrak{lie}_C^{push}$ is closed under the *Dari*-bracket. Since we saw above that

$$\Delta^{-1} : ARI_{Dari} \to ARI_{ari},$$

it is equivalent to show that $ARI^{\Delta}_{al+push*circneut}$ is a Lie subalgebra of ARI_{ari}.

Step 1. Since $\mathfrak{lie}_C^{push}$ is the space of push-invariant Lie polynomials, we have

$$ma(\mathfrak{lie}_C^{push}) = ARI^{pol}_{al+push}.$$

But we saw in Proposition III.3.9 that $\mathfrak{lie}_C^{push}$ is a Lie algebra under $\langle\ ,\ \rangle$, so $ARI^{pol}_{al+push}$ is a Lie algebra under $Dari$.

Step 2. The space $ARI^{\Delta}_{al+push}$ is a Lie algebra under *ari*. Indeed, the definition of Δ shows that this operator does not change the properties of push-invariance or alternality, i.e. $\Delta^{-1}(ARI_{al+push}) = ARI_{al+push}$. Restricted to polynomial-valued moulds, we have $\Delta^{-1}(ARI^{pol}_{al+push}) = ARI^{\Delta}_{al+push}$. Since Δ is an isomorphism from ARI_{ari} to ARI_{Dari} by virtue of (III.3.11) and $ARI^{pol}_{al+push}$ is a Lie subalgebra of ARI_{Dari} by Step 1, its image $ARI^{\Delta}_{al+push}$ under Δ^{-1} is thus a Lie subalgebra of ARI_{ari}.

Step 3. We can now complete the proof of Theorem III.3.4 by showing that the space $ARI^{\Delta}_{al+push*circneut}$ is a Lie algebra under *ari*.

Proof. Let A, B lie in $ARI^{\Delta}_{al+push*circneut}$, and let us show that $ari(A,B)$ lies in the same space. By Step 2, we know that $ari(A,B) \in ARI^{\Delta}_{al+push}$, so we only need to show that $swap\big(ari(A,B)\big)$ is *circ-neutral. But we will show that in fact this mould is actually circ-neutral.
To see this, let A_0 and B_0 be the constant-valued moulds such that $swap(A) + A_0$ and $swap(B) + B_0$ are circ-neutral.
By Lemma III.3.10, we have

$$\overline{ari}\big(swap(A) + A_0, swap(B) + B_0\big) \in \overline{ARI}_{circneut}.$$

Using the identity $swap\big(ari(M,N)\big) = \overline{ari}\big(swap(M), swap(N)\big)$, valid whenever M and N are push-invariant moulds (cf. [S], (2.5.6)), as well as the fact that constant-valued moulds are both push and swap invariant, we have

$$\begin{aligned}\overline{ari}\big(swap(A) + A_0, swap(B) + B_0\big) &= \overline{ari}\big(swap(A + A_0), swap(B + B_0)\big)\\ &= swap \cdot ari(A + A_0, B + B_0)\\ &= swap \cdot ari(A,B) + swap \cdot ari(A, B_0) + swap \cdot ari(A_0, B) + swap \cdot ari(A_0, B_0)\\ &= swap \cdot ari(A,B)\end{aligned}$$

since the definition of the *ari*-bracket shows that $ari(C, M) = 0$ whenever C is a constant-valued mould. Thus $swap \cdot ari(A,B)$ is circ-neutral, which completes the proof of Theorem III.3.4. □

The following easy corollary provides the promised proof of Proposition II.3.17 stating that $\mathfrak{lkv}$ is a Lie algebra.

Corollary III.3.11. *The subspace*

$$ARI^{pol}_{al+push/circneut} \subset ARI^{\Delta}_{al+push*circneut}$$

is a Lie algebra under the ari-bracket. Thus, by the correspondance between Poisson and ari bracket ((II.1.33)), the space

$$\mathfrak{lkv} = ma^{-1}\big(ARI^{pol}_{al+push/circneut}\big)$$

a Lie algebra under the Poisson bracket.

Proof. By the definition of *ari*, ARI^{pol} is a Lie subalgebra of ARI. Also, Lemma III.3.10 shows that the space $\overline{ARI}_{circneut}$ of circ-neutral moulds is a Lie subalgebra of $\overline{ARI}_{*circneut}$.
Thus $ARI^{\Delta}_{al+push/circneut}$ is a Lie algebra inside $ARI^{\Delta}_{al+push*circneut}$. So the intersection

$$ARI^{pol} \cap ARI^{\Delta}_{al+push/circneut} = ARI^{pol}_{al+push/circneut}$$

is one as well. □

III.3.3 The map from $\mathfrak{krv} \to \mathfrak{krv}_{ell}$

In this subsection we prove our next main result on the elliptic Kashiwara-Vergne Lie algebra, which is analogous to known results on the elliptic Grothendieck-Teichmüller Lie algebra of [E] and the elliptic double shuffle Lie algebra of [S3]. The subsection III.3.4 below is devoted to connections between these three situations.

Theorem III.3.12. *([RS1])There is an injective Lie algebra morphism*

$$\mathfrak{krv} \hookrightarrow \mathfrak{krv}_{ell} \tag{III.3.13}$$

The proof constructs the morphism from $\mathfrak{krv}$ to $\mathfrak{krv}_{ell}$ in four main steps as follows.

Step 1. We first consider a twisted version of the Kashiwara-Vergne Lie algebra, or rather of the associated polynomial space $V_{\mathfrak{krv}}$ of Definition II.3.11, via the map

$$\begin{aligned} \nu : V_{\mathfrak{krv}} &\xrightarrow{\sim} W_{\mathfrak{krv}} && \text{(III.3.14)} \\ f &\mapsto \nu(f), && \text{(III.3.15)} \end{aligned}$$

where ν is the automorphism of $\mathbb{Q}\langle x, y\rangle$ defined by

$$\nu(x) = z = -x - y, \quad \nu(y) = y. \tag{III.3.16}$$

In paragraph III.3.3, we prove that $W_{\mathfrak{krv}}$ is a Lie algebra under the Poisson or Ihara bracket, and give a description of $W_{\mathfrak{krv}}$ via two properties, the "twisted" versions of the two defining properties of $V_{\mathfrak{krv}}$ given in Definition II.3.11.

Step 2. In paragraph III.3.3, we study the mould space $ma\big(W_{\mathfrak{krv}}\big)$. Thanks to the compatibility of the *ari*-bracket with the Poisson bracket, this space is a Lie subalgebra of ARI_{ari}. Just as we reformulated the defining properties of $\mathfrak{lkv}$ in mould terms, proving that $ma(\mathfrak{lkv}) = ARI^{pol}_{al+push/circneut}$, here we reformulate the defining properties of $W_{\mathfrak{krv}}$ in mould terms: explicitly, we show that

$$ma\big(W_{\mathfrak{krv}}\big) = ARI^{pol}_{al+sen*circconst}, \tag{III.3.17}$$

the space of polynomial-valued moulds that are alternal, satisfy a certain *senary* relation (III.3.23) introduced by Écalle (see below), and whose swap is circ-constant up to addition of a constant-valued mould.

We observe that if $B \in \overline{ARI}$ is a polynomial-valued mould of homogeneous degree n whose swap is circ-constant up to addition of a constant-valued mould, then the constant-valued mould B_0 is uniquely determined as being the mould concentrated in depth n and taking the value c/n there, where $B(v_1) = cv_1^{n-1}$.

Step 3. For this part, we use again the mould *pal* previously defined in (II.2.2) and the adjoint operator $Ad_{ari}(invpal)$ on ARI_{ari}.
Letting Ξ denote the map

$$Ad_{ari}(invpal)\circ pari : ARI_{ari} \to ARI_{ari},$$

we show that it yields an injective Lie morphism

$$\Xi : ARI^{pol}_{al+sen*circconst} \to ARI^{\Delta}_{al+push*circneut} \tag{III.3.18}$$

of subalgebras of ARI_{ari}.

Step 4. The final step is to compose (III.3.18) with the Lie morphism $\Delta : ARI_{ari} \to ARI_{Dari}$, obtaining an injective Lie morphism

$$ARI^{pol}_{al+sen*circconst} \to \Delta\big(ARI^{\Delta}_{al+push*circneut}\big),$$

where the left-hand space is a subalgebra of ARI_{ari} and the right-hand one of ARI_{Dari}. Since the right-hand space is equal to $ma(\mathfrak{krv}_{ell})$, the desired injective Lie morphism $\mathfrak{krv} \to \mathfrak{krv}_{ell}$ is obtained by composing all the maps described above, as shown in the following diagram:

$$\begin{tikzcd}
\mathfrak{krv} \arrow[d, "\text{by (II.3.2)}"'] & & \\
V_{\mathfrak{krv}} \arrow[d, "\text{by (III.3.14)}"', "\nu"] & & \\
W_{\mathfrak{krv}} \arrow[d, "\text{by (III.3.17)}"', "ma"] & & \mathfrak{krv}_{ell} \\
ARI^{pol}_{al+sen*circconst} \arrow[r, "\Xi", "\text{by (III.3.18)}"'] & ARI^{\Delta}_{al+push*circneut} \arrow[r, "\Delta"] & \Delta\big(ARI^{\Delta}_{al+push*circneut}\big) \arrow[u, "ma^{-1}", "\text{by (III.3.8)}"']
\end{tikzcd}$$

Step 1: The twisted space $W_{\mathfrak{krv}}$

Proposition III.3.13. *Let $W_{\mathfrak{krv}} = \nu(V_{\mathfrak{krv}})$. Then $W_{\mathfrak{krv}}$ is a Lie algebra under the Poisson bracket.*

Proof. The key point is the following lemma on derivations.

Lemma III.3.14. *Conjugation by ν induces an isomorphism of Lie algebras*

$$\begin{aligned} \mathfrak{sder}_2 &\xrightarrow{\sim} \mathfrak{ider}_2 \\ E_{a,b} &\mapsto d_{\nu(b)}. \end{aligned} \tag{III.3.19}$$

Proof. Recall that $E_{a,b} \in \mathfrak{sder}_2$ maps $x \mapsto [x,a]$ and $y \mapsto [y,b]$, and $d_{\nu(b)} \in \mathfrak{ider}_2$ is the Ihara derivation defined by $x \mapsto 0$, $y \mapsto [y,\nu(b)]$.

Let us first show that $d_{\nu(b)}$ is the conjugate of $E_{a,b}$ by ν, i.e. $d_{\nu(b)} = \nu \circ E_{a,b} \circ \nu$ (since ν is an involution). It is enough to show they agree on x and y, so we compute

$$\nu \circ E_{a,b} \circ \nu(x) = \nu \circ E_{a,b}(z) = 0 = d_{\nu(b)}(x)$$

and

$$\nu \circ E_{a,b} \circ \nu(y) = \nu \circ E_{a,b}(y) = \nu\big([y,b]\big) = [y,\nu(b)] = d_{\nu(b)}(y).$$

This shows that $\nu \circ E_{a,b} \circ \nu$ is indeed equal to $d_{\nu(b)}$. To show that $d_{\nu(b)}$ lies in $\mathfrak{ider}_2$, we check that $d_{\nu(b)}(z)$ is a bracket of z with another element of $\mathfrak{lie}_2$:

$$d_{\nu(b)}(z) = \nu \circ E_{a,b} \circ \nu(z) = \nu \circ E_{a,b}(x) = \nu([x,a]) = [z,\nu(a)].$$

The same argument goes the other way to show that conjugation by ν maps an element of $\mathfrak{ider}_2$ to an element of $\mathfrak{sder}_2$, which yields the isomorphism (III.3.19) as vector spaces. To see that it is also an isomorphism of Lie algebras, it suffices to note that conjugation by ν preserves the Lie bracket of derivations in $\mathfrak{der}_2$, i.e.

$$\nu \circ [D_1, D_2] \circ \nu = [\nu \circ D_1 \circ \nu, \nu \circ D_2 \circ \nu],$$

since ν is an involution. Since the Lie brackets on $\mathfrak{sder}_2$ and $\mathfrak{ider}_2$ are just restrictions to those subspaces of the Lie bracket on the space of all derivations, conjugation by ν carries one to the other. □

We use the lemma to complete the proof of Proposition III.3.13. Write

$$\mathfrak{krv}^\nu = \{\nu \circ E \circ \nu \mid E \in \mathfrak{krv}\} \subset \mathfrak{ider}_2.$$

By restricting the isomorphism (III.3.19) to the subspace $\mathfrak{krv} \subset \mathfrak{sder}_2$, we obtain a commutative diagram of isomorphisms of vector spaces

$$\begin{array}{ccc} \mathfrak{krv} & \to & \mathfrak{krv}^\nu, \\ \downarrow & & \downarrow \\ V_{\mathfrak{krv}} & \xrightarrow{\nu} & W_{\mathfrak{krv}}, \end{array}$$

where the left-hand vertical arrow is the isomorphism (II.3.2) mapping $E_{a,b} \mapsto b$, and the right-hand vertical map sends an Ihara derivation d_f to f.
Equipping $W_{\mathfrak{krv}}$ with the Lie bracket inherited from $\mathfrak{krv}^\nu$ makes this into a commutative diagram of Lie isomorphisms. But this bracket is nothing other than the Poisson bracket since $\mathfrak{krv}^\nu \subset \mathfrak{ider}_2$. □

We now give a characterization of $W_{\mathfrak{krv}}$ by two defining properties which are the twists by ν of those defining $V_{\mathfrak{krv}}$. Recall that β is the the backwards operator.

Proposition III.3.15. *The space $W_{\mathfrak{krv}}$ is the space spanned by polynomials $b \in \mathfrak{lie}_C$, of homogeneous degree $n \geq 3$, such that*

(i) $b_y - b_x$ is anti-palindromic*, i.e. $\beta(b_y - b_x) = (-1)^{n-1}(b_y - b_x)$, and*

(ii) $b + \frac{c}{n} y^n$ is circ-constant*, where $c = (b|x^{n-1}y)$.*

Proof. Let $f = \nu(b)$, so that $f \in V_{\mathfrak{krv}}$. Then the property that $b_y - b_x$ is anti-palindromic is precisely equivalent to the push-invariance of f (this is proved as the equivalence of properties (iv) and (v) of Theorem 2.1 of [S1]). This proves (i).

For (ii), we note that since $f \in V_{\mathfrak{krv}}$, $f^y - f^x$ is push-constant for the value $c = (f|x^{n-1}y) = (-1)^{n-1}(b|x^{n-1}y)$. We have

$$b(x,y) = xb^x(x,y) + yb^y(x,y),$$

so

$$f(x,y) = b(z,y) = zb^x(z,y) + yb^y(z,y) = -xb^x(z,y) - yb^x(z,y) + yb^y(z,y).$$

Thus since $f(x,y) = xf^x(x,y) + yf^y(x,y)$, this gives

$$f^x = -b^x(z,y) \text{ and } f^y = -b^x(z,y) + b^y(z,y),$$

so

$$f^y - f^x = b^y(z,y) = \nu(b^y).$$

Thus to prove the result, it suffices to prove that the following statement: if $g \in \mathbb{Q}\langle C\rangle$ is a polynomial of homogeneous degree n that is push-constant for $(-1)^{n-1}c$, then $\nu(g)$ is push-constant for c, since taking $g = f^y - f^x$ then shows that $\nu(g) = b^y$ is push-constant for c. The proof of this statement is straightforward using the substitution $z = -x - y$ (but see the proof of Lemma 3.5 in [S1] for details). Since $c = 0$ if $f \in V_{\mathfrak{krv}}$ is of even degree n, this proves (ii). □

Step 2: The mould version $ma(W_{\mathfrak{krv}})$

The space $ma(W_{\mathfrak{krv}})$ is closed under the *ari*-bracket by (**??**), since $W_{\mathfrak{krv}}$ is closed under the Poisson bracket.

Let $b \in W_{\mathfrak{krv}}$ and let $B = ma(b)$. Then since b is a Lie polynomial, B is an alternal polynomial mould. Let us give the mould reformulations of properties (i) and (ii) of Proposition III.3.15. The second property is easy since we already showed, in Proposition II.3.22, that a polynomial b is circ-constant if and only if $swap(B)$ is circ-constant.

Expressing the first property in terms of moulds is more complicated and calls for an identity discovered by Écalle. We need to use the mould operators *mantar* and *pari* defined by

$$mantar(B) = (u_1, \dots, u_r) = (-1)^{r-1} B(u_r, u_{r-1}, \dots, u_1) \tag{III.3.20}$$

$$pari(B)(u_1, \dots, u_r) = (-1)^r B(u_1, \dots, u_r). \tag{III.3.21}$$

The operator *pari* extends the operator $y \mapsto -y$ on polynomials to all moulds, and *mantar* extends the operator $f \mapsto (-1)^{n-1}\beta(f)$. Above all, we need Écalle's mould operator *teru*, defined by taking the mould $teru(B)$ to be equal to B in depths 0 and 1, and for depths $r > 1$, setting

$$\begin{aligned} &teru(B)(u_1, \dots, u_r) = \\ &\qquad B(u_1, \dots, u_r) + \frac{1}{u_r}\Big(B(u_1, \dots, u_{r-2}, u_{r-1} + u_r) - B(u_1, \dots, u_{r-2}, u_{r-1})\Big). \end{aligned} \tag{III.3.22}$$

Lemma III.3.16. *Let $b \in \mathfrak{lie}_C$. Then the following are equivalent:*

(1) $b_y - b_x$ is anti-palindromic;

(2) $B = ma(b)$ satisfies the senary relation

$$teru{\circ}pari(B) = push \circ mantar \circ teru \circ pari(B). \tag{III.3.23}$$

Proof. The statement is a consequence of the following result, proved in A.3 of the Appendix of [S1]. Let $\tilde{b} \in \mathfrak{lie}_C$ and let $\tilde{B} = ma(\tilde{b})$. Write $\tilde{b} = \tilde{b}_x x + \tilde{b}_y y$ as usual. Then for each depth part $(\tilde{b}_x + \tilde{b}_y)^r$ of the polynomial $\tilde{b}_x + \tilde{b}_y$ $(1 \le r \le n-1)$, the anti-palindromic property

$$(\tilde{f}_x + \tilde{f}_y)^r = (-1)^{n-1}\beta(\tilde{f}_x + \tilde{f}_y)^r \tag{III.3.24}$$

translates directly to the following relation on $\tilde{B}$:

$$teru(\tilde{B})(u_1, \dots, u_r) = push \circ mantar \circ teru\,(\tilde{B})(u_1, \dots, u_r). \tag{III.3.25}$$

Let us deduce the equivalence of (1) and (2) from that of (III.3.24) and (III.3.25). Let $\tilde{b}$ be defined by $\tilde{b}(x, y) = b(x, -y)$. This implies that $b_x = (-1)^r \tilde{b}_x$, $b_y = (-1)^{r-1}\tilde{b}_y$, and $\tilde{B} = pari(B)$. Thus $b_y - b_x$ is anti-palindromic if and only if $\tilde{b}_y + \tilde{b}_x$ is, i.e. if and only if (III.3.24) holds for $\tilde{b}$, which is the case if and only if (III.3.25) holds for $\tilde{B}$, which is equivalent to (III.3.23) with $\tilde{B} = pari(B)$. This proves the lemma. □

The following proposition summarizes the mould reformulations of the defining properties (i) and (ii) of $W_{\mathfrak{krv}}$.

Proposition III.3.17. *Let $ARI^{pol}_{al+sen*circconst}$ denote the space of alternal polynomial-valued moulds satisfying the senary relation (III.3.23) and having swap that is circ-constant up to addition of a constant-valued mould. Then we have the isomorphism of Lie algebras*

$$ma : W_{\mathfrak{krv}} \xrightarrow{\sim} ARI^{pol}_{al+sen*circconst} \subset ARI_{ari}.$$

Mould background: the *ganit* automorphism and Ecalle's fundamental identity

The next stage of our proof, the construction of a Lie algebra morphism

$$ARI^{pol}_{al+sen*circconst} \to ARI^{\Delta}_{al+push*circneut}, \qquad \text{(III.3.26)}$$

is the most difficult, and requires some further definitions .

Definition III.3.18. For any mould $Q \in \overline{GARI}$, we define an automorphism $\overline{ganit}(Q)$ of the Lie algebra $\overline{ARI}_{lu}$ Set $\mathbf{v} = (v_1, \ldots, v_r)$, and let $W_{\mathbf{v}}$ denote the set of decompositions $d_{\mathbf{v}}$ of $\mathbf{v}$ into chunks

$$d_{\mathbf{v}} = \mathbf{a}_1\mathbf{b}_1 \cdots \mathbf{a}_s\mathbf{b}_s \qquad \text{(III.3.27)}$$

for $s \geq 1$, where with the possible exception of $\mathbf{b}_s$, the $\mathbf{a}_i$ and $\mathbf{b}_i$ are non-empty. Thus for instance, when $r = 2$ there are two decompositions in $W_{\mathbf{v}}$, namely $\mathbf{a}_1 = (v_1, v_2)$ and $\mathbf{a}_1\mathbf{b}_1 = (v_1)(v_2)$, and when $r = 3$ there are four decompositions, three for $s = 1$: $\mathbf{a}_1 = (v_1, v_2, v_3)$, $\mathbf{a}_1\mathbf{b}_1 = (v_1, v_2)(v_3)$, $\mathbf{a}_1\mathbf{b}_1 = (v_1)(v_2, v_3)$, and one for $s = 2$: $\mathbf{a}_1\mathbf{b}_1\mathbf{a}_2 = (v_1)(v_2)(v_3)$.

Écalle's explicit expression for $\overline{ganit}(Q)$ is given by

$$\big(\overline{ganit}(Q) \cdot T\big)(\mathbf{v}) = \sum_{\mathbf{a}_1\mathbf{b}_1\cdots\mathbf{a}_s\mathbf{b}_s \in W_{\mathbf{v}}} Q(\lfloor\mathbf{b}_1)\cdots Q(\lfloor\mathbf{b}_s)\, T(\mathbf{a}_1\cdots\mathbf{a}_s), \qquad \text{(III.3.28)}$$

where if $\mathbf{b}_i$ is the chunk $(v_k, v_{k+1}, \ldots, v_{k+l})$, then we use the notation

$$\lfloor\mathbf{b}_i = (v_k - v_{k-1}, v_{k+1} - v_{k-1}, \ldots, v_{k+l} - v_{k-1}). \qquad \text{(III.3.29)}$$

We are now ready to introduce the fundamental identity of Écalle, which is the key to the construction of the desired map (III.3.26).

Definition III.3.19. Let constants $c_r \in \mathbb{Q}$, $r \geq 1$, be defined by setting $f(x) = 1 - e^{-x}$ and expanding $f_*(x) = \sum_{r\geq 1} c_r x^{r+1}$, where $f_*(x)$ is the *infinitesimal generator* of $f(x)$, defined by

$$f(x) = \Big(exp\big(f_*(x)\frac{d}{dx}\big)\Big) \cdot x.$$

Let $lopil$ be the mould in $\overline{ARI}_{\overline{ari}}$ defined by the simple expression

$$lopil(v_1, \ldots, v_r) = c_r \frac{v_1 + \cdots + v_r}{v_1(v_1 - v_2)\cdots(v_{r-1} - v_r)v_r} \qquad \text{(III.3.30)}$$

Set $pil = exp_{\overline{ari}}(lopil)$ where $exp_{\overline{ari}}$ denotes the exponential map associated to $\overline{preari}$, and set $pal = swap(pil)$.

The mould $lopil$ is easily seen to be both alternal and circ-neutral. It is also known (although surprisingly difficult to show) that the mould $lopal = log_{ari}(pal)$ is alternal (cf. [Ec2], or [S2], Chap. 4.). Thus the moulds pil and pal are both exponentials of alternal moulds and therefore symmetral. The inverses of pal (in $GARI$) and pil (in $\overline{GARI}$) are given by

$$invpal = exp_{ari}(-lopal), \quad invpil = exp_{\overline{ari}}(-lopil).$$

The key maps we will be using in our proof are the adjoint operators associated to pal and pil, given by

$$Ad_{ari}(pal) = exp\big(ad_{ari}(lopal)\big), \quad Ad_{\overline{ari}}(pil) = exp\big(ad_{\overline{ari}}(lopil)\big), \qquad \text{(III.3.31)}$$

where $ad_{ari}(P) \cdot Q = ari(P, Q)$. The inverses of these adjoint actions are given by

$$Ad_{ari}(invpal) = exp\big(ad_{ari}(-lopal)\big), \;\; Ad_{\overline{ari}}(invpil) = exp\big(ad_{\overline{ari}}(-lopil)\big). \tag{III.3.32}$$

These adjoint actions produce remarkable transformations of certain mould properties into others, and form the heart of much of Écalle's theory of multizeta values. Écalle's *fundamental identity* relates the two adjoint actions of (III.3.31). Valid for all push-invariant moulds M, it is given by

$$swap \cdot Ad_{ari}(pal) \cdot M = \overline{ganit}(pic) \cdot Ad_{\overline{ari}}(pil) \cdot swap(M), \tag{III.3.33}$$

where $pic \in \overline{GARI}$ is defined by $pic(v_1, \dots, v_r) = 1/v_1 \cdots v_r$ (see [Ec], or [S2], Theorem 4.5.2 for the complete proof).

For our purposes, it is useful to give a slightly modified version of this identity. Let $poc \in \overline{GARI}$ be the mould defined by

$$poc(v_1, \dots, v_r) = \frac{1}{v_1(v_1 - v_2) \cdots (v_{r-1} - v_r)}. \tag{III.3.34}$$

Then $\overline{ganit}(poc)$ and $\overline{ganit}(pic)$ are inverse automorphisms of $\overline{ARI}_{lu}$ (see [B], Lemma 4.37). Thus, we can rewrite the above identity (III.3.33) as

$$\overline{ganit}(poc) \cdot swap \cdot Ad_{ari}(pal) \cdot M = Ad_{\overline{ari}}(pil) \cdot swap(M), \tag{III.3.35}$$

and letting $N = Ad_{ari}(pal) \cdot M$, i.e. $M = Ad_{ari}(invpal) \cdot N$, we rewrite it in terms of N as

$$Ad_{\overline{ari}}(invpil) \cdot \overline{ganit}(poc) \cdot swap(N) = swap \cdot Ad_{ari}(invpal) \cdot N, \tag{III.3.36}$$

this identity being valid whenever $M = Ad_{ari}(invpal) \cdot N$ is push-invariant.

Step 3: Construction of the map Ξ

In this section we finally arrive at the main step of the construction of our map $\mathfrak{krv} \to \mathfrak{krv}_{ell}$, namely the construction of the map Ξ given in the following proposition.

Proposition III.3.20. *The operator $\Xi = Ad_{ari}(invpal) \circ pari$ gives an injective Lie morphism of Lie subalgebras of ARI_{ari}:*

$$\Xi : ARI^{pol}_{al+sen*circconst} \longrightarrow ARI^{\Delta}_{al+push*circneut}. \tag{III.3.37}$$

Proof. We have already shown that both spaces are Lie subalgebras of ARI_{ari}, the first in Proposition III.3.17 and the second in III.3.2. Furthermore, since $pari$ and $Ad_{ari}(invpal)$ are both invertible and respect the ari-bracket, the proposed map is indeed an injective map of Lie subalgebras. Thus it remains only to show that the image of $ARI^{pol}_{al+sen*circconst}$ under Ξ really lies in $ARI^{\Delta}_{al+push*circneut}$. We will show separately that if $B \in ARI^{pol}_{al+sen*circconst}$ and $A = \Xi(B)$, then

(i) A is push-invariant,

(ii) A is alternal,

(iii) $swap(A)$ is circ-neutral up to addition of a constant-valued mould,

(iv) $A \in ARI^{\Delta}$.

Proof of (i). Écalle proved that $Ad_{ari}(pal)$ transforms push-invariant moulds to moulds satisfying the senary relation (III.3.25) (see [Ec] (3.58); indeed this is how the senary relation arose). Since

B satisfies (III.3.23), $\tilde{B} = pari(B)$ satisfies (III.3.25), so $Ad_{ari}(invpal)(\tilde{B}) = \Xi(B) = A$ is push-invariant.

Proof of (ii). Recall that ARI_{al} is closed under ari and $exp_{ari}(ARI_{al})$ is the subgroup of symmetral moulds $GARI^{as}_{gari}$ of $GARI_{gari}$.
The *pal* is known to be symmetral (cf. [Ec2], or in more detail [S2], Theorem 4.3.4).
Thus, since $GARI^{as}_{gari}$ is a group, the $gari$-inverse mould $invpal$ is also symmetral. Therefore the adjoint action $Ad_{ari}(invpal)$ on ARI restricts to an adjoint action on the Lie subalgebra ARI_{al} of alternal moulds. If B is alternal, then $pari(B)$ is alternal, and so $A = \Xi(B)$ is alternal. This completes the proof of (ii).

For the assertions (iii) and (iv), we will make use of Écalle's fundamental identity in the version (III.3.36) given in III.3.33, with $N = pari(B)$ (recall that (III.3.36) is valid whenever $Ad_{ari}(invpal)\cdot N$ is push-invariant, which is the case for $pari(B)$ thanks to (i) above). The key point is that the operators $\overline{ganit}(poc)$ and $Ad_{\overline{ari}}(pil)$ on the left-hand side of (III.3.36) are better adapted to tracking the circ-neutrality and the denominators than the right-hand operator $Ad_{ari}(invpal)$ considered directly.

Proof of (iii). Let $b \in W_{\mathfrak{frv}}$, and assume that b is of homogeneous degree n. Let $B = ma(b)$. Then by Proposition III.3.15 and Proposition II.3.22, $swap(B)$ is circ-constant, and even circ-neutral if n is even.

We need to show that $swap \cdot Ad_{ari}(invpal) \cdot pari(B)$ is *circ-neutral. To do this, we use (III.3.36) with $N = pari(B)$, and in fact show the result on the left-hand side, which is equal to

$$Ad_{\overline{ari}}(invpil) \cdot \overline{ganit}(poc) \cdot pari \cdot swap(B)$$

(noting that $pari$ commutes with $swap$). We prove that this mould is *circ-neutral in three steps. First we show that the operator $\overline{ganit}(poc) \cdot pari$ changes a circ-constant mould into one that is circ-neutral (Proposition III.3.21). Secondly, we show that the operator $Ad_{\overline{ari}}(invpil)$ preserves the property of circ-neutrality (Proposition III.3.23). Finally, we show that if M is a mould that is not circ-constant but only *circ-constant, and if M_0 is the (unique) constant-valued mould such that $M + M_0$ is circ-constant, then

$$Ad_{\overline{ari}}(invpil) \cdot ganit(poc) \cdot pari(M) + M_0$$

is circ-neutral. Using (III.3.36), this will show that $swap \cdot Ad_{ari}(invpal) \cdot M$ is *circ-neutral.

Proposition III.3.21. *Fix $n \geq 3$, and let $M \in \overline{ARI}$ be a circ-constant polynomial-valued mould of homogeneous degree n. Then $\overline{ganit}(poc) \cdot pari(M)$ is circ-neutral.*

Proof. Let $c = \big(M(v_1)\,|\,v_1^{n-1}\big)$, and let $N = pari(M)$, so that $N(v_1) = -cv_1^{n-1}$. Let $\mathbf{v} = (v_1, \dots, v_r)$, and let $\mathbf{W_v}$ be the set of decompositions $d_\mathbf{v}$ of $\mathbf{v}$ into chunks $d_\mathbf{v} = \mathbf{a}_1\mathbf{b}_1 \cdots \mathbf{a}_s\mathbf{b}_s$ as in (III.3.27). For any decomposition $d_\mathbf{v}$, we let its $\mathbf{b}$-part be the unordered set $\{\mathbf{b}_1, \dots, \mathbf{b}_s\}$, its $\mathbf{a}$-part the unordered set $\{\mathbf{a}_1, \dots, \mathbf{a}_s\}$, and we write $l_\mathbf{a}$ for the number of letters in the $\mathbf{a}$-part, i.e. $l_\mathbf{a} = |\mathbf{a}_1| + \cdots + |\mathbf{a}_s|$.

Let

$$\mathbf{W} = \coprod_i \mathbf{W}_{\sigma_r^i(\mathbf{v})},$$

where the $\sigma_r^i(\mathbf{v})$ are the cyclic permutations of $\mathbf{v} = (v_1, \dots, v_r)$, and let $\mathbf{W^b}$ denote the subset of decompositions in $\mathbf{W}$ having identical $\mathbf{b}$-part. The decompositions in $\mathbf{W}$ having identical $\mathbf{b}$-part to a given decomposition $d_\mathbf{v} \in \mathbf{W_v}$ are as follows: there is exactly one decomposition in $\mathbf{W}_{\sigma_r^{i-1}(\mathbf{v})}$ for each i such that v_i is one of the letters in the $\mathbf{a}$-part of $\mathbf{v}$, which is obtained from $d_\mathbf{v}$ by placing dividers between the same letters. For example, if $r = 5$ and $d_\mathbf{v} = \mathbf{a}_1\mathbf{b}_1\mathbf{a}_2\mathbf{b}_2 = (v_1, v_2)(v_3)(v_4)(v_5)$ then the two other decompositions having the same **b**-part $\{(v_3),(v_5)\}$ are given by $(v_2)(v_3)(v_4)(v_5)(v_1)$ and $(v_4)(v_5)(v_1,v_2)(v_3)$. Thus if $\mathbf{b}$ denotes the **b**-part of a given decomposition $d_{\mathbf{v}}$ of $\mathbf{v}=(v_1,\dots,v_r)$, then $\mathbf{W^b}$ contains exactly $l_{\mathbf{a}}$ decompositions, more precisely exactly one decomposition of each cyclic permutation $(v_i,\dots,v_r,v_1,\dots,v_{i-1})$ with v_i in the **a**-part of $d_{\mathbf{v}}$.

Also, for each $n\geq 1$, let $\mathbf{W}_n^{\mathbf{a}}$ denote the set of monomials w of degree $n-l_{\mathbf{a}}$ in the letters lying in the **a**-part of $d_{\mathbf{v}}$. For instance in the example above $d_{\mathbf{v}}=(v_1,v_2)(v_3)(v_4)(v_5)$, the **a**-part is $\{(v_1,v_2),(v_4)\}$ and $\mathbf{W}_5^{\mathbf{a}}$ consists of all monomials of degree 2 in the three letters v_1,v_2,v_4, i.e. $\mathbf{W}_5^{\mathbf{a}}=\{v_1^2,v_2^2,v_4^2,v_1v_2,v_1v_4,v_2v_4\}$. Note in particular that $\mathbf{W}_n^{\mathbf{a}}=\{1\}$ when $|\mathbf{a}|=n$ and $\mathbf{W}_n^{\mathbf{a}}=\emptyset$ when $r>n$.

We now consider the mould $N=pari(M)$, of fixed homogeneous degree n, with $N(v_1)=-cv_1^{n-1}$. Since M is circ-constant for c, we have

$$N(v_1,\dots,v_r)+\cdots+N(v_r,v_1,...,v_{r-1})=(-1)^r c\sum_w w. \tag{III.3.38}$$

By the explicit formula (III.3.28), we have

$$\big(\overline{ganit}(poc)\cdot N\big)(v_1,\dots,v_r)=\sum_{\mathbf{W_v}} poc(\lfloor\mathbf{b}_1)\cdots poc(\lfloor\mathbf{b}_s)N(\mathbf{a}_1\cdots\mathbf{a}_s), \tag{III.3.39}$$

so adding up over the cyclic permutations of $\mathbf{v}$, we have

$$\begin{aligned}\sum_{i=0}^{r-1}\big(\overline{ganit}(poc)\cdot N\big)\big(\sigma_r^i(\mathbf{v})\big) &= \sum_{\mathbf{W}} poc(\lfloor\mathbf{b}_1)\cdots poc(\lfloor\mathbf{b}_s)N(\mathbf{a}_1\cdots\mathbf{a}_s)\\ &= \sum_{\mathbf{b}=\{\mathbf{b}_1,\dots,\mathbf{b}_s\}}\ \sum_{\mathbf{W^b}} poc(\lfloor\mathbf{b}_1)\cdots poc(\lfloor\mathbf{b}_s)N(\mathbf{a}_1\cdots\mathbf{a}_s)\\ &= \sum_{\mathbf{b}=\{\mathbf{b}_1,\dots,\mathbf{b}_s\}} poc(\lfloor\mathbf{b}_1)\cdots poc(\lfloor\mathbf{b}_s)\sum_{j=0}^{l_{\mathbf{a}}-1} N\big(\sigma_{l_{\mathbf{a}}}^j(\mathbf{a}_1\cdots\mathbf{a}_s)\big)\\ &= (-1)^{l_{\mathbf{a}}}c\sum_{\mathbf{b}=\{\mathbf{b}_1,\dots,\mathbf{b}_s\}}(-1)^{l_{\mathbf{a}}}poc(\lfloor\mathbf{b}_1)\cdots poc(\lfloor\mathbf{b}_s)\sum_{w\in\mathbf{W}_n^{\mathbf{a}}} w\end{aligned} \tag{III.3.40}$$

where the last equality follows from (III.3.38).

If $c=0$, the expression (III.3.40) is trivially equal to zero in all depths $r>1$, so we obtained the desired result that $ganit(poc)\cdot pari(M)$ is circ-neutral. In order to deal with the case where M is circ-constant for a value $c\neq 0$, we use a trick and subtract off a known mould that is also circ-constant for c.

Lemma III.3.22. *For $n>1$ and any constant c, let T_c^n be the homogeneous polynomial mould of degree n defined by*

$$T_c^n(v_1,\dots,v_r)=\frac{c}{r}P_r^n,$$

where P_n^r is the sum is over all monomials of degree $n-r$ in the variables $v_1,\dots,v_r$ for $1\leq r\leq n$. Then T_c^n is circ-constant and $\overline{ganit}(poc)\cdot pari(T_c^n)$ is circ-neutral.

The proof of this lemma is annoyingly technical, so we have relegated it to Appendix 2. Consider the mould $N=M-T_c^n$. The mould N is circ-constant since M and T_c^n both are, but $N(v_1)=0$, so by the result above, we know that $ganit(poc)\cdot pari(N)$ is circ-neutral. But Lemma III.3.22 shows that $ganit(poc)\cdot pari(T_c^n)$ is circ-neutral, so the mould $ganit(poc)\cdot pari(M)$ is also circ-neutral, as desired. □

We now proceed to the second step, showing that the operator $Ad_{\overline{ari}}(invpil)$ preserves circ-neutrality.

Proposition III.3.23. *If $M \in \overline{ARI}$ is circ-neutral then $Ad_{\overline{ari}}(invpil)\cdot M$ is also circ-neutral.*

Proof. By (III.3.32), we have

$$Ad_{\overline{ari}}(invpil) = exp\big(ad_{\overline{ari}}(-lopil)\big) = \sum_{n\geq 0} \frac{(-1)^n}{n} ad_{\overline{ari}}(lopil)^n. \tag{III.3.41}$$

The definition of $lopil$ in (III.3.30) shows that $lopil$ is trivially circ-neutral. Thus, since M is circ-neutral, $ad_{\overline{ari}}(lopil)\cdot M = \overline{ari}(lopil, M)$ is also circ-neutral by Lemma III.3.10, and successively so are all the terms $ad_{\overline{ari}}(lopil)^n(M)$. Thus $Ad_{\overline{ari}}(invpil)\cdot M$ is circ-neutral. □

Finally, we now assume that $swap(B)$ is a *circ-neutral polynomial-valued mould in $\overline{ARI}$ of homogeneous degree n. Let B_0 be the (unique) constant-valued mould such that $swap(B) + B_0$ is circ-neutral. Then by Propositions III.3.21 and III.3.23, the mould

$$Ad_{\overline{ari}}(invpil)\cdot ganit(poc)\cdot pari(B + B_0)$$

is circ-neutral. This mould breaks up as the sum

$$Ad_{\overline{ari}}(invpil)\cdot \overline{ganit}(poc)\cdot pari(B) + Ad_{\overline{ari}}(invpil)\cdot \overline{ganit}(poc)\cdot pari(B_0),$$

but the operator $Ad_{\overline{ari}}(invpil)\cdot\overline{ganit}(poc)$ preserves constant-valued moulds (cf. [S], Lemma 4.6.2 for the proof). Thus

$$Ad_{\overline{ari}}(invpil)\cdot ganit(poc)\cdot pari(B + B_0) = Ad_{\overline{ari}}(invpil)\cdot ganit(poc)\cdot pari(B) + B_0,$$

so

$$Ad_{\overline{ari}}(invpil)\cdot ganit(poc)\cdot pari(B) = swap\cdot \Xi(B)$$

is *circ-neutral, completing the proof of (iii).

Proof of (iv). We will again use the left-hand side of (III.3.36), this time to track the denominators that appear in the right-hand side. By (III.3.36), if B is a polynomial-valued mould satisfying the senary relation, and if $A = \Xi(B) = Ad_{ari}(invpal)\cdot pari(B)$, then A lies in ARI^{Δ} if and only if

$$swap\cdot Ad_{\overline{ari}}(invpil)\cdot \overline{ganit}(poc)\cdot swap\big(pari(B)\big) \in ARI^{\Delta}. \tag{III.3.42}$$

We will prove that this is the case, by studying the denominators that are produced, first by applying $\overline{ganit}(poc)$ to a polynomial-valued mould, and then by applying $Ad_{\overline{ari}}(invpil)$. The first result is that the denominators introduced by applying $\overline{ganit}(poc)$ are at worst of the form $(v_1 - v_2)\cdots(v_{r-1} - v_r)$.

Lemma III.3.24. *Let $M \in \overline{ARI}^{pol}$. Then*

$$swap\cdot \overline{ganit}(poc)\cdot M \in ARI^{\Delta}.$$

Proof. The explicit expression for $\overline{ganit}(Q)$ given in (III.3.28) shows that the only denominators that can occur in $\overline{ganit}(poc)\cdot M$ come from the factors

$$poc(\lfloor \mathbf{b}_1)\cdots poc(\lfloor \mathbf{b}_s) \tag{III.3.43}$$

for all decompositions $d_{\mathbf{v}} = \mathbf{a}_1\mathbf{b}_1\cdots\mathbf{a}_s\mathbf{b}_2$ of $\mathbf{v} = (v_1,\ldots,v_r)$ into chunks as in (III.3.27), and

$$\lfloor \mathbf{b}_i = (v_k - v_{k-1}, v_{k+1} - v_{k-1}, \ldots, v_{k+l} - v_{k-1})$$

(for $k > 1$) as in (III.3.29). Since poc is defined as in (III.3.34), the only factors that can appear in (III.3.43) are $(v_l - v_{l-1})$ where v_l is a letter in one of $\mathbf{b}_i$, and these factors appear in each term with multiplicity one. Since the sum ranges over all possible decompositions, the only letter of $\mathbf{v}$ that never belongs to any $\mathbf{b}_i$ is v_1; all the other factors $(v_i - v_{i-1})$ appear. Thus $(v_1 - v_2)(v_2 - v_3)\cdots(v_{r-1} - v_r)$ is a common denominator for all the terms in the sum defining $\overline{ganit}(poc)\cdot M$. The swap of this common denominator is equal to $u_2\cdots u_r$, so this term is a common denominator for $swap\cdot\overline{ganit}(poc)\cdot M$, which proves the lemma. □

Lemma III.3.25. *Let $M, N \in \overline{ARI}_{*circneut}$ be two moulds such that $swap(M)$ and $swap(N)$ lie in ARI^Δ. Then $swap\big(\overline{ari}(M,N)\big)$ also lies in ARI^Δ.*

Proof. In Proposition A.1 of the Appendix of [BS], it is shown that if M and N are alternal moulds in $\overline{ARI}$ such that $swap(M)$ and $swap(N)$ lie in ARI^Δ, then $swap\big(\overline{ari}(M,N)\big)$ also lies in ARI^Δ. In fact, it is shown in Proposition A.2 of that appendix that alternal moulds M whose swap lies in ARI^Δ satisfy the following property: setting

$$\check{M}(v_1,\ldots,v_r) = v_1(v_1-v_2)\cdots(v_{r-1}-v_r)v_r\, M(v_1,\ldots,v_r),$$

we have

$$\check{M}(0,v_2,\ldots,v_r) = \check{M}(v_2,\ldots,v_r,0). \tag{III.3.44}$$

In fact, the proof that $swap\big(\overline{ari}(M,N)\big)$ lies in ARI^Δ does not use the full alternality of M and N, but only (III.3.44). Therefore, the same proof goes through when M and N are *circ-neutral moulds such that $swap(M)$ and $swap(N)$ lie in ARI^Δ, as long as we check that every *circ-neutral mould M such that $swap(M)\in ARI^\Delta$ satisfies (III.3.44).

To check this, let M be such a mould; by additivity, we may assume that M is concentrated in a single depth $r > 1$. This means that there is a constant C_M such that

$$M(v_1,\ldots,v_r) + M(v_2,\ldots,v_r,v_1) + \cdots + M(v_r,v_1,\ldots,v_{r-1}) = C_M,$$

which we can also write as

$$\frac{\check{M}(v_1,\ldots,v_r)}{v_1(v_1-v_2)\cdots(v_{r-1}-v_r)v_r} + \frac{\check{M}(v_2,\ldots,v_r,v_1)}{v_2(v_2-v_3)\cdots(v_{r-1}-v_r)(v_r-v_1)v_1} +$$

$$\cdots + \frac{\check{M}(v_r,v_1,\ldots,v_{r-1})}{v_r(v_r-v_1)\cdots(v_{r-2}-v_{r-1})v_{r-1}} = C_M$$

where the numerators are polynomials. If we multiply the entire equality by v_1 and set $v_1 = 0$, only the first two terms do not vanish, and they yield precisely the desired relation (III.3.44). □

Corollary III.3.26. *If $N \in \overline{ARI}$ is a *circ-neutral mould such that $swap(N)\in ARI^\Delta$, then also*

$$swap\cdot Ad_{\overline{ari}}(invpil)\cdot N \in ARI^\Delta. \tag{III.3.45}$$

Proof. The lemma shows that $swap\cdot\overline{ari}(lopil,N)\in ARI^\Delta$ since the mould $lopil$ is circ-neutral and $swap\cdot lopil \in ARI^\Delta$ by (III.3.30). In fact, applying the lemma successively shows that $swap\cdot ad_{\overline{ari}}(lopil)^n(N)\in ARI^\Delta$ for all $n\geq 1$. Since $Ad_{\overline{ari}}(invpil)\cdot N$ is obtained by summing these terms by (III.3.41), we obtain (III.3.45). □

To conclude, we set $M = swap\cdot pari(B)$; then by Lemma III.3.24 we have

$$swap\cdot\overline{ganit}(poc)\cdot swap\cdot pari(B)\in ARI^\Delta.$$

By Proposition III.3.21 this mould is *circ-neutral, so we can apply Corollary III.3.26 with $N = \overline{ganit}(poc) \cdot swap \cdot pari(B)$ to conclude that

$$swap \cdot Ad_{\overline{ari}}(invpil) \cdot \overline{ganit}(poc) \cdot swap \cdot pari(B) \in ARI^{\Delta}.$$

Thus thus by (III.3.36) with $N = pari(B)$, we finally find that

$$Ad_{ari}(invpal) \cdot pari(B) = \Xi(B) \in ARI^{\Delta},$$

which completes the proof of (iv).

We have thus finished proving Proposition III.3.20. Backtracking, this means we have completed the details of Step 3 of the proof of Theorem III.3.12. Step 4, the final step in the proof, is very easy and was explained completely just before paragraph III.3.3. Thus we have now completed the proof of Theorem III.3.12, i.e. we have completed the construction of the injective Lie algebra morphism $\mathfrak{krv} \hookrightarrow \mathfrak{krv}_{ell}$. □

III.3.4 Relations with elliptic Grothendieck-Teichmüller and double shuffle

Our next result is the proof of the commutativity of the diagram

$$\begin{array}{ccccc} \mathfrak{grt} & \hookrightarrow & \mathfrak{ds} & \hookrightarrow & \mathfrak{krv} \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{\mathfrak{grt}}_{ell} & \hookrightarrow & \mathfrak{ds}_{ell} & \hookrightarrow & \mathfrak{krv}_{ell} \\ & \diagdown & \downarrow & \diagup & \\ & & \mathfrak{oder}_2. & & \end{array}$$

. In fact, this result is simply a consequence of putting together the results of the previous sections with known results. Indeed, the commutativity of the diagram

$$\begin{array}{ccc} \mathfrak{grt} & \hookrightarrow & \mathfrak{ds} \\ \downarrow & & \downarrow \\ \widetilde{\mathfrak{grt}}_{ell} & \hookrightarrow & \mathfrak{ds}_{ell} \\ & \diagdown \quad \diagup & \\ & \mathfrak{oder}_2, & \end{array} \qquad \text{(III.3.46)}$$

where $Ad_{ari}(invpal) : \mathfrak{ds} \to \mathfrak{ds}_{ell}$ is the right-hand vertical map as shown in [S3].

By (10), the injective map $\mathfrak{ds} \hookrightarrow \mathfrak{krv}$ is given by

$$b(x, y) \mapsto \hat{b} = b(z, -y)$$

, or more precisely to the derivation in $\mathfrak{krv}$ given by $a \mapsto \hat{b}$, $[a, b] \mapsto 0$.

If $b(x, y) \in \mathfrak{ds}$, then $b(x, -y)$ lies in $W_{\mathfrak{krv}}$ and $b(z, -y)$ lies in $V_{\mathfrak{krv}}$, so this map unpacks to

$$\mathfrak{ds} \overset{y \mapsto -y}{\to} W_{\mathfrak{krv}} \overset{x \mapsto z}{\to} V_{\mathfrak{krv}} \to \mathfrak{krv},$$

where the last map comes from (II.3.2). We can thus construct a commutative square

$$\begin{array}{ccc} \mathfrak{ds} & \rightarrow & \mathfrak{krv} \\ \downarrow & & \downarrow \\ \mathfrak{ds}_{ell} & \subset & \mathfrak{krv}_{ell} \end{array} \tag{III.3.47}$$

given in detail by

$$\begin{array}{ccc} \mathfrak{ds} & \overset{y\mapsto -y}{\rightarrow} & W_{\mathfrak{krv}} \simeq \mathfrak{krv} \\ ma \downarrow & & \downarrow ma \\ ARI^{pol}_{al*il} & \overset{pari}{\rightarrow} & ARI^{pol}_{al+sen*circconst} \\ Ad_{ari}(invpal) \downarrow & & \downarrow Ad_{ari}(invpal)\circ pari \\ ARI^{\Delta}_{al*al} & \subset & ARI^{\Delta}_{al+push*circneut} \\ \Delta \downarrow & & \downarrow \Delta \\ \Delta(ARI^{\Delta}_{al*al}) & \subset & \Delta(ARI^{\Delta}_{al+push*circneut}) \\ ma^{-1} \downarrow & & \downarrow ma^{-1} \\ \mathfrak{ds}_{ell} & \subset & \mathfrak{krv}_{ell}. \end{array}$$

The first line of this diagram comes from the injection $\mathfrak{ds} \hookrightarrow \mathfrak{krv}$ and the definition of $W_{\mathfrak{krv}}$. The second line is the direct mould translation of the top one, as the left-hand space is exactly $ma(\mathfrak{ds})$, the right-hand space is $ma(W_{\mathfrak{krv}})$ by (III.3.17), and the map *pari* restricted to polynomials is nothing other than $y \mapsto -y$. The vertical morphism

$$Ad_{ari}(invpal) : ARI^{pol}_{al*il} \rightarrow ARI^{\Delta}_{al*al}$$

is proven in [S3], and the vertical morphism

$$Ad_{ari}(invpal)\circ pari : ARI^{pol}_{al+sen*circconst} \rightarrow ARI^{\Delta}_{al+push*circneut}$$

comes from Proposition III.3.20. Since *pari* is an involution, this proves that the horizontal injection in the third line of the diagram is nothing but an inclusion. Finally, the last line of the diagram comes from the definitions $\mathfrak{ds}_{ell} = \Delta(ARI^{\Delta}_{al*al})$ [S3]) and $\mathfrak{krv}_{ell} = \Delta(ARI^{\Delta}_{al+push*circneut})$ by Definition III.3.1.

This diagram shows that the diagram (III.3.46) above can be completed by the diagram (III.3.47) to the commutative diagram of III.3.4.

III.3.5 Comparison of the two independently defined elliptic Kashiwara-Vergne Lie algebras

This last part follows the exposition given by Leila Schneps and the author given in the article [RS2]. After defining the other elliptic construction by Alekseev-Kawazumi-Kuno-Naef from [AKKN2], we reformulate their defining properties to reformulate it in terms similar to the ones use in Chapter II Section 3.

The elliptic Kashiwara-Vergne Lie algebra from Aleksee-Kawazumi-Kuno-Naef

Let $\mathfrak{lie}^{(1,1)}$ be the free Lie algebra on two generators Lie$[x, y]$, to be thought of as the Lie algebra associated to the fundamental group of the once-punctured torus.
We set $c = [x, y]$ so that the relation $[x, y] = c$ holds in $\mathfrak{lie}^{(1,1)}$. Let $\mathfrak{lie}_n^{(1,1)}$ denote the weight n part of $\mathfrak{lie}^{(1,1)}$, where the weight is the total degree in x and y, and let $\mathfrak{lie}_{n,r}^{(1,1)}$ denote the weight n, depth r part of $\mathfrak{lie}^{(1,1)}$, where the depth is the y-degree.

For any element $f \in \mathfrak{lie}^{(1,1)}$, we decompose it as

$$f = f_x x + f_y y = x f^x + y f^y = x f_x^x x + x f_y^x y + y f_x^y x + y f_y^y y.$$

Let $\mathfrak{der}^{(1,1)}$ denote the Lie subalgebra of Der $\mathfrak{lie}^{(1,1)}$ of derivations u such that $u(c) = 0$. Let $\mathfrak{der}_n^{(1,1)}$ denote the subspace of $\mathfrak{der}^{(1,1)}$ of derivations u such that $u(x), u(y) \in \mathfrak{lie}_n^{(1,1)}$.

Let $f \in \mathfrak{lie}_n^{(1,1)}$ for $n > 1$. Then f is the value on x of a derivation $u \in \mathfrak{der}_n^{(1,1)}$ if and only if f is push-invariant, in which case $u(y)$ is uniquely defined. If $u \in \mathfrak{der}^{(1,1)}$ and $u(x) = f$, $u(y) = g$, we sometimes write $u = D_{f,g}$.

Let $\mathfrak{tr}_2$ be the quotient of $\mathfrak{lie}^{(1,1)}$ by the equivalence relation: two words w and w' are equivalent if one can be obtained from the other by cyclic permutation of the letters.

The *elliptic divergence* $div : \mathfrak{der}^{(1,1)} \to \mathfrak{tr}_2$ is defined by

$$div(u) = tr(f_x + g_y)$$

where $u = D_{f,g}$. Since $u([x, y]) = [x, g] + [f, y] = 0$, we have

$$x g_x x + x g_y y - x g^x x - y g^y y = y f_x x + y f_y y - x f^x y - y f^y y.$$

Comparing the terms on both sides that start with x and end with y shows that $g_y = -f^x$. Thus we can write the divergence condition as a function of just f:

$$div(u) = tr(f_x - f^x).$$

Definition III.3.27. The elliptic Kashiwara-Vergne Lie algebra $\mathfrak{krv}^{(1,1)}$ defined in ([AKKN2]) is the $\mathbb{Q}$-vector space spanned by the derivations $u \in \mathfrak{der}_n^{(1,1)}$, $n \geq 3$, such that

$$div(u) = \begin{cases} K\, tr\left(c^{\frac{n-1}{2}}\right) \text{ for some } K \in \mathbb{Q} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases} \tag{1}$$

It is closed under the bracket of derivations.

A reformulation of the div condition.

Observe that the Lie algebra $\mathfrak{krv}^{(1,1)}$ is bigraded by the weight and depth. We write $\mathfrak{krv}_{n,r}^{(1,1)}$ for the vector subspace of $\mathfrak{krv}_{ell}$ spanned by derivations u such that $u(x) \in \mathfrak{lie}_{n,r}^{(1,1)}$ and $u(y) \in \mathfrak{lie}_{n,r+1}^{(1,1)}$. Let $u \in \mathfrak{krv}_{n,r}^{(1,1)}$, let $f = u(x)$ and $g = u(y)$, and write

$$f = \sum_{\underline{i}=(i_0,\ldots,i_r)} c_{\underline{i}}\, x^{i_0} y x^{i_1} y \cdots y x^{i_r}.$$

Then

$$f_x = \sum_{\underline{i} \text{ s.t. } i_r \neq 0} c_{\underline{i}}\, x^{i_0} y x^{i_1} y \cdots y x^{i_r - 1}$$

and

$$f^x = \sum_{\underline{i} \text{ s.t. } i_0 \neq 0} c_{\underline{i}}\, x^{i_0-1} y x^{i_1} y \cdots y x^{i_r}.$$

For any word w in x, y, let $C(w)$ denote the trace class of w, i.e. the set of words obtained by cyclically permuting the letters of w. The trace of a polynomial h is given by

$$\sum_{C(w)} \big(tr(h) \,|\, C(w)\big) \cdot C(w)$$

where the sum runs over the trace classes of words and the coefficient $\big(tr(h) \,|\, C(w)\big)$ of the class $C(w)$ in $tr(h)$ is given by

$$\big(tr(h) \,|\, C(w)\big) = \sum_{v \in C(w)} (h|v),$$

where $(h|v)$ denotes the coefficient of v in h.

Fix a word w of weight $n-1$ and depth r and consider the coefficient of $C(w)$ in $tr(f_x - f^x)$:

$$\begin{aligned} \big(tr(f_x - f^x) \,|\, C(w)\big) &= \sum_{v \in C(w)} (f_x|v) - (f^x|v) \\ &= \sum_{v \in C(w)} (f_x x|vx) - (x f^x|xv) \\ &= \sum_{v \in C(w)} (f|vx) - (f|xv). \end{aligned}$$

(2) For all words $v \in C(w)$ that start in x, the word $v' \in C(w)$ obtained from v by taking the first x of v and putting it at the end satisfies $vx = xv'$ and thus $(f|vx) = (f|xv')$. Thus the corresponding terms in (2) cancel out, i.e. writing $C^x(w)$ (resp. $C^y(w)$, $C_x(w)$, $C_y(w)$) for the terms in $C(w)$ that start with x (resp. start with y, end with x, end with y), we have

$$\sum_{v \in C^x(w)} (f|vx) - \sum_{v \in C_x(w)} (f|xv) = 0,$$

so (2) reduces to

$$\big(tr(f_x - f^x) \,|\, C(w)\big) = \sum_{v \in C^y(w)} (f|vx) - \sum_{v \in C_y(w)} (f|xv).(3)$$

Since for any word v of length $n-2$, if $v = yu \in C^y(w)$ then $uy \in C_y(w)$, we can write (3) as

$$\big(tr(f_x - f^x) \,|\, C(w)\big) = \sum_{u \text{ s.t. } yu \in C^y(w)} (f|yux) - \sum_{u \text{ s.t. } uy \in C_y(w)} (f|xuy).(4)$$

For any word $u = x^{i_0} y \cdots y x^{i_r}$ of depth r, we define

$$push(u) = x^{i_r} y x^{i_0} y \cdots y x^{i_{r-1}}$$

and

$$pushsym(u) = \sum_{i=0}^{r} push^i(u),$$

and extend the operators $push$ and $pushsym$ to polynomials by linearity. If u is a word of weight $n-2$ and depth $r-1$ such that $yu \in C^y(w)$, then $uy \in C_y(w)$ and we have

$$\begin{cases} C^y(w) = \{y\, push^i(u) \,|\, 0 \leq i \leq r-1\} \\ C_y(w) = \{push^i(u)\, y \,|\, 0 \leq i \leq r-1\}. \end{cases}$$

Note that $C^y(w)$ may be of order less than r (in fact strictly dividing r) when w has a symmetry under the push.
In this case, summing over the set of pushes of u from 0 to $r-1$ comes down to summing $r/|C^y(w)|$ times over the set $C^y(w)$ or $C_y(w)$.
Using this, we rewrite (4) for $w = uy$ as

$$\begin{aligned}
&\sum_{yu\in C^y(w)} (f|yux) - \sum_{uy\in C_y(w)} (f|xuy)\\
&\sum_{yu\in C^y(w)} (f_x^y|u) - \sum_{uy\in C_y(w)} (f_y^x|u)\\
&= \frac{|C^y(w)|}{r}\sum_{i=0}^{r-1}\big(f_x^y \mid push^i(u)\big) - \frac{|C_y(w)|}{r}\sum_{i=0}^{r-1}\big(f_y^x \mid push^i(u)\big)\\
&= \frac{|C_y(w)|}{r}\big(pushsym(f_x^y - f_y^x) \mid u\big).
\end{aligned}$$

This allows us to rewrite the divergence condition (1) on an element $f \in \mathfrak{krv}_{n,r}^{(1,1)}$ as the following family of relations for all words u of weight $n-2$ and depth $r-1$:

$$\begin{aligned}
&\big(pushsym(f_x^y - f_y^x)|u\big)\\
&= \begin{cases} \frac{Kr}{|C_y(uy)|}\sum_{v\in C(uy)}\big([x,y]^r \mid v\big) \text{ for some } K\in\mathbb{Q} & \text{if } n = 2r+1\\ 0 & \text{if } n \neq 2r+1.\end{cases}
\end{aligned} \tag{5}$$

This is the version of the divergence condition that we will use for comparison with the Lie algebra $\mathfrak{krv}_{ell}$.

The comparison

Let $F \in \mathfrak{krv}_{ell}$ be a mould of depth r and degree d, so that it corresponds under the bijection ma to a polynomial $f \in \mathfrak{lie}_{n,r}^{(1,1)}$ with $n = d + r$. The polynomial push-invariance of f implies that there exists a unique polynomial $g \in \mathfrak{lie}_{n,r+1}^{(1,1)}$ such that setting $u(x) = f$, $u(y) = g$, we obtain a derivation $u \in \mathfrak{der}_n^{(1,1)}$. The Lie bracket on $\mathfrak{krv}_{ell}$ corresponds to the Lie bracket on $\mathfrak{krv}^{(1,1)}$, namely bracketing of the derivations u.
Thus, in order to prove that $\mathfrak{krv}_{ell}$ is in bijection with $\mathfrak{krv}^{(1,1)}$, it remains only to prove that the circ-constance condition on $\Delta^{-1}\big(swap(F)\big)$ is equivalent to the divergence condition (5) on f.

Since

$$\Delta^{-1}(F)(u_1,\dots,u_r) = \frac{1}{(u_1+\cdots+u_r)u_1\cdots u_r)}F(u_1,\dots,u_r),$$

we have

$$swap\big(\Delta^{-1}(F)\big)(v_1,\dots,v_r) = \frac{1}{v_1(v_1-v_2)\cdots(v_{r-1}-v_r)v_r}swap(F)(v_1,\dots,v_r),$$

so the circ-constance condition is given explicitly by

$$\begin{aligned}
&\frac{swap(F)(v_1,\dots,v_r)}{v_1(v_1-v_2)\dots(v_{r-1}-v_r)v_r}\\
&\frac{swap(F)(v_2,\dots,v_1)}{v_2(v_2-v_3)\dots(v_r-v_1)v_1} + \dots\\
+&\frac{swap(F)(v_r,\dots,v_{r-1})}{v_r(v_r-v_1)\dots(v_{r-2}-v_{r-1})v_{r-1}} = Kr
\end{aligned}$$

for all depths $r \geq 2$. Putting this over a common denominator gives the equivalent equality

$$\begin{aligned} &swap(F)(v_1, v_2, \dots, v_r)v_2 \dots v_{r-1}(v_r - v_1) \\ &+swap(F)(v_2, \dots, v_r, v_1)v_3 \cdots v_r(v_1 - v_2) + \cdots \\ &+swap(F)(v_r, \dots, v_{r-1})v_1 \dots v_{r-2}(v_{r-1} - v_r) \\ &= Krv_1 \cdots v_r(v_1 - v_2) \cdots (v_r - v_1). \end{aligned}$$

The left-hand side expands to

$$\begin{aligned} &v_2 \dots v_{r-1}v_r\, swap(F)(v_1, \dots, v_r) - v_1v_2 \dots v_{r-1}\, swap(F)(v_1, \dots, v_r) \\ &+v_1v_3 \dots v_r\, swap(F)(v_2, \dots, v_r, v_1) - v_2v_3 \dots v_r\, swap(F)(v_2, .., v_r, v_1) + \dots \\ &+v_1 \dots v_{r-1}\, swap(F)(v_r, \dots, v_{r-1}) - v_1 \dots v_{r-2}v_r\, swap(F)(v_r, \dots v_{r-1}). \end{aligned}$$

Fix a monomial $v_1^{i_1+1}v_2^{i_2+1} \dots v_r^{i_r+1}$. Calculating its coefficient in the above expression yields

$$\begin{aligned} &\big(swap(F)(v_1, \dots, v_r)|v_1^{i_1+1}v_2^{i_2} \dots v_r^{i_r}\big) - \big(swap(F)(v_1, \dots, v_r)|v_1^{i_1}v_2^{i_2} \dots v_r^{i_r+1}\big) \\ &+\big(swap(F)(v_2, \dots, v_r, v_1)|v_1^{i_1}v_2^{i_2+1} \dots v_r^{i_r}\big) - \big(swap(F)(v_2, \dots, v_r, v_1)|v_1^{i_1+1}v_2^{i_2} \dots v_r^{i_r}\big) + \cdots \\ &+\big(swap(F)(v_r, v_1, \dots, v_{r-1})|v_1^{i_1} \dots v_r^{i_r+1}\big) - \big(swap(F)(v_r, v_1, \dots, v_{r-1})|v_1^{i_1} \dots v_{r-1}^{i_{r-1}+1}v_r^{i_r}\big) \\ =&\big(swap(F)(v_1, \dots, v_r)|v_1^{i_1+1}v_2^{i_2} \dots v_r^{i_r}\big) - \big(swap(F)(v_1, \dots, v_r)|v_1^{i_1}v_2^{i_2} \dots v_r^{i_r+1}\big) \\ &+\big(swap(F)(v_1, \dots, v_r)|v_1^{i_2+1}v_2^{i_3} \dots v_r^{i_1}\big) - \big(swap(F)(v_1, \dots, v_r)|v_1^{i_2}v_2^{i_3} \dots v_r^{i_1+1}\big) + \dots \\ &+\big(swap(F)(v_1, \dots, v_r)|v_1^{i_r+1} \dots v_r^{i_{r-1}}\big) - \big(swap(F)(v_r, v_1, \dots, v_{r-1})|v_1^{i_r} \dots v_r^{i_{r-1}+1}\big), \end{aligned}$$

where the equality is obtained by bringing every term back to a coefficient of a word in $swap(F)(v_1, \dots, v_r)$.

The circ-constance condition on $swap\big(\Delta^{-1}(F)\big)$ can thus be expressed by the family of relations for every tuple $(i_1, \dots, i_r)$:

$$\begin{aligned} &\big(swap(F)(v_1, \dots, v_r)|v_1^{i_1+1}v_2^{i_2} \dots v_r^{i_r}\big) - \big(swap(F)(v_1, \dots, v_r)|v_1^{i_1}v_2^{i_2} \dots v_r^{i_r+1}\big) \\ &+\big(swap(F)(v_1, \dots, v_r)|v_1^{i_2+1}v_2^{i_3} \dots v_r^{i_1}\big) - \big(swap(F)(v_1, \dots, v_r)|v_1^{i_2}v_2^{i_3} \dots v_r^{i_1+1}\big) + \cdots \\ &+\big(swap(F)(v_1, \dots, v_r)|v_1^{i_r+1} \dots v_r^{i_{r-1}}\big) - \big(swap(F)(v_r, v_1, \dots, v_{r-1})|v_1^{i_r} \dots v_r^{i_{r-1}+1}\big) \\ =&Kr\left((v_1 - v_2) \dots (v_r - v_1) \mid v_1^{i_1} \dots v_r^{i_r}\right). \end{aligned}$$

We can now translate this equality directly back into terms of the polynomial $f(x, y)$ corresponding to the mould F. We can write the right-hand side of the above equation as

$$Kr\left((v_1 - v_2) \cdots (v_{r-1} - v_r)v_r \mid v_1^{i_1} \cdots v_r^{i_r}\right) - Kr\left((v_1 - v_2) \cdots (v_{r-1} - v_r)v_1 \mid v_1^{i_1} \cdots v_r^{i_r}\right), \quad \text{(III.3.48)}$$

or equivalently, setting $B(v_1, \dots, v_r) = (v_1 - v_2) \cdots (v_{r-1} - v_r)v_r$, as

$$Kr\left(B \mid v_1^{i_1} \cdots v_r^{i_r}\right) - (-1)^{r-1}Kr\left(B \mid v_1^{i_r} \cdots v_r^{i_1}\right), \quad \text{(III.3.49)}$$

by numbering the v_i in the second term of (III.3.48) in the opposite order.

We have $[x, y] = ad(x)(y) = C_2$, so $[x, y]^r = C_2^r$, so the polynomial-valued mould corresponding to $[x, y]^r$ is given by

$$A(u_1, \dots, u_r) = (-1)^r u_1 \cdots u_r.$$

The swap of this mould is given by

$$swap(A)(v_1, \dots, v_r) = -(v_1 - v_2) \cdots (v_{r-1} - v_r)v_r.$$

Thus the mould B of (III.3.49) satisfies $B = -swap(A)$, so the expression (III.3.49) reformulating the RHS translates back to polynomials as to

$$-Kr\left([x,y]^r \mid x^{i_1}y\cdots x^{i_r}y\right) + (-1)^{r-1}Kr\left([x,y]^r \mid x^{i_r}y\cdots x^{i_1}y\right). \tag{III.3.50}$$

Using (7) to directly translate the left-hand side of (10) in terms of the polynomial f, we thus obtain the following expression equivalent to the circ-neutrality property (10):

$$\begin{aligned}
&\big(f \mid x^{i_1+1}yx^{i_2}y\dots yx^{i_r}y\big) - \big(f \mid x^{i_1}yx^{i_2}y\cdots yx^{i_r+1}y\big)\\
+&\big(f \mid x^{i_2+1}yx^{i_3}y\dots yx^{i_1}y\big) - \big(f \mid x^{i_2}yx^{i_3}y\cdots yx^{i_1+1}y\big) + \cdots\\
+&\big(f \mid x^{i_r+1}yx^{i_1}y\dots yx^{i_{r-1}}y\big) - \big(f \mid x^{i_r}yx^{i_1}y\cdots yx^{i_{r-1}+1}y\big)\\
=&-Kr\left([x,y]^r \mid x^{i_1}y\cdots x^{i_r}y\right) + (-1)^{r-1}Kr\left([x,y]^r \mid x^{i_r}y\cdots x^{i_1}y\right).
\end{aligned}$$

Since f is push-invariant, we have $(f|uy) = (f|yu)$ for every word u, so we can modify the negative terms in (14):

$$\begin{aligned}
&\big(f \mid x^{i_1+1}yx^{i_2}y\dots yx^{i_r}y\big) - \big(f \mid yx^{i_1}yx^{i_2}y\cdots yx^{i_r+1}\big)\\
+&\big(f \mid x^{i_2+1}yx^{i_3}y\dots yx^{i_1}y\big) - \big(f \mid yx^{i_2}yx^{i_3}y\cdots yx^{i_1+1}\big) + \cdots\\
+&\big(f \mid x^{i_r+1}yx^{i_1}y\dots yx^{i_{r-1}}y\big) - \big(f \mid yx^{i_r}yx^{i_1}y\cdots yx^{i_{r-1}+1}\big)\\
=&-Kr\left([x,y]^r \mid x^{i_1}y\cdots x^{i_r}y\right) + (-1)^{r-1}Kr\left([x,y]^r \mid x^{i_r}y\cdots x^{i_1}y\right).
\end{aligned}$$

Now all words in the positive terms start in x and end in y, and all words in the negative terms start in y and end in x, so we can remove these letters and write

$$\begin{aligned}
&\big(f_y^x \mid x^{i_1}yx^{i_2}y\dots yx^{i_r}\big) - \big(f_x^y \mid x^{i_1}yx^{i_2}y\cdots yx^{i_r}\big)\\
+&\big(f_y^x \mid x^{i_2}yx^{i_3}y\dots yx^{i_1}\big) - \big(f_x^y \mid x^{i_2}yx^{i_3}y\cdots yx^{i_1}\big) + \cdots\\
+&\big(f_y^x \mid x^{i_r}yx^{i_1}y\dots yx^{i_{r-1}}\big) - \big(f_x^y \mid x^{i_r}yx^{i_1}y\cdots yx^{i_{r-1}}\big)\\
=&-Kr\left([x,y]^r \mid x^{i_1}y\cdots x^{i_r}y\right) + (-1)^{r-1}Kr\left([x,y]^r \mid x^{i_r}y\cdots x^{i_1}y\right).
\end{aligned}$$

The left-hand side of this equal to

$$\big(pushsym(f_y^x - f_x^y) \mid x^{i_1}y\cdots yx^{i_r}\big),$$

so to the left-hand side of the divergence condition (5) for the weight $n-2$ word $u = x^{i_1}y\cdots yx^{i_r}$. If $n \neq 2r+1$, then the right-hand sides of (5) and (16) are both equal to 0. To prove that (5) and (16) are identical, it remains only to check that the right-hand sides are equal when $n = 2r+1$, which, cancelling the factor Kr from both sides, reduces to the following lemma.

Lemma III.3.28. *For each word u of depth $r-1$ and weight $2r-1$, we have*

$$\frac{1}{|C_y(uy)|}\sum_{v\in C(uy)}\big([x,y]^r \mid v\big) = \big([x,y]^r \mid uy\big) - (-1)^{r-1}\big([x,y]^r \mid u'y\big),$$

where u' denotes the word u written backwards.

Proof. Observe that if $([x,y]^r|uy) \neq 0$, then uy must satisfy the *parity property* that, writing $uy = u_1\cdots u_{2r}$ where each u_i is letter x or y, the pair $u_{2i-1}u_{2i}$ must be either xy or yx for $0 \leq i \leq r$. The coefficient of the word uy in $[x,y]^r$ is equal to $(-1)^j$ where j is the number of pairs $u_{2i-1}u_{2i}$ in uy that are equal to yx. In other words, if a word w appears with non-zero coefficient in $[x, y]^r$, then letting $U = yx$ and $V = xy$, we must be able to write w as a word in U, V, and the coefficient of w in $[x, y]^r$ is $(-1)^m$ where m denotes the number of times the letter U occurs.

If $w = uy = V^r = (xy)^r$, then $u'y = uy$. The coefficient of V^r in $[x, y]^r$ is equal to 1, so the right-hand side of (III.3.28) is equal to 2 if r is even and 0 if r is odd. For the left-hand side, $C(uy) = \{V^r, U^r\}$ and $C_y(uy) = \{V^r\}$, so $|C_y(uy)| = 1$. The coefficient of U^r in $[x, y]^r$ is equal to $(-1)^r$, so the left-hand side is again equal to 2 if r is even and 0 if r is odd. This proves (III.3.28) in the case $uy = V^r$.

Suppose now that $uy \neq V^r$ but that it satisfies the parity property. Write $uy = U^{a_1}V^{b_1} \cdots U^{a_s}V^{b_s}$ in which all the $a_i, b_i \geq 1$ except for a_1, which may be 0. Then $u'y$ is equal to $xU^{b_s-1}V^{a_s} \cdots U^{b_1}V^{a_1}y$. If $b_s > 1$, then the pair $u_{2(b_s-1)+1}u_{2(b_s-1)+2}$ is xx, so $([x, y]^r|u'y) = 0$. If $b_s = 1$, then the word $u'y$ begins with xx and thus does not have the parity property, so again $([x, y]^r|u'y) = 0$. This shows that if $([x, y]^r|uy) \neq 0$ then $([x, y]^r|u'y) = 0$ and vice versa.

This leaves us with three possibilities for $uy \neq V^r$.

Case 1: $([x, y]^r|uy) \neq 0$. Then uy has the parity property, so we write $uy = U^{a_1}V^{b_1} \cdots U^{a_s}V^{b_s}$ as above. The right-hand side of (III.3.28) is then equal to $(-1)^j$ where $j = a_1 + \cdots + a_s$. For the left-hand side, we note that the only words in the cyclic permutation class $C(uy)$ that have the parity property are the cyclic shifts of uy by an even number of letters, otherwise a pair xx or yy necessarily occurs as above. These are the same as the cyclic permutations of the word uy written in the letters U, V. All these cyclic permutations obviously have the same number of occurrences j of the letter U. Thus, the words in $C(uy)$ for which $[x, y]^r$ has a non-zero coefficient are the cyclic permutations of the word uy in the letters U, V, and the coefficient is always equal to $(-1)^j$. These words are exactly half of the all the words in $C(uy)$, so the sum in the left-hand side is equal to $(-1)^j|C(uy)|/2$. But $|C_y(uy)| = |C(uy)|/2$, so the left-hand side is equal to $(-1)^j$, which proves (III.3.28) for words uy having the parity property.

Case 2: $([x, y]^r|u'y) \neq 0$. In this case it is $u'y$ that has the parity property, and the right-hand side of (III.3.28) is equal to $(-1)^{r+j'}$ where j' is the number of occurrences of U in the word $u'y = U^{a_1}V^{b_1} \cdots U^{a_s}V^{b_s}$, i.e. $j' = a_1 + \cdots + a_s$. We have $uy = xU^{b_s-1}V^{a_s} \cdots U^{b_1}V^{b_1}y$. The word $w = U^{b_s-1}V^{a_s} \cdots U^{b_1}V^{b_1}U$ then occurs in $C(uy)$, and the number of occurrences of the letter U in uy is equal to $j = b_1 + \cdots + b_{s-1} + b_s$. Since $a_1 + b_1 + \cdots + a_s + b_s = r$, we have $j + j' = r$ so $j' = r - j$ and the right-hand side of (III.3.28) is equal to $(-1)^j$. The number of words in $C(uy)$ that have non-zero coefficient in $[x, y]^r$ is $|C(uy)|/2 = |C_y(uy)|$ as above, these words being exactly the cyclic permutations of w written in U, V, and the coefficient is always equal to $(-1)^j$. So the left-hand side of (III.3.28) is equal to $(-1)^j$, which proves (III.3.28) in the case where $u'y$ has the parity property.

Case 3: $([x, y]^r|uy) = ([x, y]^r|u'y) = 0$. The right-hand side of (III.3.28) is zero. For the left-hand side, consider the words in $C(uy)$. If there are no words in $C(uy)$ whose coefficient in $[x, y]^r$ is non-zero, then the left-hand side of (III.3.28) is also zero and (III.3.28) holds. Suppose instead that there is a word $w \in C(uy)$ whose coefficient in $[x, y]^r$ is non-zero. Then as we saw above, w is a cyclic shift of uy by an odd number of letters, and since all cyclic shifts of w by an even number of letters then have the same coefficient in $[x, y]^r$ as w, we may assume that w is the cyclic shift of uy by one letter, i.e. taking the final y and putting it at the beginning. Since w has non-zero coefficient in $[x, y]^r$, we can write $w = U^{a_1}V^{b_1} \cdots U^{a_s}V^{b_s}$, where $a_1 > 0$ since w now starts with y, but b_s may be equal to 0 since w may end with x. Then $uy = xU^{a_1-1}V^{b_1} \cdots U^{a_s}V^{b_s}y$, so we can write $u'y = U^{b_s}V^{a_s} \cdots U^{b_1}V^{a_1-1}xy = U^{b_s}V^{a_s} \cdots U^{v_1}V^{a_1}$. But then $u'y$ satisfies the parity property, so its coefficient in $[x, y]^r$ is non-zero, contradicting our assumption. Thus under the assumption, all words in $C(uy)$ have coefficient zero in $[x, y]^r$, which completes the proof of the Lemma. □

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Appendices

A Appendix : Proof of Lemma III.3.22

Let us recall the statement of the technical lemma 31.

Lemma 31. *For $n > 1$ and any constant $c \neq 0$, let T_c^n be the homogeneous polynomial mould of degree n defined by*

$$T_c^n(v_1, \dots, v_r) = \frac{c}{r} P^n_r,$$

where P_n^r is the sum is over all monomials of degree $n - r$ in the variables $v_1, \dots, v_r$. Then T_c^n is circ-constant and $\overline{ganit}(poc) \cdot pari(T_c^n)$ is circ-neutral.

Proof. The mould T_c^n is trivially circ-constant. Consider the proof of Proposition III.3.21 with $M = T_c^n$, $N = pari(M)$. In order to show that $ganit(poc) \cdot N$ is circ-neutral, we start by recalling from the proof of Proposition III.3.21 that for each $r > 1$, the cyclic sum

$$\overline{ganit}(poc) \cdot N(v_1, \dots, v_r) + \cdots + \overline{ganit}(poc) \cdot N(v_r, v_1, \dots, v_{r-1})$$

is equal to the expression (III.3.40). Thus we need to show that (III.3.40) is equal to zero for all $r > 1$. To show this, we will break up the sum

$$\sum_{\mathbf{b}=\{\mathbf{b}_1,\dots,\mathbf{b}_s\}} (-1)^{l_\mathbf{a}} poc(\lfloor \mathbf{b}_1) \cdots poc(\lfloor \mathbf{b}_s) \sum_{w \in \mathbf{W}_n^\mathbf{a}} w \tag{.0.51}$$

into parts that are simpler to express.

We need a little notation. Let us write $V_j = \{1, \dots, j\}$. If $B \subset V_r$, let P_B^d denote the sum of of all monomials of degree d in the variables $v_i \in B$. We write $P_B^0 = 1$ for all B.

We will break up the sum (.0.51) as the sum of partial sums $S_0 + \cdots + S_r$, where S_0 is the term of (.0.51) corresponding to the empty set and S_i is the sum over the $\mathbf{b}$-parts containing v_i but not $v_{i+1}, \dots, v_r$, for each $i \in \{1, \dots, r\}$. Notice that the $\mathbf{b}$-parts containing v_i but not $v_{i+1}, \dots, v_r$ are in bijection with the 2^{i-1} subsets $B \subset V_{i-1}$, by taking $\mathbf{b}$ to be the set $B' = B \cup \{v_i\}$, divided into chunks consisting of consecutive integers. For example, if $i = 5$ and $B = \{1, 3\}$ then $B' = \{1, 3, 5\}$ and the associated $\mathbf{b}$-part is $(v_1)(v_3)(v_5)$; if $B = \{1, 2\}$ then $B' = \{1, 2, 5\}$ and the $\mathbf{b}$-part is $(v_1, v_2)(v_5)$, and if $B = \{1, 4\}$ then $B' = \{1, 4, 5\}$ and the $\mathbf{b}$-part is $(v_1)(v_4, v_5)$.

Setting $v_0 = v_r$, this means that $S_0 = P_{V_r}^{n-r}$ and for $1 \leq i \leq r$,

$$S_i = \sum_{B \subseteq \{1,\dots,i-1\}} \frac{(-1)^{r-|B'|} P^{n-r+|B'|}_{V_r \setminus B'}}{\prod_{j \in B'} (v_{j-1} - v_j)}. \tag{.0.52}$$

In order to prove that (.0.51) is zero, we will give simplified expressions for $S_1, \dots, S_{r-1}$ in Claim 1, a simplified expression for S_r in Claim 2, and then show how to sum them up in Claim 3.

Claim 1. For $1 \leq i \leq r - 1$, we have $v_{i+1}, \dots, v_r$. Let $v_0 = v_r$. Then we have

$$S_i = \frac{(-1)^{r-i} P^{n-r+i}_{\{i-1,i+1,\dots,r-1\}}}{(v_r - v_1)(v_1 - v_2) \cdots (v_{i-1} - v_i)}. \tag{.0.53}$$

Proof. We will use the following trivial but useful identity. Let $B \subsetneq V_r$, let $v_j \notin B$, and let $B' = B \cup \{v_j\}$. Then

$$P^d_{B'} = P^d_B + v_j P^{d-1}_{B'}. \tag{.0.54}$$

Multiplying by the common denominator, we write (.0.52) as

$$(-1)^{r-i}\prod_{j=1}^{i}(v_{j-1}-v_j)S_i = \sum_{B\subseteq V_{i-1}}(-1)^{i-|B'|}\prod_{j\in V_{i-1}\setminus B}(v_{j-1}-v_j)P^{n-r+|B'|}_{V_r\setminus B'}. \qquad (.0.55)$$

We will show below that for each $1 \le k \le i-1$, the right-hand side of (.0.55) is equal to the expression

$$Q_k = \sum_{v_1,\ldots,v_k\notin B\subset V_{i-1}}(-1)^{i-|B|+k-1}\left(\prod_{j\in V_{i-1}\setminus(B\cup\{v_1,\ldots,v_k\}}(v_{j-1}-v_j)\right)P^{n-r+|B'|+k}_{V_{r-1}\setminus B'\cup\{v_1,\ldots,v_{k-1}\}}.$$

Taking $k=i-1$ in this expression, the sum Q_{i-1} reduces to the single term corresponding to $B=\emptyset$, which is just $P^{n-r+i}_{v_{i-1},v_{i+1},\ldots,v_{r-1}}$. Thus by (.0.55), we obtain

$$(-1)^{r-i}\prod_{j=1}^{i}(v_{i-1}-v_i)S_i = P^{n-r+i}_{v_{i-1},v_{i+1},\ldots,v_{r-1}},$$

which proves (.0.53).

Let us prove that the right-hand side of (.0.55) is equal to Q_k for all $1\le k\le i-1$. We will use induction on k. Let us do the base case $k=1$ by showing that the right-hand side of (.0.55) is equal to Q_1. We start by breaking the right-hand side of (.0.55) into $v_1\in B$ and $v_1\notin B$, and compute

$$\sum_{v_1\in B}(-1)^{i-|B'|}\prod_{j\in V_{i-1}\setminus B}(v_{j-1}-v_j)P^{n-r+|B'|}_{V_r\setminus B'} + \sum_{v_1\notin B}(-1)^{i-|B'|}\prod_{j\in V_{i-1}\setminus B}(v_{j-1}-v_j)P^{n-r+|B'|}_{V_r\setminus B'}$$

then setting $C = B\setminus\{v_1\}$ in the first sum

$$= \sum_{v_1\notin C}(-1)^{i-|C|}\prod_{j\in V_{i-1}\setminus(C\cup\{v_1\})}(v_{j-1}-v_j)P^{n-r+|C|+2}_{V_r\setminus(C\cup\{v_1,v_i\})} + \sum_{v_1\notin B}(-1)^{i-|B|-1}\prod_{j\in V_{i-1}\setminus B}(v_{j-1}-v_j)P^{n-r+|B'|}_{V_r\setminus B'}$$

then renaming $C=B$ and writing $B_1 = B\cup\{v_1\}$ and $B_1' = B\cup\{v_1,v_i\}$,

$$= \sum_{v_1\notin B}(-1)^{i-|B|}\prod_{j\in V_{i-1}\setminus B_1}(v_{j-1}-v_j)P^{n-r+|B|+2}_{V_r\setminus B_1'} - \sum_{v_1\notin B}(-1)^{i-|B|}\prod_{j\in V_{i-1}\setminus B_1}(v_{j-1}-v_j)\cdot(v_r-v_1)P^{n-r+|B'|}_{V_r\setminus B'}$$

$$= \sum_{v_1\notin B}(-1)^{i-|B|}\left(\prod_{j\in V_{i-1}\setminus B_1}(v_{j-1}-v_j)\right)\left(P^{n-r+|B'|+1}_{V_r\setminus B_1'} - (v_r-v_1)P^{n-r+|B'|}_{V_r\setminus B'}\right)$$

$$= \sum_{v_1\notin B}(-1)^{i-|B|}\left(\prod_{j\in V_{i-1}\setminus B_1}(v_{j-1}-v_j)\right)P^{n-r+|B'|+1}_{V_{r-1}\setminus B'},$$

which is exactly Q_1. The last equality is obtained by using (.0.54) twice on the right-hand factor. This proves the base case $k=1$.

Now fix $k<i-1$ and assume that $Q_1=\cdots=Q_k$. We will show by the same method that $Q_k=Q_{k+1}$. We break the expression for Q_k into $v_k\in B$ and $v_k\notin B$, and compute

$$\sum_{v_1,\ldots,v_k\notin B,v_{k+1}\in B}(-1)^{i-|B|+k-1}\left(\prod_{j\in V_{i-1}\setminus(B\cup\{v_1,\ldots,v_k\})}(v_{j-1}-v_j)\right)P^{n-r+|B'|}_{V_{r-1}\setminus(B'\cup\{v_1,\ldots,v_{k-1}\}))}$$

$$+\sum_{v_1,\ldots,v_{k+1}\notin B}(-1)^{i-|B|+k-1}\left(\prod_{j\in V_{i-1}\setminus(B\cup\{v_1,\ldots,v_k\})}(v_{j-1}-v_j)\right)P^{n-r+|B'|}_{V_{r-1}\setminus(B'\cup\{v_1,\ldots,v_{k-1}\})}$$

then setting $C = B \setminus \{v_{k+1}\}$, $C_{k+1} = C \cup \{v_{k+1}\} = B$, $C'_k = C \cup \{v_{k+1}, v_i\} = B'$ in the first sum,

$$= \sum_{v_1,\ldots,v_{k+1}\notin C} (-1)^{i-|C|+k} \Bigg(\prod_{j\in V_{i-1}\setminus(C\cup\{v_1,\ldots,v_{k+1}\})} (v_{j-1} - v_j) \Bigg) P^{n-r+|C|+2}_{V_{r-1}\setminus(C'\cup\{v_1,\ldots,v_{k-1},v_{k+1}\})}$$

$$+ \sum_{v_1,\ldots,v_{k+1}\notin B} (-1)^{i-|B|+k-1} \Bigg(\prod_{j\in V_{i-1}\setminus(B\cup\{v_1,\ldots,v_k\})} (v_{j-1} - v_j) \Bigg) P^{n-r+|B'|}_{V_{r-1}\setminus(B'\cup\{v_1,\ldots,v_{k-1}\})}$$

and renaming $C = B$,

$$= \sum_{v_1,\ldots,v_{k+1}\notin B} (-1)^{i-|B|+k} \Bigg(\prod_{j\in V_{i-1}\setminus(B\cup\{v_1,\ldots,v_{k+1}\})} (v_{j-1} - v_j) \Bigg) P^{n-r+|B|+2}_{V_{r-1}\setminus(B'\cup\{v_1,\ldots,v_{k-1},v_{k+1}\})}$$

$$+ \sum_{v_1,\ldots,v_{k+1}\notin B} (-1)^{i-|B|+k-1} \Bigg(\prod_{j\in V_{i-1}\setminus(B\cup\{v_1,\ldots,v_{k+1}\})} (v_{j-1} - v_j) \Bigg) (v_k - v_{k+1}) P^{n-r+|B'|}_{V_{r-1}\setminus(B'\cup\{v_1,\ldots,v_{k-1}\})}$$

$$= \sum_{v_1,\ldots,v_{k+1}\notin B} (-1)^{i-|B|+k} \Bigg(\prod_{j\in V_{i-1}\setminus(B\cup\{v_1,\ldots,v_{k+1}\})} (v_{j-1} - v_j) \Bigg) \times$$

$$\Bigg(P^{n-r+|B|+2}_{V_{r-1}\setminus(B'\cup\{v_1,\ldots,v_{k-1},v_{k+1}\})} - (v_k - v_{k+1}) P^{n-r+|B'|}_{V_{r-1}\setminus(B'\cup\{v_1,\ldots,v_{k-1}\})} \Bigg)$$

$$= \sum_{v_1,\ldots,v_{k+1}\notin B} (-1)^{i-|B|+k} \Bigg(\prod_{j\in V_{i-1}\setminus(B\cup\{v_1,\ldots,v_{k+1}\})} (v_{j-1} - v_j) \Bigg) P^{n-r+|B|+2}_{V_{r-1}\setminus(B'\cup\{v_1,\ldots,v_k\})},$$

again by (.0.54) applied twice. Thus $Q_1 = \cdots = Q_{k+1}$, so by induction, $Q_1 = Q_{i-1}$ and thus the right-hand side of (.0.55) equals Q_{i-1} as desired. □

Unfortunately, the expression in Claim 1 for S_i does not work for $i = r$ due to the fact that when $i = r$ in (.0.55), the subset $B = V_{r-1}$ occurs in the sum and the corresponding polynomial $P_{V_r\setminus B'} = 0$. It turns out that the expression for S_r is actually simpler.

Claim 2. The term S_r is given by

$$S_r = \frac{v_{r-1}^{n-1}}{(v_r - v_1)(v_1 - v_2)\cdots(v_{r-2} - v_{r-1})}. \tag{.0.56}$$

Proof. We will show that

$$\prod_{j=1}^{r-1} (v_{j-1} - v_j) S_r = v_{r-1}^{n-1} \tag{.0.57}$$

starting from the equality (.0.55) for $i = r$, slightly rewritten as

$$\prod_{j=1}^{r} (v_{j-1} - v_j) S_r = \sum_{B\subseteq V_{r-1}} (-1)^{r-|B|-1} \prod_{j\in V_{r-1}\setminus B} (v_{j-1} - v_j) P^{n-r+|B|+1}_{V_{r-1}\setminus B}. \tag{.0.58}$$

Let us write $C = V_{r-1} \setminus B$; this becomes

$$\prod_{j=1}^{r} (v_{j-1} - v_j) S_r = \sum_{B\subseteq V_{r-1}} (-1)^{r-|B|-1} \prod_{j\in C} (v_{j-1} - v_j) P^{n-r+|B|+1}_{C}. \tag{.0.59}$$

We have $P_\emptyset = 0$, and we may as well sum over the subsets C, so it becomes

$$\prod_{j=1}^{r}(v_{j-1} - v_j)S_r = \sum_{\emptyset \neq C \subseteq V_{r-1}} (-1)^{|C|-1} \prod_{j \in C}(v_{j-1} - v_j)P_C^{n-|C|-1}. \tag{.0.60}$$

We will prove the following formula, valid for $1 \leq i \leq r-1$ and $n \geq 1$:

$$R_i^n = \sum_{\emptyset \neq B \subseteq V_i} (-1)^{|B|-1} \prod_{j \in B}(v_{j-1} - v_j)P_B^{n-|B|} = (v_r - v_i)v_i^{n-1}, \tag{.0.61}$$

where we set $P_B^0 = 1$ and $P_B^m = 0$ if $m < 0$.

This equality suffices to prove the desired result (.0.57). Indeed, taking $i = r-1$, we see that R_{r-1}^n is equal to the right-hand side of (.0.60), so

$$\prod_{j=1}^{r}(v_{j-1} - v_j)S_r = R_{r-1}^n = (v_r - v_{r-1})v_{r-1}^{n-1},$$

and canceling out the factor $(v_r - v_{r-1})$ from both sides yields (.0.57).

Let us prove (.0.61) by induction on i. When $i = 1$, we have $B = \{v_i\}$ and for all $n \geq 1$, we have $R_1^n = (v_1 - v_r)v_1^{n-1}$, proving the base case. Assume (.0.61) holds for $i-1$ for all $n \geq 1$. Fix n. We break R_i^n into the sum over B containing v_i and B not containing V_i, as follows:

$$\begin{aligned}
R_i^n &= \sum_{\emptyset \neq B \subseteq V_{i-1}} (-1)^{|B|-1} \prod_{j \in B}(v_{j-1} - v_j)P_B^{n-|B|} \\
&\quad - \sum_{B \subseteq V_{i-1}} (-1)^{|B|-1} \prod_{j \in B}(v_{j-1} - v_j)(v_{i-1} - v_i)P_{B,v_i}^{n-|B|-1} \\
&= R_{i-1}^n - \sum_{B \subseteq V_{i-1}} (-1)^{|B|-1} \prod_{j \in B}(v_{j-1} - v_j)(v_{i-1} - v_i)P_{B,v_i}^{n-|B|-1} \\
&= R_{i-1}^n + (v_{i-1} - v_i)v_i^{n-1} - (v_{i-1} - v_i) \sum_{\emptyset \neq B \subseteq V_{i-1}} (-1)^{|B|-1} \prod_{j \in B}(v_{j-1} - v_j)P_{B,v_i}^{n-|B|-1}
\end{aligned}$$

The key point is that for B not containing v_i, we can write

$$P_{B,v_i}^{n-|B|-1} = P_B^{n-|B|-1} + v_i P_B^{n-|B|-2} + v_i^2 P_B^{n-|B|-3} + \cdots + v_i^{n-|B|-2} P_B^1 + v_i^{n-|B|-1}.$$

Using this, the equality becomes

$$\begin{aligned}
&= R_{i-1}^n + (v_{i-1} - v_i)v_i^{n-1} \\
&\quad - (v_{i-1} - v_i) \sum_{\emptyset \neq B \subseteq V_{i-1}} (-1)^{|B|-1} \prod_{j \in B}(v_{j-1} - v_j)\left(P_B^{n-|B|-1} + v_i P_B^{n-|B|-2} + \cdots + v_i^{n-|B|-1}\right) \\
&= R_{i-1}^n + (v_{i-1} - v_i)v_i^{n-1} - (v_{i-1} - v_i) \sum_{k=0}^{n-2} v_i^k R_{i-1}^{n-k-1} \\
&= (v_r - v_{i-1})v_{i-1}^{n-1} + (v_{i-1} - v_i)v_i^{n-1} - (v_{i-1} - v_i) \sum_{k=0}^{n-2} v_i^k (v_r - v_{i-1})v_{i-1}^{n-k-2}
\end{aligned}$$

$$
\begin{aligned}
&= (v_r - v_{i-1})v_{i-1}^{n-1} + (v_{i-1} - v_i)v_i^{n-1} - (v_r - v_{i-1})(v_{i-1} - v_i)\sum_{k=0}^{n-2} v_i^k v_{i-1}^{n-k-2} \\
&= (v_r - v_{i-1})v_{i-1}^{n-1} + (v_{i-1} - v_i)v_i^{n-1} - (v_r - v_{i-1})(v_{i-1}^{n-1} - v_i^{n-1}) \\
&= (v_r - v_{i-1})v_i^{n-1} + (v_{i-1} - v_i)v_i^{n-1} \\
&= (v_r - v_i)v_i^{n-1}.
\end{aligned}
$$

This proves (.0.61) and thus completes the proof of Claim 2. □

We can now prove that the expression (.0.51) is equal to zero by showing that $S_0 + \cdots + S_r = 0$.

Claim 3. We have $S_0 + \cdots + S_r = 0$.

Proof. The key point is the following computation of partial sums for $i < r$:

$$
S_0 + \cdots + S_i = \frac{(-1)^{r-i} P^{n-r+i}_{v_i,\dots,v_{r-1}}}{(v_r - v_1)(v_1 - v_2)\cdots(v_{i-1} - v_i)}. \tag{.0.62}
$$

We prove it by induction on i. The base case $i = 0$ is just given by the formula for S_0 (with $v_0 = v_r$). Assume (.0.62) up to $i - 1$. Then

$$
\begin{aligned}
S_0 + \cdots + S_i &= (S_0 + \cdots + S_{i-1}) + S_i && (.0.63) \\
&= \frac{(-1)^{r-i+1} P^{n-r+i-1}_{v_{i-1},\dots,v_{r-1}}}{(v_r - v_1)\cdots(v_{i-2} - v_{i-1})} + \frac{(-1)^{r-i} P^{n-r+i}_{v_{i-1},v_{i+1},\dots,v_{r-1}}}{(v_r - v_1)\cdots(v_{i-1} - v_i)}, && (.0.64)
\end{aligned}
$$

Using (.0.54) and multiplying (.0.63) by the denominator, we find

$$
\begin{aligned}
(-1)^{r-i}(v_r - v_1)\cdots(v_{i-1} - v_i)(S_0 + \cdots + S_i) &= P^{n-r+i}_{v_{i-1},v_{i+1},\dots,v_{r-1}} - (v_{i-1} - v_i)P^{n-r+i-1}_{v_{i-1},\dots,v_{r-1}} \\
&= P^{n-r+1}_{v_{i-1},v_{i+1},\dots,v_{r-1}} + v_i P^{n-r+i-1}_{v_{i-1},\dots,v_{r-1}} - v_{i-1} P^{n-r+i-1}_{v_{i-1},\dots,v_{r-1}} \\
&= P^{n-r+1}_{v_{i-1},\dots,v_{r-1}} - v_{i-1} P^{n-r+i-1}_{v_{i-1},\dots,v_{r-1}} \\
&= P^{n-r+1}_{v_i,\dots,v_{r-1}}.
\end{aligned}
$$

Now, taking this equality for $i = r - 1$ yields

$$
S_0 + \cdots + S_{r-1} = \frac{-P^{n-1}_{v_{r-1}}}{(v_r - v_1)(v_1 - v_2)\cdots(v_{r-2} - v_{r-1})} = \frac{-v_{r-1}^{n-1}}{(v_r - v_1)\cdots(v_{r-2} - v_{r-1})},
$$

which is equal to $-S_r$ by Claim 2. This proves Claim 3. □

Since $S_0 + \cdots + S_r$ is equal to (.0.51), we have finally shown that whatever the value of c, $\overline{ganit}(poc)\cdot N$ is circ-neutral, completing the proof of Proposition III.3.21. □

B A Mould bestiary

A moulds bestiary			
Who ?	What?	Tell me more !	Lives in...
alternal(ity)	property	equivalent to being a Lie polynomial, related to shuffle	II.1.39
alternil(ity)	property	related to stuffle	II.2.7
ARI	Lie algebra (with *ari* bracket)	moulds in variables u_i with $M(\emptyset) = 0$	II.1.18
$\overline{ARI}$	Lie algebra (with $\overline{ari}$ bracket)	moulds in variables v_i with $M(\emptyset) = 0$	II.1.18
amit(M)	operator	derivation of ARI_lu	II.1.5
anit(M)	operator	derivation of ARI_lu	II.1.4
arit(M)	operator	derivation of ARI_lu	II.1.6
ari	Lie bracket	defined with flexions, closely related to the Poisson bracket	II.1.27
$ARI_{a/b}$	space	moulds in ARI having property a whose swap have property b	
ARI_{a*b}	space	moulds in ARI having property a whose swap have property b up to adding on a constant-valued mould	
$ARI_{\underline{a}/\underline{b}}$	space	subspace of $ARI_{a/b}$ of moulds that are even functions of u_1 in depth 1	
$BARI$	space	bimoulds with $B(\emptyset) = 0$	II.1.18
$BIMU$	space	all bimoulds	II.1.3
Dari	Lie bracket	another bracket used on ARI	III.3.7
Darit (M)	operator	derivation of ARI_lu	III.3.5
dar	operator	multiply by $u_1 \dots u_r$, corresponds to the transformation $y \to [y, x]$ in $f(x, y) \in \mathbb{Q}\langle C\rangle$.	III.2.1
Δ	operator	multiply by $u_1 \dots u_r(u_1 + \cdots + u_r)$, key to the construction of $\mathfrak{ds}_{ell}$ and $\mathfrak{krv}_{ell}$.	III.2.3
dur	operator	multiply by $(u_1 + \cdots + u_r)$, corresponds to Lie bracketing with x	III.2.2
dupal	mould in ARI	used for the recursive construction of *pal*	II.2.11
$ganit(Q)$	operator	automorphism of $\overline{GARI}_{lu}$	III.3.28
GARI	group	moulds in variables u_i with $M(\emptyset) = 1$	II.1.35
$\overline{GARI}$	group	moulds in variables v_i with $M(\emptyset) = 1$	II.1.35
invmu	operation	takes the inverse of a mould for the *mu* multiplication	
invpal	mould in $GARI$	inverse of *pal* for the group law *gari*	II.2.2
limu	Lie bracket	commutator for the multiplication *mu*. Also denoted *lu*.	II.1.18
lu	Lie bracket	see *limu*	
mu	binary operation	multiplication of moulds	II.1.15
pal	mould in $GARI$	key to Ecalle's fundamental identity	II.2.2
pari	operator	multiplies by $(-1)^r$	III.3.21
pic	mould		III.3.33
pil	mould	$= swap(pal)$	
poc	mould		III.3.34
senary	relation	key to the construction of the map $\mathfrak{krv} \hookrightarrow \mathfrak{krv}_{ell}$	III.3.23
symmetral(ity)	property	$expari(M) \in GARI$ is symmetral if M is alternal	II.1.39
teru	operator	main ingredient of the senary relation	III.3.22