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Applications of the Batalin-Vilkovisky Geometry: Dualities and Chern-Simons Theory with Boundary

Thèse

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Résumé

La présente thèse est consacrée à l'étude d'aspects géométriques du formalisme de Batalin-Vilkovisky pour les théories des champs dont l'action est dégénérée, aussi appelées théories de jauge. Elle est basée sur deux articles de l'auteur, "T-duality through BV Morphisms and BV Pushforwards in Topological Field Theories" [7], reproduit dans le chapitre 2, et "Chern-Simons theory with Wilson lines and boundary in the BV-BFV formalism" écrit en collaboration avec Anton Alekseev et Pavel Mnev [2], reproduit ici dans le chapitre 3.

A ce jour, la méthode la plus complète pour étudier la quantification des théories de jauge reste le formalisme de Batalin et Vilkovisky (abrégé *formalisme BV* par la suite), en ce sens qu'il permet le traîtement de théories dont les transformations de jauge infinitésimales forment une distribution non-involutive. Le principe général est de construire autour de l'espace des champs classiques $\mathcal{F}_{\mathrm{cl}}$ d'une théorie de jauge un espace des champs BV, $\mathcal{F}_{\mathrm{BV}}$, une variété $\mathbb{Z}$-graduée équipée d'une forme symplectique $\Omega_{\mathrm{BV}} \in \Omega^2(\mathcal{F}_{\mathrm{BV}})$ de degré -1 appelée structure BV. Le crochet de Poisson impair associé $\{\cdot,\cdot\}$, évidemment de degré 1, est également appelé crochet BV ou anti-crochet. Par la suite, il faut trouver une action BV, S_{BV}, qui satisfait l'équation master classique $\{S_{\mathrm{BV}}, S_{\mathrm{BV}}\} = 0$ et telle que sa restriction à l'espace des champs classiques reproduit l'action classique S_{cl}. Pour fixer la jauge, il suffit de déterminer un sous-espace Lagrangien $\mathcal{L}$ de $\mathcal{F}_{\mathrm{BV}}$ tel que $S_{\mathrm{BV}}|_{\mathcal{L}}$ est non-dégénérée.

Si l'espace $\mathcal{F}_{\mathrm{BV}}$ est équipé d'une mesure d'intégration μ qui satisfait certaines conditions, on peut étendre cette construction à la théorie quantique grâce à un théorème de Batalin et Vilkovisky sur les intégrales de chemin définies sur des sous-espaces Lagrangiens.

Ces dernières années, Cattaneo, Mnev et Reshetikhin [**?**] on entamé l'étude de la quantification BV de théories de jauge sur des variétés avec bord. A la théorie BV dans le volume, ils associent un modèle BFV sur le bord qui, au niveau classique, découle des effets de bord lors de la variation de l'action BV. Un modèle BFV, pour Batalin, Fradkin et Vilkovisky, est en fait très similaire à

un modèle BV, si ce n'est que la structure BFV est une structure symplectique paire de degré zéro, et que l'action BFV est elle impaire de degré 1. L'idée de la quantification de cette construction est que les fonctions de corrélation du volume, au lieu d'être complexes, prennent valeur dans un espace de Hilbert des états quantiques associé au bord. Pour construire cet espace, Cattaneo, Mnev et Reshetikhin proposent d'appliquer une quantification géométrique au modèle BFV du bord telle que l'action BFV devient un opérateur, la charge BFV, dont le carré vaut zéro. L'espace des états quantiques du bord est ensuite réalisé par la cohomologie de degré zéro de la charge BFV.

Le chapitre 1 de cette thèse résume les principaux concepts du formalisme BV en insistant sur leur interprétation géométrique, et introduit également le formalisme BV-BFV de Cattaneo, Mnev et Reshetikhin.

Après cette introduction, le chapitre 2 se focalise sur un aspect jusqu'ici négligé du formalisme BV, les dualités. On dit que deux théories des champs sont duales l'une à l'autre si bien qu'au premier abord elles semblent différentes, elles décrivent néanmoins la même physique, ou les mêmes invariants dans le cas de théories topologiques. Forts du constat que la géométrie du formalisme BV est en fait symplectique, nous proposons une première méthode pour exprimer les dualités basée sur des symplectomorphismes des structures BV, que nous appelons morphismes BV, et que nous généralisons au cas quantique. A titre d'exemple, nous considérons un cas particulier du sigma modèle de Courant proposé par Roytenberg [38], où le morphisme BV que nous obtenons reproduit un isomorphisme d'algébroïdes de Courant découvert par Cavalcanti et Gualtieri [21] en suivant des arguments géométriques inspirés de la théorie des cordes.

Une deuxième méthode implique des intégrations BV partielles et conduit à des théories effectives duales les unes aux autres. A l'aide de cette méthode combinée à la première, nous montrons que le contenu topologique des règles de Buscher [14] pour la T-dualité [11] [12] [13] peut être reproduit par des transformations de sigma modèles topologiques qui représentent le secteur topologique de l'action de Polyakov pour la théorie des cordes.

Le chapitre 3, basé sur une collaboration avec Anton Alekseev et Pavel Mnev, traîte deux exemples de théories des champs définies sur des variétés avec bord dans le formalisme BV-BFV de Cattaneo, Mnev et Reshetikhin. Inspirés de résultats d'Alekseev et Mnev sur la quantification du modèle de Chern-Simons à une dimension [5], nous appliquons d'abord la méthode BV-BFV à cette théorie. En particulier, nous obtenons une charge BFV qui coïncide avec l'opérateur cubique de Dirac introduit par Kostant [33]. Nous nous concentrons ensuite sur le modèle de Chern-Simons à trois dimensions. L'action BFV du bord est celle d'un modèle BF impair où le rôle du champ B est joué par le fantôme du modèle dans le volume. Après quantification canonique du modèle BFV du bord, nous trouvons un espace des états quantiques isomorphe à l'espace des blocs conformes du modèle de Wess-Zumino-Witten sur le bord. Pour introduire des boucles de Wilson (et ensuite des lignes de Wilson ouvertes) dans le volume, nous nous basons sur la formule d'Alekseev, Faddeev et Shatashvili [3] qui permet de supprimer l'ordonnancement des chemins au prix d'une intégration fonctionnelle supplémentaire, et montrons comment cette intégration peut être traitée dans le formalisme BV. Cette méthode de construction d'observables basée sur des termes auxiliaires pour l'action BV a été généralisée par Mnev [37]. Dans notre cas, l'intérêt d'avoir une action auxiliaire BV pour les boucles de Wilson est qu'on peut dès lors également considérer des lignes de Wilson ouvertes qui se terminent au bord du volume du modèle de Chern-Simons dans le formalisme BV-BFV. L'introduction de lignes de Wilson a pour effet d'ajouter des sources ponctuelles à l'action BFV impaire du bord situées aux extrémités des lignes, qui se traduisent au niveau des fonctions de corrélation du modèle WZW associé par des insertions de champs.

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Preface

This thesis is focusing on geometrical aspects of the Batalin-Vilkovisky formalism and is built around two articles treating the subject, "T-duality through BV Morphisms and BV Pushforwards in Topological Field Theories", which is reproduced in chapter 2, and "Chern-Simons theory with Wilson lines and boundary in the BV-BFV formalism", a joint work with Anton Alekseev and Pavel Mnev, reproduced in chapter 3.

The Batalin-Vilkovisky formalism represents at this time the most advanced method to treat quantum field theories with gauge symmetries in the path integral approach. These symmetries arise when the mathematical description of a field theory contains continuous families of configurations that describe a unique point in the physical phase space of the theory. At the classical level first, if we want to determine this phase space, we need to find the gauge equivalence classes. At the quantum level, this problem is much more critical, as path integrals over a space of fields subject to gauge symmetries are ill-defined, in particular the propagators do not exist. The remedy is to fix the gauge by choosing a single configuration in each equivalence class before computing the path integral.

Faddeev-Popov Gauge Fixing The first solution to this problem was given by Faddeev and Popov [24], in the particular case where the gauge symmetry can be represented by the action of a gauge group. Once a gauge condition that picks a single configuration in each orbit is chosen, they impose it as a constraint to the path integral with the use of Lagrange multipliers. This restriction also requires the introduction of a Jacobian determinant, which they interpret as an additional path integral over Grassmanian fields, the so-called *ghosts* and *antighosts*, of a Gaussian function. The exponent in this function together with the product of the Lagrange multipliers with the constraints can thus be interpreted as an additional term to the classical action S_{cl}, which is called the *gauge-fixing term* S_{gf}, and together they form the Faddeev-Popov gauge-fixed action $S_{\mathrm{FP}} = S_{\mathrm{cl}} + S_{\mathrm{gf}}$, which can be used as a starting point for well-behaved path integrals.

BRST method Becchi, Rouet, Stora and Tyutin [10] gave a mathematical interpretation of this gauge-fixing term in the framework of cohomology theory. Their idea was to associate to gauge transformations a cohomological operator Q_{BRST} (i.e. an operator that squares to zero) such that gauge invariant functionals sit in $\mathrm{Ker}(Q_{\mathrm{BRST}})$, the kernel of this operator, and gauge equivalent functionals differ by a BRST-exact term, i.e. a term that sits in its image $\mathrm{Im}(Q_{\mathrm{BRST}})$. In other words, the gauge equivalence classes of gauge invariant functionals are represented by the cohomology $H^\bullet(Q_{\mathrm{BRST}}) = \mathrm{Ker}(Q_{\mathrm{BRST}})/\mathrm{Im}(Q_{\mathrm{BRST}})$. In the BRST formalism, the classical action and the Faddeev-Popov gauge-fixed action belong to the same cohomology class, $[S_{\mathrm{FP}}] = [S_{\mathrm{cl}}] \in H^\bullet(Q_{\mathrm{BRST}})$, so the gauge fixing procedure can be interpreted as the choice of a so-called gauge-fixing fermion Ψ so that the BRST gauge-fixed action $S_{\mathrm{BRST}} = S_{\mathrm{cl}} + Q_{\mathrm{BRST}}(\Psi)$ is nondegenerate (see chapter 1 for more details).

Zinn-Justin's antifields formalism At a later stage, Zinn-Justin [44] showed that the BRST operator Q_{BRST} can be associated to a Hamiltonian cohomological vector field Q on some $\mathbb{Z}$-graded symplectic manifold $\mathcal{F}$ with an odd symplectic structure Ω. Furthermore, Q is generated by a functional S_{ZJ} that satisfies a master equation $\{S_{\mathrm{ZJ}}, S_{\mathrm{ZJ}}\} = 0$ with the braces denoting the odd Poisson bracket (also called antibracket) generated by Ω. The gauge-fixing fermion Ψ from the BRST formalism is used to define a Lagrangian submanifold $\mathcal{L}_\Psi$ of $\mathcal{F}$, and the BRST gauge-fixed action corresponds to the restriction of the Zinn-Justin action to this submanifold, $S_{\mathrm{BRST}} = S_{\mathrm{ZJ}}|_{\mathcal{L}_\Psi}$.

Geometry of the BV formalism This construction motivated Batalin and Vilkovisky [9] in the development of a formalism that can be applied to a very large class of gauge theories, much larger than the BRST formalism. Their method roughly consists of constructing a BV space of fields $\mathcal{F}$, a similar $\mathbb{Z}$-graded symplectic manifold as in the Zinn-Justin interpretation, that contains the classical fields, and finding a BV action S_{BV} which reproduces the classical action when restricted to the classical fields and which satisfies the classical master equation $\{S_{\mathrm{BV}}, S_{\mathrm{BV}}\} = 0$, where the braces denote the antibracket generated by the symplectic structure on $\mathcal{F}$, simply called the BV structure in this setting. Gauge equivalence classes of gauge invariant functionals, in other words observables, are described by the cohomology of the Hamiltonian vector field $Q_{\mathrm{BV}} = \{S_{\mathrm{BV}}, \cdot\}$. Gauge fixing in the BV formalism is achieved through the restriction of the BV action to a Lagrangian submanifold $\mathcal{L} \subset \mathcal{F}$ where it is not degenerate. Provided $\mathcal{F}$ is equipped with a path integration measure μ fulfilling certain conditions, Batalin and Vilkovisky showed under which further conditions path integrals defined on a Lagrangian submanifold $\mathcal{L}$ are invariant under deformations of $\mathcal{L}$. In particular, the classical action corresponds to the restriction of the BV action to a certain Lagrangian submanifold $\mathcal{L}_{\mathrm{cl}}$ (where it is degenerate in the case of gauge theories), and it remains to find a deformation of $\mathcal{L}_{\mathrm{cl}}$ where the BV action is nondegenerate to be able to compute path integrals. We refer to chapter 1 for more details.

AKSZ construction To be more accurate, this geometric interpretation of the BV formalism in terms of odd symplectic geometry with a cohomological Hamiltonian vector field is due to Schwarz [40]. Together with Alexandrov, Kontsevich and Zaboronsky [6], he used these results to construct a systematic way of finding BV spaces of fields with BV actions satisfying the classical master equation, a method known as the *AKSZ construction*. It heavily relies on supergeometry, and most notable examples include the BV formulation of Chern-Simons theory or the Poisson sigma model, which can be used as a starting point to find BV actions for the topological A- and B-models of topological string theory.

While the relation of AKSZ sigma models with physical theories is not always immediately obvious, except perhaps for Chern-Simons theory, they can be linked to various mathematical structures such as Poisson structures, complex structures, Lie algebroids or Courant algebroids. A useful application is the derivation of Kontsevich's star product for deformation quantization [32] through perturbative quantization of the Poisson sigma model on a disk by Cattaneo and Felder [15].

Dualities An aspect of the study of quantum field theories in the BV formalism that had been so far neglected was the issue of dualities. Two quantum field theories are said to be dual to each other if their correlation functions coincide. The most notable example is perhaps T-duality in string theory. The odd symplectic geometry interpretation of the BV formalism offers a description of dualities in terms of symplectomorphisms. While symplectic flows generated by Hamiltonian vector fields describe gauge transformations, one can find symplectomorphisms between different BV models which then describe the same physics. Our article "T-duality through BV Morphisms and BV Pushforwards in Topological Field Theories" reproduced in chapter 2 explores this idea and relates it to known results from string theory. In particular, we show that an isomorphism of Courant algebroids [21] inspired by string theory coincides with a BV duality between two Courant sigma models [38] based on the corresponding algebroids.

BV-BFV formalism The last couple of years have seen a renewed interest for the geometry of the BV formalism. It turns out that the BV formalism works best for field theories constructed on closed manifolds. The presence of a boundary requires a careful application of boundary conditions (see for instance [15] [16] [17] for boundary conditions in the Poisson sigma model) to obtain solutions of the classical master equation. In the search for a more systematic way to deal with gauge theories on manifolds with boundaries, Cattaneo, Mnev and Reshetikhin [**?**] came up with the *BV-BFV formalism*. Batalin, Fradkin and Vilkovisky developed the BFV formalism as a *Hamiltonian* counterpart [8] to the *Lagrangian* BV formalism. The BV-BFV combines both. While the BV formalism is still used to describe the bulk degrees of freedom, it is complemented with the BFV formalism for the boundary degrees of freedom. The main difference with the BV formalism is that the *BFV structure* is an even symplectic structure on the BFV space of fields.

If the boundary values of the fields are left unconstrained, the BV cohomological vector field Q_{BV} is no longer Hamiltonian. The boundary fields add a correction to the Hamiltonian relation $\imath_{Q_{\mathrm{BV}}}\Omega_{\mathrm{BV}} = \delta S_{\mathrm{BV}} + \pi^*\alpha^{\partial}_{\mathrm{BFV}}$ described by a one-form (or possibly a connection) $\alpha^{\partial}_{\mathrm{BFV}}$ defined on some boundary space of fields $\mathcal{F}^{\partial}_{\mathrm{BFV}}$ and pulled back to the BV space of fields by a projection map $\pi : \mathcal{F}_{\mathrm{BV}} \to \mathcal{F}^{\partial}_{\mathrm{BFV}}$. In the BV-BFV formalism, the two-form $\Omega^{\partial}_{\mathrm{BFV}} := \delta\alpha^{\partial}_{\mathrm{BFV}}$ is a symplectic structure on the boundary space of fields $\mathcal{F}^{\partial}_{\mathrm{BFV}}$, it plays the role of the BFV structure. The pushforward of the cohomological vector field Q_{BV} by the projection map, $Q^{\partial}_{\mathrm{BFV}} = \pi_* Q_{\mathrm{BV}}$, happens to be Hamiltonian with respect to the BFV structure, which justifies the application of a formalism initially designed for field theories in the Hamiltonian formulation.

The main motivation behind the development of this BV-BFV formalism is to find a way to define path integral quantization for field theories on manifolds with boundary, so that one can apply cutting-gluing methods to topological field theories aiming at computing topological invariants, such as Chern-Simons theory or BF theory. Physical problems as well will benefit from this formalism. Transition amplitudes such as the ones used to compute cross sections in scattering experiments involve path integrals of quantum field theories defined on pieces of space-time delimited by two constant time slices. The main reason to apply the BV formalism to this kind of problems is that BV path integrals are compatible with renormalization [22].

The method proposed in [**?**] to construct a BV-BFV quantum theory involves the application of geometric quantization to the boundary BFV model to build a boundary space of states, and path integrals of the BV bulk theory should take value in this space. For details, we refer to the short review of the BV-BFV formalism in chapter 1.

Outline of the thesis and main results This thesis is structured as follows. The first chapter gives a short account of the BV formalism and the BV-BFV formalism. It begins with an introduction to supergeometry, focusing on the structures that arise in the BV formalism, the so-called QP-manifolds and SP-manifolds. It continues with a discussion of the BV formalism, showing how these mathematical structures describe and generalize the BRST method for gauge fixing, and explaining two useful constructions of effective theories and observables. Finally it gives a brief review of the main results of the BV-BFV formalism developed in [**?**].

As already mentioned, chapter 2 reproduces an article [7] focusing on the study of dualities in the BV formalism. We propose two methods to construct dual field theories, one based on BV morphisms and the other on dual BV pushforwards. Actually, both can be combined into a third composite method. We illustrate them with topological field theories inspired from string theory. In particular, starting with a special example of Roytenberg's Courant sigma model [38], we find an isomorphism of Courant algebroids constructed on principal torus bundles that coincides with the one discovered by Cavalcanti and Gualtieri [21], based on geometrical arguments inspired by string theory. We also find that the topological content of the Buscher rules [14] of T-duality is encoded in a duality of two-dimensional sigma models that represent the topological sectors of some string theories.

Finally, chapter 3 reproduces a joint work with Anton Alekseev and Pavel Mnev [2]. Inspired by their results on the quantization of the one-dimensional Chern-Simons theory [5], we applied the BV-BFV approach to this problem and found that the associated BFV charge reproduces the Kostant cubic Dirac operator. We considered next the three-dimensional Chern-Simons theory. As the boundary BFV action, we found an odd version of the two-dimensional BF-model, where the role of the B-field is played by an odd Lie alegebra-valued scalar, the ghost of the bulk theory. The BFV boundary space of quantum states turned out to be related to the space of conformal blocks of the associated Wess-Zumino-Witten model. To add Wilson loops (and open Wilson lines) to the bulk model, we considered the formula for Wilson loops of Alexeev, Faddeev and Shatashvili [3] based on Kirillov's orbit method. This formula allows to remove the path-ordering at the price of an additional path integration. We showed that this path integration can be carried out in the BV formalism. In effect, we can introduce Wilson loops observables to a gauge theory by augmenting the space of fields in a certain way and adding an auxiliary term to the action. A general treatment of this method is due to Mnev [37]. The main advantage of the BV formulation of Wilson loops is that it can be easily modified to take into account open Wilson lines that end on the boundary of the underlying manifold. Of course, we then need to apply the BV-BFV formalism. In our example of Chern-Simons theory, the effect on the boundary BFV action was to add singular source terms to the odd BF-model, and the corresponding boundary WZW correlation functions would receive field insertions at the extremities of the Wilson lines.

Contents

Chapter 1

Preliminaries

1.1 Mathematical Preliminaries

In this section, we introduce the notions of supergeometry that we need to express the Batalin-Vilkovisky formalism for quantum field theory. We mostly follow [36], that partly follows results of Schwarz regarding the geometry of the BV formalism [40], [41].

1.1.1 Supergeometry

Definition 1 (Mnev, [36]) *A $\mathbb{Z}$-graded manifold or supermanifold is defined as a direct sum of vector bundles $\mathcal{M} = \bigoplus_{k\in\mathbb{Z}} \mathcal{M}^k$ over a smooth manifold $\mathcal{M}_0$, where the rank of $\mathcal{M}^k$ vanishes for all but finitely many values of k. The ring of functions on $\mathcal{M}$ is defined as the graded (super-)commutative algebra of sections*

$$\mathrm{Fun}(\mathcal{M}) := \Gamma(\mathcal{M}_0, S^\bullet \mathcal{M}^*_{\mathrm{even}} \otimes \Lambda^\bullet \mathcal{M}^*_{\mathrm{odd}})$$

where $\mathcal{M}_{\mathrm{even}} = \bigoplus_{k\in\mathbb{Z}} \mathcal{M}^{2k}$ and $\mathcal{M}_{\mathrm{odd}} = \bigoplus_{k\in\mathbb{Z}} \mathcal{M}^{2k+1}$ are the even and odd parts of $\mathcal{M}$, and $S^\bullet$ and $\Lambda^\bullet$ denote the sums of symmetric powers and of exterior powers of the bundle respectively. Grading is defined by assigning degree $-k$ to sections of $(\mathcal{M}^k)^*$.

We will refer to this grading as the internal degree (as opposed to the exterior degree of a differential form) or the ghost number, and write $gh(\cdot)$ for the ghost number of a given function or coordinate.

Most of the examples of supermanifolds we will be dealing with involve shifted tangent or cotangent bundles.

Definition 2 (Mnev, [36]) *The tangent bundle $T[s]\mathcal{M}$ shifted by s of a supermanifold $\mathcal{M}$ is a $\mathbb{Z}$-graded manifold $T[s]\mathcal{M} = \bigoplus_{k\in\mathbb{Z}}(T[s]\mathcal{M})^k$, with components defined as*

$$(T[s]\mathcal{M})^k = \begin{cases} T\mathcal{M}_0 \oplus \mathcal{M}^{-s} \oplus \mathcal{M}^0, & \text{if} \quad k = -s, \\ \mathcal{M}^k \oplus \mathcal{M}^{k+s}, & \text{if} \quad k \neq -s. \end{cases}$$

An example we will encounter regularly is the odd tangent bundle of a manifold $T[1]M$, whose only nonvanishing component is $(T[1]M)^{-1} = TM$. By definition, functions on this odd tangent bundle can be identified with differential forms on its base manifold, $\mathrm{Fun}(T[1]M) = \Omega^\bullet(M)$. As a byproduct of this identification, we find a canonical measure on $T[1]M$, where functions are interpreted as differential forms on M, among which we extract the top form, that we may integrate over M.

We can generalize this result to define differential forms on a supermanifold,

$$\Omega^\bullet(\mathcal{M}) := \mathrm{Fun}(T[1]\mathcal{M}).$$

Vector fields on a supermanifold $\mathcal{M}$, on the other hand, can still be defined like vector fields on real manifolds as derivations of the algebra $\mathrm{Fun}(\mathcal{M})$,

$$\mathrm{Vect}(\mathcal{M}) = \mathrm{Der}(\mathrm{Fun}(\mathcal{M})).$$

1.1.2 QP-manifolds

Supergeometry allows to define an interesting class of vector fields:

Definition 3 *A cohomological vector field $Q \in \mathrm{Vect}(\mathcal{M})$ on a supermanifold $\mathcal{M}$ is a vector field with internal degree one, $gh(Q) = 1$, that squares to zero, $[Q, Q] = 0$. A supermanifold that carries such a vector field is called a Q-manifold, and the vector field is sometimes referred to as the Q-structure.*

An important example is the de Rham vector field $D \in \mathrm{Vect}(T[1]M)$ defined on the odd tangent bundle of a real manifold M. If M carries local coordinates x^μ, $\mu = 1, \ldots, \dim(M)$, there are natural Grassmannian coordinates θ^μ on the fibers, with $gh(\theta^\mu) = 1$, and D can be written as $D = \theta^\mu \frac{\partial}{\partial x^\mu}$, with implicit summation over repeated indices. Under the identification of functions on $T[1]M$ with differential forms on M, this vector field actually corresponds to the de Rham differential.

After tangent bundles, it is natural to introduce shifted cotangent bundles as well.

Definition 4 (Mnev, [36]) *The cotangent bundle $T^*[s]\mathcal{M}$ shifted by s of a supermanifold $\mathcal{M}$ is a $\mathbb{Z}$-graded manifold $T^*[s]\mathcal{M} = \bigoplus_{k\in\mathbb{Z}}(T^*[s]\mathcal{M})^k$, with components defined as*

$$(T^*[s]\mathcal{M})^k = \begin{cases} T^*\mathcal{M}_0 \oplus \mathcal{M}^{-s} \oplus (\mathcal{M}^0)^*, & \text{if} \quad k = -s, \\ \mathcal{M}^k \oplus (\mathcal{M}^{-k-s})^*, & \text{if} \quad k \neq -s. \end{cases}$$

We will encounter the case $s = 2$ in the construction of AKSZ sigma models in chapter 2. Of particular interest for the BV formalism is the case $s = -1$ relevant for P-manifolds.

Definition 5 *We call a P-manifold a supermanifold $\mathcal{M}$ carrying a symplectic form $\omega \in \Omega^2(\mathcal{M})$ with $gh(\omega) = -1$. This odd symplectic form is sometimes referred to as a P-structure.*

Not only can we locally find Darboux coordinates on a P-manifold, but there is also a much stronger statement:

Theorem 1 (Schwarz, [40]) *Every P-manifold $\mathcal{M}$ is symplectomorphic to a P-manifold of the form $T^*[-1]\mathcal{N}$, where $\mathcal{N}$ is a supermanifold that can be chosen to be purely even.*

This result allows us to find Lagrangian submanifolds of a P-manifold, namely maximally isotropic submanifolds. The first observation is that $\mathcal{N}$ itself is a Lagrangian submanifold of $T^*[-1]\mathcal{N}$. The idea is now to try to deform it to another Lagrangian submanifold. To do so, we introduce coordinates x^i on $\mathcal{N}$, and induced coordinates ξ_i on the fibers of $T^*[-1]\mathcal{N}$. We see that $gh(\xi_i) = -1 - gh(x^i)$. With a function Ψ on $\mathcal{N}$ of degree -1, i.e. $gh(\Psi) = -1$, we can deform $\mathcal{N}$ to the Lagrangian submanifold

$$\mathcal{L}_\Psi = \left\{ (x, \xi) \in \mathcal{L} | \xi_i = -\frac{\partial}{\partial x^i}\Psi(x) \right\}. \tag{1.1}$$

We can naturally combine the two notions of P-manifold and Q-manifold:

Definition 6 *A QP-manifold $\mathcal{M}$ is a P-manifold carrying a Q-structure compatible with its P-structure in the sense that $\mathcal{L}_Q\omega = 0$, where $\mathcal{L}$ denotes the Lie derivative.*

In case the cohomological vector field Q is in addition Hamiltonian, namely if there exists a function $S_0 \in \mathrm{Fun}(\mathcal{M})$ such that $\imath_Q\omega = dS_0$, then we can write $Q = \{S_0, \cdot\}$, where $\{\cdot,\cdot\}$ denotes the odd Poisson bracket associated to ω (also called Gerstenhaber bracket or antibracket).

1.1.3 SP-manifolds

Ultimately, we will need to integrate functions over supermanifolds, which requires a measure μ that we can consider as a section of the Berezinian bundle.

Definition 7 (Mnev, [36]) *The Berezinian bundle of a graded manifold $\mathcal{M}$ is defined as the line bundle*

$$\mathrm{Ber}(\mathcal{M}) = \Lambda^{\dim \mathcal{M}_0} T^* [-1] \mathcal{M}_0 \otimes \Lambda^{\mathrm{rk} \mathcal{M}_{\mathrm{even}}} \mathcal{M}^*_{\mathrm{even}} [-1] \otimes \Lambda^{\mathrm{rk} \mathcal{M}_{\mathrm{odd}}} \mathcal{M}_{\mathrm{odd}}$$

over $\mathcal{M}_0$, whose sections are called Berezin measures on $\mathcal{M}$.

To define integration on the supermanifold $\mathcal{M}$,

$$\int_{\mathcal{M}} : \quad \mathrm{Fun}(\mathcal{M}) \otimes_{C^\infty(\mathcal{M}_0)} \Gamma(\mathcal{M}_0, \mathrm{Ber}(\mathcal{M})) \to \mathbb{R},$$

that assigns a real number to any choice of (integrable) function and Berezin measure, we need to separate even and Grassmannian integration variables by decomposing $f = f_{\mathrm{even}} \otimes f_{\mathrm{odd}}$ and $\mu = \mu_{\mathrm{even}} \otimes \mu_{\mathrm{odd}}$ with $f_{\mathrm{even}} \in \mathrm{Fun}(\mathcal{M}_{\mathrm{even}})$, $f_{\mathrm{odd}} \in \mathrm{Fun}(\mathcal{M}_{\mathrm{odd}})$, $\mu_{\mathrm{even}} \in \Gamma(\mathcal{M}_0, \mathrm{Ber}(\mathcal{M}_{\mathrm{even}}))$ and $\mu_{\mathrm{odd}} \in \Gamma(\mathcal{M}_0, \Lambda^{\mathrm{rk}\mathcal{M}_{\mathrm{odd}}} \mathcal{M}_{\mathrm{odd}})$, and then we can set

$$\int_{\mathcal{M}} f \mu := \int_{\mathcal{M}_{\mathrm{even}}} f_{\mathrm{even}} \langle f_{\mathrm{odd}}, \mu_{\mathrm{odd}} \rangle \mu_{\mathrm{even}},$$

where $\langle \cdot, \cdot \rangle$ is the canonical pairing between the exterior algebra of $\mathcal{M}_{\mathrm{odd}}$ and the one of its dual $\mathcal{M}^*_{\mathrm{odd}}$.

A Berezin measure can be restricted to a Lagrangian submanifold $\mathcal{L}$ of $\mathcal{M}$ as follows. We first choose local coordinates (x^i, ξ_i) of $\mathcal{M}$ near $\mathcal{L}$ such that $\mathcal{L}$ is specified by the condition $\xi_i = 0$. We can express the Berezin measure μ in these coordinates with the help of a density function ρ on $\mathcal{M}$, $\mu = \rho(x, \xi) \prod_i \mathcal{D}x^i \mathcal{D}\xi_i$. The induced measure is locally defined as $\sqrt{\mu}_{\mathcal{L}} = \sqrt{\rho(x,0)} \prod_i \mathcal{D}x^i$.

With a Berezin measure on $\mathcal{M}$, we can also define a divergence operator $\mathrm{div}_\mu : \mathrm{Vect}(\mathcal{M}) \to \mathrm{Fun}(\mathcal{M})$ that satisfies the relation

$$\int_{\mathcal{M}} X(f) \mu = \int_{\mathcal{M}} f \, \mathrm{div}_\mu(X) \mu$$

for any test function f on $\mathcal{M}$.

Definition 8 (Schwarz, [40], [41]) *Given a Berezin measure μ on a P-manifold $\mathcal{M}$, we can define a Laplace operator on $\mathcal{M}$ as*

$$\Delta_\mu f = \frac{1}{2} \mathrm{div}_\mu \{f, \cdot\}.$$

We say that the measure μ determines an SP-structure on $\mathcal{M}$ if $\Delta_\mu^2 = 0$, and then $\mathcal{M}$ is called an SP-manifold.

We are finally in a position to state the theorem that lays at the foundation of the Batalin-Vilkovisky formalism.

Theorem 2 (Batalin-Vilkovisky [9], Schwarz [40]) *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are two Lagrangian submanifolds of an SP-manifold $\mathcal{M}$ that are connected by a continuous family $\mathcal{L}_t$ of Lagrangian submanifolds and if $f \in \mathrm{Fun}(\mathcal{M})$ is a function satisfying $\Delta_\mu f = 0$, then*

$$\int_{\mathcal{L}_1} f\sqrt{\mu}_{\mathcal{L}_1} = \int_{\mathcal{L}_2} f\sqrt{\mu}_{\mathcal{L}_2}.$$

Moreover, for any function $g \in \mathrm{Fun}(\mathcal{M})$ and any Lagrangian submanifold $\mathcal{L}$, we have

$$\int_{\mathcal{L}} \Delta_\mu g\sqrt{\mu}_{\mathcal{L}} = 0.$$

In fact, Schwarz proved that a sufficient condition for the first part of the BV theorem is that the bodies $\mathcal{L}_1 \cap \mathcal{M}_0$ and $\mathcal{L}_2 \cap \mathcal{M}_0$ of the Lagrangian submanifolds are homologous in $\mathcal{M}_0$.

1.2 Physical Motivations

1.2.1 Field Theory

In physics, a convenient way to express a classical field theory is the Lagrangian formalism. In this formalism, one first needs to determine a classical space of fields $\mathcal{F}_{\mathrm{cl}}$, usually an infinite-dimensional manifold, and a classical action $S_{\mathrm{cl}} \in \mathrm{Fun}(\mathcal{F}_{\mathrm{cl}})$. In many theories of interest, $\mathcal{F}_{\mathrm{cl}}$ is the space of sections of some bundle over a manifold N, or the space of connections of a principal bundle P over N, or the mapping space $\mathrm{Map}(N, M)$ between N (often called in this case 'worldline', 'worldsheet' or 'worldvolume', depending on its dimension) and another manifold M in the case of sigma models. The action S_{cl} is then defined as an integral over N of a Lagrangian density L. We will first assume that $\partial N = \emptyset$ and consider the case with boundary only in section 1.4.

The classical equations of motion are specified according to the principle of stationary action by the critical points of S_{cl} on $\mathcal{F}_{\mathrm{cl}}$, namely the points where $\delta S_{\mathrm{cl}} = 0$. Here δ denotes the de Rham differential operator on $\mathcal{F}_{\mathrm{cl}}$. If we introduce coordinate fields ϕ^i on $\mathcal{F}_{\mathrm{cl}}$, we can write

$$\delta S_{\mathrm{cl}} = \int_N \delta\phi^i \frac{\overrightarrow{\delta} S_{\mathrm{cl}}}{\delta\phi^i} = \int_N \frac{S_{\mathrm{cl}} \overleftarrow{\delta}}{\delta\phi^i} \delta\phi^i =: \int_N EL, \tag{1.2}$$

where summation over repeated indices is implicit, and the left- and right-functional derivatives $\frac{\overrightarrow{\delta} A}{\delta\phi^{i_0}}$ and $\frac{A\overleftarrow{\delta}}{\delta\phi^{i_0}}$ of a functional A are defined through the formula

$$\frac{d}{dt}A\left[\phi^{i_0}+t\epsilon\right]|_{t=0}=\int_N \epsilon\frac{\overrightarrow{\delta} A}{\delta\phi^{i_0}}=\int_N \frac{A\overleftarrow{\delta}}{\delta\phi^{i_0}}\epsilon.$$

The difference between left- and right-functional derivatives is needed for the situations where a field is a differential form or Grassmannian-valued and could anti-commute with the differentiated functional. Note that these functional derivatives are regular when A is defined as an integral over N, otherwise they yield a result proportional to a Dirac distribution.

The condition $\delta S_{\rm cl}=0$ is thus equivalent to the set of equations $\frac{\overrightarrow{\delta} S_{\rm cl}}{\delta\phi^{i}}=0$ called Euler-Lagrange equations, hence the notation EL for the integrand in (1.2).

A possible way to define a quantum theory starting from a field theory in the Lagrangian formalism is through the use of path integrals. One needs to define an integration measure $\mu_{\rm cl}$ on $\mathcal{F}_{\rm cl}$, and we can formally define correlation functions as integrals of the form

$$\langle f\rangle=\frac{1}{Z}\int_{\mathcal{F}_{\rm cl}}\mu_{\rm cl}\, f\, e^{\frac{i}{\hbar}S_{\rm cl}}, \tag{1.3}$$

where $f\in \mathrm{Fun}(\mathcal{F}_{\rm cl})$ is what is called an observable, $\hbar$ is a formal constant (Planck's constant divided by 2π in physics problems), and Z is the partition function of the theory,

$$Z=\int_{\mathcal{F}_{\rm cl}}\mu_{\rm cl}\, e^{\frac{i}{\hbar}S_{\rm cl}}. \tag{1.4}$$

1.2.2 Gauge Symmetries

The trouble with this formalism is that in the presence of gauge symmetries, the classical action $S_{\rm cl}$ is degenerate and path integrals are intrisically ill-defined. For instance, perturbative calculations are excluded due to the impossibility to construct propagators.

Geometrically, infinitesimal gauge symmetries form a distribution in $T\mathcal{F}_{\rm cl}$,

$$\mathfrak{D}=\left\{X\in\Gamma(T\mathcal{F}_{\rm cl})|\ X(S_{\rm cl})=0\right\}.$$

The classical observables of a gauge theory should also be invariant under gauge transformations, which is why we consider only the invariant functions on the space of fields, $\mathrm{Fun}(\mathcal{F}_{\rm cl})^{\mathfrak{D}}$.

To achieve a quantization of a gauge theory, one should implement some gauge fixing procedure and thus reduce the classical space of fields to a phase space where path integrals can be defined. The fact that observables are invariant under gauge transformations should ensure that the correlation functions do not depend on a choice of gauge.

If the distribution $\mathfrak{D}$ of the infinitesimal gauge transformations is involutive, the classical phase space of the theory is defined as the space of leaves of a foliation generated by the distribution. In this case, Becchi, Rouet and Stora, and Tyutin [10] showed that the gauge symmetries can be encoded in a Q-manifold, the BRST space of fields $\mathcal{F}_{\mathrm{BRST}}$, whose Q-structure is denoted by Q_{BRST}.

The so-called minimal BRST space of fields (i.e. the smallest one with the required property) is the supermanifold

$$\mathcal{F}_{\mathrm{BRST,min}} = \mathfrak{D}\,[1] \subset T\,[1]\,\mathcal{F}_{\mathrm{cl}}, \tag{1.5}$$

and it carries a cohomological vector field $Q_{\mathrm{BRST,min}} \in \mathrm{Vect}(\mathcal{F}_{\mathrm{BRST,min}})$ built from the infinitesimal gauge transformations. The idea is to replace the infinitesimal gauge parameters by odd ghost fields that belong to sections of $\mathfrak{D}\,[1]$, which explains the terminology of ghost number for the internal grading, and extend these shifted gauge transformations to a vector field on $\mathfrak{D}\,[1]$ that squares to zero. This construction will be made explicit in the example below.

Note that in order to preserve locality, we might be forced to use a representation of $\mathfrak{D}\,[1]$ that is itself degenerate, in which case we would need to introduce fields of ghost number 2 (or *ghosts for ghosts*) to lift the new degeneracy, and repeat the process until all degeneracies have been lifted. We will be confronted with ghosts for ghosts when we study Courant Sigma models in chapter 2.

Gauge invariant functions on $\mathcal{F}_{\mathrm{BRST}}$ are the ones that sit in the kernel of Q_{BRST}. In particular, we have $Q_{\mathrm{BRST}}(S_{\mathrm{cl}}) = 0$. Moreover, two functions are gauge equivalent if they differ by a Q_{BRST}-exact term. The idea behind the BRST construction is to add some BRST exact term to the classical action so as to obtain a gauge-fixed BRST action

$$S_{\Psi} = S_{\mathrm{cl}} + Q_{\mathrm{BRST}}(\Psi) \tag{1.6}$$

which is no longer degenerate in the BRST space of fields. Degree counting shows that Ψ should be a fermion (i.e. a Grassmannian valued function) of ghost number -1, which is impossible in the minimal BRST space of fields. For this reason, we need additional fields, typically antighosts sitting in $\mathfrak{D}^*\,[-1]$ and Lagrange multipliers in $\mathfrak{D}^*$, and we obtain the full BRST space of fields

$$\mathcal{F}_{\mathrm{BRST}} = \mathfrak{D}\,[1] \oplus \mathfrak{D}^*\,[-1] \oplus \mathfrak{D}^*.$$

The fact that Ψ should be chosen in such a way that the degeneracy is lifted by a gauge-fixing motivates the name of *gauge-fixing fermion*.

Example: Yang-Mills theory. The motivating example of gauge symmetries is the situation where they arise under action of a gauge group $\mathcal{G} = \mathrm{Map}(N, G)$, where G is a Lie group with Lie algebra $\mathfrak{g}$ and N is the manifold on which the classical action is defined. Taken locally, such maps can be interpreted as transition functions of a local trivialization of some principal G-bundle P over N. The classical field content of a so-called pure gauge field theory consists of *gauge fields* A, which are defined only locally as the pullback by local trivialisation maps $s : U \to P$ of a connection $\mathcal{A} \in \Omega^1(P, \mathfrak{g})$. For simplicity, we will assume that P is a trivial bundle, so that A can be taken as a globally defined one-form with value in $\mathfrak{g}$, so that we can write

$$\mathcal{F}_{\mathrm{cl}} = \Omega^1(N, \mathfrak{g}).$$

The action of a given local gauge transformation $g \in \mathcal{G}$ is defined as

$$A \mapsto gAg^{-1} - dg\, g^{-1}.$$

In quantum field theory, perhaps the most interesting example of a gauge theory is given by Yang-Mills theory, which seeks to describe the dynamics of pure gauge fields, an important sector of the Standard Model of particle physics. Its classical action is given by

$$S_{\mathrm{YM,cl}} = \frac{1}{4g^2} \int_N (F, *F), \tag{1.7}$$

where g is a coupling constant, $(\cdot, \cdot)$ is an invariant nondegenerate bilinear form on the Lie algebra $\mathfrak{g}$, $*$ denotes the Hodge operator and F is the curvature of the connection A,

$$F = dA + \frac{1}{2}\,[A, A]\,.$$

Under gauge transformations, the curvature transforms as $F \mapsto gFg^{-1}$, and it is clear why the classical action $S_{\mathrm{YM,cl}}$ is invariant under gauge transformations.

As the infinitesimal version of the local gauge transformations $\mathcal{G}$, we find the action of the Lie algebra $\mathrm{Lie}(\mathcal{G}) = \mathrm{Map}(N, \mathfrak{g})$ on $\Omega^1(N, \mathfrak{g})$, namely for a given $\epsilon \in \mathrm{Map}(N, \mathfrak{g})$,

$$A \mapsto -d\epsilon - [A, \epsilon] = -d_A \epsilon,$$

where we introduced the covariant derivative $d_A = d + [A, \cdot]$. In the language of geometry, this corresponds to the Lie algebra homomorphism

$$\begin{array}{rcl} \mathcal{X} : \mathrm{Lie}(\mathcal{G}) = \Omega^0(N, \mathfrak{g}) & \to & \Gamma(T\mathcal{F}_{\mathrm{cl}}) = \Gamma(T\Omega^1(N, \mathfrak{g})) \\ \epsilon & \mapsto & \mathcal{X}_\epsilon = -d_A \epsilon \frac{\delta}{\delta A}. \end{array}$$

The distribution of gauge transformations corresponds to the image of this homomorphism, $\mathfrak{D} = \mathcal{X}(\Omega^0(N,\mathfrak{g})) \simeq \Omega^0(N,\mathfrak{g})$. As a model for the minimal BRST space of fields, we can thus simply shift the space of zero-forms with value in $\mathfrak{g}$ by 1,

$$\mathcal{F}_{\mathrm{BRST,min}} = \Omega^1(N,\mathfrak{g}) \oplus \Omega^0(N,\mathfrak{g}\,[1]).$$

The even gauge parameter ϵ is replaced by a ghost field $\alpha \in \Omega^0(N,\mathfrak{g}\,[1])$. The Q-structure is found by adding a term to $\mathcal{X}_\alpha$ in the direction of $\frac{\delta}{\delta\alpha}$ so that $[Q,Q]=0$, namely

$$Q_{\mathrm{BRST,min}} = -d_A\alpha\frac{\delta}{\delta A} + \frac{1}{2}\,[\alpha,\alpha]\,\frac{\delta}{\delta\alpha}. \tag{1.8}$$

1.2.3 Antifields

Zinn-Justin made an observation that simplifies the implementation of the BRST gauge fixing procedure [44]. He suggested to assign an antifield ϕ_i^+ to each field or ghost ϕ^i (collectively denoted by ϕ) of the BRST space of fields, such that $\mathrm{gh}(\phi_i^+) = -1 - \mathrm{gh}(\phi^i)$, and to add an antifield term to the classical action,

$$S_0\left[\phi,\phi^+\right] = S_{\mathrm{cl}}\left[\phi\right] + \int_N \sum_i \phi_i^+ Q_{\mathrm{BRST}}(\phi^i) = S_{\mathrm{cl}}\left[\phi\right] + S_{\mathrm{antifields}}\left[\phi,\phi^+\right]. \tag{1.9}$$

The gauge-fixed BRST action (1.6) then corresponds to the restriction

$$S_\Psi\left[\phi\right] = S_0\left[\phi,\phi^+ = \frac{\delta\Psi}{\delta\phi}\right]. \tag{1.10}$$

Geometrically, the antifields can be interpreted as the fiber coordinates of the odd cotangent bundle

$$T^*\left[-1\right]\mathcal{F}_{\mathrm{BRST}} =: \mathcal{F}_{\mathrm{BV}}, \tag{1.11}$$

which is defined as the *BV space of fields*. It is evidently a P-manifold. For this reason, we can compute the Hamiltonian vector field generated by the additional term $S_{\mathrm{antifields}}$,

$$\tilde{Q}_{\mathrm{BRST}} = \left\{S_{\mathrm{antifields}},\cdot\right\},$$

which reproduces Q_{BRST} under pushforward by the fiber projection $\mathcal{F}_{\mathrm{BV}} \to \mathcal{F}_{\mathrm{BRST}}$. Here $\{\cdot,\cdot\}$ is the antibracket on $T^*\left[-1\right]\mathcal{F}_{\mathrm{BRST}}$, i.e. the odd Poisson bracket generated by its canonical odd symplectic structure. Since Q_{BRST} is cohomological, we find that $\{S_{\mathrm{antifields}}, S_{\mathrm{antifields}}\} = 0$. Moreover, we have $\{S_{\mathrm{antifields}}, S_{\mathrm{cl}}\} = Q_{\mathrm{BRST}}(S_{\mathrm{cl}}) = 0$ due to gauge invariance of the classical action, and $\{S_{\mathrm{cl}}, S_{\mathrm{cl}}\} = 0$ since the classical action does not depend on the antifields, so that we get $\{S_0, S_0\} = 0$. The Hamiltonian vector field $Q = \{S_0,\cdot\}$

is thus cohomological, and it can serve as a Q-structure that makes $\mathcal{F}_{\mathrm{BV}}$ a QP-manifold. Remembering the construction of a Lagrangian submanifold of a P-manifold (1.1), we see that the gauge-fixing fermion specifies a Lagrangian submanifold $\mathcal{L}_\Psi$ in $\mathcal{F}_{\mathrm{BV}}$ so that $S_0|_{\mathcal{L}_\Psi}$ is nondegenerate.

Example: Yang-Mills theory, continued. We continue with the example of Yang-Mills theory. First of all, we need to complete the minimal BRST space of fields with antighosts and Lagrange multipliers if we want to be able to implement gauge fixing. We obtain the BRST space of fields

$$\mathcal{F}_{\mathrm{BRST}} = \Omega^1(N, \mathfrak{g}) \oplus \Omega^0(N, \mathfrak{g}\,[1]) \oplus \Omega^4(N, \mathfrak{g}^*\,[-1]) \oplus \Omega^4(N, \mathfrak{g}^*),$$

consisting of a gauge field A, a ghost α, an antighost $\overline{\alpha}$ and a Lagrange multiplier λ respectively. The gauge symmetry in this full BRST space of fields is encoded by the cohomological vector field

$$Q_{\mathrm{BRST}} = -d_A\alpha\frac{\delta}{\delta A} + \frac{1}{2}\,[\alpha,\alpha]\,\frac{\delta}{\delta\alpha} - \lambda\frac{\delta}{\delta\overline{\alpha}}.$$

In the antifields formalism, we find the BV space of fields

$$\mathcal{F}_{\mathrm{BV}} = \mathcal{F}_{\mathrm{BRST}} \oplus \Omega^3(N, \mathfrak{g}^*\,[-1]) \oplus \Omega^4(N, \mathfrak{g}^*\,[-2]) \oplus \Omega^0(N, \mathfrak{g}) \oplus \Omega^0(N, \mathfrak{g}\,[-1]),$$

with the antifields A^+, α^+, $\overline{\alpha}^+$ and λ^+ in this order. Note that the antifield of a ghost should not be mistaken for the antighost.

The action (1.9) that properly encodes the symmetry in this example is

$$S_{\mathrm{YM},0} = \frac{1}{4g^2}\int_N (F, *F) + \int_N \left(\langle A^+, d_A\alpha\rangle - \langle \alpha^+, \frac{1}{2}\,[\alpha,\alpha]\rangle + \langle \overline{\alpha}^+, \lambda\rangle \right).$$

To fix the gauge, we need to find a local map $G_{\mathrm{gf}} : \Omega^1(N, \mathfrak{g}) \to \Omega^0(N, \mathfrak{g})$ such that the condition $G_{\mathrm{gf}}(A) = 0$ picks one gauge field in each gauge equivalence class. For instance, $G_{\mathrm{gf}}(A) = \partial^\mu A_\mu$ corresponds to the often-used Einstein gauge. This gauge condition can be implemented with the gauge-fixing fermion

$$\Psi = \int_N \langle \overline{\alpha}, G_{\mathrm{gf}}(A)\rangle + i\int_N (\overline{\alpha}, *\lambda),$$

where $(\cdot,\cdot)$ now denotes the scalar product on $\mathfrak{g}^*$ induced by the invariant non-degenerate bilinear form on $\mathfrak{g}$. It specifies a Lagrangian submanifold $\mathcal{L}_\Psi$ of $\mathcal{F}_{\mathrm{BV}}$ through the constraints

$$A^+ = \frac{\delta\Psi}{\delta A}, \quad \alpha^+ = \frac{\delta\Psi}{\delta\alpha} = 0, \quad \overline{\alpha}^+ = \frac{\delta\Psi}{\delta\overline{\alpha}}, \quad \lambda^+ = \frac{\delta\Psi}{\delta\lambda}.$$

The corresponding gauge fixed action is given by

$$S_{\mathrm{YM},0}|_{\mathcal{L}_\Psi} = \frac{1}{4g^2}\int_N (F, *F) + \int_N \left(\langle \overline{\alpha}, \frac{\delta G_{\mathrm{gf}}}{\delta A} d_A \alpha \rangle + \langle \lambda, G_{\mathrm{gf}}(A) \rangle + i(\lambda, *\lambda) \right).$$

The first term in the gauge fixing part yields the usual Faddeev-Popov determinant after integration of the Grassmannian ghosts and antighosts, while the last two terms implement the gauge fixing through a Gaussian averaging around the zero of G_{gf}. We see that the gauge-fixing fermion encodes a gauge condition as well as a way to implement it. Note that if we dropped the second (imaginary) term in Ψ, it would replace the Gaussian distribution by a Dirac distribution.

1.3 BV Formalism

1.3.1 Classical BV Formalism

This formulation with antifields allowed Batalin and Vilkovisky [9] to treat field theories with a non-involutive distribution of infinitesimal gauge transformations $\mathfrak{D}$. In this case, while it is impossible to find a cohomological vector field on $\mathcal{F}_{\mathrm{BRST}}$ that encodes the gauge transformations, it is often possible to construct one on $\mathcal{F}_{\mathrm{BV}} = T^*[-1]\,\mathcal{F}_{\mathrm{BRST}}$. The procedure to fix the gauge is therefore the same as the one outlined in the previous section: we first need to find a BV action $S_0 \in \mathrm{Fun}(\mathcal{F}_{\mathrm{BV}})$ that satisfies the classical master equation

$$\{S_0, S_0\} = 0 \tag{1.12}$$

and that reproduces the classical action when restricted to the classical space of fields, $S_0|_{\mathcal{F}_{\mathrm{cl}}} = S_{\mathrm{cl}}$, and then we need to find a Lagrangian submanifold $\mathcal{L}$ in $\mathcal{F}_{\mathrm{BV}}$ where S_0 is nondegenerate. We call $S_0|_{\mathcal{L}}$ the gauge-fixed action.

The classical master equation ensures that the Hamiltonian vector field

$$Q = \{S_0, \cdot\} \in \mathrm{Vect}(\mathcal{F}_{\mathrm{BV}}) \tag{1.13}$$

is cohomological, and this is the one that encodes $\mathfrak{D}$. Equivalently, we have the relation

$$\imath_Q \Omega = \delta S_0, \tag{1.14}$$

which will be easier to generalize to the case with boundary.

With this new cohomological vector field, we can generalize the definition of observables from the BRST formalism.

Definition 9 *Classical observables in the BV formalism are defined as the cohomology $H_Q(\mathrm{Fun}(\mathcal{F}_{\mathrm{BV}}))$ of the Q-structure generated by the BV action S_0.*

We see that the geometrical description of the classical BV formalism really consists of a special class of QP-manifolds where the Q-structure is Hamiltonian. To emphasize the role of the BV action in the BV formalism, in contrast to the definition 6 of a QP-manifold, we introduce the concept of classical *BV manifolds*:

Definition 10 *A classical BV manifold* $(\mathcal{F}, \Omega, S, Q)$ *is a graded manifold* $\mathcal{F}$ *equipped with an odd symplectic structure* $\Omega \in \Omega^2(\mathcal{F})$ *of degree -1 and supporting a function* $S \in \mathrm{Fun}(\mathcal{F})$ *of degree zero satisfying the classical master equation* $\{S, S\} = 0$*, which generates the Hamiltonian vector field* Q*.*

In this setting, the notion of classical duality discussed in chapter 2 takes the form of an isomorphism of BV manifolds, that we call classical BV morphism:

Definition 11 *A classical BV morphism* $\Phi : (\mathcal{F}, \Omega, S, Q) \to (\mathcal{F}', \Omega', S', Q')$ *between two classical BV manifolds is a symplectomorphism* $\Phi : (\mathcal{F}, \Omega) \to (\mathcal{F}', \Omega')$ *with respect to the BV structures that satisfies the additional requirement that* $S = \Phi^*(S')$*. In particular, it follows that* $Q' = \Phi_* Q$*.*

Note that not all classical BV morphisms actually correspond to dualities. In particular, symplectic flows of Hamiltonian vector fields generated by a functional R of degree -1 on $\mathcal{F}$ that acts on S as $S \mapsto S + \{S, R\}$ describe gauge transformations. Genuine dualities are BV morphisms between different spaces of fields, or automorphisms that cannot be deformed into the identity.

1.3.2 Quantum BV Formalism

We already mentioned that the issue of degeneracies of the classical action is particularly critical for the calculation of path integrals such as the partition function (1.4) or correlation functions (1.3). In the previous section, we saw that, at least classically, we can lift the degeneracy if we find a BV action S_0 extending S_{cl} and a Lagrangian submanifold $\mathcal{L}$ of the BV space of fields where S_0 is nondegenerate.

This was the main motivation that led Batalin and Vilkovisky to theorem 2. The difference from the classical theory is that we need an integration measure μ on $\mathcal{F}_{\mathrm{BV}}$ that should be invariant under gauge transformations. From the hypotheses of the BV theorem, we see that it should be such that Δ_μ squares to zero, or in other words that $\mathcal{F}_{\mathrm{BV}}$ takes the structure of an SP-manifold. Moreover, when restricted to the classical space of fields $\mathcal{F}_{\mathrm{cl}}$, μ should reproduce the measure μ_{cl}.

As a direct corrolary of the BV theorem, if we find a quantum BV action $W \in \mathrm{Fun}(\mathcal{F}_{\mathrm{BV}})\,[[\hbar]]$, which can depend on $\hbar$ as a formal power series, such that the quantum master equation

$$\Delta_\mu e^{\frac{i}{\hbar}W} = 0 \tag{1.15}$$

is satisfied and $W|_{\mathcal{F}_{\mathrm{BRST}}} = S_{\mathrm{cl}}$ (this restriction corresponds to setting the antifields to zero) as well as a Lagrangian submanifold $\mathcal{L}$ where W is nondegenerate, we obtain a well-behaved expression for the path integral of the partition function,

$$Z = \int_{\mathcal{L}} e^{\frac{i}{\hbar}W} \sqrt{\mu}_{\mathcal{L}}. \tag{1.16}$$

The important fact is that $\mathcal{F}_{\mathrm{BRST}}$ itself is a Lagrangian submanifold of $\mathcal{F}_{\mathrm{BV}}$, and one has to assume that $\mathcal{L}$ is a deformation of it if one wants to keep the same physical content.

The choice of a Lagrangian submanifold where W is nondegenerate corresponds to a choice of gauge and of gauge-fixing procedure, as already mentioned in section 1.2.3.

Note that $\mathcal{F}$ being usually infinite-dimensional renders the definition of a BV measure and then of a BV Laplacian quite difficult, but the study of finite-dimensional problems can give precious insights into gauge-fixing and BV integration [1].

The quantum master equation (1.15) is equivalent to

$$\frac{1}{2}\{W, W\} - i\hbar\Delta_\mu W = 0 \tag{1.17}$$

and can be solved order by order in $\hbar$, which explains why we take W as a formal power series in $\hbar$ of functions on $\mathcal{F}_{\mathrm{BV}}$,

$$W = \sum_{k\geq 0} S_k \hbar^k.$$

With this development, the quantum master equation can be decomposed order by order into

$$\begin{aligned} \{S_0, S_0\} &= 0, \\ \{S_0, S_1\} - i\Delta_\mu S_0 &= 0, \\ \{S_0, S_2\} + \tfrac{1}{2}\{S_1, S_1\} - i\Delta_\mu S_1 &= 0, \\ &\cdots \end{aligned}$$

Note that at zeroth order, we obtain the classical master equation (1.12), which explains the choice of notation for the classical BV action.

In general, the existence of a solution W of the quantum master equation does not follow from the existence of a solution S_0 of the classical master equation. We speak of *anomaly* when a solution of the CME exists, but not of the QME. This happens when the quantum theory does not have the same gauge symmetry as the classical theory.

At each order in $\hbar$, we find an obstruction for the existence of a solution of the quantum master equation in terms of the cohomology $H^1_Q(\mathrm{Fun}(\mathcal{F}_{\mathrm{BV}}))$ of $Q = \{S_0, \cdot\}$. For instance, we need $[i\Delta_\mu S_0] = [0]$ and $\left[-\frac{1}{2}\{S_1, S_1\} + i\Delta_\mu S_1\right] = [0]$ at the first two orders. In particular, if $H^1_Q(\mathrm{Fun}(\mathcal{F}_{\mathrm{BV}}))$ is trivial, then all obstructions vanish automatically and the existence of a solution of the quantum master equation is guaranteed for each solution of the classical master equation [36].

We just saw that the zeroth order of the quantum master equation defines the same cohomological vector field Q as in the classical BV theory. Actually, if W satisfies both the classical master equation and the quantum master equation (i.e. if $\{W, W\} = 0$ and $\Delta W = 0$), which means that the BV space of fields is at the same time an SP-manifold and a QP-manifold, then the integration measure μ of the BV space of fields seen as an SP-manifold is invariant under gauge transformations, in the sense that the divergence of Q vanishes, $\mathrm{div}_\mu Q = 0$ [36].

Note that in the setting of infinite-dimensional spaces of fields of quantum field theory, the BV Laplacian Δ_μ is singular and should be regularized. We refer to [22] for details.

Nevertheless, many interesting theories involve a BV Laplacian that can be regularized in such a way that S_0 is BV harmonic, $\Delta_\mu S_0 = 0$, so that $W = S_0$ already satisfies the quantum master equation.

We can compute correlation functions in a similar way as the partition function (1.16),

$$\langle f \rangle = \int_{\mathcal{L}} f\, e^{\frac{i}{\hbar}W} \sqrt{\mu}_{\mathcal{L}}, \tag{1.18}$$

but the classical gauge invariance condition is replaced by the quantum gauge invariance condition

$$\Delta_\mu (f\, e^{\frac{i}{\hbar}W}) = 0 \tag{1.19}$$

required by the assumptions of the BV theorem. Evidently, a quantum observable should depend on $\hbar$ as a formal power series, $f \in \mathrm{Fun}(\mathcal{F}_{\mathrm{BV}})\,[[\hbar]]$. If we take into account the quantum master equation, the quantum gauge invariance condition is equivalent to

$$\delta_{\mathrm{BV}} f := -i\hbar\Delta_\mu f + \{W, f\} = 0, \tag{1.20}$$

where we introduced the quantum BV cohomology operator $\delta_{\rm BV}$, which is a deformation of the classical cohomological vector field Q. Formally, observables should sit in the cohomology of $\delta_{\rm BV}$.

Geometrically, we see that the quantum BV formalism can be expressed in terms of an SP-manifold (the BV space of fields) and an action W satisfying the quantum master equation. We can regroup these two elements to form a quantum BV manifold:

Definition 12 *A quantum BV manifold is a quadruple $(\mathcal{F}, \Omega, \mu, W)$ where $\mathcal{F}$ is a $\mathbb{Z}$-graded manifold equipped with an odd symplectic structure $\Omega \in \Omega^2(\mathcal{F})$ of degree -1 and a Berezin measure μ such that the induced BV Laplacian Δ_μ squares to zero, and supporting a formal power series of functions $W \in \mathrm{Fun}(\mathcal{F})\,[[\hbar]]$ of degree zero satisfying the quantum master equation $\Delta_\mu e^{\frac{i}{\hbar}W} = 0$.*

In the classical limit $\hbar \to 0$, W goes to S_0, which satisfies the classical master equation and thus generates a Hamiltonian cohomological vector field, turning the quantum BV manifold into a classical BV manifold.

Finally, we can define quantum duality in analogy with classical duality as a quantum BV morphism.

Definition 13 *A quantum BV morphism $\hat{\Phi} : (\mathcal{F}, \Omega, \mu, W) \to (\mathcal{F}', \Omega', \mu', W')$ is a formal power series $\hat{\Phi} = \sum_k \Phi_k \hbar^k \in \mathrm{Map}(\mathcal{F}, \mathcal{F}')\,[[\hbar]]$ of maps between $\mathcal{F}$ and $\mathcal{F}'$ such that $\hat{\Phi}$ is a symplectomorphism with respect to the BV structures, $\hat{\Phi}^*(\Omega') = \Omega$, that satisfies the additional requirements that $\hat{\Phi}^*(W') = W$ and $\hat{\Phi}_*(\mu) = \mu'$.*

We will see in chapter 2 that quantum BV morphisms preserve correlation functions of the related quantum field theories.

Example: Yang-Mills theory, continued. To illustrate the BV formalism and the quantum master equation, we can get back to our example of Yang-Mills theory. With the space of fields previously introduced, we can write the BV structure in Darboux coordinates,

$$\Omega_{\rm YM} = \int_N \big(\langle \delta A^+, \delta A \rangle + \langle \delta \alpha^+, \delta \alpha \rangle + \langle \delta \overline{\alpha}^+, \delta \overline{\alpha} \rangle + \langle \delta \lambda^+, \delta \lambda \rangle \big) \, .$$

These Darboux coordinates allow to formally define a canonical integration measure on the BV space of fields,

$$\mu = \mathcal{D}A \mathcal{D}A^+ \mathcal{D}\alpha \mathcal{D}\alpha^+ \mathcal{D}\overline{\alpha} \mathcal{D}\overline{\alpha}^+ \mathcal{D}\lambda \mathcal{D}\lambda^+ ,$$

and we can write the associated BV Laplace operator as

$$\Delta_\mu = \int_N \left(\frac{\delta^2}{\delta A \delta A^+} + \frac{\delta^2}{\delta \alpha \delta \alpha^+} + \frac{\delta^2}{\delta \overline{\alpha} \delta \overline{\alpha}^+} + \frac{\delta^2}{\delta \lambda \delta \lambda^+} \right).$$

One can show in two steps that the Yang-Mills BV action

$$S_{\mathrm{YM},0} = \frac{1}{4g^2} \int_N (F, *F) + \int_N \left(\langle A^+, d_A \alpha \rangle - \langle \alpha^+, \frac{1}{2} [\alpha, \alpha] \rangle + \langle \overline{\alpha}^+, \lambda \rangle \right)$$

satisfies both the classical and the quantum master equations.

We begin with the classical master equation,

$$\begin{aligned} \frac{1}{2} \{S_{\mathrm{YM},0}, S_{\mathrm{YM},0}\} = & \int_N \langle \frac{S_{\mathrm{YM},0} \overleftarrow{\delta}}{\delta A}, \frac{\overrightarrow{\delta} S_{\mathrm{YM},0}}{\delta A^+} \rangle + \langle \frac{S_{\mathrm{YM},0} \overleftarrow{\delta}}{\delta \alpha}, \frac{\overrightarrow{\delta} S_{\mathrm{YM},0}}{\delta \alpha^+} \rangle \\ & + \langle \frac{S_{\mathrm{YM},0} \overleftarrow{\delta}}{\delta \overline{\alpha}}, \frac{\overrightarrow{\delta} S_{\mathrm{YM},0}}{\delta \overline{\alpha}^+} \rangle + \langle \frac{S_{\mathrm{YM},0} \overleftarrow{\delta}}{\delta \lambda}, \frac{\overrightarrow{\delta} S_{\mathrm{YM},0}}{\delta \lambda^+} \rangle \\ = & - \int_N \left(\langle \left[\alpha, A^+\right], d_A \alpha \rangle + \langle d_A A^+ + \left[\alpha, \alpha^+\right], \frac{1}{2} [\alpha, \alpha] \rangle \right) \\ & + \frac{1}{4g^2} \int_N (d_A * F, d_A \alpha) \\ = & \, 0, \end{aligned}$$

where we use integration by part, the invariance of the bilinear form on $\mathfrak{g}$ and the Jacobi identity to see that the various terms cancel each other. In particular, the last term is equal to $\frac{1}{4g^2} \int_N (*F, [F, \alpha]) = 0$ and vanishes due to invariance of the bilinear form on $\mathfrak{g}$. To see this, we can use generators of the Lie algebra $t^a \in \mathfrak{g}$ with structure constants f_{abc}. The integrand is $*F^a \wedge F^b \, \alpha^c f_{abc}$, which is symmetric under the exchange of F^a with F^b, but f_{abc} is totally antisymmetric. Note that the bracket operation between an element of $\mathfrak{g}$ and one of its dual $\mathfrak{g}^*$ is defined via the coadjoint action.

Actually, the BV action in the minimal BV formulation of Yang-Mills theory already satisfies the classical master equation. This action is the same as $S_{\mathrm{YM},0}$, but with its last term removed. The addition of this term does not affect the way the BV action satisfies the classical master equation, since neither field involved in it sees its BV conjugate among the fields on which $S_{\mathrm{YM},0}$ depends. For this reason, the fields $\overline{\alpha}^+$ and λ form what we call a trivial pair. The addition of such trivial pairs is often convenient to be able to fix the gauge. This mechanism is a generalization of what we alreay mentioned about the BRST gauge fixing in section 1.2.2.

In the second step, we see that $\Delta_\mu S_{\mathrm{YM},0} = 0$, so that $S_{\mathrm{YM},0}$ immediately satisfies the quantum master equation as well. More accurately, we obtain an expression of the form $(\beta - \beta)\delta(0)$ that technically needs to be regularized before we can claim it vanishes. We refer to [22] for details on renormalization in the BV formalism. Nevertheless, once Δ_μ has been properly regularized, the fact that the action $S_{\mathrm{YM},0}$ respects the quantum master equation ensures, according to the BV theorem, that the gauge-fixed action $S_{\mathrm{YM},0}|_{\mathcal{L}_\Psi}$ we gave in the second part of this example can be used to compute correlation functions of the quantum Yang-Mills theory in the path integral formalism.

1.3.3 BV Pushforwards

In quantum field theory, one is sometimes interested only in the dynamics of the low energy (infrared) fields, which is described by an effective action, once one has integrated out the high energy (ultraviolet) fields. Losev suggested an implementation of this process in the framework of the BV formalism [35], which was properly defined by Mnev in [36].

The starting point is a BV space of fields $\mathcal{F}$ that can be decomposed into two SP-manifolds, $\mathcal{F} = \mathcal{F}_{\mathrm{IR}} \oplus \mathcal{F}_{\mathrm{UV}}$, the infrared and ultraviolet sectors of the space of fields. Their measures μ_{IR} and μ_{UV} can be combined into a measure $\mu = \mu_{\mathrm{IR}} \otimes \mu_{\mathrm{UV}}$ of $\mathcal{F}$. We assume we know a solution $W \in \mathrm{Fun}(\mathcal{F})\,[[\hbar]]$ of the quantum master equation $\Delta_\mu e^{\frac{i}{\hbar}W} = 0$ on $\mathcal{F}$ and a Lagrangian submanifold $\mathcal{L} = \mathcal{L}_{\mathrm{IR}} \times \mathcal{L}_{\mathrm{UV}} \subset \mathcal{F}$ where W is nondegenerate, such that $\mathcal{L}_{\mathrm{IR}} \subset \mathcal{F}_{\mathrm{IR}}$ and $\mathcal{L}_{\mathrm{UV}} \subset \mathcal{F}_{\mathrm{UV}}$ are both Lagrangian.

We define the infrared effective action $W_{\mathrm{eff}} \in \mathrm{Fun}(\mathcal{F}_{\mathrm{IR}})\,[[\hbar]]$ through the formula

$$e^{\frac{i}{\hbar}W_{\mathrm{eff}}} := \int_{\mathcal{L}_{\mathrm{UV}}} e^{\frac{i}{\hbar}W} \sqrt{\mu_{\mathrm{UV}}}_{\mathcal{L}_{\mathrm{UV}}}. \tag{1.21}$$

One can then show [36] that this effective action satisfies the quantum master equation $\Delta_{\mu_{\mathrm{IR}}} e^{\frac{i}{\hbar}W_{\mathrm{eff}}} = 0$ in the infrared sector of the space of fields.

We call the BV integration on $\mathcal{L}_{\mathrm{UV}}$ a *BV pushforward* by the projection map $\pi : \mathcal{F} \to \mathcal{F}_{\mathrm{IR}}$. In chapter 2, we use BV pushforwards as a way to construct dual theories.

1.3.4 Observables

Mnev showed in [37] (see also [20]) that in addition to effective actions, BV pushforwards can also be used to construct observables, i.e. (formal power series in $\hbar$ of) functions that satisfy the condition (1.19). The idea is to augment the BV space of fields of the theory under consideration with an auxiliary space of fields, $\mathcal{F}_{\mathrm{BV}} \to \mathcal{F}_{\mathrm{BV}} \times \mathcal{F}^{\mathrm{aux}}$, an SP-manifold with measure μ_{aux}, and look for an auxiliary action

$$W^{\text{aux}} \in \text{Fun}(\mathcal{F}_{\text{BV}} \times \mathcal{F}^{\text{aux}})\,[[\hbar]]$$

such that the total action $W + W^{\text{aux}}$ satisfies the quantum master equation

$$(\Delta_\mu + \Delta_{\mu_{\text{aux}}}) e^{\frac{i}{\hbar}(W+W^{\text{aux}})} = 0.$$

If $\mathcal{L}^{\text{aux}} \subset \mathcal{F}^{\text{aux}}$ is a Lagrangian submanifold, we can define an observable f through the formula

$$f\, e^{\frac{i}{\hbar}W} := \int_{\mathcal{L}^{\text{aux}}} e^{\frac{i}{\hbar}(W+W^{\text{aux}})} \sqrt{\mu_{\text{aux}}}_{\mathcal{L}^{\text{aux}}}, \tag{1.22}$$

and as a corollary of the construction of effective actions, it readily satisfies the condition (1.19).

The first step in the search of a suitable auxiliary action happens at the classical level, with the classical master equation $\{S_0 + S_0^{\text{aux}}, S_0 + S_0^{\text{aux}}\} = 0$. Mnev suggested in [37] an adaptation of the AKSZ formalism to find a classical BV auxiliary action S_0^{aux} that we use in chapter 3 to express Wilson lines in the BV formalism.

1.4 Boundary BV-BFV Formalism

1.4.1 Classical Action with Boundary

So far, we were assuming that the underlying manifold N was closed. It is time to consider the case with boundary $\partial N \neq \emptyset$. The first difficulty appears already in the computation (1.2) of the variation of the classical action, explicitly when we are confronted with the variation of the derivative of a field, such as $\delta d\phi^i$ (note that d is the de Rham differential operator on N, while δ is the one on $\mathcal{F}$). To isolate $\delta\phi^i$, it is natural to perform an integration by parts, which is harmless when $\partial N = \emptyset$, but creates a boundary term otherwise, that we can write as

$$\delta S_{\text{cl}} = \int_N EL + \pi^*(\alpha_{\text{cl},\partial}). \tag{1.23}$$

The additional term involves a one-form $\alpha_{\text{cl},\partial} \in \Omega^1(C_{\partial N})$ defined on the space of boundary Cauchy data $C_{\partial N}$. In many cases, such as the ones considered in this thesis, for a given $(n-1)$-dimensional manifold Σ, the space of Cauchy data C_Σ can be seen as the fields on Σ that determine a unique solution to the Euler-Lagrange equations on $N = \Sigma \times [0, \epsilon]$ for ϵ sufficiently small. For the general case, we refer to [18]. The one-form $\alpha_{\text{cl},\partial}$ can be pulled back to the classical space of fields by the projection map $\pi : \mathcal{F}_{\rm cl} \to C_{\partial N}$. We refer to [18] for details.

If we are interested in the quantum theory, we can first observe that $\omega_{\partial N} = \delta\alpha_{\partial N} \in \Omega^2(C_{\partial N})$ is a (pre)symplectic structure, symplectic if $S_{\rm cl}$ is regular.

In this symplectic case, Cattaneo, Mnev and Reshetikhin suggest in [19] to construct through geometric quantization a vector space $H_{\partial N}$ associated to $C_{\partial N}$. In the same spirit as the Atiyah-Segal axioms for topological field theories, where a Hilbert space is associated to the boundaries, we should assign states (and/or operators) to manifolds with boundaries, for instance in the path integral formalism, with a modification of the partition function formula (1.4),

$$\psi_N = \int_{\mathcal{F}_{\rm cl}} e^{\frac{i}{\hbar}S_{\rm cl}}\mu_{\rm cl} \in H_{\partial N}. \tag{1.24}$$

If we can separate the boundary into ingoing and outgoing components, $\partial N = \partial_{\rm in} N \sqcup \partial_{\rm out} N$ with different orientations, we get

$$\psi_N \in H^*_{\partial_{\rm in} N} \otimes H_{\partial_{\rm out} N} \simeq \mathrm{Hom}(H_{\partial_{\rm in} N}, H_{\partial_{\rm out} N}),$$

in the same spirit as the path integral representation of transition amplitudes in quantum mechanics.

1.4.2 BFV Formalism

In case ω_Σ is degenerate, we could apply symplectic reduction before quantization, but we might end up with a singular space. Instead, according to [19] [18], it is safer to embed C_Σ into a symplectic space of fields F_Σ as a coisotropic submanifold, and to quantize the algebra $\mathrm{Fun}(C_\Sigma)^{\mathfrak{D}_\Sigma}$ of functions invariant under the distribution $\mathfrak{D}_\Sigma = \ker(\omega_\Sigma)$.

We saw that in the framework of Lagrangian field theories, it is possible to encode the distribution $\mathfrak{D}$ of infinitesimal gauge transformations in a cohomological vector field Q in the BV formalism. Batalin, Fradkin and Vilkovisky [8] carried this idea over to Hamiltonian field theories, and showed that $\mathfrak{D}_\Sigma$ can also be encoded in a cohomological vector field.

Theorem 3 (Batalin-Fradkin-Vilkovisky) *One can embed F^∂_Σ in a graded symplectic manifold $\mathcal{F}^\partial_\Sigma$ and find a function S^∂_Σ of degree 1 satisfying $\{S^\partial_\Sigma, S^\partial_\Sigma\} = 0$ such that $\mathrm{Fun}(C_\Sigma)^{\mathfrak{D}_\Sigma}$ is isomorphic, as a Poisson algebra, to the degree zero cohomology $H^0_{Q^\partial_\Sigma}(\mathrm{Fun}(\mathcal{F}^\partial_\Sigma))$ of the cohomological Hamiltonian vector field $Q^\partial_\Sigma = \{S^\partial_\Sigma, \cdot\}$.*

Mathematically, we can introduce the notion of BFV manifold, by analogy with BV manifolds:

Definition 14 *A BFV manifold* $(\mathcal{F}^\partial, \Omega^\partial, Q^\partial)$ *is a graded manifold* $\mathcal{F}^\partial$ *carrying an even symplectic structure* Ω^∂ *of degree zero and a cohomological symplectic vector field* Q^∂ *of degree one.*

In order to find the quantum states associated to C_Σ, we can apply geometric quantization to the symplectic space $\mathcal{F}^\partial_\Sigma$ to construct a graded Hilbert space $\mathcal{H}^\partial_\Sigma$. However, this space is still too large, and we need to extract its gauge invariant states and mod out the gauge equivalent ones. In the quantization process, the function S^∂_Σ becomes an operator $\hat{S}^\partial_\Sigma \in \mathrm{End}(\mathcal{H}^\partial_\Sigma)$, called the BFV charge. If we find a polarization of $\mathcal{F}^\partial_\Sigma$ such that $(\hat{S}^\partial_\Sigma)^2 = 0$, then we can define the Hilbert space quantizing C_Σ as its degree 0 cohomology $H^0_{\hat{S}^\partial_\Sigma}(\mathcal{H}^\partial_\Sigma)$. Note that the condition $\left\{S^\partial_\Sigma, S^\partial_\Sigma\right\} = 0$ implies $(\hat{S}^\partial_\Sigma)^2 = 0$ only up to order $\hbar^2$ [19].

1.4.3 Classical BV-BFV Formalism

This construction of a BFV model at the boundary of a classical field theory is actually related to the BV formulation of the bulk theory. If the underlying manifold N has a boundary, the cohomological vector field Q_N on the BV space of fields that encoded the infinitesimal gauge transformations is no longer Hamiltonian (note that we added a subscript N in Q_N to emphasize the difference between bulk and boundary theories, and we will do the same with $\mathcal{F}_N$, Ω_N and S_N, instead of S_0). The variation of the BV action S_N contains boundary terms and the relation (1.14) must be corrected with a one-form, much like (1.23),

$$\imath_{Q_N}\Omega_N = \delta S_N + \tilde{\pi}^*\tilde{\alpha}^\partial_{\partial N}. \tag{1.25}$$

The one-form $\tilde{\alpha}^\partial_{\partial N} \in \Omega^1(\tilde{\mathcal{F}}^\partial_{\partial N})$ is defined on the the space $\tilde{\mathcal{F}}^\partial_{\partial N}$ of preboundary fields, roughly the restriction to the boundary ∂N of the fields in $\mathcal{F}_N$, and the projection map $\tilde{\pi} : \mathcal{F}_N \to \tilde{\mathcal{F}}^\partial_{\partial N}$ corresponds to this restriction. If $\tilde{\Omega}^\partial_{\partial N} = \delta\tilde{\alpha}^\partial_{\partial N}$ is degenerate, we define the space of boundary fields as the symplectic reduction $(\mathcal{F}^\partial_{\partial N}, \Omega^\partial_{\partial N})$ of $(\tilde{\mathcal{F}}^\partial_{\partial N}, \tilde{\Omega}^\partial_{\partial N})$, and we denote by $\pi : \mathcal{F}_N \to \mathcal{F}^\partial_{\partial N}$ the composition of $\tilde{\pi}$ with this reduction, but sometimes we are lucky and $\tilde{\Omega}^\partial_{\partial N}$ is already symplectic.

The symplectic manifold $(\mathcal{F}^\partial_{\partial N}, \Omega^\partial_{\partial N})$ actually coincides with the BFV space of fields described in the previous section. Moreover, we can pushforward Q by π to the BFV space of fields,

$$\pi_*(Q_N) =: Q^\partial_{\partial N},$$

to obtain the cohomological vector field that makes $(\mathcal{F}^\partial_{\partial N}, \Omega^\partial_{\partial N}, Q^\partial_{\partial N})$ a BFV manifold. This shows how the gauge transformations on the boundary are related to the ones in the bulk.

If the BFV structure $\Omega^\partial_{\partial N}$ happens to be exact with a Liouville form $\alpha^\partial_{\partial N} \in \Omega^1(\mathcal{F}^\partial_{\partial N})$, namely if $\Omega^\partial_{\partial N} = \delta\alpha^\partial_{\partial N}$, we can refine the relation (1.25),

$$\imath_{Q_N}\Omega_N = \delta S_N + \pi^*\alpha^\partial_{\partial N}. \tag{1.26}$$

Geometrically, we can interpret it as a modification of a BV manifold:

Definition 15 (Cattaneo, Mnev, Reshetikhin [18] [19]) *A BV-BFV manifold over a given exact BFV manifold* $(\mathcal{F}^\partial, \Omega^\partial = \delta\alpha^\partial, Q^\partial)$ *is a quintuple* $(\mathcal{F}, \Omega, S, Q, \pi)$ *where* $\mathcal{F}$ *is a* $\mathbb{Z}$*-graded manifold,* $\Omega \in \Omega^2(\mathcal{F})$ *is an odd symplectic structure of degree -1,* S *is an even function on* $\mathcal{F}$ *of degree 0,* $Q \in \mathrm{Vect}(\mathcal{F})$ *is a cohomological vector field of degree 1 and* $\pi : \mathcal{F} \to \mathcal{F}^\partial$ *is a surjective submersion that satisfy the relations* $\imath_Q\Omega = \delta S + \pi^*\alpha^\partial$ *and* $\pi_*(Q) = Q^\partial$*.*

Note that if Ω^∂ is not exact, we can often replace α^∂ by a connection of a complex line bundle over $\mathcal{F}^\partial$ and view $e^{\frac{i}{\hbar}S}$ as a section of the pullback bundle over $\mathcal{F}$ [19].

1.4.4 Quantum BV-BFV Formalism

This BV-BFV relation between bulk and boundary can be extended to the quantum level. The boundary BFV model can be quantized in the way described at the end of section 1.4.2, where we fix a polarization of $\mathcal{F}^\partial_{\partial N}$ such that the BFV charge $\hat{S}^\partial_{\partial N}$ squares to zero. For simplicity, we assume that we have a submanifold $\mathcal{K} \subset \mathcal{F}^\partial_{\partial N}$ transversal to the polarization. The graded Hilbert space of boundary states can be identified with the functions on this transversal submanifold, $\mathcal{H}^\partial_{\partial N} = \mathrm{Fun}(\mathcal{K})$.

If we have global Darboux coordinates ξ^a, λ_a on $\mathcal{F}^\partial_{\partial N}$ with $\Omega^\partial_{\partial N} = \sum_a \delta\lambda_a\delta\xi^a$, we can use canonical quantization, and the submanifold $\mathcal{K}$ is the subset where the 'momenta' vanish, $\lambda_a = 0$, so that $\mathcal{H}^\partial_{\partial N}$ consists of the functions of the 'positions' ξ^a. The canonical commutation relations can be imposed by replacing the momenta with functional derivatives by the positions, $\lambda_a \mapsto -i\hbar\frac{\delta}{\delta\xi^a}$.

Now the BV path integrals of the bulk theory should take value in the space $\mathcal{H}^\partial_{\partial N}$, much like in the nondegenerate case (1.24). In particular, the partition function becomes a boundary state $\psi_N \in \mathcal{H}^\partial_{\partial N} = \mathrm{Fun}(\mathcal{K})$ defined as

$$\psi_N(\phi) = \int_{\mathcal{L}_\phi} e^{\frac{i}{\hbar}S}\sqrt{\mu}_{\mathcal{L}_\phi}, \tag{1.27}$$

where $\mathcal{L}_\phi$ is a Lagrangian submanifold of the fiber over $\phi \in \mathcal{K}$ in the BV-BFV manifold $\mathcal{F}_N$ over $\mathcal{F}^\partial_{\partial N}$.

One can show that this state satisfies the condition $\hat{S}^\partial_{\partial N}\psi_N = 0$ for gauge invariant states [19], so that it defines a cohomology class

$$[\psi_N] \in H^0_{\hat{S}^\partial_{\partial N}}(\mathcal{H}^\partial_{\partial N})$$

in the Hilbert space of BFV boundary quantum states.

The construction of this Hilbert space thus becomes an essential problem in the quantization of gauge theories on manifold with boundaries. We treat it for the case of Chern-Simons theory in chapter 3, based on a joint work with Alekseev and Mnev [2]. Polarization involves the introduction of a complex structure, and the choice of a transversal submanifold $\mathcal{K}$ singles out a chiral component of the gauge field on the boundary. The space of BFV boundary quantum states is shown to coincide with the space of conformal blocks of the Wess-Zumino-Witten model coupled to an external chiral gauge field. In other words, path integrals in the bulk should coincide with path integrals of a related theory on the boundary, in a BV-BFV manifestation of the holographic principle, first discovered by Witten for the CS-WZW relation [43]. In particular, we are able to include Wilson lines in the BV formulation of the bulk Chern-Simons theory with the method discussed in section 1.3.4, and the associated BFV model is a case where the BFV structure is not exact. The effect on the BFV boundary quantum states is to add insertions to the WZW model at the extremities of the Wilson lines.

Chapter 2

T-duality through BV Morphisms and BV Pushforwards in Topological Field Theories

2.1 Introduction

T-duality (short for target space duality) was first discovered within the framework of string theory, where two theories compactified on circles (or more generally on tori) are equivalent under certain conditions. If the geometry of the target space is given by a principal circle bundle endowed with background metric and H-flux, gauging of the S^1 symmetry and integration of the newly established gauge connection leads to a T-dual model on a different circle bundle with different background fields, related to the ones of the initial theory through the Buscher rules [14]. The geometrical and topological content was formalized by Bouwknegt, Mathai and others [11], [12], [13]. The main result relates the H-flux of a model with the first Chern class of the circle bundle geometry of its T-dual. From there on, one can show that the twisted cohomologies on the target spaces of the two T-dual theories are isomorphic. Furthermore, the S^1 invariant differential forms underlying these cohomologies can be interpreted as Clifford modules for certain Courant algebroids, so it is possible to find an isomorphism of the Courant algebroids that is compatible with this isomorphism of the Clifford modules [21]. In the realm of generalized geometry, T-duality is therefore defined as an isomorphism of two Courant algebroids in a purely mathematical way, that is without reference to any physical model, and several results can be re-derived independently on the initial physical theory. For instance, the Buscher rules can be deduced from the behavior of a generalized metric on the Courant algebroid under this T-duality isomorphism.

In parallel, these topological structures have found themselves at the heart of several topological field theories (models that do not depend on a metric), such as the two-dimensional twisted Poisson sigma model [39] [31] or the three-dimensional Courant sigma model [38]. It might thus be tempting to repeat the procedure initially used to derive the Buscher rules on these models and retrieve only the topological content of T-duality. However, the price to pay for the topological nature of these models is a complicated set of symmetries (the kinetic terms of string sigma models with background fields break these symmetries) that renders their action degenerate. The Batalin-Vilkovisky formalism is thus ideally suited to deal with these theories. In this formalism, the space of classical fields is extended with so-called ghosts and antifields to a much richer BV space of fields equipped with an odd symplectic form (the BV structure) and a BV Laplacian. Quantization then requires the classical action to be extended to a BV action that has to satisfy a master equation. Even though the BV machinery is usually hard to apply, it provides us with a natural method to construct effective theories (through so-called "BV pushforwards") and a nice interpretation of T-duality as either a BV morphism, namely a symplectomorphism with respect to the BV structures, or the result of dual BV pushforwards. Our goal is therefore to express the topological aspects of T-duality as BV dualities of *ad hoc* topological models: the twisted Poisson sigma model with a trivial Poisson structure, which describes the topological sector of the sigma model for a string theory with background fields, and the Courant sigma model, built from a Courant algebroid following the AKSZ prescription.

We begin our exposition with a short introduction to the basic aspects of the BV formalism and the AKSZ construction in section 2.2. In section 2.3, we describe three different approaches to dualities offered by the BV formalism. As a first example, we show in section 2.4 how a BV morphism applied to a certain type of Courant sigma models reproduces an isomorphism of Courant algebroids first constructed by Cavalcanti and Gualtieri from geometric considerations. In section 2.5, we briefly review the main results regarding T-duality in physics and geometry in order to understand the motivation behind this isomorphism and to provide material for our second example involving dual BV pushforwards, that we treat in section 2.6. Finally, we extend this discussion to the case of general principal torus bundles in section 2.7.

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2.2 The Batalin-Vilkovisky Formalism

2.2.1 Basics

The Batalin-Vilkovisky formalism [9] was developed in the 80's as a tool for the quantization of degenerate quantum field theories, for instance gauge theories. Given a theory described in the Lagrangian formalism by a classical action S_{cl} defined over a space $\mathcal{F}_{\mathrm{cl}}$ of classical fields, the mathematical data for its classical BV formulation, the so-called *classical BV manifold*, contains three elements $(\mathcal{F}, \Omega, S)$. First, a BV space of fields $\mathcal{F}$, which is a $\mathbb{Z}$-graded (or sometimes $\mathbb{Z}_2$-graded) infinite-dimensional manifold, extends the space $\mathcal{F}_{\mathrm{cl}}$ of classical fields. The internal degree associated to this grading is called "ghost number" and is set to zero for the classical fields. Second, this BV space of fields carries a symplectic structure Ω of ghost number -1 (the BV structure), whose induced odd Poisson bracket $\{\cdot,\cdot\}$ has ghost number 1 (the BV bracket). Third, a BV action S defined on $\mathcal{F}$ extends the classical action in the sense that $S|_{\mathcal{F}_{\mathrm{cl}}} = S_{\mathrm{cl}}$ and satisfies the classical master equation

$$\{S, S\} = 0.$$

As a consequence of this classical master equation, the Hamiltonian vector field generated by S, $Q = \{S, \cdot\}$, is cohomological, $[Q, Q] = 0$.

If $\mathcal{F}$ is furthermore equipped with an integration measure μ, we can define a Laplace operator Δ (of degree 1) acting on a function f on $\mathcal{F}$ as

$$\Delta f = \frac{1}{2}\mathrm{div}_\mu \{f, \cdot\},$$

namely the divergence with respect to μ of the Hamiltonian vector field generated by f. To define the quantum BV theory, we need a measure μ such that this Laplacian squares to zero, $\Delta^2 = 0$. We will call the measures with this property *BV measures*. If this condition is satisfied, a theorem by Batalin and Vilkovisky [9] states that the integral over a Lagrangian submanifold $\mathcal{L}$ of $\mathcal{F}$ of a given function f on $\mathcal{F}$ is constant under continuous deformations of $\mathcal{L}$,

$$\int_{\mathcal{L}} \sqrt{\mu}_{\mathcal{L}}\, f = \int_{\mathcal{L}'} \sqrt{\mu}_{\mathcal{L}'}\, f,$$

provided f is BV-harmonic, $\Delta f = 0$. Here $\sqrt{\mu}_{\mathcal{L}}$ is the measure on $\mathcal{L}$ induced by μ, and $\mathcal{L}'$ is a deformation of $\mathcal{L}$. Furthermore, the integral of the BV Laplacian of a function vanishes,

$$\int_{\mathcal{L}} \sqrt{\mu}_{\mathcal{L}}\, \Delta g = 0.$$

The problem with a degenerate theory is that integrals such as the one of the partition function

$$Z = \int_{\mathcal{F}_{\mathrm{cl}}} \sqrt{\mu}_{\mathrm{cl}}\, e^{\frac{i}{\hbar} S_{\mathrm{cl}}}$$

do not make sense due to the degeneracy. As a remedy, we can find a Lagrangian submanifold $\mathcal{L}_{\mathrm{cl}}$ of $\mathcal{F}$ that contains $\mathcal{F}_{\mathrm{cl}}$, a BV measure μ on $\mathcal{F}$ that reproduces μ_{cl} when restricted to $\mathcal{F}_{\mathrm{cl}}$, and finally a quantum BV action W on $\mathcal{F}$ such that $W|_{\mathcal{L}_{\mathrm{cl}}} = S_{\mathrm{cl}}$ and

$$\Delta e^{\frac{i}{\hbar} W} = 0, \tag{2.1}$$

along with a Lagrangian submanifold $\mathcal{L}$ deformed from $\mathcal{L}_{\mathrm{cl}}$ on which W is non-degenerate, and the Batalin-Vilkovisky theorem ensures that

$$Z = \int_{\mathcal{F}_{\mathrm{cl}}} \mu_{\mathrm{cl}}\, e^{\frac{i}{\hbar} S_{\mathrm{cl}}} = \int_{\mathcal{L}} \sqrt{\mu}_{\mathcal{L}}\, e^{\frac{i}{\hbar} W}, \tag{2.2}$$

where the expression on the right-hand side can be computed, usually perturbatively.

The difficult task is to find a solution W of the quantum master equation (2.1), which is equivalent to

$$\frac{1}{2} \{W, W\} = i\hbar\, \Delta W, \tag{2.3}$$

where the relation with the classical master equation is obvious. In practice, one first looks for a solution S of the classical master equation, and then proceeds to solve the QME order by order in $\hbar$, with the expansion $W = S + \sum_{k=1}^{\infty} \hbar^k W_k$, provided no anomaly prevents the existence of a solution. In other words, we work with formal power series in $\hbar$ of functions, $W \in \mathrm{Fun}(\mathcal{F})\,[[\hbar]]$.

The mathematical data for the quantum BV formulation of a QFT is the *quantum BV manifold* $(\mathcal{F}, \Omega, \mu, W)$, which contains one more element than its classical counterpart, namely a BV measure μ, and where the classical BV action S satisfying the classical master equation has been replaced by the quantum

BV action W satisfying the quantum master equation (see for instance [40] for details).

Note that the BV Laplacian is singular, and its proper definition requires the use of some regularization procedure. Nevertheless, in the theories we are dealing with in this paper, the BV Laplacian can be regularized in such a way that the solution S of the CME will be BV harmonic and will thus already satisfy the QME.

Finally, the algebra of observables in the quantum BV formalism $\mathcal{O}_{\text{quant}}$ is defined as the subset of functions f on $\mathcal{F}$ that satisfy the condition

$$\Delta(f e^{\frac{i}{\hbar} W}) = 0, \tag{2.4}$$

which is equivalent to

$$-i\hbar\Delta f + \{W, f\} = 0. \tag{2.5}$$

This allows us to define their expectation value in a similar way as the partition function,

$$\langle f \rangle = \frac{1}{Z} \int_{\mathcal{L}} \sqrt{\mu}_{\mathcal{L}}\, f\, e^{\frac{i}{\hbar} W}. \tag{2.6}$$

Notice that in the classical limit, we obtain the condition

$$\{S, f\} = 0 \tag{2.7}$$

for the algebra of classical observables $\mathcal{O}_{\text{clas}}$, and that quantum observables may also be constructed as formal power series in $\hbar$, starting at zeroth order with a classical observable.

2.2.2 The AKSZ Construction

Generalities

While it is often a tedious work to find the BV extension of a given classical degenerate action, the AKSZ construction [6], based on geometrical considerations, leads to solutions of the classical master equation that sometimes involve interesting models, such as the Chern-Simons theory or the Poisson sigma model used to derive Kontsevich's formula for deformation quantization [16]. This construction is well suited to implement T-duality, as we will see that a symplectomorphism of the target space of an AKSZ model can be naturally lifted to a full BV morphism of the AKSZ space of fields. The AKSZ construction has been extensively treated in the literature, so here we will just give a brief explanation of the Courant sigma model [38], which we will use to illustrate a T-duality BV morphism in the AKSZ formalism.

We know that due to the classical master equation, the Hamiltonian (with respect to the BV structure) vector field $Q = \{S, \cdot\}$ generated by the BV action S is cohomological. The idea behind the AKSZ construction is therefore to build a cohomological vector field on a graded symplectic manifold, and see if it is Hamiltonian.

The AKSZ space of fields consists of maps from $T[1]\,\Sigma_{n+1}$, the tangent bundle of some $(n+1)$-dimensional closed[1] manifold Σ_{n+1} with the degree of its fibers shifted by one, to a graded symplectic manifold Y equipped with a symplectic structure ω_Y of degree n (this is the internal degree, also called ghost number, as opposed to the degree as a differential form, which is of course 2) and a cohomological Hamiltonian vector field Q_Y generated by a function Θ_Y of degree $n+1$ on Y, i.e. $\imath_{Q_Y}\omega_Y = \delta\Theta_Y$, where δ denotes the exterior derivative on Y. Moreover, we assume that a Liouville form α_Y is associated to $\omega_Y = \delta\alpha_Y$. Differential forms on the target space (such as ω_Y, α_Y or Θ_Y) can be lifted to the AKSZ space of fields

$$\mathcal{F} = \mathrm{Map}(T\,[1]\,\Sigma_{n+1}, Y)$$

via a pullback by the evaluation map

$$\mathrm{ev} : T\,[1]\,\Sigma_{n+1} \times \mathcal{F} \to Y$$

followed by an integration on the source space $T\,[1]\,\Sigma_{n+1}$ (see [16] for details). For the integration of a function on an odd tangent bundle, one normally uses the canonical measure μ. These functions are identified with differential forms on the base manifold Σ_{n+1}, and the top-form is extracted and integrated over Σ_{n+1}.

Explicitly, the AKSZ BV structure is defined as

$$\Omega = \int_{T[1]\Sigma_{n+1}} \mu\,\mathrm{ev}^*(\omega_Y). \tag{2.8}$$

Note that the exterior derivative on Y becomes the exterior variational derivative in the space of fields, which explains the choice of δ for its notation.

Before we can give the AKSZ action, we need to define the de Rham vector field on $\mathcal{F}$,

$$Q_D = \sum_a D\phi^a \frac{\delta}{\delta\phi^a},$$

[1] The case of manifolds with boundaries requires a careful treatment of the boundary conditions [16],[17] or can be treated in the BV-BFV formalism of Cattaneo, Mnev and Reshetikhin [18],[19].

where the sum runs over all the (super)fields of $\mathcal{F}$ (the fields ϕ^a can be interpreted as coordinate-fields of $\mathcal{F}$) and D is the de Rham vector field on the source space. Explicitly, if $u^\mu,\ \mu = 1, \ldots, n+1$ are some coordinates on Σ_{n+1} and θ^μ the corresponding odd coordinates (with ghost number 1) along the fibers of $T[1]\,\Sigma_{n+1}$, we define the de Rham vector field on $T[1]\,\Sigma_{n+1}$ as $D = \theta^\mu \frac{\partial}{\partial u^\mu}$.

The AKSZ action is then defined as

$$S = \int_{T[1]\Sigma_{n+1}} \mu \left(\imath_{Q_D} \mathrm{ev}^*(\alpha_Y) + \mathrm{ev}^*(\Theta_Y) \right). \tag{2.9}$$

We mention in passing that if $n = 1$, we can set $Y = T^*[1]\,M$ for some manifold M with ω being the canonical symplectic structure, and we obtain the AKSZ construction of the Poisson sigma model.

The Courant Sigma Model

Of interest in this paper is the case $n = 2$, which leads to the so-called Courant sigma model, based on a Courant algebroid.

We recall that a Courant algebroid over a manifold M is a vector bundle $\mathcal{E}$ over M equipped with a fiber-wise non-degenerate symmetric scalar product $\langle \cdot, \cdot \rangle_{\mathcal{E}} : \mathcal{E} \times \mathcal{E} \to M \times \mathbb{R}$, a bracket of sections $[\cdot, \cdot] : \Gamma\mathcal{E} \times \Gamma\mathcal{E} \to \Gamma\mathcal{E}$ and an anchor map $\rho : \mathcal{E} \to TM$ satisfying the axioms

$$\begin{aligned} [\phi, [\chi, \psi]] &= [[\phi, \chi], \psi] + [\chi, [\phi, \psi]], \\ [\phi, f\psi] &= \rho(\phi) f\, \psi + f\, [\phi, \psi], \\ [\phi, \phi] &= \tfrac{1}{2} D\langle \phi, \phi \rangle, \\ \rho(\phi)\langle \psi, \psi \rangle &= 2\langle [\phi, \psi], \psi \rangle, \end{aligned} \tag{2.10}$$

where ϕ, ψ, χ are sections of $\mathcal{E}$ and f a smooth function on M, and $D = \kappa \circ \rho^T \circ d$ with $\kappa : \mathcal{E}^* \to \mathcal{E}$ being the isomorphism induced by the inner product.

Given a closed three-form H, the direct sum $TM \oplus T^*M$ of a tangent and cotangent bundles of the same manifold M equipped with the canonical product

$$\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(\alpha(Y) + \beta(X))$$

and the Courant bracket

$$[X + \alpha, Y + \beta]_H = [X, Y] + \mathcal{L}_X \beta - \mathcal{L}_Y \alpha - \frac{1}{2} d(\imath_X \beta - \imath_Y \alpha) + \imath_X \imath_Y H$$

provides the standard example of a Courant algebroid, called an exact Courant algebroid, since the sequence

$$0 \to T^*M \xrightarrow{\rho^*} \mathcal{E} \xrightarrow{\rho} TM \to 0$$

is in this case exact, where we introduced the dual map of ρ, $\rho^* : T^*M \to \mathcal{E}^* \simeq \mathcal{E}$, and used the invariant scalar product on $\mathcal{E}$ to identify it with its dual. Furthermore, it has been shown [42] that any Courant algebroid that fits in such an exact sequence is isomorphic to $TM \oplus T^*M$ twisted by some closed three-form H, and that the corresponding equivalence classes are in one-to-one correspondence with the third de Rham cohomology classes of these twists, $[H] \in H^3(M)$, called Ševera's classes.

Back to the AKSZ construction, if $\mathcal{E}$ is a Courant algebroid over M, we need to take as the target space Y a subbundle of $T^*[2]\,\mathcal{E}[1]$ that corresponds to the isometric embedding $\mathcal{E} \hookrightarrow \mathcal{E}^* \oplus \mathcal{E}$ with respect to the Courant algebroid inner product on the left-hand side and the canonical product on the right-hand side. If $\mathcal{E}$ is an exact Courant algebroid, we can simply take $Y = T^*[2]\,T^*[1]\,M$.

In the superfield formalism, the field content of this model is given by a base map $X \in \mathrm{Map}(T[1]\,N \to M)$ and sections p of the pullback of the shifted cotangent bundle $X^*T^*[2]\,M$ and Ξ of the pullback of the shifted Courant algebroid $X^*\mathcal{E}[1]$. The space of fields supports the canonical BV structure

$$\Omega = \int_{T[1]N} \mu\, \left(\langle \delta p, \delta X \rangle + \frac{1}{2} \langle \delta\Xi, \delta\Xi \rangle_{\mathcal{E}} \right), \tag{2.11}$$

where the first product is the canonical pairing between tangent and cotangent vectors. The AKSZ construction leads to the action

$$S = \int_{T[1]N} \mu\, \left(\langle p, DX \rangle + \frac{1}{2} \langle \Xi, D\Xi \rangle_{\mathcal{E}} - \langle p, \rho(\Xi) \rangle + \langle \Xi, [\Xi,\Xi]_{\mathcal{E}} \rangle_{\mathcal{E}} \right), \tag{2.12}$$

built with the constructing blocks of a Courant algebroids: the anchor map, the Courant bracket and the fiber scalar product. One can show that the classical master equation is satisfied if and only if these elements satisfy the integrability conditions (2.10), and that Courant algebroids are uniquely encoded (up to Courant algebroid isomorphisms) in these Courant sigma models [38].

We illustrate the computations with the somewhat simpler (but relevant) case of an exact Courant algebroid $T^*M \oplus TM$ twisted by H. First, we may decompose the section Ξ of $X^*\mathcal{E}[1]$ into its tangent and cotangent parts, ξ and Θ respectively. If M locally admits coordinates x^i, we can use them to express the superfields, X^i, Θ_i, ξ^i and p_i of ghost number 0, 1, 1 and 2 respectively. It is straightforward to write the BV structure

$$\Omega = \int_{T[1]N} \mu\, \left(\delta p_i\, \delta X^i - \delta\xi^i\, \delta\Theta_i \right) \tag{2.13}$$

and the action

$$S = \int_{T[1]N} \mu \left(p_i(DX^i - \xi^i) + \frac{1}{2}\xi^i\, D\Theta_i + \frac{1}{2}\Theta_i\, D\xi^i + \frac{1}{6}H_{ijk}(X)\xi^i\xi^j\xi^k \right). \tag{2.14}$$

Since integration along the odd dimensions of $T\,[1]\,N$ lowers the ghost number by $\dim(N) = 3$, the BV structure has a ghost number -1 and the BV action 0 as expected.

To verify the classical master equation, we first compute the functional derivatives of the BV action in the superfield formalism. Due to different commutation rules, we need to define left- and right-derivatives separately. The trick is to compute the exterior derivative in the space of fields $\mathcal{F}$. If we denote by ϕ^a a generic superfield in $\mathcal{F}$ and assume that a runs over all of them, we may define these derivatives as

$$\delta S = \int_{T[1]N} \mu \sum_a \delta\phi^a \frac{\overrightarrow{\delta}S}{\delta\phi^a} = \int_{T[1]N} \mu \sum_a \frac{S\overleftarrow{\delta}}{\delta\phi^a}\delta\phi^a. \tag{2.15}$$

Note that in the superfield formalism with an odd-dimensional N on the source side and a functional of even ghost number (such as an action), $\delta\phi^a$ always commutes with the corresponding functional derivative, and we have

$$\frac{\overrightarrow{\delta}S}{\delta\phi^a} = \frac{S\overleftarrow{\delta}}{\delta\phi^a},$$

so we could as well drop the small arrows.

Explicitly, we find

$$\begin{aligned}
\frac{\overrightarrow{\delta}S}{\delta p_i} &= DX^i - \xi^i, \\
\frac{\overrightarrow{\delta}S}{\delta X^i} &= -Dp_l + \tfrac{1}{6}\partial_l H_{ijk}(X)\xi^i\xi^j\xi^k, \\
\frac{\overrightarrow{\delta}S}{\delta\Theta_i} &= D\xi^i, \\
\frac{\overrightarrow{\delta}S}{\delta\xi^i} &= D\Theta_i - p_i + \tfrac{1}{2}H_{ijk}(X)\xi^j\xi^k,
\end{aligned}$$

where we had to perform a few integrations by parts, using the fact that $\partial N = \emptyset$.

Finally, to compute the BV bracket of S with itself, we need to invert the

BV structure, which we can do in the superfield formalism,

$$\begin{aligned}
\frac{1}{2}\{S,S\} &= \int_{T[1]N} \mu \left(\frac{S\overleftarrow{\delta}}{\delta X^i}\frac{\overrightarrow{\delta}S}{\delta p_i} - \frac{S\overleftarrow{\delta}}{\delta \Theta_i}\frac{\overrightarrow{\delta}S}{\delta \xi^i} \right) \\
&= \int_{T[1]N} \mu \left(\left(-Dp_l + \frac{1}{6}\partial_l H_{ijk}(X)\xi^i\xi^j\xi^k \right) \left(DX^l - \xi^l \right) \right. \\
&\qquad \left. -D\xi^i \left(D\Theta_i - p_i + \frac{1}{2}H_{ijk}(X)\xi^j\xi^k \right) \right) \\
&= \int_{T[1]N} \mu \, \frac{1}{6}\partial_l H_{ijk}(X)\xi^l\xi^i\xi^j\xi^k \\
&= 0.
\end{aligned}$$

In the second to last line, we used Stokes' theorem to eliminate D-exact terms since $\partial N = \emptyset$, and the last equality follows from the fact that $dH = 0$.

2.3 Dualities in the BV Formalism

Two quantum field theories are called dual to each other if they describe equivalent physics, or equivalent topological invariants in the case of topological field theories, even though they are seemingly different. The BV formalism provides two natural ways to obtain dual theories, that we will describe here. The first one involves effective field theories, derived through a process called "BV pushforward", and the second one involves BV morphisms. Of course, one can combine BV pushforwards with BV morphisms to obtain a third composite way.

2.3.1 BV Pushforwards

The procedure to construct effective actions in the BV formalism was first suggested by Losev [35] and later used in [36] and in [20].

Suppose that the space of fields admits a splitting

$$\mathcal{F} = \mathcal{F}_{\mathrm{IR}} \oplus \mathcal{F}_{\mathrm{UV}}$$

into infrared and ultraviolet degrees of freedom, compatible with a decomposition of the BV structure

$$\Omega = \Omega_{\mathrm{IR}} + \Omega_{\mathrm{UV}}$$

in the sense that Ω_{IR} is a BV structure on $\mathcal{F}_{\mathrm{IR}}$ and Ω_{UV} is one on $\mathcal{F}_{\mathrm{UV}}$, and that we have a solution W of the QME on $\mathcal{F}$. Then we can integrate $e^{\frac{i}{\hbar}W}$ over a Lagrangian submanifold of the ultraviolet sector of the space of fields to find an effective BV action in the infrared sector,

$$e^{\frac{i}{\hbar}W_{\text{eff}}} = \int_{\mathcal{L}_{\text{UV}}\subset\mathcal{F}_{\text{UV}}} \sqrt{\mu}_{\mathcal{L}_{\text{UV}}}\, e^{\frac{i}{\hbar}W}. \tag{2.16}$$

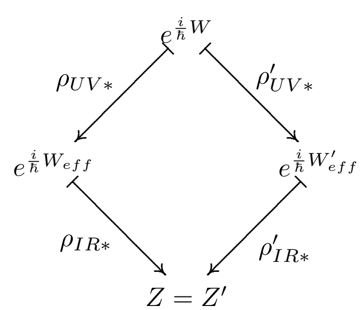
One can show that the effective action W_{eff} satisfies the QME associated to $\mathcal{F}_{\text{IR}}$. In the physics language, one says that the ultraviolet degrees of freedom have been integrated out. In mathematics, one also talks about a "BV pushforward" by the projection map $\rho_{\text{UV}} : \mathcal{F} \to \mathcal{F}_{\text{IR}}$ onto the infrared sector of the space of fields,

$$e^{\frac{i}{\hbar}W_{\text{eff}}} = \rho_{\text{UV}*}(e^{\frac{i}{\hbar}W}). \tag{2.17}$$

One step farther, we can pick a Lagrangian submanifold $\mathcal{L}_{\text{IR}}$ of the infrared sector $\mathcal{F}_{\text{IR}}$ and perform the functional integration of $e^{\frac{i}{\hbar}W_{\text{eff}}}$ thereon to compute the partition function of the full model. This integration can also be represented by a pushforward map $\rho_{\text{IR}*}$,

$$Z = \rho_{\text{IR}*}(e^{\frac{i}{\hbar}W_{\text{eff}}}) = \rho_{\text{IR}*} \circ \rho_{\text{UV}*}(e^{\frac{i}{\hbar}W}).$$

This last step corresponds to the computation of the partition function on the Lagrangian submanifold $\mathcal{L}_{\text{IR}} \times \mathcal{L}_{\text{UV}}$ of the full space of fields $\mathcal{F}$. As a consequence of the Batalin-Vilkovisky theorem, the value of Z does not depend on the particular splitting $\mathcal{F} = \mathcal{F}_{\text{IR}} \oplus \mathcal{F}_{\text{UV}}$, and a different choice $\mathcal{F} = \mathcal{F}'_{\text{IR}} \oplus \mathcal{F}'_{\text{UV}}$ leads to the same result,



(2.18)

as long as the Lagrangian submanifolds on the whole of $\mathcal{F}$ are can be deformed one into the other. The models with action W_{eff} and W'_{eff} hence have the same partition function.

We should also include observables, if we want to compare correlation functions of the two models. We mention this only for the sake of completeness, and we will not go into many details, as the examples we will be treating below do not involve observables, the topological information we are interested in being completely encoded in the action.

Starting with an observable $f \in \mathcal{O}_{\text{quant}}$ of the BV model on $\mathcal{F}$, we can pushforward $f\, e^{\frac{i}{\hbar}W}$ by ρ_{UV} and ρ'_{UV} to find $f_{eff} e^{\frac{i}{\hbar}W_{eff}}$ and $f'_{eff} e^{\frac{i}{\hbar}W'_{eff}}$ with identical correlation functions $\langle f_{eff}\rangle_{W_{eff}} = \langle f'_{eff}\rangle_{W'_{eff}} = \langle f\rangle_W$ (the subscript denotes the action of the model in which the corresponding correlation function is to be computed), and thus construct the algebras of observables $\mathcal{O}_{\text{quant},eff}$ and $\mathcal{O}'_{\text{quant},eff}$. Now in order to obtain truly dual theories, we should choose splittings of $\mathcal{F}$ and take $\mathcal{O}_{\text{quant}}$ to be only a subalgebra of all the possible observables of the whole model in such a way that the algebras $\mathcal{O}_{\text{quant},eff}$ and $\mathcal{O}'_{\text{quant},eff}$ are isomorphic.

2.3.2 BV Morphisms

Classical BV Morphisms

Given two BV spaces of fields $(\mathcal{F}, \Omega)$ and $(\mathcal{F}', \Omega')$, we call a map

$$\Phi : \mathcal{F} \to \mathcal{F}'$$

a classical BV morphism if it is a symplectomorphism with respect to the BV structures Ω and Ω', namely if Φ is a diffeomorphism and

$$\Phi^*(\Omega') = \Omega, \tag{2.19}$$

where Φ^* stands for the pullback by Φ of differential forms. If we know a classical BV action S', solution of the classical master equation on $\mathcal{F}'$, $\{S', S'\}_{\Omega'} = 0$, we immediately get a solution of the classical master equation on $\mathcal{F}$ by pulling it back with Φ, because

$$\{\Phi^*(S'), \Phi^*(S')\}_{\Phi^*(\Omega')} = \Phi^*\left(\{S', S'\}_{\Omega'}\right) = 0,$$

and we obtain a morphism of classical BV manifolds,

$$\Phi : (\mathcal{F}, \Omega, S) \to (\mathcal{F}', \Omega', S').$$

For the same reason, a classical observable $f' \in \mathcal{O}'_{\text{clas}}$ on $\mathcal{F}'$ that satisfies $\{S', f'\}_{\Omega'} = 0$ can be pulled back by Φ to a function on $\mathcal{F}$ that automatically satisfies $\{\Phi^*(S'), \Phi^*(f')\}_{\Phi^*(\Omega')} = 0$.

The two actions S and S' then describe the same dynamics and symmetries and isomorphic algebras of classical observables. In the case of topological field theories, this formalism can encode the diffeomorphism invariance, and Φ would be an automorphism of $\mathcal{F}$, that could be continuously transformed into the identity. Of greater interest is the situation where $\mathcal{F}$ and $\mathcal{F}'$ are different, or when Φ cannot be cast into a continuous family of automorphisms of $\mathcal{F}$ of the identity. In these cases, we say that the models related by Φ are classically dual to each other. We will give examples farther.

When the duality arises out of the mathematical structures on the target space of a sigma model, we speak of target space duality, or shorter T-duality.

Quantum BV Morphisms

One step farther, if $\mathcal{L} \subset \mathcal{F}$ is a Lagrangian submanifold with respect to the symplectic structure $\Omega = \Phi^*(\Omega')$, then $\Phi(\mathcal{L}) \subset \Phi(\mathcal{F}) = \mathcal{F}'$ is also Lagrangian, with respect to Ω', so this classical duality might be extended to the quantum level.

A quantum BV morphism

$$\hat{\Phi} : (\mathcal{F}, \Omega, \mu, W) \to (\mathcal{F}', \Omega', \mu', W')$$

between two quantum BV manifolds is defined as a formal power series of maps $\hat{\Phi} \in \mathrm{Map}(\mathcal{F}, \mathcal{F}')\,[[\hbar]]$ such that $\hat{\Phi}^*(\Omega') = \Omega$, $\hat{\Phi}_*(\mu) = \mu'$ and $\hat{\Phi}^*(W') = W$.

In this case, the partition function (as well as all other correlation functions) is left invariant by $\hat{\Phi}$,

$$\begin{aligned} Z = \int_{\mathcal{L}\subset\mathcal{F}} \sqrt{\mu}_{\mathcal{L}} e^{\frac{i}{\hbar}W} &= \int_{\mathcal{L}\subset\mathcal{F}} \sqrt{\mu}_{\mathcal{L}} e^{\frac{i}{\hbar}\hat{\Phi}^*(W')} \\ &= \int_{\hat{\Phi}(\mathcal{L})\subset\hat{\Phi}(\mathcal{F})} \sqrt{\hat{\Phi}_*(\mu)}_{\hat{\Phi}(\mathcal{L})} e^{\frac{i}{\hbar}W'} = Z'. \end{aligned} \tag{2.20}$$

Unfortunately, quantum duality between W and W' does not necessarily follow from classical duality of their tree-level part, $W|_{\hbar=0} = \Phi^*(W'|_{\hbar=0})$, since most of the time a regularization scheme enters the game. Moreover, μ, Ω and their duals might need to receive quantum corrections in $\hbar$. Consequently, quantum duality should always be checked independently from classical duality, which is possible order by order in $\hbar$.

Nevertheless, in many interesting examples, the BV Laplacian can be regularized in such a way that the classical BV action is BV harmonic and thus already satisfies the quantum master equation, so that quantum duality actually follows from classical duality with $\hat{\Phi} = \Phi$.

Target Space Duality

With the AKSZ construction, we see that a trick to build BV morphisms is to find symplectomorphisms of target spaces of AKSZ models,

$$\Phi : (Y, \omega_Y) \to (Y', \omega_{Y'}).$$

Such a symplectomorphism can be lifted to a BV morphism between the two AKSZ spaces of fields,

$$\Phi : \mathrm{Map}(T[1]\,N, Y) \to \mathrm{Map}(T[1]\,N, Y').$$

Since the duality between the two resulting BV theories comes from a symplectomorphism of their target spaces, we call this duality target space duality, or T-duality. We will see a concrete example in section 2.4.

2.3.3 Combination of BV Morphisms and BV Pushforwards

In a last step, we can naturally combine BV pushforwards with BV morphisms to construct dual BV effective theories. We start with a quantum BV morphism

$$\hat{\Phi} : (\mathcal{F}, \Omega, W) \to (\mathcal{F}', \Omega', W')$$

Essentially we get the same example of quantum duality as in the previous section, in particular the partition functions Z and Z' are equal. On each side of this BV morphism $\hat{\Phi}$, we choose a splitting of the space of fields into infrared and ultraviolet sectors, $\mathcal{F} = \mathcal{F}_{\mathrm{IR}} \oplus \mathcal{F}_{\mathrm{UV}}$ and $\mathcal{F} = \mathcal{F}'_{\mathrm{IR}} \oplus \mathcal{F}'_{\mathrm{UV}}$. Through respective BV pushforwards onto the infrared sectors, we obtain effective theories that admit the same partition function,

$$\begin{array}{ccc} e^{\frac{i}{\hbar}W} & \xleftarrow{\ \hat{\Phi}^*\ } & e^{\frac{i}{\hbar}W'} \\ \rho_{UV*} \downarrow & & \downarrow \rho'_{UV*} \\ e^{\frac{i}{\hbar}W_{eff}} & & e^{\frac{i}{\hbar}W'_{eff}} \\ & \rho_{IR*} \searrow \quad \swarrow \rho'_{IR*} & \\ & Z = Z'. & \end{array} \tag{2.21}$$

Note that a natural choice of UV sectors makes use of the BV morphism,

$$\mathcal{F}'_{\mathrm{UV}} = \hat{\Phi}(\mathcal{F}_{\mathrm{UV}}).$$

This ensures that the effective theories are truly dual to each other, in particular we obtain isomorphic algebras of observables.

2.4 T-duality and Courant Sigma Models

One of the simplest examples of duality expressed as a BV morphism involves the Courant sigma model based on a Courant algebroid of the form $(TE \oplus T^*E)/S^1$, where E is a principal circle bundle, and actually reproduces the Courant algebroid isomorphism first discussed by Cavalcanti and Gualtieri [21].

2.4.1 The Courant Sigma Model Based on $(TE \oplus T^*E)/S^1$

We start with a principal circle bundle

$$\begin{array}{ccc} S^1 & \longrightarrow & E \\ & & \Big\downarrow \rho \\ & & M \end{array}$$

over a manifold M, supporting a closed three-form $H \in \Omega^3_{\text{closed}}(E)$ and equipped with a connection $\mathcal{A} \in \Omega^1(E)$ with curvature $F = d\mathcal{A}$. Evidently, we can use H to define a Courant bracket on $TE \oplus T^*E$ and make it an exact Courant algebroid over E, as described in section 2.2.2. Now, if H is S^1-invariant, we can actually restrict the structures defining the Courant algebroid (i.e. scalar product, Courant bracket and anchor map) to S^1-invariant sections of TE and T^*E, and thus obtain a Courant algebroid over M with total space $(TE \oplus T^*E)/S^1$, which is no longer exact.

Once we get the BV structure and BV action of the associated Courant sigma model, it will be more or less clear how to define a BV morphism to a dual Courant sigma model, but to reach this goal, we first need explicit expressions for the three structures defining a Courant algebroid.

A couple of tricks simplify this task. First, since H is S^1-invariant, we can use the connection $\mathcal{A}$ to express its component along the fibers of E and decompose it into two terms

$$H = H_{(3)} + \mathcal{A} \wedge H_{(2)},$$

where $H_{(3)} \in \Omega^3(M)$ and $H_{(2)} \in \Omega^2(M)$ are two basic forms.

Second, we notice that the involved quotient bundles can be decomposed[2] as

$$TE/S^1 \cong TM \oplus \langle \partial_{\mathcal{A}} \rangle, \quad T^*E/S^1 \cong T^*M \oplus \langle \mathcal{A} \rangle, \tag{2.22}$$

[2]This can be shown with an Atiyah exact sequence.

where $\partial_{\mathcal{A}}$ is an invariant period-1 generator of the circle action. Under this splitting, sections of $(TE \oplus T^*E)/S^1$ can be identified with the following expressions,

$$\begin{aligned}\mathcal{X} &= X + f\,\partial_{\mathcal{A}} + \alpha + s\,\mathcal{A},\\ \mathcal{Y} &= Y + g\,\partial_{\mathcal{A}} + \beta + t\,\mathcal{A},\end{aligned}$$

where X and Y are vector fields on M, α and β one-forms on M and f, g, s and t real-valued functions on M.

We are now in a position to give an explicit expression of their scalar product,

$$\langle \mathcal{X}, \mathcal{Y} \rangle = \frac{1}{2}(\alpha(Y) + \beta(X) + s\,g + t\,f), \tag{2.23}$$

as well as of their Courant bracket [29],

$$\begin{aligned}[\mathcal{X}, \mathcal{Y}] = &[X, Y] + (X(g) - Y(f) + \imath_X \imath_Y F)\,\partial_{\mathcal{A}}\\ &+ \mathcal{L}_X\beta - \mathcal{L}_Y\alpha + t\imath_X F - s\imath_Y F - \frac{1}{2}\,(\imath_X\beta - \imath_Y\alpha)\\ &+ \frac{1}{2}\,(df\,t + gds - fdt - dg\,s) + \imath_X\imath_Y H_{(3)} + g\imath_X H_{(2)} - f\imath_Y H_{(2)}\\ &+ \big(X(t) - Y(s) + \imath_X\imath_Y H_{(2)}\big)\,\mathcal{A},\end{aligned} \tag{2.24}$$

and finally of the anchor map,

$$\rho(\mathcal{X}) = X. \tag{2.25}$$

Knowing the constituents of this Courant algebroid, we may construct a Courant sigma model for $(TE \oplus T^*E)/S^1$. The procedure only requires a small adaptation from the sigma model associated to an exact Courant algebroid. The space of fields is the mapping space between the odd tangent bundle of a three-dimensional manifold N and the degree 2 cotangent bundle of the quotiented degree 1 cotangent bundle of the circle bundle E, namely

$$\mathcal{F} = \mathrm{Map}\left(T\,[1]\,N, T^*\,[2]\,(T^*\,[1]\,E/S^1)\right).$$

We can use the decomposition (2.22) to identify this space of fields with

$$\mathcal{F} \cong \mathrm{Map}\,(T\,[1]\,N, T^*\,[2]\,M \oplus T\,[1]\,M \oplus T^*\,[1]\,M \oplus \langle\partial_{\mathcal{A}}\rangle\,[1] \oplus \langle\mathcal{A}\rangle\,[1])\,. \tag{2.26}$$

Like in the example of the Courant sigma model based on an exact Courant algebroid, we will work with superfields, namely a base map $X \in \mathrm{Map}(T\,[1]\,N, M)$, that we complete with fiber maps p, ξ and Θ such that

$$\begin{aligned}(X, p) &\in \mathrm{Map}(T\,[1]\,N, T^*\,[2]\,M),\\ (X, \xi) &\in \mathrm{Map}(T\,[1]\,N, T\,[1]\,M),\\ (X, \Theta) &\in \mathrm{Map}(T\,[1]\,N, T^*\,[1]\,M),\end{aligned}$$

and two odd functions $\phi, \psi \in \mathrm{Fun}(T\,[1]\,N, \mathbb{R}\,[1])$ that we can combine with $\partial_{\mathcal{A}}$ and $\mathcal{A}$ respectively to obtain superfields associated to the last two components of the target space (2.26). In effect, we may identify the space of fields with the mapping space

$$\mathcal{F} \cong \mathrm{Map}\left(T\,[1]\,N, T^*\,[2]\,M \oplus T\,[1]\,M \oplus T^*\,[1]\,M \oplus \mathbb{R}\,[1] \oplus \mathbb{R}\,[1]\right),$$

with coordinate-fields $(X, p, \xi, \Theta, \phi, \psi)$.

The target space being a cotangent bundle, the space of fields carries the canonical BV structure

$$\Omega = \int_{T[1]N} \mu\,\left(\delta p_i\,\delta X^i - \delta\xi^i\,\delta\Theta_i - \delta\phi\,\delta\psi\right). \tag{2.27}$$

The AKSZ action can be constructed by using the Courant bracket (2.24) and anchor map (2.25) in the formula (2.12),

$$\begin{aligned} S = \int_{T[1]N} \mu\,\Big(& p_i\,DX^i + \frac{1}{2}\xi^i\,D\Theta_i + \frac{1}{2}\Theta_i\,D\xi^i + \frac{1}{2}\phi\,D\psi + \frac{1}{2}\psi\,D\phi \\ & -p_i\xi^i + \psi\,\frac{1}{2}F_{ij}\xi^i\xi^j + \frac{1}{6}H_{(3)ijk}\xi^i\xi^j\xi^k + \phi\,\frac{1}{2}H_{(2)ij}\xi^i\xi^j\Big). \end{aligned} \tag{2.28}$$

In a similar way as the calculation ran in section 2.2.2 for exact Courant algebroids, one can check that the classical master equation

$$\begin{aligned} \frac{1}{2}\,\{S, S\} = \int_{T[1]N} \mu\,\Big(& -\psi\,\frac{1}{2}\partial_l F_{ij}\xi^l\xi^i\xi^j - \phi\,\frac{1}{2}\partial_l H_{(2)ij}\xi^l\xi^i\xi^j \\ & +\frac{1}{6}\partial_l H_{(3)ijk}\xi^l\xi^i\xi^j\xi^k - \frac{1}{2}F_{ij}\xi^i\xi^j\frac{1}{2}H_{(2)kl}\xi^k\xi^l\Big) = 0 \end{aligned}$$

is satisfied provided

$$\begin{aligned} dF &= 0, \\ dH_{(2)} &= 0, \\ dH_{(3)} - F \wedge H_{(2)} &= 0. \end{aligned} \tag{2.29}$$

The first condition follows from the fact that the curvature of a connection on a principal circle bundle is closed, and the other two from the fact that the twist H is also closed.

2.4.2 A T-duality BV Morphism

If we take a closer look at the BV structure (2.27), we see that it is invariant under the exchange $\phi \leftrightarrow \psi$. The first obvious step in defining a BV morphism

$\Phi : \mathcal{F} \to \hat{\mathcal{F}}$ would therefore be to require that the dual space of fields $\hat{\mathcal{F}}$ admits the same identification as the original space of fields $\mathcal{F}$,

$$\hat{\mathcal{F}} \simeq \mathrm{Map}\left(T\left[1\right]N, T^*\left[2\right]M \oplus T\left[1\right]M \oplus T^*\left[1\right]M \oplus \mathbb{R}\left[1\right] \oplus \mathbb{R}\left[1\right]\right) \tag{2.30}$$

with coordinate-fields $(X, p, \xi, \Theta, \hat{\phi}, \hat{\psi})$, then set

$$\Phi(\phi) = \hat{\phi} = \psi \quad \text{and} \quad \Phi(\psi) = \hat{\psi} = \phi,$$

and ask that it leaves the other fields invariant,

$$\Phi|_{\mathrm{Map}(T[1]N, T^*[2](T^*[1]M))} = \mathrm{Id}.$$

The dual space of fields $\hat{\mathcal{F}}$ carries the natural BV structure

$$\hat{\Omega} = \int_{T[1]N} \mu \, \left(\delta p_i \, \delta X^i - \delta \xi^i \, \delta \Theta_i - \delta \hat{\phi} \, \delta \hat{\psi}\right), \tag{2.31}$$

and Φ is automatically a BV morphism, $\Phi^*(\hat{\Omega}) = \Omega$.

In order to determine the dual BV theory, we still need to find an action $\hat{S}$ on $\hat{\mathcal{F}}$ such that $\Phi^*(\hat{S}) = S$. A functional of the coordinate fields of $\hat{\mathcal{F}}$ that satisfies this requirement is

$$\begin{aligned} \hat{S} = \int_{T[1]N} \mu \, \Bigg(& p_i \, DX^i + \frac{1}{2}\xi^i \, D\Theta_i + \frac{1}{2}\Theta_i \, D\xi^i + \frac{1}{2}\hat{\phi} \, D\hat{\psi} + \frac{1}{2}\hat{\psi} \, D\hat{\phi} \\ & - p_i \xi^i + \hat{\psi} \, \frac{1}{2}\hat{F}_{ij}\xi^i\xi^j + \frac{1}{6}\hat{H}_{(3)ijk}\xi^i\xi^j\xi^k + \hat{\phi} \, \frac{1}{2}\hat{H}_{(2)ij}\xi^i\xi^j \Bigg), \end{aligned} \tag{2.32}$$

provided Φ acts also on the background fields (by 'background fields', we understand fields defined on the target space of the model),

$$F \mapsto \Phi(F) = \hat{F}, \quad H_{(2)} \mapsto \Phi(H_{(2)}) = \hat{H}_{(2)} \quad H_{(3)} \mapsto \Phi(H_{(3)}) = \hat{H}_{(3)},$$

such that

$$\hat{F} = H_{(2)}, \quad \hat{H}_{(2)} = F, \quad \hat{H}_{(3)} = H_{(3)}. \tag{2.33}$$

In other words, the roles of the curvature F and the component $H_{(2)}$ of the twist H have been exchanged. If the twist H is chosen in such a way that $H_{(2)} = \hat{F}$ has integral periods (which is the case of the ones relevant to physics, due to the Wess-Zumino consistency condition), it can be related to the first Chern class of a principal circle bundle $\hat{E}$ with connection $\hat{\mathcal{A}}$ satisfying $\hat{F} = d\hat{\mathcal{A}}$. Then one can see that the dual action $\hat{S}$ describes a Courant sigma model for the Courant algebroid $(T\hat{E} \oplus T^*\hat{E})/S^1$ twisted by

$$\hat{H} = \hat{H}_{(3)} + \hat{H}_{(2)} \wedge \hat{\mathcal{A}} = H_{(3)} + F \wedge \hat{\mathcal{A}}.$$

The target space of the corresponding AKSZ construction

$$\hat{\mathcal{F}} = \mathrm{Map}\left(T[1]\,N, T^*[2]\,(T^*[1]\,\hat{E}/S^1)\right)$$

can be decomposed in a similar way as the original model,

$$\hat{\mathcal{F}} \simeq \mathrm{Map}\left(T[1]\,N, T^*[2]\,M \oplus T[1]\,M \oplus T^*[1]\,M \oplus \langle\partial_{\hat{\mathcal{A}}}\rangle\,[1] \oplus \langle\hat{\mathcal{A}}\rangle\,[1]\right),$$

which allows us to describe the geometric structure of Φ,

$$\begin{aligned}\Phi(\mathrm{Map}\,(T[1]\,N, \langle\partial_{\mathcal{A}}\rangle\,[1])) &= \mathrm{Map}(T[1]\,N, \langle\hat{\mathcal{A}}\rangle\,[1]),\\ \Phi(\mathrm{Map}\,(T[1]\,N, \langle\mathcal{A}\rangle\,[1])) &= \mathrm{Map}\left(T[1]\,N, \langle\partial_{\hat{\mathcal{A}}}\rangle\,[1]\right),\end{aligned}$$

and which ensures that the identification (2.30) is valid.

So if we assume that both spaces of fields are identified with the same model space of fields,

$$\mathcal{F} \simeq \mathrm{Map}\,(T[1]\,N, T^*[2]\,M \oplus T[1]\,M \oplus T^*[1]\,M \oplus \mathbb{R}\,[1] \oplus \mathbb{R}\,[1]) \simeq \hat{\mathcal{F}},$$

we can interpret the BV structures Ω and $\hat{\Omega}$ and BV actions S and $\hat{S}$ as functionals on this model space of fields, on which we even have the equalities $\hat{\Omega} = \Omega$ and $\hat{S} = S$. These ensure that the Courant algebroids $(TE \oplus T^*E)/S^1$ and $(T\hat{E} \oplus T^*\hat{E})/S^1$ twisted by H and $\hat{H}$ respectively are actually isomorphic, as the Courant algebroid structures are encoded in the associated Courant sigma model actions [38].

This isomorphism is the same as the one derived by Cavalcanti and Gualtieri in [21] through arguments solely based on geometrical considerations inspired by T-duality in string theory. Our field theoretic approach, on the other hand, follows the same spirit as the derivation of the Buscher rules from a duality of sigma models. We now give a short review of T-duality in physics and geometry to illustrate the difference between the two approaches and to motivate our next example of duality in the BV formalism, based this time on BV pushforwards combined with BV morphisms.

2.5 T-duality in Physics and Geometry

In this review, we focus on the results that can be expressed as examples of dual BV theories. References for standard material are provided.

2.5.1 Periodic Scalar Field

The simplest example of T-duality (see [28] for details) arises when one considers the bosonic theory of a single scalar field ϕ of periodicity 2π on a worldsheet Σ, with action

$$S_\phi = \frac{R^2}{2}\int_\Sigma d\phi \wedge *d\phi,$$

where the Hodge star operator is denoted by $*$. We assume the worldsheet metric g_Σ to be of Lorentzian signature so that $*$ squares to one when applied on one-forms. We recognize the action of a sigma model whose target space is a circle of radius R. We can introduce an auxiliary one-form field η to construct the first order action

$$S' = \frac{1}{2R^2}\int_\Sigma \eta \wedge *\eta + \int_\Sigma \eta \wedge d\phi.$$

If we complete the square to integrate out η, we recover the original action.

On the other hand, if we first integrate over ϕ, it imposes the constraint $d\eta = 0$, which can be shown to be equivalent [28] to

$$\eta = d\vartheta$$

for some dual periodic scalar ϑ, also of period 2π. Inserting this condition into the extended action S' leads to the T-dual action

$$S_\vartheta = \frac{1}{2R^2}\int_\Sigma d\vartheta \wedge *d\vartheta,$$

another sigma model with a circle for its target space, but with radius $1/R$. One can also find a direct relation between ϕ and θ,

$$Rd\phi = \frac{1}{R} * d\vartheta. \tag{2.34}$$

Since $Rd\phi$ and $R * d\phi$ are the conserved currents of the theory with action S_ϕ that count the momentum and the winding number respectively, the relation (2.34) means that T-duality not only transforms the radius $R \leftrightarrow 1/R$, but also exchanges the momentum and the winding number around the circle [28].

2.5.2 Principal Circle Bundles, Buscher Rules, Curvature and H-flux

The sigma models of the previous section can be interpreted as string theories. However, a consistent string theory cannot admit a one-dimensional target space such as S^1. It has to be completed with a nine-dimensional manifold M to form a ten-dimensional target space $\mathcal{M} = S^1 \times M$ compatible with superstring theories. More generally, one can also consider the ten-dimensional total space E of some principal circle bundle over M,

$$\begin{array}{ccc} S^1 & \longrightarrow & E \\ & & \big\downarrow \rho \\ & & M. \end{array}$$

In this case, the simple exchange $R \leftrightarrow 1/R$ for a single scalar field ϕ and its T-dual ϑ is replaced by the more complicated Buscher rules [14].

The starting point for their derivation is the Polyakov action of the string sigma model with background fields. In this model, strings are parametrized by maps $X : \Sigma \to \mathcal{M}$ from a two-dimensional worldsheet Σ to a space-time manifold $\mathcal{M}$. If $(x^I)_{I=1}^N$ is a set of coordinates on a patch U of $\mathcal{M}$, we can decompose X in its components X^I for convenience. The worldsheet Σ supports a metric g_Σ while the target space $\mathcal{M}$ comes equipped with a metric $G = G_{IJ}(x)dx^I \otimes dx^J$ and a B-field $B = \frac{1}{2}B_{IJ}(x)dx^I \wedge dx^J$ (in mathematical terms, the connection of a two-gerbe) with curvature $H = dB$, the H-flux, both corresponding to massless modes of the string spectrum. The dynamics is described by the action

$$S_{\text{string}} = \int_\Sigma \frac{1}{2}G_{IJ}dX^I \wedge *dX^J + \frac{1}{2}B_{IJ}dX^I \wedge dX^J. \tag{2.35}$$

It is easy to see how S_ϕ is a one-dimensional version of this action, with the single component of the metric corresponding to the compactification radius R.

To find the Buscher rules, one can thus consider S_{string} with a target space given by the total space E of a principal circle bundle, and T-dualize along its fibers.

If $\mathcal{A}$ is a connection of E, we choose coordinates on E such that it is written as $\mathcal{A} = dx^0 + A_i\, dx^i$, where x^0 is a coordinate along the fibers and the x^i's are coordinates on the base manifold M. The part $A = A_i\, dx^i$ is sometimes called the gauge potential.

The connection $\mathcal{A}$ allows us to write a canonical invariant metric as well as a B-field on E,

$$\begin{aligned} G &= \mathcal{A} \otimes \mathcal{A} + g_{ij}dx^i \otimes dx^j, \\ B &= \hat{A} \wedge \mathcal{A} + \tfrac{1}{2}b_{ij}dx^i \wedge dx^j, \end{aligned} \tag{2.36}$$

where $g = g_{ij}dx^i \otimes dx^j$ is a metric on the base manifold M, and $\hat{A}$ and $b = \frac{1}{2}b_{ij}dx^i \wedge dx^j$ are local forms on M, only local since B itself is actually the connection of a gerbe and thus also only locally defined on E.

The sigma model obtained when these structures are introduced in the action (2.35) possesses a global $U(1)$ symmetry $X^0 \to X^0 + C$ that can be gauged [14] by introducing a $U(1)$ connection θ on Σ and a Lagrange multiplier $\hat{X}_0$ to enforce flatness of θ. If one integrates out this Lagrange multiplier, one retrieves the original model, but if one integrates out the connection θ, one obtains a T-dual sigma model based on a T-dual circle bundle $\hat{E}$ with fiber coordinate $\hat{x}_0$ (defined via the Lagrange multiplier field $\hat{X}_0$) supporting background fields $\hat{G}$ and $\hat{B}$, related to the original ones through the Buscher rules [14].

The topological content of these rules has been formalized by Bouwknegt, Evslin and Mathai [11] [12] [13]. Essentially, the potential $\hat{A}$ defined through the relation $B = b + \hat{A} \wedge \mathcal{A}$ turns out to enter the definition of a connection for the dual bundle, $\hat{\mathcal{A}} = d\hat{x}_0 + \hat{A}$, and the T-dual B-field is given by

$$\hat{B} = b + A \wedge d\hat{x}^0 = b + A \wedge \hat{\mathcal{A}} - A \wedge \hat{A}. \tag{2.37}$$

From there on, it is easy to calculate the H-flux and its T-dual,

$$\begin{array}{rcl} H & = & db - \hat{A} \wedge F + \hat{F} \wedge \mathcal{A}, \\ \hat{H} & = & db - \hat{A} \wedge F + F \wedge \hat{\mathcal{A}}. \end{array} \tag{2.38}$$

Note that out of (2.37), we find can also find the T-dual partner of b,

$$\hat{b} = b - A \wedge \hat{A}. \tag{2.39}$$

This relation can be symmetrized to obtain a (local) two-form

$$\hat{b} - \frac{1}{2}\hat{A} \wedge A = b - \frac{1}{2}A \wedge \hat{A} =: b_{\text{inv}} \tag{2.40}$$

which is invariant under T-duality. The relation (2.37) then corresponds to the exchange of the circle bundles E and $\hat{E}$ in the formulas

$$b = b_{\text{inv}} + \frac{1}{2}A \wedge \hat{A} \leftrightarrow \hat{b} = b_{\text{inv}} + \frac{1}{2}\hat{A} \wedge A. \tag{2.41}$$

The topological information of each of these sigma models is contained in the H-fluxes H and $\hat{H}$ and the curvatures F and $\hat{F}$ (we recall that the first Chern class $[F] \in H^2(M;\mathbb{Z})$ of the associated line bundle uniquely characterizes a circle bundle E). From the relations (2.38) between these four differential forms, we can therefore extract the topological content of the Buscher rules and define geometric T-duality as follows: T-duality for principal circle bundles relates two pairs (E, H) and $(\hat{E}, \hat{H})$ of principal circle bundles E and $\hat{E}$ over a mutual base-manifold M and H-fluxes $[H] \in H^3(E;\mathbb{Z})$ and $\left[\hat{H}\right] \in H^3(\hat{E};\mathbb{Z})$. If $\mathcal{A}$ is a connection of E with curvature F and one assumes H to be the S^1-invariant representative of $[H]$, and the same yields for $\hat{\mathcal{A}}$, $\hat{E}$, $\hat{F}$ and $\hat{H}$, one can decompose these fluxes as

$$H = H_{(3)} + \mathcal{A} \wedge H_{(2)} \text{ and } \hat{H} = \hat{H}_{(3)} + \hat{\mathcal{A}} \wedge \hat{H}_{(2)}.$$

The pairs (E, H) and $(\hat{E}, \hat{H})$ are then called T-dual if [11] [12] [13]

$$F = \hat{H}_{(2)}, \; \hat{F} = H_{(2)} \text{ and } H_{(3)} = \hat{H}_{(3)}. \tag{2.42}$$

Note that these relations coincide with the action of our BV morphism on background fields of the Courant sigma model (2.33).

More formally, we can pullback the twists H and $\hat{H}$ to the correspondence space $E \times_M \hat{E}$, namely the fiber product of the two bundles, by the dual projection maps $p : E \times_M \hat{E} \to E$ and $\hat{p}$ to compare them,

$$p^*(H) - \hat{p}^*(\hat{H}) = d(\hat{p}^*(\hat{\mathcal{A}}) \wedge p^*(\mathcal{A})).$$

This motivates the more general definition of T-duality that states that (E, H) and $(\hat{E}, \hat{H})$ are T-dual if there exists a non-degenerate two-form $\mathcal{B} \in \Omega(E \times_M \hat{E})$ on the correspondence space such that

$$p^*(H) - \hat{p}^*(\hat{H}) = d\mathcal{B}.$$

This definition can be more easily extended to principal torus bundles and it can be justified by exact sequences in topology [13], independently from the Buscher rules.

2.5.3 Courant Algebroids

This T-duality relation between pairs of principal circle bundles with H-flux can be used to define an isomorphism of complexes of S^1-invariant differential forms with twisted differential [12],

$$\tau : (\Omega^\bullet_{S^1}(E), d_H) \to (\Omega^\bullet_{S^1}(\hat{E}), d_{\hat{H}}),$$

where $d_H = d + H \wedge \cdot$. Explicitly, if $\omega \in \Omega^\bullet_{S^1}(E)$, we can write it as $\omega = \omega' + \mathcal{A} \wedge \omega''$, and its image as $\tau(\omega) = \hat{\omega} = \hat{\omega}' + \hat{\mathcal{A}} \wedge \hat{\omega}''$, with $\hat{\omega}' = \omega''$ and $\hat{\omega}'' = -\omega'$. The isomorphism of complexes of twisted differentials means that $d_H \omega = 0$ if and only if $d_{\hat{H}} \hat{\omega} = 0$.

Gualtieri and Cavalcanti use this isomorphism as the starting point for the construction of an isomorphism of Courant algebroids [21], that actually coincides with the one underlying our BV morphism.

Their first observation is that the space $\Omega^\bullet_{S^1}(E)$ of invariant differential forms on E has the structure of a Clifford module for the Courant algebroid $(TE \oplus T^*E)/S^1$ over M that we already encountered in our construction of a BV morphism. The Clifford action of a section of the Courant algebroid on a differential form is defined as the addition of the contraction with the vector field part and the exterior multiplication with the differential form part. An isomorphism of Courant algebroids

$$\Phi : (TE \oplus T^*E)/S^1 \to (T\hat{E} \oplus T^*\hat{E})/S^1,$$

with $\hat{E}$ being the dual circle bundle, can then be constructed in such a way that the map $\tau : \Omega^\bullet_{S^1}(E) \to \Omega^\bullet_{S^1}(\hat{E})$ is promoted to an isomorphism of Clifford modules, namely such that

$$\tau(v \cdot \rho) = \Phi(v) \cdot \tau(\rho)$$

for any section $v \in \Gamma((TE \oplus T^*E)/S^1)$ and any invariant differential form $\rho \in \Omega^\bullet_{S^1}(E)$. The Clifford action is denoted by the dot.

Their construction follows geometric considerations which are inspired from a duality of field theories, yet it could be based on purely topological and geometrical arguments. Our construction somehow closes the gap by providing a field theoretic derivation of the same isomorphism. In a next step, it is tempting to try to express the topological content of the Buscher rules as a BV duality of two-dimensional topological sigma models.

2.6 T-duality and Twisted Poisson Sigma Models

This time, we will have to combine a BV morphism with dual BV pushforwards, a method explained in section 2.3.3. The BV morphism will involve a model constructed on the same correspondence space where we compared H and $\hat{H}$. The pushforwards will be needed to go down to the individual principal circle bundles, they will somehow correspond to the path integrations that led to the Buscher rules.

2.6.1 A Sigma Model for the Topological Sector of a String with Background Fields

In order to find a topological sigma model related to string theory and subject to T-duality transformations, it is natural to start with a WZ Poisson sigma model [31], whose action we recall is

$$S = \int_\Sigma \eta_i \wedge dX^i + \frac{1}{2}\pi^{ij}(X)\eta_i \wedge \eta_j + \int_N X^*(H). \tag{2.43}$$

Here the X^i's denote the coordinate components of a map $X \in \mathrm{Map}(\Sigma \to \mathcal{M})$, $\eta \in \Gamma(T^*\Sigma \otimes X^*(T^*\mathcal{M}))$ is a one form on the closed worldsheet Σ with value in the pullback by X of the cotangent bundle of the target space, $H \in \Omega^3_{\mathrm{closed}}(\mathcal{M})$ is the twist (or in the language of strings the H-flux) of the target space, and N is a handlebody for $\Sigma = \partial N$. The Wess-Zumino consistency condition requires the twist to have integral period, $[H] \in H^3(\mathcal{M}; \mathbb{Z})$.

This choice of sigma model is motivated by the fact that when $\pi = 0$, this model represents the topological sector of a sigma model for a string in background fields G (metric on the target space $\mathcal{M}$) and B (connection of a two-gerbe with curvature $dB = H$). Indeed, if we add a term $-\frac{1}{2}\int_\Sigma G^{ij}(X)\eta_i \wedge *\eta_j$ to the action, where $*$ denotes the Hodge star operator on the worldsheet Σ, we may integrate out the η fields and recover the desired action.

The minimal BV extension of this simplified model with a zero Poisson structure requires to augment the classical space of fields spanned by (X, η) with the algebra of symmetries shifted by one, that is the space of ghost fields $\beta \in \Gamma(X^*T^*[1]\mathcal{M})$, to form the BRST space of fields $\mathcal{F}_{\mathrm{BRST}}$ spanned by (X, η, β), and then take its cotangent bundle shifted by minus one, $\mathcal{F} = T^*[-1]\mathcal{F}_{\mathrm{BRST}}$. This BV space of fields admits the canonical BV structure

$$\Omega = \int_\Sigma \delta\eta^{+i} \wedge \delta\eta_i + \delta X_i^+ \, \delta X^i + \delta\beta^{+i} \, \delta\beta_i, \tag{2.44}$$

where $\eta^+ \in \Gamma(T^*\Sigma \otimes X^*T[-1]\mathcal{M})$, $X^+ \in \Gamma(\bigwedge^2 T^*\Sigma \otimes X^*T^*[-1]\mathcal{M})$ and $\beta^+ \in \Gamma(\bigwedge^2 T^*\Sigma \otimes X^*T[-2]\mathcal{M})$ are cotangent fiber coordinates of $\mathcal{F}$. The BV action is obtained through addition to the classical one of a term that encodes its symmetry under infinitesimal transformations $\delta_\epsilon \eta_i = d\epsilon_i$,

$$S = \int_\Sigma \eta_i \wedge dX^i - \eta^{+i} \wedge d\beta_i + \int_N X^*(H), \tag{2.45}$$

and it remains to check that the classical master equation is indeed satisfied,

$$\begin{aligned} \frac{1}{2}\{S, S\} &= \int_\Sigma \left(\frac{S\overleftarrow{\delta}}{\delta\eta_i} \frac{\overrightarrow{\delta} S}{\delta\eta^{+i}} + \frac{S\overleftarrow{\delta}}{\delta X^i} \frac{\overrightarrow{\delta} S}{\delta X_i^+} + \frac{S\overleftarrow{\delta}}{\delta\beta_i} \frac{\overrightarrow{\delta} S}{\delta\beta^{+i}} \right) \\ &= \int_\Sigma dX^i \wedge d\beta_i = -\int_\Sigma d(\beta_i dX^i) = 0, \end{aligned}$$

as the integral of an exact form over a closed surface vanishes. Here functional left- and right-derivatives are defined in a similar way as in the case (2.15) of superfields, the only difference being that one integrates here differential forms over a manifold instead of functions over a supermanifold with a canonical measure.

Note that the space of fields admits the structure of a mapping space *à la* AKSZ, $\mathcal{F} = \mathrm{Map}(T[1]\Sigma, T^*[1]\mathcal{M})$, similar to the one of the regular Poisson sigma model, however the twist H prevents the application of the full AKSZ procedure.

To investigate T-duality, we are interested in the situation where the target space is a principal circle bundle E over M,

$$\begin{array}{ccc} S^1 & \longrightarrow & E \\ & & \downarrow \rho \\ & & M. \end{array}$$

We choose similar coordinates on E as before, so that the connection is written as $\mathcal{A} = dx^0 + A$.

From covariant considerations, we would expect an action of the form

$$S_E = \int_\Sigma \eta_i \wedge dX^i + \eta_0 \wedge (dX^0 + X^*(A)) + \int_N X^*(H), \tag{2.46}$$

where $dX^0 + X^*(A)$ is the pullback by the field X of the connection $\mathcal{A}$ on E. If we add the term

$$-\int_\Sigma \left(G^{ij}(X)\eta_i \wedge *\eta_j + G^{00}(X)\eta_0 \wedge *\eta_0\right)$$

to this topological action and integrate out the one-form fields η, we recover the string sigma model action with background fields given by the canonical metric and B-field (2.36) on the circle bundle E.

Furthermore, we want H to be S^1-invariant, for reasons that will become clearer.

The price to pay for the introduction of the pullback of the connection $\mathcal{A}$ on E is that the action is no longer invariant under transformations $\delta_{\epsilon_0}\eta_0 = d\epsilon_0$. It would pick up a term $\delta_{\epsilon_0} S_E = \int_\Sigma \epsilon_0 F$ proportional to the curvature of the connection $\mathcal{A}$. One way to recover the symmetry would be to add a term $\int_\Sigma \hat{X}_0 F = \int_\Sigma d\hat{X}_0 \wedge \mathcal{A}$ to the action and require that the new field $\hat{X}_0$ transforms as $\delta_{\epsilon_0}\hat{X}_0 = \epsilon_0$. Note that we have performed an integration by parts to make the dependence on the connection of E obvious. The new symmetry is obviously abelian and one-dimensional, so it is natural to interpret $\hat{X}_0$ as induced by the fiber coordinate $\hat{x}_0$ of a dual bundle $\hat{E}$, to which we assign a connection $\hat{\mathcal{A}} = d\hat{x}_0 + \hat{A}$ with curvature $\hat{F} = d\hat{\mathcal{A}}$. Here dual means that the Lie algebras $\mathfrak{u}(1) \simeq \mathbb{R}$ and $\hat{\mathfrak{u}}(1) \simeq \mathbb{R}$ of the fibers of E and $\hat{E}$ are dual to each other, in the same way as for the circle bundles of section 2.5.

2.6.2 A BV Model on the Correspondence Space

We are apparently building a model on the correspondence space $E \times_M \hat{E}$, the fiber product of both circle bundles E and $\hat{E}$. It is therefore natural to gauge the $U(1)$ symmetry along fibers of E by adding a connection one-form $\hat{\eta}^0$ for the pullback bundle $X^*(E)$ on Σ. We are now in possession of all the elements to write down a $U(1) \times U(1)$ invariant (classical) action on the correspondence space,

$$S^{\mathrm{cl}}_{E\times_M\hat{E}} = \int_\Sigma \eta_i \wedge dX^i + \Big(\eta_0 + d\hat{X}_0 + X^*(\hat{A})\Big) \wedge \big(\hat{\eta}^0 + dX^0 + X^*(A)\big) + \int_N X^*(h). \tag{2.47}$$

The three-form h that enters the last term (a Wess-Zumino type term) is yet to be determined. The $U(1) \times U(1)$ invariance actually requires h to be basic (i.e. $h \in \Omega^3_{\mathrm{closed}}(M)$).

Three geometrical structures of the target space $E \times_M \hat{E}$ enter the formulation of this model, namely the three-form h and the two connections $\mathcal{A}$ and $\hat{\mathcal{A}}$. We will see how they behave under a T-duality transformation, but before we can study this example of a BV morphism, we obviously need to determine the BV formulation of the model.

Symmetry transformations are described by local parameters $\epsilon = (\epsilon_i, \epsilon_0, \hat{\epsilon}^0)$,

$$\begin{aligned} \delta_\epsilon \eta_i &= d\epsilon_i, \\ \delta_\epsilon \eta_0 &= d\epsilon_0, \\ \delta_\epsilon \hat{\eta}^0 &= d\hat{\epsilon}^0, \\ \delta_\epsilon X^0 &= \hat{\epsilon}^0, \\ \delta_\epsilon \hat{X}_0 &= \epsilon_0. \end{aligned}$$

Again, to obtain the minimal BV formulation, we replace the gauge parameters with odd fields of ghost number 1, namely β_i, β_0 and $\hat{\beta}^0$, we assign an antifield to each field or ghost, we construct the canonical BV structure on the resulting space of fields $\mathcal{F}_{E\times_M\hat{E}} = \mathrm{Map}(T\,[1]\,\Sigma, T^*\,[1]\,(E \times_M \hat{E}))$,

$$\begin{aligned} \Omega_{E\times_M\hat{E}} = \int_\Sigma & \delta\eta^{+i} \wedge \delta\eta_i + \delta X_i^+ \,\delta X^i + \delta\beta^{+i}\,\delta\beta_i \\ & + \delta\eta^{+0} \wedge \delta\eta_0 + \delta X_0^+ \,\delta X^0 + \delta\beta^{+0}\,\delta\beta_0 \\ & + \delta\hat{\eta}_0^+ \wedge \delta\hat{\eta}^0 + \delta\hat{X}^{+0}\,\delta\hat{X}_0 + \delta\hat{\beta}_0^+\,\delta\hat{\beta}^0, \end{aligned} \tag{2.48}$$

and finally we add corresponding terms that encode the symmetries to the action,

$$S_{E\times_M\hat{E}} = S^{\mathrm{cl}}_{E\times_M\hat{E}} + \int_\Sigma -\eta^{+i} \wedge d\beta_i - \eta^{+0} \wedge d\beta_0 - \hat{\eta}_0^+ \wedge d\hat{\beta}^0 + X_0^+ \wedge \hat{\beta}^0 + \hat{X}^{+0} \wedge \beta_0. \tag{2.49}$$

Note that even though the space of fields is similar to the usual mapping space of the AKSZ construction with its canonical BV structure, the BV action is not of the AKSZ type, like in the case of the twisted Poisson sigma model with trivial Poisson structure we considered above.

The BV action $S_{E\times_M\hat{E}}$ readily satisfies the quantum master equation, provided a suitable regularization of the BV Laplacian is adopted. It therefore corresponds to the action W in the diagram (2.21).

2.6.3 The Original Model as an Effective Theory

We started from a sigma model on the circle bundle E to build this action on the correspondence space. We thus expect to be able to re-derive the initial model as an effective action, as described in section 2.3.1. This will correspond to the ultraviolet BV pushforward $\rho_{\mathrm{UV}*}$ in the diagram (2.21).

Obviously, to obtain an effective action on E, the ultraviolet sector of the space of fields needs to contain the components associated to $\hat{E}$, namely $\hat{X}_0$, $\hat{\eta}^0$ and $\hat{\beta}^0$. However, we saw that the symmetry associated to β_0 is also broken in the action (2.46), consequently this ghost field should also belong to the UV sector. In summary, the ultraviolet sector $\mathcal{F}_{UV}$ is spanned by

$$\left(\hat{X}_0, \hat{\eta}^0, \hat{\beta}^0, \beta_0, \hat{X}^{+0}, \hat{\eta}_0^+, \hat{\beta}_0^+, \beta^{+0}\right).$$

We specify the Lagrangian subspace of $\mathcal{F}_{\mathrm{UV}}$ by setting $\beta_0 = 0$ and $\hat{\beta}^0 = 0$, which breaks the $U(1) \times U(1)$ symmetry of the total model, as well as $\hat{X}_0 = 0$ and $\hat{\eta}^0 = 0$, which selects only the fibers of E in the model and removes the ones of the dual $\hat{E}$. Since the restriction of the action to this Lagrangian subspace of the ultraviolet sector does not depend on the other fields of this sector, the functional integration yields only trivial constants of no interest, and the resulting effective theory is described by the action

$$S_E^{\mathrm{eff}} = \int_\Sigma \eta_i \wedge dX^i + \eta_0 \wedge X^*(\mathcal{A}) + X^*(\hat{A} \wedge \mathcal{A}) + \eta^{+i} \wedge d\beta_i + \int_N X^*(h). \quad (2.50)$$

At first glance, the term $X^*(\hat{A} \wedge \mathcal{A})$ seems awkward, but it actually contributes to the WZ term. If we take its exterior derivative, we can write it as an integral over the handlebody N and the connection $\mathcal{A}$ will bring an invariant contribution along the fibers of E to a twist H that would otherwise remain basic,

$$S_E^{\mathrm{eff}} = \int_\Sigma \eta_i \wedge dX^i + \eta_0 \wedge X^*(\mathcal{A}) + \eta^{+i} \wedge d\beta_i + \int_N X^*(h - F \wedge \hat{A} + \hat{F} \wedge \mathcal{A}). \quad (2.51)$$

We immediately see that the twist on the target space E is given by

$$H = h - F \wedge \hat{A} + \hat{F} \wedge \mathcal{A}, \quad (2.52)$$

and we understand why we had to assume that H was S^1-invariant. Furthermore, by comparison with (2.38), we see that we can interpret h as the curvature $h = db$ of the basic component of the B-field (2.36).

2.6.4 A T-duality BV Morphism

The next step is to determine the T-duality BV morphism Φ, the one that sits at the top of the diagram (2.21), which will give us a dual action on the full space.

We saw that on two-dimensional models, T-duality involves the Hodge star operator. If we consider the term

$$\int_\Sigma \left(\eta_0 + d\hat{X}_0 + X^*(\hat{A})\right) \wedge \left(\hat{\eta}^0 + dX^0 + X^*(A)\right)$$

of the action (2.47) of the sigma model on the correspondence space and apply on it the prescription (2.34) we found for T-duality[3], we find the T-dual term

$$\begin{aligned} &\int_\Sigma * \left(\eta_0 + d\hat{X}_0 + X^*(\hat{A})\right) \wedge * \left(\hat{\eta}^0 + dX^0 + X^*(A)\right) \\ = &\int_\Sigma \left(\hat{\eta}^0 + dX^0 + X^*(A)\right) \wedge \left(\eta_0 + d\hat{X}_0 + X^*(\hat{A})\right), \end{aligned} \tag{2.53}$$

where we used the symmetry of the product $\cdot \wedge *\cdot$ of two one-forms and the involutivity of the Hodge star operator induced by a metric of Lorentzian signature when acting on one-forms. We see that T-duality can be interpreted as the exchange of the roles of the two circle bundles E and $\hat{E}$, and that the Hodge star operator disappears from the formula, which is good when one considers topological field theories (of the Schwarz type).

We can use this first hint to start constructing a BV morphism Φ, by requiring

$$\begin{array}{rclrcl} \Phi(\eta_0) &=& \hat{\eta}^0, & \Phi(\hat{\eta}^0) &=& \eta_0, \\ \Phi(X^0) &=& \hat{X}_0, & \Phi(\hat{X}_0) &=& X^0, \\ \Phi(\beta_0) &=& \hat{\beta}^0, & \Phi(\hat{\beta}^0) &=& \beta_0. \end{array}$$

The last six terms of the BV structure (2.48) of the model on the correspondence space then tell us how to define the action of Φ on the corresponding antifields in such a way that Φ becomes a BV morphism of $(\mathcal{F}_{E\times_M\hat{E}}, \Omega_{E\times_M\hat{E}})$, namely

$$\begin{array}{rclrcl} \Phi(\eta^{+0}) &=& \hat{\eta}_0^+, & \Phi(\hat{\eta}_0^+) &=& \eta^{+0}, \\ \Phi(X_0^+) &=& \hat{X}^{+0}, & \Phi(\hat{X}^{+0}) &=& X_0^+, \\ \Phi(\beta^{+0}) &=& \hat{\beta}_0^+, & \Phi(\hat{\beta}_0^+) &=& \beta^{+0}. \end{array}$$

[3] Note that in this topological model, the radius is normalized to $R = 1$. The corresponding information is actually contained in the metric component G_{00} that we left aside.

The swap induced by the Hodge operator (2.53) involves the background fields A and $\hat{A}$, too. Therefore, the BV morphism Φ should also affect them, namely

$$\Phi(A) = \hat{A}, \quad \Phi(\hat{A}) = A.$$

Together with the action of Φ on the fiber coordinates X^0 and $\hat{X}_0$, this is equivalent to swapping the connections $\mathcal{A}$ and $\hat{\mathcal{A}}$.

In effect, Φ exchanges the two circle bundles E and $\hat{E}$. In other words, it maps the space of fields $\mathcal{F}_{E\times_M \hat{E}}$ to its dual,

$$\Phi : (\mathcal{F}_{E\times_M \hat{E}}, \Omega_{E\times_M \hat{E}}) \to (\hat{\mathcal{F}}_{\hat{E}\times_M E}, \hat{\Omega}_{\hat{E}\times_M E}).$$

So far, we considered only the second term of (2.47). The first term $\int_\Sigma \eta_i \wedge dX^i$ involves only the base manifold M of the circle bundles E and $\hat{E}$ and should therefore be left unaffected by Φ, which is why its action on the corresponding fields X^i, η_i and β_i and their antifields is trivial.

It immediately follows by construction that

$$\Phi^*(\hat{\Omega}_{\hat{E}\times_M E}) = \Omega_{E\times_M \hat{E}}.$$

It remains to consider the last term $\int_N X^*(h)$, on which the action of Φ is a bit subtle. Since Φ affects the background fields A and $\hat{A}$, there is *a priori* no reason why it should leave h, the third background field of the model, invariant. Actually, h is the curvature three-form of the basic part b of the B-field introduced in equation (2.36), $h = db$. As a result, Φ should follow the T-duality transformation rule (2.41) for this field,

$$\Phi(h) = \Phi(db) = d\hat{b} = h - d(A \wedge \hat{A}) = \hat{h}. \tag{2.54}$$

Note that the application of Φ follows the prescription for the exchange of the roles of the two circle bundles E and $\hat{E}$ and leaves the invariant part b_{inv} untouched.

Finally the action of the T-dual model on the correspondence space is the one that fulfills the condition

$$\Phi^*(\hat{S}_{\hat{E}\times_M E}) = S_{E\times_M \hat{E}},$$

namely

$$\begin{aligned}\hat{S}^{\mathrm{cl}}_{\hat{E}\times_M E} = \int_\Sigma \eta_i \wedge dX^i + \big(\hat{\eta}^0 + dX^0 + X^*(A)\big) \wedge \Big(\eta_0 + d\hat{X}_0 + X^*(\hat{A})\Big) \\ + \int_N X^*(\hat{h}).\end{aligned} \tag{2.55}$$

As a side remark, note that in three dimensions, the T-duality morphism exchanged factors in the term $\delta\phi\delta\psi$ of the BV structure, which were symmetric, whereas here it swaps pairs of terms of Ω.

2.6.5 The T-dual Effective Model

In the last step, we look for a dual BV pushforward $\rho'_{\mathrm{UV}*}$ that will give us a dual effective action $S^{\mathrm{eff}}_{\hat{E}}$, out of which we will be able to read the dual topological information, namely the dual connection $\hat{\mathcal{A}}$, its curvature $\hat{F}$ and the dual twist $\hat{H}$.

This time, the ultraviolet sector $\hat{\mathcal{F}}_{UV}$ is spanned by

$$\left(X^0, \eta_0, \beta_0, \hat{\beta}^0, X_0^+, \eta^{+0}, \beta^{+0}, \hat{\beta}_0^+\right)$$

and is nothing but the image of $\mathcal{F}_{UV}$ under the BV morphism Φ. We choose a similar Lagrangian subspace as before, namely by setting $\hat{\beta}^0 = 0$, $\beta_0 = 0$ and $X^0 = 0$, which keeps the fibers of $\hat{E}$ but not of E as expected. The T-dual effective action we obtain is similar to the original effective action, but the fields associated to the fibers and the background fields have been exchanged,

$$S^{\mathrm{eff}}_{\hat{E}} = \int_{\Sigma} \eta_i \wedge dX^i + \hat{\eta}^0 \wedge X^*(\hat{\mathcal{A}}) + \eta^{+i} \wedge d\beta_i + \int_N X^*(\hat{h} - \hat{F} \wedge A + F \wedge \hat{\mathcal{A}}). \quad (2.56)$$

From the WZ term, we find the dual twist

$$\hat{H} = \hat{h} - \hat{F} \wedge A + F \wedge \hat{\mathcal{A}} \quad (2.57)$$

on the dual circle bundle. From this relation as well as its T-dual (2.52), we can infer the first two equalities of (2.42),

$$H_{(2)} = \hat{F}, \quad \hat{H}_{(2)} = F.$$

We see that these follow directly from the exchange of the two circle bundles in the topological sigma model on the correspondence space with action $S_{E \times_M \hat{E}}$. The change of topology of a circle bundle under a T-duality transformation is thus entirely encoded in the topological sector of the string sigma model and does not rely on the full set of Buscher rules.

On the other hand, the third equality

$$H_{(3)} = \hat{H}_{(3)},$$

which states that the basic part of the T-dual twist $\hat{H}$ coincides with the one of the original twist H, holds only if

$$\hat{h} - \hat{F} \wedge A = h - F \wedge \hat{A},$$

a condition satisfied by the prescription (2.54) for the action of the BV morphism Φ on the background fields, which is reminiscent from the Buscher rules. But this actually ensures that the Wess-Zumino consistency condition for the dual effective model, namely $\left[\hat{H}\right] \in H^3(\hat{E};\mathbb{Z})$, follows from the one on the initial model $[H] \in H^3(E;\mathbb{Z})$. To see this, one needs to keep in mind that being the curvatures of their dual connections, the vertical parts of the H-fluxes satisfy $\left[H_{(2)}\right] = \left[\hat{F}\right] \in H^2(M;\mathbb{Z})$ and $\left[\hat{H}_{(2)}\right] = [F] \in H^2(M;\mathbb{Z})$, and that the connections $\mathcal{A}$ and $\hat{\mathcal{A}}$ are normalized to have an integral period around the fibers.

In summary, the topological content of the Buscher rules is encoded in the relation between the topological sigma models with action S_E^{eff} and $S_{\hat{E}}^{\text{eff}}$, which represent the topological sectors of the involved string sigma models with background fields. Due to the broken $U(1)$ symmetry, a BV morphism could not be readily defined between them, and we had to consider an augmented model on the correspondence space $E \times_M \hat{E}$ to introduce a T-duality transformation, and the two effective models could be retrieved through BV pushforwards, the usual prescription in the BV formalism for effective field theories.

2.7 Principal Torus Bundles

Our discussion can be generalized to higher dimensional principal torus bundles,

$$\begin{array}{ccc} \mathbb{T}^n & \longrightarrow & E \\ & & \big\downarrow \rho \\ & & M. \end{array}$$

In particular, the connection $\mathcal{A}$ is now a $\mathfrak{t}^n$-valued one-form on E, where $\mathfrak{t}^n \simeq \mathbb{R}^n$ is the Lie algebra of $\mathbb{T}^n$, and the corresponding period one generator of the $\mathbb{T}^n$-action $\partial_{\mathcal{A}}$ is $\mathfrak{t}^{*n}$-valued.

Before going into technical details, we should analyze once more the results for a circle bundle. In the case of the Courant algebroid, once the spaces of fields for both Courant sigma models based on $(TE \oplus T^*E)/S^1$ and $(T\hat{E} \oplus T^*\hat{E})/S^1$ were identified with a model space of fields, we were able to interpret the T-duality BV morphism as an automorphism of this model space of fields that left the AKSZ action invariant. In other words, the duality was readily present as a discrete symmetry of the AKSZ action and BV structure, out of which one could find the relations (2.42). We then applied an adaptation of this morphism to a two-dimensional model. The T-duality was in this case a morphism of the spaces of fields $\mathcal{F}_{E\times_M\hat{E}} \simeq \hat{\mathcal{F}}_{\hat{E}\times_M E}$, but this time the action was not invariant.

Nevertheless it still encoded the change of topology induced by the Buscher rules.

For higher-dimensional torus bundles, the procedure is similar. The main difference is that not all H-fluxes allow a T-duality transformation, those are called T-dualizable, and we will see they are precisely the ones that make the Courant sigma model "T-symmetric". These T-duality symmetry transformations will form a group, the usual $O(n, n; \mathbb{Z})$ known from string theory. Due to the multiplicity of T-dual models for $n > 1$, the construction of morphisms of two-dimensional topological sigma models will be a bit more complicated.

Since the construction of these T-duality BV morphisms for torus bundles closely follows the case of circle bundles, we will not go into all the details, but rather focus on the subtleties implied by a larger T-duality group.

2.7.1 Courant Sigma Models

Given an H-flux $[H] \in H^3(E; \mathbb{Z})$, we can again construct a quotient Courant algebroid on E if we choose an invariant representative H for this cohomology class. Mathematically, it means that $\mathcal{L}_X(H) = 0$ for any Killing vector of the torus action on E. Moreover, this H-flux is called T-dualizable [12] if it satisfies the additional requirement that $\imath_X \imath_Y H = 0$ for any two Killing vector fields X and Y of the torus action. In practice, this means that the H-flux can be decomposed as

$$H = H_{(3)} + \mathcal{A}^a \wedge H^a_{(2)},$$

where addition over repeated superscripts a is implicit, $\mathcal{A}^a$, $a = 1, \ldots, n$, denotes an individual component of the $\mathfrak{t}^n$-valued connection $\mathcal{A}$, and $H_{(3)}$ and $H^a_{(2)}$ are basic three- and two-forms respectively. In particular, no terms quadratic or cubic in the connection $\mathcal{A}$ enters this formula. It appears that $H_{(2)}$ can be interpreted as a $\mathfrak{t}^{*n}$-valued two form on the base manifold M, and we may write $\mathcal{A}^a \wedge H^a_{(2)} = \mathcal{A} \wedge H_{(2)}$.

The space of fields of the Courant sigma model based on the Courant algebroid $(TE \oplus T^*E)/\mathbb{T}^n$ is sensibly the same as the one in the case $n = 1$, see equation (2.26). A small difference appears for the superfields ϕ and ψ, which are now defined as

$$\phi \in \mathrm{Map}(T[1]N, \mathfrak{t}^n[1]) \quad \text{and} \quad \psi \in \mathrm{Map}(T[1]N, \mathfrak{t}^{*n}[1]),$$

with components $\phi^a, \psi^a \in \mathrm{Fun}(T[1]N, \mathbb{R}[1])$.

With this notation for the fields, the AKSZ BV structure and action remain the same as in the case $n = 1$, but now (2.27) and (2.28) really mean

$$\Omega = \int_{T[1]N} \mu \, \left(\delta p_i \, \delta X^i - \delta\xi^i \, \delta\Theta_i - \delta\phi^a \, \delta\psi^a\right).$$

and

$$\begin{aligned} S = \int_{T[1]N} \mu \, \Big(& p_i \, DX^i + \frac{1}{2}\xi^i \, D\Theta_i + \frac{1}{2}\Theta_i \, D\xi^i + \frac{1}{2}\phi^a \, D\psi^a + \frac{1}{2}\psi^a \, D\phi^a \\ & -p_i\xi^i + \psi^a \, \frac{1}{2}F^a_{ij}\xi^i\xi^j + \frac{1}{6}H_{(3)ijk}\xi^i\xi^j\xi^k + \phi^a \, \frac{1}{2}H^a_{(2)ij}\xi^i\xi^j \Big). \end{aligned} \tag{2.58}$$

From now on, we will identify $\mathfrak{t}^n \simeq \mathfrak{t}^{*n} \simeq \mathbb{R}^n$ with $\mathbb{R}^n$. We can therefore combine the $\mathbb{R}^n$ [1]-valued superfields ϕ and ψ into a $\mathbb{R}^{2n}$ [1]-valued superfield

$$\Xi = \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \mathrm{Map}(T\,[1]\,N, \mathbb{R}^{2n}\,[1]).$$

This allows us to re-write the last term in the AKSZ BV structure Ω as

$$\delta\phi^a \, \delta\psi^a = \frac{1}{2}\,(\delta\phi \;\, \delta\psi) \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} \delta\phi \\ \delta\psi \end{pmatrix} = \frac{1}{2}\,\delta\Xi^T \, K \, \delta\Xi = \frac{1}{2}\langle\delta\Xi, \delta\Xi\rangle_K,$$

where $K = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$ is a $2n \times 2n$ symmetric matrix used to define the scalar product $\langle v, w\rangle_K = v^T \, K \, w$. We can thus write the BV structure

$$\Omega = \int_{T[1]N} \mu \, \left(\delta p_i \, \delta X^i - \delta\xi^i \, \delta\Theta_i - \frac{1}{2}\langle\delta\Xi, \delta\Xi\rangle_K\right). \tag{2.59}$$

We see that Ω is invariant under linear transformations of the Ξ superfield,

$$\Xi \mapsto \mathcal{O}\Xi = \hat{\Xi},$$

such that $\mathcal{O}^T \, K \, \mathcal{O} = \mathbf{1}$. Since K can be diagonalized to $\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$, this transformations form the group $O(n, n; \mathbb{R})$. Such a linear transformation can be lifted to a BV automorphism

$$\Phi_{\mathcal{O}} : \mathcal{F} \to \mathcal{F}.$$

To see how it affects the action S, it is best to combine the $\mathbb{R}^n$-valued two-forms F and $H_{(2)}$ into an $\mathbb{R}^{2n}$-valued two-form

$$\mathbf{F} = \begin{pmatrix} F \\ H_{(2)} \end{pmatrix},$$

so that one can write the BV action

$$S = \int_{T[1]N} \mu \left(p_i \, DX^i + \frac{1}{2}\xi^i \, D\Theta_i + \frac{1}{2}\Theta_i \, D\xi^i + \frac{1}{2}\langle \Xi, D\Xi \rangle_K \right. \\ \left. -p_i\xi^i + \langle \Xi, \frac{1}{2}\mathbf{F}_{ij}\xi^i\xi^j \rangle_K + \frac{1}{6}H_{(3)ijk}\xi^i\xi^j\xi^k \right). \tag{2.60}$$

By comparison with the circle bundle case, we see that we can construct a dual action

$$\hat{S} = \int_{T[1]N} \mu \left(p_i \, DX^i + \frac{1}{2}\xi^i \, D\Theta_i + \frac{1}{2}\Theta_i \, D\xi^i + \frac{1}{2}\langle \hat{\Xi}, D\hat{\Xi} \rangle_K \right. \\ \left. -p_i\xi^i + \langle \hat{\Xi}, \frac{1}{2}\hat{\mathbf{F}}_{ij}\xi^i\xi^j \rangle_K + \frac{1}{6}H_{(3)ijk}\xi^i\xi^j\xi^k \right). \tag{2.61}$$

that satisfies the duality requirement $\Phi^*_{\mathcal{O}}(\hat{S}) = S$ provided $\Phi_{\mathcal{O}}(\mathbf{F}) = \hat{\mathbf{F}}$, and that we have actually $\hat{S} = S$ if $\Phi_{\mathcal{O}}(\mathbf{F}) = \mathcal{O}\mathbf{F}$, so that the Courant algebroids encoded in the Courant sigma models actions S and $\hat{S}$ are actually isomorphic.

It is essential to note that the $2n$ components of $\mathbf{F}$ define integral cohomology classes,

$$[F^a]\,, \; \left[H^a_{(2)}\right] \in H^2(M;\mathbb{Z}), \; a = 1, \ldots, n,$$

because the curvature of a circle bundle has integral periods and H needs to satisfy the Wess-Zumino consistency condition. The $2n$ components of $\hat{\mathbf{F}}$ are subject to the same constraint. This is ensured if $\mathcal{O}$ is taken in the subgroup $O(n,n;\mathbb{Z})$ of $O(n,n;\mathbb{R})$. This is actually the T-duality group for the torus T^n, which leads us to interpret $\Phi_{\mathcal{O}}$ as a transformation of the fibers of the torus bundle E into a dual bundle $\Phi_{\mathcal{O}}(E)$.

This formulation of toroidal T-duality in the Courant sigma model has the additional advantage to cast some new light on the condition of the twist H to be T-dualizable. Had H not satisfied this condition, so would the Courant sigma model action for the torus bundle (2.58) have contained terms quadratic or cubic in ϕ, and we would not have been able to re-write it in the symmetric form (2.60).

2.7.2 Two-dimensional Sigma Models

A similar duality between two-dimensional models as in section 2.6 is a bit more complicated to work out for higher-dimensional tori. When $n = 1$, we have $O(1,1;\mathbb{Z}) = \mathbb{Z}_2$, which makes the T-dual of a certain circle bundle with twist unique. We used this unicity to combine the connections associated to both bundles in a generalization of the Poisson sigma model action, on which the T-duality group $\mathbb{Z}_2$ acted by exchange of the two bundles. For $n > 1$, the group is larger, but $\{\mathbf{1}, K\}$ is a canonical $\mathbb{Z}_2$ subgroup of $O(n, n; \mathbb{Z})$ that we may use to pick a specific dual principal circle bundle.

As in the case $n = 1$, given a principal $\mathbb{T}^n$-bundle E with T-dualizable twist H, we start with a classical action

$$S_E^{\mathrm{cl}} = \int_\Sigma \eta_i \wedge dX^i + \theta \wedge X^*(\mathcal{A}) + \int_N X^*(H), \tag{2.62}$$

where $\mathcal{A}$ is a connection on E, therefore a $\mathfrak{t}^n$-valued one-form on E. This implies in turn that θ (which replaces η_0 from the $n = 1$ case) is a $\mathfrak{t}^{*n}$-valued one-form on Σ.

Like in three dimensions, we cast the curvature F of the connection $\mathcal{A}$ and the component $H_{(2)}$ into an $\mathbb{R}^{2n}$-valued two-form $\mathbf{F}$. We can interpret $H_{(2)}$ as the curvature of the connection $\hat{\mathcal{A}}$ of a dual torus bundle $\hat{E}$. In that case, $\mathbf{F}$ corresponds to the curvature of the connection

$$\mathbf{A} = \begin{pmatrix} \mathcal{A} \\ \hat{\mathcal{A}} \end{pmatrix}$$

of the $\mathbb{T}^{2n}$-bundle $E \times_M \hat{E}$. To gauge the whole $\mathbb{T}^n \times \mathbb{T}^n$ symmetry, we need to introduce a connection $\hat{\theta}$ (a generalization of $\hat{\eta}^0$), locally a $\mathfrak{t}^n$-valued one-form on Σ. We may regroup it with θ into an $\mathbb{R}^{2n}$-valued connection

$$\Theta = \begin{pmatrix} \hat{\theta} \\ \theta \end{pmatrix}.$$

With the matrix $J = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$, we can write an action on the correspondence space that generalizes (2.47),

$$S_{E \times_M \hat{E}}^{\mathrm{cl}} = \int_\Sigma \left(\eta_i \wedge dX^i + \frac{1}{2} \left(\Theta + X^*(\mathbf{A})\right)^T \wedge J \left(\Theta + X^*(\mathbf{A})\right) \right) + \int_N X^*(h). \tag{2.63}$$

The basic twist h is a generalization of (2.52) for the $n = 1$ situation,

$$h = H + d(\mathcal{A}^a \wedge \hat{A}^a)$$

where $\hat{A}$ is the gauge potential associated to the connection $\hat{\mathcal{A}}$.

We write only the classical part of the action, the full BV action can easily be inferred from its version (2.49) for circle bundles.

To retrieve the original action associated to the topological sector of a string on the torus bundle E, we treat as UV degrees of freedom the fiber coordinates of the torus bundle $\hat{E}$, the first half of Θ and of course all the ghosts. The functional integration still yields a constant, and the effective action on the IR sector is our initial action (2.62), as expected, with $d(\mathcal{A} \wedge \hat{A})$ completing h to the whole twist H.

The $O(n,n;\mathbb{Z})$ T-duality group acts on $\mathbf{A}$ and Θ by matrix multiplication. For $n > 1$, we evidently obtain more than one T-dual model. In this case, each

$$\mathcal{O}\mathbf{F} = \mathbf{F}' = \begin{pmatrix} F' \\ H'_{(2)} \end{pmatrix}$$

will determine a pair of torus bundles E' and $\hat{E}'$, and the BV morphism $\Phi_{\mathcal{O}}$ associated to the T-duality transformation $\mathcal{O} \in O(n,n;\mathbb{Z})$ maps the space of fields of the topological model on the correspondence space $E \times_M \hat{E}$ to the one of the model on $E' \times_M \hat{E}'$,

$$\Phi_{\mathcal{O}} : \mathcal{F}_{E \times_M \hat{E}} \to \mathcal{F}_{E' \times_M \hat{E}'}.$$

It is also required to map h to $h' = \Phi_{\mathcal{O}}(h)$ in such a way that the basic part of the twist of the effective models based on the torus bundles E and E' is invariant. The idea is to use the fact that $h = db$ and to adapt the transformation (2.41), but instead of just exchanging A with $\hat{A}$, we can form an $\mathbb{R}^{2n}$-valued gauge potential $(A, \hat{A})^T$, act on it with $\mathcal{O}$,

$$\begin{pmatrix} A' \\ \hat{A}' \end{pmatrix} = \mathcal{O} \begin{pmatrix} A \\ \hat{A} \end{pmatrix},$$

and find

$$h' = h + \frac{1}{2} d(A' \wedge \hat{A}' - A \wedge \hat{A}).$$

The process to find the effective model based on E' is similar as the one for E. One starts with the action $S_{E' \times_M \hat{E}'}$ and chooses the UV sector to be made of the fiber coordinates of $\hat{E}'$, the first half of $\Theta' = \mathcal{O}\Theta$ and the ghosts.

Note that if we choose for $\mathcal{O}$ the particular element $K = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$, we obtain the same swap of the bundles E and $\hat{E}$ that we had in the case of circle bundles, namely $E' = \hat{E}$ and $\hat{E}' = E$. This was to be expected, as K is the only non-trivial element of $O(1,1;\mathbb{Z}) = \mathbb{Z}_2$.

To summarize, we saw that in the case of principal torus bundles, the T-duality group acts on topological field theories by BV morphisms. For three-dimensional TFTs, we recovered isomorphisms of Courant algebroids when the H-flux was T-dualizable. In two dimensions, through BV pushforwards from the various T-dual models, we were able to find the topological sectors of the T-dual string sigma models with background fields associated to these principal torus bundles and their relations corresponding to the topological content of the Buscher rules.

Chapter 3

Chern-Simons Theory with Wilson Lines and Boundary in the BV-BFV Formalism

3.1 Introduction

In the case of complicated space-time topology, a promising approach to quantization of general field theories involves cutting the space-time manifold into simple pieces, where the problem is more easily solved, and then gluing back the individual elements to obtain the final answer. This method was proposed by Atiyah and successfully applied by Witten in [43] to study the quantization of the Chern-Simons theory with Wilson lines (cf. also [26]). Following this idea, a systematic program to understand quantization in the Batalin-Vilkovisky formalism for field theories with degeneracies on manifolds with boundaries has been initiated in [18],[19]. As a part of the construction, a Batalin-Fradkin-Vilkovisky model is associated to the boundary of the space-time manifold. The canonical quantization of this boundary BFV model provides a space of boundary states, together with a cohomological invariance condition that defines the admissible quantum states of the theory among all boundary states. In the case of quantum field theories on manifolds without boundary, the partition function and other correlation functions are complex-valued. In the presence of a boundary, the correlators of the bulk theory take values in this boundary space of states.

The aim of this paper is to apply the BV-BFV formalism of [18],[19] to the

Chern-Simons theory on manifolds with boundary, with Wilson lines ending on the boundary. The BV formulation of this theory on closed manifolds is well understood (and served as the motivating example for the AKSZ construction). We will also consider the one-dimensional Chern-Simons model, obtained when the AKSZ construction is carried out in one dimension.

We include in our construction Wilson lines which may end on the boundary of the manifold. This requires some extra work in BV-BFV formalism. Our treatment is based on the path integral representation for Wilson loops suggested in [3], [23]. This is also an example of a more general construction of observables for AKSZ sigma models proposed in [37].

We compare our answers with those obtained using the geometric quantization for the boundary [43] and using canonical quantization [27], [25].

One of our main results is the boundary BFV action for Chern-Simons theory with Wilson lines, which has the form of an odd (degree 1) version of BF action modified by source terms for the B field at points where the Wilson lines meet the boundary. We also consider the toy model of one-dimensional Chern-Simons theory and derive the corresponding boundary action. Its quantization coincides with Kostant cubic Dirac operator. We compare the BV-BFV results for the one-dimensional model with the ones obtained in [5] on segments, and also see how Wilson lines can be added to the one-dimensional model. In the three-dimensional case, the boundary space of states, arising as the cohomology of the quantized BFV action, coincides with the space of conformal blocks of the WZW model on the boundary (in the picture of [27]).

We begin in section 3.2 with the treatment of Wilson lines in the BV formalism. We also provide a short introduction to the relevant aspects of the BV formalism. In section 3.3 we describe the BV formulation of the Chern-Simons theory with Wilson lines, applying the AKSZ construction to this special setting. Then we proceed to explain how the bulk model gets supplemented with a boundary BFV theory if the underlying manifold has a boundary. We repeat this procedure in section 3.4 for the one-dimensional Chern-Simons model. The $\mathbb{Z}_2$-grading that replaces the usual $\mathbb{Z}$-grading in this case leads to certain subtleties with the master equation. In section 3.5 we present the quantization of the boundary BFV models and describe the arising spaces of quantum states, that we compare with known results for the quantization of the involved models.

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3.2 Wilson lines in the BV formalism

In this section, we start with a brief introduction to the BV formalism. The main point is to incorporate the Wilson line observables in this approach.

3.2.1 A short introduction to the BV formalism

We know that the path integral in quantum field theories is not well defined if the classical action S_{cl} defined over the space of classical fields $\mathcal{F}_{cl}$ is degenerate, for instance due to gauge symmetries. The Batalin-Vilkovisky formalism provides a general method for the perturbative calculation of partition functions and correlators.

In the BV formalism, the space of fields is augmented to a BV space of fields $\mathcal{F}_{BV}$, a graded infinite-dimensional manifold equipped with a symplectic structure Ω_{BV} of degree -1 called the BV structure. The grading (usually $\mathbb{Z}$, sometimes $\mathbb{Z}_2$) is commonly referred to as "ghost number", in relation with the Faddeev-Popov prescription. The BV bracket is defined as the Poisson bracket obtained by inverting the BV structure,

$$\{F, G\} = \Omega_{BV}^{-1}(\delta F, \delta G),$$

and obviously has a ghost number 1. Note that the variational operator δ can be interpreted as a de Rham differential in the space of fields. In many cases of interest, the BV space of fields is a cotangent bundle where the degree of the fibers is shifted to -1, which ensures its canonical symplectic form has the proper degree. Coordinates along the cotangent fibers are then called antifields. In the case of gauge theories, where the degeneracy arises under the action of a gauge group, the BV space of fields is simply the shifted cotangent bundle of the BRST space of fields which contains all classical fields as well as the ghosts parametrizing the gauge symmetries (basically the infinitesimal gauge parameters with a ghost number shifted by one), $\mathcal{F}_{BV} = T^*[-1]\,\mathcal{F}_{BRST}$.

At the classical level, infinitesimal gauge transformations and the classical action can be used to construct a differential acting on the functionals on the BV space of fields. Geometrically, this differential corresponds to a cohomological vector field Q on $\mathcal{F}_{BV}$. Moreover, Q is a Hamiltonian vector field,

$$\imath_Q \Omega_{BV} = \delta S_{BV},$$

with the Hamiltonian function being the BV action S_{BV} that reduces to the classical action when all antifields are set to zero. The condition $Q^2 = 0$ follows from the classical master equation $\{S_{BV}, S_{BV}\} = 0$, and determining the BV formulation of a given theory amounts to determining an extension of the classical action to the BV space of fields that satisfies this classical master equation.

3.2.2 The AKSZ construction

While it is usually tedious to find the BV formulation of a given field theory with a degenerate action, the study of the geometric interpretation of the classical master equation in [6] led to an insightful procedure to construct solutions thereof, called the AKSZ construction after its authors. A formalized more recent treatment can be found in [16].

In this construction, the target space of the theory is a graded manifold Y equipped with a symplectic structure ω_Y of degree $n-1$ and a compatible cohomological vector field Q_Y of degree 1, in the sense that it preserves the symplectic structure, $\mathcal{L}_{Q_Y}\omega_Y = 0$. We also want ω_Y to be associated to a Liouville one-form α_Y (of degree $n-1$ as well), namely $\omega_Y = \delta\alpha_Y$. Here δ denotes the de Rham exterior derivative on Y, while we keep the usual d for the one on the source manifold N of the model. For $n \neq 0$ (see for instance [38] for details), Q_Y can be shown to be Hamiltonian, i.e. there exists a function Θ_Y of degree n on Y such that $\imath_{Q_Y}\omega_Y = \delta\Theta_Y$. Like in the BV formalism, the nilpotency of Q_Y follows from the condition $\{\Theta_Y, \Theta_Y\}_Y = 0$, where the curly braces with a subscript Y denote the Poisson bracket on Y associated to its symplectic structure.

The BV space of fields is then given by maps between the odd tangent bundle of some n-dimensional manifold N and the graded manifold Y,

$$\mathcal{F}_{AKSZ} = \mathrm{Map}(T\,[1]\,N, Y).$$

The odd tangent bundle is naturally equipped with a cohomological vector field, the de Rham vector field, which can be expressed as

$$D = \theta^\mu \frac{\partial}{\partial x^\mu}$$

in coordinates x^μ of the base manifold N and θ^μ of the odd fibers. Notice also that real-valued functions on $T\,[1]\,N$ can be interpreted as differential forms on N (by expanding the function in powers of θ^μ),

$$C^\infty(T\,[1]\,N, \mathbb{R}) \simeq \Omega^\bullet(N),$$

which allows to define a canonical measure μ on $T\,[1]\,N$: the Berezinian integration along all odd fibers simply extracts the top-form out of this expansion and it remains to integrate it over the base N.

Roughly, the idea behind the AKSZ construction involves lifting the symplectic structure ω_Y from the target space to define the BV structure Ω_{AKSZ} on the space of fields. In effect, we replace functions and differential forms on Y by functionals on the space of fields with values in differential forms on N and their variations (which also explains the choice of δ to denote the exterior derivative on Y), and we integrate over the source space $T\,[1]\,N$ using its canonical measure μ,

$$\Omega_{\text{AKSZ}} = \int_{T[1]N} \mu\, \tilde{\omega}_Y. \tag{3.1}$$

The tilde denotes the extension from function on Y to functional on the space of fields. The Berezinian integration along the fibers will lower the ghost number of ω_Y from $n-1$ to -1 as required.

In a second stage we need to lift the cohomological vector field Q_Y on the target-space Y as well as the de Rham vector field D on the source-space $T\,[1]\,N$ to the space of fields, and combine them to form the BV cohomological vector field Q discussed above that will happen to be Hamiltonian. Its generating functional is nothing but the BV-AKSZ action

$$S_{\text{AKSZ}} = \int_{T[1]N} \mu\ \left(\imath_{Q_D}\tilde{\alpha}_Y + \tilde{\Theta}_Y\right), \tag{3.2}$$

where $Q_D = \Sigma_i D\phi^i \frac{\delta}{\delta\phi^i}$ is the lift of the de Rham vector field (ϕ^i denotes generic coordinates on the space of fields). This AKSZ action automatically solves the classical master equation as a consequence of the integrability condition on Θ_Y and the fact that the integral of exact forms vanishes provided $\partial N = \emptyset$.

As an example of the AKSZ construction, we derive here the BV formulation of the Chern-Simons theory, which corresponds to the special case $n = 3$ with $Y = \mathfrak{g}\,[1]$, where $\mathfrak{g}$ is a Lie algebra equipped with an invariant scalar product. As required, the target space $\mathfrak{g}\,[1]$ supports a symplectic structure of degree 2 and a Hamiltonian cohomological vector field of degree 1 (sometimes called a Q-structure). If we denote with δ the exterior derivative on $\mathfrak{g}\,[1]$ and ψ a generic element, the symplectic form, its Liouville potential and the Hamiltonian of the cohomological vector field respectively can be written as

$$\omega_{\mathfrak{g}[1]} = -\frac{1}{2}(\delta\psi, \delta\psi), \tag{3.3}$$

$$\alpha_{\mathfrak{g}[1]} = -\frac{1}{2}(\psi, \delta\psi), \tag{3.4}$$

$$\Theta_{\mathfrak{g}[1]} = -\frac{1}{6}(\psi, [\psi, \psi]). \tag{3.5}$$

Note that Grassmanian variables ψ anticommute, but so do differential forms of odd degree, which explains why $\omega_{\mathfrak{g}[1]}$ may be built out of a symmetric product.

If we use coordinates $x^\mu, \mu = 1, 2, 3$ on N and corresponding Grassmanian coordinates θ^μ on the odd fibers of $T[1]N$, we can decompose the fields $\mathbf{A} \in \mathrm{Map}(T[1]N, \mathfrak{g}[1])$ into $\mathfrak{g}$-valued differential forms of various degrees and grading,

$$\mathbf{A} = \gamma + A_\mu\theta^\mu + \frac{1}{2}A^+_{\mu\nu}\theta^\mu\theta^\nu + \frac{1}{6}\gamma^+_{\mu\nu\sigma}\theta^\mu\theta^\nu\theta^\sigma, \tag{3.6}$$

specifically

$$\begin{array}{lcl} \gamma & \in & \mathrm{Map}(N, \mathfrak{g}[1]), \\ A & \in & \Gamma(T^*N \otimes \mathfrak{g}), \\ A^+ & \in & \Gamma(\bigwedge^2 T^*N \otimes \mathfrak{g}[-1]), \\ \gamma^+ & \in & \Gamma(\bigwedge^3 T^*N \otimes \mathfrak{g}[-2]). \end{array}$$

These fields are endowed with two gradings, namely the ghost-grading (that stands in square brackets when non-zero) and the degree as a differential form. Their sum, the total degree, should amount to 1, since each θ^μ has a ghost number 1, and all terms in the decomposition (3.6) should have the same total ghost number of 1. As usual, the fields of ghost number 0 are the classical fields, here a $\mathfrak{g}$-valued connection A, and the fields of ghost number 1 are simply called ghosts. The other two fields are their antifields (which in the BV formalism means canonically conjugated), as is clear when one computes the BV structure,

$$\begin{aligned} \Omega^{\mathrm{CS}}_{\mathrm{BV}} &= \int_{T[1]N} \mu\, \tilde{\omega}_{\mathfrak{g}[1]} = -\int_{T[1]N} \mu\, (\delta\mathbf{A}, \delta\mathbf{A}) \\ &= \int_N \big((\delta\gamma^+, \delta\gamma) - (\delta A^+, \delta A)\big) = \int_N \big(-(\delta\gamma, \delta\gamma^+) + (\delta A, \delta A^+)\big). \end{aligned} \tag{3.7}$$

Note that the commutation rules for the fields (which are simultaneously functions on $\mathcal{F}$ and differential forms on N) are determined by the total degree (the de Rham degree of the differential form plus the ghost number): two fields of odd total degree anticommute, and commute if at least one field has even total degree.

It remains to compute the BV action, which is straightforward in the AKSZ scheme,

$$
\begin{aligned}
S_{\mathrm{BV}}^{\mathrm{CS}} &= \int_{T[1]N} \mu \left(\imath_{Q_D} \tilde{\alpha}_{\mathfrak{g}[1]} + \tilde{\Theta}_{\mathfrak{g}[1]} \right) \\
&= \int_{T[1]N} \mu \left((\mathbf{A}, D\mathbf{A}) + \frac{1}{6} (\mathbf{A}, [\mathbf{A}, \mathbf{A}]) \right) \\
&= \int_N \left(\frac{1}{2} (A, dA) + \frac{1}{6} (A, [A, A]) - \left(A^+, d\gamma + [A, \gamma] \right) + \left(\gamma^+, \frac{1}{2} [\gamma, \gamma] \right) \right).
\end{aligned}
\tag{3.8}
$$

In the two terms involving only physical fields, we recognize the classical action of the Chern-Simons theory. The other terms complete the BV action, and by construction, it is clear that it satisfies the classical master equation. Nevertheless, we will show it explicitly, mainly to present an example of calculations in the space of fields, on which we will rely in the rest of this paper.

3.2.3 Calculations in the BV formalism

First of all, we need to find the BV bracket, simply by inverting the BV structure (3.7), without forgetting that the product between two differential forms in the space of fields really means the exterior product,

$$
\delta\phi^+ \delta\phi = \delta\phi^+ \wedge \delta\phi = \delta\phi^+ \otimes \delta\phi \pm \delta\phi \otimes \delta\phi^+,
$$

where the sign depends on the commutation rules between ϕ and ϕ^+. We find the following expression for the BV bracket of two functionals F_1 and F_2,

$$
\begin{aligned}
\{F_1, F_2\} = \int_N \Bigg(&\left(\frac{F_1 \overleftarrow{\delta}}{\delta\gamma}, \frac{\overrightarrow{\delta} F_2}{\delta\gamma^+} \right) - \left(\frac{F_1 \overleftarrow{\delta}}{\delta\gamma^+}, \frac{\overrightarrow{\delta} F_2}{\delta\gamma} \right) \\
&- \left(\frac{F_1 \overleftarrow{\delta}}{\delta A}, \frac{\overrightarrow{\delta} F_2}{\delta A^+} \right) + \left(\frac{F_1 \overleftarrow{\delta}}{\delta A^+}, \frac{\overrightarrow{\delta} F_2}{\delta A} \right) \Bigg)
\end{aligned}
\tag{3.9}
$$

where the functional derivatives $\frac{\overrightarrow{\delta}}{\delta\phi}$ and $\frac{\overleftarrow{\delta}}{\delta\phi}$ for $\phi \in \{A, A^+, \gamma, \gamma^+, g^+\}$ are the duals of the differentials $\delta\phi$ in the space of fields (which can be interpreted as variations of fields in the framework of variational calculus). We need to make the difference between right- and left-derivatives due to the commutation rules that depend on ghost numbers and degrees of differential forms on N.

All the fields of the Chern-Simons model are $\mathfrak{g}$-valued, so taking the functional derivative of a real-valued functional F on $\mathcal{F}$ by one of these fields should produce a $\mathfrak{g}^*$-valued result, but we can use the non-degenerate scalar product

$(\cdot,\cdot)$ to identify $\mathfrak{g}$ with its dual. If F is constructed as an integral, like an action, the left- and right-derivatives by a field $\phi \in \{A, A^+, \gamma, \gamma^+, g^+\}$ can be defined as the components of the exterior derivative with respect to the local frame induced by these coordinate-fields of $\mathcal{F}$,

$$\delta F(\phi_1, \ldots, \phi_n) = \int_{\partial N} \sum_{j=1}^{n} \left(\delta\phi_j, \frac{\overrightarrow{\delta} F}{\delta \phi_j} \right) = \int_{\partial N} \sum_{j=1}^{n} \left(\frac{F \overleftarrow{\delta}}{\delta \phi_j}, \delta\phi_j \right).$$

If on the other hand F depends only of the value of the fields at a given point, we may still express it as an integral provided we filter its position with a Dirac distribution, a distribution that will stick to the functional derivative.

At a later stage, we will need to consider Lie group-valued fields of the form $g \in \mathrm{Map}(N, G)$. The problem with such a field is that its variation does not take value in $\mathfrak{g}$, but rather in the tangent space at g of the Lie group G. Natural coupling with the other $\mathfrak{g}$-valued fields via the invariant scalar product involves the right multiplication by g^{-1} to bring it back to the Lie algebra, explicitly $\delta g\, g^{-1}$. The dual derivative $\frac{\delta}{\delta g}$ assumes its value in $T^*_{g^{-1}}G$, which is isomorphic to $T_{g^{-1}}G$ thanks to the invariant non-degenerate scalar product, and we need to apply this time left-multiplication by g to get back to the Lie algebra. If the functional F depends also on g, we find for the derivative

$$\delta F(\phi_1, \ldots, \phi_n, g) = \int_{\partial N} \sum_{j=1}^{n} \left(\delta\phi_j, \frac{\overrightarrow{\delta} F}{\delta \phi_j} \right) + \left(\delta g\, g^{-1}, g \frac{\overrightarrow{\delta} F}{\delta g} \right).$$

To compute $\left\{ S^{\mathrm{CS}}_{\mathrm{BV}}, S^{\mathrm{CS}}_{\mathrm{BV}} \right\}$, we need the derivatives of the Chern-Simons BV action. We find

$$\begin{aligned} \delta S^{\mathrm{CS}}_{\mathrm{BV}} = \int_N \Bigg(& \left(\delta A, dA + \frac{1}{2}[A, A] + \left[A^+, \gamma\right] \right) \\ & + \left(\delta\gamma, -d_A A^+ - \left[\gamma^+, \gamma\right] \right) \\ & + \left(\delta A^+, -d_A \gamma \right) + \left(\delta\gamma^+, \frac{1}{2}[\gamma, \gamma] \right) \Bigg), \end{aligned} \tag{3.10}$$

where we introduced the covariant derivative $d_A = d + [A, \cdot]$. Note that we need to integrate by parts to find the contribution of the exterior derivatives, such as

$$\int_N (A, \delta dA) = \int_N (A, d\delta A) = -\int_{\partial N} (A, \delta A) + \int_N (dA, \delta A),$$

where the boundary term vanishes for a closed manifold N. When we consider source spaces with boundaries, these terms will no longer vanish, and they will contribute to a one-form on the boundary space of fields. For now we can check the classical master equation,

$$\frac{1}{2}\left\{S_{\mathrm{BV}}^{\mathrm{CS}}, S_{\mathrm{BV}}^{\mathrm{CS}}\right\} = \int_N \left(\left(\frac{S_{\mathrm{BV}}^{\mathrm{CS}}\overleftarrow{\delta}}{\delta\gamma}, \frac{\overrightarrow{\delta}S_{\mathrm{BV}}^{\mathrm{CS}}}{\delta\gamma^+}\right) - \left(\frac{S_{\mathrm{BV}}^{\mathrm{CS}}\overleftarrow{\delta}}{\delta A}, \frac{\overrightarrow{\delta}S_{\mathrm{BV}}^{\mathrm{CS}}}{\delta A^+}\right)\right)$$
$$= \int_N \left(\left(d_A A^+ + \left[\gamma^+,\gamma\right], \frac{1}{2}\left[\gamma,\gamma\right]\right) - \left(dA + \frac{1}{2}\left[A,A\right] + \left[A^+,\gamma\right], -d_A\gamma\right)\right)$$
$$= 0.$$

In the last step we make repeated use of the invariance of the scalar product, the Jacobi identity for $\mathfrak{g}$, and the Stokes theorem.

We are now fully prepared to describe Wilson lines in the BV formalism.

3.2.4 Wilson Lines

In gauge theories, a degeneracy arises under the local action of a Lie group on the space of fields, the gauge group. In what follows, we will denote by G the gauge group, and $\mathfrak{g}$ its associated Lie algebra. The so-called gauge field is a connection A in a principal G-bundle over some manifold N. The gauge symmetry is parametrized in the BV (and BRST) formalism by a ghost field $\gamma \in \mathrm{Map}(N, \mathfrak{g}[1])$. The BV variation of these two fields depends only on their behavior under gauge transformations and not on the specific type of the underlying ambient theory. We assume that the dynamics and the gauge structure of this ambient theory is encoded in the BV action S^{amb} and the corresponding BV structure Ω^{amb} (both defined as integrals over N), and of course that S^{amb} solves the ambient classical master equation $\left\{S^{\mathrm{amb}}, S^{\mathrm{amb}}\right\}_{\mathrm{amb}} = 0$. The part of this ambient BV bracket involving the gauge connection and the ghost field relevant for our further investigation is defined by the BV variation of these fields, namely

$$\begin{array}{rcccl} \left\{S^{\mathrm{amb}}, A\right\} &=& Q\,A &=& d_A\gamma, \\ \left\{S^{\mathrm{amb}}, \gamma\right\} &=& Q\,\gamma &=& \frac{1}{2}\left[\gamma,\gamma\right]. \end{array} \tag{3.11}$$

Natural non-local observables to consider in gauge theories are given by Wilson-loops, traces of the holonomy of the connection A along a curve Γ embedded in N in given representations of the Lie algebra,

$$W_{\Gamma,R}\left[A\right] = \mathrm{Tr}_R \mathrm{Pexp}\left(\int_\Gamma A\right),$$

where P stands for the path-ordering and R labels the representation of $\mathfrak{g}$.

This cumbersome path-ordering can be removed at the price of integrating over all gauge transformations along the loop [3],

$$W_{\Gamma,R}\,[A] = \int \mathcal{D}g \exp\left(\int_\Gamma \langle T_0, g^{-1}Ag + g^{-1}dg\rangle\right). \tag{3.12}$$

The dual algebra element $T_0 \in \mathfrak{g}^*$ encodes the representation R, along the lines of the orbit method [30] that links unitary irreducible representations of Lie groups and their coadjoint orbits. This expression for the Wilson loop can be absorbed into an extended action by adding the auxiliary term

$$S_{\text{Wilson}} = \int_\Gamma \langle \text{Ad}^*_g(T_0), A + dg\,g^{-1}\rangle \tag{3.13}$$

to the ambient action S^{amb} of the model under consideration. In this last step we replaced the adjoint action on the second factor of the product by the coadjoint action on the first factor, to emphasize the role of the coadjoint orbit $\mathcal{O}$ of T_0.

Now we would like to find a BV formulation of this contribution, so as to obtain a BV action of the full model with Wilson loops. The partition function of such a model with an action extended to take into account a Wilson line as an auxiliary term actually corresponds to the expectation value of this Wilson line in the pure theory,

$$Z_{S^{\text{amb}}+S^{\text{aux}}} = \langle W_{\Gamma,R}\rangle_{S^{\text{amb}}}.$$

We note that the coadjoint orbit $\mathcal{O}$ supports the Kirillov symplectic structure $\omega_{\mathcal{O}}$, of ghost number 0, and that the curve Γ carrying the Wilson line has dimension 1, the first two main ingredients for the AKSZ construction for $n = 1$. It is thus tempting to try to apply the prescription proposed in [37] to construct observables within the AKSZ formalism. Nonetheless, [37] treats exclusively the case of an ambient theory of the AKSZ type, whereas we want to consider gauge theories, with the sole requirement that their space of fields contains a gauge connection and an associated ghost field obeying the relations (3.11). The obvious solution is to study a gauge theory of the AKSZ type, a condition fulfilled by the Chern-Simons model, the main subject of this paper. The BV formulation of the Wilson line contribution will happen to remain valid for other gauge theories.

So following [37], the auxiliary fields are the maps between the odd tangent bundle of the curve Γ and the coadjoint orbit,

$$\mathcal{F}^{\text{aux}} = \text{Map}(T\,[1]\,\Gamma, \mathcal{O}).$$

This auxiliary space of fields needs to be equipped with its own BV structure, Ω^{aux}, that once added to the BV structure Ω^{amb} of the ambient theory will provide the BV structure $\Omega = \Omega^{\mathrm{amb}} + \Omega^{\mathrm{aux}}$ of the full model with space of fields $\mathcal{F} = \mathcal{F}^{\mathrm{amb}} \oplus \mathcal{F}^{\mathrm{aux}}$. Then it will be possible to add to the ambient action S^{amb} an auxiliary term S^{aux} that obeys certain constraints to obtain a solution of the master equation of the full model.

The definition of Ω^{aux} is similar to the one of the AKSZ-BV structure (3.1), we just need to change the source space and the symplectic structure of the target,

$$\Omega^{\mathrm{aux}} = \int_{T[1]\Gamma} \mu_\Gamma \, \tilde{\omega}_{\mathcal{O}}.$$

Here μ_Γ is obviously the canonical measure on $T\,[1]\,\Gamma$.

Unfortunately, the Kirillov symplectic form on the coadjoint orbit is in general not exact, so it is in general not possible to find a Liouville one-form, which we would normally use to construct the kinetic term of the auxiliary action. However, in the case of integrable orbits, we may pick a line bundle (the pre-quantum line bundle in the language of geometric quantization) and a connection $\alpha_{\mathcal{O}}$ there with curvature $\omega_{\mathcal{O}}$,

$$\delta \alpha_{\mathcal{O}} = \omega_{\mathcal{O}}$$

(we recall that in the target spaces of AKSZ theories, we denote by δ the exterior derivative), and we can simply use this connection to construct the kinetic term of the auxiliary action.

This formulation is not very practical to carry out calculations. To find expressions easier to deal with, we apply the defining property of the Kirillov symplectic form, that the pullback by the projection map

$$\pi : G \to \mathcal{O} \simeq G/\mathrm{Stab}(T_0) \tag{3.14}$$

brings it to an explicit presymplectic form ω_G on G,

$$\pi^*(\omega_{\mathcal{O}}) = \omega_G = -\langle \mathrm{Ad}^*_g(T_0), \frac{1}{2}\left[\delta g\, g^{-1}, \delta g\, g^{-1}\right] \rangle. \tag{3.15}$$

This two-form is the contraction of T_0 with the exterior derivative of the Maurer-Cartan one-form on G. It thus admits a potential

$$\alpha_G = -\langle \mathrm{Ad}^*_g(T_0), \delta g\, g^{-1} \rangle. \tag{3.16}$$

As it happens, the pullback by the projection map π brings the connection $\alpha_{\mathcal{O}}$ over to the one-form α_G,

$$\pi^*(\alpha_{\mathcal{O}}) = \alpha_G,$$

that we will use in the place of the more cumbersome connection to compute certain quantities. Since ω_G is degenerate, it is not possible to construct a Poisson bracket out of it, unless we restrict it to invariant functions on G, such as the ones obtained by pullback of functions on the coadjoint orbit $\mathcal{O}$ by the projection map π.

It remains to define the interaction term. The idea of [37] is to construct a function $\Theta_{\mathcal{O}}$ on $\mathfrak{g}\,[1] \times \mathcal{O}$ that will generate together with $\Theta_{\mathfrak{g}[1]}$ a cohomological vector field on $\mathfrak{g}\,[1] \times \mathcal{O}$. While $\Theta_{\mathfrak{g}[1]}$ already satisfies an integrability condition on its own and generates a cohomological vector field $\mathcal{Q}_{\mathfrak{g}[1]}$, the integrability condition for $\Theta_{\mathcal{O}}$ needs to be slightly adapted to account for the mixed term, namely

$$\mathcal{Q}_{\mathfrak{g}[1]}\Theta_{\mathcal{O}} + \frac{1}{2}\left\{\Theta_{\mathcal{O}}, \Theta_{\mathcal{O}}\right\}_{\mathcal{O}} = 0.$$

As it happens, the function

$$\Theta_{\mathcal{O}} = \langle \mathrm{Ad}_g^*(T_0), \psi \rangle \tag{3.17}$$

satisfies this requirement and naturally extends the term $\langle \mathrm{Ad}_g^*(T_0), A \rangle$ that already appeared in the classical part (3.13). This integrability condition is most easily checked by pulling it back by π to a function on $\mathfrak{g}\,[1] \times G$, where the Poisson bracket $\{\cdot,\cdot\}_{\mathcal{O}}$ becomes $\{\cdot,\cdot\}_G$, which can be explicitly determined by inverting ω_G.

We now have all the ingredients to construct the auxiliary BV action, that we just need to combine into a formula similar to the usual AKSZ action (3.2),

$$S^{\mathrm{aux}} = \int_{T[1]\Gamma} \mu_\Gamma \, \left(\imath_{Q_D} \tilde{\alpha}_{\mathcal{O}} + \tilde{\Theta}_{\mathcal{O}} \right).$$

By construction, if S^{amb} is the AKSZ action of the Chern-Simons model, the total action

$$S = S^{\mathrm{amb}} + S^{\mathrm{aux}}$$

automatically satisfies the classical master equation generated by the total BV structure

$$\Omega = \Omega^{\mathrm{amb}} + \Omega^{\mathrm{aux}}.$$

We claimed that it remains true when S^{amb} is the BV action of a generic gauge theory with gauge group G. To verify this assertion, we need to compute

$$\frac{1}{2}\{S,S\} = \frac{1}{2}\left\{S^{\mathrm{amb}}, S^{\mathrm{amb}}\right\} + \left\{S^{\mathrm{amb}}, S^{\mathrm{aux}}\right\} + \frac{1}{2}\left\{S^{\mathrm{aux}}, S^{\mathrm{aux}}\right\}. \tag{3.18}$$

The first two terms involve only the ambient BV structure, since S^{amb} does not depend on the auxiliary fields. The first one vanishes due to the master equation of the BV ambient model. To compute the second term, we should know the exact dependence of the auxiliary term S^{aux} on the ambient fields A and γ, and to compute the last one, we need an expression of the auxiliary BV structure Ω^{aux} that we know how to invert.

These two issues can be addressed by using the projection map (3.14) to define an extended space of fields,

$$\hat{\mathcal{F}}^{\mathrm{aux}}_G = \pi^*(\mathcal{F}^{\mathrm{aux}}) = \left\{(g, g^+)|g \in \mathrm{Map}(\Gamma, G), g^+ \in \Omega^1(\Gamma) \otimes g^*(T\mathcal{O})\,[-1]\right\}.$$

The subscript G emphasizes the fact that the coadjoint orbit is replaced by the whole group.

This projection map, now seen as a map between spaces of fields,

$$\pi : \hat{\mathcal{F}}^{\mathrm{aux}}_G \to \mathcal{F}^{\mathrm{aux}},$$

acts on the group-valued component g by sending it to its image $\mathrm{Ad}^*_g(T_0)$ in the coadjoint orbit of T_0. It can be used to pull back differential forms on the auxiliary space of fields (such as the auxiliary BV structure, a two-form, or the auxiliary BV action, a zero-form) to this extended space of fields, where it is easier to compute BV brackets of G-invariant functionals given the explicit formulas for the pullbacks of the auxiliary BV structure and of the auxiliary action.

In $\hat{\mathcal{F}}^{\mathrm{aux}}_G$, both fields g and g^+ can be combined into a superfield of total degree 0 that we can use to express the pullback of the auxiliary BV structure and action,

$$\mathbf{H}(x, \theta) = \mathrm{Ad}^*_{g(x)}(T_0) - \theta g^+(x).$$

Here x is a coordinate of Γ and θ a Grassmanian coordinate on the odd fibers of $T\,[1]\,\Gamma$, and $g^+(x)$ is the component of the one-form g^+ expressed in this coordinate system, $g^+ = g^+(x)dx$.

We now have all the tools to compute the pullback of the auxiliary BV structure,

$$\begin{aligned}\hat{\Omega}^{\mathrm{aux}}_G = \pi^*(\Omega^{\mathrm{aux}}) &= \int_{T[1]\Gamma} \mu_\Gamma\ \pi^*(\tilde{\omega}_{\mathcal{O}}) = \int_{T[1]\Gamma} \mu_\Gamma\ \tilde{\omega}_G \\ &= -\int_{T[1]\Gamma} \mu\, \delta\langle \mathbf{H}, \delta g\, g^{-1}\rangle = \int_\Gamma \delta\langle g^+, \delta g\, g^{-1}\rangle \\ &= \int_\Gamma \langle \delta g^+, \delta g\, g^{-1}\rangle + \langle g^+, \frac{1}{2}\left[\delta g\, g^{-1}, \delta g\, g^{-1}\right]\rangle,\end{aligned} \tag{3.19}$$

and of the auxiliary BV action,

$$\begin{aligned}\hat{S}^{\text{aux}}_G = \pi^*(S^{\text{aux}}) &= \int_{T[1]\Gamma} \mu_\Gamma \ \left(\imath_{Q_D}\tilde{\alpha}_G + \pi^*(\tilde{\Theta}_{\mathcal{O}})\right) \\ &= \int_{T[1]\Gamma} \mu_\Gamma \ \left(\langle \mathbf{H}, Dg\, g^{-1}\rangle + \langle \mathbf{H}, \mathbf{A}\rangle\right) \\ &= \int_\Gamma \left(\langle \mathrm{Ad}^*_g(T_0), A + dg\, g^{-1}\rangle - \langle g^+, \gamma\rangle\right).\end{aligned} \tag{3.20}$$

The additional term $\langle g^+, \gamma\rangle$ encodes the action of gauge transformations of the ambient model on the auxiliary classical field $\mathrm{Ad}^*_g(T_0)$.

We insist on the fact that $\hat{\Omega}^{\text{aux}}_G$ is only a pre-BV structure in $\hat{\mathcal{F}}^{\text{aux}}_G$, it is closed but degenerate. Nevertheless, like its finite dimensional counterpart ω_G, it can be used to define a BV bracket on invariant functionals, such as the ones obtained by pullback from $\mathcal{F}^{\text{aux}}$, for instance $\hat{S}^{\text{aux}}_G$.

We can turn our attention to the pullback of the last two terms in (3.18). To compute the second one, we notice that $\left\{S^{\text{amb}}, \cdot\right\}$ acts as a differential (namely Q) on the fields of the ambient model, of which only A and γ appear in the auxiliary action (3.20), so that we may simply use the relations (3.11) to find

$$\begin{aligned}\left\{S^{\text{amb}}, \hat{S}^{\text{aux}}_G\right\} &= \int_\Gamma \left(\langle \mathrm{Ad}^*_g(T_0), Q(A)\rangle - \langle g^+, Q(\gamma)\rangle\right) \\ &= \int_\Gamma \left(\langle \mathrm{Ad}^*_g(T_0), d_A\gamma\rangle - \langle g^+, \frac{1}{2}\left[\gamma, \gamma\right]\rangle\right).\end{aligned} \tag{3.21}$$

Next, the bracket of the third term contains only contributions from the auxiliary structure, and we may compute

$$\begin{aligned}\frac{1}{2}\left\{\hat{S}^{\text{aux}}_G, \hat{S}^{\text{aux}}_G\right\} &= \int_\Gamma \left(\langle g \frac{\hat{S}^{\text{aux}}_G \overleftarrow{\delta}}{\delta g}, \frac{\overrightarrow{\delta}\hat{S}^{\text{aux}}_G}{\delta g^+}\rangle + \langle g^+, \frac{1}{2}\left[\frac{\hat{S}^{\text{aux}}_G \overleftarrow{\delta}}{\delta g^+}, \frac{\overrightarrow{\delta}\hat{S}^{\text{aux}}_G}{\delta g^+}\right]\rangle\right) \\ &= \int_\Gamma \left(\langle (d + \mathrm{ad}^*_A)\mathrm{Ad}^*_g(T_0), \gamma\rangle + \langle g^+, \frac{1}{2}\left[\gamma, \gamma\right]\rangle\right).\end{aligned} \tag{3.22}$$

The first line displays the BV bracket of invariant functionals on $\hat{\mathcal{F}}^{\text{aux}}_G$ constructed out of the pre-BV structure $\hat{\Omega}^{\text{aux}}_G$.

The sum of these two terms yields the integral of an exact term that vanishes since Γ is closed (for now), and the pullback of the classical master equation is satisfied,

$$\pi^*\left(\frac{1}{2}\left\{S^{\text{amb}} + S^{\text{aux}}, S^{\text{amb}} + S^{\text{aux}}\right\}\right) = 0.$$

Furthermore, since the left-hand side of this last equality is G-invariant, it behaves nicely enough under the projection π so that $S^{\text{amb}} + S^{\text{aux}}$ still solves the classical master equation at the level of $\mathcal{F} = \mathcal{F}^{\text{amb}} \oplus \text{Map}(T[1]\Gamma, \mathcal{O})$, also for generic gauge theories.

3.2.5 Quadratic Lie algebras

In many cases of interest, the Lie algebra $\mathfrak{g}$ is equipped with a non-degenerate scalar product $(\cdot, \cdot)$, which we can use to define an isomorphism $\beta : \mathfrak{g}^* \to \mathfrak{g}$, that we can apply to T_0, H and g^+. The relation $\beta(\text{Ad}_g^*(T_0)) = \text{Ad}_g(\beta(T_0))$ will be very useful, in particular we can replace the canonical pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ with the scalar product,

$$\langle \text{Ad}_g^*(T_0), \cdot \rangle = (\text{Ad}_g(\beta(T_0)), \cdot)\,,$$

and thus identify coadjoint orbits with adjoint orbits.

In the rest of this article, we will assume that $\mathfrak{g}$ admits such a non-degenerate scalar product. We will make use of it to write down all actions and BV structures, and we will simply consider T_0, H and g^+ to be elements of $\mathfrak{g}$ instead of its dual, $\mathfrak{g}^*$ (we drop the β for simplicity). Coadjoint orbits will therefore be identified with adjoint orbits. To summarize the main results of this section with this new convention, we can re-write the auxiliary BV structure (3.19) as

$$\hat{\Omega}_G^{\text{aux}} = \int_\Gamma (\delta g^+, \delta g\, g^{-1}) + (g^+, \frac{1}{2}\left[\delta g\, g^{-1}, \delta g\, g^{-1}\right]) \qquad (3.23)$$

and the auxiliary BV action (3.20) as

$$\hat{S}_G^{\text{aux}} = \int_\Gamma \big((\text{Ad}_g(T_0), A + dg\, g^{-1}) - (g^+, \gamma)\big)\,. \qquad (3.24)$$

3.3 3D Chern-Simons Theory with a Wilson line

Our main goal in this paper is to study the behavior of Chern-Simons models on a manifold N with boundaries and Wilson lines ending on these boundaries, both in three and one dimension, in the BV formalism. The presence of a boundary requires either a careful choice in the boundary conditions for the fields, so as to keep the classical master equation under control, or the application of the recently developed BV-BFV formalism for gauge theories with boundaries [18],[19], which presents the big advantage that it allows to glue pieces together along their boundary.

In order to obtain the BV-BFV formulation of the three-dimensional Chern-Simons model with a boundary supporting some Wilson lines, we first need to determine the BV theory of the bulk. Actually we already know all its ingredients. Obviously, we will treat the Wilson lines as an auxiliary part of the action, as described in section 3.2, added to the ambient Chern-Simons action (3.8),

$$S^{\text{amb}} = S^{CS}_{BV}.$$

To include Wilson lines to our model, say n of them, we need to extend the ambient space of fields

$$\mathcal{F}^{\text{amb}} = \text{Map}(T\,[1]\,N, \mathfrak{g}\,[1])$$

carrying the ambient BV structure $\Omega^{\text{amb}} = \Omega^{CS}_{BV}$ with an auxiliary part

$$\mathcal{F}^{\text{aux}} = \bigoplus_{k=1}^{n} \text{Map}(T\,[1]\,\Gamma_k, \mathcal{O}_k)$$

made of n components, one for each Wilson line labeled by k. We recall that we denote by $\mathcal{O}_k$ the (co)adjoint orbit of a Lie algebra element $T_{0,k}$ encoding the representation in which the k-th Wilson line is computed and by Γ_k the curve embedded in N supporting this Wilson line. The BV structure of this auxiliary space of fields is the sum of N copies of the auxiliary BV structure of a single Wilson line,

$$\Omega^{\text{aux}} = \sum_{k=1}^{n} \int_{T[1]\Gamma_k} \mu_{\Gamma_k}\, \tilde{\omega}_{\mathcal{O}_k}.$$

The auxiliary BV action is similarly constructed as a sum,

$$S^{\text{aux}} = \sum_{k=1}^{n} \int_{T[1]\Gamma_k} \mu_{\Gamma_k} \, \left(\imath_{Q_D} \tilde{\alpha}_{\mathcal{O}_k} + \tilde{\Theta}_{\mathcal{O}_k}\right).$$

From now on, unless specified otherwise, a superscript "amb" will always describe a quantity associated to the BV formulation of the bare Chern-Simons model, be it a BV structure, a BV action or a BV space of fields, "aux" will always describe a quantity associated to the BV formulation of the auxiliary contribution of n Wilson lines, and no superscript will mean a BV quantity of the full model, namely $\mathcal{F} = \mathcal{F}^{\text{amb}} \oplus \mathcal{F}^{\text{aux}}$, $\Omega = \Omega^{\text{amb}} + \Omega^{\text{aux}}$ and $S = S^{\text{amb}} + S^{\text{aux}}$.

Before we consider the case of a source manifold with boundary, we need to compute the Hamiltonian vector field Q generated by S, i.e. satisfying

$$\imath_Q \Omega = \delta S. \tag{3.25}$$

Once the BV structure Ω is inverted to form the BV bracket $\{\cdot, \cdot\}$, this is equivalent to

$$Q = \{S, \cdot\}\,.$$

This relation is linear in Q and S, namely $\imath_{Q^{\text{amb}}}\Omega = \delta S^{\text{amb}}$ and $\imath_{Q^{\text{aux}}}\Omega = \delta S^{\text{aux}}$. We will again use the projection maps $\pi_k : G \to \mathcal{O}_k$ to pull back the auxiliary action and the auxiliary BV structure to the space of fields $\bigoplus_{k=1}^{n} \hat{\mathcal{F}}_G^{\text{aux}}$ where the calculations are easier. Note that we need one copy of $\hat{\mathcal{F}}_G^{\text{aux}}$ for each Wilson line. The results can then be easily brought over to the actual auxiliary space of fields by the n projections π_k.

For the ambient Hamiltonian vector field we obtain

$$\begin{aligned} Q^{\text{amb}} = \left\{S^{\text{amb}}, \cdot\right\} &= \left(\left(d_A A^+ + \left[\gamma^+, \gamma\right]\right), \frac{\overrightarrow{\delta}}{\delta \gamma^+} \right) - \frac{1}{2}\left([\gamma, \gamma], \frac{\overrightarrow{\delta}}{\delta \gamma} \right) \\ &\quad - \left(\left(dA + \frac{1}{2}[A, A] + \left[A^+, \gamma\right] \right), \frac{\overrightarrow{\delta}}{\delta A^+} \right) + \left(d_A \gamma, \frac{\overrightarrow{\delta}}{\delta A} \right), \end{aligned} \tag{3.26}$$

and for its auxiliary counterpart

$$\begin{aligned} \hat{Q}_G^{\text{aux}} = \left\{\hat{S}_G^{\text{aux}}, \cdot\right\} &= -\sum_k \left(\text{Ad}_g\,(T_{0,k})\delta(\Gamma_k), \frac{\overrightarrow{\delta}}{\delta A^+} \right) - \left(d_A(\text{Ad}_g\,(T_{0,k})), \frac{\overrightarrow{\delta}}{\delta g^+} \right) \\ &\quad - \left(\gamma, g\frac{\overrightarrow{\delta}}{\delta g} \right) - \sum_k \left(g^+ \delta(\Gamma_k), \frac{\overrightarrow{\delta}}{\delta \gamma^+} \right). \end{aligned} \tag{3.27}$$

Here $\delta(\Gamma_k)$ denotes a Dirac distribution two-form centered on Γ_k to filter the curve out of the whole manifold N. The fields g and g^+ that appear in front of these Dirac two-forms are defined only on the Wilson lines. The functional derivatives appearing right after them act on functionals in the bulk, but their results are zero- and one-forms that make sense on the curves Γ_k. Since the ambient and the total actions solve classical master equations, we know that Q^{amb} and $Q = Q^{\text{amb}} + Q^{\text{aux}}$ are cohomological, but not Q^{aux}.

If the source space has a boundary, on which might end an open Wilson line along one or more curves Γ_k (for simplicity, we will assume all of them), these cohomological vector fields cease to be Hamiltonian due to boundary effects affecting the variation of the differentiated terms in the action. The integration by part required to compute the contribution of these terms to the variation of the action now contains a surface integral. The BV-BFV formalism is based on the observation that this correction can be seen as a one-form in the boundary space of fields $\mathcal{F}_\partial$. This boundary space of fields contains the restriction of the fields of the bulk BV theory to their value on the boundary of the source manifold, ∂N in the case of our ambient theory, $\bigcup_{k=1}^{n} \partial\Gamma_k$ for the auxiliary model describing the Wilson lines. We denote by

$$\pi_\partial : \mathcal{F} \to \mathcal{F}_\partial$$

the projection corresponding to this restriction. The correction to the Hamiltonian condition (3.25) can be expressed as

$$\delta S = \imath_Q \Omega + \pi_\partial^*(\alpha_\partial). \tag{3.28}$$

The exterior derivative of this one-form,

$$\Omega_\partial = \delta \alpha_\partial,$$

happens to be symplectic and is called the BFV structure. It is a two-form of ghost number 0 in the boundary space of fields. The corrected Hamiltonian condition (3.28) is linear in α_∂, too, so we may decompose the boundary BFV structure

$$\Omega_\partial = \Omega_\partial^{\mathrm{amb}} + \Omega_\partial^{\mathrm{aux}}$$

and compute it in two parts. The ambient one corresponds to the Chern-Simons model,

$$\Omega_\partial^{\mathrm{amb}} = \int_{\partial N} \left(\frac{1}{2} \left(\delta A, \delta A\right) + \left(\delta\gamma, \delta A^+\right) \right), \tag{3.29}$$

and could actually be derived in a two-dimensional adaptation of the AKSZ construction. To compute the auxiliary part, we make use of the usual trick to do calculations in the augmented space of fields. The result

$$\hat{\Omega}_{\partial,G}^{\mathrm{aux}} = \sum_k \int_{\partial\Gamma_k} \left(\mathrm{Ad}_g(T_{0,k}), \frac{1}{2} \left[\delta g\, g^{-1}, \delta g\, g^{-1}\right] \right) \tag{3.30}$$

is the sum of $2k$ copies of the symplectic form ω_G of the target space of the augmented space of fields $\hat{\mathcal{F}}_{\partial,G}^{\mathrm{aux}}$, one carried by each extremity of every Wilson line. Using the relation (3.15), we immediately find the BFV structure on the actual auxiliary boundary space of fields $\mathcal{F}_\partial^{\mathrm{aux}} = \bigoplus_{k=1}^n \mathrm{Map}(\partial\Gamma_k, \mathcal{O}_K)$,

$$\Omega_\partial^{\mathrm{aux}} = \sum_k \int_{\partial\Gamma_k} \tilde{\omega}_{\mathcal{O}_k}, \tag{3.31}$$

where $\tilde{\omega}_{\mathcal{O}_k}$ is evidently the Kirillov symplectic form on the k-th (co)adjoint orbit $\mathcal{O}_k$ lifted to the space of fields.

The curve Γ_k being one dimensional, we have $\partial\Gamma_k = \{z_k, z_k'\} \subset \partial N$, and what the last integral really means is $\int_{\partial\Gamma_k} \tilde{\omega}_{\mathcal{O}_k} = \tilde{\omega}_{\mathcal{O}_k}(z_k) - \tilde{\omega}_{\mathcal{O}_k}(z_k')$.

In the last step of the construction of the boundary BFV model, we know that the restriction of Q to the boundary surface ∂N is Hamiltonian with respect to the BFV structure, and the boundary BFV action is defined as its generating functional,

$$\imath_{Q_\partial}\Omega_\partial = \delta S_\partial.$$

Ghost number counting shows that the BFV action has ghost number 1.

In the case of the Chern-Simons model with Wilson lines, we calculate the restriction of $\hat{Q}_G = Q^{\mathrm{amb}} + \hat{Q}_G^{\mathrm{aux}}$ in the extended space of fields,

$$\begin{aligned}\hat{Q}_{\partial,G} = &-\left(\frac{1}{2}\left[\gamma,\gamma\right], \frac{\overrightarrow{\delta}}{\delta\gamma}\right) - \left(\left(dA + \frac{1}{2}\left[A,A\right] + \left[A^+,\gamma\right]\right), \frac{\overrightarrow{\delta}}{\delta A^+}\right) \\ &+ \left(d_A\gamma, \frac{\overrightarrow{\delta}}{\delta A}\right) - \sum_k \left(\mathrm{Ad}_g(T_{0,k})\delta(\Gamma_k), \frac{\overrightarrow{\delta}}{\delta A^+}\right) - \left(\gamma, g\frac{\overrightarrow{\delta}}{\delta g}\right),\end{aligned} \tag{3.32}$$

which leads to the two contributions

$$S_\partial^{\mathrm{amb}} = -\int_{\partial N}\left(\left(dA + \frac{1}{2}\left[A,A\right], \gamma\right) + \left(A^+, \frac{1}{2}\left[\gamma,\gamma\right]\right)\right) \tag{3.33}$$

and

$$\hat{S}_{\partial,G}^{\mathrm{aux}} = -\sum_k \int_{\partial\Gamma_k} (\mathrm{Ad}_g(T_{0,k}), \gamma) \tag{3.34}$$

to the boundary BFV action. As expected, the G-valued field g appears only in a (co)adjoint action, so the projection to the auxiliary space of fields is straightforward, and we obtain the BFV action

$$\begin{aligned}S_\partial = -\int_{\partial N}\Bigg(&\left(dA + \frac{1}{2}\left[A,A\right], \gamma\right) + \left(A^+, \frac{1}{2}\left[\gamma,\gamma\right]\right) \\ &+ \sum_k (\mathrm{Ad}_g(T_{0,k}), \gamma)\,(\delta^{(2)}(z_k) - \delta^{(2)}(z'_k))\Bigg).\end{aligned} \tag{3.35}$$

In the last line, we have cast everything into the integral over ∂N by making use of Dirac distributions centered on the extremities of the Wilson lines.

We recognize in the boundary BFV action of the Chern-Simons model with Wilson lines an odd version of the two-dimensional BF model with sources, where the role of the B field is taken over by the restriction to the boundary of the ghost field γ of the bulk theory.

We conclude this section with a short remark regarding the insertions (labeled by z_k and z'_k) of the boundary model. In our setting, with Wilson lines ending on the boundary, these insertions always come in pairs of points carrying the same representation, one insertion at each end of a Wilson line. If we consider Wilson graphs in the bulk model, which are a natural generalizations of the Wilson lines, we can obtain any configuration of points and representations as insertions. Wilson graphs are observables modeled after Wilson lines, but based on oriented graphs instead of curves. Each edge carries a representation of $\mathfrak{g}$ and contributes with a similar term as a Wilson line to the total action, while each vertex carries an intertwining operator between the representations of the attached edges. If the formulation of these intertwining operators is straightforward in the operator formalism, their description is more involved in the path-integral formalism and goes beyond the scope of this paper, where we will for simplicity consider only Wilson loops and open Wilson lines.

3.4 1D Chern-Simons Theory with a Wilson line

The AKSZ construction for the Chern-Simons model can also be carried out in one dimension [5]. In this section, we will see how to add a Wilson line to this model, by following the same procedure as in the previous section. The main difference comes from the fact that the Wilson line is now a space-filling observable, and that the BV bracket of the auxiliary term with itself will pick up terms from the ambient part of the BV structure. Moreover, as stated before, we will now use a $\mathbb{Z}_2$-grading, since a $\mathbb{Z}$-grading is not possible in one dimension, so instead of denoting the ghost number in square brackets, we will use the parity-reversing operator Π.

Given a one-dimensional manifold Γ (in general a disjoint union of circles and open segments), the space of fields is

$$\mathcal{F}^{\text{amb}} = \text{Map}(\Pi T\Gamma, \Pi\mathfrak{g}), \tag{3.36}$$

where $\mathfrak{g}$ is again assumed to be equipped with an invariant scalar product $(\cdot, \cdot)$.

The target space $\Pi\mathfrak{g}$ supports the same geometric structures as before, that we may again transpose to the space of fields.

If x is a coordinate on Γ and θ a Grassmanian coordinate on the odd fibers of $\Pi T\Gamma$, we can decompose the fields $\Psi \in \text{Map}(\Pi T\Gamma, \Pi\mathfrak{g})$ into a $\mathfrak{g}$-valued fermion ψ and a $\mathfrak{g}$-valued one-form $A = A(x)dx$,

$$\Psi = \psi + \theta A(x). \tag{3.37}$$

We repeat the same procedure to find the BV structure

$$\Omega^{\text{amb}} = -\int_{\Pi\Gamma} \mu\,(\delta\Psi, \delta\Psi) = \int_{\Gamma} (\delta\psi, \delta A) \tag{3.38}$$

and the BV action

$$S^{\text{amb}} = \int_{\Pi\Gamma} \mu \left(\frac{1}{2}\,(\Psi, D\Psi) + \frac{1}{6}\,(\Psi, [\Psi, \Psi]) \right) = \int_{\Gamma} \frac{1}{2}\,(\psi, d_A\psi)\,. \tag{3.39}$$

The $\mathfrak{g}$-valued one-form A can be interpreted as a connection for the trivial principal G-bundle over Γ, where G is a Lie group integrating $\mathfrak{g}$. The odd $\mathfrak{g}$-valued scalar ψ serves simultaneously as a ghost for the gauge symmetry and an antifield for A.

In order to add Wilson lines to this model, we need to extend the space of fields, the BV structure and the action with precisely the same auxiliary structure Ω^{aux} and action S^{aux} as in the three-dimensional case, except that they are now supported directly by the base manifold of the ambient source space Γ. We consider a single Wilson line for simplicity, it is easy to add similar terms for additional lines. Furthermore we assume it covers the whole source space Γ. Actually it could involve only some of the connected components of Γ, and the other ones would support a bare (in the sense that there are no Wilson lines) one-dimensional Chern-Simons model. Notice also that instead of γ we write ψ to emphasize the fact that it plays simultaneously the role of γ and A^+ of the previous model.

We insist once more that since the Wilson line is a space-filling observable, we need to check that the classical master equation is solved, a result which is not guaranteed by the AKSZ construction due to the term $\{S^{\text{aux}}, S^{\text{aux}}\}_{\text{ambient}}$ coming from the auxiliary part of the action and the ambient part of the BV structure. If we use again the projection map $\pi : G \to \mathcal{O}$ to pull differential forms from the auxiliary space of fields $\mathcal{F}^{\text{aux}}$ to the extended one $\hat{\mathcal{F}}^{\text{aux}}_G$, we can calculate

$$\begin{aligned}
\frac{1}{2}\left\{\hat{S}_G, \hat{S}_G\right\} &= \int_\Gamma \left(\left(\frac{\hat{S}_G \overleftarrow{\delta}}{\delta A}, \frac{\overrightarrow{\delta}\hat{S}_G}{\delta\psi}\right) + \left(g\frac{\hat{S}_G \overleftarrow{\delta}}{\delta g}, \frac{\overrightarrow{\delta}\hat{S}_G}{\delta g^+}\right)\right.\\
&\qquad\left. -\frac{1}{2}\left(\frac{\hat{S}_G \overleftarrow{\delta}}{\delta g^+}, \left[g^+, \frac{\overrightarrow{\delta}\hat{S}_G}{\delta g^+}\right]\right)\right)\\
&= \int_\Gamma \left(-\frac{1}{2}[\psi,\psi] + \mathrm{Ad}_g T_0, d_A\psi + g^+\right) + (-d_A(\mathrm{Ad}_g T_0), -\psi)\\
&\qquad - \frac{1}{2}\left(\psi, \left[g^+, -\psi\right]\right)\\
&= \int_\Gamma \left(\mathrm{Ad}_g T_0, g^+\right)\\
&= -\frac{1}{2}\int_{\Pi T\Gamma} \mu\left(\mathbf{H}, \mathbf{H}\right)\\
&= -\frac{1}{2}\int_{\Pi T\Gamma} \mu\left(\frac{S^{\text{aux}}\overleftarrow{\delta}}{\delta\Psi}, \frac{\overrightarrow{\delta}S^{\text{aux}}}{\delta\Psi}\right),
\end{aligned} \tag{3.40}$$

and the last line shows it explicitly. Nevertheless, this term vanishes, since g^+ takes value in the tangent space $T_H\mathcal{O}$ at $H = \mathrm{Ad}_g T_0$ to the adjoint orbit, which is easily seen to be orthogonal to H with respect to the invariant scalar product on $\mathfrak{g}$.

Again, before we turn to the case of a source space with a boundary, we need to compute the Hamiltonian cohomological vector field Q generated by S, or more accurately its counterpart in the extended space of fields, namely

$$\begin{aligned}\hat{Q}_G = &\left(\left(d_A\psi + g^+\right), \frac{\delta}{\delta A}\right) + \left(\left(-\frac{1}{2}\left[\psi,\psi\right] + \mathrm{Ad}_g(T_0)\right), \frac{\delta}{\delta \psi}\right) \\ &- \left(\psi, g\frac{\delta}{\delta g}\right) - \left(\left(\left[\psi, g^+\right] + d_A(\mathrm{Ad}_g(T_0))\right), \frac{\delta}{\delta g^+}\right).\end{aligned} \tag{3.41}$$

If the source space has a boundary, in other words if some of its components are segments, we can repeat the procedure to construct the BFV boundary model. We first calculate the image of the symplectic potential of the boundary BFV structure in the augmented space of fields from the variation of the BV action,

$$\hat{\alpha}_{\partial,G} = \int_{\partial\Gamma}\left(\frac{1}{2}\left(\psi,\delta\psi\right) + \left(T_0, g^{-1}\delta g\right)\right), \tag{3.42}$$

and the corresponding pre-BFV structure,

$$\hat{\Omega}_{\partial,G} = \int_{\partial\Gamma}\left(\frac{1}{2}\left(\delta\psi,\delta\psi\right) - \frac{1}{2}\left(\mathrm{Ad}_g(T_0), \left[\delta g\, g^{-1}, \delta g\, g^{-1}\right]\right)\right). \tag{3.43}$$

We see that the second term is connected to the pullback by the projection map $\pi : G \to \mathcal{O}$ of the Kirillov-Kostant-Souriau symplectic structure, and we obtain as a BFV structure in the proper boundary space of fields

$$\Omega_\partial = \int_{\partial\Gamma}\left(\frac{1}{2}\left(\delta\psi,\delta\psi\right) + \tilde{\omega}_{\mathcal{O}}\right). \tag{3.44}$$

In all these expressions, the integral over $\partial\Gamma$ is nothing but a sum over the boundary points, with each term carrying a sign given by the orientation of its segment.

The restriction of the cohomological vector field $\hat{Q}_G$ to the boundary,

$$\hat{Q}_{\partial,G} = \left(\left(-\frac{1}{2}\left[\psi,\psi\right] + \mathrm{Ad}_g(T_0)\right), \frac{\delta}{\delta\psi}\right) - \left(\psi, g\frac{\delta}{\delta g}\right), \tag{3.45}$$

is Hamiltonian with respect to the BFV structure, and it is generated by the BFV action of the boundary model,

$$S_\partial = \int_{\partial I}\left(-\frac{1}{6}\left(\psi,\left[\psi,\psi\right]\right) + \left(\mathrm{Ad}_g(T_0),\psi\right)\right). \tag{3.46}$$

Finally we show that S_∂ solves the master equation of the BFV model,

$$\{S_\partial, S_\partial\} = Q_\partial S_\partial = \int_{\partial\Gamma} (T_0, T_0) = 0. \tag{3.47}$$

As stated before, the last integral is really a sum over the boundary elements of $\partial\Gamma$ with a sign assigned to their orientation, and since they come in pairs, at each end of every segment, the overall sum vanishes.

3.5 Boundary Quantum States

Upon quantization, the partition function and correlators of a field theory defined on a manifold N without boundary are complex numbers. In the presence of a boundary, one rather has quantum states, elements of a Hilbert space associated to each component of the boundary ∂N, according to the Atiyah-Segal picture of quantum field theory. The disjoint union of boundary components corresponds to the tensor product of the associated Hilbert spaces. Then gluing together a pair of components of ∂N corresponds to taking the scalar product of the two corresponding factors of the tensor product.

For instance, if $N = [0, 1]$ is an interval, the partition function of the BV-BFV model should take value in some Hilbert space of the form $\mathcal{H}\otimes\mathcal{H}$, with one factor for each component of the boundary $\partial N = \{0, 1\}$, and upon gluing the two ends, contracting this tensor product using the scalar product on $\mathcal{H}$ should yield the BV partition function of the same model constructed on the circle S^1.

If the bulk theory is studied in the BV formalism, the boundary information is encoded in the associated BFV model, at least at the classical level, as we saw in the particular cases of the Chern-Simons theory in one and three dimensions, possibly with Wilson lines.

To pass to the quantum level, we first observe that a BFV boundary model can be canonically quantized. The BFV structure, a symplectic structure of ghost number 0 in the space of fields, is used to define the (anti)commutation rules for the quantized fields. These act on the Hilbert space $\mathcal{H}_\partial^{\mathrm{BFV}}$ associated to the boundary where the partition function of the bulk takes value. This Hilbert space inherits a grading from the ghost number of the classical fields. In this picture, the BFV action S_∂, which was the generator of the cohomological vector field on the boundary Q_∂, can be quantized by replacing the classical fields with their quantized counterparts, and we obtain the quantized BFV charge $\hat{S}_\partial$. Its action on the boundary space of states squares to zero and it roughly encodes the gauge transformations. At the classical level, a physical observable is a functional annihilated by the cohomological vector field Q generated by the BV action in the bulk and the BFV action S_∂ on the boundary, and two observables are gauge equivalent if they differ by a Q-exact term. At the quantum level, the role of Q is taken over by the BFV charge $\hat{S}_\partial$: a gauge invariant boundary state should be annihilated by the BFV charge, and two states are gauge-equivalent if they differ by a BFV-exact term. Moreover, we require physical states to depend only on physical quantum fields, and not on the ghosts or the antifields. In other words, the space of boundary quantum states should correspond to the BFV-cohomology at ghost number zero $H^0_{\hat{S}_\partial}(\mathcal{H}^{\mathrm{BFV}}_\partial)$.

The relation with the quantized bulk theory is that the partition function (and all the other correlators) should be gauge invariant and therefore belong to this cohomology $H^0_{\hat{S}_\partial}(\mathcal{H}^{\mathrm{BFV}}_\partial)$. Its determination thus becomes a subject of interest.

We will start with the one-dimensional Chern-Simons theory, a simpler model where all calculations can be done until the end, before we study the more interesting three-dimensional model.

3.5.1 1D Chern-Simons Theory

The zero-dimensional boundary model of the one-dimensional Chern-Simons theory contains $\mathfrak{g}$-valued fermions and bosonic fields $H = \mathrm{Ad}_g(T_0)$ which take value in the (co)adjoint orbit $\mathcal{O}$. Once quantized, the fermions form a Clifford algebra $Cl(\mathfrak{g})$. If $(t^a)_{a=1}^{\dim\mathfrak{g}}$ is an orthonormal basis of $\mathfrak{g}$ with structure constants f_{abc}, we obtain the anticommutation rules

$$\left[\hat{\psi}_a, \hat{\psi}_b\right] = \hbar\delta_{ab} \tag{3.48}$$

for the quantized fermions.

For the bosonic content of the model, the Kirillov symplectic form on the (co)adjoint orbits is the inverse of the restriction from $\mathfrak{g}^* \simeq \mathfrak{g}$ to $\mathcal{O}$ of the Kirillov-Kostant-Souriau Poisson structure, so that the commutator of two $\mathcal{O}$-valued quantized fields is simply given by their Lie bracket. If we use the basis (t_a) of the Lie algebra to write

$$H = \mathrm{Ad}_g(T_0) = X_a\, t_a,$$

we can express the commutation rules with the structure constants of the Lie algebra,

$$\left[\hat{X}_a, \hat{X}_b\right] = \hbar f_{abc}\hat{X}_c. \tag{3.49}$$

The corresponding sector of the algebra of quantum operators is a representation of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$, namely $\rho_R(\mathcal{U}(\mathfrak{g})) \subset \mathrm{End}(V_R)$. This representation is simply the representation R in which we computed the Wilson loops in the previous sections.

We can use these operators $\hat{\psi}$ and $\hat{X}$ to construct the expectation value of the Wilson line $\langle W_{\Gamma,R}\rangle$ in the operator formalism, such as in [5], where the partition function for the one-dimensional Chern-Simons model is derived in both the path-integral and the operator formalism. If the curve Γ is open, this expectation value takes values in the boundary space of quantum states which is the cohomology in degree 0 of the quantum BFV charge,

$$\langle W_{I,R}\rangle \in H^0_{\hat{S}_\partial}(\mathcal{H}_\partial).$$

We need to find this cohomology.

The BFV charge

$$\hat{S}_\partial = \int_{\partial\Gamma} \hat{X}_a\hat{\psi}_a - \frac{1}{6} f_{abc}\hat{\psi}_a\hat{\psi}_b\hat{\psi}_c \tag{3.50}$$

carries one copy of the cubic Dirac operator [4]

$$\mathfrak{D} = \hat{X}_a\hat{\psi}_a - \frac{1}{6} f_{abc}\hat{\psi}_a\hat{\psi}_b\hat{\psi}_c$$

at each boundary point of Γ. This operator squares to

$$\mathfrak{D}^2 = \frac{1}{2}[\mathfrak{D},\mathfrak{D}] = \frac{1}{2}\hat{X}_a\hat{X}_a - \frac{1}{48} f_{abc}f_{abc},$$

a central element in the quantum Weil algebra $\mathcal{U}(\mathfrak{g}) \otimes Cl(\mathfrak{g})$, which guarantees that the action of the BFV charge squares to zero.

For $\mathfrak{g}$ simple, $\mathfrak{D}^2$ is non vanishing for all irreducible representations (including the trivial representation). Hence, the corresponding cohomology vanishes [4]. In other situations, $\mathfrak{D}^2$ may vanish. One example is $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^*$ the semi-direct sum of a Lie algebra $\mathfrak{h}$ and its dual. Then, every irreducible representation of $\mathfrak{h}$ gives rise to an irreducible representation of $\mathfrak{g}$ ($\mathfrak{h}^*$ acts by zero), and $\mathfrak{D}^2$ vanishes for all such representations.

3.5.2 3D Chern-Simons Theory

We may now repeat the same procedure for the three-dimensional model. The first observation is that the treatment of the part coming from the extremities of the Wilson lines, namely the terms in the insertion points labeled by z_k

and z'_k, is essentially the same as in the one-dimensional model. Each insertion contributes to the overall BFV structure with a term in (3.31), that when canonically quantized gives the algebra of operators (3.49) we encountered in the quantization of the one-dimensional model. We can formally express the quantization map

$$\mathrm{Ad}_{g(z_k)}(T_{k,0}) = X_{k,a}(z_k)t_a \mapsto \rho_k(\hat{X}_a(z_k))t^a.$$

Even though the orbits might be different for different insertions, the commutation rules (3.49) are identical for all of them, only the representation ρ_k differs, as we emphasized on the right-hand side.

In the next step, if we choose a complex structure on the boundary surface $\Sigma = \partial N$, we get a polarization of the connection

$$A = A_z dz + A_{\overline{z}} d\overline{z}, \tag{3.51}$$

which allows us to re-write the ambient part of the BFV structure (3.29) in Darboux coordinates of the corresponding sector $\mathcal{F}_\partial^{\mathrm{amb}}$ of the BFV space of boundary fields,

$$\Omega_\partial^{\mathrm{amb}} = \int_{\partial N} dz d\overline{z} \left((\delta A_z \delta A_{\overline{z}}) + \left(\delta\gamma, \delta A^+\right)\right). \tag{3.52}$$

Consequently, we may perform the canonical quantization by choosing among each pair of conjugated fields one quantum field and replace the other one by the corresponding functional differential, for instance

$$\begin{array}{rcl} A_{\overline{z}} & \to & a, \\ A_z & \to & -\frac{\delta}{\delta a}, \\ \gamma & \to & \gamma, \\ A^+ & \to & \frac{\delta}{\delta\gamma}, \end{array} \tag{3.53}$$

so as to obtain canonical (anti)commutation rules. Note that a is a boson and γ a fermion.

The Hilbert space $\mathcal{H}_\partial^{\mathrm{BFV}}$ of boundary states on which all these operators act is therefore the space of functionals in a and γ with values in the tensor product of all the representation spaces associated to each insertion,

$$\mathcal{H}_\partial^{\mathrm{BFV}} = \mathrm{Fun}\left(a, \gamma; \bigotimes_k V_{\rho_k} \otimes V_{\rho_k}\right).$$

We recall that it is graded by the ghost number.

Among these states, we want to determine the cohomology $H^0_{\hat{S}_\partial}(\mathcal{H}^{\mathrm{BFV}}_\partial)$ of the BFV charge at ghost number zero, made up of the quantum states of the BV-BFV model. At degree zero, we are considering functionals ψ of the $\mathfrak{g}$-valued $(0,1)$-form a, independent of the ghosts γ, which take value in the tensor product of all representation spaces associated to the extremities of the Wilson lines of the models.

The BFV charge

$$\begin{aligned}\hat{S}_\partial = -\int_{\partial N} dz d\overline{z} \Bigg(&\left(\left(\partial a + \overline{\partial}\frac{\delta}{\delta a} + \left[a, \frac{\delta}{\delta a}\right], \gamma\right) \right. \\ &- \sum_k \left(\rho_k(\hat{X}_a(z_k))\delta(z - z_k) - \rho_k(\hat{X}_a(z_{k'}))\delta(z - z_{k'})\right)(t^a, \gamma) \\ &+ \left. \left(\frac{1}{2}[\gamma,\gamma], \frac{\delta}{\delta\gamma}\right)\right)\Bigg)\end{aligned} \tag{3.54}$$

acts on the Hilbert space $\mathcal{H}^{\mathrm{BFV}}_\partial$ via multiplication and differentiation by the quantum fields a and γ and via the obvious action of the representation ρ_k on its representation space V_{ρ_k}.

At ghost number zero, BFV quantum states ψ are therefore subject to the condition

$$\left(\partial a + \overline{\partial}\frac{\delta}{\delta a} + \left[a, \frac{\delta}{\delta a}\right] - \sum_k \left(\rho_k(\hat{X}_a(z_k))\delta_{z_k}(z) - \rho_k(\hat{X}_a(z_{k'}))\delta_{z_k}(z)\right) t^a\right)\psi = 0. \tag{3.55}$$

This actually coincides with the constraint (1) in [27] imposed on the Schrödinger picture states in the canonical quantization of the Chern-Simons model on $\Sigma\times\mathbb{R}$ in genus 0, or the constraint (2.2) in [25] in the same situation in genus 1, where it is found that the cohomology $H^0_{\hat{S}_\partial}(\mathcal{H}^{\mathrm{BFV}}_\partial)$ coincides with the space of conformal blocks in the WZW model for a correlator of fields inserted at the extremities of the Wilson lines.

The condition (3.55) for quantum states also appears in the geometric quantization framework, see for instance constraint (3.4) in [43], therefore the space of states in geometric quantization coincides with the space of quantum boundary states in BV-BFV quantization.

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