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THÈSE

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par

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«On some Aspects of the Behaviour of Paths and Interfaces in Discrete and Continuous Models: Random-Cluster Model, Self-repelling Polymers and Brownian Motion»

La Faculté des sciences, sur le préavis de Monsieur H. DUMINIL-COPIN, professeur ordinaire et directeur de thèse (Section de mathématiques), Monsieur A. ALEXEEV, professeur ordinaire (Section de mathématiques), Monsieur Y. VELENIK, professeur ordinaire (Section de mathématiques), Monsieur I. MANOLESCU, professeur (Université de Fribourg, Chemin du Musée 23, 1700 Fribourg, Suisse), autorise l'impression de la présente thèse, sans exprimer d'opinion sur les propositions qui y sont énoncées.

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N.B. - La thèse doit porter la déclaration précédente et remplir les conditions énumérées dans les "Informations relatives aux thèses de doctorat à l'Université de Genève".

Introduction

This work is composed of three self-contained parts, where the different models of statistical physics are discussed.

In Chapter 1 we discuss the random-cluster model. It plays a key role in studies of models on lattices, as it is connected to many of them. The results obtained for the random-cluster model give information about other models, especially the Ising model, describing the behaviour of ferromagnetic materials. Thus all conclusions made for the random-cluster model have an impact in statistical physics. In this thesis we present another proof of the well-known fact that for $\mathbb{Z}^2$ the critical probability of the random-cluster model p_{cr} is equal to $\frac{\sqrt{q}}{1+\sqrt{q}}$ for $q \in [1, 4]$. This proof involves the method of parafermionic observables applied to exploration paths in boxes and strips of growing size. It was presented in a joint paper with E. Mukoseeva [MS18].

In Chapter 3 we study the behaviour of random walks on $\mathbb{Z}^2$ under self-repelling polymers measure. It is a generalisation of a model called self-avoiding walks. Both of them are models to describe the behaviour of polymer chains. Due to the expected universality of these models the results obtained on $\mathbb{Z}^d$ hold for any discrete lattice, and for the physical case, which take place in continuous space. We show that, as for self-avoiding walks, self-repelling polymers are sub-ballistic in $\mathbb{Z}^d$ with $d \geq 2$, i.e that the probability for the walk to go linearly (on the number of steps) far is exponentially small. This result was presented in [Smi17].

Both these models are discrete and defined on a lattice, here we restrict ourselves to $\mathbb{Z}^2$ in both cases. We are interested in limiting behaviour of these models, but the methods we use are strongly linked to the fact that the models are discrete. The paths defined during the discussion (self-repelling polymers for Chapter 3 and exploration paths at criticality for the random-cluster model in Chapter 1) are believed to be conformally invariant in the scaling limit. Moreover, both exploration paths for random-cluster model and self-avoiding walks (an extreme case of self-repelling polymers) are conjectured to converge to the process called the Schramm-Loewner Evolution SLE_κ, although for different values of a parameter κ (for self-avoiding walks, κ should be equal to $\frac{8}{3}$). In the random-cluster model, κ is related to the parameter q through the equation $\kappa = \frac{4\pi}{\arccos(-\frac{1}{2}\sqrt{q})}$ if $q \leq 4$. There are few cases, when this convergence is proven. It is the case of Bernoulli percolation with $q = 1$ and $\kappa = 6$, and for the Ising model $q = 2$, and corresponding $\kappa = 3$. When $q > 4$ the behaviour at criticality is completely different and does not converge to the Schramm-Loewner Evolution.

In the remaining chapter (Chapter 2) the model and the spaces we work with are continuous. Also we work in three dimensions, while for other chapters the proofs are based on the planar lattice. Moreover, we are not restricted to the Euclidian space. We compare the behaviour of the Brownian motion in the Euclidian space $\mathbb{R}^3$ and in the spaces of constant non-zero curvature, namely a three-dimensional sphere and a three-dimensional hyperbolic space. Projections of these distributions under certain moment maps corresponds to the Duistermaat-Heckmann measure.

Résumé

Cette these est composé de trois parties séparées, ou on discute des trois modèles different de la physique statistique.

Dans le premier chapitre on discute sur le modèle qui s'appelle le modèle de random cluster. Il a un rôle principal dans la théorie des modèles statistiques sur des réseaux, car il est lié au grand nombre de tels modèles. Les résultats qu'on peut obtenir à cette base peuvent être transmis aux autres modèles, particulièrement au modèle d'Ising. Alors les conclusions on peut faire sur le modèle des clusters aléatoires nous donnent l'information sur le comportement de matériels ferromagnétiques.

Ici on présente la nouvelle preuve du fait bien connu que la probabilité critique du modèle des clusters aléatoires à $\mathbb{Z}^2$ est égale à la probabilité de l'auto-dualité: $p_{cr}(q) = p_{sd}(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$ pour $q \in [1,4]$. Cette preuve est basée sur la méthode qui s'appelle l'observable parafermionique. On étudie les chemins d'exploration dans les boîtes et des bandes dont taille converge à l'infini.

L'autre partie de ce these est aussi consacrée au modèle défini sur $\mathbb{Z}^2$. C'est une mesure sur les marches aléatoires de longueur fixe, qui s'appelle des polymères auto-repoussants. Elle nous intéresse comme un generalisation des marches auto-évitantes, qui sont des objets du grand intérêt pour des mathématiciens. Ces deux modèles sont utilisés pour prédire la forme des fibres polymériques. Grâce à l'universalité de ces modèles les résultats obtenus pour $\mathbb{Z}^2$ sont préservés aux autres réseaux planaires, ou même pour le cas des objets réels dans l'espace continu.

Dans ce chapitre on montre que le modèle des polymères auto-repoussants est (comme les marches auto-evitantes) sous-balistique sur $\mathbb{Z}^2$. C'est à dire que la probabilité d'aller à la distance linéaire par rapport au nombre de pas est exponentiellement petite. Cela implique aussi que des polymères auto-repoussants sont sous-balistiques sur $\mathbb{Z}^d$ pour les dimensions plus grandes que deux.

Les modèles précédents sont discrètes et définis sur des réseaux, ini on travaille sur $\mathbb{Z}^2$ pour les deux cas. On s'intéresse à la limite avec la taille convergente à l'infini, mais les méthodes de preuves sont liés à la nature discrète des modèles. Au plus les techniques appliquées sont sur la base du fait qu'on travaille dans espaces planaires.

Les objets définis ont des autres similarités. On croit que les marches correspondants aux modèles (des polymères auto-repoussants ils-mêmes et des chemins d'exploration pour le modèle des clusters aléatoires) ont l'invariance conforme dans la limite d'échelle. Au plus il y a une conjecture que les deux des chemins d'exploration et des marches auto-évitantes (le cas extreme du modèle des polymères auto-repoussants) convergent au processus s'appelé l'evolution de Schramm-Loewner SLE_κ, mais au valeur du parameter κ different. La valeur de κ doit être $\frac{8}{3}$ pour des marches auto-évitantes, et dans le modèle des clusters aléatoires $\kappa(q) = \frac{4\pi}{\arccos(-\frac{1}{2}\sqrt{q})}$ pour $q \in [1,4]$.

D'autre part dans le chapitre substituant le modèle est continu et des espaces sont de trois dimensions. Au plus des espaces sont même pas strictement Euclidiens. On compare le mouvement Brownien sur $\mathbb{R}^3$ et sur l'espaces de la courbure constante non-nulle (la sphère et l'espace hyperbolique de trois dimensions). Les projections de les distributions engendrées par ces mouvements Browniens correspondent aux mesures dew Duistermaat-Heckmann.

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Chapter 1

Another computation of p_c for random-cluster model with $q \in [1, 4]$

The random-cluster model, or Fortuin-Kasteleyn percolation (FK percolation), is a probability model on edge configurations of a lattice. It was defined in 1969 by K. Fortuin and P. Kasteleyn [FK72, For72a, For72b] as a two-parametric generalisation of Bernoulli percolation, one of the most simple and well-known lattice models. Bernoulli percolation model is described using only the edge parameter $p \in [0, 1]$: edges are open with probability p independently of each other. For the random-cluster model, the probability of the configuration also takes into account the number of connected components in it, this relation being described using a positive weight parameter q. In the random-cluster model there is a general dependence between the state of different edges via the probability of the configuration depends also of the number of the connected components.

1.1 Definition of a random-cluster configuration

Let us consider a finite graph $G = \{V(G), E(G)\}$ with defined set of boundary vertices $\partial G \subset V$. We will work with a subset G of the square lattice $\mathbb{Z}^d$, mostly $\mathbb{Z}^2$, with naturally defined ∂G.

Similarly to edge Bernoulli percolation, a *configuration* $\omega \in \{0, 1\}^{E(G)}$ is defined as a set of *"closed"* and *"open"* labels, associated to each edge e of the graph, that are denoted by $\omega(e) = 0$ and $\omega(e) = 1$ correspondingly. Vertices $x \in V(G)$ and $y \in V(G)$ are called connected in some particular configuration ω if there is a set of open edges that forms a path from x to y. This event is denoted $x \longleftrightarrow y$. The notation $x \underset{G'}{\longleftrightarrow} y$ stays for the event that x and y are connected through open edges, that belong to some subgraph G' of G. These definitions are easily extended to connect edges. A connected set of open edges C is called a *cluster* if it is maximal (i.e any $e \notin C$ is either closed or not connected to edges of C). If no open edges are attached to a vertex, we count it as a cluster without edges.

The difference between Bernoulli percolation and the random-cluster model lay in the definition of the probability of each configuration. In Bernoulli percolation the formula is the following:

$$\phi_{G,p}(\omega) = \prod_{e \in E(G)} p^{\omega(e)}(1-p)^{(1-\omega(e))} = p^{o(\omega)}(1-p)^{c(\omega)},$$

where $o(\omega)$ and $c(\omega)$ is the number of open and closed edges in the configuration. The single parameter $p \in [0,1]$ stands for probability of each edge to be open, independently of the other edges.

In FK percolation one more parameter $q > 0$ is added. It plugs into the probability formula the number of clusters in a configuration, denoted $K^{\#}(\omega)$, and

{FKdef}

$$\phi^{\xi}_{G,p,q}(\omega) \sim p^{o(\omega)}(1-p)^{c(\omega)} q^{K^{\xi}(\omega)}. \tag{1.1}$$

To become a probability measure, this function needs to be normalised. The normalisation coefficient

{FKdef2}

$$Z^{\xi}(G,p,q) = \sum_{\omega \in \{0,1\}^{E(G)}} p^{o(\omega)}(1-p)^{c(\omega)} q^{K^{\xi}(\omega)} \tag{1.2}$$

is called a *partition function.*

The symbol ξ in (1.1)–(1.2) denotes different ways to compute the number of clusters in a configuration, depending on *boundary conditions* (see Figure 1.1). In the case of *free* boundary conditions ($\xi = 0$) we just compute the number of the clusters in the configuration. Another widely used case is *wired* boundary conditions ($\xi = 1$), when all clusters touching the boundary are counted as one. Another case we would like to mention takes place for planar graphs (although it can be extended to bigger dimensions). Suppose we can treat ∂G as a simple circuit. For any two vertices $a, b \in \partial G$ we define *Dobrushin* boundary conditions, denoted (a, b), as wired on the boundary arc from a to b clockwise and free for the rest of ∂G. In a general case we can define boundary conditions ξ as a partition of a set ∂G into subsets which are seen as connected in one cluster. One more special example called *periodic* boundary conditions can be defined on a regular box $\Lambda^d \in \mathbb{Z}^d$ by identifying and gluing together vertices on the opposite boundaries. One-dimensional domain becomes a circle, two-dimensional box becomes a torus, and so on.

1.2 Properties of the model

1.2.1 Domain Markov property

Let us look at G' a subgraph of G. Domain Markov property describes the influence that a configuration outside of G' has on the probability measure in G'. This property is crucial in studying of random-cluster model because it allows to decouple events taking place in the different parts of the graph and treat them as independent.

Say ω is a configuration on G, and denote $\omega_{|G'}$ its restriction to G'. Fix some boundary conditions ξ, and let ψ be some fixed configuration on $G \backslash G'$.

Let us define boundary conditions on G' as follows: two clusters linked to $\partial G'$ are counted as one if they are connected in ψ (including the connections in ξ). These boundary

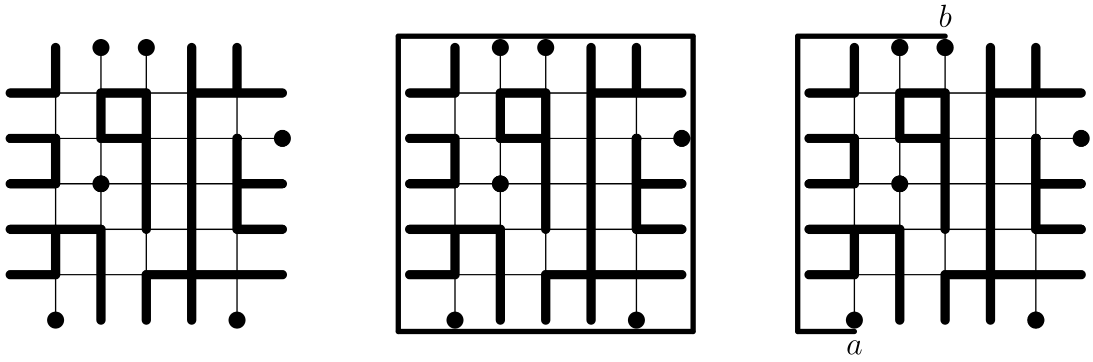

Figure 1.1: A configuration ω with different boundary conditions — free boundary conditions ($K^0(\omega) = 12$ including ones with no edges), wired boundary conditions ($K^1(\omega) = 3$), and (a, b)-Dobrushin boundary conditions ($K^{(a,b)}(\omega) = 7$). From now on, open edges will be pictured in thick lines.

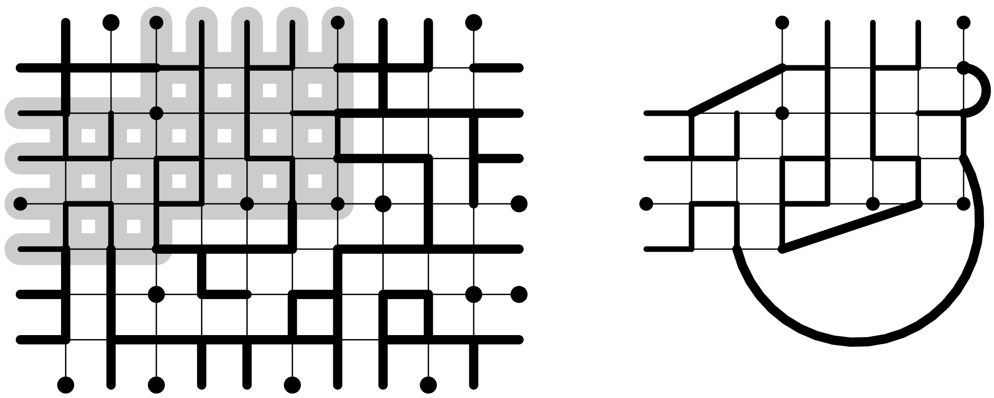

Figure 1.2: Influence of an outer configuration and outer boundary conditions (free in this case) on a configuration on a subgraph (highlighted in grey).

conditions are called *induced* by the configuration ψ with boundary conditions ξ, and denoted by (ψ, ξ).

The Domain Markov property states that

$$\phi^{\xi}_{G,p,q}(\omega \,|\, \omega_{|G\setminus G'} = \psi) = \phi^{(\psi,\xi)}_{G',p,q}(\omega_{|G}) \tag{1.3}$$ {DM}

for any $p \in [0, 1]$ and $q > 0$. The configuration outside the subgraph impacts the inside only via induced boundary conditions(see Figure 1.2).

1.2.2 Finite energy property

This property allows to compare the probability of two configurations if they are somehow close to each other.

Let us define the *Hamming distance* between two configurations ω and ω' as the number of edges where these configurations differ:

$$H(\omega, \omega') = |\{e \in E(G) : \omega(e) \neq \omega'(e)\}|.$$

Fix $q > 0$ and an interval $[\varepsilon, 1-\varepsilon]$ of possible values of p. Then the following relation holds

{finen}

$$c^{-H(\omega,\omega')} \leq \frac{\phi^{\xi}_{G,p,q}(\omega)}{\phi^{\xi}_{G,p,q}(\omega')} \leq c^{H(\omega,\omega')}. \tag{1.4}$$

The constant $c = c(\varepsilon, q) > 0$ does not depend on G, choice of the configurations, choice of p inside the interval or the boundary conditions ξ.

Then for a fixed value of p, this property can also be rewritten in the following way. Let ω_e (ω^e) be the configurations obtained from some configuration ω by forcing the edge e to be closed (open). Then,

lltoclosed}

$$\frac{p}{q(1-p)} \leq \frac{\phi^{\xi}_{G,p,q}(\omega^e) + \phi^{\xi}_{G,p,q}(\omega_e)}{\phi^{\xi}_{G,p,q}(\omega_e)} \leq \frac{p}{(1-p)} \tag{1.5}$$

for any G, p, ξ and any $q \geq 1$.

1.2.3 FKG inequality

It can be easily seen from (1.1), that if $q = 1$, number of clusters does not play any role, and FK percolation turns into Bernoulli percolation. In fact the value $q = 1$ is a crucial border between different behaviour of the model.

Let us call the event A *increasing* if for any ω where A holds, opening any edge in ω cannot break the realisation of A. The examples of such events are:

- the fixed edge e is open
- two vertices are connected, $x \longleftrightarrow y$
- there is a cluster of a size bigger then some constant
- intersection of any increasing events
- union of any increasing events

Increasing events play the key role in studying of the random-cluster model because they can be composed together through the following inequality:

Theorem 1.1 (Fortuin-Kasteleyn-Ginibre or FKG inequality, [FKG71])**.** *If A and B are two increasing events, and $q \geq 1$, then*

{FKG}

$$\phi^{\xi}_{G,p,q}(A \cap B) \geq \phi^{\xi}_{G,p,q}(A)\phi^{\xi}_{G,p,q}(B) \tag{1.6}$$

for any $p \in [0,1]$, and any G and boundary conditions ξ.

This is also called positive association, meaning that for $q \geq 1$ random-cluster measures are positively correlated (see [Gri06] for more details).

We can define decreasing events. In this case, closing an edge will favour an event. The FKG inequality can be written for this case as well. Many things we can deduce for increasing events hold for decreasing events as well.

The FKG inequality does not always hold in the case $q < 1$. Moreover, random-cluster model with small values of q has some negative correlation properties [Pem04]. Yet, almost nothing is understood for $q < 1$. From now on, we work only with the case $q \geq 1$.

1.2.4 Comparison inequalities

In this subsection, we provide some inequalities allowing to compare the probabilities of increasing events for different probability measures.

Comparison for different values of q and p

It is somehow natural that bigger edge probability parameter p is, more edges should be open. This fact can be written as a following inequality.

Statement 1.2 (Comparison in p)**.** *For any increasing event A and any $0 \leq p_1 \leq p_2 \leq 1$*

$$\phi^{\xi}_{G,p_1,q}(A) \leq \phi^{\xi}_{G,p_2,q}(A) \tag{1.7}$$ {ineqp}

for any G, any boundary conditions ξ and any $q \geq 1$.

If we allow q to vary as well, we can write two inequalities [For72b, Gri06], including a generalisation of the previous one:

- If $q_1 \geq q_2 \geq 1$ and $p_1 \leq p_2$, then

  $$\phi^{\xi}_{G,p_1,q_1}(A) \leq \phi^{\xi}_{G,p_2,q_2}(A)$$

  for any G, any boundary conditions ξ and any increasing event A.

- If $q_1 \geq q_2 \geq 1$ and $\frac{1}{q_1}\frac{p_1}{1-p_1} \geq \frac{1}{q_2}\frac{p_2}{1-p_2}$, then

  $$\phi^{\xi}_{G,p_1,q}(A) \geq \phi^{\xi}_{G,p_2,q}(A)$$

  for any G, any boundary conditions ξ and any increasing event A.

Different boundary conditions

We can define a partial order on the set of all possible boundary conditions as follows. For boundary conditions ξ and χ, say that $\xi \geq \chi$ if any two boundary points seen as parts of one cluster in χ have the same property in ξ. Then, for any increasing A and any G, p and $q \geq 1$,

$$\phi^{\xi}_{G,p,q}(A) \geq \phi^{\chi}_{G,p,q}(A). \tag{1.8}$$ {boundcond}

This inequality, together with the definition of induced boundary conditions, can be used to compare the measures in different domains. For example, if $G' \subset G$ and A is defined on G', then

$$\phi^{1}_{G',p,q}(A) \geq \phi^{\xi}_{G,p,q}(A) \geq \phi^{0}_{G',p,q}(A) \tag{1.9}$$ {boundcond1

for any p, any $q \geq 1$ and any ξ.

1.2.5 Pivotal edges

The edge e is called *pivotal* for some increasing event A in a given configuration ω if A holds for ω with e forced to be open and does not hold for ω with e forced to be close. Note that the fact that the edge is pivotal does not depend on the state of this edge in ω. The following inequality holds for any increasing event A, boundary conditions ξ and graph G [Gri06]:

$$\frac{d}{dp}\phi^{\xi}_{G,p,q}(A) \geq c \sum_{e\in E(G)} \phi^{\xi}_{G,p,q}(e \text{ is pivotal for } A), \tag{1.10}$$

{difineq}

where c does not depend on G, A or p.

1.3 Infinite-volume measures

Although the finite case can pose problems, which are non-trivial and useful for mathematics, the behaviour of the random-cluster model on infinite graphs attracts a lot more attention and interest. In a finite domain, the probabilities of all events are expressed as quotients of some polynomials of p and q, therefore change smoothly when these parameters vary. It is not the case for infinite-volume measures. Some crucial qualitative properties of the model can change sharply, providing so called *phase transitions*, which pose one of the main interest for the researchers.

1.3.1 Construction of infinite-volume measures

Let us restrict ourself to regular infinite graphs, or *lattices*, especially to $\mathbb{Z}^d$. The generalisation of (1.1)–(1.2) for the infinite case seems to be nontrivial, and there are different approaches to construct these measures.

First approach is to take a limit of finite graphs of some regularity, for example square boxes of increasing size. Let us define a box Λ_n in a trivial way:

$$\Lambda^d_n = \{x = (x_i)^d_{i=1} \in \mathbb{Z}^d : \forall i \ |x_i| \leq n\},$$

with a boundary

$$\partial\Lambda^d_n = \{x \in \Lambda^d_n : \exists\, i \text{ such that } |x_i| = n\}.$$

Later we will omit the indication of the dimension d, if it will be clear.

Let us study the limit of measures $\phi^{\xi_n}_{p,q,\Lambda^d_n}$ as n goes to infinity. We will first look only on free and wired boundary conditions, $\xi \in \{0,1\}$. We will define the limit of measures looking at the convergence of the probabilities of all events (not necessarily increasing) defined on finite boxes.

Theorem 1.3 (Thermodynamic limit [For72b, Gri06])**.** *For any $p \in [0,1]$ and $q \geq 1$, the limits*

nfmeas1eq0}

$$\phi^0_{\mathbb{Z}^d,p,q} = \lim_{n\longrightarrow\infty} \phi^0_{\Lambda^d_n,p,q} \quad \textit{and} \quad \phi^1_{\mathbb{Z}^d,p,q} = \lim_{n\longrightarrow\infty} \phi^1_{\Lambda^d_n,p,q} \tag{1.11}$$

exist.

Moreover, they are extreme in the sense that for any increasing event A defined in a finite box

{infmeasex}

$$\phi^0_{\mathbb{Z}^d,p,q}(A) \le \phi^\xi_{\mathbb{Z}^d,p,q}(A) \le \phi^1_{\mathbb{Z}^d,p,q}(A) \tag{1.12}$$

for any existing limit

$$\phi^\xi_{\mathbb{Z}^d,p,q} = \lim_{n\longrightarrow\infty} \phi^{\xi_n}_{\Lambda^d_n,p,q}. \tag{1.13}$$

{infmeas1ec

Restrictions on boxes in Theorem 1.3 can be made weaker. The limit does not depend on the shape and the central points of the boxes, the only things required is for them to be nested and have not very large boundary.

We should mention one important property of measures (1.11).

Theorem 1.4. *Measures $\phi^0_{\mathbb{Z}^d,p,q}$ and $\phi^1_{\mathbb{Z}^d,p,q}$ have translational invariance. Moreover, they are* ergodic *[Kre85], i.e all translational invariant events have probability either* 0 *or* 1*.*

Also, let us note that all the properties mentioned in Section 1.2 are preserved for these measures as well.

The second approach is due to Dobrushin, Lanford and Ruelle [Dob68, LR69]. It uses the definition of induced boundary conditions and (1.3). The measure ϕ is called *DLR infinite-volume* random cluster measure [Gri06, Gri95] for some $p \in [0,1]$ and $q \ge 1$ if for any finite $G \subset \mathbb{Z}^d$

$$\phi(\omega|_G = \eta \mid w|_{\mathbb{Z}^d\setminus G} = \psi) = \phi^\psi_{G,p,q}(\eta) \tag{1.14}$$

{infmeas2ec

for any η a configuration on G and ψ a configuration on $\mathbb{Z}^d\setminus G$. The set of all infinite-volume measures satisfying (1.14) is denoted $\mathcal{R}_{\mathbb{Z}^d,p,q}$.

The relation between these two definitions is the following:

Theorem 1.5 ([Gri95])**.** *Any measure defined as in* (1.13)*, that has translational invariance and finite energy property, is included in $\mathcal{R}_{\mathbb{Z}^d,p,q}$ [BK89, BK91]. In particular,*

$$\phi^0_{\mathbb{Z}^d,p,q}, \phi^1_{\mathbb{Z}^d,p,q} \in \mathcal{R}_{\mathbb{Z}^d,p,q}.$$

Moreover, $\phi^0_{\mathbb{Z}^d,p,q}$ and $\phi^1_{\mathbb{Z}^d,p,q}$ are extreme in the sense as in (1.12)*.*

1.3.2 Uniqueness of infinite-volume measures

Theorem 1.6 ([Gri06])**.** *If $q \ge 1$, for any increasing event A*

$$\phi^0_{\mathbb{Z}^d,p,q}(A) = \phi^1_{\mathbb{Z}^d,p,q}(A) \tag{1.15}$$

{measeqeq}

for any $p \in [0,1]$ except countably many values of p.

The proof of this theorem is based on differentiability of a quantity called *free energy*:

$$f(p,q) = \lim_{n\longrightarrow\infty} \frac{1}{|E(\Lambda_n)|} \log Z^\xi_{\Lambda_n,p,q},$$

it exists and does not depend on the choice of boundary conditions or sequence of boxes. The equality (1.15) breaks in the points where $f(p,q)$ is not differentiable.

1.3.3 Phase transition

Theorem 1.7. *There exists $p_c \in (0,1)$, such that*

- *there is almost surely no cluster of infinite size if $p < p_c$*
- *there exists almost surely a cluster of infinite size if $p > p_c$*

Theorem 1.7 is deduced easily from Theorems 1.3, 1.4 and 1.6 with

{pcdef} $$p_c = \inf\{p \in [0,1] : \phi^0_{\mathbb{Z}^d,p,q}(0 \longleftrightarrow \infty) > 0\}. \tag{1.16}$$

The fact that p_c does not equal to 0 or 1 can be proven via Peierls argument, presented in [Pei36].

Moreover, due to Theorem 1.4 and [BK89], there will be almost surely either no infinite clusters at all, or unique infinite cluster.

Rising questions

Apart of existence of a phase transition, which is expected of be true and is proven to occur for many models of statistical physics, there are other more complicated questions to study about it.

The next most natural question is the exact value of p_c. Yet it was found only for two-dimensional case [BDC12]. In bigger dimension we have nothing more that computer estimations for some particular cases [Has10].

Another interesting problem is the behaviour of the model at the critical point, for example the uniqueness or non-uniqueness of infinite measure, or the existence of the infinite cluster for this measure (or measures). The similarity of these questions for $\mathbb{Z}^2$ is proven in [DCST17]. Another question close to these ones is the continuity of the phase transition. It is formulated as the verity of the equation

{contdef} $$\lim_{p \searrow p_c} \phi_{p,q}(0 \longrightarrow \infty) = 0. \tag{1.17}$$

These questions are solved in two-dimensional case. For $q \in [1,4]$ the model is continuous, the infinite-volume measure at critical point is unique and there is no infinite cluster almost surely [DCST17]. When $q > 4$, the phase transition is discontinuous [DGH$^+$16].

The phase transition is called to be sharp if for $p < p_c$ the probability for two vertices to be connected decreases exponentially fast with increase of the distance between them

{sharpdef} $$\phi_{p,q}(x \longleftrightarrow y) < e^{-C|x-y|} \text{ for some } C > 0. \tag{1.18}$$

This property holds for random-cluster model on $\mathbb{Z}^d$ for $q \geq 1$ [DRT17].

1.4 Connection to other models

The most important property of the random-cluster model is its relation to many other models of statistical physics via some couplings or limiting convergence. In this section we will describe some of them.

1.4.1 Ising and Potts models

Ising model, defined by Lenz and his student Ising in [Isi25], and its generalisation for more than two values of spins called Potts model [Pot52], are very important for statistical physics. They have some fields of applications, like behaviour of ferromagnetic materials or solid-state physics.

Definition of the model

Let us take G a finite subgraph of $\mathbb{Z}^d$. We define a configuration $\sigma \in \{1, \dots, q\}^{V(G)}$ for some $q \in \mathbb{N}$. The Ising model corresponds to $q = 2$, any greater value of q leads to the Potts model. Note that spin values are defined on vertices. For any vertex x, its spin for a given configuration is denoted σ_x. The *Hamiltonian* of a configuration is defined as

$$H(\sigma) = - \sum_{(x,y)\in E(G)} \mathbb{I}_{\sigma_x = \sigma_y}, \tag{1.19}$$

{Ham}

Note that we look only at the situation with no external magnetic field and with equal interaction strength between all neighbouring vertices. Then, the probability of a configuration σ is defined as

$$\pi_{G,\beta,q}(\sigma) = \frac{1}{Z_P(G,\beta,q)} \cdot e^{-\beta H(\sigma)}, \tag{1.20}$$

{pottsmeas

where the partition function $Z_P(G,\beta,q)$ is defined in the same way as in (1.2). The parameter $\beta \in [0,\infty)$ has the physical interpretation as the inverse of the temperature.

As well as for random-cluster, we can define infinite-volume measures and study values of β providing a phase transition between different behaviours of the system. Also we can define the wired measure $\pi^1_{G,\beta,q}(\sigma)$ in the same way as in (1.20) with all vertices of ∂G forced to have the spin equal to 1 (or any other measure $\pi^i_{G,\beta,q}$ for $i \in \{1, \dots, q\}$).

Edwards-Sokal coupling

The connection between random-cluster and Potts models was observed from the very beginning of the development of the random-cluster model. There are several ways to couple them, but the most used is one defined by R. Edwards and A. Sokal [ES88]. Let us consider the product space

$$(\omega, \sigma) \in \{0,1\}^{E(G)} \times \{1, \dots, q\}^{V(G)},$$

the measure on it is defined as

$$\mu(\omega,\sigma) \sim \prod_{e=(x,y)\in E(G)} \Big((1-p) \cdot \mathbb{I}_{w(e)=0} + p \cdot \mathbb{I}_{w(e)=1} \cdot \mathbb{I}_{\sigma_x=\sigma_y} \Big) \tag{1.21}$$

{EScoupmeas

with proper normalisation coefficient. Let us note that this measure is nonzero only in situation when for any edge $e = (x,y)$ which is open in random-cluster configuration, the spins on the ends of this edge do not differ, i.e

$$\omega(x,y) = 1 \Rightarrow \sigma_x = \sigma_y.$$

Theorem 1.8 (Marginal measures)**.** *Let $q \in \mathbb{N}$ be not smaller than 2, and $p \in [0,1)$ admit the relation*

$$p = 1 - e^{-\beta}. \tag{1.22}$$ {ESpbeta}

Then the following statements hold for $\mu(\omega, \sigma)$ defined as in (1.21)

- *marginal measure on Potts configurations coincides with Potts measure*
- *marginal measure on random cluster configurations coincides with random cluster measure*

Also, there is a following link between partition functions of random-cluster and Potts models

$$Z^0_{RC}(G, p, q) = e^{-\beta|E(G)|} Z_P(G, \beta, q)$$

Here is an algorithm to construct a random configuration for one model based on a given configuration for the other model.

Theorem 1.9 (Conditional measures)**.** *Let us fix some q, p and β as in Theorem 1.8. Then*

- *For given $\omega \in \{0,1\}^{E(G)}$ the conditional measure $\mu(\sigma \,|\, \omega)$ is obtained by attributing a uniformly independently taken spin from $\{1 \dots q\}$ to each cluster*
- *For given $\sigma \in \{1 \dots q\}^{V(G)}$ the conditional measure $\mu(\omega \,|\, \sigma)$ is obtained by forcing edges (x, y) to be closed if $\sigma_x \neq \sigma_y$ and opening them independently with probability p otherwise*

All these constructions and theorems hold for wired boundary conditions as well.

Infinite-volume measures and phase transition

Infinite-volume Ising and Potts measures can be defined in the same way as in Theorem 1.3. Such limits exist and Edwards-Sokal coupling holds for them as well. Moreover, the behaviour of these models also admits a phase transition.

Theorem 1.10 ([Pei36])**.** *There exists such $\beta_{cr} \in (0, \infty)$, that*

- *for any $\beta < \beta_{cr}$*

$$\pi^1_{\mathbb{Z}^2,\beta,q}(\sigma(0) = 1) > \frac{1}{q},$$

i.e the value of the spin at zero depends on the boundary conditions at infinity.

- *for any $\beta > \beta_{cr}$*

$$\pi^1_{\mathbb{Z}^2,\beta,q}(\sigma(0) = 1) = \frac{1}{q}.$$

In the case when $\beta > \beta_{cr}$, there is a unique infinite-volume measure $\pi_{\mathbb{Z}^2,\beta,q}$ [Ons44], but for $\beta < \beta_{cr}$ this statement clearly does not hold. In the planar case all possible infinite-volume measures could be obtained as a linear combination of measures $\pi^i_{\mathbb{Z}^2,\beta,q}$ [Aiz80, CDCIV14], but this is not true in higher dimensions [Dob73].

Despite the importance of these models, especially the Ising model, in larger dimensions, the exact value of β_{cr} was rigorously obtained only in two dimensional case. The value of β_{cr} for Ising model on $\mathbb{Z}^2$ was conjectured to be equal to $\frac{1}{2}\log(1+\sqrt{2})$ in [KW41a, KW41b] and then proven in [Ons44, ABF87]. In $\mathbb{Z}^3$ it is yet estimated only numerically [Has10]. The critical value for two-dimensional Potts model was found due to the coupling with random-cluster model [BDC12].

The continuity property (1.17) is equal to the verity of the following inequality

$$\pi^1_{\mathbb{Z}^2,\beta_{cr}(q),q}(\sigma_0 = 1) > \frac{1}{q}.$$

The results [DCST17, DGH$^+$16] imply the continuity for planar Ising model and Potts model with $q = 3$ or $q = 4$, and discontinuity for Potts model with bigger q. The phase transition of the Ising model is continuous for higher dimensions as well [AF86, ADS15]. The Potts model for higher dimensions is conjectured to be discontinuous, but this is proven only for big values of q or in large dimensions [KS82, BC03].

The sharpness for the Potts model is defined as the exponential decay of correlations, i.e. the existence of such $c = c(\beta) > 0$, that

$$\pi_{\mathbb{Z}^2,\beta,q}\big[\big(\sigma_0 - \pi_{\mathbb{Z}^2,\beta,q}(\sigma_0)\big)\cdot\big(\sigma_x - \pi_{\mathbb{Z}^2,\beta,q}(\sigma_x)\big)\big] \leq e^{-c\|x\|} \text{ for any } x \in \mathbb{Z}^d.$$

Through [DRT17], Ising and Potts phase transitions are known to be sharp for any $\beta < \beta_{cr}$.

1.4.2 Six-vertex model

The two-dimensional random-cluster model is related to one more statistical model called *six-vertex* or ice-type model. It was introduced by L. Pauling in 1931 to study properties of ice crystals. Later this model was solved using the method called coordinate Bethe ansatz [Bet31, Bax89, DGH$^+$18]. Due to the exact solution and to the relation between the models, expressed in [TL71], one can obtain some results on planar random-cluster model, for example, the discontinuity of the phase transition for $q > 4$ [DGH$^+$16].

The model is defined as follows. Let us look at the regular box $\Lambda_{m,n} \in \mathbb{Z}^2$ with boundary glued together to obtain a torus. We can attribute a direction to all edges ($\leftarrow$ and $\rightarrow$ for horizontal edges and $\uparrow$ and $\downarrow$ for vertical ones) and look at the set of all possible configurations satisfying the rule called the *ice rule*, that states that for any vertex of $\Lambda_{m,n}$ there must be only two arrows coming in and only two arrows going out.

For any configuration we could set the probability of the six-vertex configuration to be proportional to $a^{n_1+n_2}\cdot b^{n_3+n_4}\cdot c^{n_5+n_6}$, where n_i is the number of vertices satisfying the i-th possible arrow configuration. Usually we set the weight parameters a and b to be equal to one, and c remains to be positive.

For any random-cluster configuration in a box on $\mathbb{Z}^2$ we can construct a corresponding loop configuration (see further section, especially Subsection 1.5.2). It can be seen as a

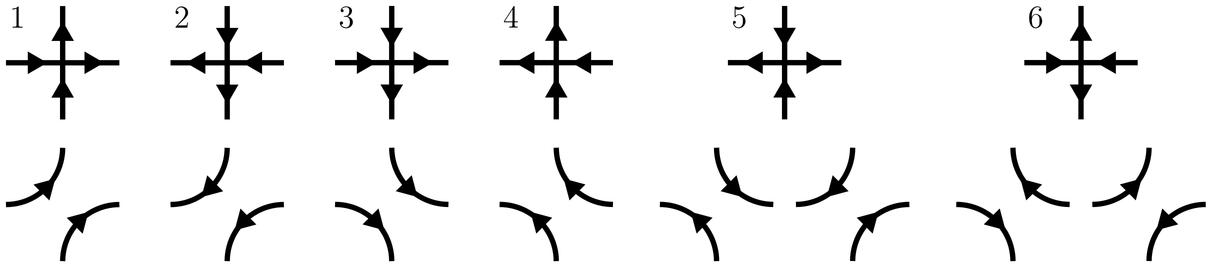

Figure 1.3: Possible arrow configurations for six-vertex model and their possible splits into parts of loops.

uniquely determined six-vertex configuration. On the other hand, one six-vertex configuration corresponds to multiple loop ensembles, and thus, random-cluster configurations, depending on how local arrow configurations are splitted (see Figure 1.3).

If we fix $a = b = 1$ and $c = \sqrt{2 + \sqrt{q}}$, then the probability of any six-vertex configuration is equal to the sum of the probabilities of the random-cluster configurations leading to this six-vertex configuration.

1.4.3 Limits when q converges to 0

Let us consider a finite connected graph G and look at limiting measures when q converges to 0. Negative association, which is expected for random-cluster measures with small q, is observed for all this models [GW03, Pem04].

Uniform spanning tree

Spanning tree is a connected subgraph of G containing no circles and covering all vertices of G. The uniform measure on the set of all spanning trees is called uniform spanning tree or *UST* measure [BLPS01]. This measure is obtained as a limiting measure for measures $\phi^0_{G,p,q}$ with both p, q converging to 0 with a restriction that p decreases slower than q [H95].

Uniform spanning forest

Spanning forest is a subgraph of G containing no circles and covering all vertices of G. The difference from spanning tree is that connection property is not needed, graph can have more than one "tree". *USF* measure is the uniform measure on the set of all possible spanning forests. It can be obtained as a limiting measure $\lim_{q \longrightarrow 0} \phi^0_{G,p=q,q}$ [H95, BLPS01].

Uniform connected graph

Here we look on the uniform measure on the set of all graphs which are just connected, without any other restrictions. It can be obtained when q decreases to 0 and p is fixed to be equal to $\frac{1}{2}$ [Gri06].

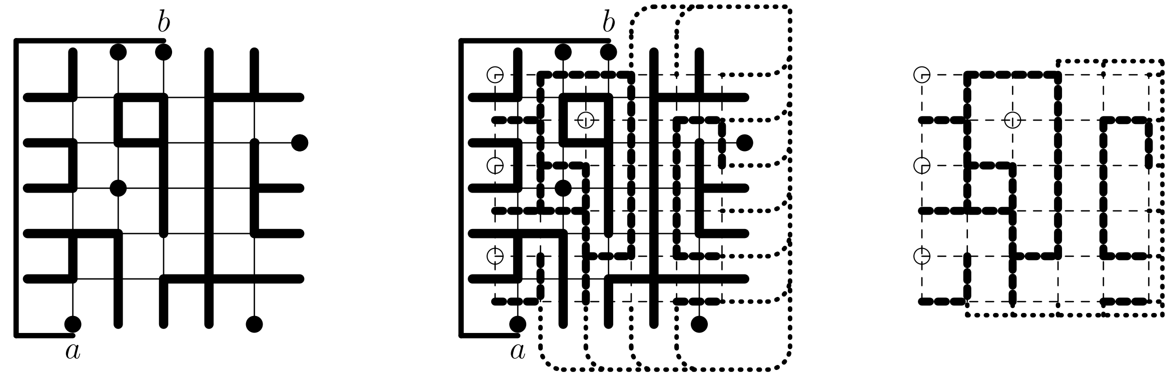

Figure 1.4: Primal and dual graph, primal and dual configurations. Here an further everything on the dual lattice will be pictured with dashed lines. Dotted line denotes wired dual boundary conditions.

1.5 Two-dimensional case

The key method of studying random-cluster model on $\mathbb{Z}^2$ is through so-called dual configurations (similar methods can be developed for higher dimensions, see [Gri06]).

Let us define the *dual lattice* $(\mathbb{Z}^2)^*$ of $\mathbb{Z}^2$ as follows. Vertices of $(\mathbb{Z}^2)^*$ correspond to faces of $\mathbb{Z}^2$ as if they are put in the middle of these faces. Each pair of vertices corresponding to adjacent faces is connected by an edge. Thus, any edge of $(\mathbb{Z}^2)^*$ intersects one and only one edge of $\mathbb{Z}^2$.

For a finite graph $G \subset \mathbb{Z}^2$, let G^* denote the subgraph of $(\mathbb{Z}^2)^*$ with edges corresponding to edges of G and vertices to the endpoints of these edges.

The configuration ω^* on G^* *dual* to a configuration ω on G (in this case w is called the *primal* configuration) is defined as follows:

$$\omega^*(e^*) = 1 - \omega(e),$$

where e is the edge of the primal lattice intersecting e^*. The probability of ω^* is set to be the probability of the corresponding primal configuration ω.

The event that x is connected to y by an open path in dual configuration is denoted $x \stackrel{*}{\longleftrightarrow} y$.

The configuration ω^* can be studied as a random-cluster configuration on G^*. It is easy to observe that if ω is distributed according to a measure $\phi^{\xi}_{G,p,q}$, then the distribution of ω^* corresponds to the random-cluster measure $\phi^{\xi^*}_{G^*,p^*,q}$ (see [Gri06, Dum13] for the details). The parameter q stays the same, and p^* satisfies the equation

$$\frac{p^*}{1-p^*} = q\,\frac{1-p}{p}, \tag{1.23}$$ {dual}

let us note that $(p^*)^* = p$. The boundary conditions should be changed to the opposite ones. Wired boundary conditions lead to free boundary conditions on the dual graph and vice versa. The (a,b)-Dobrushin boundary conditions turn to (b,a) (see Figure 1.4).

The value satisfying $p = p^*$ is called the *self-dual point* and is denoted p_{sd},

$$p_{sd}(q) = \frac{\sqrt{q}}{1+\sqrt{q}}. \tag{1.24}$$ {psd}

Later we will often omit the notation q if it is clear.

The uniqueness of the self-dual point leads to the natural conjecture that it is the point of criticality. Moreover, otherwise one should expect two critical points, that are dual in the sense as in (1.23), because the qualitative change of the behaviour of the dual model should affects on the primal model.

It was well-known that this conjecture holds for the percolation case $q = 1$ [Kes80], for the Ising case $q = 2$ [Ons44] and if q is large enough, i.e $q > 25.72$ [LMMS$^+$91]. In the general case it was not proven until recently.

Theorem 1.11 ([BDC12]). *On $\mathbb{Z}^2$, we have that for every $q \geq 1$,*

$$p_c(q) = p_{sd}(q) = \frac{\sqrt{q}}{1+\sqrt{q}}. \tag{1.25}$$

The proof presented in [BDC12] uses the technique based on the properties of the planar random-cluster model of criticality called crossing probabilities (see Subsection 1.5.1). Another proofs of this result follow from [DRT17] and [DCM16]. Here we will present the proof, that does not use crossing probabilities, but uses another planar technique called the parafermionic observable (see Subsection 1.5.2).

1.5.1 Crossing probabilities

The crossing events play the important role in studying the critical probability mode.

The similarity between primal and dual configuration distributions implies the following simple property

Statement 1.12. *For any $q \geq 1$ and $p = p_{sd}(q)$ the box Λ_n with periodic boundary conditions has a* horizontal crossing *(i.e the event $\{0\} \times [0, n] \longleftrightarrow \{n\} \times [0, n]$ holds) with probability $\frac{1}{2}$ for any positive n (see Figure 1.5).*

However, it is not trivial to prove the more general statement, that holds for rectangles with different aspect ratio.

Theorem 1.13 ([BDC12]). *For any $q \geq 1$, any $\alpha \geq 1$ and any $m \geq \alpha n > 0$ there exists $c = c(\alpha) > 0$ such that*

crossperbc}
$$c < \phi^{per}_{\Lambda_m, p_{sd}, q}(0 \times [0, n] \underset{[0,\alpha n]\times[0,n]}{\longleftrightarrow} \{\alpha n\} \times [0, n]) < 1 - c. \tag{1.26}$$

This naturally implies the same bound for the wired infinite measure $\phi^1_{p_{sd},q}$ via comparison between boundary conditions (1.9). The question whether the bound similar to (1.26) holds for measure $\phi^0_{p_{sd},q}$ is strongly linked with the uniqueness of the infinite measure at criticality and the existence of infinite cluster at criticality [DCST17].

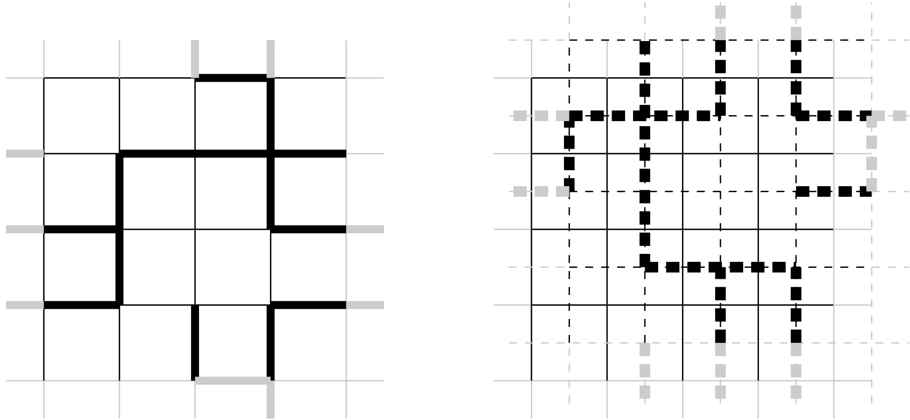

Figure 1.5: A primal cluster connecting two boundaries in a box with periodic boundary conditions (i.e a horizontal crossing), and a dual cluster that blocks possible primal connections and has the same probability as the primal cluster on the first picture.

The construction of the crossings is linked to the idea, first presented for Bernoulli percolation and known under the name of Russo-Seymour-Welsh theorem or RSW [Rus78, SW78]. It bounds the probability of the existence of a circuit that surrounds zero and lies in $\Lambda_{2n} \setminus \Lambda_n$ away from 0 and from 1. This circuit is constructed from crossings through FKG inequality.

1.5.2 Parafermionic observable

Another powerful tool in the planar case is called the *parafermionic observable*. It was inspired by physics in [FK80], and then started to be applied widely after [Smi06, RC06]

This approach was used in many other lattice models to deduce the critical point (see [DS12, Gla14, DGPS17]). For the random-cluster model, this method was used in [BDS15] in the case $q \geq 4$.

Medial lattice, exploration path and loop configurations

Let us follow [Dum17] to define all objects required for the definition of the parafermionic observable. We work on the so called *medial lattice*. For $\mathbb{Z}^2$ the medial lattice is denoted by $(\mathbb{Z}^2)^\diamond$ and defined as follows. The set of its vertices corresponds to edges of $\mathbb{Z}^2$, as if we put a vertex in the middle of every edge. There is an edge between two vertices of $(\mathbb{Z}^2)^\diamond$ if the corresponding edges have one common end vertex and are adjacent to the same face. Note that the faces of $(\mathbb{Z}^2)^\diamond$ correspond to $V(\mathbb{Z}^2) \cup V((\mathbb{Z}^2)^*)$. We orient the edges of $(\mathbb{Z}^2)^\diamond$ counterclockwise around faces corresponding to vertices of $\mathbb{Z}^2$, and, thus, clockwise around faces corresponding to vertices of $(\mathbb{Z}^2)^*$. Note that the lattice $(\mathbb{Z}^2)^\diamond$ is a rescaled and rotated version of $\mathbb{Z}^2$.

For the finite subgraph $G \subset \mathbb{Z}^2$, the medial graph $G^\diamond$ is the subgraph of $(\mathbb{Z}^2)^\diamond$ made of all faces corresponding to the vertices of G and G^*, all edges surrounding these faces and all vertices incident to these edges (see Figure 1.6).

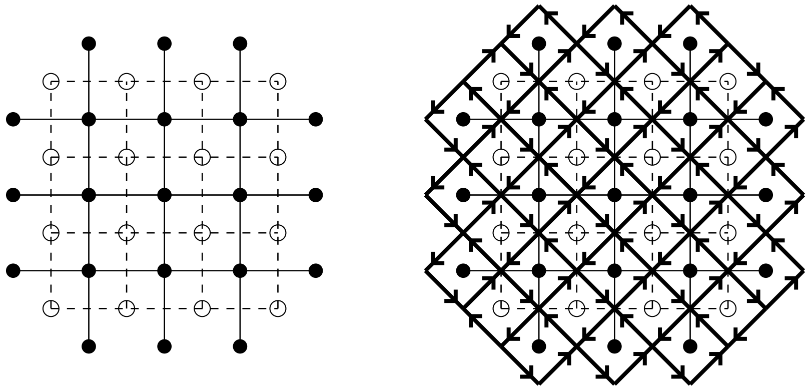

Figure 1.6: A graph with the corresponding dual graph, and the corresponding medial graph with marked orientation of the medial edges.

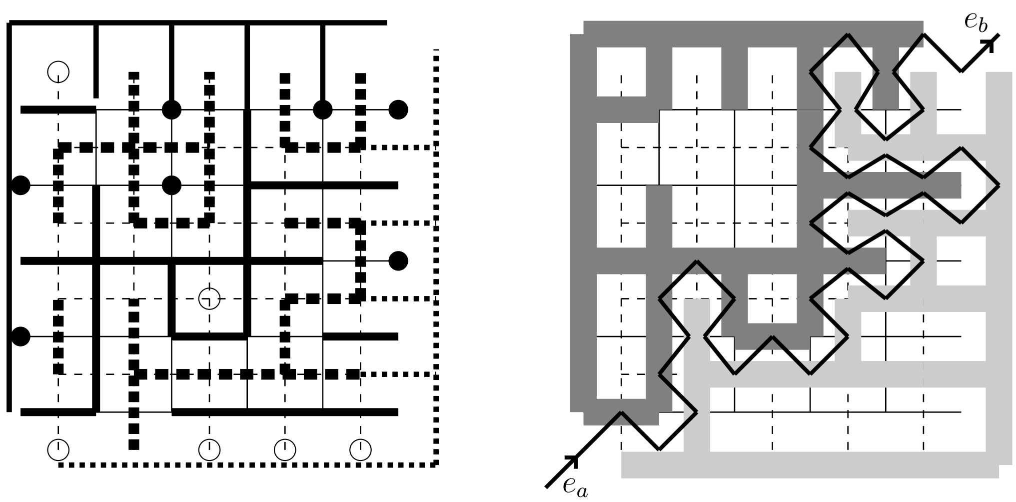

Figure 1.7: Primal and dual configurations, and the corresponding exploration path between the primal (dark grey) and the dual (light grey) clusters.

Let γ be a connected path on $G^\diamond$ and let e and e' be two edges included in γ. On each step from edge to edge this path can turn on $\frac{\pi}{2}$ or $-\frac{\pi}{2}$. Then, the *winding* $W_\gamma(e, e')$ is the total rotation done by γ on the way from the middle of e to the middle of e'. If e or e' do not belong to γ, we set $W_\gamma(e, e') = 0$.

Let us pick two vertices a and b on the boundary of G and look at configurations with Dobrushin boundary conditions (a, b). There is a primal cluster attached to the wired boundary part and a dual cluster attached to the free boundary part (in the dual configuration, it corresponds to a wired arc). The curve on the medial lattice going between these two clusters is called an *exploration path* (see Figure 1.7). Note that it is oriented. Let us also add two edges e_a and e_b to begin and to end γ outside of G, these edges are oriented according to the orientation rules stated above.

We use slightly similar construction to obtain the coupling between random-cluster and six-vertex models. For periodic boundary conditions we can obtain a collection of oriented loops surrounding and separating all primal and dual clusters, it is called a *loop configuration.*

Definition of the parafermionic observable and conjectures for random-cluster model

Now we are able to define the parafermionic observable itself. Let us pick a finite subgraph $G \subset \mathbb{Z}^2$, fix (a,b)-Dobrushin boundary conditions, pick some $q \geq 1$ and the corresponding p_{sd}. The parafermionic observable is defined on the edges of the medial lattice $e \in E(G^{\diamond})$ as

$$\hat{F}(e) = \hat{F}_{G,a,b,q,p_{sd}}(e) = \phi^{a,b}_{G,p_{sd},q}(e^{i\hat{\sigma}W_{\gamma}(e,e_b)}\mathbb{I}_{e\in\gamma}),$$

where γ is the exploration path in ω and $\hat{\sigma}$ satisfies the following relation:

$$\cos(\tfrac{\hat{\sigma}\pi}{2}) = \tfrac{\sqrt{q}}{2}.$$

For $q \in [1,4]$ the value $\sigma \in [0,1]$ satisfying the condition $\sin(\frac{\sigma\pi}{2}) = \frac{\sqrt{q}}{2}$ (it is equal to $1-\hat{\sigma}$) plays the role of a spin. In the case of the Ising model $\sigma = \frac{1}{2}$, that corresponds to the spin of the basic fermion.

Let us take a set $V \in V(G^{\diamond})$ such that any vertex $v \in V$ has four incident edges in $E(G^{\diamond}) \cup \{e_a, e_b\}$ and define its outer boundary

$$\delta V = \{e \in E(G^{\diamond}) \cup \{e_a, e_b\}, \text{only one end of } e \text{ belongs to V}\}, \qquad (1.27)$$ {boundedge

let us remind that these edges are oriented according to the rules of the medial lattice. For edge $e \in \delta V$, define the function $\eta_V(e)$ to be equal to 1 if e is pointing to a vertex of V and -1 otherwise. Then, the following property holds

Theorem 1.14 ([DCST17])**.** *Choose V with restrictions as above. Then,*

$$\sum_{e\in\delta V} \eta_V(e)\hat{F}(e) = 0. \qquad (1.28)$$ {parobssum

1.6 Another proof of Theorem 1.11 for $q \in [1,4]$ using the parafermionic observable

1.6.1 Preliminary: the graphs used in the proof

For this proof, we work not only on $\mathbb{Z}^2$, but also on other infinite graphs. The half-plane $\{(x,y) \in \mathbb{Z}^2, y \geq 0\}$ is denoted by $\mathbb{H}$ (its boundary $\partial\mathbb{H}$ is equal to $\{(x,0) \in \mathbb{Z}^2\}$).

We can extend the definition of boxes Λ_n for $\mathbb{H}$. Free and wired boundary conditions are defined for $\mathbb{H}$ in the same way, as for $\mathbb{Z}^2$, as a limit of measures defined for boxes $\Lambda_n \subset \mathbb{H}$ with free (resp. wired) boundary conditions on $\partial\Lambda_n$ and on $\partial\mathbb{H}$. Also we work with $0\backslash 1$ boundary conditions, obtained by taking a limiting measure for boxes Λ_n with Dobrushin $((0,n),(0,0))$ boundary conditions.

The *strip* of height n in $\mathbb{Z}^2$ (which can also be seen as a strip in $\mathbb{H}$) is defined as:

$$S_n = \{(x,y) \in \mathbb{Z}^2 : y \in [0,n]\}.$$

Let us call the left and the right parts of its boundary the subsets defined as:

$$\partial^+ S_n = \{(x,y) \in \mathbb{Z}^2 : y \in \{0,n\}, x \geq 0\},$$
$$\partial^- S_n = \{(x,y) \in \mathbb{Z}^2 : y \in \{0,n\}, x \leq 0\}.$$

The bottom left part of ∂S_n will be denoted by

$$\partial_b^- S_n = \{(x,0) \in \mathbb{Z}^2 : x < 0\}.$$

Free, wired or $0\backslash 1$ boundary condition measures on S_n are defined as a limit of measures on rectangles $\{(x,y) \in \mathbb{Z}^2 : x \in [-m,m], y \in [0,n]\}$ with free, wired or $((0,n),(0,0))$ boundary conditions respectively.

For some lemmas, we work on the universal cover of $\mathbb{Z}^2 \backslash F_0$ where F_0 is the face $[0,1] \times [-1,0]$. The *universal cover* $\mathbb{U}$ is a graph defined as follows:

$$\begin{aligned} V(\mathbb{U}) &= \{(x,y,z), x,y,z \in \mathbb{Z}\} = V(\mathbb{Z}^3), \\ E(\mathbb{U}) &= \{((x,y,z),(x,y+1,z))\, \forall x,y,z \in \mathbb{Z}\} \\ &\quad \cup \{((x,y,z),(x+1,y,z))\, \forall x,y,z \in \mathbb{Z} \text{ such that } x \neq 0\} \\ &\quad \cup \{((0,y,z),(1,y,z))\, \forall y,z \in \mathbb{Z} \text{ such that } y \geq 0\} \\ &\quad \cup \{((0,y,z),(1,y,z+1))\, \forall y,z \in \mathbb{Z} \text{ such that } y \leq 0\}. \end{aligned}$$

We also introduce the *truncated universal cover* $\mathbb{U}_k$ to be the subgraph of $\mathbb{U}$ with vertices of the type (x,y,z) with $|z| \leq k$. Let us define its boundary $\partial \mathbb{U}_k$ as $\{(0,-y,-k), y \in \mathbb{N}\} \cup \{(0,-y,k), y \in \mathbb{N}\}$.

We can define the analogue of a box Λ_n in $\mathbb{U}_k$ as follows:

$$\Lambda_{n,k} = \{(x,y,z) \in \mathbb{U}_k : \max(|x|,|y|) \leq n\},$$
$$\partial \Lambda_{n,k} = \{(x,y,z) \in \mathbb{U}_k : \max(|x|,|y|) = n\}.$$

Free and wired boundary conditions are defined for $\mathbb{U}_k$ in the same way as for $\mathbb{Z}^2$. Let us also note that we can extend the notions of dual and medial graphs to $\mathbb{U}_k$.

1.6.2 Zhang argument

The inequality

{Zhang} $$p_c \geq p_{sd} \tag{1.29}$$

can be easily obtained using so-called Zhang argument, originally appeared in the percolation case. Here we will briefly present it.

Suppose that this inequality does not hold, thus at criticality we have both primal and dual infinite clusters. For fixed $\varepsilon > 0$ we can choose n big enough that the primal

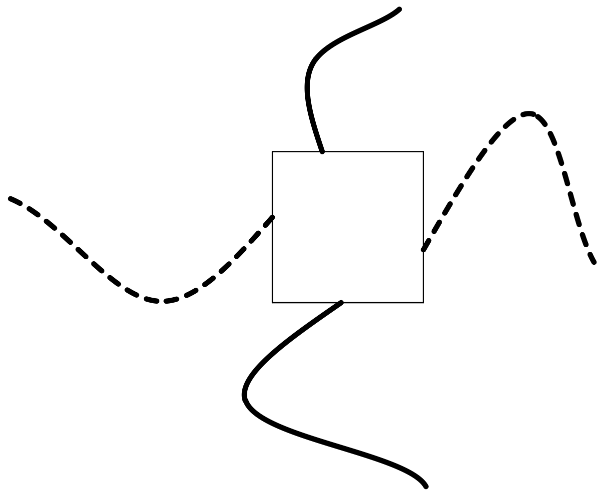

Figure 1.8: The construction of infinite clusters for Zhang argument.

infinite cluster touches Λ_n with probability bigger than $1 - \varepsilon$. An application of FKG inequality called the square-root trick bounds from below the probability of the infinite cluster to touch the upper side of the boundary of the box by $1 - \sqrt[4]{\varepsilon}$. This implies that with a probability bigger than $1 - 4\sqrt[4]{\varepsilon}$, we will find primal and dual clusters alternatingly touching the different sides of the boundary of Λ_n (see Figure 1.8). Finite energy property (1.4) allows to close all edges of Λ_n and for sure disconnect two primal infinite clusters. This contradicts with the fact that the infinite cluster is almost surely unique. Thus, (1.29) holds.

1.6.3 The case $q \in [1, 3]$

The proof is simpler when $q \leq 3$. We therefore begin by discussing this case.

Let us take a set S such that $S \subset \Lambda_n \subset \mathbb{Z}^2$ and $0 \in S$ and define its edge boundary $\Delta S = \{(x, y) \in E(\Lambda_n) : x \in S, y \notin S\}$. Then, define the following auxiliary function:

$$\varphi_{p,q,n}(S) = \sum_{(x,y)\in\Delta S} \phi^0_{S,p,q}[0 \longleftrightarrow x]. \tag{1.30}$$ {phi3}

The result is the consequence of the following two lemmas.

Lemma 1.15. *Let $p \geq p_{sd}$, $q > 1$ and $n \geq 1$. Then, for any G such that $\Lambda_n \subset G \subset \mathbb{Z}^2$, we have that*

$$\frac{d}{dp}\phi^\xi_{G,p,q}[0 \longleftrightarrow \partial\Lambda_n] \geq \frac{c}{1-p}\left(\inf_{S:\, 0\in S\subset\Lambda_n} \varphi_{p,q,n}(S)\right)(1 - \phi^\xi_{G,p,q}[0 \longleftrightarrow \partial\Lambda_n]) \tag{1.31}$$ {q3dphidpec

where c does not depend on p, G or ξ.

Proof. In this proof, we follow [DT16, DCT16], see also [Dum17]. The differential inequality (1.31) is a consequence of (1.5), (1.10) and the characterisation of pivotal edges:

$$\frac{d}{dp}\phi^\xi_{G,p,q}[0 \longleftrightarrow \partial\Lambda_n] \geq \frac{c}{1-p}\sum_{e\in E(G)} \phi^\xi_{G,p,q}[e \text{ is pivotal for } \{0 \longleftrightarrow \partial\Lambda_n\} \text{ and } \{0 \nleftrightarrow \partial\Lambda_n\}].$$

Let us call $\mathcal{S}$ the set $\{x \in \Lambda_n : x \nleftrightarrow \partial\Lambda_n\}$. The event $\{0 \nleftrightarrow \partial\Lambda_n\}$ can be rewritten as $\{0 \in \mathcal{S}\}$. Note that, when the event $\{0 \longleftrightarrow \partial\Lambda_n\}$ does not occur, all the pivotal edges are closed and lie in $\Delta\mathcal{S}$. Then,

$$\begin{aligned}\frac{d}{dp}\phi^{\xi}_{G,p,q}[0 \longleftrightarrow \partial\Lambda_n] &\geq \frac{c}{1-p}\sum_{S:\,0\in S\subset\Lambda_n}\sum_{e\in E(G)}\phi^{\xi}_{G,p,q}[e \text{ is pivotal for } \{0 \longleftrightarrow \partial\Lambda_n\}, \mathcal{S}=S]\\ &\geq \frac{c}{1-p}\sum_{S:\,0\in S\subset\Lambda_n}\sum_{x:(x,y)\in\Delta\mathcal{S}}\phi^{\xi}_{G,p,q}[\{0 \underset{S}{\longleftrightarrow} x\}, \mathcal{S}=S]\\ &\geq \frac{c}{1-p}\sum_{S:\,0\in S\subset\Lambda_n}\varphi_{p,q,n}(S)\phi^{\xi}_{G,p,q}[\mathcal{S}=S]\end{aligned}$$

where the last inequality follows from the Domain Markov property. Then, we can conclude that

$$\begin{aligned}\frac{d}{dp}\phi^{\xi}_{G,p,q}[0 \longleftrightarrow \partial\Lambda_n] &\geq \frac{c}{1-p}\left(\inf_{S:\,0\in S\subset\Lambda_n}\varphi_{p,q,n}(S)\right)\phi^{\xi}_{G,p,q}[0\in\mathcal{S}]\\ &= \frac{c}{1-p}\left(\inf_{S:\,0\in S\subset\Lambda_n}\varphi_{p,q,n}(S)\right)(1-\phi^{\xi}_{G,p,q}[0 \longleftrightarrow \partial\Lambda_n]).\end{aligned}$$

□

Lemma 1.16. *There exists $C>0$ such that for any $n \geq 1$ and S such that $0 \in S \subset \Lambda_n$,*

$$\varphi_{p_{sd},q,n}(S) > C. \tag{1.32}$$

Proof. Consider a set S such that $0 \in S \subset \Lambda_n$.

We can define the graph Λ'_n as a box Λ_n with vertices $\{(0,k), 1 \leq k \leq n\}$ removed. Then, let us denote the connected component of S that contains 0 as S_0 and work with the domain $S'_0 = S_0 \cap \Lambda'_n$. Let us call (0) the boundary conditions that are free everywhere. These boundary conditions can be seen as Dobrushin boundary conditions with the wired arc collapsed to one point (i.e. $a=b=0$). This observation allows to define the exploration path γ for a configuration in this domain. Its beginning edge $e_a = \big((0,\frac{1}{2}),(-\frac{1}{2},0)\big)$ is adjacent to the edge $e_b = \big((\frac{1}{2},0),(0,\frac{1}{2})\big)$, so γ forms a loop around 0, which bounds the open cluster in S'_0 that contains 0.

Let us call V the set of all vertices in $V((S'_0)^\diamond)$ that have four incident edges in $E((S'_0)^\diamond) \cup \{e_a, e_b\}$, then, δV defined as in (1.27) can be split into three parts: first, $\delta_0 = \{e_a, e_b\}$, second, the edges adjacent to the slit from the right and from the left (it can be written as $\delta_1 = \delta_1^+ \cup \delta_1^-$, where edges in δ_1^+ are of the form $\big((\frac{1}{2},k),(1,k+\frac{1}{2})\big)$ or $\big((\frac{1}{2},k+1),(1,k+\frac{1}{2})\big)$ and edges in δ_1^- are of the form $\big((-\frac{1}{2},k),(-1,k+\frac{1}{2})\big)$ or $\big((-\frac{1}{2},k+1),(-1,k+\frac{1}{2})\big)\big)$, third, other edges neighbouring the boundary of $(S'_0)^\diamond$.

Let us also look at the domain $\overline{S}$ defined as the reflection of S with respect to the y-axis. We can define $\overline{S'_0}$, $\overline{V}, \overline{\delta V}, \overline{\delta_1^+}, \overline{\delta_1^-}$ and $\overline{\delta_2}$ in the same way as for S.

Then, (1.28) can be written as

$$\sum_{e\in\delta_2}\eta_V(e)\hat{F}_{S'_0}(e) + \sum_{e\in\overline{\delta_2}}\eta_{\overline{V}}(e)\hat{F}_{\overline{S'_0}}(e) + \sum_{e\in\delta_1\cup\delta_0}\eta_V(e)\hat{F}_{S'_0}(e) + \sum_{e\in\overline{\delta_2}\cup\delta_0}\eta_{\overline{V}}(e)\hat{F}_{\overline{S'_0}}(e) = 0,$$

or

{lemq3par}
$$\Big|\sum_{e\in\delta_1\cup\delta_0}\eta_V(e)\hat F_{S'_0}(e)+\sum_{e\in\overline{\delta_2}\cup\delta_0}\eta_{\overline{V}}(e)\hat F_{\overline{S'_0}}(e)\Big|\le\Big|\sum_{e\in\delta_2}\eta_V(e)\hat F_{S'_0}(e)\Big|+\Big|\sum_{e\in\overline{\delta_2}}\eta_{\overline{V}}(e)\hat F_{\overline{S'_0}}(e)\Big|. \quad (1.33)$$

The first term in the right part of (1.33) is bounded as follows:

$$\begin{aligned}\Big|\sum_{e\in\delta_2}\eta_V(e)\hat F_{S'_0}(e)\Big| &\le \sum_{e\in\delta_2}|\hat F_{S'_0}(e)| = \sum_{e\in\delta_2}|\phi^{(0)}_{S'_0,p,q}(e^{i\sigma W_\gamma(e,e_b)}\mathbb{I}_{e\in\gamma})| \\ &= \sum_{e\in\delta_2}\phi^{(0)}_{S'_0,p,q}(e\in\gamma) = 2\sum_{x\in\partial S_0}\phi^0_{S',p,q}(0\longleftrightarrow x),\end{aligned}$$

because any boundary vertex corresponds to two edges from δ_2 that do or do not belong to γ simultaneously. The second term is bounded by the same value because of the symmetry between S and S':

$$\Big|\sum_{e\in\delta_2}\eta_{\overline{V}}(e)\hat F_{\overline{S'_0}}(e)\Big|\le 2\sum_{x\in\partial\overline{S}_0}\phi^0_{\overline{S'_0},p,q}(0\longleftrightarrow x)\le 2\sum_{x\in\partial S_0}\phi^0_{S',p,q}(0\longleftrightarrow x).$$

Together, this gives the following bound on the right part of (1.33):

$$\Big|\sum_{e\in\delta_2}\eta_V(e)\hat F_{S'_0}(e)\Big|+\Big|\sum_{e\in\overline{\delta_2}}\eta_{\overline{V}}(e)\hat F_{\overline{S'_0}}(e)\Big|\le 4\sum_{x\in\partial S_0}\phi^0_{S',p,q}(0\longleftrightarrow x)\le 4\sum_{x\in\partial S_0}\phi^0_{S,p,q}(0\longleftrightarrow x). \quad (1.34)$$
{leftC3}

Before writing the inequality for the right part of (1.33), let us define the vertices of S'_0 and $\overline{S'_0}$ adjacent to the slit:

$$\begin{aligned}\partial^+_{slit}&=\{(1,k),1\le k\le n\}\cup S_0, &\overline{\partial^+_{slit}}&=\{(1,k),1\le k\le n\}\cup\overline{S_0},\\ \partial^-_{slit}&=\{(1,k),1\le k\le n\}\cup S_0, &\overline{\partial^-_{slit}}&=\{(1,k),1\le k\le n\}\cup\overline{S_0}.\end{aligned}$$

Each of these points corresponds to two edges of δ_1 or $\overline{\delta_1}$. Then, the sums in the left part of (1.33) are written as follows:

$$\begin{aligned}\sum_{e\in\delta_1\cup\delta_0}\eta_V(e)\hat F_{S'_0}(e) &= (e^{i\hat\sigma\frac{3\pi}{2}}-1)+\sum_{x\in\partial^+_{slit}}\phi^0_{S'_0,p_{sd},q}(x\longleftrightarrow 0)(e^{-i\hat\sigma\pi}-e^{-i\hat\sigma\frac{\pi}{2}})\\ &\quad+\sum_{x\in\partial^-_{slit}}\phi^0_{S'_0,p_{sd},q}(x\longleftrightarrow 0)(e^{2\pi i\hat\sigma}-e^{i\hat\sigma\frac{5\pi}{2}}),\\ \sum_{e\in\overline{\delta_2}\cup\delta_0}\eta_{\overline{V}}(e)\hat F_{\overline{S'_0}}(e) &= (e^{i\hat\sigma\frac{3\pi}{2}}-1)+\sum_{x\in\overline{\partial^+_{slit}}}\phi^0_{\overline{S'_0},p_{sd},q}(x\longleftrightarrow 0)(e^{-i\hat\sigma\pi}-e^{-i\hat\sigma\frac{\pi}{2}})\\ &\quad+\sum_{x\in\overline{\partial^-_{slit}}}\phi^0_{\overline{S'_0},p_{sd},q}(x\longleftrightarrow 0)(e^{2\pi i\hat\sigma}-e^{i\hat\sigma\frac{5\pi}{2}}).\end{aligned}$$

Because of the symmetry of S_0' and $\overline{S_0'}$, these equations can be summed to give

$$
\begin{aligned}
\Big|\sum_{e\in\delta_1\cup\delta_0}\eta_V(e)\hat F_{S_0'}(e)+\sum_{e\in\overline{\delta_2}\cup\delta_0}\eta_{\overline{V}}(e)\hat F_{\overline{S_0'}}(e)\Big| &= \big|2(e^{i\hat\sigma\frac{3\pi}{2}}-1)\\
&+\sum_{x\in\partial^+_{slit}\cup\partial^-_{slit}}\phi^0_{S_0',p_{sd},q}(x\longleftrightarrow 0)(e^{-\pi i\hat\sigma}-e^{-i\hat\sigma\frac{\pi}{2}}+e^{2\pi i\hat\sigma}-e^{i\hat\sigma\frac{5\pi}{2}})\big|\\
&=\big|2(e^{i\hat\sigma\frac{3\pi}{2}}-1)+\sum_{x\in\partial^+_{slit}\cup\partial^-_{slit}}2\phi^0_{S_0',p_{sd},q}(x\longleftrightarrow 0)(e^{i\hat\sigma\frac{\pi}{2}}-e^{\pi i\hat\sigma})\cos(\hat\sigma\tfrac{3\pi}{2})\big|\\
&=2\big|e^{\pi i\hat\sigma}-e^{i\hat\sigma\frac{\pi}{2}}\big|\cdot\big|\big((e^{i\hat\sigma\frac{\pi}{2}}+1+e^{-i\hat\sigma\frac{\pi}{2}})-\sum_{x\in\partial^+_{slit}\cup\partial^-_{slit}}\phi^0_{S_0',p_{sd},q}(x\longleftrightarrow 0)\cos(\hat\sigma\tfrac{3\pi}{2})\big)\big|\\
&=2\big|e^{i\hat\sigma\frac{\pi}{2}}\big|\cdot\big|e^{i\hat\sigma\frac{\pi}{2}}-1\big|\cdot\big|\big(1+2\cos(\hat\sigma\tfrac{\pi}{2})-\sum_{x\in\partial^+_{slit}\cup\partial^-_{slit}}\phi^0_{S_0',p_{sd},q}(x\longleftrightarrow 0)\cos(\hat\sigma\tfrac{3\pi}{2})\big)\big|\\
&=2\big|e^{i\hat\sigma\frac{\pi}{2}}-1\big|\cdot\big|\big(1+\sqrt{q}+\sum_{x\in\partial^+_{slit}\cup\partial^-_{slit}}\phi^0_{S_0',p_{sd},q}(x\longleftrightarrow 0)\tfrac{\sqrt{q}}{2}(3-q)\big)\big|\\
&\geq 2\big|e^{i\hat\sigma\frac{\pi}{2}}-1\big|(1+\sqrt{q}). \qquad (1.35)
\end{aligned}
$$

{rightC3}

Together, (1.34) and (1.35) conclude the proof with $C=\frac{1}{2}|e^{i\hat\sigma\frac{\pi}{2}}-1|(1+\sqrt{q})$. □

Proof of Theorem 1.11 for $q\in[1,3]$. Let us take $p'\geq p_{sd}$. By monotonicity, we can extend the result of Lemma 1.16 to all values of p in the interval $[p_{sd},p']$. Thus, for all $p\in[p_{sd},p']$, (1.31) takes the form

$$\frac{d\phi^\xi_{G,p,q}[0\longleftrightarrow\partial\Lambda_n]}{(1-\phi^\xi_{G,p,q}[0\longleftrightarrow\partial\Lambda_n])}\geq c\frac{dp}{1-p}$$

or, written differently,

prooft3din}

$$-d\log(1-\phi^\xi_{G,p,q}[0\longleftrightarrow\partial\Lambda_n])\geq -d\log(1-p)^c. \qquad (1.36)$$

We can integrate (1.36) on $[p_{sd},p']$ to obtain

$$\frac{1-\phi^\xi_{G,p_{sd},q}[0\longleftrightarrow\partial\Lambda_n]}{1-\phi^\xi_{G,p',q}[0\longleftrightarrow\partial\Lambda_n]}\geq\left(\frac{1-p_{sd}}{1-p'}\right)^c$$

which gives

$$\phi^\xi_{G,p',q}[0\longleftrightarrow\partial\Lambda_n]\geq 1-\Big(1-\phi^\xi_{G,p_{sd},q}[0\longleftrightarrow\partial\Lambda_n]\Big)\left(\frac{1-p'}{1-p_{sd}}\right)^c\geq 1-\left(\frac{1-p'}{1-p_{sd}}\right)^c \qquad (1.37)$$

where c does not depend on G or n. We can send G to $\mathbb{Z}^2$ and n to infinity to finally obtain

$$\phi^\xi_{\mathbb{Z}^2,p',q}[0\longleftrightarrow\infty]\geq 1-\left(\frac{1-p'}{1-p_{sd}}\right)^c>0. \qquad (1.38)$$

The probability to have an infinite cluster is therefore positive for any $p'>p_{sd}$, a fact which immediately implies that $p_c\leq p_{sd}$. Together with (1.29) it gives Theorem 1.11.

□

1.6.4 The sketch of the proof for $3 < q \leq 4$

The global strategy is almost the same in this case. We work in the strip S_n rather than in the box Λ_n. Let us define the event $A_n^* = \{(0,0) \underset{S_n^*}{\longleftrightarrow} (0,n)\}$ on the dual lattice and call P^* the left-most dual-open path connecting $(0,0)$ to $(0,n)$. We will also call A_n the event complement to A_n^*.

We can the set $\mathcal{S} = \{x \in S_n : x \nleftrightarrow \partial^+ S_n\}$. The event A_n^* is equal to the event $\{\partial^- S_n \subset \mathcal{S}\}$.

We define the auxiliary function $\overline{\varphi}_{p,q,n}(S)$ as follows. Let us take a set S such that $\partial^- S_n \subset S \subset S_n$ and define $\Delta S = \{(x,y) \in E(S_n) : x \in S, y \notin S\}$. Then

$$\overline{\varphi}_{p,q,n}(S) = \sum_{\{x,y\} \in \Delta S} \phi^1_{S_n,p,q}[\partial^- S_n \underset{S}{\longleftrightarrow} x \mid \mathcal{S} = S]. \tag{1.39}$$ {phi4}

Lemma 1.17. *[Proven in Subsection 1.6.5] Let $p \geq p_{sd}$, $q > 1$ and $n \geq 1$. Then, for any G such that $S_n \subset G \subset \mathbb{Z}^2$, we have that*

$$\frac{d}{dp}\phi^1_{G,p,q}[A_n] \geq \frac{c}{1-p} \left(\inf_{S:\, \partial^- S_n \subset S \subset S_n} \overline{\varphi}_{p,q,n}(S) \right) (1 - \phi^1_{G,p,q}[A_n]) \tag{1.40}$$ {q4dphidpec

where c does not depend on p, G or ξ.

The analogue of Lemma 1.16 for $\overline{\varphi}_{p,q,n}(S)$ is the key point of the proof and requires several additional statements. Firstly, we will show the following lemma:

Lemma 1.18. *[Proven in Subsection 1.6.6] There exists a constant $c > 0$ such that for any $n \geq 1$ and for any $S \in S_n$ with the properties that $\partial^- S_n \subset S$ and $\partial^+ S_n \cap S = \emptyset$, we have*

$$\overline{\varphi}_{p,q,n}(S) \geq \sum_{x \in \partial_b^- S_n} \phi^1_{S_n,p,q}[x \longleftrightarrow P^*]. \tag{1.41}$$

In order to state the other lemma, we work on the truncated universal cover $\mathbb{U}_k$. The reason why we use the universal cover is the following. In the proof of Subsection 1.6.3 (see (1.35)), we used that $\cos(\frac{3\pi}{2}\hat{\sigma}) < 1$ to show that the contribution of any slit has the same sign as the one of e_a and e_b. In order to extend this property to $q > 3$, one has to consider larger opening between strips. Then, the proof is very similar (Lemma 1.19). One key observation will be that there is no infinite cluster in the universal case (Lemma 1.20). Combining these two facts will lead (with some work, done in Lemma 1.21) to an estimate on the plane (Lemma 1.22) This estimate will finally be used to show that the probability of a certain event decays very fast as p moves away from p_{sd}, a fact which is known to imply a bound in p_c (Lemma 1.24).

This lemma is the analogue of Lemma 1.16 in $\mathbb{U}_k$:

Lemma 1.19. *[Proven in Subsection 1.6.6] For any choice of $q \in (3,4]$, there exists a constant $C > 0$ and $k \in \mathbb{N}$ such that for any $n \geq 1$ and any set S such that $0 \in S \subset \Lambda_{n,k}$, we have that*

$$\varphi_{p_{sd},q,n,k}(S) > C, \tag{1.42}$$

where $\varphi_{p,q,n,k}$ is defined in the same way as in (1.30) for sets included in $\Lambda_{n,k}$.

We complement this lemma with the following result.

Lemma 1.20. *[Proven in Subsection 1.6.7] For any $k \geq 0$, there is almost surely no infinite cluster in $\mathbb{U}_k$, i.e.*

$$\phi^0_{\mathbb{U}_k,p_{sd},q}(0 \longleftrightarrow \infty) = 0. \tag{1.43}$$

Combined together, these lemmas give the following technical estimate on $\mathbb{Z}^2$:

Lemma 1.21. *[Proven in Subsection 1.6.8] For any $M > 0$, there exists R large enough such that for every $n > R$ and for any γ connecting $\partial\Lambda_n$ and $\partial\Lambda_R$,*

$$\sum_{(-i,-n),i\in[0,n]} \phi^{\xi}_{\Lambda_n\setminus(\Lambda_R\cup\gamma),p_{sd},q}[x \longleftrightarrow \Lambda_R \cup \gamma] \geq M,$$

where ξ denotes free boundary conditions on Λ_n and wired boundary conditions on $\Lambda_R \cup \gamma$ (see Figure 1.9).

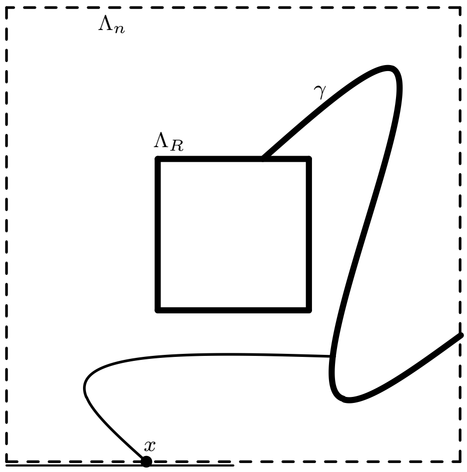

Figure 1.9: Boundary point connected to $\Lambda_R \cup \gamma$ (Lemma 1.21).

In this figure and the further pictures the free boundary conditions are represented by the dashed lines and the wired boundary conditions are represented by the bold lines.

We use this lemma to obtain the following result.

Lemma 1.22. *[Proven in Subsection 1.6.9] Let us fix $\varepsilon > 0$ and $R > 0$. Then, for any n large enough, one of the following statements should hold:*

Case 1

{q4phiCeq1}

$$\sum_{x\in\partial_b^- S_n} \phi^1_{S_n,p_{sd},q}[x \longleftrightarrow 0] > R\log n. \tag{1.44}$$

Case 2

{case2eq}

$$\phi^0_{S_n,p_{sd}-\varepsilon,q}(0 \longleftrightarrow \partial\Lambda_n) < n^{-20}. \tag{1.45}$$

For the second case, we can rewrite (1.45) using the dual model on the dual lattice to get:

{T42eq} $$\phi^{1^*}_{S_n,(p_{sd}+\delta)^*,q}[A^*_n] \leq \phi^{1^*}_{S_n,(p_{sd}+\delta)^*,q}[0 \overset{*}{\longleftrightarrow} \partial\Lambda_n] \leq n^{-20}. \tag{1.46}$$

If the first case holds, then combined with Lemma 1.18, it gives that for any $R > 0$, for n large enough

$$\overline{\varphi}_{p,q,n}(S) > R \log n \tag{1.47}$$ {q4phiCeq}

for any $S \in S_n$ with the properties $\partial^- S_n \subset S$ and $\partial^+ S_n \cap S = \emptyset$.

The combination of this inequality with Lemma 1.17 implies the following proposition.

Proposition 1.23. *[Proven in Subsection 1.6.5] For n such that* (1.44) *holds and for any $\delta > 0$, we have that*

$$\phi^{1^*}_{S_n,(p_{sd}+\delta)^*,q}[A^*_n] \leq n^{-20}. \tag{1.48}$$ {T41eq}

The fact that $p_{sd} \geq p_c$ for $3 < q \leq 4$ is the immediate consequence of (1.46) and (1.48) and the following lemma:

Lemma 1.24 ([Dum13])**.** *Suppose $p' < p_c$. Then, for infinitely many $n \in \mathbb{N}$, we have that*

$$\phi^{1^*}_{S_n,(p')^*,q}[A^*_n] > n^{-20}. \tag{1.49}$$

1.6.5 Proofs of Lemma 1.17 and Proposition 1.23.

These proofs use the same strategies as in the case $q \leq 3$.

Proof of Lemma 1.17. (see also [DT16, DCT16]). Let us remind that the event complement to A_n is equal to $\{\partial^- S_n \subset \mathcal{S}\}$. Then,

$$\begin{aligned}
\frac{d}{dp}\phi^1_{S_n,p,q}[A_n] &\geq \frac{c}{1-p}\sum_{e\in E(S_n)} \phi^1_{S_n,p,q}[e \text{ is pivotal for } A_n, A_n \text{ does not occur}] \\
&\geq \frac{c}{1-p}\sum_{S:\,\partial^- S_n\subset S\subset S_n}\ \sum_{(x,y)\in\Delta S} \phi^1_{S_n,p,q}[\partial^- S_n \underset{S}{\longleftrightarrow} x, \mathcal{S} = S] \\
&\geq \frac{c}{1-p}\sum_{S:\,\partial^- S_n\subset S\subset S_n} \overline{\varphi}_{p,q,n}(S)\phi^1_{S_n,p,q}[\mathcal{S} = S] \\
&\geq \frac{c}{1-p}\left(\inf_{S:\,\partial^- S_n\subset S\subset S_n} \overline{\varphi}_{p,q,n}(S)\right)(1 - \phi^1_{S_n,p,q}[A_n]).
\end{aligned}$$

□

Proof of Proposition 1.23. This proof uses the same method as for the proof given in Subsection 1.6.3. Fix $p' = p_{sd} + \delta$ for some $\delta > 0$. By monotonicity, (1.47) holds for any $p \in [p_{sd}, p']$. This together with (1.40) gives that

$$-d\left(\log(1 - \phi^1_{G,p,q}[A_n])\right) \geq \frac{R}{1-p}\log n\, dp \geq R \log n\, dp.$$

After integrating we obtain that

$$\phi^1_{G,p_{sd}+\delta,q}[A_n] \geq 1 - n^{-R\delta}.$$

Let us choose R large enough to have $R\delta > 20$. Then, when G goes to S_n, we obtain

$$\phi^1_{S_n,p_{sd}+\delta,q}[A_n] \geq 1 - n^{-20}$$

or, written in the dual model,

$$\phi^{1^*}_{S_n,(p_{sd}+\delta)^*,q}[A_n^*] \leq n^{-20}.$$

□

1.6.6 Proof of Lemmas 1.18 and 1.19.

Proof of Lemma 1.19. This proof uses the same strategy as in Lemma 1.16, but on $\mathbb{U}_k$ instead of $\mathbb{Z}^2$. Let us look at a set $S \in \Lambda_{n,k}$ containing 0 and denote $\overline{S} = \{(-x,y,-z),(x,y,z) \in S\}$ its reflection. We are interested in the connected component containing zero, denoted by S_0. Let us look at the set $\Lambda'_{n,k} = \Lambda_{n,k} \backslash \partial\mathbb{U}_k$ and study the sets $S'_0 = \Lambda'_{n,k} \cap S_0$ and $\overline{S'_0} = \Lambda'_{n,k} \cap \overline{S_0}$.

For boundary conditions (0) defined as before we can define the exploration path both for S'_0 and $\overline{S'_0}$. Its initial and final edges are of the form $e_a = \big((0,-\frac{1}{2},-k)(\frac{1}{2},0,-k)\big)$ and $e_b = \big((-\frac{1}{2},0,k)(0,-\frac{1}{2},k)\big)$. The sets $V, \delta V, \delta_0, \delta_1 = \delta_1^+ \cup \delta_1^-$ and δ_2 (resp. $\overline{V}, \overline{\delta V}, \overline{\delta_1} = \overline{\delta_1^+} \cup \overline{\delta_1^-}$ and $\overline{\delta_2}$) are defined as in previous proof.

Then, (1.28) enables us to write exactly the same equation as in (1.33)

{lemq4par} $$\Big|\sum_{e\in\delta_1\cup\delta_0} \eta_V(e)\hat{F}_{S'_0}(e) + \sum_{e\in\overline{\delta_1}\cup\delta_0} \eta_{\overline{V}}(e)\hat{F}_{\overline{S'_0}}(e)\Big| \leq \Big|\sum_{e\in\delta_2} \eta_V(e)\hat{F}_{S'_0}(e)\Big| + \Big|\sum_{e\in\overline{\delta_2}} \eta_{\overline{V}}(e)\hat{F}_{\overline{S'_0}}(e)\Big|. \quad (1.50)$$

As in (1.34), the right-hand side of (1.50) is bounded as follows:

{leftC4} $$\Big|\sum_{e\in\delta_2} \eta_V(e)\hat{F}_{S'_0}(e)\Big| + \Big|\sum_{e\in\overline{\delta_2}} \eta_{\overline{V}}(e)\hat{F}_{\overline{S'_0}}(e)\Big| \leq 4\sum_{x\in\partial S_0} \phi^0_{S,p_{sd},q}(0 \longleftrightarrow x). \quad (1.51)$$

The left-hand side of (1.50) can be written as:

$$
\begin{aligned}
\Big|\sum_{e\in\delta_1\cup\delta_0}\eta_V(e)\hat F_{S'_0}(e)+\sum_{e\in\overline{\delta_1}\cup\delta_0}\eta_{\overline V}(e)\hat F_{\overline{S'_0}}(e)\Big| &= \big|2(e^{i\hat\sigma(4\pi k\frac{3\pi}{2})}-1)\\
&+\sum_{x\in\partial^+_{slit}\cup\partial^-_{slit}}\phi^0_{S'_0,p_{sd},q}(x\longleftrightarrow 0)(e^{-\pi i\hat\sigma}-e^{-i\hat\sigma\frac{\pi}{2}}+e^{4k\pi i\hat\sigma}(e^{2\pi i\hat\sigma}-e^{i\hat\sigma\frac{5\pi}{2}}))\big|\\
&= \big|2(e^{i\hat\sigma(4\pi k\frac{3\pi}{2})}-1)\\
&+\sum_{x\in\partial^+_{slit}\cup\partial^-_{slit}}2\phi^0_{S'_0,p_{sd},q}(x\longleftrightarrow 0)(e^{i\hat\sigma(2\pi k+\frac{\pi}{2})}-e^{(2k+1)\pi i\hat\sigma})\cos((4\pi k+3)\tfrac{\hat\sigma}{2})\big|\\
&= 2\big|(e^{(2k+1)\pi i\hat\sigma}-e^{i\hat\sigma(2\pi k+\frac{\pi}{2})})\big|\\
&\big|1+\sum_{m=1}^{4k+1}2\cos(\tfrac{\pi m}{2}\hat\sigma)-\sum_{x\in\partial^+_{slit}\cup\partial^-_{slit}}2\phi^0_{S'_0,p_{sd},q}(x\longleftrightarrow 0)\cos((4\pi k+3)\tfrac{\hat\sigma}{2})\big|\\
&\geq 2\big|(e^{i\hat\sigma\frac{\pi}{2}}-1)\big|.
\end{aligned}
$$

The last bound holds if we pick an integer k in such a way that $\cos(\frac{\pi m}{2}\hat\sigma)\geq 0$ for any integer $m\in[0,4k+1]$ and $\cos((4k+3)\frac{\pi}{2}\hat\sigma)\leq 0$. These inequalities give the constraint

$$k\in\left[\frac{\frac{2}{\hat\sigma}-3}{2},\frac{\frac{2}{\hat\sigma}-1}{2}\right].$$

The length of this interval is equal to one so such k can always be found.

□

Proof of Lemma 1.18. Look at the exploration path γ from $(0,0)$ to $(n,0)$ in the strip S_n with $0\backslash 1$ boundary conditions. Take S such that $\partial^-S_n\subset S\subset S_n$ and its reflection $\overline{S}$ from the middle line of the strip. We call V (correspondingly $\overline{V}$) the vertices of $S^\diamond$ (correspondingly $\overline{S^\diamond}$), and $\delta_b^-,\delta_t^-,\delta_{P^*}$ and $\delta_{\overline{P^*}}$ the medial edges corresponding to the bottom and top left boundaries of S_n and to the paths P^* and $\overline{P^*}$.

The equation (1.28) can be written as

$$\Big|\sum_{e\in\delta_b^-\cup\delta_t^-}\eta_V(e)\hat F_S(e)+\eta_{\overline V}(e)\hat F_{\overline S}(e)\Big|\leq\Big|\sum_{e\in\delta_{P^*}}\eta_V(e)\hat F_S(e)\Big|+\Big|\sum_{e\in\delta_{\overline{P^*}}}\eta_{\overline V}(e)\hat F_{\overline S}(e)\Big|. \quad (1.52)$$ {lemq4Spar}

The right part of the inequality is written as

$$
\begin{aligned}
\Big|\sum_{e\in\delta_{P^*}}\eta_V(e)\hat F_S(e)\Big|+\Big|\sum_{e\in\delta_{\overline{P^*}}}\eta_{\overline V}(e)\hat F_{\overline S}(e)\Big| &= 2\Big|\sum_{e\in\delta_{P^*}}\eta_V(e)\hat F_S(e)\Big|\\
&\leq 2\sum_{e\in\delta_{P^*}}\phi^{0\backslash 1}_{S_n,p_{sd},q}(e\in\gamma)\\
&= 4\sum_{e\in P^*}\phi^{0\backslash 1}_{S_n,p_{sd},q}(e\longleftrightarrow\partial^-S_n)\\
&= 8\sum_{e\in P^*}\phi^{0\backslash 1}_{S_n,p_{sd},q}(e\longleftrightarrow\partial_b^-S_n).
\end{aligned}
$$

due to the symmetry of S and $\overline{S}$. Also, because of this symmetry, the left-hand side of (1.52) is bounded by

$$\begin{aligned}\left|(e^{i\frac{\pi}{2}\hat{\sigma}}-1+e^{-i\frac{\pi}{2}\hat{\sigma}}-e^{-i\pi\hat{\sigma}})\sum_{x\in\partial^- S_n}\phi^{0\backslash 1}_{S_n,p_{sd},q}(x\overset{*}{\longleftrightarrow}P^*)\right| &= \tfrac{\sqrt{q}}{2}\left|1-e^{-i\frac{\pi}{2}\hat{\sigma}}\right|\sum_{x\in\partial^- S_n}\phi^{0\backslash 1}_{S_n,p_{sd},q}(x\overset{*}{\longleftrightarrow}P^*)\\ &= \sqrt{q}\left|1-e^{-i\frac{\pi}{2}\hat{\sigma}}\right|\sum_{x\in\partial^-_b S_n}\phi^{0\backslash 1}_{S_n,p_{sd},q}(x\overset{*}{\longleftrightarrow}P^*).\end{aligned}$$

The combination of these two bounds finishes the proof. □

1.6.7 Proof of Lemma 1.20

To prove this lemma we introduce new definitions and prove one intermediate lemma. For every integer r, call ℓ_r the axis in $\mathbb{U}$ obtained by the rotation of $\ell_0 = \{(0,y,0) : y \geq 0\}$ by the angle $\frac{r\pi}{2}$. Let us denote $\mathrm{sect}_{i,j}$ the part of $\mathbb{U}$ between ℓ_i and ℓ_j (in particular $\mathbb{U}_k = \mathrm{sect}_{-2-4k,4k+2}$).

Let us call the domain $\Omega \subset \mathbb{U}$ *symmetric* if it is invariant under the reflection with respect to ℓ_0, i.e, if $(x,y,z)\in\Omega$ implies that $(x,-y,-z)\in\Omega$. Let us also call Ω *simple* if $0\in\Omega$ and if for any sector $\mathrm{sect}_{i,i+1}$ the domain $\mathrm{sect}_{i,i+1}\backslash\Omega$ is connected.

Lemma 1.25. *For any $k>0$ and any simple symmetric domain Ω such that $\Omega\subset\mathrm{sect}_{-k,k}$ the following is true:*

$$\phi^{0,0\backslash 1}_{\mathrm{sect}_{-k,k}\backslash\Omega,p_{sd},q}\big(\ell_1 \underset{\mathrm{sect}_{-1,1}}{\overset{*}{\longleftrightarrow}} (\partial\Omega\cap\mathrm{sect}_{-1,0})\big) \geq \frac{1}{2}. \tag{1.53}$$

where $0,0\backslash 1$ boundary conditions denote free boundary conditions at infinity, ℓ_k and ℓ_{-k}, free boundary conditions on $\mathrm{sect}_{-k,0}\cap\partial\Omega$ and wired boundary conditions on $\mathrm{sect}_{0,k}\cap\partial\Omega$.

Proof. The events

$$A_1 = \{\ell_1 \underset{\mathrm{sect}_{-1,1}\backslash\Omega}{\overset{*}{\longleftrightarrow}} (\partial\Omega\cap\mathrm{sect}_{-1,0})\}$$

and

$$A_2 = \{\ell_{-1} \underset{\mathrm{sect}_{-1,1}\backslash\Omega}{\longleftrightarrow} (\partial\Omega\cap\mathrm{sect}_{0,1})\}$$

are disjoint. Let us denote $1,0\backslash 1$ the boundary conditions which are wired at infinity, ℓ_k and ℓ_{-k}, free at $\mathrm{sect}_{-k,0}\cap\partial\Omega$ and wired at $\mathrm{sect}_{0,k}\cap\partial\Omega$. Then, by duality and by comparison between free and wired boundary conditions (which favour primal paths to appear),

$$\phi^{0,0\backslash 1}_{\mathrm{sect}_{-k,k}\backslash\Omega,p_{sd},q}(A_1) = \phi^{1,0\backslash 1}_{\mathrm{sect}_{-k,k}\backslash\Omega,p_{sd},q}(A_2) \geq \phi^{0,0\backslash 1}_{\mathrm{sect}_{-k,k}\backslash\Omega,p_{sd},q}(A_2).$$

Let us look at the events

$$B_1 = \{(\partial\Omega\cap\mathrm{sect}_{0,1}) \underset{\mathrm{sect}_{-1,1}\backslash\Omega}{\longleftrightarrow} \infty\}$$

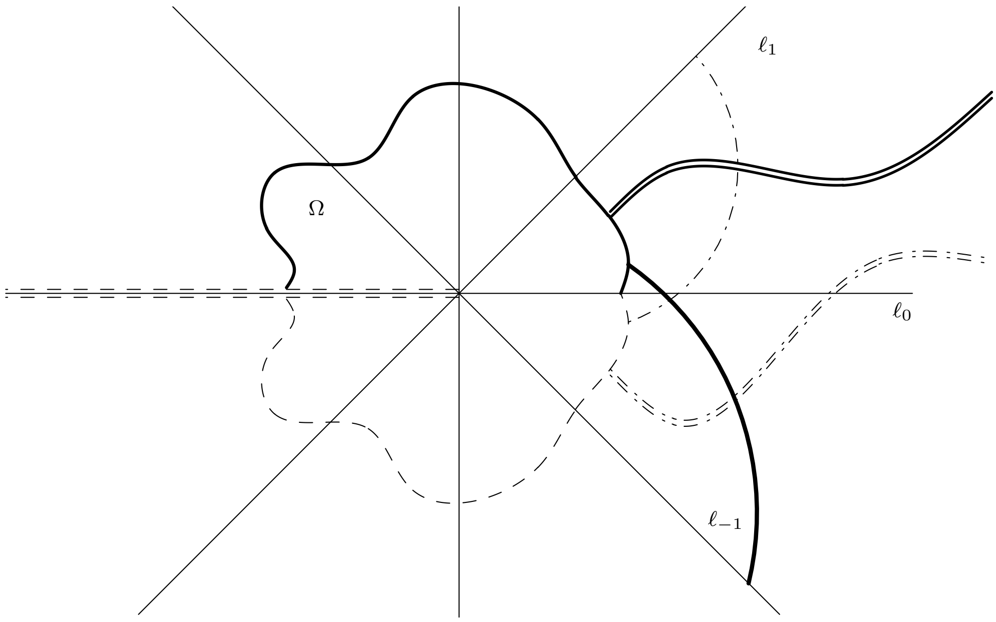

Figure 1.10: A domain Ω with corresponding boundary conditions, events A_1 and A_2, and B_1 and B_2 (doubled lines). Dual paths are represented by dash-dotted lines.

and

$$B_2 = \{(\partial\Omega \cap \mathrm{sect}_{-1,0}) \overset{*}{\underset{\mathrm{sect}_{-1,1}\setminus\Omega}{\longleftrightarrow}} \infty\}.$$

The realisations of A_2 and B_1 are the only paths blocking the event A_1 (see Figure 1.10).

Thus, A_1, A_2 and $B_1 \cap B_2$ are disjoint, and moreover, $A_1 \cup A_2 \cup (B_1 \cap B_2)$ is equal to the full probability space:

{fullspace}

$$\phi^{0,0\setminus 1}_{\mathrm{sect}_{-k,k}\setminus\Omega,p_{sd},q}(A_1) + \phi^{0,0\setminus 1}_{\mathrm{sect}_{-k,k}\setminus\Omega,p_{sd},q}(A_2) + \phi^{0,0\setminus 1}_{\mathrm{sect}_{-k,k}\setminus\Omega,p_{sd},q}(B_1 \cap B_2) = 1 \tag{1.54}$$

Let us bound the probability of $B_1 \cap B_2$. We can compare it to the event

$$\tilde{B} = \{0 \overset{*}{\underset{\mathrm{sect}_{-1,1}}{\longleftrightarrow}} \infty, 0 \underset{\mathrm{sect}_{-1,1}}{\longleftrightarrow} \infty \text{ and the dual cluster comes before the primal one on the way from } \ell_{-1} \text{ to } \ell_1\}.$$

We can open all edges of $\Omega \cap \mathrm{sect}_{0,1}$ and close all edges of $\Omega \cap \mathrm{sect}_{-1,0}$, this connects the primal cluster from the event B_1 and the dual cluster from the event B_2 to zero. Then by finite energy property there exists a positive constant c depending only on Ω, such that

$$\begin{aligned} c\phi^{0,0\setminus 1}_{\mathrm{sect}_{-k,k}\setminus\Omega,p_{sd},q}(B_1 \cap B_2) &\le \phi^{0}_{\mathrm{sect}_{-k,k},p_{sd},q}(B_1 \cap B_2 \cap \{(\Omega \cap \mathrm{sect}_{-1,0}) \text{ closed, } (\Omega \cap \mathrm{sect}_{-1,0}) \text{ open}\}) \\ &\le \phi^{0}_{\mathrm{sect}_{-k,k},p_{sd},q}(\tilde{B}). \end{aligned}$$

From now on the proof will require that $k \ge 3$, but it can be easily modified for $k \in \{1, 2\}$.

Let us look at the probability of the event

$$C = \{0 \overset{*}{\underset{\mathrm{sect}_{1,3}}{\longleftrightarrow}} \infty\}$$

conditioned on $\tilde{B}$, and compare it to the event $\tilde{B}$ itself. The existence of the primal cluster from 0 to infinity in $\text{sect}_{-1,1}$ has less influence on C, than a primal cluster from 0 to infinity in $\text{sect}_{1,-3}$ (closer to ℓ_1, than the dual cluster). The free boundary conditions at ℓ_k are two $\frac{\pi}{2}$-turns closer to the dual cluster from C than the free boundary conditions at ℓ_{-k} for the dual cluster of $\tilde{B}$. The comparison between boundary conditions concludes that

$$\phi^0_{\text{sect}_{-k,k},p_{sd},q}(C|\tilde{B}) \geq \phi^0_{\text{sect}_{-k,k},p_{sd},q}(0 \overset{*}{\longleftrightarrow} \infty \text{ in } \tilde{B} | 0 \longleftrightarrow \infty \text{ in } \tilde{B}) \geq \phi^0_{\text{sect}_{-k,k},p_{sd},q}(\tilde{B}),$$

and, by comparison between boundary conditions,

$$\phi^0_{\text{sect}_{-3,1},p_{sd},q}(C \cap \tilde{B}) \geq \phi^0_{\text{sect}_{-k,k},p_{sd},q}(C \cap \tilde{B}) \geq \big(\phi^0_{\text{sect}_{-k,k},p_{sd},q}(\tilde{B})\big)^2.$$

Let us look at the existence of the primal infinite cluster in $C \cap \tilde{B}$ conditioned on the existence of two separated infinite dual clusters (let us call this event $(0 \overset{*}{\longleftrightarrow} \infty)^2$). The probability of this event will not change if the boundaries ℓ_{-1} and ℓ_3 are glued together to obtain $\mathbb{Z}^2$. The probability for primal infinite cluster to exist increases if we remove two dual clusters. Thus,

$$\begin{aligned}
\big(c\phi^{0,0\setminus 1}_{\text{sect}_{-k,k}\setminus\Omega,p_{sd},q}(B_1 \cap B_2)\big)^2 &\leq \phi^0_{\text{sect}_{-3,1},p_{sd},q}(C \cap \tilde{B}) \\
&\leq \phi^0_{\text{sect}_{-3,1},p_{sd},q}(0 \longleftrightarrow \infty \text{ in } C \cap \tilde{B} | (0 \overset{*}{\longleftrightarrow} \infty)^2 \text{ in } C \cap \tilde{B}) \\
&\leq \phi^0_{\mathbb{Z}^2,p_{sd},q}(0 \longleftrightarrow \infty \text{ in } C \cap \tilde{B} | (0 \overset{*}{\longleftrightarrow} \infty)^2 \text{ in } C \cap \tilde{B}) \\
&\leq \phi^0_{\mathbb{Z}^2,p_{sd},q}(0 \longleftrightarrow \infty),
\end{aligned}$$

and the last probability is equal to zero because of Zhang's argument. Combined with (1.54), this implies the result.

□

Proof of Lemma 1.20. We are going to prove that, with positive probability, there exists a dual path in $\mathbb{U}_k \setminus \Lambda_{n,k}$ disconnecting 0 from infinity and that this probability does not depend on n. This fact implies the statement of the proposition.

Let us look at the event $\{\ell_{-4k-2} \underset{\mathbb{U}_k \setminus \Lambda_{n,k}}{\overset{*}{\longleftrightarrow}} \ell_{-4k-1}\}$ not in $\mathbb{U}_k = \text{sect}_{-4k-2,4k+2}$, but in a bigger domain $\text{sect}_{-12k-6,4k+2}$. The domain $\Omega = \Lambda_{n,3k+1} \cap \text{sect}_{-12k-6,4k+2}$ is simple and symmetric with respect to the symmetry line ℓ_{-4k-2}. Putting free boundary conditions on ℓ_{-12k-6} and on $\partial\Omega \cap \text{sect}_{-12k-6,-4k-2}$ instead of ℓ_{-4k-2} decreases the probability of dual path to appear. Then, by comparison between boundary conditions

$$\begin{aligned}
&\phi^0_{\mathbb{U}_k,p_{sd},q}(\ell_{-4k-2} \underset{\mathbb{U}_k \setminus \Lambda_{n,k}}{\overset{*}{\longleftrightarrow}} \ell_{-4k-1}) \\
&\qquad \geq \phi^0_{\text{sect}_{-12k-6,4k+2},p_{sd},q}(\ell_{-4k-2} \underset{\text{sect}_{-12k-6,4k+2}\setminus\Omega}{\overset{*}{\longleftrightarrow}} \ell_{-4k-1} | \Omega \cap \text{sect}_{-12k-6,-4k-2} \text{ closed}) \\
&\qquad \geq \phi^{0,0\setminus 1}_{\text{sect}_{-12k-6,4k+2}\setminus\Omega,p_{sd},q}(\ell_{-4k-2} \overset{*}{\longleftrightarrow} \ell_{-4k-1}) \geq \tfrac{1}{2},
\end{aligned}$$

where the last bound is a direct consequence of Lemma 1.25.

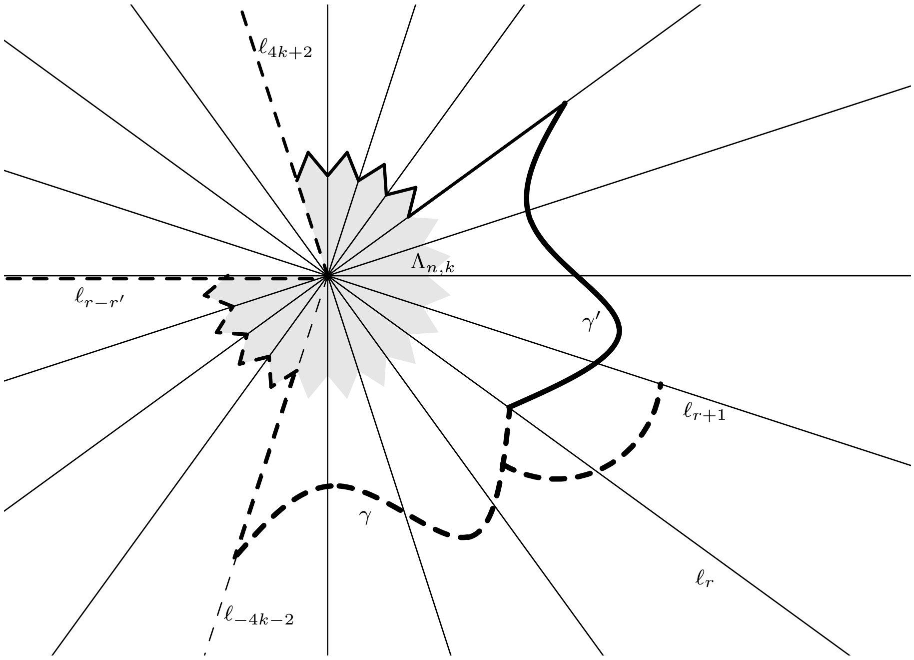

Figure 1.11: Construction of the dual connection between γ and ℓ_{r+1}.

Suppose now that for some integer $r \in [-4k-1, 4k+1]$ lines ℓ_{-4k-2} and ℓ_r are already connected in $\mathbb{U}_k \backslash \Lambda_{n,k}$ by a dual path γ (see Figure 1.11). Consider ℓ_r as a symmetry line and reflect γ with respect to it (let us call the result γ'). The domain Ω' defined as the area of $\mathrm{sect}_{-4k-2,2r+4k+2}$ bounded by $\gamma \cup \gamma'$ is a simple symmetric domain. We are going to work in a sector $\mathrm{sect}_{r-r',r+r'}$, where $r' = 4k+2+|r|$. Then $\Omega = (\Lambda_{n,3k+1} \cap \mathrm{sect}_{r-r',r+r'}) \cup \Omega'$ is also a simple symmetric domain and, using the same strategy as before, we obtain

$$\begin{aligned}\phi^0_{\mathbb{U}_k,p_{sd},q}(\ell_r \underset{\mathbb{U}_k\backslash\Lambda_{n,k}}{\overset{*}{\longleftrightarrow}} \ell_{r+1} \mid \gamma) &\geq \phi^0_{\mathrm{sect}_{r-r',r+r'},p_{sd},q}(\ell_r \underset{\mathrm{sect}_{r-r',r+r'}\backslash\Omega}{\overset{*}{\longleftrightarrow}} \ell_{r+1} \mid \Omega \cap \mathrm{sect}_{r-r',r} \text{ closed}) \\ &\geq \phi^{0,0\backslash 1}_{\mathrm{sect}_{r-r',r+r'}\backslash\Omega,p_{sd},q}(\ell_r \overset{*}{\longleftrightarrow} \ell_{r+1}) \geq \tfrac{1}{2}.\end{aligned}$$

This implies the result since, by iterative conditioning, we find

$$\phi^0_{\mathbb{U}_k,p_{sd},q}(\ell_{-4k-2} \underset{U_k\backslash\Lambda_{n,k}}{\overset{*}{\longleftrightarrow}} \ell_{4k+2}) \geq \left(\tfrac{1}{2}\right)^{8k+4}.$$

□

1.6.8 Proof of Lemma 1.21

Proof. Fix k and C as in Lemma 1.18 so that:

$$\sum_{x \in \partial\Lambda_{n,k}} \phi^0_{\Lambda_{n,k},p_{sd},q}(0 \longleftrightarrow x) \geq C.$$

Divide the boundary of $\Lambda_{n,k}$ into $8(2k+1)$ pieces by splitting each side of each boundary layers into two halfs at the midpoint. Then, there exists at least one piece $I_{n,k}$ such that

$$\sum_{x\in I_{n,k}} \phi^0_{\Lambda_{n,k},p_{sd},q}(0 \longleftrightarrow x) \geq \frac{C}{17k}. \tag{1.55}$$

{1sq}

By the finite energy property, there exists a constant $c' = c'(q,k,\xi) \geq 1$ such that for any $j \in \mathbb{Z} \cap [-k,k]$ and for any $x \in \mathbb{U}_k$ the following holds:

{fepsq}

$$\frac{1}{c'} \geq \frac{\phi^{\xi}_{\mathbb{U}_k,p,q}[x \longleftrightarrow (0,0,0)]}{\phi^{\xi}_{\mathbb{U}_k,p,q}[x \longleftrightarrow (0,0,j)]} \geq c'. \tag{1.56}$$

Combining (1.55) and (1.56), we obtain that for some positive constant c, independent of n,

$$\sum_{x\in I_{n,k}} \phi^0_{\Lambda_{n,k},p_{sd},q}[(0,0,j) \longleftrightarrow x] \geq c,$$

where j is the height of $I_{n,k}$. Using a bigger domain with $2k$ layers to centre the j-th layer and comparing boundary conditions as in (1.9), we conclude that

$$\sum_{x\in I_n} \phi^0_{\Lambda_{n,2k},p_{sd},q}[0 \longleftrightarrow x] \geq c,$$

where I_n is the projection of $I_{n,k}$ onto the layer of height 0.

Let us open all edges in a smaller box $\Lambda_{R,2k}$ for some $R \in (0,N)$. Then, we can write the following inequality

$$\begin{aligned}\sum_{x\in I_n} \phi^0_{\Lambda_{n,2k},p_{sd},q}[x \longleftrightarrow \Lambda_{R,2k} | \Lambda_{R,2k} \text{ is all open}] &\geq \frac{\sum_{x\in I_n} \phi^0_{\Lambda_{n,2k},p_{sd},q}[0 \longleftrightarrow x]}{\phi^0_{\Lambda_{n,2k},p_{sd},q}[0 \longleftrightarrow \partial\Lambda_{R,2k}]} \\ &\geq \frac{c}{\phi^0_{\Lambda_{n,2k},p_{sd},q}[0 \longleftrightarrow \partial\Lambda_{R,2k}]}.\end{aligned}$$

Also, we can choose any γ from $\partial\Lambda_n$ to $\partial\Lambda_R$, project it on all layers of $\mathbb{U}_k$ as $\tilde{\gamma}$ and make $\tilde{\gamma}$ open. Then, we obtain

$$\begin{aligned}\sum_{x\in I_n} \phi^{\xi}_{\Lambda_{n,2k}\setminus(\Lambda_{R,2k}(\tilde{\gamma})),p_{sd},q}[x \longleftrightarrow \Lambda_{R,2k}(\tilde{\gamma})] &= \sum_{x\in I_n} \phi^0_{\Lambda_{2n,k},p_{sd},q}[x \longleftrightarrow \Lambda_{R,2k} | \Lambda_{R,2k} \cup \tilde{\gamma} \text{ are open}] \\ &\geq \frac{c}{\phi^0_{\Lambda_{n,2k},p_{sd},q}[0 \longleftrightarrow \partial\Lambda_{R,2k}]}.\end{aligned}$$

where $\Lambda_{R,2k}(\tilde{\gamma})$ is a union of $\Lambda_{R,2k}$ and $\tilde{\gamma}$), and ξ denotes the boundary conditions wired on $\partial\Lambda_{R,2k}$ and $\tilde{\gamma}$ and free on $\Lambda_{n,2k}$. Notice that $\{x : x \underset{\Lambda_{n,2k}\setminus(\Lambda_{R,2k}\cup\tilde{\gamma})}{\longleftrightarrow} I_n\}$ can be projected on $\mathbb{Z}^2$ without multiple projections on one point (if not, an open path between two points with the same projection would cross $\tilde{\gamma}$). Thus, we can conclude that

inasquare1}

$$\sum_{x\in I_n} \phi^{\xi}_{\Lambda_n\setminus(\Lambda_R\cup\gamma),p_{sd},q}[x \longleftrightarrow \Lambda_R \cup \gamma] \geq \frac{c}{\phi^0_{\Lambda_{n,2k},p_{sd},q}[0 \longleftrightarrow \partial\Lambda_{R,2k}]}. \tag{1.57}$$

By Lemma 1.20, for any $\varepsilon > 0$, one may choose R large enough that for n large enough

$$\phi^0_{\Lambda_{n,2k},p_{sd},q}[0 \longleftrightarrow \partial\Lambda_{R,2k}] < \varepsilon.$$

Together with (1.57), this gives the result.

□

1.6.9 Proof of Lemma 1.22

Let us call $p_{m,n} = p_{m,n}(C)$ the probability that the box $[-m, m] \times [0, Cm] \in \mathbb{H}$ with wired boundary conditions on ∂S_n is crossed from top to bottom, i.e.

$$p_{m,n} = \phi^1_{S_n,p_{sd},q}([-m, m] \times \{0\} \underset{[-m,m]\times[0,Cm]}{\longleftrightarrow} [-m, m] \times \{Cm\}). \tag{1.58}$$

Let us fix $C \in \mathbb{N}$ and take $k, m, n \in \mathbb{N}$ such that $2Ckm < n$. For $i \geq 0$, define the domains L^b_i and L^t_i as follows:

$$\begin{aligned}
L^b_0 &= [-m, m] \times [0, Cm],\\
L^t_0 &= [-m, m] \times [n - Cm, n],\\
L^b_i &= L^b_{i-1} \cup \big(L^b_{i-1} + (-2m, 0)\big) \cup \big([-m, m] \times [0, Cm] + (-im, iCm)\big),\\
L^t_i &= L^t_{i-1} \cup \big(L^t_{i-1} + (-2m, 0)\big) \cup \big([-m, m] \times [n - Cm, n] + (-im, -iCm)\big).
\end{aligned}$$

For $i \leq k$, the domains L^t_i and L^b_i stay in the strip S_n and do not intersect. We are going to study the events

$$\begin{aligned}
A^b_{m,n,i} &= \big\{[0, m] \times \{0\} \underset{L^b_i}{\longleftrightarrow} \{-(i + 1)m\} \times \mathbb{Z}\big\},\\
A^t_{m,n,i} &= \big\{[0, m] \times \{n\} \underset{L^t_i}{\longleftrightarrow} \{-(i + 1)m\} \times \mathbb{Z}\big\}.
\end{aligned}$$

Lemma 1.26. *For every $k, m, n \in \mathbb{N}$ such that $2Ckm < n$, we have that*

$$\phi^{0\backslash 1}_{S_n,p_{sd},q}(A^b_{m,n,k} \cap A^t_{m,n,k}) \geq (\tfrac{1}{2} - p_{m,n-2Ckm})^{2(k+1)}. \tag{1.59}$$

Proof. We prove the estimate by induction. The event $A^b_{m,n,0}$ is rewritten as follows:

$$\big\{([0, m] \times \{0\} \underset{[-m,m]\times[0,Cm]}{\longleftrightarrow} \{-m\} \times [0, Cm])\big\}$$

.

Under $0\backslash 1$ boundary conditions, the events

$$\big\{[0, m] \times \{0\} \underset{[-m,m]\times[0,Cm]}{\longleftrightarrow} ([-m, m] \times \{Cm\}) \cup (\{-m\} \times [0, Cm]\big\}$$

and

$$\big\{[-m, 0] \times \{0\} \underset{[-m,m]\times[0,Cm]}{\overset{*}{\longleftrightarrow}} ([-m, m] \times \{Cm\}) \cup (\{m\} \times [0, Cm])\big\}$$

have the same probability. Since by duality and symmetry at least one of them should occur in any configuration, we find that

$$\phi^{0\backslash 1}_{S_n,p_{sd},q}([0,m]\times\{0\} \underset{[-m,m]\times[0,Cm]}{\longleftrightarrow} ([-m,m]\times\{Cm\})\cup(\{-m\}\times[0,Cm])) \geq \frac{1}{2}.$$

Adding wired boundary conditions increase the probability to have a vertical crossing of the box so

$$\phi^{0\backslash 1}_{S_n,p_{sd},q}([-m,m]\times\{0\} \underset{[-m,m]\times[0,Cm]}{\longleftrightarrow} [-m,m]\times\{Cm\}) \leq p_{m,n}$$

which implies that

$$\phi^{0\backslash 1}_{S_n,p_{sd},q}(A^b_{m,n,0}) = \phi^{0\backslash 1}_{S_n,p_{sd},q}([0,m]\times\{0\} \underset{[-m,m]\times[0,Cm]}{\longleftrightarrow} \{-m\}\times[0,Cm]) \geq \frac{1}{2} - p_{m,n}.$$

The same estimation is true also for $A^t_{m,n,0}$. Then, the FKG inequality gives that

$$\phi^{0\backslash 1}_{S_n,p_{sd},q}(A^b_{m,n,0}\cap A^t_{m,n,0}) \geq \big(\frac{1}{2} - p_{m,n}\big)^2.$$

Suppose now that for some i,

$$\phi^{0\backslash 1}_{S_n,p_{sd},q}(A^b_{m,n,i}\cap A^t_{m,n,i}) \geq (\tfrac{1}{2} - p_{m,n-2Cim})^{2(i+1)}.$$

Let us first notice that $p_{m,n}$ decreases with respect to the second argument because of the Domain Markov property (narrowing the strip by adding wired boundary conditions increases the probability to have a crossing in a box inside it). Thus, we can write that

{hugoadd} $$\phi^{0\backslash 1}_{S_n,p_{sd},q}(A^b_{m,n,i}\cap A^t_{m,n,i}) \geq (\tfrac{1}{2} - p_{m,n-2C(i+1)m})^{2(i+1)}. \tag{1.60}$$

Let us look at the event $A^b_{m,n,i+1}$ conditioned on the events $A^b_{m,n,i}$ and $A^t_{m,n,i}$ (see Figure 1.12). Let γ_b be the uppermost path from $[0,m]\times\{0\}$ to $\{-(i+1)m\}\times\mathbb{Z}$ satisfying $A^b_{m,n,i}$ and γ_t be the lowermost path from $[0,m]\times\{n\}$ to $\{-(i+1)m\}\times\mathbb{Z}$ satisfying $A^t_{m,n,i}$. Let γ'_b and γ'_t be the reflections of γ_b and γ_t with respect to the line $\{-(i+1)m\}\times\mathbb{Z}$. Note that $\{-(i+1)m\}\times\mathbb{Z}$ is an axis of symmetry for L^b_{i+1} and L^t_{i+1}.

Set the boundary conditions to be free on γ'_b and γ'_t (this can only decrease the probability of $A^b_{m,n,i+1}$). Then, by the same reasons as for $A_{m,n,0}$, the probability to have an open path from $([0,n]\times\{0\})\cup\gamma$ to the top or to the left boundary of L^b_{i+1} is bigger than $\frac{1}{2}$.

The probability for this path to hit the top part of ∂L^b_{i+1} (i.e. $[-(i+2)m,-im]\times\{(i+2)Cm\}$) is smaller than the probability for $[-m,m]\times[0,Cm]+(-(i+1)m,(i+1)Cm)$ to be crossed from top to bottom, which can be bounded above using the comparison between boundary conditions and the Domain Markov property. If we restrict our strip to $\mathbb{Z}\times[(i+1)Cm,n-(i+1)Cm]$ (all the paths will be left outside the strip) and put wired boundary conditions on its boundary, the probability $[-(i+2)m,-im]\times[(i+1)Cm,(i+2)Cm]$ to have a vertical crossing is equal to $p_{m,n-2C(i+1)m}$. The initial domain is bigger

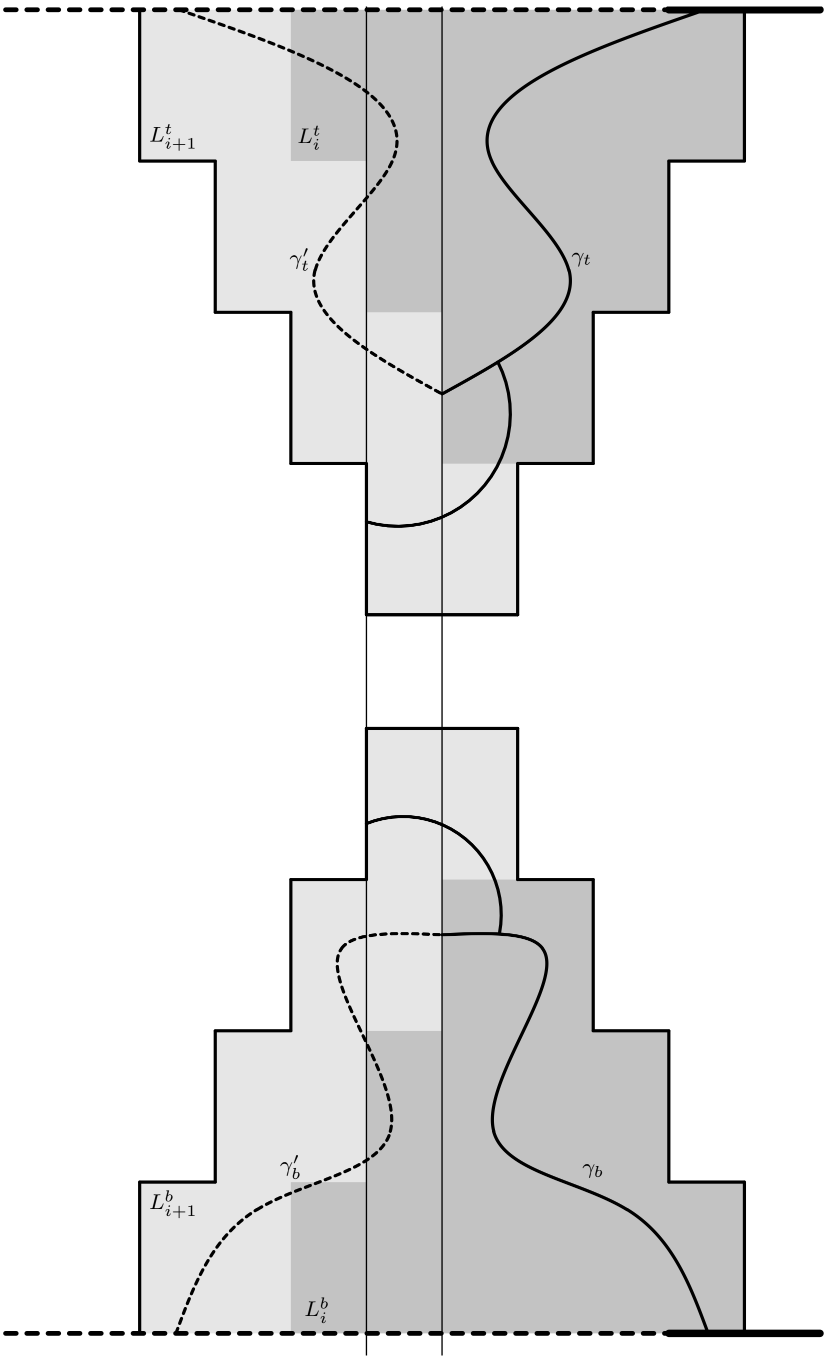

Figure 1.12: Events $A^b_{m,n,i+1}$ and $A^t_{m,n,i+1}$ with corresponding boundary conditions.

and has smaller boundary conditions, so the probability of this event is smaller in this context. This leads to

$$\phi^{0\backslash 1}_{S_n,p_{sd},q}(A^b_{m,n,i+1}|A^b_{m,n,i}\cap A^t_{m,n,i}) \geq \tfrac{1}{2} - p_{m,n-2C(i+1)m},$$

and the same bound holds for $A^t_{m,n,i+1}$. By FKG inequality, we deduce that

$$\phi^{0\backslash 1}_{S_n,p_{sd},q}(A^b_{m,n,i+1}\cap A^t_{m,n,(i+1)}|A^b_{m,n,i}\cap A^t_{m,n,i}) \geq \left(\tfrac{1}{2} - p_{m,n-2C(i+1)m}\right)^2.$$

Combining it with (1.60), we deduce that

$$\phi^{0\backslash 1}_{S_n,p_{sd},q}(A^b_{m,n,i+1} \cap A^t_{m,n,(i+1)}) \geq \left(\tfrac{1}{2} - p_{m,n-2C(i+1)m}\right)^{2(i+2)}.$$

Letting i be equal to k gives the result. □

Corollary 1.27. *Let us fix $C \geq 4$ and $m, n \in \mathbb{N}$ such that*

{relmn}
$$n \geq 9C^2 m. \tag{1.61}$$

Let us call A^b_C the event that $[0,m] \times \{0\}$ is connected either to $\{-4Cm - m\} \times [0, 8Cm]$ or to $[-4Cm - m, m] \times \{8Cm\}$ in $[-4Cm - m, 4Cm - m] \times [0, 8Cm] \cap L^b_{4C}$.

Then,

$$\phi^{0\backslash 1}_{S_n,p_{sd},q}(A^b_C) \geq (\tfrac{1}{2} - p_{m,C^2m})^{2(4C+1)}. \tag{1.62}$$

Proof. The result follows directly from the fact that $p_{m,n}$ is decreasing in the second variable and from the previous lemma applied to $A_{m,n,4C}$. The built path either ends at $\{-4Cm - m\} \times [0, 8Cm]$ or leaves the box from the top boundary. □

Lemma 1.28. *For any choice of $C \geq 4$ and $M > 0$, we have that*

$$\sum_{x \in [-(4C+1)m, -m] \times \{0\}} \phi^{0\backslash 1}_{S_n,p_{sd},q}(x \longleftrightarrow 0) \geq M(\tfrac{1}{2} - p_{m,C^2m})^{2(4C+1)} \tag{1.63}$$

for m large enough and $n \in \mathbb{N}$ such that (1.61) *holds.*

Proof. Let us choose R according to Lemma 1.21, $m \geq R$ and $n \geq 9C^2 m$ and look at the square box $\Lambda = \Lambda_{4Cm} + (-m, 4Cm)$ with the smaller box $\Lambda^0 = \Lambda_R + (-m, 4Cm)$ inside.

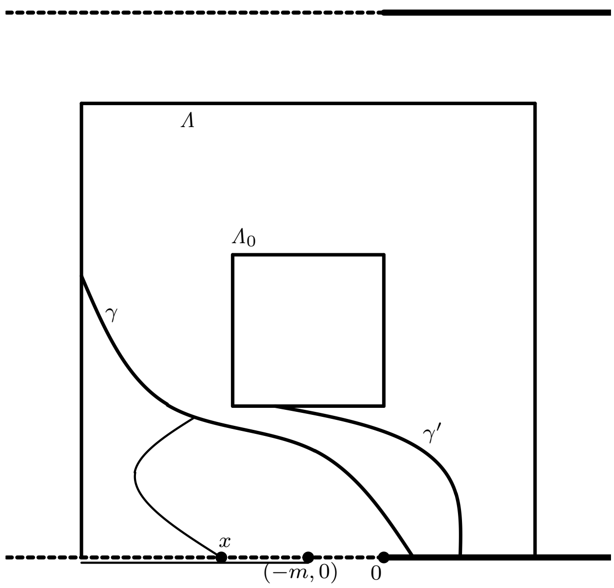

Figure 1.13: Connection of a point x to γ is more probable than its connection to $\gamma' \cup \Lambda_0$.

Then, conditioned on A_C^b, let γ be the uppermost realisation of the path going from $[0,m]\times\{0\}$ to the left or upper parts of $\partial\Lambda$ in the domain described in Corollary 1.27 (see Figure 1.13). Note that γ lies in the annulus $\Lambda\backslash\Lambda^0$. Let us choose γ' connecting $\partial\Lambda$ and Λ^0 and going to the right from γ. Then, by Lemma 1.21,

$$\sum_{x\in[-(4C+1)m,-m]\times\{0\}}\phi^{0\backslash 1}_{S_n,p_{sd},q}(x\longleftrightarrow\gamma)\geq\sum_{x\in[-(4C+1)m,-m]\times\{0\}}\phi^{0}_{\Lambda,p_{sd},q}(x\longleftrightarrow\Lambda^0\cup\gamma\,|\,\Lambda^0\cup\gamma\text{ are open})\geq M.$$

Increasing the domain to S_n and putting wired boundary conditions on ∂^+S_n only increases the probabilities in the sum. Together with Corollary 1.27, this gives the result. □

Let us now fix any $\varepsilon>0$ and define $\rho=\rho(\varepsilon)>1$ and $C=C(\rho)>4$ large enough that

$$(C+1)\leq\rho^C \tag{1.64}$$ {Cro1}

and

$$C\geq 2\left(\tfrac{\rho}{\rho-1}\right)^2 \tag{1.65}$$ {Cro2}

and look at the set

$$K_n=\{k\in\mathbb{N},k\leq\log_\rho\tfrac{n}{9C^2},p_{\rho^k(1-\rho),n}\leq\tfrac{1}{4}\}.$$

We will prove that we are either in Case 1 or in Case 2 of Lemma 1.22 depending on whether $|K_n|$ is large or not.

Lemma 1.29. *Fix $R>0$. For any choice of $\rho>1$ and for any n large enough, if $|K_n|\geq\frac{1}{2}\log_\rho\frac{n}{9C^2}$, then*

$$\sum_{x\in[-n,0)}\phi^{0\backslash 1}_{S_n,p_{sd},q}((x,0)\longleftrightarrow 0)\geq R\log n.$$

Proof. Inequality (1.64) together with the condition $|K_n|\geq\frac{1}{2}\log_\rho\frac{n}{9C^2}$ allows to pick $\frac{1}{10C}\log_\rho\frac{n}{9C^2}$ indices $k\in K_n$ such that the intervals

$$I_k=[-(4C+1)\rho^k,-\rho^k]$$

are disjoint. Let us choose $M=20RC4^{2(4C+1)}$ and apply Lemma 1.28 to each I_k. Then, we obtain

$$\begin{aligned}\sum_{x\in[-n,0)}\phi^{0\backslash 1}_{\mathbb{H},p_{sd},q}((x,0)\longleftrightarrow 0)&\geq\sum_k\sum_{x\in I_k}\phi^{0\backslash 1}_{\mathbb{H},p_{sd},q}((x,0)\longleftrightarrow 0)\\&\geq\frac{M}{10C4^{2(4C+1)}}\log_\rho\tfrac{n}{9C^2}\geq R\log n.\end{aligned}$$

Notice that R is positive and can be picked arbitrary large for n big enough. □

Lemma 1.30. *Take n large enough and suppose that $|K_n|\leq\frac{1}{2}\log_\rho\frac{n}{C}$, then*

$$\phi^0_{\mathbb{H},p_{sd}-\varepsilon,q}(0\longleftrightarrow\partial\Lambda_n)<n^{-20}.$$

To prove this theorem we need an additional result. Let us define the *Hamming distance*between two configurations ω and ω' by

$$H(\omega,\omega') := \sum_{e\in E(\Omega)} |\omega(e)-\omega'(e)|,$$

and the Hamming distance between an event A and a configuration ω] by

$$H_A(\omega) := \inf_{\omega'\in A} H(\omega,\omega').$$

Then, we can write an inequality similar to (1.10) in the terms of the expected Hamming distance [Gri06]:

{Hamdist} $$\frac{d}{dp}\log\phi^{\xi}_{G,p,q}(A) \geq \frac{1}{p(1-p)}\phi^{\xi}_{G,p,q}(H_A). \tag{1.66}$$

We now turn to the proof.

Proof. Let us study the probability of the following event:

$$\mathcal{A}_k = \left\{[-2\rho^{k+1},-2\rho^k]\times\{0\} \longleftrightarrow [2\rho^k,2\rho^{k+1}]\times\{0\} \text{ in } A_n\right\} \tag{1.67}$$

where A_k is the half-annulus

$$A_k = [-2\rho^{k+1},2\rho^{k+1}]\times[0,C\rho^k(\rho-1)]\backslash[-2\rho^k,2\rho^k]\times[0,C\rho^{k-1}(\rho-1)]$$

(see Figure 1.14).

The rectangles $[-2\rho^{k+1},-2\rho^k]\times[0,C\rho^k(\rho-1)]$ and $[2\rho^k,2\rho^{k+1}]\times[0,C\rho^k(\rho-1)]$ are crossed in the vertical direction with probability $p_{\rho^k(1-\rho)}$ each. Inequality (1.65) implies that A_k contains a square of size $4\rho^{k+1}$ that is crossed with probability bigger than $\frac{1}{2}$.

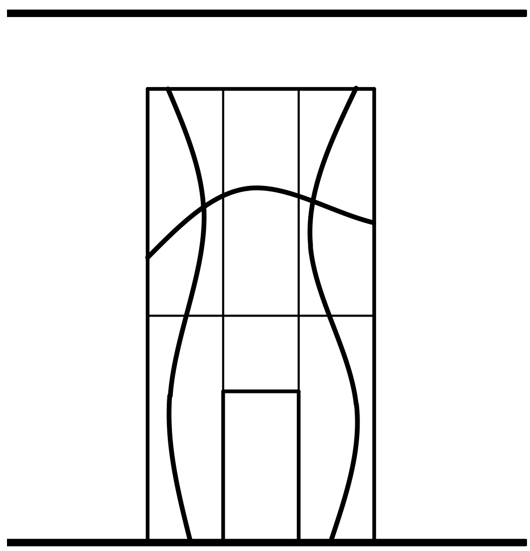

Figure 1.14: Realisation of $\mathcal{A}_k$.

Altogether, this gives a bound $\phi^1_{S_n,p_{sd},q}(\mathcal{A}_k) \geq \frac{1}{2}p^2_{\rho^k(1-\rho)}$. The assumption on the size of K_n implies that $\phi^1_{S_n,p_{sd},q}(\mathcal{A}_k) \geq \frac{1}{32}$ for at least $\frac{1}{2C}\log_\rho \frac{n}{9C^2}$ values of k. Notice also that for all these indices, A_k lies in Λ_n and that the A_k are disjoint.

We study this probability for $p = p_{sd}$ so $\mathcal{A}_k$ implies the existence of dual-open circuit in $\mathbb{H}^*$ disconnecting 0 from $\partial\Lambda_n$. The expected number of such disjoint circuits is bigger than $\frac{1}{64}\log_\rho \frac{n}{C}$, since the A_k are disjoint. This number bounds from below the expectation of the Hamming distance of the event that 0 is connected to $\partial\Lambda_n$ in the dual lattice:

$$\phi^{1^*}_{\mathbb{H},(p_{sd})^*,q}(H_{0\overset{*}{\longleftrightarrow}\partial\Lambda_n}) = \phi^0_{\mathbb{H},p_{sd},q}(H_{0\longleftrightarrow\partial\Lambda_n}) \geq \tfrac{1}{64}\log_\rho \tfrac{n}{C}.$$

Here, we changed to the primal lattice with free boundary conditions using the self-duality of the model for p_{sd}. We apply (1.66) to obtain

$$\frac{d}{dp}\log\phi^0_{\mathbb{H},p_{sd},q}[0 \longleftrightarrow \partial\Lambda_n] > \frac{1}{64}\log_\rho \frac{n}{C}.$$

By monotonicity, this inequality extends to every $p \in [p_{sd} - \varepsilon', p_{sd}]$ for $\varepsilon > 0$. Integrating the previous inequality gives:

$$\int_{p_{sd}-\varepsilon}^{p_{sd}} \frac{d}{dp}\log\phi^0_{\mathbb{H},p,q}[0 \longleftrightarrow \partial\Lambda_n] \geq \frac{\varepsilon}{64}\log_\rho \frac{n}{C},$$

or, put differently,

$$\phi^0_{\mathbb{H},p_{sd}-\varepsilon,q}[0 \longleftrightarrow \partial\Lambda_n] \leq \left(\tfrac{C}{n}\right)^{\frac{\varepsilon}{64\log\rho}}.$$

The value of ρ was chosen close enough to 1 that

$$\frac{\varepsilon}{64\log\rho} > 20.$$

This concludes the proof. $\square$

Chapter 2

Brownian motion on three-dimensional spaces and the behaviour of its projections

2.1 Definition of Brownian motion

2.1.1 Stochastic processes

A *random process* X_t is a set of random variables, indexed by some set, usually by an interval $t \in [0, T]$, the positive part of the real line $t \in \mathbb{R}^+$ or, in discrete case, $t \in \mathbb{N}$. In these case this index is often called *time*.

Markov property [Mar54] (compare with Subsection 1.3) is one of the most wanted properties for stochastic processes.

A stochastic process X_t defined on $[0, T]$, $\mathbb{R}^+$ or $\mathbb{N}$ is called a *Markov process* if for any subset $\tau = (t_i)_{i=1}^n$ of an index set, that has a strictly increasing order

$$(X_{t_n} | X_{t_{n-1}}) \perp \left((X_{t_i})_{i=1}^{n-2} \right). \tag{2.1}$$ {Mprop}

That is, for any time $t > t_0$ the process X_t depends only on the value X_{t_0}, and not on the values for previous moments.

This property is the most important preserved in the definition of Brownian motion.

2.1.2 Definition itself

The *standard Brownian motion* or the *Wiener process* is a stochastic process W_t (or B_t) on $t \in \mathbb{R}^+$ with following properties:

- It starts at zero

  $$B_0 = 0 \tag{2.2}$$ {BMdef_zero

  (we will sometimes translate B_t to start at some other point $B_0 = x \in \mathbb{R}$, then this Brownian motion is not called standard)

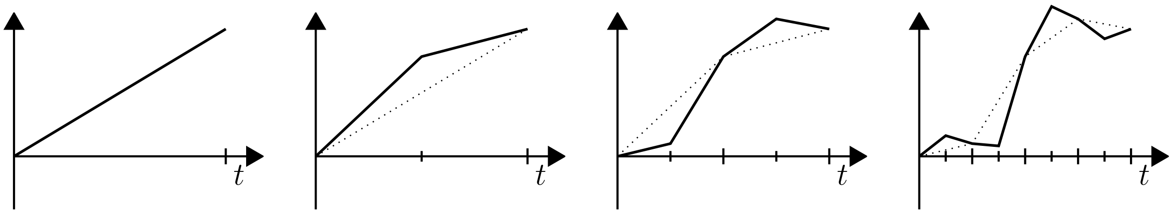

Figure 2.1: Iterative Levy's construction of a Brownian motion

- all its increments are independent, i.e for any times $t' > t$ and any $s < t$

$$(B_{t'} - B_t) \perp B_s, \tag{2.3}$$ {BMdef_inde

  this property makes Brownian motion a Markov process

- the increments are distributed as Gaussian random variables

Mdef_gauss} $$(B_{t'} - B_t) \sim \mathcal{N}(0, |t' - t|) \tag{2.4}$$

- almost surely the path of B_t is continuous

Note that property (2.4) can be made weaker. Suppose all increments $(B_{t'} - B_t)$ are distributed identically with zero mean and variance $\mathrm{Var}(B_{t'} - B_t) = |t' - t|$, and independent due to (2.3). Then the central limit theorem leads directly to (2.4). Moreover, if we bound any higher moment of distributions of $(B_{t'} - B_t)$, they even do not need to be identical due to Lyapunov central limit theorem.

2.1.3 Construction of Brownian motion

The definition above has no importance without guaranties that such process exists. This is the statement of the following theorem.

Theorem 2.1 ([Wie23]). *There exists a continuous stochastic process (called Brownian motion) that satisfies properties* (2.2)–(2.4)*.*

The proof is done by constructing Brownian motion on a time interval $[0, 1]$. Brownian motion on bigger intervals is built by concatenation (in the sense as in (3.2)) of needed number of independently sampled Brownian motions on unit intervals. There are several possible constructions leading to the result:

Levy's construction

This is step-by-step construction of the values of the process B in dyadic points (see Figure 2.1).

Let us define the set of dyadic points as follows:

$$\mathcal{D} = \bigcup_{n=0}^{\infty} \mathcal{D}_n = \bigcup_{n=0}^{\infty} \{ \frac{k}{2^n} \text{ for integer } k \in [0, 2^n] \}$$

and independently attribute a normally distributed random variable $\mathcal{Z}_t \sim \mathcal{N}(0,1)$ to each point $\{t \in \mathcal{D}\}$.

For the first step $n = 0$, set $B(0) = 0$ and $B(1) = \mathcal{Z}_1$. For the further steps define

$$B(t) = \frac{B(t-2^{-n}) + B(t+2^{-n})}{2} + \frac{\mathcal{Z}_t}{2^{\frac{1}{2}(n+1)}} \quad \forall t \in \mathcal{D}_n \backslash \mathcal{D}_{n-1}.$$

The process constructed in the following way is indeed well-defined, continuous and preserves Properties (2.2)–(2.4) (see [MP10] for the details).

Limit of random walks

Another way to construct a Brownian motion is as a limit of random walks.

The generalisation of the Central Limit Theorem called functional central limit theorem or Donsker's invariance principle [Don52] states that if $\{X_i\}_{i=1}^{\infty}$ are independently identically distributed with zero mean and variance 1, then the processes $B^{(n)}(t)$ defined as

$$B^{(n)}(t) = \frac{\sum_{i=1}^{\lfloor nt \rfloor} X_i}{\sqrt{n}}$$

converge in distribution, and the resulting process possess all properties stated in Subsection 2.1.2.

The most simple distribution of steps is to take them equal to $+1$ or -1 with probability $\frac{1}{2}$ (compare with Section 3.1), but complifications are also possible.

2.1.4 Properties of Brownian motion

Let us mention several properties of Brownian motion that will be useful for further discussions (let us for example refer to [MP10]).

- It follows from the definition that Brownian paths are continuous almost surely. On the other hand, they are almost surely non-differentiable at any point.

- It follows from the definition that Brownian motion is a Markov process. Moreover, this property can be generalised in the following sense. Let us define the stopping time τ of a random variable on $[0,T] \cup \{\infty\}$, such that for any $t < T$ the event $\{\tau \geq t\}$ depends only on the process up to time t, and not on its further values. It is usually seen as a rule (called stopping rule) to stop the process if some conditions are satisfied, for example is the process reaches some fixed value etc.

  For Brownian motion the strong Markov property, stated as

  $$(B_{t'} - B_\tau) \perp B_s \quad \forall t' > \tau, s < \tau \quad \text{for any stopping time } \tau, \tag{2.5}$$ {SM_BM}

  holds as well.

- We will call the *transition probability density* the probability density of the process leaving the point x at time t to arrive to the point $y = x + \Delta x$ at time $t + \Delta t$. The transition probability density of Brownian motion does not depend on the time nor the point of the departure and is described by the Gaussian law

$$p(x, y = x + \Delta x, t, t + \Delta t) = \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{(\Delta x)^2}{2\Delta t}}. \tag{2.6}$$ {probden}

- Let us define the process $M_t = \min_{0\leq s\leq t} B_s$. Then, the following processes have the same distribution

{reflprinc} $$(M_t - B_t) \stackrel{d}{=} |B_t|. \tag{2.7}$$

  The process $(M_t - B_t)$ can be seen as a Brownian motion reflecting from the line $x = 0$, so (2.7) is called the *reflection principle*. Moreover, the set of all times where Brownian motion reaches its minimum has the same distribution as the set of points where the value of Brownian motion is zero.

2.1.5 Generalisation to $\mathbb{R}^d$

Brownian motion in $\mathbb{R}^d$ is defined as $B_t^d = (B_t^{(i)})_{i=1}^d$ with d independent Brownian motions $B_t^{(i)}$ (note that for one-dimensional Brownian motion the index d is usually omitted). This definition is rotational-invariant and does not depend on the choice of orthonormal basis. Also it preserves all important properties from the previous subsection, such as continuity, non-differentiability, Markov property etc. The transitional probability density in this case is

$$p(x, y, t, t + \Delta t) = \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{||x-y||^2}{2\Delta t}},$$

where $||\cdot||$ denotes the Euclidian distance on $\mathbb{R}^d$.

2.2 SDE's, Itô diffusion

2.2.1 Itô stochastic integral

We can define the integration with respect to Brownian motion (in Itô sense as an opposite to Stratonovich definition) as the following limit (see [Øk03] for details). Suppose that $\pi_t = \left\{(t_i)_{i=0}^n : 0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = t\right\}$ is a partition of $[0, t]$ and denote $|\pi_t| = \max_{1\leq i\leq n}(t_i - t_{i-1})$. Then, for any process H_t which is right-continuous, locally bounded and adapted (informally it means that H_t cannot depend on anything happening in the further time $t > s$). we can define

{Itoint} $$\int_0^t H_s dB_s = \lim_{|\pi_t|\longrightarrow 0} \sum_{i=1}^n H_{t_{i-1}}(B_{t_i} - B_{t_{i-1}}). \tag{2.8}$$

This limit exists in the sense that it converges in probability, the resulting stochastic process is called Itô stochastic integral.

Note that the value of H_t in the sum (2.8) is taken in the initial point of the interval $[t_{i-1}, t_i]$, thus it is independent from $(B_{t_i} - B_{t_{i-1}})$. Here lies the difference between this definition and the Stratonovich stochastic integral [Str66].

2.2.2 The definition of Itô process

For given functions $\mu = \mu(X_t, t)$ and $\sigma = \sigma(X_t, t)$ an *Itô process* is a random process satisfying

$$X_t = X_0 + \int_0^t \mu(X_s, s)ds + \int_0^t \sigma(X_s, s)dB_s. \tag{2.9}$$ {Itodef_int

Its existence and uniqueness is guarantied under some regularity conditions on stochastic processes $\mu(X_t, t)$ and $\sigma(X_t, t)$. It is usually written in the shorten form

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t. \tag{2.10}$$ {Itodef}

Here $\mu = \mu(X_t, t)$ is called the *drift*, and $\sigma = \sigma(X_t, t)$ is the *diffusion*.

If coefficients does not directly depend on time, then (2.10) is called *Itô diffusion*

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t. \tag{2.11}$$ {Itodifdef

In this case the regularities required for μ and σ are Lipschitz continuity condition

$$\exists C > 0 : \quad |\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| < C|x - y| \quad \forall x, y, \tag{2.12}$$ {Lipcond}

and not more than linear growth:

$$\exists D > 0 : \quad |\mu(x)| + |\sigma(x)| < D(1 + |x|) \quad \forall x. \tag{2.13}$$ {growcond}

Then the solution of (2.11) for any initial position X_0 (either fixed point or random variable with bounded second moment) exists and is unique [Ito46, Øk03].

Later on we will work with Itô's diffusion only.

This definition is naturally generalised to the multidimensional case:

$$dX_t = \mu(X_t)dt + \Sigma(X_t)dB_t. \tag{2.14}$$ {Itodifdef_

with a drift vector $\mu^T(X_t) = (\mu_i(X_t))_{i=1}^d$ and a diffusion matrix $\Sigma(X_t) = (\Sigma_{ij}(X_t))_{i,j=1}^d$. As before there exists a unique solution for this equation, if $\mu^T(X_t)$ and $\Sigma(X_t)$ satisfy conditions (2.12)–(2.13).

2.2.3 Properties of Itô diffusion

- As Brownian motion, Itô diffusion is a Markov process in the sense as in (2.3). Moreover, the strong Markov property, same as (2.5) holds as well.

  The paths of Itô diffusion are continuous

- Let us mention one more property we will refer to later.

Proposition 2.2. *Suppose $B_t^{(1)}$ and $B_t^{(2)}$ are two independent one-dimensional Brownian motions. Then the process defined as*

$$dX_t = \mu(X_t)dt + \sigma_1(X_t)dB_t^{(1)} + \sigma_2(X_t)dB_t^{(2)} \tag{2.15}$$ {dbsum_temp

is an Itô diffusion with drift $\mu(X_t)$ and diffusion $\sigma(X_t)$ of the form

$$\sigma(X_t) = \sqrt{\sigma_1^2(X_t) + \sigma_2^2(X_t)}$$

This statement follows from the definition of the stochastic integral in Itô sense, when the values of diffusion coefficients do not depend on the increments of Brownian motions. Random variables coming from the sum in (2.8) thus have the following distribution:

$$\sigma_1(X_{t_{i-1}})(B_{t_i}^{(1)} - B_{t_{i-1}}^{(1)}) + \sigma_2(X_{t_{i-1}})(B_{t_i}^{(2)} - B_{t_{i-1}}^{(2)}) \sim \mathcal{N}(0, [\sigma_1^2(X_{t_{i-1}}) + \sigma_2^2(X_{t_{i-1}})](t_i - t_{i-1})).$$

Let us emphasise that if (2.15) is written as

$$dX_t = \mu(X_t)dt + \left(\sqrt{\sigma_1^2(X_t) + \sigma_2^2(X_t)}\right) dB_t,$$

the used Brownian motion B_t is not independent of $B_t^{(1)}$ nor $B_t^{(2)}$.

In the multidimensional case this proposition can be stated as follows.

Proposition 2.3. *The d-dimensional process defined as $dX_t = \mu(X_t)dt + \tilde{\Sigma}(X_t)dB_t^k$ with non-square $k \times d$ matrix $\tilde{\Sigma}(X_t)$ and k independent Brownian motions can be rewritten in the form* (2.14).

The diffusion matrix depend only on $\tilde{\Sigma}(X_t)\tilde{\Sigma}^T(X_t)$ the matrix of correlations. Note that although the correlations between components are preserved, the choice of the diffusion matrix is not unique.

2.2.4 Infinitesimal generator

Let us take a stochastic process X_t and define the *infinitesimal generator* as

infgen_def}

$$\mathcal{A}(f) = \lim_{t \downarrow 0} \frac{\mathbb{E}^x(f(X_t)) - f(x)}{t}, \tag{2.16}$$

here $\mathbb{E}^x$ denotes the expectation defined for the process X_t starting at x.

If the process X_t is defines as (2.14), then this limit exists for any compactly-supportable $f : \mathbb{R}^d \mapsto \mathbb{R}$ from C^2, and, moreover,

{AtoSDE}

$$\mathcal{A}(f(X)) = \sum_i \mu_i(X) \frac{\partial}{\partial x_i} f(x) + \frac{1}{2} \sum_{i,j} (\sigma(X)\sigma^T(X))_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} f(X). \tag{2.17}$$

In the case of the d-dimensional Brownian motion

{ALapl}
$$\mathcal{A} = \frac{1}{2}\sum_{i=1}^{d}\frac{d^2}{dx_i^2} = \frac{1}{2}\Delta, \tag{2.18}$$

here Δ is the Laplace operator.

This is the core property of Brownian motion. It relates Brownian motion to harmonic functions and allows to use it for the solution of the Dirichlet problem posed as $\Delta f = 0$ in some bounded (and regular enough) $D \subset \mathbb{R}^d$ with values of f given on the boundary ([Kak44b], see [MP10] for further discussion). This is the powerful tool to analyse the behaviour of Brownian motion, such as the question of whether it is transient (i.e almost surely $\lim_{t\longrightarrow\infty} ||B_t|| = \infty$) or recurrent ($B_t$ returns to any given point or a neighbourhood of any given point after infinitely big time) [MP10].

2.2.5 Fokker-Plank equation

Let us look at the probability density of the process defined by (2.11) or (2.14). If (2.12)–(2.13) hold and the diffusion coefficient is non-zero, then at any time $t > 0$ the probability density is a well-defined continuous function. Moreover, we can write the following differential equation describing it.

Theorem 2.4 (Fokker-Plank or Kolmogorov forward equation, [Fok14, Pla17])**.** *If X_t is an Itô process defined by* (2.10) *with Conditions* (2.12)–(2.13) *held, then its probability density $p(x,t)$ evolves from the given initial probability density $p_0(x) = p(x,0)$ according the following equation:*

$$\frac{\partial}{\partial t}p(x,t) = -\frac{\partial}{\partial x}\big(\mu(x,t)p(x,t)\big) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\big(\sigma^2(x,t)p(x,t)\big). \tag{2.19}$$
{FP1}

In the case of several dimensions, this equation is written as

$$\frac{\partial}{\partial t}p(X,t) = -\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\big(\mu_i(X,t)p(X,t)\big) + \frac{1}{2}\sum_{i=1}^{d}\frac{\partial^2}{\partial x_i \partial x_j}\big((\Sigma(X,t)\Sigma^T(X,t))_{i,j}\, p(X,t)\big). \tag{2.20}$$
{FP_dim}

Although the analytical solution can be found only in a small number of cases, it is often approximated numerically [RH89]. Also even without solving it the equation of the type (2.19)–(2.20) can give some information about the behaviour of the process.

Let us also mention Kolmogorov backward equation, [Kol31], which is in some sense dual to the previous theorem. It says that the evolution of the probability density of the Itô process before it ends up at the given final probability density is described by equation $-\frac{\partial}{\partial t}p(X,t) = \mathcal{A}(p(X,t))$ with $\mathcal{A}$ the infinitesimal generator of the process.

2.2.6 Itô's lemma

Theorem 2.5 (Itô's lemma)**.** *For X_t defined as in* (2.10) *and for a twice differentiable function $f(x,t)$ the process $f(X_t,t)$ is an Itô process with the following stochastic differential equation*

$$d\,f(X_t,t) = \Big(\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}\Big)dt + \sigma\frac{\partial f}{\partial x}dB_t \qquad (2.21)$$ {Itoform}

This formula is obtained by applying to $f(X_t,t)$ the Tailor expansion together with the formal identity $(dB_t)^2 = dt$.

The equation (2.21) can be also generalised to the multidimensional case. Let it take the d-dimensional stochastic Itô diffusion defined as in (2.14). Suppose $f(X,t) : \mathbb{R}^d \times \mathbb{R}^+ \mapsto \mathbb{R}$ (the case of multidimensional is threaten in the same way). We will denote $\nabla_X f$ its gradient vector and $\mathrm{Hess}_X f$ its Hessian matrix. Then

toform_dim} $$d\,f(X_t,t) = \big(\tfrac{\partial f}{\partial t} + (\nabla_X f)^T \cdot \mu + \frac{1}{2}\mathrm{Tr}\left(\Sigma^T \cdot \mathrm{Hess}_X f \cdot \Sigma\right)\big)dt + (\nabla_X f)^T \cdot \Sigma \cdot dB_t^d, \qquad (2.22)$$

where dB_t^d is a vector of d independent Brownian increments.

2.2.7 Example: the Bessel process

Consider the d-dimensional Brownian motion B_t^d and look at the process $Y_t^d = ||B_t^d||$. It is called a d-dimensional Bessel process ([McK60], see also [RY91] for other details).

The process B_t^d is an Itô diffusion with zero drift vector μ and the covariance matrix $\Sigma = \mathbb{I}_d$ the unit d-dimensional matrix. The function $f(X) = ||X|| = \sqrt{\sum_{i=1}^d x_i^2}$ does not explicitly depend on time ($\frac{\partial f}{\partial t} = 0$), its gradient and Hessian are of the form

$$\nabla_X^T f = \left(\frac{x_i}{(\sum_{k=1}^d x_k^2)^{\frac{1}{2}}}\right)_{i=1}^d, \qquad \mathrm{Hess}_X f = \left(\frac{\delta_{i,j}}{(\sum_{k=1}^d x_k^2)^{\frac{1}{2}}} - \frac{x_i x_j}{(\sum_{k=1}^d x_k^2)^{\frac{3}{2}}}\right)_{i,j=1}^d$$

Then the formula (2.22) applied to this case, together with Property 2.2 to reduce the number of independent Brownian motions, gives the following SDE for the process:

{BessSDE} $$dY_t^d = \frac{d-1}{2}\frac{1}{Y_t^d}dt + dB_t \qquad (2.23)$$

with one-dimensional Brownian motion B_t.

2.3 Differential geometry preliminaries

An *n-dimensional manifold* is a set such that a small neighbourhood of any of its points is homeomorphic to an open ball in $\mathbb{R}^n$, the function ϕ_M mapping the neighbourhood of M into a ball in $\mathbb{R}^n$ is called a *chart*. If such neighbourhoods of two different points M, M' intersect, the function $\phi_M \cdot \phi_{M'}^{-1}$ called a transitional map determines the structure of the manifold. If all transitional maps are smooth, then the manifold is called *smooth* as well. Smooth manifolds allows to define the operation of differentiation, and thus to construct many other related structures, such that tangent spaces or metric (for the details let us refer for example to [Hir76, dR84, Lee09]).

Any smooth n-dimensional manifold $\mathcal{M}$ can be embedded into a Euclidian space $\mathbb{R}^N$ of finite dimension $N \geq n$, i.e. $\mathcal{M}$ can be seen as a sub-manifold of $\mathbb{R}^N$ and its inner topology coincides with one induced by $\mathbb{R}^N$ (this statement is called Whitney's embeddinng theorem, see [dR84] for further details).

Another construction called Riemann curvature tensor defined in each point of a manifold also arises from the metric. As for more than two-dimensional manifold it is indeed a tensor, we can define its restriction to any two linearly independent tangent vectors from the corresponding to x tangent space. This is called the *sectional curvature*. We will work with manifolds with constant sectional curvature, that does not depend on the choice of x or tangent vectors, such manifolds are called *space forms.* [Spi99]

The natural example of a such manifold is a Euclidian space $\mathbb{R}^n$ itself. The standard metric of $\mathbb{R}^n$ is defined by metric tensor $g = \mathbb{I}_n$, a unit matrix of dimension d. Its curvature is equal to zero.

A bit less trivial example is an n-dimensional sphere S^n with a positive curvature (usually for more generality it is taken to be equal to 1). The connected n-dimensional manifold with the curvature equal to -1 is called $\mathbb{H}^n$ the hyperbolic manifold.

First example — S^3

The n-dimensional sphere is naturally seen embedded in $\mathbb{R}^{n+1}$ as

$$S^n = \{x = (x_i)_{i=1}^{n+1} \in \mathbb{R}^{n+1} : ||x|| = \sum_{i=1}^{n+1} x_i^2 = 1\}.$$

We will study the 3-dimensional sphere $S^3 \subset \mathbb{R}^4$.

Also S^3 is isomorphic to the Lie group of unitary matrices with determinant equal to one:

$$SU(2) = \left\{ u = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}; |a|^2 + |b|^2 = 1 \right\}. \tag{2.24}$$ {S3_matr}

Each point $x = (x_i)_{i=1}^4 \in S^3$ corresponds to $\begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix}$.

Let us denote the unit element $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The tangent space of $SU(2)$ at e can be seen as a Lie algebra $\mathfrak{su}(2)$ generated by orthonormal basis (with respect to the product $\frac{1}{2}\mathrm{tr}(g_1 g_2^*)$)

$$e_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}; \quad e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \quad e_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

For any other element $u \in SU(2)$ its tangent space is generated by $\{ue_1, ue_2, ue_3\}$.

It is sometimes convenient to introduce a Cartesian and polar coordinate systems and denote

$$\begin{array}{rcl} Re(a) &=& x = \cos(\theta) \\ Im(a) &=& y = \sin(\theta)\cos(\psi) \\ Re(b) &=& \sin(\theta)\sin(\psi)\cos(\phi) \\ Re(b) &=& \sin(\theta)\sin(\psi)\sin(\phi) \end{array} \qquad (2.25)$$

{thetadef}

In the case of polar coordinates (θ, ψ, ϕ) the metric tensor g_{S^3} is written as

{S3_metric}

$$g_{S^3} = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \sin^2(\theta) & 0 \\ 0 & 0 & \sin^2(\theta)\sin^2(\psi) \end{array} \right), \qquad (2.26)$$

or, written in another way, $ds^2 = d\theta^2 + \sin^2(\theta) d\psi^2 + \sin^2(\theta)\sin^2(\psi)\, d\phi^2$.

Later we use these definitions of x, y and θ.

Another example — $\mathbb{H}^3$

A good model of $\mathbb{H}^3$ is the set of positive definite Hermitian matrices of unit determinant:

{H3_matr}

$$\mathbb{H}^3 \cong \left\{ h = \left(\begin{array}{cc} a & b \\ \bar{b} & c \end{array} \right); a, c \in \mathbb{R}_+, b \in \mathbb{C}, ac - |b|^2 = 1 \right\}. \qquad (2.27)$$

As before, the unit element $e = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$ belongs to $\mathbb{H}^3$.

Any matrix $g \in \mathbb{H}^3$ has positive eigenvalues Λ, Λ^{-1} with $\Lambda \geq 1$. We denote $\lambda = \log(\Lambda)$

We can write an analogue of the polar coordinates (2.25) for the case of $\mathbb{H}^3$ as well [Cos01]. The entries (2.27) of a matrix $h \in \mathbb{H}^3$ will be then written as

{lambdadef}

$$\begin{array}{rcl} a &=& \cosh(\lambda) + \sinh(\lambda)\cos(\psi), \\ b &=& -\sinh(\lambda)\sin(\psi)e^{i\phi}, \\ c &=& \cosh(\lambda) - \sinh(\lambda)\cos(\psi). \end{array} \qquad (2.28)$$

The metric tensor $g_{\mathbb{H}^3}$ in this case is written

{H3_metric}

$$g_{\mathbb{H}^3} = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \sinh^2(\lambda) & 0 \\ 0 & 0 & \sinh^2(\lambda)\sin^2(\psi) \end{array} \right), \qquad (2.29)$$

for the coordinates (λ, ψ, ϕ).

2.4 Duistermaat-Heckmann measure. The case of $\mathbb{R}^3$

One more construction we can define on a smooth manifold $\mathcal{M}$ is called a *Poisson structure* [Lic77]. It is a Lie bracket $\{\cdot,\cdot\} : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \mapsto C^\infty(\mathcal{M})$ on the space of all smooth functions $C^\infty(\mathcal{M})$, that also satisfies the Leibniz rule $\{f, gh\} = \{f, g\}h + g\{f, h\}$. Let us remind that Lie brackets satisfy skew symmetry, bi-linearity and Jacobi identity. The Poisson bracket determine the foliation of $\mathcal{M}$ by symplectic sub-manifolds called *symplectic leaves*. As symplectic manifolds, they are equipped with closed non-degenerate differential 2-form w. (for the details let us refer to [Vai94, DZ05])

The push-forward of w under a moment map is called the *Duistermaat-Heckman measure*. First time it was discussed in [DH82], and in [BV82, AB84] it was shown that this measure is piecewise polynomial on a convex polytope (see [AMW02, dS05] for further discussions).

2.4.1 The case of $\mathbb{R}^3$

Let us construct these structures on $\mathbb{R}^3$. This space carries the Poisson structure named after Kirillov, Kostant and Souriau [Kir99, Sou70].

$$\{x, y\} = z, \quad \{y, z\} = x, \quad \{z, x\} = y.$$

In this case, the symplectic leaves are 2-dimensional spheres centred at the origin. They carry symplectic forms which are given by formula

$$\omega = d\phi \wedge dz, \tag{2.30}$$ {volform_R3

where $x + iy = \sqrt{r^2 - z^2}e^{i\phi}$ and r is the radius of the sphere. Note that they are volume forms and they are rotational invariant.

Its push-forward to the z-axis is the Duistermaat-Heckman measure:

$$\mathrm{DH}_r = 2\pi\chi_{[-r,r]}(z)dz. \tag{2.31}$$ {DH_R3}

Here $\chi_{[-r,r]}(z)$ is the characteristic function of the segment $[-r, r]$. The mass of this measure is equal to the symplectic volume of the sphere given by $\mathrm{Vol}(S^2, \omega) = 4\pi r$. The normalised measure

$$\frac{1}{\mathrm{Vol}(S^2, \omega)}\mathrm{DH}_r = \frac{1}{2r}\chi_{[-r,r]}(z)dz \tag{2.32}$$ {DH_R3_norm

is a probability measure.

2.4.2 Connection with Brownian motion on $\mathbb{R}^3$

Let us look at the Wiener process on $\mathbb{R}^3$ starting at the origin. We will consider two projections of this process. One is under the map

$$r : M = (x, y, z) \mapsto ||M|| = \sqrt{x^2 + y^2 + z^2},$$

the stochastic differential equation of this process is the following (see Subsection 2.2.7):

$$dr_t = dB_t + \frac{1}{r_t}dt, \tag{2.33}$$ {Bes3}

Another projection is the map $(x, y, z) \mapsto (r, z)$. Under this projection, the process is also a Markov process described by the following system of stochastic differential equations

{system_R^3

$$\begin{aligned} dr_t &= \tfrac{\sqrt{r_t^2 - z_t^2}}{r_t} dB_t^{(1)} + \tfrac{z_t}{r_t} dB_t^{(2)} + \tfrac{1}{r_t} dt, \\ dz_t &= dB_t^{(2)}, \end{aligned} \tag{2.34}$$

where $B_t^{(1,2)}$ are two independent Wiener processes on $\mathbb{R}$. This can also be obtained with Itô's lemma (2.22).

The following theorem establishes a relation between the system of stochastic differential equation (2.34) and the Duistermaat-Heckman measure.

Theorem 2.6. *The conditional probability density for z_t for a fixed value of r_t is given by the normalized Duistermaat-Heckman measure:*

al_R^3_eq}

$$\rho_{z_t}(z | r_t = r) = \frac{1}{\mathrm{Vol}(S^2, \omega)} \mathrm{DH}_r. \tag{2.35}$$

Proof. Let us note that for $r_t = 0$ the equations (2.34), as well as the equation (2.33), do not satisfy the conditions (2.12) –(2.13) that guarantee the existence and the uniqueness of the solution. Although we are sure the solutions exists and are unique because they are well-defined projections of a unique and existing process.

Applying Theorem 2.4 to (2.33) we obtain the equation

{FP1_R}

$$\frac{d}{dt} p_r = \frac{1}{2} \frac{d^2}{d^2 r} p_r - \frac{1}{r} \frac{d}{dr} p_r + \frac{1}{r^2} p_r \tag{2.36}$$

on the probability density function of r_t. The joint probability density for (r_t, z_t) is described by

{FP2_R}

$$\frac{d}{dt} p_{r,z} = \frac{1}{2} \frac{d^2}{d^2 r} p_{r,z} + \frac{1}{2} \frac{d^2}{d^2 z} p_{r,z} + \frac{z}{r} \frac{d^2}{dr dz} p_{r,z} - \frac{z}{r^2} p_{r,z}. \tag{2.37}$$

Let us plug in (2.37) the probability density function of the form presuming (2.49), i.e.

{FPanz_R}

$$p_{r,z}(r, z, t) = p_r(r, t) \times \tfrac{1}{2r} \chi_{[-r,r]}(z), \tag{2.38}$$

then (2.36) and (2.37) coincide. Thus for initial conditions $p_0(r, z) = \delta_{r_0}(r) \chi_{[-r_0, r_0]}(z)$ the solution of (2.37) is indeed of the form (2.38) with p_r the solution of (2.36). So does hold when r converges to 0. □

Let us shortly discuss one more relation of the type (2.35). If we apply the transform called Pitman transform [Pit75]

{Pitdef}

$$\mathcal{P}(X)_t = X_t - 2 \min_{0 \le s \le t} X_s \tag{2.39}$$

to one-dimensional Brownian motion, the resulting process is equal in distribution to three-dimensional Bessel process (2.23) (compare this result with the reflection principle (2.7) for $Y_t^1 = |B_t|$). Through the original proof by Pitman, based on approximation of Brownian motion by discrete random walks (see Subsection 2.1.3), it is also shown that the distribution of B_t conditioned on a fixed value of $\mathcal{P}(B)_t = r_0$ is uniform over the segment $[-r_0, r_0]$.

This approach inspired the studies of the behaviour of Brownian motion in more complicated spaces and under the transform constructed similarly to (2.39) using reflections described by a Coxeter group [BBO05, BBO08, Bia09, LLP10]. There, the analogues of Theorem 2.6, that relate projections of the distribution of Brownian motion to corresponding Duistermaat-Heckmann measures, were obtained as well.

Later in this chapter we obtain analogues of Theorem 2.6 for spaces S^3 and $\mathbb{H}^3$.

2.5 Brownian motion on Riemannian manifolds

2.5.1 General definition

A Riemannian manifold allows to differentiate the functions on it, so it is convenient to define Brownian motion using its infinitesimal generator [Hsu02]. In Euclidian case we have $\mathcal{A} = \frac{1}{2}\Delta$ (2.18), and for arbitrary Riemannian manufold we replace the Laplace operator by its more general analogue called the Laplace-Beltrami operator $\Delta_{LB} = \mathrm{div}(\mathrm{grad})$. In local coordinates it is written as

$$\Delta_{\mathrm{LB}} = \frac{1}{\sqrt{\det(g)}} \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \left(\sqrt{\det(g)} \sum_{j=1}^{m} g^{ij} \frac{\partial}{\partial x_j} \right) \tag{2.40}$$ {LB}

for metric tensor g, here g^{ij} denotes (i,j)-th element of g^{-1} .

2.5.2 Brownian motion on a sphere S^3

When we plug the metric tensor (2.26) into Laplace-Beltrami operator, we obtain the stochastic equation for Brownian motion on S^3

$$\begin{aligned} d\theta_t &= dB_t^{(1)} + \cot(\theta_t)dt, \\ d\psi_t &= \tfrac{1}{\sin^2(\theta)} dB_t^{(2)} + \tfrac{\cot(\psi_t)}{2\sin^2(\theta_t)} dt, \\ d\phi_t &= \tfrac{1}{\sin^2(\theta)\sin^2(\psi)} dB_t^{(3)} \end{aligned} \tag{2.41}$$ {BM_S_polar

with three independent Brownian motions $B_t^{(1,2,3)}$.

In this work we use another way of the definition. At any point $u \in SU(2)$ the diffusion term is a sum of three independent one-dimension Brownian motions performed in the directions of three (orthonormal) tangent vectors. The drift term appears to restrict the process to stay in S^3. Thus, we call the process u_t Brownian motion if it satisfies the following stochastic differential equation:

$$dg_t = g_t\Big(\sum_{i=1}^{3} e_i dB_t^{(e_i)}\Big) - \frac{3}{2} g_t dt, \tag{2.42}$$ {BM_S}

here $dB^{(e_1,e_2,e_3)}$ denote three independent one-dimensional Brownian motions. The drift coefficient $-\frac{3}{2}e$ corresponds to one half of the Casimir element of $\mathfrak{su}(2)$.

Written in polar coordinates, the stochastic differential equation (2.42) coincides with the definition through Laplace-Beltrami operator (2.41).

2.5.3 Definition for $\mathbb{H}^3$

As before we plug the metric tensor (2.29) into (2.40) to obtain the following evolution:

BM_H_polar}

$$\begin{aligned} d\lambda_t &= dB_t^{(1)} + \coth(\theta_t)dt, \\ d\psi_t &= \tfrac{1}{\sinh^2(\lambda)} dB_t^{(2)} + \tfrac{\cot(\psi_t)}{2\sinh^2(\lambda_t)} dt, \\ d\phi_t &= \tfrac{1}{\sinh^2(\lambda)\sin^2(\psi)} dB_t^{(3)} \end{aligned} \tag{2.43}$$

with three independent Brownian motions $B_t^{(1,2,3)}$.

2.6 The analogue of Theorem 2.6 for S^3

2.6.1 Duistermaat-Heckman measure in the case of S^3

The conjugacy classes on $SU(2)$ are the points e and $-e$ and two-dimensional spheres of the fixed trace [AMM98]

$$\mathcal{C}_\theta = \{g \in SU(2); \mathrm{Tr}(g) = 2\cos(\theta)\} \quad \forall \theta \in [0, \pi].$$

They carry canonical volume forms ω_θ similar to (2.30), that can be written as $d\cos(\psi) \wedge d\phi$ in polar coordinates. The total volume of the conjugacy class $\mathcal{C}_\theta$ is equal to $\mathrm{Vol}(\mathcal{C}_\theta) = 4\pi\sin(\theta)$.

Let us consider projections $a : SU(2) \longrightarrow D \subset \mathbb{C}$, to the unit disc D, and $\theta : SU(2) \longrightarrow [0, \pi]$, defined as in (2.25). Then

$$a(\mathcal{C}_\theta) = \{\cos\theta + iy; y \in [-\sin(\theta), \sin(\theta)]\}.$$

Furthermore, the Duistermaat-Heckman measure being a push-forward of ω_θ satisfies the following equation

{DH_SU(2)}

$$\mathrm{DH}_\theta := a_*(\omega_\theta) = 2\pi\chi_{[-\sin(\theta), \sin(\theta)]} dy. \tag{2.44}$$

We obtain a probability measure from DH_θ normalising it:

$$\frac{1}{\mathrm{Vol}(\mathcal{C}_\theta)} \mathrm{DH}_\theta = \frac{1}{2\sin(\theta)} \chi_{[-\sin(\theta), \sin(\theta)]} dy.$$

2.6.2 The behaviour of the projections of Brownian motion on S^3

Let us consider $B_t^{S^3}$ Brownian motion on S^3 defined by (2.42) and starting at e.

The projections a and θ defined in the previous subsection have the following properties:

Proposition 2.7. *The projection θ of $B_t^{S^3}$ is a Markov process described by the stochastic differential equation*

{thetaSDE}
$$d\theta_t = dB_t + \cot(\theta_t)dt, \tag{2.45}$$

where B_t is a standard one-dimensional Brownian motion.

Note that the equation (2.45) coincides with one from (2.41) obtained from polar coordinates and Laplace-Beltrami operator.

Proposition 2.8. *The projection $a = x + iy$ of $B_t^{S^3}$ is a Markov process described by the following system of stochastic differential equations*

$$\begin{array}{rcl} dx_t & = & \frac{y_t}{\sqrt{x_t^2+y_t^2}}dB_t^{(1)} + \frac{x_t\sqrt{1-x_t^2-y_t^2}}{\sqrt{x_t^2+y_t^2}}dB_t^{(2)} - \frac{3}{2}x_t dt, \\ dy_t & = & -\frac{x_t}{\sqrt{x_t^2+y_t^2}}dB_t^{(1)} + \frac{y_t\sqrt{1-x_t^2-y_t^2}}{\sqrt{x_t^2+y_t^2}}dB_t^{(2)} - \frac{3}{2}y_t dt, \end{array} \tag{2.46}$$
{system_S^3

where $B_t^{(1,2)}$ are two independent Brownian motions on $\mathbb{R}$.

Proof. As the process $B_t^{S^3}$ is described by (2.24), the evolution of a_t and b_t is written in the following differential equations:

$$\begin{array}{rcl} da_t & = & b_t(idB_t^{(e_1)} + dB_t^{(e_2)}) + ia_t dB_t^{(e_3)} - \frac{3}{2}a_t dt, \\ db_t & = & a_t(idB_t^{(e_1)} - dB_t^{(e_2)}) - ib_t dB_t^{(e_3)} - \frac{3}{2}b_t dt. \end{array} \tag{2.47}$$
{system_S^3

The evolution of $x_t = Re(a_t)$ is described as follows:

$$\begin{array}{rcl} dx_t & = & -Im(b_t)dB_t^{(e_1)} + Re(b_t)dB_t^{(e_2)} - Im(a_t)dB_t^{(e_3)} - \frac{3}{2}x_t dt \\ & = & \sqrt{1-x_t^2}dB_t - \frac{3}{2}x_t dt, \end{array} \tag{2.48}$$
{system_S^3

the last equality follows from Proposition 2.2.

Applying Itô's lemma (2.21) to this equation with $\theta = \arccos(x)$ we obtain Theorem 2.7.

The system (2.46) of equations for x_t and $y_t = Im(a_t)$ is derived from (2.47) as well. The correlation matrix of these two processes is equal to $\begin{pmatrix} 1-x_t^2 & x_t y_t \\ x_t y_t & 1-y_t^2 \end{pmatrix}$, thus by Proposition 2.3 the equations (2.46) indeed describe the evolution of (x_t, y_t).

□

Let us note that similarly to the case of $\mathbb{R}^3$, both (2.45) and (2.46) do not satisfy the conditions (2.12)-(2.13) that guarantee the existence and the uniqueness of the solution. These guarantees are provided as these processes are projections of well-defined Brownian motion on S^3.

The following theorem establishes a relation between the system of stochastic differential equation (2.46) and Duistermaat-Heckman measures of conjugacy classes:

Theorem 2.9. *The conditional probability density for y_t for a fixed value of x_t is given by the normalized Duistermaat-Heckman measure:*

$$\rho_{y_t}(y|x_t = \cos(\theta)) = \frac{1}{\mathrm{Vol}(\mathcal{C}_\theta)}\mathrm{DH}_\theta. \tag{2.49}$$
{conditiona

Proof of Theorem 2.9. To prove it, we compare two Fokker-Plank equations (2.19)–(2.20) on evolution of the probability densities of x_t and y_t. One derived from (2.48) gives

$$\frac{d}{dt}p_x = \frac{1}{2}(1-x^2)\frac{\partial^2}{\partial x^2}p_x - \frac{1}{2}x\frac{\partial}{\partial x}p_x + \frac{1}{2}p_x \qquad (2.50)$$ {FP1_S3}

for $p_x = p_x(x,t)$ the probability density of x_t.

One describing the distribution $p_{x,y} = p_{x,y}(x,y,t)$ for both x and y and derived from (2.46) is the following:

{FP2_S3}
$$\begin{aligned}\tfrac{d}{dt}p_{x,y} &= \tfrac{1}{2}(1-x^2)\tfrac{\partial^2}{\partial x^2}p_{x,y} + \tfrac{1}{2}(1-y^2)\tfrac{\partial^2}{\partial y^2}p_{x,y} - xy\tfrac{\partial^2}{\partial x \partial y}p_{x,y} \\ &\quad -\tfrac{3}{2}x\tfrac{\partial}{\partial x}p_{x,y} - \tfrac{3}{2}y\tfrac{\partial}{\partial y}p_{x,y}.\end{aligned} \qquad (2.51)$$

Equations (2.50) and (2.51) coincide when $p_{x,y}$ is of the form presuming (2.49), i.e.

{FPanz_S3}
$$p_{x,y}(x,y,t) = p_x(x,t) \times \frac{1}{2\sqrt{1-x^2}}\chi_{[-\sqrt{1-x^2},\sqrt{1-x^2}]}(y), \qquad (2.52)$$

so there is a solution of this form. Let us take the initial condition

$$p_0(x,y) = \delta_{\sqrt{1-\varepsilon^2}}(x) \times \tfrac{1}{2\varepsilon}\chi_{[-\varepsilon,\varepsilon]}(y),$$

they correspond to (2.52). Thus together with the equation (2.51) it gives the unique solution, and it is of the type (2.52) as well. It holds when $\varepsilon \longrightarrow 0$ as well, this lead to the result. □

2.7 The analogue of Theorem 2.6 for $\mathbb{H}^3$

2.7.1 Duistermaat-Heckman measure in the case of $\mathbb{H}^3$ and moment maps in the sense of Lu

The hyperbolic space $\mathbb{H}^3$ carries a Lu-Weinstein Poisson structure [LW90] and a quasi-Poisson structure [AMM98]. The leaves determined by both structures are conjugacy classes under the $SU(2)$-action. The conjugacy class of the unit matrix e consists of one point, other leaves are two-dimensional spheres of elements of $h \in \mathbb{H}^3$ with fixed trace

$$\mathcal{C}_\lambda = \{h =\in \mathbb{H}^3; \mathrm{Tr}(g) = e^\lambda + e^{-\lambda} = 2\cosh\lambda\}, \quad \forall \lambda \in [0,\infty).$$

As in the previous sections, $SU(2)$ conjugacy classes carry canonical volume forms ω_λ, similar to ones for $SU(2)$. The total volume of $\mathcal{C}_\lambda$ is equal to $\mathrm{Vol}(\mathcal{C}_\lambda) = 4\sinh(\lambda)$.

Let us look at the map $(w,c) : \mathbb{H}^3 \mapsto \mathbb{R}^2$ mapping an element h to a pair that consists of the half of it trace $w = \frac{a+c}{2}$ and of its lower diagonal entry c. Also let us define the map $\lambda : \mathbb{H}^3 \mapsto \mathbb{R}$

$$\lambda(h) = \mathrm{arccosh}\big(\tfrac{\mathrm{Tr}(h)}{2}\big) = \mathrm{arccosh}\big(\tfrac{a(h)+c(h)}{2}\big).$$

Then the projection of the leaf under the first map is of the following form:

$$(w,c)(\mathcal{C}_\lambda) = \{(\cosh(\lambda), c); c \in [e^{-\lambda}, e^\lambda]\}.$$

The analogue of the Duistermaat-Heckman measure is given by

{DH_H^3}
$$\mathrm{DH}_\lambda := c_*(\omega_\lambda) = 2\pi\chi_{[e^{-\lambda},e^{\lambda}]}dc. \tag{2.53}$$

The corresponding normalised measure is of the form

$$\frac{1}{\mathrm{Vol}(\mathcal{C}_\lambda)}\mathrm{DH}_\lambda = \frac{1}{2\sinh(\lambda)}\chi_{[e^{-\lambda},e^{\lambda}]}dc.$$

2.7.2 The behaviour of the projections of Brownian motion on $\mathbb{H}^3$

Let us consider $B_t^{\mathbb{H}^3}$ Brownian motion on $\mathbb{H}^3$ as in (2.43).

The maps (w,c) and λ defined in previous subsection have the following properties:

Proposition 2.10. *The projection λ of $B_t^{\mathbb{H}^3}$ is a Markov process described by the stochastic differential equation*

$$d\lambda_t = dB_t + \coth(\lambda_t)dt, \tag{2.54}$$
{lambdaSDE

where B_t is the standard Brownian motion on $\mathbb{R}$.

This proposition follows immediately from (2.43).

Proposition 2.11. *The projection (w,c) of $B_t^{\mathbb{H}^3}$ is a Markov process described by the following system of stochastic differential equations*

$$\begin{array}{rcl} dw_t &=& \sqrt{w_t^2-1}\,dB_t^{(1)} + \frac{3}{2}w_t dt,\\ dc_t &=& \frac{c_t w_t - 1}{\sqrt{w_t^2-1}}dB_t^{(1)} + \frac{\sqrt{2c_t w_t - c_t^2 - 1}}{\sqrt{w_t^2-1}}dB_t^{(2)} + \frac{3}{2}c_t dt, \end{array} \tag{2.55}$$
{system_H^3

where $B_t^{(1,2)}$ are two independent Brownian motions on $\mathbb{R}$.

The first equation is obtained from (2.54) by Itô lemma (2.21) applied to $w = \cosh(\lambda)$. The second inequality as well follows from (2.21), as $c = c(\lambda,\psi)$ defined in (2.28), and we can plug the evolution of coordinates λ and ψ from (2.43) into this definition.

The following theorem establishes a relation between the system of stochastic differential equation (2.55) and Duistermaat-Heckman measures of conjugacy classes:

Theorem 2.12. *The conditional probability density for c_t for a fixed value of $w_t = \cosh(\lambda)$ is given by the normalised Duistermaat-Heckman measure:*

$$\rho_{c_t}(c|w_t = \cosh(\lambda)) = \frac{1}{\mathrm{Vol}(\mathcal{C}_\lambda)}\mathrm{DH}_\lambda. \tag{2.56}$$
{conditiona

Proof of Theorem 2.12. We use the same method as for Theorem 2.9, comparing two Fokker-Plank equations (2.19)–(2.20).

One obtained from the first equation of (2.55) and describing the evolution of the probability density of w_t only is the following:

$$\frac{d}{dt}p_w = \frac{1}{2}(w^2-1)\frac{\partial^2}{\partial w^2}p_w + \frac{1}{2}w\frac{\partial}{\partial w}p_w - \frac{1}{2}p_w, \tag{2.57}$$
{FP1_H}

here $p_w = p_w(w,t)$.

One that includes both w and c via the evolution of $p_{w,c} = p_{w,c}(w,c,t)$ for is derived from (2.55) as follows:

$$\begin{aligned} \tfrac{d}{dt}p_{w,c} &= \tfrac{1}{2}(w^2-1)\tfrac{\partial^2}{\partial w^2}p_{w,c} + \tfrac{1}{2}c^2\tfrac{\partial^2}{\partial c^2}p_{w,c} + (cw-1)\tfrac{\partial^2}{\partial w\partial c}p_{w,c} \\ &\quad + \tfrac{3}{2}w\tfrac{\partial}{\partial w}p_{w,c} + \tfrac{3}{2}c\tfrac{\partial}{\partial c}p_{w,c}. \end{aligned} \qquad (2.58)$$ {FP2_H}

If $p_{w,c}$ is of the form

{FPanz_H} $$p_{w,c}(w,c,t) = p_w(w,t) \times \frac{1}{2\sqrt{w^2-1}}\chi_{[w-\sqrt{w^2-1},w+\sqrt{w^2-1}]}(c), \qquad (2.59)$$

implied by (2.56), then (2.57) and (2.58) coincide. The existence and the uniqueness of the solution of (2.57) and (2.58) is guaranteed as they are the projections of Brownian motion. The equation (2.58) together with initial conditions

$$p_0(x,y) = \delta_{\sqrt{1+\varepsilon^2}}(x) \times \tfrac{1}{2\varepsilon}\chi_{[\sqrt{1+\varepsilon^2}-\varepsilon,\sqrt{1+\varepsilon^2}+\varepsilon]}(y),$$

gives rise to the unique solution, it is of the form (2.59). Thus the solution is of this form for $\varepsilon \longrightarrow 0$ as well, this finishes the proof. $\square$

Chapter 3

Sub-ballisticity of the model of self-repelling polymers

3.1 Simple random walks on $\mathbb{Z}^d$

Let us start with a short discussion about the simple random walk. As one of the most simple and natural models it has been studied for many years and it is now considered to be well known. This model plays a key role in many fields of studies as it converges to the Brownian motion (see Section 2.1.2). This allows to make conclusions about the model of simple random walks based on results for the Brownian motion, sometimes the other way can be used as well (for example see the discussion in Subsection 2.4.2).

Self-avoiding walks, as well as self-repelling polymers, can be considered as modifications and complications of simple random walks on a lattice. Thus, to determine when their behaviour is different is on of the key questions while studying these models.

3.1.1 Nearest neighbour simple random walks on $\mathbb{Z}^d$

Let us fix a lattice $\mathbb{Z}^d$ of d dimensions (when d is not restricted to be two or three, it is sometimes called the *hypercubic lattice*). We will call its basis vectors $\{e_i\}_{i=1}^d$, and its coordinates $\{x_i\}_{i=1}^d$, its first coordinate x_1 will be usually denoted x.

A *walk of length* n in $\mathbb{Z}^d$ is a sequence $\gamma = (\gamma(i))_{i=0}^n$ of $n+1$ vertices in $\mathbb{Z}^d$, such that $|\gamma(i) - \gamma(i-1)| = 1$ for every $0 < i \leq n$ (i.e. $\gamma(i)$ and $\gamma(i-1)$ are connected by an edge of $\mathbb{Z}^d$). Let us set $\gamma(0) = 0$. We denote $|\gamma|$ the length of a given walk γ.

The set of all walks with a fixed length n is denoted W_n and we will consider the uniform measure on it (note that the index d is omitted). The random path γ of length n distributed according to this measure is called a *(nearest neighbour) (simple) random walk*,

$$\mathbb{P}_{\mathrm{SRW}_n}(\gamma) = \frac{1}{|\mathrm{W}_n|} = \frac{1}{2^{dn}} \quad \forall \gamma \in \mathrm{W}_n. \qquad (3.1)$$ {SAWmeas}

It can be seen as a sequence of steps from $\gamma(i-1)$ to $\gamma(i)$, each step chosen randomly (uniformly and independently on other steps) from the set of $2d$ possible steps (namely $\{+1, -1\} \times \{e_i\}_{i=1}^d$).

Later we will study the properties of this distribution for n sufficiently big or even converging to infinity.

3.1.2 Simple properties of the measure

The property, that holds for simple random walks, and does not hold for further models, is Markov property (2.1).

As all steps of a simple random walk are sampled independently, we can say that for $m < n$ the behaviour of the walk after m-th step does not depend on the first $(m-1)$ steps, but only on the position γ_m:

$$\mathbb{P}_{\mathrm{SRW}_n}\big((\gamma_i)_{m+1}^n \bigm| (\gamma_i)_0^m \text{ fixed}\big) = \mathbb{P}_{\mathrm{SRW}_n}\big((\gamma_i)_{m+1}^n \bigm| \gamma_m\big).$$

This property can be also rewritten as follows. Consider $\gamma_1 \in \mathrm{W}_m$ and $\gamma_2 \in \mathrm{W}_{n-m}$. The *concatenation* $\gamma_1 \circ \gamma_2$ of γ_1 and γ_2 is the walk from W_n defined by

{concat}
$$\gamma_1 \circ \gamma_2 = \Big((\gamma_1(i))_{i=0}^m,\, (\gamma_1(m) + \gamma_2(i))_{i=0}^{n-m} \Big). \tag{3.2}$$

It clearly holds that

concatprob}
$$W_{n+m} = W_n\, W_m; \quad \mathbb{P}_{\mathrm{SRW}_n}(\gamma_1 \circ \gamma_2) = \mathbb{P}_{\mathrm{SRW}_m}(\gamma_1) \cdot \mathbb{P}_{\mathrm{SRW}_{n-m}}(\gamma_2). \tag{3.3}$$

Let us look at the behaviour of this measure under the symmetries of the lattice. For $\gamma \in \mathrm{W}_n$, the reflection of γ under the hyperplane $\{x_i = 0\}$ is denoted $\mathcal{R}_{x_i}(\gamma)$. If $\mathbb{Z}^d$ is invariant under the rotation α (i.e if α is an $\frac{n\pi}{2}$ rotation around an axis of $\mathbb{Z}^d$), then this rotation of γ around the origin is denoted $r_\alpha(\gamma)$. It is clear that $\mathcal{R}_{x_i}(\gamma), r_\alpha(\gamma) \in \mathrm{W}_n$, and

eflrotprob}
$$\mathbb{P}_{\mathrm{SRW}_n}(\gamma) = \mathbb{P}_{\mathrm{SRW}_n}(\mathcal{R}_{x_i}(\gamma)), \quad \mathbb{P}_{\mathrm{SRW}_n}(\gamma) = \mathbb{P}_{\mathrm{SRW}_n}(r_\alpha(\gamma)). \tag{3.4}$$

3.1.3 Ballistic assumption

The question we are studying in this chapter is to bound the probability for a walk to go far enough from the initial point. More precisely, we are interested in the question of whether the probability model μ_n defined on the sets of walks W_n is *ballistic*, meaning that $\exists v > 0$ such that $\mathbb{E}_{\mu_n}[|x(\gamma_n)|] \geq n$ for any n. We will in fact prove something stronger, and therefore introduce the following assumption.

Assumption 3.1 (Weak ballistic assumption)**.** *There exists $v > 0$ such that:*

{BA}
$$\limsup_{n \longrightarrow \infty} \frac{1}{n} \log \mathbb{P}_{\mu_n}(x(\gamma_n) > vn) = 0. \tag{3.5}$$

For any model preserved by the symmetries of the lattice it is equivalent to the statement

$$\limsup_{n \longrightarrow \infty} \tfrac{1}{n} \log \mathbb{P}_{\mu_n}(||\gamma_n|| > vn) = 0.$$

Note that if a model is not weakly ballistic, than it is not ballistic.

It follows from the central limit theorem (see Subsection 2.1.3) that as n converges to infinity, the distribution of $x(\gamma_n)$ converges to $\mathcal{N}(0, \frac{n}{2^{d-1}})$. Together with Cramér's theorem of the theory of large deviations it leads to the fact that

$$\mathbb{P}_{\mathrm{SRW}_n}(x(\gamma_n) > vn) \longrightarrow \int_{vn}^{\infty} e^{-x^2 \frac{2^{d-1}}{n}} dx \leq e^{-Cn}$$

for some $C > 0$.

Thus, the model of simple random walks is sub-ballistic in any dimension.

3.1.4 Spread-out random walk

Here we slightly generalise the model above.

Suppose Ω is a finite subset of vertices of $\mathbb{Z}^d$ which is preserved under the symmetries of $\mathbb{Z}^d$ and does not contain zero. We can define a walk γ as a set of steps, all of them in Ω. As before, the set of walks of length n beginning at 0 with all steps $(\gamma(i) - \gamma(i-1)) \in \Omega$ is denoted W_n^{Ω}.

In the general case, the probability of each step can be different. This can be formally written as follows. Let us define the jump-distribution ρ a probability mass function on Ω, restricted to be invariant under the symmetries of the lattice. Then, the weight of a walk γ is defined as a product of the weights of all its steps:

$$\sigma(\gamma) = \prod_{i=1}^{n} \rho(\gamma(i) - \gamma(i-1)) \quad \forall \gamma \in \mathrm{W}_n^{\Omega}, \tag{3.6}$$

{SORWweight

and the probability for a walk is proportional to its weight

$$\mathbb{P}_{\mathrm{SAW}_n}(\gamma_0) = \frac{\sigma(\gamma_0)}{\sum_{\gamma \in \mathrm{W}_n^{\Omega}} \sigma(\gamma)} \quad \forall \gamma_0 \in \mathrm{W}_n^{\Omega}. \tag{3.7}$$

{SORWmeas}

All the properties mentioned for nearest neighbour simple random walks hold for this model as well. It is Markov, invariant under symmetries of the lattice and sub-ballistic in all dimensions.

3.2 Self-avoiding walks on $\mathbb{Z}^d$

Let us now turn to the model of (nearest neighbours) self-avoiding walks. They were introduced by chemist P. Flory [Flo49] in the middle of the twentieth century to describe the geometrical shape of polymer chains. Flory conjectured universality of the model and the independence of the general behaviour of this model from the lattice.

3.2.1 Definition of the model

As for simple random walks, we will work on the set W_n of all walks on length n equipped with weights attributed to each walk. As it follows from the name of the model, the weight of a walk γ is equal to zero if γ intersects itself and is nonzero (let us say it is equal to one) if γ does not, or in other words if it is *self-avoiding*.

More formally, let

$$l_x(\gamma) = \sum_{k=0}^{|\gamma|} \mathbb{I}_{\gamma(k)=x} \tag{3.8}$$

{numbvis}

denote the number of times γ visits the point $x \in \mathbb{Z}^d$. Then the self-avoiding measure on W_n is defined as

$$\mathbb{P}_{\mathrm{SAW}_n}(\gamma_0) = \frac{\sigma(\gamma)}{\sum_{\gamma \in \mathrm{W}_n} \sigma(\gamma)} \quad \forall \gamma_0 \in \mathrm{W}_n.$$

It can be also rewritten as

{SAWmeas}

$$\mathbb{P}_{\mathrm{SAW}_n}(\gamma_0) = \frac{\mathbb{I}_{\gamma_0 \in \mathrm{SAW}_n}}{|\mathrm{SAW}_n|} \quad \forall \gamma_0 \in \mathrm{SAW}_n, \tag{3.9}$$

here SAW_n is the set of all self-avoiding walks of length n.

This model is clearly invariant under symmetries of $\mathbb{Z}^d$, as neither rotations nor reflections can change the number of intersections of a path so that if $\gamma \in \mathrm{SAW}_n$, then $\mathcal{R}_{x_i}(\gamma), r_\alpha(\gamma) \in \mathrm{SAW}_n$ for all α and x_i.

The Markov property does not hold for this model. Two self-avoiding walks γ_1 and γ_2 may intersect after concatenation. Thus,

SAWsetsize}

$$|\mathrm{SAW}_{n+m}| \leq |\mathrm{SAW}_n| \cdot |\mathrm{SAW}_m|, \tag{3.10}$$

and the sequence $\{\log |\mathrm{SAW}_n|\}$ is sub-additive.

On the other hand the Domain Markov property, defined similarly as in Subsection 1.2.1, does hold, if we define the domain of all possible steps as the whole lattice except the part surrounded and occupied by already visited vertices.

3.2.2 Connective constant

The size of the set SAW_n and its behaviour for n converging to infinity is the most naturally question about the model. It plays very important role in understanding of the behaviour of self-avoiding walks. Unfortunately, not many results have yet been proven.

Let us define the *connective constant* μ_c as

{SAWmu}

$$\mu_c = \lim_{n \longrightarrow \infty} \sqrt[n]{|\mathrm{SAW}_n|}. \tag{3.11}$$

The existence of this limit was first time shown in [HM54], it directly follows from (3.10) and Fekete's sub-additive lemma [Fek23]. The exact value of the connective constant is known to be equal to $\sqrt{2+\sqrt{2}}$ only for two-dimensional hexagonal lattice. This result was first conjectured from Renormalisation Group Theory in [Nie82], and not long time ago proven in [DS12] using the method of parafermionic observable (see Subsection 1.5.2).

Also the exact value is known for some particular infinite graphs (see [GL17]). Also in [GL17] the case of graphs that does not preserve some properties of Riemannian space is discussed.

Let us now sum up what is known for hypercubic lattices. It is clear that if a walk in $\mathbb{Z}^d$ is composed only of steps from $\{e_i\}_{i=1}^d$ (only positive directions can be taken), then it does not intersect itself. On the other hand for any self-avoiding walk γ each step is chosen from $\{e_i, -e_i\}_{i=1}^d$ barring the direction, that coincides with the previous step. Thus,

$$d \leq \mu_c(\mathbb{Z}^d) \leq (2d-1) < \lim_{n\longrightarrow\infty} \sqrt[n]{|\mathrm{W}_n|} = 2d.$$

For more then one dimension the better bounds were yet obtained only by numerical methods, such as Monte-Carlo simulations or simple enumeration, or with the use of the technique called the lace expansion [BS85] (for more details about its application see [HS92, MS96]).

In two dimensions we know that $2.6256 \leq \mu_c(\mathbb{Z}^2) \leq 2.6792$ [Noo00, PT00, Jen04]. Another type of computation gives the approximation $\mu_c(\mathbb{Z}^2) \approx 2.63815$ [Jen03]. For three-dimensional space we know that $\mu_c(\mathbb{Z}^3) \approx 4.684043$ [CLS07]. For higher dimensions we have $\mu_c(\mathbb{Z}^4) \in [6.742945, 6.8179]$, $\mu_c(\mathbb{Z}^5) \in [8.828529, 8.8602]$, $\mu_c(\mathbb{Z}^6) \in [10.874038, 10.8886]$ (see [BW72, HSS93, Fin98, Noo00, PT00]).

When d converges to infinity, we can expand $\mu_c(\mathbb{Z}^d) - (2d-1)$ as a power series of $\frac{1}{2d}$ [HS91] (see also [Kes64], further discussions and computation of the first eleven coefficients of the series can be found in [CLS07]).

Let us now discuss some results that we can obtain on $|\mathrm{SAW}_n|$ due to the value of μ_c.

Theorem 3.2. *The following bounds on the size of* SAW_n *holds*

$$\mu_c^n < |\mathrm{SAW}_n| < \mu_c^n e^{\kappa\sqrt{n}} \forall n \in \mathbb{N} \tag{3.12}$$ {SAW_HW}

for some constant $\kappa > 0$.

The lower bound of (3.12) follows directly from (3.10), the upper bound is less trivial. It was shown in [HW62] using the decomposition of a walk by bridges (bridges are defined Subsection 3.2.5, and the similar proof is presented in Subsection 3.3.2). For dimensions $d \geq 3$ this result was improved in [Kes63].

In [Kes63] it was as well proven that $\frac{|\mathrm{SAW}_{n+2}|}{|\mathrm{SAW}_n|} \longrightarrow \mu_c^2$ as n grows to infinity. It naturally leads to the conjecture that for n converging to infinity we have $\frac{|\mathrm{SAW}_{n+1}|}{|\mathrm{SAW}_n|} \longrightarrow \mu_c$ for n as well, but yet it is proven only for $d \geq 5$ [HS91]. Moreover, even the inequality $|\mathrm{SAW}_{n+1}| > |\mathrm{SAW}_n|$ proven in [O'B90] was found to be nontrivial.

We expect that for sufficiently big n the size of the set SAW_n behaves as follows [MS96]:

$$|\mathrm{SAW}_n| \sim A\mu_c^n n^{\gamma-1}, \tag{3.13}$$ {SAWsizeexp

the constants A and γ should depend only on the dimension of the lattice.

3.2.3 Different behaviour in different dimensions

Since the model was defined it is believed that its behaviour has no strong dependence on the shape of the lattice (see discussions in [MS96, BDGS12]). For example in (3.13) the parameter γ is one of the critical exponents and is believed to stay the same for all lattices of similar dimension. This property is called universality. It is also conjectured that if we define the model of self-avoiding walks in spread-out case, as in Subsection 3.1.4, this behaviour does not change as well.

On the other hand the behaviour of the model differs a lot in different dimensions. In one dimension the model of self-avoiding walks is trivial. The set SAW_n contains only two walks, one in positive and one in negative direction. The ballistic assumption holds in this case for any speed $v \leq 1$. The same results hold in the case of spread-out self-avoiding walks as well.

A famous hypothesis [LSW04] states that the distribution of self-avoiding walks on two-dimensional lattice converges to Schramm-Loewner Evolution of parameter $\frac{8}{3}$. The validity of this conjecture would imply many properties describing the behavior of self-avoiding walk when its length tends to infinity. One of the corollaries would be that for any lattice of dimension at least 2, the mean-squared distance between the beginning and the end of the self-avoiding walk of length n behaves like $n^{2\nu+o(1)}$ with $\nu < 1$ [MS96, LSW04]. The latter would imply sub-ballisticity for any self-avoiding walks in two or more dimensions.

In higher dimensions ($d \geq 5$) the behaviour of the model is known to behave as the model of simple random walks [HS91] (note that this implies sub-ballisticity). This similarity seems natural as simple random walks converge to Brownian motion, and in $d \geq 5$ it almost surely has no points of self-intersection [Kak44a]. The model of self-avoiding walks in high dimensions converges to Brownian motion as well. The dispersion coefficient is not equal to 1, as for simple random walks, but is slightly bigger, numerical estimates in [HS92] bound it by $[1.098, 1.803]$.

3.2.4 Sub-ballisticity in $\mathbb{Z}^d$

Recently, H. Duminil-Copin and A. Hammond gave a rigorous proof of sub-ballisticity for the lattices $\mathbb{Z}^d$ when $d \geq 2$.

Theorem 3.3 ([DH13])**.** *For the model of self-avoiding walks on $\mathbb{Z}^d$ the following holds for any positive v:*

$$\lim_{n \longrightarrow \infty} \frac{1}{n} \log \mathbb{P}_{\mathrm{SAW}_n} (|\gamma(n)| > vn) < 0.$$

Later in this chapter we extend this proof to a more general case

3.2.5 Bridges

The following definitions plays a role in the proof of sub-ballisticity presented in [DH13].

The walk $\gamma \in \mathrm{W}_n$ (or W_n^Ω) is called a *bridge* if

{bridgedef}

$$x(\gamma(0)) < x(\gamma(i)) \leq x(\gamma(n)) \quad \forall i : 1 \leq i \leq n. \tag{3.14}$$

We will call the set of all bridges $\mathrm{B}_n \subset \mathrm{W}_n$ (or $\mathrm{B}_n^\Omega \subset \mathrm{W}_n^\Omega$). Note that in general it is not restricted to be self-avoiding.

Suppose that γ is a bridge of length n. Then, the number $i : 0 < i < n$ is called a *renewal time* of γ if $x(\gamma(i)) < x(\gamma(k))$ for any $k > i$ and $x(\gamma(i)) \geq x(\gamma(k))$ for any $k < i$. We sometimes say that $\gamma(i)$ is a renewal point if i is a renewal time.

The increasing sequence of all renewal times of the bridge γ will be denoted R_γ.

The bridge γ is called *irreducible* if $R_\gamma = \emptyset$. The set of all irreducible bridges of arbitrary lengths is denoted iB.

A renewal time r splits γ into two bridges $(\gamma(i))_{i=0}^{r}$ and $(\gamma(i))_{i=r}^{n}$. From this point of view, an irreducible bridge is a bridge that does not admit any decomposition into shorter bridges.

Let us call SAB_n the set of all self-avoiding bridges of length n and iSAB the set of irreducible self-avoiding bridges of any length. Then

Theorem 3.4 (Kesten's Lemma). *For μ_c defined as in* (3.11) *it holds that*

$$\sum_{\gamma \in \mathrm{iSAB}} \mu_c^{-|\gamma|} = 1 \tag{3.15}$$

This allows to define the probability measure $\mathbb{P}_{\mathrm{iSAB}}(\gamma)$ on the set of self-avoiding irreducible bridges. The proof of this lemma will be in more general case presented in Subsection 3.3.3.

3.3 Self-repelling polymers on $\mathbb{Z}^d$

This model can be seen as an interpolation between simple random walks and self-avoiding walks. The walk intersecting itself can appear, but is less probable, than a similar walk with no intersections. The model of self-repelling polymers can be used as a better approximation for polymer chains, as it can take into account monomer-monomer connections of different length and a possibility that different parts of the chain can have quite small distance between them. This model can be used as a better approximation for polymer chains, as it takes into account monomer-monomer connections of different length and a possibility that different parts of the chain can have quite small distance between them.

3.3.1 Definition of the model

We will follow [IV08] to define this measure. Let us work in the general case of spread-out walks.

Fix some set of steps Ω equipped with jump-function ρ and define W_n^Ω as in Subsection 3.1.4. Consider $\phi : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ (called the *potential*) such that for any $a, b \in \mathbb{Z}_+$,

$$\phi(a+b) \geq \phi(a) + \phi(b) \tag{3.16}$$ {phi}

and $\phi(0) = \phi(1) = 0$ (thus, ϕ is non-decreasing). Let $l_x(\gamma)$ be defined as in (3.8). Then to any $\gamma \in \mathrm{W}_n^\Omega$ we can associate the weight $\sigma(\gamma)$ defined by

$$\sigma(\gamma) = \Big(\prod_{x \in \mathbb{Z}^d} e^{-\phi(l_x(\gamma))}\Big)\Big(\prod_{i=1}^{n} \rho(\gamma(i) - \gamma(i-1))\Big). \tag{3.17}$$ {weight}

As usual, the measure of self-repelling polymers is defined by

$$\mathbb{P}_{\mathrm{SRP}_n}(\gamma_0) = \frac{\sigma(\gamma_0)}{\sum_{\gamma \in \mathrm{W}_n} \sigma(\gamma)} \quad \forall \gamma_0 \in \mathrm{W}_n. \tag{3.18}$$

{SRPmeas}

The case $\phi(a) = 0$ for any $a \geq 0$ corresponds to the simple random walk (3.7). If $\phi(a) = +\infty$ for any $a \geq 2$, then no intersection is allowed and (3.18) coincides with the measure of self-avoiding walks (3.9). For any intermediate potential, intersections of γ are allowed but decrease its probability. The case $\phi(a) = k \cdot (a-1)_+$ is called weakly self-avoiding walks [Sla05].

Let A be a subset of W_n^{Ω}. Introduce the analogue of the size of the set as

$$Z(A) := \sum_{\gamma \in A} \sigma(\gamma). \tag{3.19}$$

If $A = \mathrm{W}_n^{\Omega}$, then we simply write Z_n. With this notation, for any event $A \subset \mathrm{W}_n^{\Omega}$,

$$\mathbb{P}_{\mathrm{SRP}_n}(A) = \frac{Z(A)}{Z_n}. \tag{3.20}$$

Bridges and irreducible bridges are defined as in Subsection 3.2.5. We denote

$$H_n = \sum_{\gamma \in \mathrm{B}_n} \sigma(\gamma) \tag{3.21}$$

the weight of the set of all bridges of length n.

Let us now generalise the results of the previous section to the model of self-repelling polymers. Let us from now on omit the index Ω of the set of all possible steps.

3.3.2 Connective constant

Let us now introduce the analogue of connective constant μ_c for self-repelling polymers.

Theorem 3.5. *The sequence* $(\frac{1}{n}\log Z_n)$ *converge. Furthermore,* $Z^n \geq e^{\lambda_0 n}$*, where*

lambda0def}

$$\lambda_0 = \lim_{n \longrightarrow \infty} \frac{1}{n} \log Z_n \tag{3.22}$$

is called the connective constant.

In the case of SAW, λ_0 is the logarithm of the connective constant of the lattice μ_c.

Proof. Any walk $\gamma \in \mathrm{W}_{n+m}$ can be decomposed in a unique way into two walks $\gamma_1 \in \mathrm{W}_n$ and $\gamma_2 \in \mathrm{W}_m$ so that $\gamma = \gamma_1 \circ \gamma_2$. The definition of the potential implies that

$$\phi(l_x(\gamma)) = \phi\big(l_x(\gamma_1) + l_x(\gamma_2)\big) \geq \phi(l_x(\gamma_1)) + \phi(l_x(\gamma_2)).$$

Thus, $\sigma(\gamma) \leq \sigma(\gamma_1)\sigma(\gamma_2)$ and

{cnadd}

$$Z_{n+m} \leq \sum_{\substack{\gamma_1 \in \mathrm{W}_n \\ \gamma_2 \in \mathrm{W}_m}} \sigma(\gamma_1)\sigma(\gamma_2) = Z_n Z_m. \tag{3.23}$$

The sequence $(\log Z_n)$ is therefore sub-additive and non-negative. Fekete's lemma thus implies the existence of a non-negative limit

$$\lambda_0 = \lim_{n \longrightarrow \infty} \tfrac{1}{n} \log Z_n$$

and the inequality $Z_n \geq e^{n\lambda_0}$ for any n. $\square$

Define $W(\lambda) = \sum_{n=0}^{\infty} Z_n e^{-\lambda n}$.

It follows from Cauchy-Hadamard Theorem that $W(\lambda)$ converges for any $\lambda > \lambda_0$ and diverges for any $\lambda < \lambda_0$. Moreover, due to Theorem 3.5, the sum diverges at λ_0

$$\sum_{n=0}^{\infty} Z_n e^{-\lambda_0 n} = \infty. \tag{3.24}$$

{sumwalks}

Theorem 3.6 (The analogue of Theorem 3.2)**.** *There exists a constant $C > 0$ such that for any $n \in \mathbb{N}$*

$$e^{-C\sqrt{n}} Z_n \leq H_n \leq Z_n. \tag{3.25}$$

Proof. The second inequality follows directly from definition the set of bridges H_n. To obtain the first inequality, observe that any walk of length n can be decomposed into bridges by the following procedure.

Start with the walks that lie in the upper half-space after the first step, i.e.

$$\gamma \in \mathrm{W}_n^+ := \{\gamma \in \mathrm{W}_n,\, x(\gamma(i)) > 0 \text{ for any } i > 0\}.$$

Find the latest step α_1 such that $(\gamma(i))_0^{\alpha_1}$ is a bridge and cut the walk γ at this point: $\gamma = \gamma_1 \circ \gamma_2$. Note that value of $\alpha_1 = |\gamma_1|$ can be expressed as follows:

$$\alpha_1 = \max \Big[i : x(\gamma(i)) = \max_{0 \leq i \leq n} \big(x(\gamma(i)) \big) \Big]. \tag{3.26}$$

{alpha1}

Due to (3.26), all points of γ after the α_1-th step have a smaller x-coordinate than $\gamma(\alpha_1)$. Hence, γ_2 belongs to the reflection of $\mathrm{W}_{n-\alpha_1}^+$ and therefore can be decomposed using the same method. We continue applying this procedure until the remaining walk is itself a bridge.

We obtain the sequence of bridges related to the initial walk. Their widths $h_i = |x(\gamma_i(\alpha_i)) - x(\gamma_i(0))|$ are ordered in a strictly decreasing manner and the sum of their lengths n_i is equal to n. Also, the height of the walk is bounded by $h_i \leq D n_i$, where D is the size of the set Ω of all possible steps, i.e.

$$D = \max_{\tilde{\gamma} \in \Omega} \big(x(\tilde{\gamma}) \big). \tag{3.27}$$

{D-nlines}

(Note that D is fixed by the definition of the model)

Let us denote $H_{n,h}$ the weight of the set of all bridges of length n and width h. The initial walk is uniquely determined by the set of bridges that composed it. Moreover, any set of bridges with decreasing widths corresponds to a walk in $\mathrm{W}^+ = \cup_{n\in\mathbb{N}}\mathrm{W}_n^+$. Thus, we can conclude that

$$Z(\mathrm{W}_n^+) \leq \sum_{\substack{(n_i)_{i=1}^k \\ \sum_{i=1}^k n_i = n \\ h_1<h_2<\ldots<h_k}} \prod_{i=1}^k H_{n_i,h_i} \leq \sum_{\substack{(n_i)_{i=1}^k \\ \sum_{i=1}^k n_i = n}} \prod_{i=1}^k H_{n_i} \leq H_n \sum_{\substack{(n_i)_{i=1}^k \\ \sum_{i=1}^k n_i = n}} 1. \tag{3.28}$$

{walk+2bric

The last inequality follows from the fact that the composition of brigdes is a bridge and its weight is a product of weights of the initial bridges.

It is known that the number of partitions of the integer n is of the order $e^{\tilde{C}\sqrt{n}}$ for some constant $\tilde{C}$ [Ram18]. This, together with (3.28), implies

$$Z(\mathrm{W}_n^+) \leq e^{\tilde{C}\sqrt{n}} H_n. \tag{3.29}$$

To extend the proof from W_n^+ to W_n, we cut the walk $\gamma \in \mathrm{W}_n$ at the first point with minimal x-coordinate, i.e write $\gamma = \gamma_1 \circ \gamma_2$, where

$$|\gamma_1| = \alpha_0 = \min\Big[i : x(\gamma(i)) = \min_{0\leq i\leq n}\big(x(\gamma(i))\big)\Big]. \tag{3.30}$$

Then, the walk $\overline{\gamma_1} = (\gamma_1(\alpha_0 - i))_{i=0}^{\alpha_0}$ is a translation of a walk in $\mathrm{W}_{\alpha_0}^+$.

The rest of the walk is allowed to visit the initial hyperplane $\{x = 0\}$ more than once so to make the decomposition into bridges possible we should add one step at the beginning of the walk. By the symmetry of the set of all possible steps, there exists at least one $\tilde{\gamma} \in \mathrm{W}_1^+$ and $\tilde{\gamma} \circ \gamma_2 \in \mathrm{W}_{n-\alpha_0+1}^+$.

Both walks $\overline{\gamma_1}$ and $\tilde{\gamma} \circ \gamma_2$ admit the decomposition into bridges as above so the final bound on Z_n is

lks2walks+}

$$Z_n \leq \sum_{\alpha=0}^n Z(\mathrm{W}_\alpha^+) Z(\mathrm{W}_{(n-\alpha+1)}^+) \leq e^{2\tilde{C}\sqrt{n}} \sum_{\alpha=0}^n H_\alpha H_{n-\alpha+1} \leq e^{C\sqrt{n}} H_{n+1}. \tag{3.31}$$

Since $H_{n+1} \geq cH_n$, the result follows.

□

Corollary 3.7. *There exists an analogue of the connective constant defined for bridges* $\lambda_{bridge} = \lim_{n\longrightarrow\infty} \frac{1}{n}\log H_n$ *and it is equal to* λ_0.

Corollary 3.8. *The series* $H(\lambda) = \sum_{n=0}^\infty H_n e^{-\lambda n}$ *converge for any* $\lambda > \lambda_0$. *Moreover,*

sumbridges}

$$\sum_{n=0}^\infty H_n e^{-\lambda_0 n} = \infty. \tag{3.32}$$

Let us note that these results holds for the model of self-avoiding walks as well.

Proof. The first part of the statement follows directly from the previous corollary. To prove the second part, one should use the intermediate steps of the proof of Proposition

3.6. We can use the first inequality of (3.31) to bound the generating function $W(\lambda)$ as follows:

$$\sum_{n=0}^{\infty} Z_n e^{-\lambda n} \le \sum_{n=0}^{\infty}\sum_{\alpha=0}^{n} e^{\lambda n}\left(Z(W_\alpha^+)e^{-\lambda\alpha}\right)\left(Z(W_{n-\alpha+1}^+)e^{-\lambda(n-\alpha+1)}\right)$$

{cntocn+} $$= e^{\lambda}\left(\sum_{n=0}^{\infty} Z(W_n^+)e^{-\lambda n}\right)^2. \qquad (3.33)$$

Due to the first inequality of (3.28), this sum can be rewritten as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} Z(W_n^+)e^{-\lambda n} &\le \sum_{\substack{(n_i)_{i=1}^k \\ h_1<h_2<....<h_k}} \prod_{i=1}^{k} H_{n_i,h_i} e^{-\lambda n_i} \\ &= \prod_{h_1<h_2<...}\left(1+\sum_{n=0}^{\infty} H_{n,h_i}e^{-\lambda n}\right) \\ &= \prod_{h_1<h_2<...}\exp\left(\left(\sum_{n=0}^{\infty} e^{-\lambda n}H_{n,h_i}\right)\right) \\ &= \exp\left(\left(\sum_{n=0}^{\infty}\left(\sum_{h=0}^{\infty} H_{n,h}\right)\right)e^{-\lambda n}\right) \\ &= \exp\left(\left(\sum_{n=0}^{\infty} H_n e^{-\lambda n}\right)\right). \qquad (3.34)\end{aligned}$$ {cn+tobn}

The divergence of the sum $\sum_{n=0}^{\infty} Z_n e^{-\lambda n}$ as $\lambda \longrightarrow \lambda_0$ (by (3.24)) implies the divergence of the sum $\sum_{n=0}^{\infty} Z(W_n^+)e^{-\lambda n}$ which itself together with (3.33) and (3.34) implies that $\sum_{n=0}^{\infty} H_n e^{-\lambda_0 n}$ diverges.

□

3.3.3 Kesten's lemma

Let us now state and proof the analogue of Theorem 3.4 for the case of self-repelling polymers.

Theorem 3.9. *For λ_0 defined as in (3.22) the following holds:*

$$\sum_{\gamma\in \mathrm{iB}} e^{-\lambda_0|\gamma|}\sigma(\gamma) = 1. \qquad (3.35)$$ {keslemeq}

Proof. Each bridge can be seen as a concatenation of $K = |R_\gamma| + 1$ irreducible bridges $(\gamma_k)_{k=1}^K$. These bridges have no common point except the points $\gamma(R_\gamma(i))$ where one bridge begins and another one ends. Therefore, according to the definition of σ, we obtain that

$$\sigma(\gamma) = \prod_{k=1}^{K}\sigma(\gamma_k). \qquad (3.36)$$

Thus, we can rewrite the generating function of bridges in the following way:

$$H(\lambda) = \sum_{K=0}^{\infty} \prod_{k=1}^{K} \sum_{\gamma_k \in \mathrm{iB}} \sigma(\gamma_k) e^{-\lambda|\gamma_k|} = \frac{1}{1 - \sum_{\gamma \in \mathrm{iB}} \sigma(\gamma) e^{-\lambda|\gamma|}}. \tag{3.37}$$ {keslem1}

From Corollary 3.8 we obtain that $H(\lambda)$ exists for $\lambda > \lambda_0$ and converges to infinity when $\lambda \searrow \lambda_0$. Comparing this result with (3.37) gives that

$$\lim_{\lambda \searrow \lambda_0} \sum_{\gamma_k \in \mathrm{iB}} \sigma(\gamma_k) e^{-\lambda|\gamma_k|} = 1.$$

This equality implies the result of the theorem. □

We define a probability measure on the set of irreducible bridges based on Theorem 3.9. For all $\gamma \in \mathrm{iB}$,

$$\mathbb{P}_{\mathrm{iB}}(\gamma) = \sigma(\gamma) e^{-\lambda_0|\gamma|}. \tag{3.38}$$

Also, we define the probability measure $\mathbb{P}_{\mathrm{iB}}^{\otimes \mathbb{N}}$ on the set B_∞^+ of semi-infinite random walks $\gamma : \mathbb{Z}_+ \longrightarrow (\mathbb{Z}_+) \times \mathbb{Z}^{d-1}$ that begin at zero and lie in the upper-half space after the first step by considering a concatenation of irreducible bridges chosen independently according to the probability distribution $\mathbb{P}_{\mathrm{iB}}$, i.e. for any $\gamma \in \mathrm{B}_\infty^+$ and $\tilde{\gamma} \in \mathrm{iB}$,

$$\mathbb{P}_{\mathrm{iB}}^{\otimes \mathbb{N}}(\exists \gamma_1 \in \mathrm{B}_\infty^+ \, : \, \gamma = \tilde{\gamma} \circ \gamma_1) = \mathbb{P}_{\mathrm{iB}}(\tilde{\gamma}). \tag{3.39}$$

If such γ_1 exists, then the value $r_1(\gamma) = |\tilde{\gamma}|$ is called the first renewal time of γ, and all other renewal times are defined by $r_{k+1}(\gamma) = r_k(\gamma_1)$. Also we set $r_0 = 0$. The sequence of all renewal times $R_\gamma = (r_k(\gamma))$ is infinite almost surely.

Finite random bridges are related to this model in the following way:

Lemma 3.10. *The distribution $\mathbb{P}_{\mathrm{SRP}_n}$ is equal to $\mathbb{P}_{\mathrm{iB}}^{\otimes \mathbb{N}}\big(\,\cdot\,\big|\gamma_n \in R_\gamma\big)$.*

Proof. To prove this lemma, we use almost the same decomposition as in Lemma 3.9. We have

$$\begin{aligned} H_n = \sum_{\gamma \in \mathrm{B}_n} \sigma(\gamma) &= e^{\lambda_0 n} \sum_{K=0}^{\infty} \sum_{\substack{(\gamma^i \in \mathrm{iB})_{i=1}^K \\ \sum_{i=1}^K |\gamma^i| = n}} \prod_{i=1}^{K} \sigma(\gamma^i) e^{-\lambda_0|\gamma^i|} \\ &= \mathbb{P}_{\mathrm{iRB}}^{\otimes \mathbb{N}}\big(\exists K \in \mathbb{N} : \gamma_n \text{ is a } K\text{-th renewal point of } \gamma\big) \\ &= \mathbb{P}_{\mathrm{iRB}}^{\otimes \mathbb{N}}\big(\gamma_n \in R_\gamma\big). \end{aligned}$$

For any event A we can write the following equality:

$$\mathbb{P}_{\mathrm{B}_n}(A) = \frac{1}{H_n} \sum_{\gamma \in \mathrm{B}_n} \sigma(\gamma) \mathbb{I}_{\gamma \in A} = \frac{\mathbb{P}_{\mathrm{iB}}^{\otimes \mathbb{N}}\big((\gamma_i)_{i=0}^n \in A, \, \gamma_n \in R_\gamma\big)}{\mathbb{P}_{\mathrm{iB}}^{\otimes \mathbb{N}}\big(\gamma_n \in R_\gamma\big)} = \mathbb{P}_{\mathrm{iB}}^{\otimes \mathbb{N}}\big(A \,\big|\gamma_n \in R_\gamma\big).$$

□

The probability measure $\mathbb{P}_{\text{iB}}^{\otimes\mathbb{N}}$ can be extended to bi-infinite bridges $\gamma : \mathbb{Z} \longrightarrow \mathbb{Z}^d$ with the restriction that $\gamma_0 = 0$ and is a renewal point, i.e.

$$x(\gamma(-n)) \leq 0 \text{ and } x(\gamma(n)) > 0 \quad \forall n \in \mathbb{N}.$$

For this type of walks we define the two-sided sequence of renewal points $R_\gamma = (r_k)_{k=-\infty}^{\infty}$ similarly as before.

For any renewal point $r_k \in R_\gamma$, define the operation of shift $\tau(\gamma)$ that sets $\gamma(r_1)$ to zero: $\tau(\gamma) = (\gamma(i - r_1))_{i=-\infty}^{\infty}$. By construction, the probability measure $\mathbb{P}_{\text{iB}}^{\otimes\mathbb{Z}}$ is invariant under τ.

Lemma 3.11. $\mathbb{P}_{\text{iB}}^{\otimes\mathbb{Z}}$ *is ergodic under* τ*.*

Proof. Let A be a shift-invariant measurable event. Then, for any choice of $\varepsilon > 0$ we can pick a positive integer M and an event A_M depending only on $(\gamma(i))_{i=-M}^{M}$ such that $\mathbb{P}_{\text{iRB}}^{\otimes\mathbb{Z}}(A \triangle A_M) \leq \varepsilon$. We can express $\mathbb{P}_{\text{iRB}}^{\otimes\mathbb{Z}}(A)$ as $\mathbb{P}_{\text{iRB}}^{\otimes\mathbb{Z}}(A \cap A) = \mathbb{P}_{\text{iRB}}^{\otimes\mathbb{Z}}(A \cap \tau^{4M}(A))$. This probability is bounded in the following way:

$$\begin{aligned} \left|\mathbb{P}_{\text{iRB}}^{\otimes\mathbb{Z}}(A\cap\tau^{4M}(A)) - \mathbb{P}_{\text{iRB}}^{\otimes\mathbb{Z}}(A_M\cap\tau^{4M}(A_M))\right| &\leq \mathbb{P}_{\text{iRB}}^{\otimes\mathbb{Z}}\big((A\cap\tau^{4M}(A))\triangle(A_M\cap\tau^{4M}(A_M))\big) \\ &\leq \mathbb{P}_{\text{iRB}}^{\otimes\mathbb{Z}}(A\triangle A_M) + \mathbb{P}_{\text{iRB}}^{\otimes\mathbb{Z}}(\tau^{4M}(A)\triangle\tau^{4M}(A_M)) \\ &\leq 2\varepsilon. \end{aligned} \tag{3.40}$$ {tauerg1}

All irreducible pieces of γ are sampled independently and $r_{2M} > M$ so the events A_M and $\tau^{4M}(A_M)$ depend on the independent collection of irreducible bridges $(\gamma_m)_{m=-M}^{M}$ and $(\gamma_m)_{m=3M}^{5M}$ so we can write the following estimation:

$$\begin{aligned} \left|\mathbb{P}_{\text{iRB}}^{\otimes\mathbb{Z}}(A_M\cap\tau^{4M}(A_M)) - \big(\mathbb{P}_{\text{iRB}}^{\otimes\mathbb{Z}}(A)\big)^2\right| &\leq \left|\big(\mathbb{P}_{\text{iRB}}^{\otimes\mathbb{Z}}(A)\big)^2 + 2\varepsilon\big(\mathbb{P}_{\text{iRB}}^{\otimes\mathbb{Z}}(A)\big) + \varepsilon^2 - \big(\mathbb{P}_{\text{iRB}}^{\otimes\mathbb{Z}}(A)\big)^2\right| \\ &\leq 3\varepsilon. \end{aligned} \tag{3.41}$$ {tauerg2}

The bounds (3.40) and (3.41) give that

$$|\mathbb{P}_{\text{iRB}}^{\otimes\mathbb{Z}}(A) - \big(\mathbb{P}_{\text{iRB}}^{\otimes\mathbb{Z}}(A)\big)^2| \leq 5\varepsilon \tag{3.42}$$

which is true for any choice of ε. This implies that $\mathbb{P}_{\text{iRB}}^{\otimes\mathbb{Z}}(A) \in \{0, 1\}$. □

3.3.4 Sub-ballisticity of the model

The analogue of Theorem 3.3 holds for the model of self-avoiding polymers as well.

Theorem 3.12. *Consider the self-repelling polymers on* $\mathbb{Z}^d$ *with a jump-distribution* ρ *invariant under the symmetries of the lattice, then for any positive* n

$$\lim_{n\longrightarrow\infty} \frac{1}{n} \log \mathbb{P}_{\text{SRP}_n}(|\gamma(n)| > vn) < 0.$$

This theorem is proven by contradiction in two steps. In Section 3.4, we show that if Theorem 3.12 does not hold (i.e the Ballistic Assumption 3.5 holds), then the mean length of an irreducible bridge is finite. In Section 3.5, we show that the mean length of irreducible bridge is infinite, which altogether with Section 3.4 proves Theorem 3.12.

3.4 Consequences of Ballistic Assumption

Suppose that Assumption 3.5 holds an look at the results it can imply.

Let us work only with bridges long enough that they play a role in the ballisticity of the model:

$$\mathrm{RB}_{n,v} = \{\gamma \in \mathrm{RB}_n : x(\gamma_n) > vn\},$$

and define on this set the measure $\mathbb{P}_{\mathrm{SRB}_{u_n,v}}$.

The Ballistic Assumption puts the following restriction on the number of renewal points.

Theorem 3.13. *If* (3.5) *holds, then for any increasing sequence of positive integers* (u_n)*, there exists* $\delta > 0$ *such that*

{T1eq}
$$\limsup_{n\longrightarrow\infty} \frac{1}{u_n} \log \mathbb{P}_{\mathrm{SRB}_{u_n,v}}(|R_\gamma| > \delta u_n) = 0. \tag{3.43}$$

To prove this theorem, we generalise the idea of renewal points and look at the hyperplanes $\pi_{x_0+1/2} = \{(x_1, x_2, \dots x_d), x_1 = x_0 + 1/2\}$ that have not many crossings with segments, corresponding to the steps of γ:

$$Rl_\gamma^m = \Big\{x_0 \in \mathbb{N} : 1 \le \Big|\{i : (\gamma(i,)\gamma(i+1)) \cap \pi_{x_0+1/2} \neq \emptyset\}\Big| \le m\Big\}.$$

It is easy to see that $Rl_\gamma^m \subset Rl_\gamma^{m+1}$ for any positive m.

For nearest-neighbor walks there is a bijection between R_γ and Rl_γ^1. It is not true in the more general case, but nonetheless there exists $D > 0$ such that

{R2}
$$|R_\gamma| \le |Rl_\gamma^1| \le D|R_\gamma| \tag{3.44}$$

where D is defined as in (3.27) and depends only on Ω.

Define the following subsets of $\mathrm{RB}_{n,v}$:

$$\mathrm{RB}_{n,v,\delta}^m = \{\gamma \in \mathrm{RB}_{n,v} : |Rl_\gamma^m| > \delta n\}.$$

Theorem 3.13 is a consequence of the following lemma.

Lemma 3.14. *If* (3.5) *holds, then for any* v*,* $\delta > 0$ *and* $m \ge 2$ *and for any sequence* (u_n) *in* Z_+*, there exists* $\delta' > 0$ *and a subsequence* (t_n) *of* (u_n) *such that*

{L1eq}
$$\limsup_{t_n\longrightarrow\infty} \frac{1}{t_n} \log \left(\frac{Z(\mathrm{RB}_{t_n,v,\delta}^m)}{Z(\mathrm{RB}_{t_n,v,\delta'}^{m-1})}\right) \le 0. \tag{3.45}$$

The proof of this lemma is based on the unfolding operation defined below.

Fix $\gamma \in \mathrm{RB}_n$. The pair of integers (i, j), $0 < i \le j < n$ is called a *zigzag* of γ if

$$\begin{aligned} x(\gamma_k) &\le x(\gamma_i) && \forall\, k < i, \\ x(\gamma_j) \le x(\gamma_k) &< x(\gamma_i) && \forall\, i < k < j, \\ x(\gamma_j) &< x(\gamma_k) && \forall\, k > j. \end{aligned}$$

The set of all zigzags of the walk will be denoted by ZigZag_γ. Note that all zigzags in the set ZigZag_γ are disjoint.

We can define the *unfolding* of a zigzag as follows. Suppose that $\gamma \in \mathrm{RB}_n$ and $(i,j) \in \mathrm{ZigZag}_\gamma$. Then, define the new bridge:

$$\mathrm{Unf}_{(i,j)}(\gamma) := (\gamma_k)_{k=0}^{i} \circ \mathcal{R}_x\big((\gamma_k)_{k=i}^{j}\big) \circ (\gamma_k)_{k=j}^{n}. \tag{3.46}$$

Let us recall some elementary properties of this operation.

Lemma 3.15. *The following properties hold for any bridge $\gamma \in \mathrm{RB}_n$ and for any $(i,j) \in \mathrm{ZigZag}_\gamma$:*

- $\sigma(\gamma) \leq \sigma\big(\mathrm{Unf}_{(i,j)}(\gamma)\big)$,
- $x(\gamma_n) \leq x\big((\mathrm{Unf}_{(i,j)}(\gamma))_n\big)$,
- $\gamma(i) \in R_{\mathrm{Unf}_{(i,j)}(\gamma)}$, $\big(\mathrm{Unf}_{(i,j)}(\gamma)\big)(j) \in R_{\mathrm{Unf}_{(i,j)}(\gamma)}$,
- $\mathrm{ZigZag}_\gamma \backslash \{i,j\} \subset \mathrm{ZigZag}_{\mathrm{Unf}_{(i,j)}(\gamma)}$,
- $\forall (i,j), (\tilde{\imath}, \tilde{\jmath}) \in \mathrm{ZigZag}_\gamma :\ \mathrm{Unf}_{(i,j)}\mathrm{Unf}_{(\tilde{\imath},\tilde{\jmath})}(\gamma) = \mathrm{Unf}_{(\tilde{\imath},\tilde{\jmath})}\mathrm{Unf}_{(i,j)}(\gamma)$.

The last property allows us to define the unfolding of a set of zigzags as a row of successive unfoldings (the order in which we do the unfolding operations is irrelevant).

Proof of Lemma 3.14. Let us fix $m > 2$, $v > 0$, $\delta > 0$ and any sequence of positive integers (u_n). Look at the sequence $(\mathrm{RB}^m_{u_n,v,\delta})_{n=0}^{\infty}$. At least one of the following propositions must be true:

- The number of sets where a positive density of renewal points has quite high probability to occur is infinite:

  Case A. *There exists $\delta' > 0$ and a subsequence (t_n) of (u_n) such that*

  $$Z\big(\{\gamma \in \mathrm{RB}^m_{t_n,v,\delta} : |R_\gamma| \geq \delta' t_n\}\big) \geq \frac{1}{3} Z\big(\mathrm{RB}^m_{t_n,v,\delta}\big). \tag{3.47}$$ {case1eq}

- In any set $\mathrm{RB}_{u_n,v}$ there is a good probability that the number of zigzags in a walk is sufficiently small:

  Case B. *There exists $(\varepsilon_n) \searrow 0$ such that*

  $$Z\big(\{\gamma \in \mathrm{RB}^m_{u_n,v,\delta} : |\mathrm{ZigZag}_\gamma| \leq \varepsilon_n u_n\}\big) \geq \frac{1}{3}\, Z(\mathrm{RB}^m_{u_n,v,\delta}). \tag{3.48}$$

- The number of u_n such that a bridge with positive density of zigzags and sufficiently small number of renewal points has a high probability to occur in $\mathrm{RB}_{u_n,v}$ is infinite.

Case C. *There exist* $\varepsilon > 0$*, a sequence* $(\delta'_n) \searrow 0$ *and a subsequence* (t_n) *of* (u_n) *such that*

$$Z\big(\{\gamma \in \mathrm{RB}^m_{t_n,v,\delta} : |R_\gamma| \le \delta'_n t_n \text{ and } |\mathrm{ZigZag}_\gamma| \ge 2\varepsilon t_n\}\big) \ge \frac{1}{3} Z(\mathrm{RB}^m_{t_n,v,\delta}). \qquad (3.49)$$ {case3eq}

Proof in Case A. Inequality (3.45) follows directly from (3.47) and the fact that $|Rl^{m-1}_\gamma| \ge |Rl^1_\gamma| \ge |R_\gamma|$ for any m by (3.44). □

Proof in Case B. Let us take $\gamma \in \mathrm{RB}^m_{u_n,v,\delta}$ satisfying the property $|\mathrm{ZigZag}_\gamma| \le \varepsilon_n u_n$.

For each hyperplane $\pi \in Rl^m_\gamma$ that has exactly m crossings with γ, there exists at least one zigzag $(j,k) \in \mathrm{ZigZag}_\gamma$ such that $(\gamma(i))_{i=j}^k$ intersects π. Hence, two other parts $(\gamma(i))_{i=0}^j$ and $(\gamma(i))_{i=k}^{u_n}$ also have at least one crossing with π. If we unfold this zigzag, then all points of γ after step j will have a larger x-coordinate than in π. Then, π has no more than $m-2$ crossings with $\mathrm{Unf}_{(i,j)}(\gamma)$. For any other hyperplane in Rl^m_γ, there is a corresponding hyperplane in $Rl^m_{\mathrm{Unf}_{(j,k)}(\gamma)}$ with at most the same number of crossings.

Let us repeat this operation and unfold all zigzags in ZigZag_γ. The resulting walk $\tilde\gamma = \mathrm{Unf}_{\mathrm{ZigZag}_\gamma}(\gamma)$ will satisfy the following property:

$$|Rl^{m-1}_{\tilde\gamma}| \ge |Rl^m_\gamma|. \qquad (3.50)$$

Now, let us choose a walk $\tilde\gamma \in \mathrm{RB}^{m-1}_{u_n,v,\delta}$ and bound the number of walks γ that gives $\tilde\gamma$ as a result of the unfolding operation. The number of $\gamma \in \mathrm{RB}^m_{u_n,v,\delta}$ such that $|\mathrm{ZigZag}_\gamma| \le \varepsilon_n u_n$ and $\mathrm{Unf}_{\mathrm{ZigZag}_\gamma}(\gamma) = \tilde\gamma$ is equal to the number of possible ways to pick at most $2\varepsilon_n u_n$ points of $\tilde\gamma$ to form all zigzags in γ. Thus,

{unf-1card}
$$|\{\gamma \in \mathrm{RB}^m_{u_n,v,\delta} : \mathrm{Unf}_{\mathrm{ZigZag}_\gamma}(\gamma) = \tilde\gamma\}| \le \sum_{k=0}^{2e_n u_n} \binom{u_n}{k} \le \exp\left(2u_n\varepsilon_n \log\left(\frac{1}{2\varepsilon_n}\right)\right). \qquad (3.51)$$

Here, we used the bound $\sum_{k=0}^{\varepsilon n} \binom{n}{k} \le 2^{nH(\varepsilon)}$, where $\varepsilon < 1/2$ and $H(\varepsilon) = -\varepsilon \log_2(\varepsilon) - (1-\varepsilon)\log_2(1-\varepsilon)$.

For all γ in this set, $\sigma(\gamma) \le \sigma(\tilde\gamma)$ so

$$\sigma(\tilde\gamma) \ge \sum_{\gamma \in \mathrm{RB}^m_{u_n,v,\delta},\ \mathrm{Unf}_{\mathrm{ZigZag}_\gamma}(\gamma) = \tilde\gamma} \frac{\sigma(\gamma)}{\exp\left(2u_n\varepsilon_n \log\left(\frac{1}{2\varepsilon_n}\right)\right)}.$$

The set of all $\tilde\gamma$ that have a preimage in $\mathrm{RB}^m_{u_n,v,\delta}$ is not bigger than $\mathrm{RB}^{m-1}_{u_n,v,\delta}$, so

$$Z\left(\mathrm{RB}^{m-1}_{u_n,v,\delta}\right) \ge \frac{\frac{1}{3} Z\left(\mathrm{RB}^m_{u_n,v,\delta}\right)}{\exp\left(-2u_n\varepsilon_n \log\left(\frac{1}{2\varepsilon_n}\right)\right)}. \qquad (3.52)$$

This inequality and the fact that $(\varepsilon_n) \searrow 0$ implies the statement of Lemma 3.14. □

Proof in Case C. The idea of this proof is to unfold the necessary number of small zigzags and obtain some renewal points by this unfolding.

Let us take a random bridge $\gamma \in \mathrm{RB}^m_{t_n,v,\delta}$ such that $|R_\gamma| \le \delta'_n t_n$ and $|\mathrm{ZigZag}_\gamma| \ge 2\varepsilon t_n$. We can define a set containing all small zigzags of γ:

$$\mathrm{ShortZigZag}_\gamma = \left\{(i,j) \in \mathrm{ZigZag}_\gamma : j - i \le \frac{1}{\varepsilon}\right\}. \tag{3.53}$$

The central sections of all zigzags in ZigZag_γ are disjoint and the sum of the number of steps in all central sections is not bigger than t_n. Inequality $|\mathrm{ZigZag}_\gamma| \ge 2\varepsilon t_n$ implies that

$$\left\{\gamma \in \mathrm{RB}^m_{t_n,v,\delta} : \left|\mathrm{ZigZag}_\gamma\right| \ge 2\varepsilon t_n\right\} \subset \left\{\gamma \in \mathrm{RB}^m_{t_n,v,\delta} : \left|\mathrm{ShortZigZag}_\gamma\right| \ge \varepsilon t_n\right\}. \tag{3.54}$$

Let us define

$$RB_c = \{\gamma \in \mathrm{RB}^m_{t_n,v,\delta} : |R_\gamma| \le \delta'_n t_n \text{ and } |\mathrm{ShortZigZag}_\gamma| \ge \varepsilon t_n\}. \tag{3.55}$$

It is easy to see that this set contains the set defined on the right-hand side of (3.49).

Now, let us take a subset $\mathrm{ZZ} \subset \mathrm{ShortZigZag}_\gamma$ of size $\varepsilon' t_n < \varepsilon t_n$. The precise value of ε' will be defined later. Then, unfold all zigzags in ZZ. The resulting walk $\tilde\gamma = \mathrm{Unf}_{\mathrm{ZZ}}(\gamma)$ has at least $2\varepsilon' t_n$ renewal points, i.e.

$$\tilde\gamma \in \mathrm{RB}^1_{t_n,v,2\varepsilon'}. \tag{3.56}$$

Different walks can be obtained by the different choice of ZZ. For fixed γ the number of ways to pick ZZ can be estimated as follows:

$$\left|\left\{\tilde\gamma : \exists \mathrm{ZZ} \subset \mathrm{ShortZigZag}_\gamma, |\mathrm{ZZ}| = \varepsilon' t_n, \mathrm{Unf}_{\mathrm{ZZ}}(\gamma) = \tilde\gamma\right\}\right| \ge \binom{\varepsilon t_n}{\varepsilon' t_n}. \tag{3.57}$$ {min_gt_fro

Define the set of all possible pairs (γ, ZZ):

$$\mathrm{BZ} = \{(\gamma, \mathrm{ZZ}) : \gamma \in RB_c, \mathrm{ZZ} \subset \mathrm{ShortZigZag}, |\mathrm{ZZ}| = \varepsilon' t_n\}. \tag{3.58}$$

The number of renewal points of $\tilde\gamma$ can be bounded from below.

Let us unfold one zigzag $(i,j) \in \mathrm{ZZ}$ in γ and look at the number of crossings of γ and $\tilde\gamma = \mathrm{Unf}_{(i,j)}(\gamma)$ with different hyperplanes $\pi_{x_0} = \{x = x_0 + 1/2\}$. For any $x_0 < x(\gamma(j))$, the number of crossings is preserved, so for $\tilde\gamma$ does not contain any renewal points except the points that were already presented in R_γ. For any $x_0 \ge x(\gamma(i))$, there is a correspondence between the crossings of γ and π_{x_0} and between the crossings of γ and π_{x_0} and the crossings of $\tilde\gamma$ and $\pi_{x_0+2(x(\gamma(i))-x(\gamma(j)))}$. This part of $\tilde\gamma$ will also have no new renewal points. The remaining middle part of $\tilde\gamma$ has width $3\,|x(\gamma(i)) - x(\gamma(j))|$ that can be bound by $3(j-i)D$, where D is defined in (3.27). In this gap there can be maximum $3(j-i)D$ renewal points. Note that $(i,j) \in \mathrm{ShortZigZag}_\gamma$ and that $j - i \le \frac{1}{\varepsilon}$.

This operation can be applied consequentially for all zigzags in ZZ and gives the following result:

$$|R_{\mathrm{Unf}_{\mathrm{ZZ}}(\gamma)}| \le \delta'_n t_n + \varepsilon' t_n \frac{3D}{\varepsilon}. \tag{3.59}$$ {NRenewal}

For each $\tilde\gamma$, there can be many pairs $(\gamma, \mathrm{ZZ}) \in \mathrm{BZ}$ that gives $\tilde\gamma$ after unfolding. Their number can be bounded in the following way.

The number of possible ways to make $\varepsilon' t_n$ zigzags to obtain γ from $\tilde{\gamma}$ is not bigger than the number of ways to choose $2\varepsilon' t_n$ points from all renewal points of $\tilde{\gamma}$. Then, we can use inequality (3.59) to obtain the following bound:

$$|\{(\gamma, \mathrm{ZZ}) \in \mathrm{BZ} : \ \mathrm{Unf}_{\mathrm{ZZ}}(\gamma) = \tilde{\gamma}\}| \leq \binom{\delta'_n t_n + \varepsilon' t_n \frac{3D}{\varepsilon}}{2\varepsilon' t_n}. \qquad (3.60)$$

{max_g_from

We can use inequalities (3.49), (3.57) and (3.60) and the bound $\left(\frac{a}{b}\right)^b \leq \binom{a}{b} \leq \left(\frac{e \cdot a}{b}\right)^b$ to obtain

$$\begin{aligned} Z(\mathrm{RB}^1_{t_n,v,2\varepsilon'}) &\geq Z(RB_c) \cdot \frac{\min_{\gamma \in RB_c} |\{\tilde{\gamma} : \exists (\gamma, \mathrm{ZZ}) \in \mathrm{BZ}, \mathrm{Unf}_{\mathrm{ZZ}}(\gamma) = \tilde{\gamma}\}|}{\max_{\tilde{\gamma} \in \mathrm{RB}^1_{t_n,v,2\varepsilon'}} |\{(\gamma, \mathrm{ZZ}) \in \mathrm{BZ} : \mathrm{Unf}_{\mathrm{ZZ}}(\gamma) = \tilde{\gamma}\}|} \\ &\geq \frac{1}{3} Z(\mathrm{RB}^m_{t_n,v,\delta}) \cdot \frac{\binom{\varepsilon t_n}{\varepsilon' t_n}}{\binom{\delta'_n t_n + \varepsilon' t_n \frac{3D}{\varepsilon}}{2\varepsilon' t_n}} \\ &\geq \frac{1}{3} Z(\mathrm{RB}^m_{t_n,v,\delta}) \cdot \left(\frac{\varepsilon}{\varepsilon'} \cdot \frac{4}{e^2} \cdot \left(\frac{\varepsilon'}{\delta'_n + 3D\frac{\varepsilon'}{\varepsilon}} \right)^2 \right)^{\varepsilon' t_n}. \qquad (3.61) \end{aligned}$$

The result of Lemma 3.14 holds when a constant lower bound on $\frac{\varepsilon \cdot \varepsilon'}{\left(\delta'_n + 3D\frac{\varepsilon'}{\varepsilon}\right)^2}$ is bigger than 2. We can choose $\varepsilon' = \frac{\varepsilon^3}{2(6D)^2}$ and use the fact that $\delta'_n \searrow 0$ to obtain this bound. Then,

$$Z(\mathrm{RB}^1_{t_n,v,2\varepsilon'}) \geq \frac{1}{3} Z(\mathrm{RB}^m_{t_n,v,\delta}) \left(\frac{8}{e^2}\right)^{\varepsilon' t_n},$$

which implies (3.45).

□

This finishes the proof of Lemma 3.14 in the general case. □

Proof of Theorem 3.13. For any $\gamma \in \mathrm{RB}_{n,v}$ the number of hyperplanes π_k crossed at least once is bigger than vn. Thus, at least half of them are crossed less than $\frac{2D}{v}$ times where D is defined in (3.27). Hence, we deduce that

$$\mathrm{RB}_{u_n,v} = \mathrm{RB}^{\frac{2D}{v}}_{u_n,v,\frac{v}{2}}. \qquad (3.62)$$

We can apply Lemma 3.14 $\frac{2D}{v}$ times with m chosen decreasingly from $\frac{2D}{v}$ to 2. We have to take $\delta = \frac{v}{2}$ at the first step and set it equal to δ' from the previous step afterwards. Then, we can find $\delta' > 0$ and (t_n a subsequence of (u_n)) such that

$$\lim_{t_n \longrightarrow \infty} \frac{1}{t_n} \log \left(\frac{Z(\mathrm{RB}^1_{t_n,v,\delta'})}{Z(\mathrm{RB}^{\frac{2\alpha}{v}}_{t_n,v,\frac{v}{2}})} \right) \geq 0.$$

□

Corollary 3.16. *If the Ballistic Assumption holds, then*

{Einf}
$$\mathbb{E}_{\text{iRB}}(|\gamma|) < \infty. \tag{3.63}$$

Proof. Suppose that $\mathbb{E}_{\text{iRB}}(|\gamma|) = \infty$. This implies that for any choice of $C > 0$ and $\alpha > 0$, there exists a bound $x_0(C, \alpha)$ such that for any $x > x_0$,

$$\mathbb{P}_{\text{iRB}}(|\gamma| \geq x) > \frac{C}{x^{1+\alpha}}.$$

Let us fix $C_0 = 8/9$ and $\alpha_0 = 1/2$ and define $M = \max(\frac{2}{v}, x_0(C_0, \alpha_0))$.

For any positive constant A, we can construct a three-point distribution X as follows:

$$\begin{aligned}
\mathbb{P}_X(X = 0) &= P_{\text{iRB}}(|\gamma| < M), \\
\mathbb{P}_X(X = M) &= P_{\text{iRB}}(M \leq |\gamma| < A), \\
\mathbb{P}_X(X = A) &= P_{\text{iRB}}(|\gamma| \geq A).
\end{aligned}$$

The expectation of this distribution is not smaller than M if for a pair (C, α) satisfying $x_0(C, \alpha) \leq M$:

$$(2M)^\alpha \leq A^\alpha \leq \frac{C}{2} \frac{M^\alpha}{M^{1+\alpha} - C}. \tag{3.64}$$ {MAineq}

These two inequalities can hold simultaneously if

$$C \geq \frac{(2M)^{1+\alpha}}{1 + 2^{1+\alpha}}.$$

Let us choose $\alpha = 2$ and correspondingly $C = \frac{8}{9}M^3$. Then, $\frac{C}{x_0^{1+\alpha}} = \frac{C_0}{x_0^{1+\alpha_0}}$ implies that $x_0(C, \alpha) \leq M$ and allows to choose A accordingly to (3.64) and obtain

$$\mathbb{E}_X(X) \geq \frac{2}{v}. \tag{3.65}$$ {expX}

The size of the set of renewal points can be estimated as follows:

$$\mathbb{P}_{\text{RB}_n}(|R_\gamma| > vn) = \mathbb{P}_{\text{iRB}}(\sum_{i=1}^{vn} |\gamma_i| < n) \leq \mathbb{P}_X\left(\frac{1}{vn}\sum_{i=1}^{vn} X_i < \frac{1}{v}\right), \tag{3.66}$$

where random variables X_i are independently distributed according to X.

Because of (3.65), this probability can be estimated by Cramer's Theorem [Cra38] in large deviation theory. There exists a positive constant $c > 0$ such that for any n large enough

$$\mathbb{P}_{\text{RB}_n}(|R_\gamma| > vn) \leq \mathbb{P}_X\left(\frac{1}{vn}\sum_{i=1}^{vn} X_i < \frac{1}{v}\right) < e^{-cn}. \tag{3.67}$$

This inequality contradicts the consequence of the Ballistic Assumption proved in Theorem 3.13.

□

3.5 Expectation of a length of an irreducible bridge is infinite

The object of this section is to prove the following theorem, which, combined with the result of the previous section, contradicts the Ballistic Assumption.

Theorem 3.17. *The following holds:*

{Enoninf}
$$\mathbb{E}_{\text{iRB}}(|\gamma|) = \infty. \tag{3.68}$$

Theorem 3.17 will be proven by contradiction. Let us suppose that there exists a constant ν such that

{T2not}
$$\mathbb{E}_{\text{iRB}}(|\gamma|) < \nu < \infty. \tag{3.69}$$

The main tool in this section is the operation of stickbreaking

A renewal time of a bridge $\gamma(i) \in R_\gamma$ is called a *diamond time* of γ if all other points of γ lie in the cone

$$\begin{aligned} &\{(x,y) : x > x(\gamma(i)), x - x(\gamma(i)) \geq y - y(\gamma(i)) > -(x - x(\gamma(i)))\} \cup \\ &\quad \{(x,y) : x < x(\gamma(i)), x(\gamma(i)) - x > y - y(\gamma(i))ge - (x(\gamma(i)) - x)\}. \end{aligned} \tag{3.70}$$

The set of all diamond times of the bridge γ put in increasing order will be denoted by D_γ.

The set D_γ has a positive density in R_γ under the assumptions (3.5) and (3.69).

Lemma 3.18. *Suppose that $\mathbb{E}_{\text{iRB}}(|\gamma|) < \infty$. Then, there exists a positive density $\delta > 0$ such that*

$$\liminf_{n \longrightarrow \infty} \frac{D_\gamma \cap R_\gamma \cap [0,n]}{n} \geq \delta \tag{3.71}$$

almost surely.

Proof. The probability measure $\mathbb{P}_{\text{iRB}}$ is invariant under reflection $\mathcal{R}_x(\gamma)$. Therefore the expectation of the y-coordinate of the endpoint of $\gamma \in \text{iRB}$ is equal to zero.

The finite expectation of $|\gamma|$ implies also the fact that

$$\mathbb{E}_{\text{iRB}}(\max_{t \in \gamma} |y(t)|) \leq \mathbb{E}_{\text{iRB}}(|\gamma|) < \infty. \tag{3.72}$$

We can apply the law of large numbers to find that for γ distributed accordingly to $\mathbb{P}_{\text{iRB}}^{\otimes \mathbb{N}}$, there exists a constant $\mu > 0$ such that

{ibend}
$$\left(\frac{x(\gamma(r_n))}{n}, \frac{y(\gamma(r_n))}{n} \right) \longrightarrow (\mu, 0) \text{ almost surely.} \tag{3.73}$$

This result implies that there exists a positive integer K and a non-zero probability p_K for γ to lie in a half-cone $\{(x,y) : (x+K) \geq y > -(x+K)\}$. Indeed,

{coneK}
$$\mathbb{P}_{\text{iRB}}^{\otimes \mathbb{N}} \left(\left\{ \inf_{i \geq 0} (x(\gamma(i)) + y(\gamma(i))) \geq -K \right\} \cap \left\{ \inf_{i \geq 0} (x(\gamma(i)) - y(\gamma(i))) \geq -K \right\} \right) \geq p_K. \tag{3.74}$$

Taking any step from Ω and applying all necessary reflections and turns, we can obtain a one step walk $\tilde{\gamma}$ such that $x(\tilde{\gamma}(1)) \geq y(\tilde{\gamma}(1)) \geq 0$. Then, $\gamma_0 = \tilde{\gamma} \circ \mathcal{R}_y(\tilde{\gamma})$ is located in a cone $\{x \geq y > -x\}$ and the end of γ_0 lies on the axes $y = 0$. The weight of the segment equivalent to γ_0 will be denoted by σ_0.

By construction of $\mathbb{P}_{\text{iRB}}^{\otimes\mathbb{N}}$, we can add K samples of γ_0 to the beginning of any infinite walk γ. If γ lies in a cone $\{(x+K) \geq y > -(x+K)\}$ used in (3.74), then the result of this addition will be located in a cone $\{x \geq y > -x\}$. The probability price of this operation is equal to σ_0^K.

We can combine this fact with (3.74) to obtain that

$$\mathbb{P}_{\text{iRB}}^{\otimes\mathbb{N}} \left(\gamma(0) \in D_\gamma\right) \geq \sigma_0^K \, p_K. \tag{3.75}$$

The same bound is true for the bi-infinite random bridge:

$$\mathbb{P}_{\text{iRB}}^{\otimes\mathbb{Z}} \left(\gamma(0) \in D_\gamma\right) = \delta \geq (\sigma_0^K \, p_K)^2. \tag{3.76}$$

By the invariance of $\mathbb{P}_{\text{iRB}}^{\otimes\mathbb{Z}}$ under the operation of shift, $\mathbb{P}_{\text{iRB}}^{\otimes\mathbb{Z}} \left(\gamma(r_k) \in D_\gamma\right) = \delta$ for any $r_k \in R_\gamma$. The estimated density of diamond points is then equal to

$$\lim_{n \longrightarrow \infty} \mathbb{E}_{\text{iRB}}^{\otimes\mathbb{Z}} \left(\frac{D_\gamma \cap \{r_k\}_{k=0}^n}{n} \right) = \delta.$$

We can use this fact and apply Lemma 3.11 to the shift-invariant event

$$\liminf_{n \longrightarrow \infty} \left| \frac{D_\gamma \cap \{r_k\}_{k=0}^n}{n} \right| \geq \delta$$

to conclude that it has probability equal to 1. □

The definition of the diamond point can be extended to the bridges of finite length. It is easy to see that if $\gamma_n \in \text{RB}_n$ coincides with the beginning of γ, then $D_\gamma \cap \gamma_n \subset D_{\gamma_n}$. The operation consisting in taking the finite part of the bridge can only add new diamond points but not destroy the initial ones.

For bridges γ with at least two diamond points, we can define the operation of *stick-breaking*. Suppose that $\gamma \in \text{RB}_n$ and that there exist two points $i, j \in D_\gamma$ with $i < j$. Then, define a new bridge via the formula:

$$\text{StBr}_{(i,j)}(\gamma) = (\gamma_k)_{k=0}^i \circ r_{\pi/2}\big((\gamma_k)_{k=i}^j\big) \circ (\gamma_k)_{k=j}^n$$

. This operation does not add any crossing to the walk, so the weight does not change: $\sigma(\gamma) = \sigma(\text{StBr}_{(i,j)}(\gamma))$ for any choice of diamond points i and j. Also, note that the result of this operation is not necessary a bridge.

Proof. Let us assume (3.69). From any infinite bridge $\gamma \in \text{B}_\infty$, we can take a finite beginning containing the first n irreducible bridges of the walk: $\gamma^{(n)} = (\gamma)_{i=0}^{r_n}$. Let us use the notation $\tilde{\gamma} \triangleleft \gamma$ to say that there exists a renewal point $r_n \in R_\gamma$ such that $\tilde{\gamma} = \gamma^{(n)}$.

Define the width of any finite bridge as follows:

$$W(\gamma) = \max_{0 \leq i,j, \leq |\gamma|} \left(y(\gamma(i)) - y(\gamma(j))\right). \tag{3.77}$$

Now, fix $\varepsilon > 0$ (the exact value of the constant ε will be determined later). Look at the set of infinite bridges starting with not very long and not very wide finite bridges:

$$\overline{\mathrm{RB}_{\infty}^{+}}(n,\varepsilon) = \left\{ \gamma \in \mathrm{RB}_{\infty}^{+} : |\gamma^{(n)}| < \nu n, W(\gamma^{(n)}) < \varepsilon n, |D_{\gamma^{(n)}}| \geq \frac{\delta n}{2} \right\}. \tag{3.78}$$

{WNbridge}

The exact value of the constant ε will be determined later.

The irreducible bridges that form $\gamma \in \mathrm{RB}_{\infty}^{+}$ are independent and identically distributed so we can use the law of large numbers and the formula (3.73) to conclude that

$$\lim_{n \longrightarrow \infty} \mathbb{P}_{\mathrm{iRB}}^{\otimes \mathbb{N}} \left(\gamma : r_n < \nu n, W(\gamma^{(n)}) < \varepsilon n \right) = 1. \tag{3.79}$$

The probability of the condition on the number of diamond points is the result of Lemma 3.18:

$$\lim_{n \longrightarrow \infty} \mathbb{P}_{\mathrm{iRB}}^{\otimes \mathbb{N}} \left(\gamma : |D_{\gamma^{(n)}}| \geq \frac{\delta n}{2} \right) = 1. \tag{3.80}$$

The combination of these two estimations give us

arrowprob1}

$$\lim_{n \longrightarrow \infty} \mathbb{P}_{\mathrm{iRB}}^{\otimes \mathbb{N}} \left(\overline{\mathrm{RB}_{\infty}^{+}}(n,\varepsilon) \right) = 1. \tag{3.81}$$

We obtain the contradiction with (3.81) and prove the theorem by constructing the necessary amount of wide bridges using the operation of stickbreaking.

Let us define the set of all appropriate finite opening bridges as follows:

$$(\overline{\mathrm{RB}_{\infty}^{+}}(n,\varepsilon))^{(n)} = \left\{ \tilde{\gamma} : \exists \gamma \in \overline{\mathrm{RB}_{\infty}^{+}}(n,\varepsilon), \tilde{\gamma} = \gamma^{(n)} \right\}. \tag{3.82}$$

Then, use Lemma 3.10 to estimate the probability of this set in the following way:

$$c\left((\overline{\mathrm{RB}_{\infty}^{+}}(n,\varepsilon))^{(n)} \right) = \mathbb{P}_{\mathrm{iRB}}^{\otimes \mathbb{N}} \left(\overline{\mathrm{RB}_{\infty}^{+}}(n,\varepsilon) \right) = \sum_{\tilde{\gamma} \in (\overline{\mathrm{RB}_{\infty}^{+}}(n,\varepsilon))^{(n)}} \mu_c^{-|\tilde{\gamma}|} \sigma(\tilde{\gamma}). \tag{3.83}$$

Let us take the bridge $\tilde{\gamma} \in (\overline{\mathrm{RB}_{\infty}^{+}}(n,\varepsilon))^{(n)}$ and the diamond points $d_i, d_j \in D_{\tilde{\gamma}}$ with $i \in [\frac{\delta n}{10}, \frac{2\delta n}{10}]$ and $j \in [\frac{3\delta n}{10}, \frac{4\delta n}{10}]$. The result of the stickbreaking operation $\phi = \mathrm{StBr}_{d_i,d_j}(\tilde{\gamma})$ is a bridge if the following conditions hold:

beabridge1}

$$\min_{d_i \leq k \leq d_j} x(\phi(k)) > 0, \tag{3.84}$$

beabridge2}

$$\max_{d_i \leq k \leq d_j} x(\phi(k)) \leq x(\phi(|\tilde{\gamma}|)). \tag{3.85}$$

Inequalities (3.84) and (3.85) are true if

bridgecond}

$$x(\tilde{\gamma}(d_i)) - W((\tilde{\gamma}(k))_{k=d_i}^{d_j}) > \frac{\delta n}{10} - \varepsilon n > 0. \tag{3.86}$$

To guarantee that the above is valid, choose for example $\varepsilon = \frac{\delta}{20}$.

The width of the result can be bound in the following way:

$$W(\mathrm{StBr}_{d_i,d_j}(\tilde{\gamma})) \geq \frac{\delta n}{10} \geq \varepsilon n. \tag{3.87}$$

The number of renewal points of $\phi = \mathrm{StBr}_{d_i,d_j}(\tilde{\gamma})$ has the following upper bound:

$$\begin{aligned} |R_\phi| &= |R_{\tilde{\gamma}}| + |R_\phi \cap \{\phi(k)\}_{k=d_i}^{d_j}| - |R_{\tilde{\gamma}} \cap \{\tilde{\gamma}(k)\}_{k=d_i}^{d_j}| \\ &\leq n + W\left((\tilde{\gamma}(k))_{k=d_i}^{d_j}\right) - |j-i| \\ &\leq n + \varepsilon n - \frac{\delta n}{10} \leq n. \end{aligned} \tag{3.88}$$

We can conclude that any $\gamma \in \mathrm{B}_\infty^+$ starting with ϕ does not belong to $\overline{\mathrm{B}_\infty^+}(n, \frac{\delta}{20})$ because $W(\gamma^{(n)}) > \varepsilon n$.

The length of $\tilde{\gamma}$ cannot be bigger than νn. Hence, the number of $\tilde{\gamma}$ that can form the beginning of some fixed $\gamma \in \mathrm{RB}_\infty^+$ after the stickbreaking with some choice of i and j can be bounded by the number of ways to choose i and j over νn possibilities

$$\left|\left\{(\tilde{\gamma}, i, j) \in (\overline{\mathrm{RB}_\infty^+}(n,\varepsilon))^{(n)} \times [\tfrac{\delta n}{10}, \tfrac{2\delta n}{10}] \times [\tfrac{3\delta n}{10}, \tfrac{4\delta n}{10}] : \mathrm{StBr}_{d_i,d_j}(\tilde{\gamma}) \triangleleft \gamma\right\}\right| \leq (\nu n)^2. \tag{3.89}$$ {stbr-1card

For any fixed choice of $\tilde{\gamma} \in (\overline{\mathrm{RB}_\infty^+}(n,\varepsilon))^{(n)}$, $i \in [\frac{\delta n}{10}, \frac{2\delta n}{10}]$ and $j \in [\frac{3\delta n}{10}, \frac{4\delta n}{10}]$, Lemma 3.10 implies that for $\phi = \mathrm{StBr}_{d_i,d_j}(\tilde{\gamma})$

$$\mathbb{P}_{\mathrm{iRB}}^{\otimes \mathbb{N}}(\exists \gamma \in \mathrm{RB}_\infty^+ : \phi \triangleleft \gamma) = e^{-\lambda_0 |\phi|}\sigma(\phi) = e^{-\lambda_0|\tilde{\gamma}|}\sigma(\tilde{\gamma}). \tag{3.90}$$ {weight1}

After the summation of (3.90) over all possible $\tilde{\gamma}, i$ and j and plugging in (3.89), we obtain that

$$\begin{aligned} \left(\frac{\delta n}{10}\right)^2 \sum_{\tilde{\gamma} \in (\overline{\mathrm{RB}_\infty^+}(n,\varepsilon))^{(n)}} e^{-\lambda_0|\tilde{\gamma}|}\sigma(\tilde{\gamma}) &= \left(\frac{\delta n}{10}\right)^2 \mathbb{P}_{\mathrm{iRB}}^{\otimes \mathbb{N}}(\overline{\mathrm{RB}_\infty^+}(n,\varepsilon)) \\ &\leq (\nu n)^2\, \mathbb{P}_{\mathrm{iRB}}^{\otimes \mathbb{N}}\left(\gamma : \exists (\tilde{\gamma}, i, j) : \mathrm{StBr}_{d_i,d_j}(\tilde{\gamma}) \triangleleft \gamma\right) \\ &\leq (\nu n)^2\, \mathbb{P}_{\mathrm{iRB}}^{\otimes \mathbb{N}}(W(\gamma^{(n)}) > \varepsilon n). \end{aligned} \tag{3.91}$$

This inequality contradicts (3.81), so the assumption (3.69) has to be rejected.

□

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