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OPTIMIZED SCHWARZ METHODS WITHOUT OVERLAP FOR THE HELMHOLTZ EQUATION*

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Abstract. The classical Schwarz method is a domain decomposition method to solve elliptic partial differential equations in parallel. Convergence is achieved through overlap of the subdomains. We study in this paper a variant of the Schwarz method which converges without overlap for the Helmholtz equation. We show that the key ingredients for such an algorithm are the transmission conditions. We derive optimal transmission conditions which lead to convergence of the algorithm in a finite number of steps. These conditions are, however, nonlocal in nature, and we introduce local approximations which we optimize for performance of the Schwarz method. This leads to an algorithm in the class of optimized Schwarz methods. We present an asymptotic analysis of the optimized Schwarz method for two types of transmission conditions, Robin conditions and transmission conditions with second order tangential derivatives. Numerical results illustrate the effectiveness of the optimized Schwarz method on a model problem and on a problem from industry.

Key words. optimized Schwarz methods, domain decomposition, preconditioner, iterative parallel methods, acoustics

AMS subject classifications. 65F10, 65N22

PII. S1064827501387012

1. Introduction.

The classical Schwarz algorithm has a long history. It was invented by Schwarz more than a century ago [25] to prove existence and uniqueness of solutions to Laplace's equation on irregular domains. Schwarz decomposed the irregular domain into overlapping regular ones and formulated an iteration which used only solutions on regular domains and which converged to a unique solution on the irregular domain. A century later the Schwarz method was proposed as a computational method by Miller in [23], but it was only with the advent of parallel computers that the Schwarz method really gained popularity and was analyzed in depth both at the continuous level (see, for example, [17], [18], [19]) and as a preconditioner for discretized problems (see the books by Quarteroni and Valli [24] and Smith, Bjørstad, and Gropp [26] or the survey papers by Chan and Mathew [4], Xu [28], Xu and Zou [29], and references therein). The classical Schwarz algorithm is not effective for Helmholtz problems because the convergence mechanism of the Schwarz algorithm works only for the evanescent modes, not for the propagative ones. Nevertheless, the Schwarz algorithm has been applied to Helmholtz problems by adding a relatively fine coarse mesh for the propagative modes in [3] and changing the transmission conditions from Dirichlet in the classical Schwarz case to Robin, as first done in [9], and then in [8], [1], [7], [22], [2], [21], [6]. We study in this paper the influence of the transmission conditions on the Schwarz algorithm for the Helmholtz equation. We derive optimal transmission conditions which lead to the best possible convergence of the Schwarz algorithm and which do not require overlap to be effective as in [12]. These optimal

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transmission conditions, however, are nonlocal in nature and thus not ideal for implementations. We focus in what follows on approximating the optimal transmission conditions by local transmission conditions and we optimize them for performance of the Schwarz algorithm, which leads to optimized Schwarz methods without overlap for the Helmholtz equation. Even though the optimization is performed on a model problem in the whole plane, the derived optimized transmission conditions can be used for general decompositions of arbitrary domains, as we illustrate in section 5, and they prove to be effective on both the model problem and the industrial case presented in section 6. A preliminary study of the presented approach can be found in [5] and in [20].

2. The Schwarz algorithm without overlap. We consider the Helmholtz equation

$$\mathcal{L}(u) := (-\omega^2 - \Delta)(u) = f(x,y), \qquad x, y \in \Omega.$$

Although the following analysis could be carried out on rectangular domains as well, we prefer for simplicity to present the analysis in the domain $\Omega = \mathbb{R}^2$ with the Sommerfeld radiation condition at infinity,

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} + i\omega u\right) = 0,$$

where $r = \sqrt{x^2+y^2}$. The results we obtain for the unbounded domain are valid as well on bounded domains with a suitable restriction of the spectrum, which we discuss briefly in section 3.1. We decompose the domain into two nonoverlapping subdomains $\Omega_1 = (-\infty, 0] \times \mathbb{R}$ and $\Omega_2 = [0, \infty) \times \mathbb{R}$ and consider the Schwarz algorithm

$$\begin{aligned} -\Delta u_1^{n+1} - \omega^2 u_1^{n+1} &= f(x,y), & x, y \in \Omega_1, \\ \mathcal{B}_1(u_1^{n+1})(0,y) &= \mathcal{B}_1(u_2^n)(0,y), & y \in \mathbb{R}, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} -\Delta u_2^{n+1} - \omega^2 u_2^{n+1} &= f(x,y), & x, y \in \Omega_2, \\ \mathcal{B}_2(u_2^{n+1})(0,y) &= \mathcal{B}_2(u_1^n)(0,y), & y \in \mathbb{R}, \end{aligned} \tag{2.2}$$

where $\mathcal{B}_j$, $j = 1, 2$, are two linear operators. Note that for the classical Schwarz method $\mathcal{B}_j$ is the identity, $\mathcal{B}_j = I$, and without overlap the algorithm cannot converge. However, even with overlap in the case of the Helmholtz equation, only the evanescent modes in the error are damped, while the propagating modes are unaffected by the Schwarz algorithm [11]. One possible remedy is to use a relatively fine coarse grid [3] or Robin transmission conditions; see, for example, [8] and [2]. We propose here a new type of transmission conditions which leads to a convergent nonoverlapping version of the Schwarz method. We assume that the linear operators $\mathcal{B}_j$ are of the form

$$\mathcal{B}_j := \partial_x + \mathcal{S}_j, \qquad j = 1, 2,$$

for two linear operators $\mathcal{S}_1$ and $\mathcal{S}_2$ acting in the tangential direction on the interface. Our goal is to use these operators to optimize the convergence rate of the algorithm. For the analysis it suffices to consider by linearity the case $f(x,y) = 0$ and to analyze convergence to the zero solution. Taking a Fourier transform in the y direction we obtain

$$\begin{aligned} -\frac{\partial^2 \hat{u}_1^{n+1}}{\partial x^2} - (\omega^2 - k^2)\hat{u}_1^{n+1} &= 0, & x < 0,\ k \in \mathbb{R}, \\ (\partial_x + \sigma_1(k))(\hat{u}_1^{n+1})(0,k) &= (\partial_x + \sigma_1(k))(\hat{u}_2^n)(0,k), & k \in \mathbb{R} \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} -\frac{\partial^2 \hat{u}_2^{n+1}}{\partial x^2} - (\omega^2 - k^2)\hat{u}_2^{n+1} &= 0, & x > 0,\ k \in \mathbb{R}, \\ (\partial_x + \sigma_2(k))(\hat{u}_2^{n+1})(0,k) &= (\partial_x + \sigma_2(k))(\hat{u}_1^{n})(0,k), & k \in \mathbb{R}, \end{aligned} \tag{2.4}$$

where $\sigma_j(k)$ denotes the symbol of the operator $\mathcal{S}_j$ and k is the Fourier variable, which we also call frequency. The general solution of these ordinary differential equations is

$$\hat{u}_j^{n+1} = A_j e^{\lambda(k)x} + B_j e^{-\lambda(k)x}, \qquad j = 1, 2,$$

where $\lambda(k)$ denotes the root of the characteristic equation $\lambda^2 + (\omega^2 - k^2) = 0$ with positive real or imaginary part, and

$$\lambda(k) = \sqrt{k^2 - \omega^2} \text{ for } |k| \geq \omega, \qquad \lambda(k) = i\sqrt{\omega^2 - k^2} \text{ for } |k| < \omega. \tag{2.5}$$

Since the Sommerfeld radiation condition excludes growing solutions as well as incoming modes at infinity we obtain the solutions

$$\begin{aligned} \hat{u}_1^{n+1}(x,k) &= \hat{u}_1^{n+1}(0,k)e^{\lambda(k)x}, \\ \hat{u}_2^{n+1}(x,k) &= \hat{u}_2^{n+1}(0,k)e^{-\lambda(k)x}. \end{aligned}$$

Using the transmission conditions and the fact that

$$\begin{aligned} \frac{\partial \hat{u}_1^{n+1}}{\partial x} &= \lambda(k)\hat{u}_1^{n+1}, \\ \frac{\partial \hat{u}_2^{n+1}}{\partial x} &= -\lambda(k)\hat{u}_2^{n+1} \end{aligned}$$

we obtain over one step of the Schwarz iteration

$$\begin{aligned} \hat{u}_1^{n+1}(x,k) &= \frac{-\lambda(k) + \sigma_1(k)}{\lambda(k) + \sigma_1(k)} e^{\lambda(k)x}\hat{u}_2^{n}(0,k), \\ \hat{u}_2^{n+1}(x,k) &= \frac{\lambda(k) + \sigma_2(k)}{-\lambda(k) + \sigma_2(k)} e^{-\lambda(k)x}\hat{u}_1^{n}(0,k). \end{aligned}$$

Evaluating the second equation at $x = 0$ for iteration index n and inserting it into the first equation, we get after evaluating again at $x = 0$

$$\hat{u}_1^{n+1}(0,k) = \frac{-\lambda(k) + \sigma_1(k)}{\lambda(k) + \sigma_1(k)} \cdot \frac{\lambda(k) + \sigma_2(k)}{-\lambda(k) + \sigma_2(k)} \hat{u}_1^{n-1}(0,k).$$

Defining the convergence rate ρ by

$$\rho(k) := \frac{-\lambda(k) + \sigma_1(k)}{\lambda(k) + \sigma_1(k)} \cdot \frac{\lambda(k) + \sigma_2(k)}{-\lambda(k) + \sigma_2(k)} \tag{2.6}$$

we find by induction

$$\hat{u}_1^{2n}(0,k) = \rho(k)^n \hat{u}_1^{0}(0,k)$$

and by a similar calculation on the second subdomain

$$\hat{u}_2^{2n}(0,k) = \rho(k)^n \hat{u}_2^{0}(0,k).$$

Thus choosing in the Fourier transformed domain

$$\sigma_1(k) := \lambda(k), \quad \sigma_2(k) := -\lambda(k)$$

we get $\rho(k) \equiv 0$, and the algorithm converges in two steps independently of the initial guess. Unfortunately, this choice becomes difficult to use in the real domain where computations take place, since the optimal choice of the symbols $\sigma_j(k)$ leads to nonlocal operators $\mathcal{S}_j$ in the real domain caused by the square root in the symbols. In the next section we construct local approximations for the optimal transmission conditions which lead to an algorithm in the class of optimized Schwarz methods.

3. Optimized transmission conditions. We approximate the nonlocal symbols $\sigma_j(k)$ involving the square root by polynomials $\sigma_j^{app}(k)$ which represent differential operators in physical space and are thus local. To avoid an increase in the bandwidth of the local subproblems, we take polynomials of degree at most 2, which leads to transmission operators $\mathcal{S}_j^{app}$ which are at most second order partial differential operators acting along the interface. By symmetry of the Helmholtz equation there is no interest in a first order term. We therefore approximate the operators $\mathcal{S}_j$ either by a constant, $\mathcal{S}_1^{app} = -\mathcal{S}_2^{app} = a$, $a \in \mathbb{C}$, which leads to a Robin transmission condition, or by $\mathcal{S}_1^{app} = -\mathcal{S}_2^{app} = a + b\partial_{\tau\tau}$, where τ denotes the tangent direction at the interface and $a, b \in \mathbb{C}$. A first approximation that comes to mind is a low frequency approximation using a Taylor expansion of the optimal transmission condition,

$$\mathcal{S}_1^{app} = -\mathcal{S}_2^{app} = i\left(\omega - \frac{1}{2\omega}\partial_{\tau\tau}\right),$$

which leads to the zeroth or second order Taylor transmission conditions, depending on whether one keeps only the constant term or also the second order term. However, these transmission conditions are effective only for the low frequency components of the error. In the next two subsections we develop transmission conditions which are effective for the entire spectrum.

3.1. Optimized Robin transmission conditions. We approximate the optimal operators $\mathcal{S}_j$, $j = 1, 2$, in the form

$$\mathcal{S}_1^{app} = -\mathcal{S}_2^{app} = p + qi, \qquad p, q \in \mathbb{R}^+. \tag{3.1}$$

The nonnegativity of p, q comes from the Shapiro–Lopatinski necessary condition for the well-posedness of the local subproblems (2.1)–(2.2). Inserting this approximation into the convergence rate (2.6) we find

$$\rho(p,q,k) = \begin{cases} \frac{p^2+\left(q-\sqrt{\omega^2-k^2}\right)^2}{p^2+\left(q+\sqrt{\omega^2-k^2}\right)^2}, & \omega^2 \geq k^2, \\ \frac{q^2+\left(p-\sqrt{k^2-\omega^2}\right)^2}{q^2+\left(p+\sqrt{k^2-\omega^2}\right)^2}, & \omega^2 < k^2. \end{cases} \tag{3.2}$$

First note that for $k^2 = \omega^2$ the convergence rate $\rho(p, q, \omega) = 1$ no matter what one chooses for the free parameters p and q. In the Helmholtz case one cannot uniformly minimize the convergence rate over all relevant frequencies, as in the case of positive definite problems; see [14], [11], [16]. The point $k = \omega$ represents, however, only one single mode in the spectrum, and a Krylov method will easily take care of this when the Schwarz method is used as a preconditioner, as our numerical experiments will show. We therefore consider the optimization problem

$$\min_{p,q\in\mathbb{R}^+}\left(\max_{k\in(k_{\min},\,\omega_-)\cup(\omega_+,\,k_{\max})}|\rho(p,q,k)|\right), \tag{3.3}$$

where ω_- and ω_+ are parameters to be chosen and $k_{\min}$ denotes the smallest frequency relevant to the subdomain and $k_{\max}$ denotes the largest frequency supported by the numerical grid. This largest frequency is of the order π/h. For example, if the domain Ω is a strip of height L with homogeneous Dirichlet conditions on the top and bottom, the solution can be expanded in a Fourier series with the harmonics $\sin(\frac{j\pi y}{L})$, $j \in \mathbb{N}$. Hence the relevant frequencies are $k = \frac{j\pi}{L}$. They are equally distributed with a spacing $\frac{\pi}{L}$ (the lowest one is $k_{\min} = \frac{\pi}{L}$) and choosing $\omega_- = \omega - \pi/L$ and $\omega_+ = \omega + \pi/L$ leaves precisely one frequency $k = \omega$ for the Krylov method and treats all the others by the optimization. If ω falls in between the relevant frequencies, say $\frac{j\pi}{L} < \omega < \frac{(j+1)\pi}{L}$, then we can even get the iterative method to converge by choosing $\omega_- = \frac{j\pi}{L}$ and $\omega_+ = \frac{(j+1)\pi}{L}$, which will allow us to directly verify our asymptotic analysis numerically without the use of a Krylov method. How to choose the optimal parameters p and q is given by the following theorem.

THEOREM 3.1 (optimized Robin conditions). *Under the three assumptions*

$$2\omega^2 \le \omega_-^2 + \omega_+^2, \qquad \omega_- < \omega, \tag{3.4}$$

$$2\omega^2 > k_{\min}^2 + \omega_+^2, \tag{3.5}$$

$$2\omega^2 < k_{\min}^2 + k_{\max}^2, \tag{3.6}$$

the solution to the min-max *problem* (3.3) *is unique and the optimal parameters are given by*

$$p^* = q^* = \sqrt{\frac{\sqrt{\omega^2-\omega_-^2}\;\sqrt{k_{\max}^2-\omega^2}}{2}}. \tag{3.7}$$

The optimized convergence rate (3.3) *is then given by*

$$\max_{k\in(k_{\min},\,\omega_-)\cup(\omega_+,\,k_{\max})}\rho(p^*,q^*,k) = \frac{1-\sqrt{2}\left(\frac{\omega^2-\omega_-^2}{k_{\max}^2-\omega^2}\right)^{\frac14}+\sqrt{\frac{\omega^2-\omega_-^2}{k_{\max}^2-\omega^2}}}{1+\sqrt{2}\left(\frac{\omega^2-\omega_-^2}{k_{\max}^2-\omega^2}\right)^{\frac14}+\sqrt{\frac{\omega^2-\omega_-^2}{k_{\max}^2-\omega^2}}}. \tag{3.8}$$

REMARK 3.2. *The assumptions in Theorem* 3.1 *are not restrictive: if we let* $\omega_\pm = \omega \pm \delta_\pm$, $\delta_\pm > 0$, *then assumption* (3.4) *amounts to* $2(\delta_+ - \delta_-)\omega + \delta_+^2 + \delta_-^2 \ge 0$ *which is satisfied if, for example,* $\delta_+ = \delta_- > 0$. *Assumption* (3.5) *is not restrictive either since in practice* $k_{\min}$ *is small and* ω_+ *is close to* ω. *Finally, the rule of thumb to take at least* 10 *points per wavelength leads to a typical mesh size* $h \le \frac{\pi}{5\omega}$. *If* $k_{\max} = \frac{\pi}{h}$, *this rule gives* $k_{\max} \ge 5\omega$ *so that assumption* (3.6) *is also satisfied.*

Proof. To use the symmetry in (3.2) we introduce the change of variables

$$x = \begin{cases} \sqrt{\omega^2-k^2}, & \omega^2 \ge k^2,\\ \sqrt{k^2-\omega^2}, & \omega^2 \le k^2, \end{cases}$$

so that the min-max problem (3.3) becomes

$$\min_{p\in\mathbb{R}^+,\, q\in\mathbb{R}^+}\left(\max\left(\max_{x\in[x_1,\, y_1]} g(p,q,x),\ \max_{x\in[x_2,\, y_2]} g(q,p,x)\right)\right), \tag{3.9}$$

where the function g is given by

$$g(p,q,x) = \frac{p^2+(q-x)^2}{p^2+(q+x)^2}$$

and the limits in the new maximization are

$$x_1 = \sqrt{\omega^2-\omega_-^2},\ y_1 = \sqrt{\omega^2-k_{\min}^2},\ x_2 = \sqrt{\omega_+^2-\omega^2},\ \text{and } y_2 = \sqrt{k_{\max}^2-\omega^2}.$$

The assumptions in Theorem 3.1 now are

$$x_1 \le x_2 < y_1 < y_2. \tag{3.10}$$

To prove Theorem 3.1, we need several lemmas.

Lemma 3.3. *The optimal parameters are strictly positive;* $p^* > 0$ *and* $q^* > 0$.

Proof. We need only to show that the optimal parameters cannot be zero, since negative parameters are excluded in the optimization. We prove this by contradiction. First note that $g(p,q,x) < 1$ as long as $p,q,x > 0$ so that the optimum is necessarily less than one; $\rho^* < 1$. Now suppose, for example, that $p^* = 0$. Then $g(q^*,0,x) = 1$ for any x and therefore $\rho^* = 1$. However, this cannot be an optimum, since we have just seen that for $p,q,x > 0$ we have $\rho^* < 1$. The same argument also holds for q^*. ∎

The next three lemmas are about the function g. Let $\operatorname{sgn}(x)$ denote the following signum function: $\operatorname{sgn}(x) = +1$ if $x > 0$, $\operatorname{sgn}(x) = -1$ if $x < 0$, and $\operatorname{sgn}(x) = 0$ if $x = 0$.

Lemma 3.4. *For all* $x > 0$*, we have* $\operatorname{sgn}(p-q) = \operatorname{sgn}(g(p,q,x) - g(q,p,x))$.

Proof. A direct computation gives

$$g(p,q,x) - g(q,p,x) = 4(p-q)\frac{x(p^2+q^2+x^2)}{(p^2+(q+x)^2)(q^2+(p+x)^2)}. \qquad \square$$

Lemma 3.5. *Assuming that* $p,q > 0$*, the maximum of* $g(p,q,x)$ *for* $0 < x_1 < x < y_1$ *is attained at either* x_1 *or* y_1*. Similarly, the maximum of* $g(q,p,x)$ *for* $0 < x_2 < x < y_2$ *is attained at either* x_2 *or* y_2*.*

Proof. Computing the derivative of g with respect to x we find

$$\frac{\partial g}{\partial x} = \frac{4q(x^2-(p^2+q^2))}{(p+(q+x)^2)^2},$$

and hence there is only one extremum at $x = \sqrt{p^2+q^2}$. Evaluating the second derivative of g at the extremum gives

$$\left.\frac{\partial^2 g}{\partial x^2}\right|_{x=\sqrt{p^2+q^2}} = \frac{2q(p^2+q^2)(q+\sqrt{p^2+q^2})}{(p^2+q^2+q\sqrt{p^2+q^2})^3} > 0$$

and thus the extremum is a minimum. Hence the maximum must be attained on the boundary at either x_1 or y_1. The proof for the second statement of the lemma is similar. ∎

LEMMA 3.6. *The function $g(p,q,x)$ is monotonically increasing with $p>0$ for all $x,q>0$.*

Proof. The partial derivative of $g(p,q,x)$ with respect to p is

$$\frac{\partial g}{\partial p} = \frac{8pqx}{(p^2+(q+x)^2)^2} > 0. \qquad \square$$

The following lemmas are directly related to the min-max problem (3.9). For given $p,q>0$, let ρ denote the corresponding maximum value

$$\rho := \max\left(\max_{x\in[x_1,\,y_1]} g(p,q,x),\ \max_{x\in[x_2,\,y_2]} g(q,p,x)\right).$$

Since the maximum value is likely to be attained at several locations, we define the sets where the maximum is attained by

$$E_1(p,q) := \{x \in \{x_1,\,y_1\} : g(p,q,x) = \rho\}$$

and

$$E_2(p,q) := \{x \in \{x_2,\,y_2\} : g(q,p,x) = \rho\},$$

and we denote their cardinality by $\#E_i(p,q)$, $i=1,2$. By Lemma 3.5, we know that $\#E_i(p,q) \le 2$. When there is no ambiguity, we sometimes omit the argument (p,q). An optimal choice of the parameters will be denoted by (p^*,q^*), and we note $\#E_i^* := \#E_i(p^*,q^*)$.

LEMMA 3.7. *Let (p^*,q^*) be an optimal choice of the parameters. Then the cardinality $\#E_i^* \ge 1$ for $i=1,2$.*

Proof. It is not possible to have $\#E_i^* = 0$ for some i, as one can see by contradiction: suppose, without loss of generality, that we have $\#E_1^* = 0$. Then, by Lemma 3.6, lowering q and keeping $p=p^*$ would lead to a value of ρ smaller than ρ^* and thus the choice (p^*,q^*) would not be optimal. $\square$

LEMMA 3.8. *Let (p^*,q^*) be an optimal choice of the parameters. If $\#E_1^* = \#E_2^* = 1$, then $p^*=q^*$.*

Proof. Without loss of generality, we suppose $E_1^* = \{x_1\}$ and $E_2^* = \{y_2\}$. The other cases lead to computations identical to the ones we do for this case below. We have a first relation between p^* and q^* given by

$$g(p^*,q^*,x_1) = g(q^*,p^*,y_2). \tag{3.11}$$

We consider a small variation $(\delta p,\delta q)$ of the parameters about the optimum (p^*,q^*) so that the necessary optimality condition of Lemma 3.7 is satisfied. For variations small enough, we have $\#E_i \le 1$. Therefore, we can restrict ourselves to variations of (p,q) which are such that $\#E_i = 1$ and consequently such that $g(p,q,x_1) = g(q,p,y_2)$, i.e.,

$$\partial_1 g(p^*,q^*,x_1)\delta p + \partial_2 g(p^*,q^*,x_1)\delta q = \partial_1 g(q^*,p^*,y_2)\delta q + \partial_2 g(q^*,p^*,y_2)\delta p,$$

or

$$\delta q = \frac{\partial_1 g(p^*,q^*,x_1) - \partial_2 g(q^*,p^*,y_2)}{\partial_1 g(q^*,p^*,y_2) - \partial_2 g(p^*,q^*,x_1)}\delta p, \tag{3.12}$$

where ∂_1 and ∂_2 denote the derivatives with respect to the first and second variable. For $(\delta p, \delta q)$ related by (3.12), the optimality condition is

$$\delta\rho^* = \delta g(p^*, q^*, x_1) = \partial_1 g(p^*, q^*, x_1)\delta p + \partial_2 g(p^*, q^*, x_1)\delta q = 0.$$

Note that we could have chosen as optimality condition $\delta g(q^*, p^*, y_2) = 0$ as well, which is equivalent. Using (3.12) we get

$$\delta\rho^* = \frac{(\partial_1 g(p^*, q^*, x_1)\partial_1 g(q^*, p^*, y_2) - \partial_2 g(p^*, q^*, x_1)\partial_2 g(q^*, p^*, y_2))}{(\partial_1 g(q^*, p^*, y_2) - \partial_2 g(p^*, q^*, x_1))}\delta p = 0.$$

The case $\partial_1 g(q^*, p^*, y_2) - \partial_2 g(p^*, q^*, x_1) = 0$ can be excluded since we have $\partial_1 g(q^*, p^*, y_2) > 0$ by Lemma 3.6 and $\partial_2 g(p^*, q^*, x_1) \leq 0$ because otherwise $\delta p = 0$ and $\delta q < 0$ would lower both $g(p^*, q^*, x_1)$ and $g(q^*, p^*, y_2)$ and hence ρ^*. Thus we must have

$$0 = \partial_1 g(p^*, q^*, x_1)\partial_1 g(q^*, p^*, y_2) - \partial_2 g(p^*, q^*, x_1)\partial_2 g(q^*, p^*, y_2). \tag{3.13}$$

Solving (3.11) and (3.13) for p^* and q^*, we get

$$p^* = q^* = \sqrt{\frac{x_1 y_2}{2}}. \qquad \square$$

There are the following four possibilities remaining for a solution to the min-max problem:

Case 1. $\#E_1^* = \#E_2^* = 1$.

Case 2. $\#E_1^* = 2$ and $\#E_2^* = 1$.

Case 3. $\#E_1^* = 1$ and $\#E_2^* = 2$.

Case 4. $\#E_1^* = 2$ and $\#E_2^* = 2$.

Case 4 can be excluded because it leads to a contradiction: $\#E_1^* = \#E_2^* = 2$ implies that all the maxima are equal; $g(p^*, q^*, x_1) = g(p^*, q^*, y_1) = g(q^*, p^*, x_2) = g(q^*, p^*, y_2)$. Then, for any $y \in [x_2, y_2]$, we have $\rho(q^*, p^*, y) \leq \rho(q^*, p^*, y_2) = \rho(p^*, q^*, y_1)$. In particular, for $y = y_1$, we get $\rho(q^*, p^*, y_1) \leq \rho(p^*, q^*, y_1)$. Therefore, by Lemma 3.4, $q^* \leq p^*$. Similarly, for any $x \in [x_1, y_1]$, we have $\rho(p^*, q^*, x) \leq \rho(p^*, q^*, x_1) = \rho(q^*, p^*, x_2)$. In particular, for $x = x_2$, we have $\rho(p^*, q^*, x_2) \leq \rho(q^*, p^*, x_2)$. Therefore, by Lemma 3.4, $p^* \leq q^*$. This shows that we must have $p^* = q^*$. However, from $g(p^*, p^*, x_1) = g(p^*, p^*, y_1) = g(p^*, p^*, y_2)$ together with assumption (3.10) the function $g(p^*, p^*, x)$ then must have at least three maxima on $[x_1, y_2]$, which contradicts Lemma 3.5. Now we examine the other three cases. By Lemma 3.8, Case 1 corresponds to $p^* = q^*$. By Lemma 3.5 and assumption (3.10), this gives $E_1^* = \{x_1\}$ and $E_2^* = \{y_2\}$ which corresponds to the global solution given in Theorem 3.1. Let ρ_1^* denote the corresponding value of ρ^*. In Case 2, the only possibility is $E_1^* = \{x_1, y_1\}$ and $E_2^* = \{y_2\}$ because of assumption (3.10) and we have $g(p^*, q^*, x_1) = g(p^*, q^*, y_1) = g(q^*, p^*, y_2)$. A direct computation shows that

$$p^* = \frac{\sqrt{x_1 y_1}\,(y_2^2 + x_1 y_1)}{\sqrt{y_2^4 + 4y_2^2 x_1 y_1 + x_1^2 y_1^2 + x_1^2 y_2^2 + y_1^2 y_2^2}},$$

$$q^* = \frac{\sqrt{x_1 y_1}\,(x_1 + y_1) y_2}{\sqrt{y_2^4 + 4y_2^2 x_1 y_1 + x_1^2 y_1^2 + x_1^2 y_2^2 + y_1^2 y_2^2}},$$

and we denote the corresponding value for ρ^* by ρ_2^*. In Case 3 the only possibility is $E_1^* = \{x_1\}$ and $E_2^* = \{x_2, y_2\}$ by assumption (3.10). Note that this reduces to

Case 1 if $x_1 = x_2$. Therefore, we consider Case 3 only if $x_1 < x_2$. Then from $g(p^*, q^*, x_1) = g(q^*, p^*, x_2) = g(q^*, p^*, y_2)$ we deduce by a direct computation that

$$p^* = \frac{\sqrt{y_2 x_2}\, x_1(x_2 + y_2)}{\sqrt{x_1^2 x_2^2 + 4x_1^2 y_2 x_2 + x_1^2 y_2^2 + x_1^4 + y_2^2 x_2^2}},$$

$$q^* = \frac{\sqrt{y_2 x_2}\, (x_1^2 + y_2 x_2)}{\sqrt{x_1^2 x_2^2 + 4x_1^2 y_2 x_2 + x_1^2 y_2^2 + x_1^4 + y_2^2 x_2^2}},$$

which are the same formulas as in Case 2 when exchanging p with q and x_1 with y_2 and x_1, y_1 with x_2, y_2. We denote the corresponding value for ρ^* by ρ_3^*. A direct comparison now shows that $\rho_1^* < \rho_2^*$, and, provided that $x_1 < x_2$, we have $\rho_1^* < \rho_3^*$ and thus Case 1 is the global optimum. ∎

3.2. Optimized second order transmission conditions. We approximate the operators $\mathcal{S}_j$, $j = 1, 2$, in the form $\mathcal{S}_1^{app} = -\mathcal{S}_2^{app} = a + b\partial_{\tau\tau}$ with $a, b \in \mathbb{C}$ and τ denoting the tangent direction at the interface. The design of optimized second order transmission conditions is simplified by the following lemma.

LEMMA 3.9. *Let u_1 and u_2 be two functions which satisfy*

$$\mathcal{L}(u_j) \equiv (-\omega^2 - \Delta)(u) = f \quad \text{in } \Omega_j,\ j = 1, 2,$$

and the transmission condition

$$\left(\frac{\partial}{\partial n_1} + \alpha\right)\left(\frac{\partial}{\partial n_1} + \beta\right)(u_1) = \left(-\frac{\partial}{\partial n_2} + \alpha\right)\left(-\frac{\partial}{\partial n_2} + \beta\right)(u_2) \tag{3.14}$$

with $\alpha, \beta \in \mathbb{C}$, $\alpha + \beta \neq 0$, and n_j denoting the unit outward normal to domain Ω_j. Then the second order transmission condition

$$\left(\frac{\partial}{\partial n_1} + \frac{\alpha\beta - \omega^2}{\alpha + \beta} - \frac{1}{\alpha + \beta}\frac{\partial^2}{\partial \tau_1^2}\right)(u_1) = \left(-\frac{\partial}{\partial n_2} + \frac{\alpha\beta - \omega^2}{\alpha + \beta} - \frac{1}{\alpha + \beta}\frac{\partial^2}{\partial \tau_2^2}\right)(u_2) \tag{3.15}$$

is satisfied as well.

Proof. Expanding the transmission condition (3.14) yields

$$\left(\frac{\partial^2}{\partial n_1^2} + (\alpha + \beta)\frac{\partial}{\partial n_1} + \alpha\beta\right)(u_1) = \left(\frac{\partial^2}{\partial n_2^2} - (\alpha + \beta)\frac{\partial}{\partial n_2} + \alpha\beta\right)(u_2).$$

Now using the equation $\mathcal{L}(u_1) = f$, we can substitute $-(\frac{\partial^2}{\partial \tau_1^2} + \omega^2)(u_1) - f$ for $\frac{\partial^2}{\partial n_1^2}(u_1)$ and similarly we can substitute $-(\frac{\partial^2}{\partial \tau_2^2} + \omega^2)(u_2) - f$ for $\frac{\partial^2}{\partial n_2^2}(u_2)$. Hence, we get

$$\left(-\frac{\partial^2}{\partial \tau_1^2} - \omega^2 + (\alpha + \beta)\frac{\partial}{\partial n_1} + \alpha\beta\right)(u_1) - f = \left(-\frac{\partial^2}{\partial \tau_2^2} - \omega^2 - (\alpha + \beta)\frac{\partial}{\partial n_2} + \alpha\beta\right)(u_2) - f.$$

Now the terms f on both sides cancel and a division by $\alpha + \beta$ yields (3.15). ∎

Note that Higdon has already proposed approximations to absorbing boundary conditions in factored form in [13]. In our case, this special choice of approximating $\sigma_j(k)$ by

$$\sigma_1^{app}(k) = -\sigma_2^{app}(k) = \frac{\alpha\beta - \omega^2}{\alpha + \beta} + \frac{1}{\alpha + \beta}k^2 \tag{3.16}$$

leads to a particularly elegant formula for the convergence rate. Inserting $\sigma_j^{app}(k)$ into the convergence rate (2.6) and simplifying, we obtain

$$\rho(k;\alpha,\beta) := \left(\frac{\lambda(k)-\sigma_1}{\lambda(k)+\sigma_1}\right)^2 = \left(\frac{-(\alpha+\beta)\lambda(k)+\alpha\beta+k^2-\omega^2}{(\alpha+\beta)\lambda(k)+\alpha\beta+k^2-\omega^2}\right)^2$$

$$= \left(\frac{\lambda(k)^2-(\alpha+\beta)\lambda(k)+\alpha\beta}{\lambda(k)^2+(\alpha+\beta)\lambda(k)+\alpha\beta}\right)^2 = \left(\frac{\lambda(k)-\alpha}{\lambda(k)+\alpha}\right)^2\left(\frac{\lambda(k)-\beta}{\lambda(k)+\beta}\right)^2, \tag{3.17}$$

where $\lambda(k)$ is defined in (2.5) and the two parameters $\alpha, \beta \in \mathbb{C}$ can be used to optimize the performance. By the symmetry of $\lambda(k)$ with respect to k, it suffices to consider only positive k to optimize performance. We thus need to solve the min-max problem

$$\min_{\alpha,\,\beta\in\mathbb{C}}\left(\max_{k\in(k_{\min},\,\omega_-)\cup(\omega_+,\,k_{\max})}|\rho(k;\alpha,\beta)|\right), \tag{3.18}$$

where ω_- and ω_+ are again the parameters to exclude the frequency $k=\omega$ where the convergence rate equals 1, as in the zeroth order optimization problem. The convergence rate $\rho(k;\alpha,\beta)$ consists of two factors, and λ is real for vanishing modes and imaginary for propagative modes. If we choose $\alpha \in i\mathbb{R}$ and $\beta \in \mathbb{R}$, then for λ real the first factor is of modulus one and the second one can be optimized using β. If λ is imaginary, then the second factor is of modulus one and the first one can be optimized independently using α. Hence for this choice of α and β the min-max problem decouples. We therefore consider here the simpler min-max problem

$$\min_{\alpha\in i\mathbb{R},\,\beta\in\mathbb{R}}\left(\max_{k\in(k_{\min},\,\omega_-)\cup(\omega_+,\,k_{\max})}|\rho(k;\alpha,\beta)|\right) \tag{3.19}$$

which has an elegant analytical solution. Note, however, that the original minimization problem (3.18) might have a solution with better convergence rate, an issue investigated in [10].

THEOREM 3.10 (optimized second order conditions). *The solution of the* min-max *problem* (3.19) *is unique and the optimal parameters are given by*

$$\alpha^* = i\left((\omega^2-k_{\min}^2)(\omega^2-\omega_-^2)\right)^{1/4} \in i\mathbb{R} \tag{3.20}$$

and

$$\beta^* = \left((k_{\max}^2-\omega^2)(\omega_+^2-\omega^2)\right)^{1/4} \in \mathbb{R}. \tag{3.21}$$

The convergence rate (3.19) *is then for the propagating modes given by*

$$\max_{k\in(k_{\min},\omega_-)}|\rho(k,\alpha^*,\beta^*)| = \left(\frac{(\omega^2-\omega_-^2)^{1/4}-(\omega^2-k_{\min}^2)^{1/4}}{(\omega^2-\omega_-^2)^{1/4}+(\omega^2-k_{\min}^2)^{1/4}}\right)^2 \tag{3.22}$$

and for the evanescent modes it is

$$\max_{k\in(\omega_+,\,k_{\max})}\rho(k,\alpha^*,\beta^*) = \left(\frac{(k_{\max}^2-\omega^2)^{1/4}-(\omega_+^2-\omega^2)^{1/4}}{(k_{\max}^2-\omega^2)^{1/4}+(\omega_+^2-\omega^2)^{1/4}}\right)^2. \tag{3.23}$$

Proof. For $k \in (k_{\min},\,\omega_-)$ we have $|\frac{i\sqrt{\omega^2-k^2}-\beta}{i\sqrt{\omega^2-k^2}+\beta}| = 1$ since $\beta \in \mathbb{R}$ and thus $|\rho(k;\alpha,\beta)| = |\frac{i\sqrt{\omega^2-k^2}-\alpha}{i\sqrt{\omega^2-k^2}+\alpha}|^2$ depends only on α . Similarly, for $k \in (\omega_+,\,k_{\max})$ we have

$|\frac{\sqrt{k^2-\omega^2}-\alpha}{\sqrt{k^2-\omega^2}+\alpha}| = 1$ since $\alpha \in i\mathbb{R}$ and therefore $|\rho(k;\alpha,\beta)| = |\frac{\sqrt{k^2-\omega^2}-\beta}{\sqrt{k^2-\omega^2}+\beta}|^2$ depends only on β. The solution $(\alpha,\, \beta)$ of the minimization problem (3.19) is thus given by the solution of the two independent minimization problems

$$\min_{\alpha\in i\mathbb{R},}\left(\max_{k\in(k_{\min},\,\omega_-)}\left|\frac{i\sqrt{\omega^2-k^2}-\alpha}{i\sqrt{\omega^2-k^2}+\alpha}\right|\right) \tag{3.24}$$

and

$$\min_{\beta\in\mathbb{R}}\left(\max_{k\in(\omega_+,\,k_{\max})}\left|\frac{\sqrt{k^2-\omega^2}-\beta}{\sqrt{k^2-\omega^2}+\beta}\right|\right). \tag{3.25}$$

We show the solution for the second problem (3.25) only; the solution for the first problem (3.24) is similar. First note that the maximum of $|\rho_\beta| := |\frac{\sqrt{k^2-\omega^2}-\beta}{\sqrt{k^2-\omega^2}+\beta}|$ is attained on the boundary of the interval $[\omega_+, k_{\max}]$ because the function ρ_β (but not $|\rho_\beta|$) is monotonically increasing with $k \in [\omega_+, k_{\max}]$. On the other hand, a function of β, $|\rho_\beta(\omega_+)|$ grows monotonically with β while $|\rho_\beta(k_{\max})|$ decreases monotonically with β. The optimum is therefore reached when we balance the two values on the boundary, $\rho_\beta(\omega_+) = -\rho_\beta(k_{\max})$, which implies that the optimal β satisfies the equation

$$\frac{\sqrt{k_{\max}^2-\omega^2}-\beta}{\sqrt{k_{\max}^2-\omega^2}+\beta} = -\frac{\sqrt{\omega_+^2-\omega^2}-\beta}{\sqrt{\omega_+^2-\omega^2}+\beta} \tag{3.26}$$

whose solution is given in (3.21). ☐

The optimization problem (3.25) arises also for symmetric positive definite problems when an optimized Schwarz algorithm without overlap and Robin transmission conditions is used and the present solution can be found in [27].

4. Asymptotic analysis. The classical Schwarz method with overlap and Dirichlet transmission conditions does not converge when applied to a Helmholtz problem. The propagating modes are unaffected by the classical Schwarz method and errors in that frequency range remain; the convergence rate $\rho_p = 1$ for propagating modes [11]. Without an additional mechanism, like a coarse grid fine enough to carry all the propagating modes [3], the method cannot be used. The evanescent modes of the error, however, are damped, like in the case of Laplace's equation, at a rate depending on the size of the overlap [11]. If the overlap is of order h, which is often all that one can afford in real applications, the convergence rate of the evanescent modes is $\rho_e = 1 - O(h)$. Here we analyze the asymptotic behavior of the discretized counterparts of the optimized Schwarz methods introduced in the previous section. As we have seen, there will always be one mode which is not affected by the optimized Schwarz method, namely $k = \omega$. All the other modes in the error, however (the propagating ones and the evanescent ones), are converging. The following two theorems give the asymptotic convergence rates in the mesh parameter h of the discretized zeroth and second order optimized Schwarz methods which we also call OO0 for "Optimized Order 0" and OO2 for "Optimized Order 2."

THEOREM 4.1 (asymptotic convergence rate of OO0). *The asymptotic convergence rate* (3.3) *of the nonoverlapping Schwarz method* (2.1), (2.2) *with optimized zeroth order transmission conditions* (3.7) *discretized with mesh parameter h is given*

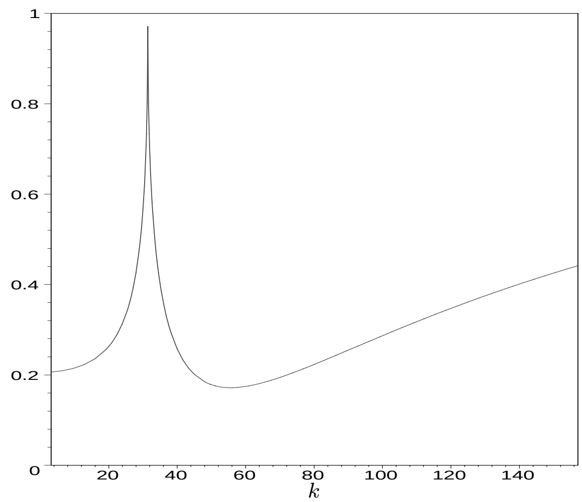

FIG. 4.1. *Convergence rate of the optimized Schwarz method with zeroth order transmission conditions in Fourier space for* $\omega = 10\pi$.

by

$$\rho = 1 - 2\frac{\sqrt{2}(\omega^2 - \omega_-^2)^{1/4}}{\sqrt{\pi}}\sqrt{h} + O(h). \tag{4.1}$$

Proof. Since on a numerical grid with grid spacing h the highest frequencies representable are of the order $k_{\max} = \pi/h$, we need only to compute the expansion in h of the optimized convergence rate $\rho(p^*, q^*, \omega_-)$ given in (3.8) for $k_{\max} = \pi/h$, which leads to the stated result. □

Figure 4.1 shows the convergence rate obtained for a model problem on the unit square with two subdomains, $\omega = 10\pi$ and $h = 1/50$. The optimal parameters were found to be $p^* = q^* = 32.462$ which gives an overall convergence rate of $\rho^* = 0.4416$.

THEOREM 4.2 (asymptotic convergence rate of OO2). *The asymptotic convergence rate* (3.19) *of the nonoverlapping Schwarz method* (2.1)*,* (2.2) *with optimized second order transmission conditions* (3.20)*,* (3.21) *discretized with mesh parameter* h *is for the propagating modes*

$$\rho_p = 1 - 4(2\Delta\omega)^{1/4}\left(\frac{1}{\omega}\right)^{1/4} + O(\sqrt{1/\omega}), \quad \Delta\omega = \omega - \omega_-, \tag{4.2}$$

and for the evanescent modes

$$\rho_e = 1 - 4\frac{(\omega_+^2 - \omega^2)^{1/4}}{\sqrt{\pi}}\sqrt{h} + O(h). \tag{4.3}$$

Proof. For the evanescent modes, using the fact that on a grid with grid spacing h the highest frequencies representable are $k_{\max} = \pi/h$, we can expand the convergence rate (3.23) in h to find (4.3). For the propagating modes, we set $\omega_- := \omega - \Delta\omega$ and perform an asymptotic expansion in ω of the convergence rate (3.22) to obtain the result (4.2). □

Figure 4.2 shows the convergence rate obtained for a model problem on the unit square with two subdomains, $\omega = 10\pi$ and $h = 1/50$. The optimal parameters

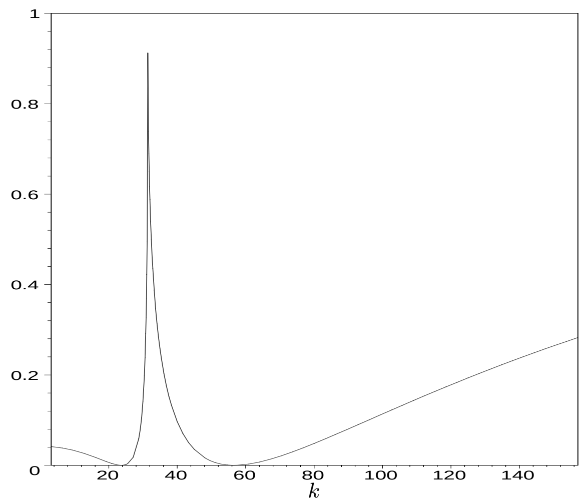

FIG. 4.2. *Convergence rate of the optimized Schwarz method with second order transmission conditions in Fourier space for $\omega = 10\pi$.*

were found to be $\alpha^* = 20.741i$ and $\beta^* = 47.071$ which gives a convergence rate of $\rho = 0.0419$ for the propagating modes and $\rho = 0.2826$ for the evanescent modes. Note how the convergence rate is uniformly faster than in the case of OO0. In addition, the propagating modes converge extremely fast. It is interesting to note that with the current practice in engineering of choosing about 10 grid points per wavelength, we have $h \approx \pi/(5\omega)$ and thus for the propagating modes the optimized Schwarz method presented here has an asymptotic convergence rate of

$$\rho_p = 1 - O(h^{1/4}).$$

5. Discretization. We now show how the new transmission conditions can be implemented in a finite element framework. Implementations using finite difference or finite volume discretizations could be considered as well. Using a reformulation of the algorithm, we show that both the transmission conditions of Robin type and the ones with second order tangential derivatives along the interface are as easy to implement as Neumann conditions. We first treat the case of a decomposition into two subdomains and then present the general case of an arbitrary decomposition of the domain into subdomains.

5.1. Two-domain decomposition. We decompose the domain Ω into two subdomains Ω_1 and Ω_2 with interface Γ_{12}. So far, we have considered the optimized Schwarz algorithm at the continuous level

$$\begin{array}{rcll} -\Delta u_1^{n+1} - \omega^2 u_1^{n+1} & = & f_1 & \text{in } \Omega_1 \\ \frac{\partial u_1^{n+1}}{\partial n_1} + \mathcal{S}_1^{app}(u_1^{n+1}) & = & -\frac{\partial u_2^n}{\partial n_2} + \mathcal{S}_1^{app}(u_2^n) & \text{on } \Gamma_{12} \\ -\Delta u_2^{n+1} - \omega^2 u_2^{n+1} & = & f_2 & \text{in } \Omega_2 \\ \frac{\partial u_2^{n+1}}{\partial n_2} + \mathcal{S}_2^{app}(u_2^{n+1}) & = & -\frac{\partial u_1^n}{\partial n_1} + \mathcal{S}_2^{app}(u_1^n) & \text{on } \Gamma_{12}. \end{array} \tag{5.1}$$

A direct discretization would require the computation of the normal derivatives along the interfaces in order to evaluate the right-hand sides in the transmission conditions of (5.1). This can be avoided by introducing two new variables,

$$\lambda_1^n = -\frac{\partial u_2^n}{\partial n_2} + \mathcal{S}_1^{app}(u_2^n) \quad \text{and} \quad \lambda_2^n = -\frac{\partial u_1^n}{\partial n_1} + \mathcal{S}_2^{app}(u_1^n).$$

The algorithm then becomes

$$\begin{aligned} -\Delta u_1^{n+1} - \omega^2 u_1^{n+1} &= f_1 && \text{in } \Omega_1, \\ \tfrac{\partial u_1^{n+1}}{\partial n_1} + \mathcal{S}_1^{app}(u_1^{n+1}) &= \lambda_1^n && \text{on } \Gamma_{12}, \\ -\Delta u_2^{n+1} - \omega^2 u_2^{n+1} &= f_2 && \text{in } \Omega_2, \\ \tfrac{\partial u_2^{n+1}}{\partial n_2} + \mathcal{S}_2^{app}(u_2^{n+1}) &= \lambda_2^n && \text{on } \Gamma_{12}, \\ \lambda_1^{n+1} &= -\lambda_2^n + (\mathcal{S}_1^{app} + \mathcal{S}_2^{app})(u_2^{n+1}), \\ \lambda_2^{n+1} &= -\lambda_1^n + (\mathcal{S}_1^{app} + \mathcal{S}_2^{app})(u_1^{n+1}). \end{aligned} \tag{5.2}$$

We can interpret this new algorithm as a fixed point algorithm in the new variables λ_j, $j = 1, 2$, to solve the substructured problem

$$\begin{aligned} \lambda_1 &= -\lambda_2 + (\mathcal{S}_1^{app} + \mathcal{S}_2^{app})(u_2(\lambda_2, f_2)), \\ \lambda_2 &= -\lambda_1 + (\mathcal{S}_1^{app} + \mathcal{S}_2^{app})(u_1(\lambda_1, f_1)), \end{aligned} \tag{5.3}$$

where $u_j = u_j(\lambda_j, f_j)$, $j = 1, 2$, are solutions of

$$\begin{aligned} -\Delta u_j - \omega^2 u_j &= f_j && \text{in } \Omega_j \\ \tfrac{\partial u_j}{\partial n_j} + \mathcal{S}_j^{app}(u_j) &= \lambda_j && \text{on } \Gamma_{12}. \end{aligned}$$

Instead of solving the substructured problem (5.3) by the fixed point iteration (5.2), one usually uses a Krylov subspace method to solve the substructured problem directly. This corresponds to using the optimized Schwarz method as a preconditioner for the Krylov subspace method. A finite element discretization of the substructured problem (5.3) leads to the linear system

$$\begin{aligned} \lambda_1 &= -\lambda_2 + (S_1 + S_2) B_2 u_2, \\ \lambda_2 &= -\lambda_1 + (S_1 + S_2) B_1 u_1, \\ \widetilde{K}_1 u_1 &= f_1 + B_1^T \lambda_1, \\ \widetilde{K}_2 u_2 &= f_2 + B_2^T \lambda_2, \end{aligned} \tag{5.4}$$

where B_1 and B_2 are the trace operators of domain Ω_1 and Ω_2 on the interface Γ_{12} and we omit the superscript *app* in the discretization S_j of the continuous operators $\mathcal{S}_j^{app}$ to reduce the notation. If the two vectors u_1 and u_2 containing the degrees of freedom have their first components corresponding to the interior unknowns

$$u_j = \begin{bmatrix} u_j^i \\ u_j^b \end{bmatrix}, \qquad j = 1, 2, \tag{5.5}$$

where the indices i and b correspond to interior and interface degrees of freedom, respectively, for domain Ω_j, then the discrete trace operators B_1 and B_2 are just the boolean matrices corresponding to the decomposition (5.5) and they can be written as

$$B_j = \begin{bmatrix} 0 & I \end{bmatrix}, \qquad j = 1, 2, \tag{5.6}$$

where I denotes the identity matrix of appropriate size. For example, $B_1 u_1 = u_1^b$ and $B_2 u_2 = u_2^b$. The matrices $\widetilde{K}_1$ and $\widetilde{K}_2$ arise from the discretization of the local Helmholtz subproblems along with the transmission conditions $\partial_n + a - b\partial_{\tau\tau}$,

$$\widetilde{K}_j = K_j - \omega^2 M_j + B_j^T (aM_{\Gamma_{12}} + bK_{\Gamma_{12}})B_j, \quad j = 1, 2. \tag{5.7}$$

Here K_1 and K_2 are the stiffness matrices, M_1 and M_2 are the mass matrices, $M_{\Gamma_{12}}$ is the interface mass matrix, and $K_{\Gamma_{12}}$ is the interface stiffness matrix

$$[M_{\Gamma_{12}}]_{nm} = \int_{\Gamma_{12}} \phi_n \phi_m d\xi \quad \text{and} \quad [K_{\Gamma_{12}}]_{nm} = \int_{\Gamma_{12}} \nabla_\tau \phi_n \nabla_\tau \phi_m d\xi. \tag{5.8}$$

The functions ϕ_n and ϕ_m are the basis functions associated with the degrees of freedom n and m on the interface Γ_{12}, and $\nabla_\tau \phi$ is the tangential component of $\nabla\phi$ on the interface. We have

$$S_j = aM_{\Gamma_{12}} + bK_{\Gamma_{12}}, \qquad j = 1, 2.$$

For given λ_1 and λ_2, the acoustic pressure u_1 and u_2 can be computed by solving the last two equations of (5.4). Eliminating u_1 and u_2 in the first two equations of (5.4) using the last two equations of (5.4), we obtain the substructured linear system

$$F\lambda = d, \tag{5.9}$$

where $\lambda = (\lambda_1, \lambda_2)$ and the matrix F and the right-hand side d are given by

$$\begin{aligned} F &= \begin{bmatrix} I & I - (S_1 + S_2)B_2\widetilde{K}_2^{-1}B_2^T \\ I - (S_1 + S_2)B_1\widetilde{K}_1^{-1}B_1^T & I \end{bmatrix}, \\ d &= \begin{bmatrix} (S_1 + S_2)B_1\widetilde{K}_1^{-1}f_1 \\ (S_1 + S_2)B_2\widetilde{K}_2^{-1}f_2 \end{bmatrix}. \end{aligned} \tag{5.10}$$

The linear system (5.9) is solved by a Krylov subspace method. The matrix vector product amounts to solving a subproblem in each subdomain and sending interface data between subdomains. Note that the optimization of the transmission conditions was performed for the convergence rate of the additive Schwarz method and not for a particular Krylov method applied to the substructured problem. In the positive definite case one can show that minimizing the convergence rate is equivalent to minimizing the condition number of the substructured problem [15]. Numerical experiments in the next section indicate that for the Helmholtz equation our optimization also leads to parameters close to the best ones for the preconditioned Krylov method.

5.2. General case. A finite element formulation of the global problem leads to the linear system

$$(K - \omega^2 M)u = f, \tag{5.11}$$

where K is the stiffness matrix and M is the mass matrix. The domain Ω is decomposed into N nonoverlapping subdomains Ω_j, $j = 1, \dots, N$. The substructured problems are

$$\lambda_{lj} = -\lambda_{jl} + (S_{lj} + S_{jl})B_{jl}u_j, \quad 1 \le l, j \le N, \quad l \ne j, \tag{5.12}$$

$$\widetilde{K}_l u_l = f_l + \sum_{j \ne l} B_{lj}^T \lambda_{lj}, \quad 1 \le l \le N, \tag{5.13}$$

where B_{lj} is the trace operator of domain Ω_l on the interface Γ_{lj}. The matrices $\widetilde{K}_l$ arise from the discretization of the local Helmholtz subproblems along with the transmission conditions $\partial_n + a - b\partial_{\tau\tau}$,

$$\widetilde{K}_l = K_l - \omega^2 M_l + \sum_{j \neq l} B_{lj}^T (aM_{\Gamma_{lj}} + bK_{\Gamma_{lj}}) B_{lj}, \tag{5.14}$$

where K_l are the local stiffness matrices and M_l are the local mass matrices. The interface stiffness matrices $K_{\Gamma_{lj}}$ and the interface mass matrices $M_{\Gamma_{lj}}$ are defined by

$$[K_{\Gamma_{lj}}]_{nm} = \int_{\Gamma_{lj}} \nabla_\tau \phi_n \nabla_\tau \phi_m d\xi \quad \text{and} \quad [M_{\Gamma_{lj}}]_{nm} = \int_{\Gamma_{lj}} \phi_n \phi_m d\xi, \tag{5.15}$$

where ϕ_n and ϕ_m are the basis functions associated with the degrees of freedom n and m on the interface Γ_{lj} and $\nabla_\tau \phi$ is the tangential component of $\nabla \phi$ on the interface as before. We have

$$S_{lj} = aM_{\Gamma_{lj}} + bK_{\Gamma_{lj}}, \qquad l \neq j. \tag{5.16}$$

LEMMA 5.1. *On any interface, the matrix $S_{lj} + S_{jl}$ is invertible.*

Proof. The matrix $S_{lj} + S_{jl}$ is square so that proving its invertibility reduces to proving its kernel is null. By (3.16) we have $S_{lj} + S_{jl} = \frac{2}{\alpha+\beta}((\alpha\beta - \omega^2)M_{\Gamma_{lj}} + K_{\Gamma_{lj}})$. Let ϕ be such that $(S_{lj} + S_{jl})\phi = 0$. Taking its hermitian scalar product with ϕ, we get

$$(\phi, (\alpha\beta - \omega^2)M_{\Gamma_{lj}}\phi) + (\phi, K_{\Gamma_{lj}}\phi) = 0.$$

From (3.19), we have $\alpha \in i\mathbb{R}$ and $\beta \in \mathbb{R}$ so that by taking the imaginary part of the above equation we get

$$\left(\phi, \left(\frac{\alpha}{i}\beta\right) M_{\Gamma_{lj}}\phi\right) = 0$$

which proves that $\phi \equiv 0$ because the mass matrix is positive definite. $\square$

THEOREM 5.2. *If the original problem* (5.11) *is well-posed, then the substructured problem* (5.12)–(5.13) *is also well-posed and the solution* $((u_l)_{1\le l\le N}, (\lambda_{jl})_{1\le j\neq l\le N})$ *is such that u_l is the restriction to subdomain Ω_l of the solution u to the original problem* (5.11).

Proof. System (5.12)–(5.13) is square. Existence for any right-hand side is thus equivalent to uniqueness for the zero right-hand side. We prove uniqueness by proving that a solution to (5.12)–(5.13) yields a solution to (5.11) which is unique by assumption. This proves the uniqueness of $(u_l)_{1\le l\le N}$ solution to (5.12)–(5.13). Uniqueness of $(\lambda_{jl})_{1\le j\neq l\le N}$ follows by (5.13). Let D_l denote the restriction operator to domain Ω_l and C_{lj} the restriction operator to the interface Γ_{lj}, $l \neq j$. Taking the equations (5.12) on any interface Γ_{lj} once for lj and once for jl, and taking the difference, we get

$$(S_{lj} + S_{jl})(B_{jl}u_j - B_{lj}u_l) = 0.$$

Using Lemma 5.1 the only solution to this system is the zero solution and therefore

$$B_{jl}u_j = B_{lj}u_l \tag{5.17}$$

which implies $u_j = u_l$ on Γ_{lj}. It is therefore natural to define the solution on the entire domain u from the solutions on the subdomains by $D_l u = u_l$, $1 \le l \le N$. Then, multiplying (5.13) from the left by D_l^T and summing over l, $1 \le l \le N$, we obtain, using (5.14) and (5.16),

$$\sum_l D_l^T (K_l - \omega^2 M_l) D_l u + \sum_{l \neq j} D_l^T B_{lj}^T S_{lj} B_{lj} u_l = \sum_l D_l^T f_l + \sum_{l \neq j} D_l^T B_{lj}^T \lambda_{lj},$$

where here $\sum_{l \neq j}$ denotes a double sum over the indices l and j. With the restriction operators to the interfaces $C_{lj} = B_{lj} D_l$ and their transposed $C_{lj}^T = D_l^T B_{lj}^T$, we get

$$\sum_l D_l^T (K_l - \omega^2 M_l) D_l u + \sum_{l \neq j} C_{lj}^T S_{lj} B_{lj} u_l = \sum_l D_l^T f_l + \sum_{l \neq j} C_{lj}^T \lambda_{lj}.$$

Using $C_{lj}^T = C_{jl}^T$ and $B_{lj} u_l = B_{jl} u_j$ from (5.17) we find

$$\sum_l D_l^T (K_l - \omega^2 M_l) D_l u + \sum_{l<j} C_{lj}^T (S_{lj} + S_{jl}) B_{jl} u_j = \sum_l D_l^T f_l + \sum_{l<j} C_{lj}^T (\lambda_{lj} + \lambda_{jl}),$$

where $\sum_{l<j}$ is a double sum over the indices l and j which corresponds to the summation over the interfaces, each interface being counted once. From (5.12), we get

$$\sum_l D_l^T (K_l - \omega^2 M_l) D_l u = \sum_l D_l^T f_l$$

which is equivalent to

$$(K - \omega^2 M) u = f. \qquad \square$$

REMARK 5.3. *It is well known that problem* (5.11) *is not necessarily well-posed since it corresponds to a Neumann problem for the Helmholtz equation. If a mixed boundary condition* $\partial_n + i\omega$ *is imposed on part of the boundary, one can show that it is well-posed. Theorem* 5.2 *is still valid in that case since its proof is purely algebraic.*

6. Numerical experiments. We show two sets of numerical experiments. The first set corresponds to the model problem analyzed in this paper, and the results obtained illustrate the analysis and confirm the asymptotic convergence results. The second numerical experiment comes from industry and consists of analyzing the noise levels in the interior of a Volvo S90.

6.1. Model problem. We study a two-dimensional cavity on the unit square Ω with homogeneous Dirichlet conditions on the top and bottom and on the left and right radiation conditions of Robin type. We thus have the Helmholtz problem

$$\begin{aligned} -\Delta u - \omega^2 u &= f, && 0 < x, y < 1, \\ u &= 0, && 0 < x < 1,\ y = 0, 1, \\ \frac{\partial u}{\partial x} - i\omega u &= 0, && x = 0,\ 0 < y < 1, \\ -\frac{\partial u}{\partial x} - i\omega u &= 0, && x = 1,\ 0 < y < 1. \end{aligned} \tag{6.1}$$

We decompose the unit square into two subdomains of equal size, and we use a uniform rectangular mesh for the discretization. We perform all our experiments directly on the error equations, $f = 0$, and choose the initial guess of the Schwarz iteration so that all the frequencies are present in the error. We show two sets of experiments. The first

TABLE 6.1
Number of iterations for different transmission conditions and different mesh parameters for the model problem.

	Order zero				Order two			
	Iterative		Krylov		Iterative		Krylov	
h	Taylor	Optimized	Taylor	Optimized	Taylor	Optimized	Taylor	Optimized
1/50	-	457	26	16	-	22	28	9
1/100	-	126	34	21	-	26	33	10
1/200	-	153	44	26	-	36	40	13
1/400	-	215	57	34	-	50	50	15
1/800	-	308	72	43	-	71	61	19

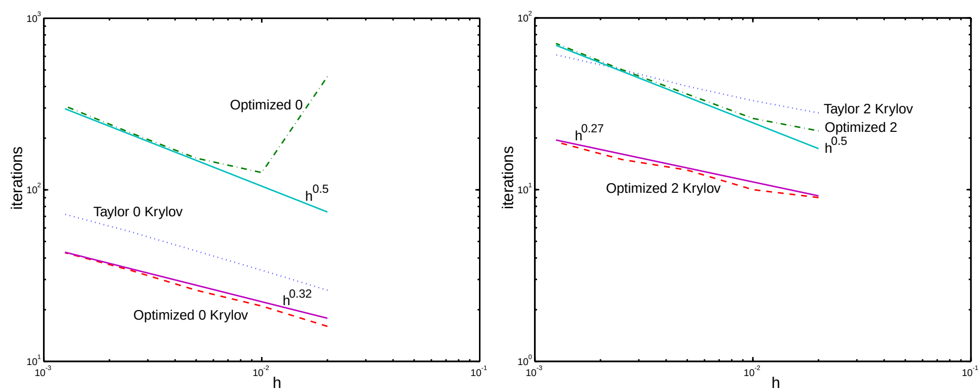

FIG. 6.1. *Asymptotic behavior on the left for the zeroth order transmission conditions and on the right for the second order transmission conditions.*

one is with $\omega = 9.5\pi$, thus excluding ω from the frequencies k relevant in this setting, $k = n\pi$, $n = 1, 2, \ldots$. This allows us to test directly the iterative Schwarz method, since with optimization parameters $\omega_- = 9\pi$ and $\omega_+ = 10\pi$ we obtain a convergence rate which is uniformly less than one for all k. Table 6.1 shows the number of iterations needed for different values of the mesh parameter h for both the zeroth and second order transmission conditions. The Taylor transmission conditions do not lead to a convergent iterative algorithm because, for all frequencies $k > \omega$, the convergence rate equals 1. However, with Krylov acceleration, GMRES in this case, the methods converge. Note, however, that the second order Taylor condition is only a little better than the zeroth order Taylor conditions. The optimized transmission conditions lead, in the case where ω lies between two frequencies, already to a convergent iterative algorithm. The iterative version even beats the Krylov accelerated Taylor conditions in the second order case. It is no wonder that the optimized conditions lead by far to the best algorithms when they are accelerated by a Krylov method; the second order optimized Schwarz method is more than a factor 3 faster than any Taylor method. Note that the only difference in cost of the various transmission conditions consists of different entries in the interface matrices, without enlarging the bandwidth of the matrices. Figure 6.1 shows the asymptotic behavior of the methods considered, on the left for zeroth order conditions and on the right for second order conditions. Note that the scale on the right for the second order transmission conditions is different by an order of magnitude. In both cases the asymptotic analysis is confirmed for the iterative version of the optimized methods. In addition, one can see that the

Table 6.2

Number of iterations for different transmission conditions and different mesh parameters for the model problem when ω lies precisely on a frequency of the problem and thus Krylov acceleration is mandatory.

	Order zero		Order two	
h	Taylor	Optimized	Taylor	Optimized
1/50	24	15	27	9
1/100	35	21	35	11
1/200	44	26	41	13
1/400	56	33	52	16
1/800	73	43	65	20

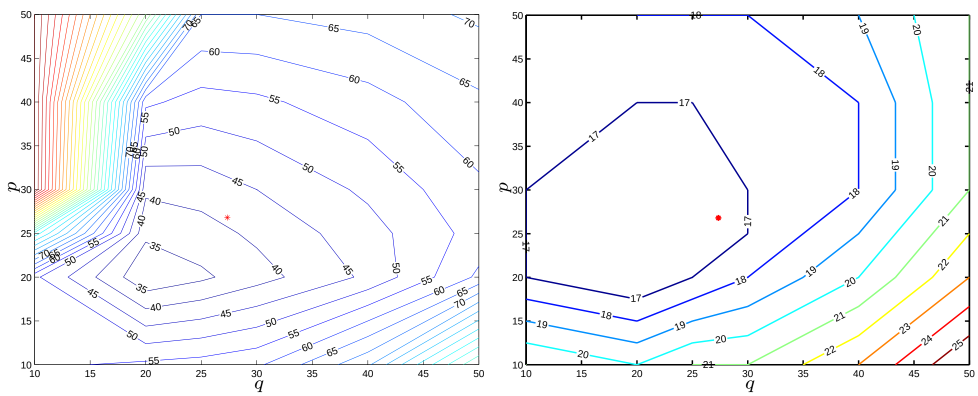

Fig. 6.2. *Number of iterations needed to achieve a certain precision as function of the optimization parameters p and q in the zeroth order transmission conditions, on the left for the iterative algorithm and on the right for the Krylov accelerated one. The star denotes the optimized parameters p^* and q^* found by our Fourier analysis.*

Krylov method improves the asymptotic rate by almost an additional square root, as expected from the analysis in ideal situations. Note the outlier of the zeroth order optimized transmission condition for $h = 1/50$. It is due to the discrepancy between the spectrum of the continuous and the discrete operator: $\omega = 9.5\pi$ lies precisely in between two frequencies 9π and 10π at the continuous level, but for the discrete Laplacian with $h = 1/50$ this spectrum is shifted to 8.88π and 9.84π and thus the frequency 9.84π falls into the range $[9\pi, 10\pi]$ neglected by the optimization. Note, however, that this is of no importance when Krylov acceleration is used, so it is not worthwhile to consider this issue further.

Now we put ω directly onto a frequency of the model problem, $\omega = 10\pi$, so that the iterative methods cannot be considered any more, since for that frequency the convergence rate equals one. The Krylov accelerated versions, however, are not affected by this, as one can see in Table 6.2. The number of iterations does not differ from the case where ω was chosen to lie between two frequencies, which shows that with Krylov acceleration the method is robust for any values of ω. We finally tested for the smallest resolution of the model problem how well Fourier analysis predicts the optimal parameters to use. Since we want to test both the iterative and the Krylov versions, we again need to put the frequency ω in between two problem frequencies, and in this case it is important to be precise. We therefore choose ω to be exactly between two frequencies of the discrete problem, $\omega = 9.3596\pi$, and optimized using $\omega_- = 8.8806\pi$ and $\omega_+ = 9.8363\pi$. Figure 6.2 shows the number of iterations the

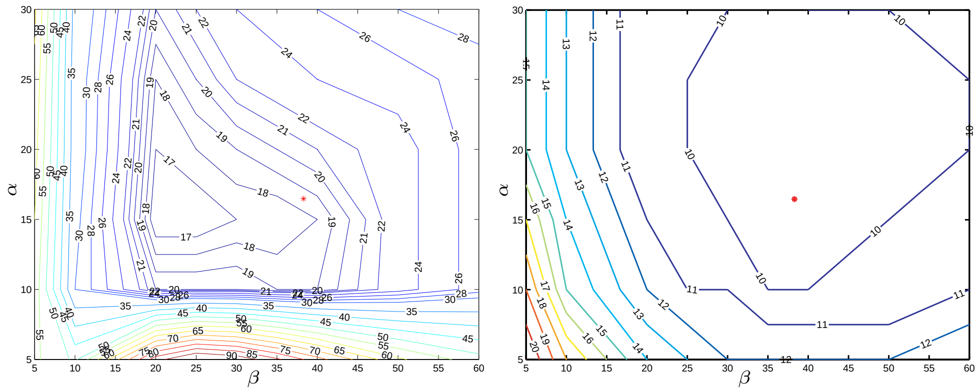

FIG. 6.3. *Number of iterations needed to achieve a certain precision as function of the optimization parameters α and β in the second order transmission conditions, on the left for the iterative algorithm and on the right for the Krylov accelerated one. The star denotes the optimized parameters α^* and β^* found by our Fourier analysis.*

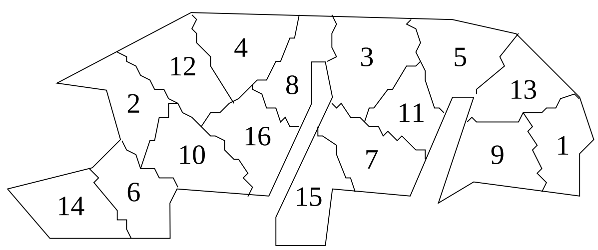

FIG. 6.4. *Decomposition of the passenger compartment into* 16 *subdomains.*

algorithm needs to achieve a residual of $10e-6$ as a function of the optimization parameters p and q of the zeroth order transmission conditions, on the left in the iterative version and on the right for the Krylov accelerated version. The Fourier analysis shows well where the optimal parameters lie, and, when a Krylov method is used, the optimized Schwarz method is very robust with respect to the choice of the optimization parameter. The same holds also for the second order transmission conditions, as Figure 6.3 shows.

6.2. Noise levels in a Volvo S90. We analyze the noise level distribution in the passenger cabin of a Volvo S90. The vibrations are stemming from the part of the car called firewall. This example is representative for a large class of industrial problems where one tries to determine the acoustic response in the interior of a cavity caused by vibrating parts. We perform a two-dimensional simulation on a vertical cross section of the car. Figure 6.4 shows the decomposition of the car into 16 subdomains. The computations were performed in parallel on a network of Sun workstations with four processors. The problem is characterized by $\omega a = 18.46$ which corresponds to a frequency of 1000 Hz in the car of length a. To solve the problem, the optimized Schwarz method was used as a preconditioner for the Krylov method ORTHODIR and as convergence criterion we used

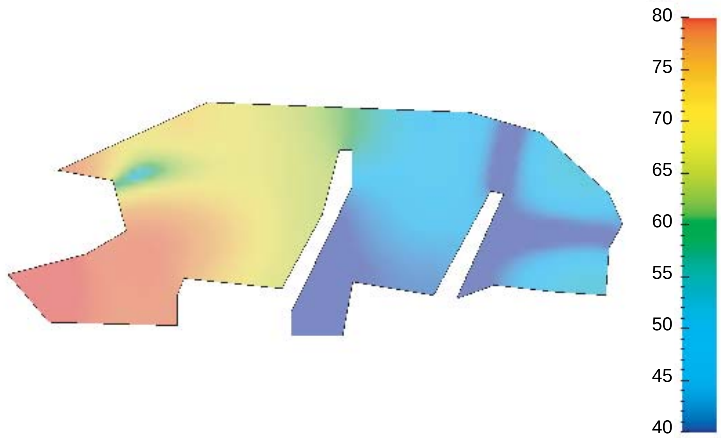

FIG. 6.5. *Acoustic field in the passenger compartment of the Volvo S90.*

$$\|\widetilde{K}u - f\|_{L_2} \leq 10^{-6} \ \|f\|_{L_2}. \tag{6.2}$$

When using zeroth order Taylor conditions and a decomposition into 16 subdomains, the method needed 105 iterations to converge, whereas when using second order optimized transmission conditions, the method converged in 34 iterations, confirming that also in real applications the optimized Schwarz method is about a factor 3 faster, as we found for the model problem earlier. Figure 6.5 shows the acoustic field obtained in the passenger compartment of the Volvo S90.

7. Conclusions. We introduced optimized Schwarz methods without overlap for Helmholtz problems. We analyzed a model problem with two subdomains and showed that the performance of the Schwarz method can be optimized using the transmission conditions employed between subdomains. We chose zeroth and second order transmission conditions and solved the corresponding optimization problems. We showed for the zeroth order conditions that the asymptotic convergence rate of the optimized Schwarz method discretized with mesh parameter h is $1-O(h^{1/2})$, whereas for the second order transmission conditions the asymptotic convergence rate for the propagating modes does not depend on the mesh parameter, while the convergence rate for the vanishing modes is asymptotically $1 - O(h^{1/2})$ for the iterative method, which gives together with Krylov acceleration an asymptotic rate of about $1-O(h^{1/4})$ for both the OO0 and OO2 methods. Numerical experiments showed that the method behaves asymptotically as predicted and that it is very effective on an industrial problem.

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