Enumerative geometry on the moduli space of rank 2 Higgs bundles

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This publication URL: https://archive-ouverte.unige.ch/unige:150031

Publication DOI: 10.13097/archive-ouverte/unige:150031

UNIVERSITÉ DE GENÈVE
Section de Mathématiques

FACULTÉ DES SCIENCES
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Enumerative geometry on the moduli space of rank 2 Higgs bundles

THÈSE

Présentée à la Faculté des Sciences de l'Université de Genève
pour obtenir le grade de Docteur ès sciences, mention Mathématiques

par

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de

Milan (Italie)

Thèse No. 5515

GENÈVE
Atelier d'impression ReproMail de l'Université de Genève
2020

UNIVERSITÉ DE GENÈVE

FACULTÉ DES SCIENCES

DOCTORAT ÈS SCIENCES, MENTION MATHEMATIQUES

Thèse de Monsieur Simone Melchiorre CHIARELLO

intitulée:

«Enumerative Geometry on the Moduli Space of Rank 2 Higgs Bundles»

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Genève, le 22 Août 2020

Thèse - 5515 -

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Chapter 1

Introduction

The P=W conjecture of De Cataldo, Hausel and Migliorini [CHM] relates two filtrations, the “weight” and the “perverse” one, on the cohomology of the moduli space of Higgs bundles. Such conjecture was proven in [CHM] in the rank 2, odd-degree case, using geometrical methods; in this thesis, we take the enumerative approach to re-prove a part of this difficult problem. Our starting point is the theory of equivariant integration developed in [HP] which, along with the intersection formulas on the moduli space of stable bundles found in [Th1] and [Z], allows us to write localization formulas on the moduli space of Higgs bundles, which are powerful enough to formulate the enumerative version of P=W (Theorem 7.0.2).

In Section 2, we briefly survey the basics of intersection theory, in order to set our notation and, most importantly, to explain what we mean when we say we are integrating a cohomology (or Chow) class. The Kodaira and Hirzebruch-Riemann-Roch theorems, which lie at the heart of Witten’s localization formulas developed as in [Th1] and [Z], are recalled at the end of the section.

In Section 3, the equivariant cohomology theory of a smooth manifold is defined, following the classical text [GS]. After that, we formulate and prove the basic localization formula for smooth compact manifolds with a torus action, following the proof given in [AB1]. Finally, important applications of such formula are presented; in particular, the intersection theory of symmetric products of Riemann surfaces will be used when writing the equivariant localization formulas for the moduli space of Higgs bundles of rank 2.

Section 4 is dedicated to vector bundles and their moduli spaces. We introduce Chern classes of vector bundles, the notion of slope-stability, and sketch the construction of the moduli space of stable bundles as in [AB]. We introduce Witten’s notation for expressing the cohomology classes of the moduli space, and explain Thaddeus’ and Zagier’s intersection formulas of [Th1] and [Z]: they will be the basis of our equivariant intersection formulas.

In Section 5 we review the definition and basic properties of Higgs bundles of rank 2 and of their moduli space $\mathcal{M}_g$. We focus on the equivariant cohomology of such space with respect to the natural $\mathbb{C}^*$-action, and on the compactification $\overline{\mathcal{M}}_g$ constructed in [Ha1]. Then, we review the definition of equivariant integration as in [HP] and see how, through Kalkman’s formula (Lemma 4), this allows in particular to compute classical integrals on the infinity divisor of $\overline{\mathcal{M}}_g$.

In Section 6 we explain the classical Narasimhan-Seshadri correspondence between vector bundles over a Riemann surface and representations of the fundamental group, and how such correspondence extends to Higgs bundles, yielding the nonabelian Hodge correspondence. The exposition follows closely [AB] and [Hi1]. We also recall the basic notions of mixed Hodge structures, and formulate the classical P=W conjecture of [CHM] for rank 2 Higgs bundles.

The new results of the research project are contained in Sections 7 and following. Parts of these sections are the topic of the paper [CHSz], written by T. Hausel, my advisor A. Szenes, and myself.

In Section 7 we compute the equivariant intersection formula on $\mathcal{M}_g$ in terms of cohomology classes in Witten's notation. Using Kalkman's formula, we compute a localization formula for the integrals on the infinity divisor of $\overline{\mathcal{M}}_g$. We also introduce the notion of defect of a cohomology class and relate it to the order of the pole appearing in the localization formula.

In Section 8 we formulate the enumerative version of P=W, and prove its equivalence with the classical one of [CHM]. We then apply the formulas found in Section 7 to formulate the matrix problem at top-defect and find the explicit solution for the classes β^k. For the more general classes $\beta^{k-h}\gamma^h$, we find and prove determinantal criterion for the existence of a solution.

Finally, in Section 9 we examine the difficulties arising when dealing with the pairing at lower defects (i.e. finding the *higher* defects of the solution to P=W). We address such problem by rationalizing our localization formulas (Proposition 9.2.1); this allows us to find the homogeneous part of the solution to P=W of defect one higher, in the case of the classes β^k.

Acknowledgements

With this thesis I have tried to collect, in a few pages, some of the work I have done in five years as a PhD student in Geneva. My advisor, Andras Szenes, has been my guide: he has taught me how to perform research and how to grow mathematically and as a person, always recalling that we should never forget the main reason we do research: have fun. I owe a lot to this amazing man and professor.

Another professor at the mathematics department, Anton Alekseev, has accepted to help me undertake some side projects, which had a great role in keeping my enthusiasm and curiosity high during the most difficult periods. Without his support and his precious advice, the road would have been much harder.

Five years is a long period of time, during which many changes in everyone's story occur. The number one protagonist in my story is my wife Angela, who has always given me love and support: I would like to thank her greatly for this and all the other things she does to make my life a marvellous adventure.

I would also like to thank all my colleagues and friends that I met in these years. They not only helped me in my studies, but most of all shared many happy moments with me throughout the PhD journey.

Finally, I want to thank my family (which has doubled during my PhD) for making me feel beloved and making me always understand that, at the end of the day, any decision I take is the right one, as long as I believe in it.

Remerciements

Avec cette thèse, j'ai essayé de rassembler dans peu de pages une partie importante du travail que j'ai fait pendant mes cinq ans comme Doctorant à l'Université de Genève. Mon directeur de thèse, Andras Szenes, a aussi bien été mon guide: il m'a enseigné le métier de la recherche en améliorant toujours moi même, en tant que mathématicien et homme. Cela sans jamais oublier le but principale de la recherche: s'amuser. Je dois beaucoup à ce marveilleux professeur et homme.

Un autre professeur de la Section de mathématiques, Anton Alekseev, m'a aidé entreprendre des projets additionnels, qui m'ont permis de garder ma passion et ma curiosité toujours hautes, aussi dans les moments les plus sombres. Sans son support et ses conseils précieux, le chemin aurait été beaucoup plus ardu.

Cinq ans c'est une longue periode de temps, pendant laquelle beaucoup de changements peuvent arriver dans l'histoire de chacun. Le protagoniste numéro un dans mon histoire, c'est Angela. Elle a toujours me donné tout l'amour et le support que j'ai desiré: je veux la remercier grandement, pour ça et toute autre attention qu'elle m'a dédié pour rendre ma vie une aventure magnifique.

Je voudrais aussi remercier tous mes collègues et amis que j'ai rencontré dans ces années. Ils m'ont aidé dans mes études, et partagé avec moi plusieurs moments joyeux tout au long du Doctorat.

Finalement, je remercie ma famille (qui est doublée pendant mon Doctorat) pour me faire sentir aimé et pour me faire comprendre que, après tout, toute décision que je prends c'est juste, si j'y croix vraiment.

Chapter 2

Intersection Theory

We can define intersection theory as the branch of algebraic geometry studying subvarieties of a given variety, and the way they intersect. Actually, one rarely wants to consider all subvarieties as distinct objects: what must be done is defining an equivalence relation $\sim$ of subvarieties, which respects intersections:

$$A' \sim A \text{ and } B' \sim B \Rightarrow A' \cap B' \sim A \cap B. \tag{2.1}$$

Our definition of intersection theory can thus be refined by saying that it deals with the properties of the set of equivalence classes $[A]$, for all subvarieties A of a given variety X. In particular, we will have different theories depending on the definition of the equivalence relation.

Once the relation has been chosen, one defines $\mathcal{F}(X)$ to be the free abelian group over all subvarieties of X, and $R_\sim(X) = \mathcal{F}(X)/\sim$, where the relation $\sim$ has been extended to $\mathcal{F}(X)$ by $\mathbb{Z}$-linearity. Since (2.1) defines a product on the set of equivalence relations, namely

$$[A] \cdot [B] := [A \cap B], \tag{2.2}$$

we have that $R_\sim(X)$ is a ring.

When it comes to a morphism $f : X \to Y$, we should recall that f does not respect intersections in general. However, the pre-image f^{-1} does, so that it is natural to say that the functor $X \mapsto R_\sim(X)$ should be *contravariant*: there exists a map $f^* : R_\sim(Y) \to R_\sim(X)$ defined by $f^*[A] = [f^{-1}(A)]$ (where we still extend f^{-1} by $\mathbb{Z}$-linearity to the free abelian group $\mathcal{F}(Y)$); of course, in order for f^* to be well-defined, the equivalence relation $\sim$ must be respected by the morphism f. In this way, we see that f^* is a ring homomorphism.

Arguably the two most important instances of intersection theory come from considering $\sim$ to be either *rational equivalence* or *(co)-boundary*. We will quickly discuss the former in the following section, while the latter, which brings to (co)-homology theory, will be the main subject of the present thesis.

2.1 The Chow ring and Chern classes

Let M be a smooth complex quasi-projective variety of dimension m [1] and let $\mathcal{F}(M)$ to be the free abelian group over its subvarieties. We say that a subvariety $\Gamma \subseteq M \times \mathbb{C}P^1$ is *graph-like* if it is not contained in any fiber $M \times \{p\}$, $p \in \mathbb{C}P^1$. For a graph-like Γ and a point $p \in \mathbb{C}P^1$, we call $\Gamma_p = \Gamma \cap (M \times \{p\})$, which has constant dimension for every p when it is non-empty. Intuitively, we can consider a graph-like $\Gamma \subseteq M \times \mathbb{C}P^1$ as defining an algebraic homotopy between Γ_p and Γ_q for any two points $p, q \in \mathbb{C}P^1$.

Choose two distinct points $\{0, \infty\} \subseteq \mathbb{C}P^1$: for X and Y subvarieties of M, we define

$$X \sim Y \iff X = \Gamma_0 \text{ and } Y = \Gamma_\infty, \text{ for some graph-like } \Gamma.$$

We say that the two subvarieties X and Y are *rationally equivalent*. We call the quotient $A(M) := \mathcal{F}(M)/\sim$ the *Chow group* of M. Since rational equivalence respect dimensions, we see that

$$A(M) = \bigoplus_{l=0}^{m} A_l(M) \tag{2.3}$$

where $A_l(M)$ is the subgroup consisting of equivalence classes of dimension l subvarieties.

Notice that when considering the ring structure (2.2), the direct sum decomposition (2.3) is not a grading. Indeed, if $\dim X = k$ and $\dim Y = l$ we expect, for general enough X and Y, to have $\dim X \cap Y = k + l - m$. Thus we define the *Chow ring* to be $A(M)$ as abelian group, but with the grading

$$A^*(M) = \bigoplus_{l=0}^{m} A^l(M), \text{ with } A^l(M) := A_{m-l}(M) \text{ for } l = 0, \dots, m. \tag{2.4}$$

This is not enough yet to have a well-defined graded ring structure on $A(M)$, since we did not define what "general enough" means when we intersect two subvarieties; moreover, we should check whether the intersection of subvarieties respect rational equivalence.

We say that X and Y are *generically transverse* if for all p in a Zariski open subset of $X \cap Y$, we have $T_pX + T_pY = T_pM$. The fundamental ingredient, that can be considered the heart of intersection theory, is then the following lemma, which allows us to say that $A^*(X)$ is a graded commutative ring, with product defined as in (2.2). For its proof we refer to Lemma A.1 in [EH], and to the references therein.

Theorem 2.1.1 (Moving Lemma). *Let $\alpha \in A^k(M)$ and $\beta \in B^l(M)$. Then there exist generically transverse subvarieties X and $Y \in \mathcal{F}(M)$ such that $[X] = \alpha$ and $[Y] = \beta$. Moreover, the class of $[X \cap Y]$ only depends on α and β.*

The issue of transversality remains when we want to define the pull-back map f^* of a morphism $f : M \to N$. We say that $Y \subseteq N$ is *generically transverse to* f if $f^{-1}(Y)$ is reduced and $\operatorname{codim}_N Y = \operatorname{codim}_M f^{-1}(Y)$. It can be shown (Lemma A.2 in [EH]) that this is equivalent to be generically tranverse to a particular finite

[1] We mention here that the richest part of this theory comes from considering much more general schemes over a field. However for our exposition this assumption will be enough.

collection of subvarieties of Y. Therefore the Moving Lemma allows us to define $f^* : A(N) \to A(M)$ by the following condition:

$$f^*(\alpha) = [f^{-1}(Y)] \text{ for } Y \text{ generically transverse to } f \text{ such that } [Y] = \alpha.$$

It is now easily shown that $\mathrm{id}^* = \mathrm{id}$ and $(g \circ f)^* = f^* \circ g^*$ when it makes sense. In this way, we have a complete definition of the intersection theory on a smooth quasi-projective variety M, based on rational equivalence.

Remark 2.1.2. Notice that $A^0(M) = A_m(M) \simeq \mathbb{Z}$ since the only dimension m subvariety is M itself. Oppositely, $A^m(M) = A_0(M)$ is the set of points up to rational equivalence, thus it can be a complicated object. If M is a smooth projective variety, $A^1(M)$ is the set of Weil divisors on M up to linear equivalence, so that $A^1(M) \simeq \mathrm{Pic}(M)$.

Notice that the Chow group also has a notion of *push-forward*: for a morphism $f : M \to N$ and a subvariety $X \subseteq M$ of dimension l, we can define $f_*[X] := [f(X)] \in A_l(N)$; it can be shown this is a map of $A^*(N)$-modules, i.e.

$$f_*(f^*\beta \cdot \alpha) = \beta \cdot f_*\alpha, \text{ for } \alpha \in A(M),\ \beta \in A(N).$$

If $\dim M = m$ and $\dim N = n$, we see that $f_* : A^l(M) \to A^{n-m+l}(N)$. In particular, if $M = \{p\}$ is a point, so that $A(M) = A^0(M) = \mathbb{Z}$, we have that $f_*[p] \in A^n(N)$ is the class of the point $f(p)$ in the Chow ring of N. More interestingly, if $f : M \to \{pt\}$ is the constant map, the push-forward of f defines the *degree* map $\deg f := f_* : A^n(M) \to \mathbb{Z}$.

The Chow ring satisfies right-exact excision and the Mayer-Vietoris properties: for $i : X \hookrightarrow M$, $Y \subseteq M$ closed embeddings, and $j : U = M \setminus X \hookrightarrow M$, then we have

$$A(X \cap Y) \to A(X) \oplus A(Y) \to A(X \cup Y) \to 0 \tag{2.5}$$

$$A(X) \xrightarrow{i_*} A(M) \xrightarrow{j^*} A(U) \to 0 \tag{2.6}$$

Example 1. Let $\mathbb{A}^n$ be the affine space of dimension n. If $X \subseteq \mathbb{A}^n$ is a proper subvariety and $p \notin X$, then letting Γ to be the projectivization of the cone of X centered in p gives a rational equivalence between $\Gamma_1 = Y$ and the empty set Γ_∞, thus $Y \sim 0$. This implies that $A(\mathbb{A}^n) = A^0(\mathbb{A}^n) = \mathbb{Z} \cdot [\mathbb{A}^n]$.

Example 2. In general, letting $U \subseteq \mathbb{A}^n$ be a non-empty open subvariety, by the previous Example and by (2.6) we have that $A(U) = A^0(U) = \mathbb{Z} \cdot [U]$.

Example 3. Using excision (2.6) and the classical stratification of $\mathbb{CP}^n$ by affine spaces, we see that $A^k(\mathbb{CP}^n) = \mathbb{Z} \cdot [L_{n-k}]$ where L_{n-k} is any $(n-k)$-dimensional linear subspace. Moreover, since L_{n-k} is the transversal intersection of k hyperplanes, we have $[L_{n-k}] = \zeta^k$ where $\zeta \in A^1(\mathbb{CP}^n)$ is the class of a hyperplane. It is actually not difficult to show that this gives the full description of the Chow ring

$$A(\mathbb{CP}^n) \cong \mathbb{Z}[\zeta]/(\zeta^{n+1}).$$

Remark 2.1.3. We used the Moving Lemma as the foundation of intersection theory. It should be mentioned here that although this approach has the advantage of being intuitive (and, most importantly, of giving the correct results!), it is not the best in terms of theoretical bases, since there are some controversies on the proofs of the strong version of it, namely the possibility to construct subvarieties generically transverse to a morphism $f : M \to N$.

The modern theoretical framework of Fulton and MacPherson [Ful], [FMP] makes the Moving Lemma unnecessary for the definition of the Chow ring and of its functorial properties; moreover, it rigorously shows that the classical statements of the Moving Lemma are indeed correct, so that they can be freely used in our geometric intuitions.

The importance of the Chow ring in the present thesis relies on the fact that it is the natural framework to define *Chern classes* of vector bundles in terms of its sections.

Definition 2.1.4. Let V be a rank r vector bundle on a smooth variety M, and let $\sigma_1, \dots, \sigma_r$ be generically independent global sections of V. For $k = 1, \dots, r$ we define the k*-th Chern class* as

$$c_k(V) := Z(\sigma_1 \wedge \cdots \wedge \sigma_{r-k+1}) \in A^k(M)$$

where we denote by $Z(\sigma)$ the subvariety of M in which the section σ is zero.

It can be shown (see [EH]) that the class of $c_k(V)$ in the Chow group $A^k(M)$ is well-defined, i.e. it does not depend on the particular choice of the sections σ_i as long as they are generically independent.

Of course, this does not define Chern classes for vector bundles with not enough sections. In order to have a comprehensive definition, we use a classical result (see for example [MSt]) which allows us to extend the construction by functoriality to all vector bundles. Thus we state, without proof, the following theorem.

Theorem 2.1.5. *[Existence of Chern classes] There is a unique way of assigning to every vector bundle V of rank r on M and every $k \in \mathbb{Z}$ a class $c_k(V) \in A^k(M)$ such that*

1. $c_0(V) = 1$ *and* $c_k(V) = 0$ *for* $k > r$.
2. *If* $0 \to U \to V \to W \to 0$ *is a short exact sequence of vector bundles on* M, *then*

$$c_k(V) = \sum_{i=0}^{k} c_i(U) c_{k-i}(W).$$

3. *If* $\varphi : N \to M$ *is a morphism, then for all* k

$$c_k(\varphi^* V) = \varphi^* c_k(V).$$

4. *For bundles generated by global sections,* $c_k(V)$ *is as in Definition 2.1.4.*

Point (2) is usually called *Whitney's formula*; it can be expressed in a compact form using the *total Chern class*

$$c(V) := 1 + c_1(V) + \ldots + c_r(V) \in A(M)$$

using which Whitney's formula simply becomes $c(V) = c(U)c(W)$.

Example 4. Letting $L = \mathcal{O}(1)$ be the line bundle on $\mathbb{CP}^n$ corresponding to a hyperplane section H, we have $c_1(L) = [H] = \zeta \in A^1(\mathbb{CP}^n)$, thus for the tautological bundle $\tau = L^*$ we will have $c_1(\tau) = -\zeta$. Denoting V the trivial $(n+1)$-bundle on $\mathbb{CP}^n$ and $Q = V/\tau$ the universal quotient bundle, Whitney's formula gives

$$c(Q) = \frac{1}{1-\zeta} = 1 + \zeta + \zeta^2 + \ldots + \zeta^n \in A(\mathbb{CP}^n).$$

Although the Chow ring is a natural framework for introducing the intersection theory of a variety, a much coarser concept is sufficient for the scope of the present thesis. We thus discuss the role of cohomology in the next section, keeping the possibility of coming back to Chow groups in the (rare) cases it will be needed.

2.2 Cohomology and intersection numbers

For us, the *cohomology* of an algebraic variety M will be the singular cohomology with complex coefficients $H^i(M) = H^i(M; \mathbb{C})$.

Remark 2.2.1. Assuming M is smooth, assigning to subvariety the corresponding non-compact cycle gives a homomorphism $A_i(M) \to H^{BM}_{2i}(M; \mathbb{Z})$, where we are considering Borel-Moore homology; under Poincaré duality, this gives a homomorphism

$$A^i(M) \to H^{2i}(M; \mathbb{Z}).$$

Notice that if M is smooth and projective, then this is an isomorphism for $i = 1$, since both objects are isomorphic to Pic(M).

With a slight abuse of notation, we denote the *Chern class* $c_k(V) \in H^{2k}(M)$ of a vector bundle V to be the image, under the homomorphism of Remark 2.2.1, of $c_k(V) \in A^k(M)$ as in Definition 2.1.4 and Theorem 2.1.5.

For the rest of this section, we assume M is a smooth compact complex algebraic variety of complex dimension n. Then, Poincaré duality allows us to describe the cohomology of M in terms of intersection data, as follows.

Definition 2.2.2. The *integral* is the map

$$\int_M : H^i(M) \longrightarrow \mathbb{C}$$

which is identically 0 for $i \neq 2n$, and for $i = 2n$ sends the class Poincaré-dual to a point to $1 \in \mathbb{C}$. For a class $\alpha \in H^{2n}(M)$, we write $\int_M \alpha$ for its integral.

Given a set $S = \{\alpha_1, \ldots, \alpha_m\}$ of generators of the cohomology ring H(M), the *intersection numbers* of M relative to the generating set S are the complex numbers of the form

$$\int_M \alpha_{i_1} \cdots \alpha_{i_k}.$$

Of course, only products of top degree will yield non-zero intersection numbers. *Enumerative geometry* can be described as the study of the structure of such intersection numbers. Typically, one is brought to consider a generating set of *integral* classes $\alpha_i \in H^i(M; \mathbb{Z})$: since Poincaré duality holds at the level of integral cohomology, such intersection numbers will be integers. However, this is not strictly necessary for the development of the theory, and one can equally well work with more general, complex intersection numbers.

The following Lemma is of key importance for enumerative geometry.

Lemma 1. *A class* $\alpha \in H^i(M)$ *is zero if and only if, for all* $\beta \in H^{2n-i}(M)$*,*

$$\int_M \alpha\beta = 0.$$

Proof. It follows from the fact that the Poincaré duality is a perfect pairing. □

This easy lemma has a striking consequence: assuming we know a set S of generators of the cohomology ring H(M), knowing the intersection numbers allows (in principle) to find all the relations, thus describing the ring H(M) completely. Enumerative geometry allows to transform any problem involving the cohomology of a compact smooth variety into one about the combinatorial or analytic structure of its intersection numbers. This is the point of view we will exploit in the present thesis.

2.3 The Kodaira vanishing and Hirzebruch-Riemann-Roch theorems

In the context of intersection theory on moduli spaces, two basic results on complex algebraic variety provide a powerful tool to obtain striking formulas such as the ones obtained by Witten originally for the moduli space of stable bundles.

We say that a holomorphic line bundle L on a complex manifold M is *positive* if the curvature associated to its Chern connection (i.e. the unique connection whose antiholomorphic part is the $\overline{\partial}$-operator defining the holomorphic structure of L) $-i\omega$ where ω is a positive definite form. We say that a holomorphic line bundle is *negative* if its dual is positive. We refer to [GH] for all these notions, which will also be surveyed in Section 6.

Theorem 2.3.1 (Kodaira vanishing)**.** *Let* M *be a compact Kähler manifold and* L *a negative line bundle. Then for* $i > 0$*, we have* $H^i(M; L) = 0$*.*

The second result, the *Hirzebruch-Riemann-Roch theorem*, allows to express the *Euler characteristic*

$$\chi(M; V) := \sum_{i \geqslant 0} (-1)^i \dim H^i(M; V)$$

of a holomorphic vector bundle V as an integral.

The statement of the theorem involves the *Chern character* ch(V) of the vector bundle V; this is defined as

$$\mathrm{ch}(V) := e^{\lambda_1} + \ldots + e^{\lambda_r}$$

where $\lambda_1, \ldots, \lambda_r$ are the Chern roots of V. Moreover, we define the *Todd class* of a bundle V as

$$\mathrm{td}(V) := \prod_{i=1}^{n} Q(\lambda_i), \text{ with } Q(x) = \frac{x}{1-e^{-x}} = 1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \ldots$$

Theorem 2.3.2 (Hirzebruch-Riemann-Roch). *If* V *is a holomorphic vector bundle over a compact complex manifold* M, *then*

$$\chi(M;V) = \int_M \mathrm{ch}(V)\mathrm{td}(M). \tag{2.7}$$

where $\mathrm{ch}(V)$ *is the Chern character of* V *and* $\mathrm{td}(M)$ *is the Todd class of* TM.

Notice, in particular, that if V is a line bundle which satisfies the hypotheses of Kodaira vanishing, then (2.7) provides a formula for the dimension of the space of sections $H^0(M;V)$ as a particular intersection number. This is a crucial step in finding the intersection formulas developed in [Th1] for the moduli space of stable bundles, which in turn are the basis for our intersection formulas on moduli spaces of Higgs bundles.

Chapter 3

Equivariant Cohomology and Localization Formulas

Our moduli spaces are particular quotients of an algebraic variety under the action of a reductive algebraic group. A fundamental tool to include the group action into the theory is looking at the *equivariant cohomology* of the variety. This is an extension of the usual cohomology theory, the use of which allows to introduce the celebrated *localization formulas* of Berline-Vergne [BV] and Atiyah-Bott [AB1]; finding and exploiting such formulas for the moduli space of Higgs bundles will be one of the goals of the present thesis.

3.1 Group actions on algebraic varieties

In this chapter we let G be a compact algebraic group, and M a compact manifold of dimension n, on which G acts in an algebraic way, i.e. the action defines an algebraic morphism $G \times M \to M$. We let M^G be the set of fixed points of the action. If $m \in M^G$, then the differential of the action induces an action of G on $T_m M$, i.e. an algebraic representation of G.

Most of the time, we will concentrate on the case of $G = T$ being a compact torus of rank r; this implies that we can describe the action of G on the tangent space to a fixed point $T_m M$ by its weight decomposition

$$T_m M = \bigoplus_{\lambda} V_\lambda$$

where $\lambda \in \Lambda \subseteq \mathfrak{t}^*$ is the weight lattice.

Remark 3.1.1. In fact, more is true. *Cartan's lemma* states that such an action can be linearized: for each fixed point m there is an invariant analytic open neighbourhood U of m, an embedding $G \hookrightarrow GL(n;\mathbb{C})$, and an analytic isomorphism $U \cong \mathbb{C}^n$ which is equivariant under that embedding. In particular, since the fixed locus on $\mathbb{C}^n$ will be a linear subspace, this implies that M^G is smooth.

If the action of G on M is free, then we know that M/G is naturally an algebraic variety; therefore we can naturally define an equivariant cohomology class to be a class of $H(M/G)$. However, in many interesting cases the action on M will not be free, yet we want a cohomology theory which deals with such actions.

The main idea is that if E is a contractible topological space, the homotopy type of $E \times M$ is the same to the one of M, so all cohomological data of the two spaces must coincide. The idea is then to find a contractible [1] space EG with a free G-action, and to define the *Borel model* of M as

$$M_G := (EG \times M)/G$$

where the G-action on the product is the diagonal one. Notice that this is a quotient of a topological space with a free G-action.

Definition 3.1.2. The *equivariant cohomology* of M is the singular cohomology of its Borel model; so we define

$$H_G(M) := H(M_G; \mathbb{C}).$$

Remark 3.1.3. Since the classifying space is typically an infinite-dimensional manifold, some care must be exercised in defining the cohomology of M_G. We will not discuss this technicality in detail, since the Weil or Cartan models which we will describe in the next section provide a good, finite-dimensional way to compute equivariant cohomology.

Example 5. If the action of G on M is free, then we can show that M_G has the same homotopy type of M/G, thus $H_G(M) \cong H(M/G)$. At the other extreme, consider the case when $M = \{pt\}$ is a point where G acts trivially. Then we have

$$M_G \simeq EG/G := BG.$$

Let M and N be two manifolds with a G-action, and let $f : M \to N$ be a G-equivariant map. Then by looking at the Borel models we deduce that there is a pull-back map

$$f_G^* : H_G(N) \to H_G(M).$$

The space BG of Example 5 is called the *classifying space* of the group G. Its cohomology ring $H(BG) \cong H_G(pt)$ is of fundamental importance in the equivariant theory: indeed, since the constant map $M \to \{pt\}$ is clearly G-equivariant, we deduce that $H_G(M)$ is always an $H(BG)$-module.

Remark 3.1.4 (Push-forwards)**.** Let $f : N \to M$ be a map of compact complex manifolds and let $\dim M - \dim N = q$. Associated to f there is a *push-foward* map

$$f_* : H^*(N) \to H^{*+q}(M),$$

which can be defined by applying Poincaré duality on both sides and applying the classical push-forward in homology. In the case f is a fibration, f_* corresponds to integration along the fibers; in the case f is an embedding, then letting $\Phi : H^{*-q}(N) \xrightarrow{\sim} H^*(M, M \setminus N)$ be the Thom isomorphism and $j^* : H^*(M, M \setminus N) \to H^*(M)$ be the map induced by the inclusion of the pair $(M, \emptyset)$, then $f_* = j^* \circ \Phi$. In the most general setting, these are the so-called Umkehr maps, an axiomatic treatment of which can be found in [CK].

[1]More precisely, we only require it to be weakly contractible, i.e. with all homotopy groups trivial.

Since all this can be made G-equivariant, we have an analogous notion of *equivariant push-forward*, which we still denote by the same symbol

$$f_* : H^*_G(N) \to H^{*+q}_G(M).$$

It satisfies the same functorial properties of the standard push-forward map: it is covariant and it is a homomorphism of $H^*_G(N)$-modules, i.e.

$$f_*(\alpha f^*\beta) = (f_*\alpha)\beta.$$

It can be shown that, if $f : N \hookrightarrow M$ is an embedding and ν_N is the normal bundle to N in M, then

$$f^* f_* 1 = e(\nu_N) \tag{3.1}$$

the Euler class of ν_N. Similarly, in the equivariant setting, we have the same formula involving the equivariant Euler class. Formula (3.1) is at the center of localization formulas, both in the classical and equivariant setting.

Remark 3.1.5. It can be shown that in the case $G = T$ is a torus of rank r (either compact or complex), then

$$H(BT) \cong \mathbb{C}[u_1, \dots, u_r].$$

We will mostly be interested, when talking about the moduli space of Higgs bundles, in the case $T = \mathbb{C}^*$, so that $H_T(M)$ will be a $\mathbb{C}[u]$-algebra.

3.2 The Weil and Cartan models for Equivariant Cohomology

Sometimes, computing the equivariant cohomology of G-manifolds by explicitly finding their Borel model can be difficult. The Weil and the Cartan models allow to describe the ring $H_G(M)$ in a more direct way. Except for Theorem 3.3.2, we refer to Chapters 1-4 of [GS] for the proofs of the theorems contained in this section.

Definition 3.2.1. Let G be a compact reductive group and let $\mathfrak{g}$ be its Lie algebra. The *Weil algebra* $W\mathfrak{g}$ is defined as

$$W\mathfrak{g} = \wedge\mathfrak{g}^* \otimes S\mathfrak{g}^*$$

i.e. the tensor product of the exterior and symmetric algebras. We make it a graded algebra by stating that the elements of $\wedge^1\mathfrak{g}^*$ have degree one, and the ones of $S^1\mathfrak{g}^*$ have degree 2.

Fixing a basis $\{e_1, \dots, e_n\}$ of $\mathfrak{g}^*$, we define the *contractions* ι_a for $a = 1, \dots, n$ as linear endomorphisms of $W\mathfrak{g}$ such that, letting $\theta^i \in \wedge^1\mathfrak{g}^*$ and $z^i \in S^1\mathfrak{g}^*$ the corresponding basis elements,

$$\iota_a\theta^b = \delta^b_a, \ \ \iota_a z^b = 0$$

and extending it to be a degree 1 derivation; i.e.

$$\iota_a(xy) = \iota_a(x)y + (-1)^{|x|}x\iota_a(y), \text{ for } x, y \in W\mathfrak{g}^*.$$

We say that an element $x \in W\mathfrak{g}^*$ is *horizontal* if $\iota_a x = 0$ for all a.

We also make $W\mathfrak{g}^*$ a *differential graded algebra* by defining the differential d as

$$d\theta^a = z^a - \frac{1}{2} f^a_{jk}\theta^j\theta^k, \ dz^a = -f^a_{jk}\theta^j z^k$$

where the Einstein's sum convention is used, and f^c_{ab} are the Lie algebra structure constants such that $[e_a, e_b] = f^c_{ab} e_c$. We extend then the differential to $W\mathfrak{g}^*$ by making it a degree 1 derivation just like ι_a. It is easily shown that $d \circ d = 0$.

Finally, we define the *Lie derivatives* L_a for $a = 1, \dots, n$ as

$$L_a := \iota_a d + d\iota_a.$$

It can be shown that L_a is a degree 0 derivation, i.e.

$$L_a(xy) = L_a(x)y + xL_a(y), \text{ for } x, y \in W\mathfrak{g}^*.$$

We say that an element $x \in W\mathfrak{g}^*$ is *basic* if it is horizontal and $L_a x = 0$ for all a. Notice that the subspace of basic elements $(W\mathfrak{g}^*)_{\text{bas}}$ forms a *differential graded subalgebra*; in particular, it is a subcomplex with respect to the differential d, and its cohomology is

$$H((W\mathfrak{g})_{\text{bas}}, d) \cong (S\mathfrak{g}^*)^G$$

Remark 3.2.2. With these definitions, $W\mathfrak{g}^*$ is a G^*-space in the sense of Guillemin-Sternberg [GS].

Notice that by standard differential geometry, if M is a manifold with an action of G, then contractions ι_a and Lie derivatives L_a are naturally defined on the space $\Omega(M)$ of differential forms. This brings us to the following definition.

Definition 3.2.3. Let M be a manifold with a G-action and let $W_G(M) := (M \otimes W\mathfrak{g}^*)_{\text{bas}}$ the basic subcomplex, which we call the *Weil model*.

Theorem 3.2.4. *The cohomology of the Weil model is isomorphic to* $H_G(M)$.

In particular, the cohomology of the basic subcomplex $(W\mathfrak{g})_{\text{bas}}$ is the equivariant cohomology of a point, thus the cohomology of the classifying space for the group G. Thus we have

$$H_G(pt) \cong (S\mathfrak{g}^*)^G$$

where the action of G on $\mathfrak{g}^*$ is the coadjoint one.

The Weil model can be written even more explicitly thanks to the following definition.

Definition 3.2.5. Let M be a manifold with a G-action, define

$$C_G(M) := (S\mathfrak{g}^* \otimes \Omega(M))^G$$

where the action on $\mathfrak{g}^*$ is the adjoint one, and make it a differential algebra by

$$d_G := 1 \otimes d_M - z^a \otimes \iota_a$$

where still we are using the sum convention on repeated indices (it can be verified that this is indeed a differential). It is easy to see that $(C_G(M), d_G)$ defines a complex, which we call the *Cartan model* for the G-manifold M.

Theorem 3.2.6. *The cohomology of the Cartan model is isomorphic to* $H_G(M)$.

3.3 Localization formulas

From now on, we will concentrate to the case in which $G = T$ is a torus. Also, in this section, we assume M is a *compact* complex manifold.

A very useful feature of equivariant cohomology is the existence of *localization formulas* which allow to compute the push-forward maps to a point. We call the *integral* such push-forward map $\int_M : H_T(M) \to H_T(pt)$.

Remark 3.3.1. Although we have used the same notation for usual cohomology and Chow groups, this integral is quite different, in that $\int_M x$ for $x \in H_T(M)$ will not be a number in general, but rather an element of the polynomial algebra $H_T(pt) \cong \mathbb{C}[u_1, \dots, u_r]$.

The main theorem of the present section is the following.

Theorem 3.3.2. *[Localization formula] Let T be a torus acting on a smooth compact complex manifold M. For each connected component $F \subseteq M^T$, let $\iota_F : F \hookrightarrow M$ be the natural embedding, and let ν_P be the normal bundle to F in M. Then for all $\varphi \in H_T(M)$, we have*

$$\int_M \varphi = \sum_{F \subseteq M^T} \int_F \frac{\iota_F^* \varphi}{E(\nu_F)} \tag{3.2}$$

where the sum runs through all connected components of M^T.

Proof. This proof is taken from [AB1] and it has a more "algebraic" flavour. Other proofs can be found in [BV] and [Wi]. Let r be the rank of the torus T, so that

$$H_T(pt) = \mathbb{C}[u_1, \dots, u_r] := R.$$

Let A be an R-module, and define its *support* to be

$$\operatorname{Supp}(A) = \bigcap_{f:\, fA=0} V_f \subseteq \mathfrak{t}^{\mathbb{C}}$$

which is nothing but the variety associated to the torsion elements of A. For $f \in R$ a non-zero polynomial, we denote by R_f the localization of R with respect to the multiplicative set made by the powers of f

$$R_f := \left\{ \frac{r}{f^n} : \; r \in R, \, n \geqslant 0 \right\}$$

and $A_f = A \otimes_R R_f$. Notice that $A_f = 0$ if and only if f is a torsion element; in particular if $\operatorname{Supp}(A) \subseteq V_f$, then $A_f = 0$.

By considering $A = H_G(M)$, we see that this is a graded R-module, with $\deg(u_i) = 2$; by considering the $\mathbb{C}^*$-action $\lambda \cdot h = \lambda^{2q} h$ if $h \in A^q$, and $\lambda \cdot u_i = \lambda^2 u_i$, then we see that $\operatorname{Supp}(A)$ is $\mathbb{C}^*$-invariant, thus it is a cone.

Let $i : M^T \hookrightarrow M$ be the embedding of the fixed locus for the T action. Then, as discussed previously, we have

$$\iota^* \iota_* 1 = e_T(\nu_{M^T}) = \prod_{F \subseteq M^T} e_T(\nu_F) \tag{3.3}$$

the equivariant Euler class of the normal bundle to F in M. To compute it, consider a connected component F of M^T and $p \in F$; then the induced action of T on $(\nu_F)_p$ has no fixed directions by definition of the fixed locus, therefore it decomposes into a direct sum of 2-dimensional nontrivial representations of T (we are considering the *real* normal bundle here). After choosing an orientation of F and M, such representations are described by characters $\lambda_i : T \to U(1)$, which can be written as

$$\lambda_i = \exp\left(2\pi\sqrt{-1}\sum_{j=1}^{r}\lambda_{ij}u_j\right)$$

if we recall that $u_j \in \mathfrak{t}^*$ are the coordinates of $\mathfrak{t}$. Then we have

$$e_T(\nu_F) = \prod_i \sum_{j=1}^{r} \lambda_{ij}u_j \in R.$$

Denote the previous polynomial by f_F and let $f := \prod_{F\subseteq M^T} f_F \in R$. Then, by definition and by (3.3), the map induced by $\iota^*\iota_*$ in the localization by f is invertible, and for all class $\phi \in H_T(M)$, we have

$$\phi = \sum_{F\subseteq M^T} \frac{\iota_*^F \iota_F^* \phi}{e_T(\nu_F)} \tag{3.4}$$

with self-evident notation for ι_F and ι^F. Applying the push-forward to a point $\pi_* : H_T(M) \to R$ (which is by definition the integral $\int_M$) to (3.4) we obtain the result. □

3.4 Examples

We summarize here a few remarkable examples which will be useful in the following sections.

3.4.1 Compact Riemann surfaces

Let $M = \Sigma_g$ be a compact Riemann surface of genus $g \geqslant 0$. Then it is well known that $H^1(\Sigma_g)$ is generated as vector space by cycles a_i and b_i for $i = 1, \ldots, g$, which satisfy the intersection properties

$$a_i a_j = b_i b_j = 0, \; a_i b_j = \delta_{i,j}\omega$$

where $\omega \in H^2(\Sigma_g)$ is the Poincaré dual to $[\Sigma_g] \in H^0(\Sigma_g)$, so that $\int_{\Sigma_g}\omega = 1$ by definition.

Notice that since there is only one generator of even cohomology, namely ω, the Chern class of any complex vector bundle over Σ_g must be an integer multiple of ω. Letting L be a line bundle over Σ_g, we have

$$c_1(L) = (\deg L)\omega,$$

so that the Chern character of L is $1 + (\deg L)\omega$. It is well-known that the degree of the tangent bundle $T\Sigma_g$ is $\chi(M) = 2 - 2g$, so that

$$c_1(T\Sigma_g) = (2 - 2g)\omega.$$

In particular the Todd class of Σ_g is $1 + (1 - g)\omega$. The Hirzebruch-Riemann-Roch formula then reads

$$H^0(\Sigma_g; L) - H^1(\Sigma_g; L) = \int_{\Sigma_g} \mathrm{ch}(L)\mathrm{td}(M) = \deg L - g + 1,$$

which is the classical Riemann-Roch formula for line bundles on a Riemann surface.

3.4.2 Grassmannians

Consider the *Grassmannian* manifold $\mathrm{Gr}(n, k)$, which is the space of k-dimensional subspaces of a fixed complex vector space V of dimension n. It is a compact complex manifold of dimension $k(n - k)$, which comes with two distinguished vector bundles: the *tautological bundle* S, whose fiber at a point $[H] \in \mathrm{Gr}(n, k)$ is the space H itself, and the trivial bundle V of rank n, of which S is naturally a subbundle. The *quotient bundle* Q is the one fitting into the exact sequence

$$0 \to S \to V \to Q \to 0. \tag{3.5}$$

We write the total Chern classes of S and Q as

$$c(S) = 1 + s_1 + \ldots + s_k,$$

$$c(Q) = 1 + q_1 + \ldots + q_{n-k},$$

which are related by $c(S)c(Q) = 1$, because of (3.5). This allows, recursively, to compute the q_i's as polynomial expressions in the s_i's: here are the first ones

$$q_1 = -s_1, \; q_2 = s_1^2 - s_2, \; q_3 = -s_1^3 + 2s_1 s_2 - s_3.$$

Notice that such expressions for q_i are completely formal, therefore are valid even if $i > n - k$, for which $q_i = 0$ trivially. This gives relations in the cohomology ring of $\mathrm{Gr}(n, k)$, and the main theorem in the study of such ring is that these are a complete set of relations, i.e.

$$H^*(\mathrm{Gr}(n, k)) \cong \mathbb{C}[s_1, \ldots, s_k]/(q_{n-k+1}, \ldots, q_n).$$

In particular, the cohomology ring of the Grassmannian is generated by the Chern classes s_i for $i = 1, \ldots, n - k$.

In order to compute geometrically the classes s_i, we need to pick generic sections $\gamma_1, \ldots, \gamma_{k-i+1}$ of S, and check the locus the subspace they generate is not of maximal dimension. To do this, let λ_i for $1 \leqslant i \leqslant k$ be integers such that

$$n - k \geqslant \lambda_1 \geqslant \ldots \geqslant \lambda_k \geqslant 0 \tag{3.6}$$

and define, for $\lambda := (\lambda_1, \dots, \lambda_k)$ and for $\mathcal{F} = (V_i)_{i=0}^n$ an arbitrary complete flag of V, the locus

$$\Sigma_\lambda(\mathcal{V}) = \{[H] \in \mathrm{Gr}(n,k) : \ \dim(V_{n-k+i-\lambda_i} \cap H) \geqslant i, \text{ for all } i \geqslant 0\}.$$

These loci are called *Schubert cells*, and their definition can be considered an assignment of incidence conditions of a subspace H against the spaces of the complete flag $\mathcal{F}$. The study of the products of the cohomology classes $\sigma_\lambda := [\Sigma_\lambda(\mathcal{F})]$ is at the core of *Schubert calculus*, one of the first and most important landmark in intersection theory. As for the Chern classes, it can be shown (see for instance [EH]) that

$$s_i = (-1)^i \sigma_{(1,\dots,1)}$$

where on the subscript there are i ones (and $k-i$ zeros, which are omitted in the writing). Similarly, we have $q_i = \sigma_i$. Notice that each $\sigma_\lambda \in H^{2|\lambda|}(\mathrm{Gr}(n,k))$, thus the top class is $\omega := \sigma_{(n-k,\dots,n-k)}$ (with k entries) so that

$$\int_{\mathrm{Gr}(n,k)} \sigma_{(n-k,\dots,n-k)} = 1.$$

Notice that since the cohomology ring of the Grassmannian is generated by the Chern classes s_i, all intersection theory on $\mathrm{Gr}(n,k)$ is encoded into the generating function

$$F_{n,k}(t_1,\dots,t_k) := \int_{\mathrm{Gr}(n,k)} (1 - t_1 s_1 - \dots - t_k s_k)^{-1}.$$

Now we choose n distinct integers

$$a_1 < a_2 < \dots < a_n$$

and let $T = \mathbb{C}^*$ act on $\mathbb{C}^n$ by

$$z \cdot (x_1, \dots, x_n) = (z^{a_1} x_1, \dots, z^{a_n} x_n),$$

an action which naturally descends to the Grassmannian $\mathrm{Gr}(n,k)$. There are $\binom{n}{k}$ fixed points corresponding to the subspaces spanned by $v_{i_1}, \dots, v_{i_k}$ where the v_i's are the vectors of the canonical basis of $\mathbb{C}^n$. We want to use the localization formula of Theorem 3.3.2 to deal with the intersection theory on $\mathrm{Gr}(n,k)$. Notice that S and Q are naturally T-equivariant bundles: we will denote the corresponding equivariant chern classes still by s_i and q_i.

Then it can be shown as in [Zi] that the localization formula gives

$$F_{n,k}(t_1,\dots,t_k) = \frac{1}{k!} \operatorname*{Res}_{z_1,\dots,z_k=\infty} \frac{\prod_{i \neq j}(z_i - z_j)}{(1 - t_1 e_1 - \dots - t_k e_k) \prod_{i=1}^n \prod_{j=1}^k (a_i - z_j)}$$

where $e_i = e_i(z_1,\dots,z_k)$ are the elementary symmetric polynomials. It is rather surprising that such expression does not depend on the particular choice of the weights a_i. Since we are taking residues at only one point, the order of the residues will not matter. As mentioned, this formula contains all intersection numbers for $\mathrm{Gr}(n,k)$, and can be used to extract information about its cohomology ring. However, the level of difficulty of Schubert calculus suggests that using the expression of $F_{n,k}$ in practice is not so easy.

3.4.3 Symmetric products

In our study of the moduli space of Higgs bundles, the symmetric products of the Riemann surface Σ_g will play a key role, they being identified with some connected components of the fixed locus with respect to the natural $\mathbb{C}^*$-action. What follows are the basic results on their cohomology ring, due to Macdonald [Mac] and also found in [Th1]. For the sake of ease of notation, we will denote the Riemann surface Σ_g by X, and its i-th symmetric product by $X^{(i)}$. The generators of the first cohomology group $H^1(X;\mathbb{Z})$ are a_i, b_i for $i = 1, \dots, g$. We define the *universal divisor* $\Delta \subseteq X_i \times X$ as

$$\Delta = [\{(S, p) \in X_i \times X : \; p \in S\}] \in H^2(X_i \times X; \mathbb{Z})$$

which we decompose in Künneth components as

$$\Delta = \eta + \sum_j (\zeta_i a_i - \xi_i b_i) + i[X]$$

thus yielding classes $\eta \in H^2(X_i;\mathbb{Z})$ and ξ_i, $\zeta_i \in H^1(X_i;\mathbb{Z})$ which generate the cohomology ring of X_i; we write $\sigma_j := \xi_j \zeta_j$ and $\sigma = \sum_j \sigma_j$. Notice that $\sigma_j^2 = 0$, thus if $k \geqslant 0$, $\sigma^k/k!$ is the k-th elementary symmetric polynomial in the σ_j's, which is a sum of $\binom{g}{k}$ monomials. Now, for any multi-index I without repeats, we have

$$\int_{X_i} \eta^{i-|I|} \sigma_I = 1$$

so that for any two formal power series $A(x)$ and $B(x)$,

$$\int_{X_i} A(\eta) \exp(B(\eta)\sigma) = \sum_{k=0}^{\infty} \int_{X_i} A(\eta) \frac{B(\eta)^k \sigma^k}{k!} =$$

$$= \sum_{k=0}^{g} \binom{g}{k} \operatorname*{Res}_{\eta=0} \left(\frac{A(\eta)B(\eta)^k}{\eta^{i-k+1}} d\eta \right) = \operatorname*{Res}_{\eta=0} \frac{d\eta}{\eta^{i+1}} A(\eta)(1+\eta B(\eta))^g. \qquad (3.7)$$

Since the even cohomology of X_i is generated by η and σ, formula (3.7) encodes all intersection numbers on X_i. It will be used in a following section to prove the residue formulas for equivariant integrals on the moduli space of Higgs bundles.

Chapter 4

Vector Bundles

In this section, we will introduce the main concepts related to vector bundles which will be used in the present thesis. In particular, the focus will be on the definition and basic properties of the moduli space of stable bundles over Riemann surfaces, in the case the rank and the degree are coprime: in this case, the moduli space will be a smooth compact complex manifold. We will then briefly investigate the intersection theory on such space through the celebrated Witten's intersection formulas, introduced in [Wi] and proved in several different ways in [Wi], [Th1] and [JK].

4.1 Chern classes

Let V be a rank r vector bundle over a compact smooth manifold M. We have seen in Definition 2.1.4 and in the following Theorem 2.1.5 that we can associate classes $c_i(V) \in A^i(M)$ in the Chow ring of M. Through the homomorphism of Remark 2.2.1 we can see that there are corresponding cohomology classes, still called *Chern classes* and still denoted by $c_i(V) \in H^{2i}(M;\mathbb{Z})$.

If $M = \Sigma_g$ is a compact Riemann surface of genus $g \geqslant 0$, we make the following definition.

Definition 4.1.1. Let V be a vector bundle over a Riemann surface Σ_g. The *degree* of V is defined as

$$\deg(V) := c_1(V) \in \mathbb{Z}$$

under the isomorphism $H^2(\Sigma_g, \mathbb{Z}) \cong \mathbb{Z}$ given by the complex structure of Σ_g.

Remark 4.1.2. One can also define the degree more in general, for a vector bundle over a projective variety. However, the definition above is sufficient for the scope of the present thesis.

One of the most important feature of a vector bundle is its *stability*, which tells us which vector bundles should be considered in the moduli problem.

Definition 4.1.3. Let V be a vector bundle over a Riemann surface Σ_g. Its *slope* is defined as the ratio between the degree and the rank of V:

$$\mu(V) := \frac{\deg(V)}{\mathrm{rk}(V)}.$$

We say that V is *semi-stable* if for all subbundles $W \subseteq V$, we have $\mu(W) \leqslant \mu(V)$. If strict inequality holds for all proper subbundles, we say that V is *stable*.

Stability is essentially a condition about automorphisms of a vector bundle.

Proposition 4.1.4. *Let V be a stable vector bundle. Then the only automorphisms of V are multiplication by a non-zero scalar.*

Proof. The essential step is proving that if U and V are semi-stable bundles and $\mu(U) > \mu(V)$, then there are no nonzero homomorphisms $\phi : U \to V$. Indeed, letting W be the image of ϕ, by (semi-)stability of V and by definition of slope we have

$$\mu(U) \leqslant \mu(W) \leqslant \mu(V) \tag{4.1}$$

thus bringing to a contradiction. Assume now that $\mu(U) = \mu(V)$; if U is stable then (4.1) forces ϕ to be an injective, and if V is stable (4.1) forces ϕ to be surjective. Thus we can conclude that *if V is stable, any endomorphism of V is either 0 or an isomorphism.* This means that $\mathrm{End}(V)$ is a finite-dimensional division algebra over $\mathbb{C}$, therefore $\mathrm{End}(V) \cong \mathbb{C}$ because $\mathbb{C}$ is algebraically closed. □

Lemma 2. *Let V be a vector bundle such that* $\deg(V)$ *and* $\mathrm{rk}(V)$ *are coprime. Then V is semi-stable if and only if it is stable.*

Proof. Assume V is not stable. Then there is a proper $U \subsetneq V$ such that $\mu(U) = \mu(V)$, which means $\mathrm{rk}(U) < \mathrm{rk}(V)$ and

$$\deg(U) \cdot \mathrm{rk}(V) = \deg(V) \cdot \mathrm{rk}(U)$$

which implies that $\deg(V)$ and $\mathrm{rk}(V)$ have a common factor. □

The set of line bundles of degree 0 over Σ_g forms a group under the tensor product; this is a dimension g variety called the *Jacobian* and denoted by $\mathrm{Jac}(\Sigma_g)$. Its importance in our setting lies in the fact that tensorizing a vector bundle E over Σ_g by an element of the Jacobian does not change the rank and degree of E, and preserves its stability.

4.2 The cohomology of the moduli space

The key result states that the moduli problem for vector bundles is "good", meaning that there exists a reasonable variety $\mathcal{N}_g^{ss}(n, d)$ parametrizing all semi-stable vector bundles over Σ_g up to isomorphism. The source of the following proof is [AB], Section 7.

Theorem 4.2.1. *There exists a coarse moduli space $\mathcal{N}_g^{ss}(n, d)$ of semi-stable vector bundles, which is a compact singular complex variety of dimension $n^2(g-1)+1$. The moduli space of stable vector bundles is a smooth subvariety.*

Sketch of the proof. Fix a vector bundle E of rank n and degree d on Σ_g. In order to describe a holomorphic structure on E, we need to specify the $\overline{\partial}$-operator so that the holomorphic sections of E will be the ones such that $\overline{\partial} u = 0$. Any such operator will be of the form

$$\overline{\partial} = \overline{\partial}_0 + B$$

with $\overline{\partial}_0$ the anti-holomorphic Cauchy-Riemann operator defined by the complex structure of Σ_g, and $B \in \Omega^{0,1}(\mathrm{End}(E))$. Therefore, letting $\mathcal{C}(E)$ be the space of such operators, then $\mathcal{C}(E)$ is an affine space based on the vector space $\Omega^{0,1}(\mathrm{End}(E))$. The automorphism group $\mathrm{Aut}(E)$ acts naturally on $\mathcal{C}(E)$: our moduli space will be, by definition, the quotient of the subset $\mathcal{C}_{ss}(E)$ corresponding to the semi-stable bundles with respect to the action of $\mathrm{Aut}(E)$.

The last step is to prove that such a quotient is a compact algebraic variety, i.e. that the conormal bundle to the semistable locus has a well-defined dimension. First of all, the normal bundle to an orbit of $\mathcal{C}(E)$ is $H^1(\Sigma_g;\mathrm{End}(E))$ (see [AB]); still for [AB], the conormal bundle to the semistable locus is also $H^1(\Sigma_g;\mathrm{End}(E))$. The complement of $\mathcal{C}_{ss}(E)$ is a union of locally closed orbits (which form the *Shatz stratification*). The dimension is computed with the Hirzebruch-Riemann-Roch theorem which says

$$h^0(\Sigma_g;\mathrm{End}(E)) - h^1(\Sigma_g;\mathrm{End}(E)) = c_1(\mathrm{End}(E)) + \mathrm{rk}(\mathrm{End}(E))(1-g).$$

For stable bundles, the proof of Proposition 4.1.4 implies that $h^0(\Sigma_g;\mathrm{End}(E)) = 1$ is given by scalar multiplications, and finally $c_1(\mathrm{End}(E)) = c_1(E^* \otimes E) = 0$. Since the stable locus is an open subset of the semi-stable one, we obtain the dimension of the statement of the theorem. Still by proposition 4.1.4, the bundles on the stable locus have no nontrivial automorphisms, thus the isotropy group is constantly $\mathbb{C}^*$ on such locus and the quotient will be smooth. □

Since we have showed in Lemma 2 that if $(n,d) = 1$, then semi-stability and stability coincide, we immediately have the following corollary.

Corollary 4.2.2. *If $(n,d) = 1$, the moduli space of semi-stable bundles $\mathcal{N}_g(n,d)$ is a smooth compact complex variety.*

Actually, when it comes to cohomology, the moduli space $\mathcal{N}_g(n,d)$ only depends on the remainder of d modulo n, since we have the following.

Proposition 4.2.3. *For any integer m, there is an isomorphism*

$$\mathcal{N}_g(n,d) \simeq \mathcal{N}_g(n,d+mn).$$

Proof. Fixing a line bundle L of degree 1 on Σ_g, and $E \in \mathcal{N}_g(n,d)$, then $E \otimes L^{\otimes m}$ has rank n and degree $d+mn$, so that tensorizing with $L^{\otimes m}$ gives the desired isomorphism. □

In this thesis, we will concentrate on the case of rank 2, odd-degree vector bundles over Riemann surfaces and with fixed determinant.

Definition 4.2.4. Fix a line bundle Λ on Σ_g of degree 1. We denote by $\mathcal{N}_g$ the subvariety of $\mathcal{N}_g(2,1)$ whose elements have determinant Λ.

Notice that $\mathcal{N}_g$ is the fiber of the determinant map $\mathcal{N}_g(2,1) \to \mathrm{Pic}^1(\Sigma_g)$. Proposition 4.2.3 says that there is nothing special about setting $d = 1$: any odd degree will yield isomorphic moduli space.

Since $\mathcal{N}_g(n,d)$ is a fine moduli space, there exists a *universal vector bundle*

$$\mathbf{E}_g \to \mathcal{N}_g(n,d) \times \Sigma_g$$

such that for all $E \in \mathcal{N}_g(n,d)$, $(\mathbf{E}_g)|_{E\times\Sigma_g} \simeq E$. Actually, such bundle is not uniquely defined: letting $\pi : \mathcal{N}_g(n,d) \times \Sigma_g \to \mathcal{N}_g(n,d)$ be the projection, and letting $\mathcal{L}$ be any line bundle over $\mathcal{N}_g(n,d)$, the tensorized $\mathbf{E}_g \otimes \pi^*\mathcal{L}$ is another universal bundle.

In the case of rank 2 stable bundles, however, we have an explicit way to describe the generators of the cohomology ring. Indeed, the vector bundle $\mathrm{End}(\mathbf{E}_g)$ is well-defined, and its first Chern class is 0. Letting $\omega \in H^2(\Sigma_g)$ and $e_i \in H^1(\Sigma_g)$ for $i = 1, \ldots, 2g$ be a standard basis of the cohomology of the Riemann surface, we can use the Künneth theorem to write

$$c_2(\mathrm{End}(\mathbf{E}_g)) = 2\alpha \otimes \omega + 4\sum_{i=1}^{2g} \psi_i \otimes e_i - \beta \otimes 1 \in H^*(\mathcal{N}_g) \otimes H^*(\Sigma_g). \tag{4.2}$$

Then we have the following central theorem, which is a particular case of a more general

Theorem 4.2.5. *The Künneth components $\alpha \in H^2(\mathcal{N}_g)$, $\beta \in H^4(\mathcal{N}_g)$ and $\psi \in H^3(\mathcal{N}_g)$ defined in* (4.2) *generate the cohomology ring of $\mathcal{N}_g$.*

This means that there is a surjective map $\mathbb{C}[\alpha,\beta] \otimes \wedge[\psi_i] \twoheadrightarrow H^*(\mathcal{N}_g)$. In order to fully describe the cohomology ring of $H^*(\mathcal{N}_g)$, one needs then to find the kernel of such map. The first result in this direction is *Mumford's conjecture*, which we state as in Zagier [Z].

Theorem 4.2.6. *Let $\pi : \mathcal{N}_{4g-3} \times \Sigma_g \to \mathcal{N}_{4g-3}$ be the first projection, then the kernel of the map $\mathbb{C}[\alpha,\beta] \otimes \wedge[\psi_i] \twoheadrightarrow H^*(\mathcal{N}_g)$ is generated by the Chern classes $c_i(\pi_!\mathbf{E}_{4g-3})$, written in terms of the α, β and ψ_j, for $i \geqslant 2g$.*

Instead of explaining the direct proof of this result, we will link it to the intersection theory on the moduli space; this approach is present in [Th1] and [Z].

4.3 Intersection formulas on $\mathcal{N}_g$

A major breakthrough in the study of the cohomology ring of $\mathcal{N}_g$ is found in [Th1], in which the intersection numbers are obtained by the use of the *Verlinde formula*.

The *Quillen determinant line bundle* $\mathcal{L}$ on the space of holomorphic structures $\mathcal{C}(E)$ on a fixed complex vector bundle $E \to \Sigma_g$ has been introduced in [Q]. Its fiber at an operator ∂ is

$$\mathcal{L}_\partial = \wedge^{\mathrm{top}}\left(\mathrm{Ker}\,\overline{\partial}\right)^* \otimes \wedge^{\mathrm{top}}\left(\mathrm{Coker}\,\overline{\partial}\right).$$

It can be shown (see [Th1]) that $\mathcal{L}^{\otimes 2}$ passes to the quotient, defining a line bundle on $\mathcal{N}_g$ which we call L.

Theorem 4.3.1 (Verlinde formula)**.** *For k even and $g \geqslant 2$, we have*

$$h^0(\mathcal{N}_g; L^{k/2}) = \left(\frac{k+2}{2}\right)^{g-1} \sum_{m=1}^{k+1} \frac{(-1)^{m+1}}{(\sin\frac{m\pi}{k+2})^{2g-2}}. \tag{4.3}$$

Notice that already the fact that the sum in (4.3) gives an integer number is not so obvious.

It can be shown [N] that the canonical bundle of $\mathcal{N}_g$ is negative, therefore the Hirzebruch-Riemann-Roch formula (2.7), together with Kodaira vanishing theorem 2.3.1, implies that

$$h^0\left(\mathcal{N}_g; L^{k/2}\right) = \int_{\mathcal{N}_g} \mathrm{ch}(L^{k/2})\mathrm{td}\left(\mathcal{N}_g\right).$$

Still by [N], we have that $c_1(L) = \alpha$ and, in the notation of (4.2),

$$\mathrm{td}(\mathcal{N}_g) = \exp(\alpha)\left(\frac{\frac{1}{2}\sqrt{\beta}}{\sinh\frac{1}{2}\sqrt{\beta}}\right)^{2g-2}$$

thus eventually, after some straightforward computing (see [Th1] for the details), we have

$$\int_{\mathcal{N}_g} \alpha^m \beta^n = (-1)^{g-1}\frac{m!}{(m-g+1)!}2^{2g-2}(2^{m-g+1}-2)B_{m-g+1} \tag{4.4}$$

where B_i is the i-th Bernoulli number; of course, this formula is valid only if $m + 2n = 3g - 3$, otherwise the integral is zero for degree reasons.

In order to compute the missing integrals, i.e. the ones involving the ψ_i's we introduce the classes $\gamma_i := \psi_i\psi_{i+g}$ for $i = 1, \dots, g$ and

$$\gamma := -2\sum_{i=1}^{g} \gamma_i. \tag{4.5}$$

The following Proposition, found in Thaddeus [Th1], allows us to compute all integrals on $\mathcal{N}_g$.

Proposition 4.3.2. *All integrals on* $\mathcal{N}_g$ *are zero except the ones of the form*

$$\int_{\mathcal{N}_g} \alpha^m \beta^n \gamma_{i_1}\cdots\gamma_{i_p}$$

with $m + 2n + 3p = 3g - 3$*. Moreover,*

$$\int_{\mathcal{N}_g} \alpha^m\beta^n\gamma_{i_1}\cdots\gamma_{i_p} = \frac{(g-p)!}{(-2)^p g!}\int_{\mathcal{N}_g}\alpha^m\beta^n\gamma^p = \frac{(g-p)!}{(-2)^{p-1}(g-1)!}\int_{\mathcal{N}_{g-1}}\alpha^m\beta^n\gamma^{p-1}$$

where the last equality is valid for $g \geqslant 3$*.*

Actually, we are interested in a slightly different way to compute integrals on $\mathcal{N}_g$. More specifically, we want a generating function which produces the integrals on $\mathcal{N}_g$ as a function of g, so that no recursive calculation as the one in Proposition 4.3.2 will be needed.

This was the approach taken by Witten in [Wi] to compute general integrals over moduli spaces of flat connections. We use the results in [Z] and modify them in order to comply with the notations we are using, that are fundamentally the ones of [Th1].

We need one more piece of notation, which will also be used and re-introduced in Section 5 when talking about equivariant integrals on the moduli space of Higgs bundles. Notice that if $P \in \mathbb{C}[y^2]$ is a polynomial, and $\widetilde{P} \in \mathbb{C}[y]$ is such that $P(y) = \widetilde{P}(-y^2)$ then we can write a Künneth component decomposition, similar to the one of (4.2):

$$\tilde{P}(c_2(\mathrm{End}(\mathbf{E}))) = P_{(2)} \otimes \omega + \sum_{i=1}^{2g} P_{(e_i)} \otimes e_i + P_{(0)} \otimes 1 \in H^*(\mathcal{N}_g) \otimes H^*(C) \qquad (4.6)$$

which defines a family of cohomology classes $P_{(2)}$, $P_{(e_i)}$ and $P_{(0)}$ for all polynomials $P \in \mathbb{C}[y^2]$. Formula (4.2) is a particular case of (4.6) in the case $P = -y^2$.

It is easy to see that

$$[-y^2/2]_{(2)} = \alpha,\ [y^2]_{(0)} = \beta,\ [-y^2/4]_{(e_i)} = \psi_i,\ [-y^4/4]_{(2)} = \alpha\beta + 4\gamma, \qquad (4.7)$$

To show this, note that, for polynomials $P, Q \in \mathbb{C}[y^2]$, we have

$$[PQ]_{(2)} = P_{(2)}Q_{(0)} + P_{(0)}Q_{(2)} + \sum_{i=1}^{g} \left(P_{(e_i)}Q_{(e_{i+g})} - P_{(e_{i+g})}Q_{(e_i)} \right).$$

Then we obtain (4.7) by setting $P = y^2/2$ and $Q = -y^2/2$ in this formula.

Remark 4.3.3. In general, for any polynomial $P \in \mathbb{C}[y^2]$, we can write the contraction $P_{(2)}$ in terms of classes α, β and γ in the following way:

$$P_{(2)} = \frac{2\gamma - \alpha\beta}{\beta\sqrt{\beta}} P'(\sqrt{\beta}) - \frac{2\gamma}{\beta} P''(\sqrt{\beta}). \qquad (4.8)$$

To show this, set $t = y^2$. Then by the Leibnitz rule we have

$$(t^k)'' = k(k-1)t^{k-2}(t')^2 + kt^{k-1}t''.$$

Now by substituting $(t')^2 \to -8\gamma$, $t'' \mapsto -2\alpha$, $t \to \beta$ we deduce that if $Q(t) = P(\sqrt{t})$, then

$$P_{(2)} = -2\alpha Q'(\beta) - 8\gamma Q''(\beta),$$

where the derivatives of Q are taken with respect to the variable t. The formula follows from the identities

$$\frac{d}{dt} = \frac{1}{2y}\frac{d}{dy}, \quad \frac{d^2}{dt^2} = \frac{1}{4y^2}\left(\frac{d^2}{dy^2} - \frac{1}{y}\frac{d}{dy}\right).$$

Using the notations just introduced, we can write the final formula for the integrals on $\mathcal{N}_g$ following the results found in [Z]. The following Theorem will be of fundamental importance for computing the equivariant integrals on the moduli space of Higgs bundles, of which $\mathcal{N}_g$ is a connected component of the fixed locus for the natural $\mathbb{C}^*$-action.

Theorem 4.3.4. *Let* $T \in \mathbb{C}[y^2]$ *and* $P \in y^2\mathbb{C}[y^2]$. *Then*

$$\int_{\mathcal{N}_g} T_{(0)} \exp(P_{(2)}) = \operatorname*{Res}_{y=0} \frac{2^{2g-1}T(y)P''(y)^g}{y^{2g-2}\left[\exp(P'(y)) - \exp(-P'(y))\right]} \qquad (4.9)$$

Proof. Changing variables via $Q(T) = y^2$ in Proposition 2 of [Z], we obtain

$$\int_{\mathcal{N}_g} f(\beta)e^{u(\beta)\alpha + w(\beta)\gamma^*} = \operatorname*{Res}_{y=0} \frac{(-4)^{g-1} f(y^2) u(y^2)^g}{y^{2g-2} \sinh(yu(y^2) + y^3 w(y^2))}. \tag{4.10}$$

Observe that substituting $\gamma^* = \alpha\beta - 2\gamma$ in (4.8) [1] one arrives at

$$P_{(2)} = -\alpha P''(\sqrt{\beta}) + \gamma^* \left(P''(\sqrt{\beta})/\beta - P'(\sqrt{\beta})/\beta\sqrt{\beta} \right)$$

(note that the coefficient of γ^* is a polynomial since P is divisible by y^2). Finally, performing the substitutions

$$f(\beta) = T(\sqrt{\beta}), \quad u(\beta) = -P''(\sqrt{\beta}), \quad w(\beta) = P''(\sqrt{\beta})/\beta - P'(\sqrt{\beta})/\beta\sqrt{\beta}$$

in (4.10), we obtain (4.9). □

[1]Notice that formula (6) in [Z] matches the one in [Th1] at page 14, but the γ's defined in the two papers differ by a sign. Therefore, we have to apply the formulas in [Z] by changing the sign of γ wherever it appears.

Chapter 5

Higgs Bundles

We now turn our attention onto more complex[1] objects on a Riemann surface, namely Higgs bundles, which in some way extend what has been discussed so far about vector bundles. We let Σ_g be a genus g compact Riemann surface and K be its canonical line bundle.

Definition 5.0.1. A *Higgs bundle* (or *Higgs pair*) of rank n, degree d over Σ_g is a pair (E, Φ) with E a rank n, degree d holomorphic vector bundle over Σ_g and a holomorphic section $\Phi \in \Gamma(\mathrm{End}(E) \otimes K)$, called the *Higgs field*. A *Higgs subbundle* of a Higgs bundle (E, Φ) is a subbundle $F \subseteq E$ such that $(F, \Phi|_F)$ is a Higgs bundle.

Notice that $F \subseteq E$ is a Higgs subbundle if and only if $\Phi(F) \subseteq F \otimes K$. Similarly to the notion of stability for a vector bundle, we say that a Higgs bundle is *semi-stable* if for all Higgs subbundles $F \subseteq E$, we have $\mu(F) \leqslant \mu(E)$. If strict inequality holds for proper subbundles, we say that the Higgs bundle (E, Φ) is *stable*. Lemma 2 implies that if the rank and the degree are coprime, then a Higgs bundle is semistable if and only if it is stable.

Similarly to the case of vector bundles, we have the following.

Theorem 5.0.2. *There exists a coarse moduli space $\mathcal{M}_g^{ss}(n, d)$ of semi-stable rank n, degree d Higgs bundles over Σ_g. It is a complex quasi-projective variety of complex dimension $2n^2(g-1)+2$. The smooth locus $\mathcal{M}_g(n, d)$ is the moduli space of stable Higgs bundles.*

Remark 5.0.3. Unlike the moduli space $\mathcal{N}_g(n, d)$ of stable bundles, $\mathcal{M}_g(n, d)$ is not compact ([Hi1]). This implies that we cannot use the standard techniques of intersection theory to investigate the cohomology of such space. However, we will see in the following sections how, through its equivariant cohomology, we can develop a suitable intersection theory on $\mathcal{M}_g$ as well.

In the present thesis, we will focus on the case of rank 2, degree 1 Higgs bundles: we call $\mathcal{M}_g(GL_2)$ the corresponding moduli space. By associating to each Higgs bundle $(E, \Phi) \in \mathcal{M}_g(GL_2)$ the determinant bundle $\wedge^2 E$, we obtain a map

$$\mathcal{M}_g(GL_2) \to \mathrm{Pic}_1(\Sigma_g).$$

[1]In all senses.

Notice that the fibers of this map are all isomorphic, with isomorphism given by tensorizing by an appropriate element of $\mathrm{Jac}(\Sigma_g)$. We fix any line bundle $L \in \mathrm{Pic}_1(\Sigma_g)$ and we define

$$\mathcal{M}_g(SL_2) := \{(E, \Phi) \in \mathcal{M}_g(GL_2) | \wedge^2 E \simeq L,\ \mathrm{Tr}\, \Phi = 0\}.$$

Finally, if $\Lambda \in \mathrm{Jac}(\Sigma_g)$ is such that $\Lambda^{\otimes 2} \simeq \mathcal{O}$, then $\wedge^2(E \otimes \Lambda) \simeq \wedge^2 E$. This defines an action of the group Γ of 2-torsion points of $\mathrm{Jac}(\Sigma_g)$ on $\mathcal{M}_g(SL_2)$: the main protagonist of what follows will be the quotient space

$$\mathcal{M}_g(PGL_2) := \mathcal{M}_g(SL_2)/\Gamma.$$

We will denote such space simply by $\mathcal{M}_g$. It is a semi-projective variety of dimension $6g - 6$, with orbifold singularities corresponding to the fixed points for the action of $\Gamma \simeq \mathbb{Z}_2^{2g}$.

Remark 5.0.4. Let E be a *stable* rank 2, degree 1 bundle. Then any traceless Higgs field $\Phi \in H^0(\Sigma_g; \mathrm{End}_0(E) \otimes K)$ gives a stable Higgs bundle $(E, \Phi) \in \mathcal{M}_g(SL_2)$. By Serre duality, this is canonically an element of the dual of $H^1(M; \mathrm{End}_0(E))$, which is the tangent bundle to $\mathcal{N}_g$. Therefore, we can see that the cotangent bundle $T^*\mathcal{N}_g$ is canonically embedded in $\mathcal{M}_g(SL_2)$.

Definition 5.0.5. The *Hitchin map* is the morphism

$$h : \mathcal{M}_g(SL_2) \to \mathcal{A} := H^0(\Sigma_g, K^{\otimes 2})$$

given by

$$h(E, \Phi) := \det \Phi.$$

This map clearly descends to the quotient $\mathcal{M}_g$: we still call the induced map the Hitchin map, and we denote it by h as well.

Notice that, by the Riemann-Roch theorem, the dimension of the target of h is $3g-3$, which is exactly half the dimension of $\mathcal{M}_g(SL_2)$. This is not a coincidence: in fact one of the main theorems in the topology of the moduli space of Higgs bundles, already present in [Hi1], states that the Hitchin map displays $\mathcal{M}_g(SL_2)$ as a *complete integrable system* over $\mathcal{A}$, whose fibers are particular *Prym varieties*.

5.1 The $\mathbb{C}^*$-action and the Białynicki-Birula decomposition

Any moduli space of Higgs bundles carries a natural action of $\mathbb{T} := \mathbb{C}^*$, given by

$$\lambda \cdot (E, \Phi) = (E, \lambda\Phi).$$

This will be the main ingredient in our study of the equivariant theory of $\mathcal{M}_g$: the torus action we will consider will be precisely the natural $\mathbb{C}^*$ action described above.

In particular, although we have seen that the moduli space $\mathcal{M}_g$ is not compact, the *fixed locus* $\mathcal{M}_g^{\mathbb{T}}$ *is compact*. This will be of fundamental importance in the definition of the equivariant integral on $\mathcal{M}_g$ and on exploiting its properties. In order to do this, we need a precise description of the fixed locus.

Proposition 5.1.1. *We have*

$$\mathcal{M}_g^{\mathbb{T}} \simeq F_0 \sqcup F_1 \sqcup \ldots \sqcup F_{g-1}$$

where $F_0 \simeq \mathcal{N}_g$ *and for* $1 \leqslant i \leqslant g-1$, $F_i \simeq S^{2g-2i-1}(\Sigma_g)$ *is a symmetric product of* Σ_g.

Proof. Clearly, all Higgs bundles of the form $(E, 0)$ for E a stable bundle are fixed for the $\mathbb{T}$ action. These give a connected component F_0 isomorphic to $\mathcal{N}_g$, which is compatible with the embedding $\mathcal{N}_g \subseteq T^*\mathcal{N}_g \subseteq \mathcal{M}_g$ corresponding to the zero section of the cotangent bundle.

Now consider a bundle of the form $E = L \oplus \Lambda L^{-1}$ with $1 \leqslant \deg L \leqslant g-1$, and a Higgs field Φ such that $\Phi|_{\Lambda L^{-1}} = 0$, and $\Phi|_L$ is a nonzero map of line bundles $\phi : L \to \Lambda L^{-1}K$. We can thus locally write Φ as a lower triangular matrix

$$\Phi = \begin{pmatrix} 0 & 0 \\ \phi & 0 \end{pmatrix}$$

with $\phi \neq 0$. Given $\lambda \in \mathbb{C}^*$, we choose one of its square roots μ, and define $D_\mu = \mathrm{Diag}(\mu^{-1}, \mu)$. Then we immediately see that

$$\lambda\Phi = D_\mu \Phi D_\mu^{-1}$$

which means that $(E, \Phi) \in \mathcal{M}_g^{\mathbb{T}}$, since the Higgs bundle $\lambda \cdot (E, \Phi)$ is isomorphic to (E, Φ) via the isomorphism induced by D_μ.

As the degree of $\Lambda L^{-2}K$ is $2g-1-2\deg L$, such a map gives us a divisor on Σ_g, which is an element of $S^{2g-2\deg L-1}(\Sigma_g)$ of the corresponding symmetric product of the curve. Of course, such a section exists if and only if $\deg(\Lambda L^{-2}K) \geqslant 0$, i.e. if $\deg L \leqslant g-1$. Therefore we have found the other components of the fixed locus as $F_i \cong S^{2g-2i-1}(C)$, $i = 1, \ldots, g-1$.

By looking at the eigenvalues of Φ, it is clear that if $\lambda\Phi$ is similar to Φ, then Φ must be nilpotent. Therefore, the Higgs bundles we found are the only ones in the fixed locus. □

The $\mathbb{C}^*$-action allows to partition $\mathcal{M}_g$ following the flows with respect to it. For $p \in \mathcal{M}_g^{\mathbb{T}}$ we define

$$U_p := \{x \in X | \lim_{\lambda \to 0} \lambda x = p\}$$

upward flow from p and

$$D_p := \{x \in X | \lim_{\lambda \to \infty} \lambda x = p\}$$

downward flow from p. Then

$$\mathcal{M}_g = \bigcup_{p \in \mathcal{M}_g^{\mathbb{T}}} U_p$$

is called the *Bialinycki-Birula decomposition* of $\mathcal{M}$ and

$$\mathcal{C} := \bigcup_{p \in \mathcal{M}_g^{\mathbb{T}}} D_p$$

is the *Bialinycki-Birula decomposition* of $\mathcal{C} \subset \mathcal{M}_g$, called the *core* of $\mathcal{M}$. Let $\mathcal{M}^{\mathbb{T}} = \bigcup_i F_i$ be the decomposition of the fixed point set into connected components as in Proposition 5.1.1; then

$$U_i = \bigcup_{p \in F_i} U_p \subset \mathcal{M}$$

and

$$D_i = \bigcup_{p \in F_i} D_p \subset \mathcal{M}$$

are affine bundles over F_i.

5.2 The (equivariant) cohomology of the moduli space

From now on, when we talk about the equivariant cohomology of $\mathcal{M}_g$, we use the $\mathbb{T}$ action defined in the previous section. Thus we see that $H_{\mathbb{T}}(\mathcal{M}_g)$ is a finitely generated module over the equivariant cohomology of a point $H_{\mathbb{T}}(pt)$, which we will identify with the polynomial ring in a single variable u:

$$H_{\mathbb{T}}(pt) = H^*(B\mathbb{T}) \cong \mathbb{C}[u].$$

It can be seen [HV2] that the Higgs moduli spaces are semi-projective with respect to the $\mathbb{T}$-action, and this, in particular, implies their *formality*: additively, we have a $H^*(B\mathbb{T}) \cong \mathbb{C}[u]$-module isomorphism

$$H^*_{\mathbb{T}}(\mathcal{M}_g) \cong H^*(\mathcal{M}_g) \otimes H^*(B\mathbb{T}) = H^*(\mathcal{M}_g)[u]. \tag{5.1}$$

In [HT1] a universal Higgs bundle endowed with a compatible $\mathbb{T}$-action over $\mathcal{M} \times C$ was constructed. While the rank-2 vector bundle $\mathbf{E}$ is only unique up to tensoring with a line bundle on $\mathcal{M}$, the rank-4, $\mathbb{T}$-equivariant vector bundle $\mathrm{End}(\mathbf{E})$ is unambiguously defined. Now we fix an appropriate basis of $H^*(\Sigma_g)$:

- we denote by 1 the canonical generator of $H^0(\Sigma_g)$;
- we denote by ω the Poincaré dual of the class of a point in $H^2(\Sigma_g)$;
- finally, we choose elements $e_1, \dots, e_{2g} \in H^1(\Sigma_g)$, which form a symplectic basis of $H^1(\Sigma_g)$, i.e. for $i < j$, they satisfy $e_i e_j = \delta_{i+g-j,0} \cdot \omega$.

The Künneth decomposition of the second $\mathbb{T}$-equivariant Chern class of $\mathrm{End}(\mathbf{E})$

$$c_2(\mathrm{End}(\mathbf{E})) = 2\alpha \otimes \omega + 4 \sum_{i=1}^{2g} \psi_i \otimes e_i - \beta \otimes 1 \in H^*_{\mathbb{T}}(\mathcal{M}_g) \otimes H^*(\Sigma_g) \tag{5.2}$$

provides us[2] with well-defined equivariant classes $\alpha \in H^2_{\mathbb{T}}(\mathcal{M}_g)$, $\psi_i \in H^3_{\mathbb{T}}(\mathcal{M}_g)$ and $\beta \in H^4_{\mathbb{T}}(\mathcal{M}_g)$. It is proved in [HT2] that α, ψ_i and β generate the $\mathbb{T}$-equivariant cohomology ring $H^*_{\mathbb{T}}(\mathcal{M}_g)$ as an $H^*(B\mathbb{T})$ algebra. Their images in ordinary cohomology, in other words, the Künneth components of the second non-equivariant

[2]Note that the definition of the universal classes in [CHM, (1.2.10)] as well as in [HV1, (5.1)] do not have the correct scalars. The correct ones are as in (5.2) and as in [HT2, (1.5)]. This discrepancy in the scalars does not effect the arguments in [CHM, HV1].

Chern class of the vector bundle $\mathrm{End}(\mathbf{E})$, generate $H^*(\mathcal{M}_g)$. One can use this observation to give an explicit embedding $H^*(\mathcal{M}_g) \to H^*_{\mathbb{T}}(\mathcal{M}_g)$ yielding (5.1). For this reason, we will use the same notation α,β and ψ_i, for the Künneth components of the second non-equivariant Chern class of $\mathrm{End}(\mathbf{E})$ as well.

We are not going to investigate closely the ring structure of $H_{\mathbb{T}}(\mathcal{M}_g)$, however we have the following classical result, whose proof can be found in [Ki].

Lemma 3. *We have a ring isomorphism* $H_{\mathbb{T}}(\mathcal{M}_g)/uH_{\mathbb{T}}(\mathcal{M}_g) \simeq H(\mathcal{M}_g)$.

5.3 Compactification of the moduli space

A compactification $\overline{\mathcal{M}}_g \supset \mathcal{M}_g$ was constructed in [Ha1]. The construction there was with symplectic cutting, producing a projective variety $\overline{\mathcal{M}}_g$. There is an algebraic version of this construction explained in [HV2] for a general semiprojective variety. This yields the following in our case: $\mathcal{M}_g$ comes with an ample line bundle $L \in \mathrm{Pic}(\mathcal{M}_g) \cong \mathbb{Z}$ the generator of the Picard group. The $\mathbb{T}$-action can be linearized, and with an appropriate linearization we can construct the GIT quotient

$$Z := \mathcal{M}_g/\!/\mathbb{T} = (\mathcal{M}_g \setminus \cup_i D_i)/\mathbb{T} \tag{5.3}$$

This is a projective orbifold of dimension $6g-7$. We can add it as divisor at infinity to compactify $\mathcal{M}$ as follows.

$$\overline{\mathcal{M}}_g := (\mathcal{M}_g \times \mathbb{C})/\!/\mathbb{T} = (\mathcal{M}_g \times \mathbb{C} \setminus (\cup_i D_i) \times \{0\})/\mathbb{T}. \tag{5.4}$$

$\overline{\mathcal{M}}_g$ is a projective orbifold of dimension $6g-6$, which has the decomposition $\overline{\mathcal{M}}_g = \mathcal{M}_g \cup Z$. From the quotient construction, we have the Kirwan map

$$\kappa : H^*_{\mathbb{T}}(\mathcal{M}_g) \cong H^*_{\mathbb{T}}(\mathcal{M}_g \times \mathbb{C}) \to H^*(\overline{\mathcal{M}}_g) \tag{5.5}$$

which is surjective [Ki]. Thus $H^*(\overline{\mathcal{M}}_g)$ is generated by $\kappa(\alpha), \kappa(\psi_i), \kappa(\beta)$ and $\kappa(u)$. By abuse of notation, we will denote these by $\alpha \in H^2(\overline{\mathcal{M}}_g))$, $\psi_i \in H^3(\overline{\mathcal{M}}_g))$, $\beta \in H^4(\overline{\mathcal{M}}_g))$, respectively, and the additional new class by $\eta = \kappa(u) \in H^2(\overline{\mathcal{M}}_g))$.

We also see that the Hitchin map of Definition (5.0.5) is $\mathbb{T}$-equivariant when $\mathbb{T}$ acts on $\mathcal{A}$ with weight 2, and thus we can extend the Hitchin map as follows [Ha1]:

$$\bar{h} : \overline{\mathcal{M}}_g \to \overline{\mathcal{A}},$$

$\overline{\mathcal{A}} \cong \mathbb{P}(\mathcal{A} \times \mathbb{C})$ is a projective space of dimension $3g-3$.

5.4 Equivariant integrals and residue formulas

We now mimic the localization formula for compact varieties to define *equivariant integrals* in our cases. The main idea behind that is the fact that although $\mathcal{M}_g$ is not compact, its fixed locus $\mathcal{M}_g^{\mathbb{T}}$ is (we say that $\mathcal{M}_g$ is a *circle-compact* manifold).

The (formal) *equivariant integral* of a class $x \in H^*_{\mathbb{T}}(\mathcal{M}_g)$ is then defined [HP] as a sum over the fixed-point components

$$\oint_{\mathcal{M}_g} x := \sum_{F \in \pi_0(\mathcal{M}_g^{\mathbb{T}})} \int_F \frac{x|_F}{e_{\mathbb{T}}(N_F)} \in \mathbb{C}[u, u^{-1}] \tag{5.6}$$

where $e_{\mathbb{T}}(N_F)$ is the equivariant Euler class of the normal bundle to $F \subseteq \mathcal{M}_g$. The equivariant integral thus is a functional of degree $-2\dim \mathcal{M}_g$ on $H^*_{\mathbb{T}}(\mathcal{M}_g)$, taking values in Laurent polynomials in u. Sometimes we will allow the class x to be in a completion of $H^*_{\mathbb{T}}(\mathcal{M}_g)$, and in this case, the values will be in Laurent series in u.

With this definition we have the following formula for (usual) integrals over the compact infinity divisor Z, following from results in [Ka], [Le] and [EG].

Lemma 4. *Let* $d_Z = \dim Z = 6g - 7$ *and* $x \in H^{2d_Z}_{\mathbb{T}}(\mathcal{M}_g)$. *Then we have*

$$\int_Z \kappa_Z(x) = -\operatorname*{Res}_{u=0}\left(\oint_{\mathcal{M}_g} x\right) du, \tag{5.7}$$

where $\kappa_Z(x)$ *is the composition of the Kirwan map 5.5 with the restriction to* Z.

In order to have nicer formulas, we reintroduce *Witten's notation* in the context of $\mathcal{M}_g$. Our equivariant intersection formulas will be written using this notation.

We recall that we defined in (5.2) universal classes as Künneth components of the second Chern class of the bundle $\mathrm{End}(\mathbf{E})$). More generally, we let the permutation group S_2 act on $\mathbb{C}[y]$ by $y \mapsto -y$ and take $P \in \mathbb{C}[y]^{S_2}$, an even polynomial. Then there is a polynomial $\tilde{P}$ such that $P(y) = \tilde{P}(-y^2)$. We form the class

$$\tilde{P}(c_2(\mathrm{End}(\mathbf{E}))) \in H^*_{\mathbb{T}}(\mathcal{M} \times C),$$

and we introduce the following notation for its Künneth components:

$$\tilde{P}(c_2(\mathrm{End}(\mathbf{E}))) = P_{(2)} \otimes \omega + \sum_{i=1}^{2g} P_{(e_i)} \otimes e_i + P_{(0)} \otimes 1 \in H^*_{\mathbb{T}}(\mathcal{M}) \otimes H^*(C);$$

this defines a family of equivariant cohomology classes $P_{(2)}, P_{(e_i)}, P_{(0)} \in H^*_{\mathbb{T}}(\mathcal{M})$.

Similarly to what has been seen in the case of vector bundles in (4.7) and Remark 4.3.3, we have

$$[-y^2/2]_{(2)} = \alpha,\ [y^2]_{(0)} = \beta,\ [-y^2/4]_{(e_i)} = \psi_i,\ [-y^4/4]_{(2)} = \alpha\beta + 4\gamma, \tag{5.8}$$

where

$$\gamma = -2\sum_{i=1}^{g} \psi_i \psi_{i+g}. \tag{5.9}$$

Also, just like in Remark 4.3.3, we have

$$P_{(2)} = \frac{2\gamma - \alpha\beta}{\beta\sqrt{\beta}} P'(\sqrt{\beta}) - \frac{2\gamma}{\beta} P''(\sqrt{\beta}). \tag{5.10}$$

This, in particular, implies that all intersection numbers involving the classes α, β and γ are encoded in the numbers of the form $\oint_{\mathcal{M}_g} T_{(0)} \exp(Q_{(2)})$. Indeed, we can choose for example $Q = -Ay^2/2 - Gy^4/4$, getting any class by subsequent derivatives with respect to the formal variables A and G. Now, we have the result analogous to 4.3.2 also in the setting of the moduli space of rank 2 Higgs bundles, if we add polynomials in the class u too. Because of this, we extend the notation $T_{(0)}$ by letting T be a polynomial in y^2 and u; in this case, we mean that $u_{(0)} = u$. Therefore, *all* equivariant intersection numbers on $\mathcal{M}_g$ are encoded in the integral $\oint_{\mathcal{M}_g} T_{(0)} \exp(Q_{(2)})$. In Theorem 7.1.1, we will find a compact formula for such integral, thus paving the way to the intersection theory on $\mathcal{M}_g$.

Chapter 6

Nonabelian Hodge Theory and P=W

One of the groundbreaking results in the study of the moduli space of vector bundles over a Riemann surface Σ_g is the relation it has with the moduli space of flat connections, which in turn can be described via representations of the fundamental group of Σ_g: this is the classical Narasimhan-Seshadri theorem 6.1.4.

An analogous statement, the *nonabelian Hodge correspondence* (NHC), is true for the Higgs moduli space. The case of rank 2 Higgs bundles, which is the one we state in Theorem 6.3.1 is already present in [Hi1], while the treatment of the general case can be found in [Si].

The NHC gives a real analytic diffeomorphism between the Higgs moduli space and the *character variety* $\mathcal{M}_B$, which is naturally a complex affine manifold. It can be seen that the NHC does *not* respect the two algebraic structures (i.e. it is not an algebraic isomorphism), thus the natural question is how the cohomology rings of these two varieties are related through the NHC.

It turns out that the cohomology of $\mathcal{M}_B$ has a nontrivial *mixed Hodge structure*, while the same structure on $\mathcal{M}_g$ is trivial. However, since the NHC gives an isomorphism between the two cohomologies, there should be some particular structure on $\mathcal{M}_g$ which reflects the mixed Hodge structure on $\mathcal{M}_B$. It turns out that the right structure to consider is the so-called *perverse filtration* on $\mathcal{M}_g$, which is closely related to the geometry of the Hitchin map. The link between these two structures is stated in the *P=W theorem*, proven for the first time in [CHM] for rank 2 Higgs bundles. In the same article, the precise relationship between the two structures has been conjectured in all ranks, leading to the *P=W conjecture*, one of the most challenging open problems in the study of Higgs moduli spaces.

6.1 Connections and the Narasimhan-Seshadri theorem

We will first examine the case of vector bundles, which gives the Narasimhan-Seshadri theorem. Let E be a vector bundle of rank r and degree k on a complex manifold M. As already discussed in Section 4, we can define different *holomorphic structures* on E: loosely speaking, one wants to specify which sections of E are holomorphic. Notice that if we have specified a set of local sections for a holomorphic frame $e_1, \dots, e_r$ on an open set U, and if $\alpha_i \in \Omega^i(U)$, then we can define the operator $\overline{\partial}$ on E-valued differential forms by defining it locally as

$$\overline{\partial} \sum_{i=1}^{r} \alpha_i \otimes e_i = \sum_{i=1}^{r} \overline{\partial} \alpha_i \otimes e_i.$$

Indeed, it can be seen that such an operator does not depend on the particular holomorphic frame chosen: it does define a global operator

$$\overline{\partial} : \Omega^{i,j}(M; E) \to \Omega^{i,j+1}(M; E). \tag{6.1}$$

Conversely, if such an operator is defined, then we can say that a section s is holomorphic if $\overline{\partial}(s) = 0$. Therefore, *specifying a holomorphic structure on* E *is equivalent to defining an operator as in* (6.1).

Since M is a complex manifold, we have a Cauchy-Riemann differential operator $\overline{\partial}_0$ on differential forms. It can be shown [AB] that if

$$\overline{\partial} = \overline{\partial}_0 + B$$

then $B \in \Omega^{0,1}(M; \mathrm{End}(E))$. In order to have the condition $\overline{\partial}^2 = 0$, we need B to satisfy the two equations

$$\overline{\partial}_0 B = 0, \quad B \wedge B = 0. \tag{6.2}$$

Notice that if $M = \Sigma_g$ is a compact Riemann surface, the conditions (6.2) are automatically satisfied, therefore B can be chosen arbitrarily. We collect the remarks done so far in a Lemma.

Lemma 5. *The space* $\mathcal{C}(E)$ *of holomoprhic structures on* E *is an affine space, with underlying vector space* $\Omega^{0,1}(M; \mathrm{End}(E))$.

Now, the automorphism group $\mathrm{Aut}(E)$ acts on $\mathcal{C}(E)$ by conjugation of $\overline{\partial}$. After specifying the proper stability conditions as in Definition 4.1.3, the moduli space of semistable bundles is $\mathcal{N}^{ss}(r, k) = \mathcal{C}_{ss}(E)/\mathrm{Aut}(E)$.

Closely related to what we discussed is the concept of connection. Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$, let M be a complex manifold, and let $\pi : P \to M$ be a principal G-bundle over M. A *connection* on P is a form

$$\omega \in \Omega^1(P; \mathfrak{g}),$$

i.e. a form with values in the Lie algebra $\mathfrak{g}$, such that

1. Letting R_g be the right action of G on P along its fibers, we have the relation $R_g^* \omega = \mathrm{Ad}_{g^{-1}} \omega$, where on the right-hand side we are using the adjoint action of G on $\mathfrak{g}$.

2. For each $\xi \in \mathfrak{g}$, letting ξ_P be the fundamental vector field of ξ on P induced by the G-action, we have $\iota_{\xi_M} \omega = \xi$.

For such a form, the commutator $[\omega, \omega] \in \Omega^2(P; \mathfrak{g})$ is well defined: its value on a pair X, Y of vector fields is simply $[\omega(X), \omega(Y)] \in \mathfrak{g}$. We define the *curvature* of the connection ω by

$$F(\omega) := d\omega + [\omega, \omega].$$

We say that a connection is *flat* if its curvature is zero.

By condition (2) of the definition of a connection, we see that ω induces a splitting of the tangent space of P as

$$TP = \operatorname{Ker}\pi_* \oplus \operatorname{Ker}\omega \tag{6.3}$$

The section of $\operatorname{Ker}\omega$ are, by definition, vector fields X such that $\iota_X\omega = 0$: we call such a vector field *horizontal*. Similarly, we call the vector fields of $\operatorname{Ker}\pi_*$ *vertical*: they are simply the ones tangent to the fibers, so that the G-action gives a natural isomorphism $(\operatorname{Ker}\pi_*)_p \simeq \mathfrak{g}$ for all $p \in P$. Notice that π_* gives an isomorphism $\operatorname{Ker}\omega \simeq TM$: we define the *horizontal lift* of a vector field $X \in \mathfrak{X}(M)$ as the corresponding $\widetilde{X} \in \operatorname{Ker}\omega$ under such isomorphism.

Now, if G acts on a vector space V, we can define the *associated vector bundle*

$$E = P \times_G V,$$

so that its sections $\Gamma(M; E)$ can be identified by the G-equivariant maps $P \to V$. Letting $s \in \Gamma(M; E)$ be a section of E, its *covariant derivative* is defined as

$$\nabla s \in \Omega^1(M; E) : \ \iota_X \nabla s = \widetilde{X} \cdot \pi^* s,$$

i.e. the derivative in the $\widetilde{X}$-direction of the function $\pi^* s = s \circ \pi$. In many books and papers, the definition of a *connection* on E coincides with what we have called the covariant derivative ∇. We denote by $\mathcal{A}(E)$ the set of connections on E.

Now assume $G = U(r)$ is the unitary group. Decomposing the covariant derivative by

$$\nabla = \nabla^{1,0} + \nabla^{0,1}$$

following the Hodge decomposition induced by the complex structure of M, one can show [GH, §0.5] that $\nabla^{0,1}$ induces a holomorphic structure on E. Therefore, we have a map

$$\mathcal{A}(E) \to \mathcal{C}(E), \ \ \nabla \mapsto \nabla^{0,1} \tag{6.4}$$

Notice however that since the decomposition of ∇ depends on the metric of M, the map 6.4 also depends on the particular metric on M.

Now we focus on the case $M = \Sigma_g$ is a Riemann surface. Assume that a connection ω is flat; it is a classical result [KN] that that such a connection is determined by the *holonomy*, i.e. a homomorphism

$$\pi_1(\Sigma_g) \to G \tag{6.5}$$

up to conjugation by G. Explicitly, since $\pi_1(\Sigma_g)$ is generated by $2g$ elements a_i, b_i for $i = 1, \dots, g$ with the only relation $\prod_{i=1}^g [a_i, b_i] = e$[1], we have that a flat connection corresponds to a $2g$-uple

$$(A_i, B_i)_{i=1}^g \in G^{2g} \text{ such that } \prod_{i=1}^g [A_i, B_i] = \mathrm{Id}.$$

In the case $G = SU(r)$, the holonomy is a unitary representation of $\pi_1(\Sigma_g)$ of dimension r.

[1]Here we mean the group commutator, i.e. $[x, y] = xyx^{-1}y^{-1}$.

Remark 6.1.1. If the bundle E on the Riemann surface Σ_g admits a flat connection, then it must have degree 0. Indeed, letting A be a connection with curvature $F(A)$, let $\omega \in \Omega^2(\Sigma_g)$ be such that $\pi^*\omega = F(A)$ (such form exists since $F(A)$ is basic), then [AB]

$$\int_{\Sigma_g} \omega = -2\pi i c_1(E).$$

Notice in particular that the integral does not depend on the particular connection A: it is a *topological* datum of the vector bundle E.

Because of Remark 6.1.1, in order to deal with bundles of different degrees, we cannot simply consider a homomorphism such as (6.5); this is why we must introduce *central extensions* of $\pi_1(\Sigma_g)$ and the corresponding notion of *Yang-Mills connection*.

Definition 6.1.2. Let $\star$ be the Hodge star operator induced by the complex structure on Σ_g. A connection A on a principal bundle over Σ_g is called *Yang-Mills*, or *harmonic*, if

$$\nabla_A \star F(A) = 0, \tag{6.6}$$

which are called *Yang-Mills equations*.

In the same way flat connections are associated to representations of the fundamental group as in (6.5), Yang-Mills connections are associated [AB] to representations of the fundamental central extension of $\pi_1(\Sigma_g)$. This allows to define a one-to-one correspondence between such representations and stable bundles of any degree, namely the Narasimhan-Seshadri correspondence.

By the presentation of $\pi_1(\Sigma_g)$, it follows that there exists a universal central extension $\Gamma_{\mathbb{R}}$:

$$0 \to \mathbb{R} \to \Gamma_{\mathbb{R}} \to \pi_1(\Sigma_g) \to 0$$

or, quotienting out by $\mathbb{Z}$,

$$0 \to \mathbb{Z} \to \Gamma_{\mathbb{R}} \to U(1) \times \pi_1(\Sigma_g) \to 0. \tag{6.7}$$

Now we fix a $U(1)$-principal bundle $Q \to \Sigma_g$ equipped with a fixed harmonic connection A with curvature $-2\pi i\omega$, with ω the volume form on M [AB]. Letting $\widetilde{\Sigma}_g \to \Sigma_g$ be the universal covering, then $Q \times_{\Sigma_g} \widetilde{\Sigma}_g$ is a $U(1) \times \pi_1(\Sigma_g)$-bundle, with a harmonic connection with curvature $-2\pi i\omega$. This can in turn seen, via the morphism of (6.7), as a $\Gamma_{\mathbb{R}}$-principal bundle, with harmonic connection of curvature $-2\pi i\omega$. Finally, if

$$\rho : \Gamma_{\mathbb{R}} \to G$$

is a representation of $\Gamma_{\mathbb{R}}$, we have an induced G-connection A_ρ on Σ_g by pushing forward along ρ. It can be shown [AB] that this is a one-to-one correspondence.

Proposition 6.1.3. *The correspondence $\rho \mapsto A_\rho$ induces a bijective correspondence between conjugacy classes of homomorphisms $\Gamma_{\mathbb{R}} \to G$ and equivalence classes of Yang-Mills connections on Σ_g.*

Now the main theorem of the present section, due to Narasimhan and Seshadri [NS], is the following.

Theorem 6.1.4. *Under the correspondence of Proposition 6.1.3, a vector bundle is stable if and only if it comes from an irreducible representation* $\rho : \Gamma_{\mathbb{R}} \to U(r)$.

This allows to represent the moduli space of stable bundles via the corresponding *character variety*, as in the following Proposition, the proof of which can be found in [AB, §6].

Proposition 6.1.5. *For* $G = U(r)$ *we have a real analytic diffeomorphism*

$$\mathcal{N}_g^{ss}(r,k) \simeq \left\{ (A_i, B_i)_{i=1}^g \in U(r)^{2g} \mid \prod_{i=1}^{g} [A_i, B_i] = e^{2k\pi i/r} \right\} / U(r)$$

where the action on the right is by diagonal conjugation.

Remark 6.1.6. Notice that the diffeomorphism of Proposition 6.1.5 is real analytic, but the space on the left is a complex variety. This allows to define a complex structure on the space on the right.

When we turn to Higgs bundles, we have a parallel of Narasimhan-Seshadri theorem, called the nonabelian Hodge correspondence. However, since the corresponding character variety is a complex variety in its own, one can compare the two complex structures.

The NHC is a real analytic diffeomorphism, but since the moduli space of Higgs bundles is not compact, a real analytic diffeomorphism does not necessarily give an algebraic isomorphism. Actually, this is indeed the case, so that the resulting diffeomorphism will transform the two mixed Hodge structures (see Section 6.4) in the two cohomology rings. The precise relation between the two structures is encoded in the P=W theorem.

6.2 Self-duality equations on Riemann surfaces

When Hitchin first introduced the concept of Higgs bundle in [Hi1], he started with the discussion of *self-duality equations* on a Riemann surfaces, showing that the solutions of such equations naturally correspond to *stable* Higgs bundles.

Let Σ_g be a compact, genus $g > 1$ Riemann surface (the condition on the genus will become clear in what follows), and let P be a principal G-bundle over Σ_g with a connection A. We let $V = \mathfrak{g}^{\mathbb{C}}$ and we let G act on it via the adjoint representation. The complex vector bundle $E = P \times_G V$ is called the *complexified adjoint bundle* associated to P. Recall that, since the metric on Σ_g is set, the $(0,1)$-component $\nabla_A^{(0,1)}$ of the covariant derivative of the connection A defines a holomorphic structure on E. A *Higgs field* is by definition a section $\Phi \in \Omega^{1,0}(\Sigma_g; E)$. Letting $F(A)$ to be the curvature of A, we say that the pair (A, Φ) satisfies the *self-duality equations* if

$$\begin{cases} F(A) + [\Phi, \Phi^*] & = 0 \\ \nabla_A^{(0,1)} \Phi & = 0 \end{cases}$$

Here Φ^* is the conjugate transpose in $G^{\mathbb{C}}$ under some unitary representation of G, and since Φ is a 1-form, we have written

$$[\Phi, \Phi^*] = \Phi\Phi^* + \Phi^*\Phi.$$

The first theorem proven in [Hi1] is the following.

Theorem 6.2.1. *[Hi1] Let $G = SO(3)$, P be a principal G-bundle over Σ_g, let E be the complexified adjoint bundle and assume the pair (A, Φ) satisfies the self-duality equations. Then if $L \subseteq E$ is a Φ-invariant subbundle, we have*

$$\deg(V) \leqslant \frac{1}{2}\deg(E).$$

In order to define a moduli space, one needs to specify the notion of automorphism of the objects we want to parametrize. In this case, we define a *gauge transformation* to be a smooth section of the bundle of groups $P \times_G G$ where G acts on itself by conjugation. A gauge transformation acts in a natural way on (A, Φ) by conjugating Φ and by classical gauge action [KN] on the connection A; such action preserves the self-duality equations. We call *gauge equivalent* two pairs which differ by a gauge transformation.

Theorem 6.2.2. *[Hi1, 2.7] If (A_1, Φ_1) and (A_2, Φ_2) are isomorphic via an automorphism of E which brings ∇_{A_1} into ∇_{A_2} and Φ_1 to Φ_2, then (A_1, Φ_1) and (A_2, Φ_2) are gauge-equivalent.*

From this basic ingredient, Hitchin deduces the following.

Theorem 6.2.3. *[Hi1, 5.7] Let E be a rank 2, odd degree vector bundle over a compact Riemann surface Σ_g of genus $g > 1$, and let $\mathcal{M}_g$ be the space of solutions to self-duality equations on E with fixed induced connection on $\bigwedge^2 E$. Then $\mathcal{M}_g$ is a smooth complex manifold of dimension $6g - 6$.*

Therefore, Theorem 6.2.3 provides a way to describe Higgs bundles over a Riemann surface, in terms of the self-duality equations.

6.3 The nonabelian Hodge correspondence

Notice that Theorem 6.2.1 allows us to use interchangebly the notion of stable Higgs bundles with fixed determinant, and the one of solutions to self-duality equations. Therefore, for this section, we are justified to switch back to Higgs bundles.

The key observation contained in [Hi1] is that if (A, Φ) is a solution to the self-duality equations, then the $PSL(2; \mathbb{C})$-connection

$$\nabla := \nabla_A + \Phi + \Phi^*$$

is flat and irreducible. Moreover, if any two such connections are equivalent under complex gauge transformations, then the original pairs are gauge-equivalent.

The last ingredient is a theorem by Donaldson [D] which states that any irreducible flat connections is complex gauge-equivalent to a connection of the form $A + \psi$ where (A, ψ) satisfy the self-duality equations.

Therefore, by repeating the argument done in the case of stable bundles, we arrive to the following theorem, which is a generalization of Proposition 6.1.5 in the case of Higgs bundles.

Theorem 6.3.1 (Nonabelian Hodge correspondence)**.** *The moduli space of stable* SL_2*-Higgs bundles of odd degree on a Riemann surface* Σ_g *with* $g \geqslant 2$*, is diffeomorphic to the moduli space of flat* $SL(2;\mathbb{C})$*-connections:*

$$\mathcal{M}_B := \left\{ (A_i, B_i)_{i=1}^g \in SL(2;\mathbb{C})^{2g} \mid \prod_{i=1}^{g} [A_i, B_i] = -\mathrm{Id} \right\} / SL(2;\mathbb{C}) \tag{6.8}$$

where the quotient is by the diagonal conjugation action.

This space $\mathcal{M}_B$ is called the *Betti moduli space*, and it is an affine variety. In the following, we will see that the algebraic structure of $\mathcal{M}_B$ is very different from the one of $\mathcal{M}$. Such a difference will bring naturally to the P=W theorem.

6.4 Mixed Hodge structures on smooth algebraic varieties

The mixed Hodge structure of an algebraic variety is a way to generalize the concept of Hodge structure to the non-compact or non-smooth cases. Since we will only be concerned to the mixed Hodge structure of $\mathcal{M}_B$ or $\mathcal{M}_g$, we will just discuss the smooth, non-compact case.

The first, fundamental theorem is due to Deligne [De]; it states the existence of mixed Hodge structures for complex algebraic varieties.

Theorem 6.4.1. *Let* X *be a complex algebraic variety. For each* j*, there exists an increasing* weight filtration

$$0 = W_{-1} \subseteq W_0 \subseteq W_1 \subseteq \ldots \subseteq W_{2j} = H^j(X;\mathbb{Q})$$

and a decreasing Hodge filtration

$$H^j(X;\mathbb{Q}) = F^0 \supseteq F^1 \supseteq \ldots \supseteq F^m \supseteq F^{m+1} = 0$$

such that the filtration induced by F *on the complexification of the graded pieces of the weight filtration* $\mathrm{Gr}_l^W := W_l/W_{l-1}$ *is such that, for all* $p = 0, \ldots, l$*, we have*

$$\mathrm{Gr}_l^{W^{\mathbb{C}}} = F^p \mathrm{Gr}_l^{W^{\mathbb{C}}} \oplus \overline{F^{l-p+1} \mathrm{Gr}_l^{W^{\mathbb{C}}}}$$

This means that the Hodge filtration $F^\bullet$ endows the graded piece Gr_l^W with a *pure Hodge structure of weight* l. It is shown in the articles [De] that such structure is compatible with maps induced by algebraic morphisms $f : X \to Y$, with the Künneth isomorphism and with cup products. As mentioned, we will focus on the case of smooth complex varieties.

Proposition 6.4.2. *Assume* X *is smooth connected of complex dimension* d*. Then we have the following.*

- *The forgetful map* $H_c^k(X) \to H^k(X)$ *is compatible with mixed Hodge structures.*
- $W_{j-1}H^j(X) = 0$*, and the* pure part

  $$PH^*(X) := \oplus_k W_k H^k(X) \subseteq H^k(X)$$

  is a subring.

- *If* $i : X \to Y$ *is a smooth compactification, then* $\mathrm{Im}(i^*) = PH^*(X)$.
- *The Poincaré duality*

$$H^k(X) \times H_c^{2d-k}(X) \to H_c^{2d}(X) \cong \mathbb{Q}(-d)$$

is compatible with mixed Hodge structures, where $\mathbb{Q}(-d)$ *is the pure mixed Hodge structure on* $\mathbb{Q}$ *with weight* $2d$ *and Hodge filtration* $F^d = \mathbb{Q}$, $F^{d+1} = 0$.

In particular, we have $W_{j+1}H_c^j(X) \simeq H_c^j(X)$.

One can apply this theory to the case $X = \mathcal{M}_B$ the Betti moduli space of (6.8). We will not explain in detail the computation of the weight and Hodge filtration for $\mathcal{M}_B$, and we refer to [HV1] for the following result.

Theorem 6.4.3. *The classes* $\alpha \in H^2(\mathcal{M}_B)$, $\beta \in H^4(\mathcal{M}_B)$ *and* $\psi_i \in H^3(\mathcal{M}_B)$ *have all homogeneous weight* 4*. Therefore,* $\gamma \in H^6(\mathcal{M}_B)$ *has homogeneous weight* 8*. Moreover, the pure part of* $H^*(\mathcal{M}_B)$ *is generated by* β.

Recall that the classes α, ψ_i and β have been defined in (5.2) on the moduli space of Higgs bundles $\mathcal{M}_g$. Since this is diffeomorphic to $\mathcal{M}_B$ via the nonabelian Hodge correspondence of Theorem 6.3.1, they are naturally cohomology classes of $\mathcal{M}_B$ as well.

6.5 The perverse filtration and the P=W theorem

The situation is very different in the case of the moduli space of Higgs bundles $\mathcal{M}_g$. Indeed, it is shown in [CM] that the mixed Hodge structure carried by $\mathcal{M}_g$ is *pure*, i.e. $H^*(\mathcal{M}_g) = PH^*(\mathcal{M}_g)$. In particular, α has weight 2, the ψ_i have weight 3 and β has weight 4. This already implies that the nonabelian Hodge correspondence is not an algebraic isomorphism.

Under the nonabelian Hodge correspondence of Theorem 6.3.1, the weight filtration of $\mathcal{M}_B$ is carried to a filtration in the cohomology ring of $\mathcal{M}_g$: the natural question is to describe geometrically such filtration. The answer to such question has been given by De Cataldo, Hausel and Migliorini in [CHM] in the case in exam, and conjectured for the moduli space of Higgs bundles of all ranks in the same article.

Definition 6.5.1. Let $h : \mathcal{M}_g \to \mathcal{A} \simeq \mathbb{C}^{3g-3}$ be the Hitchin map of Definition 5.0.5, and for $s \geqslant 0$ let $\Lambda^s \subseteq \mathcal{A}$ be a general s-dimensional subspace. The *perverse filtration* on the cohomology group $H^j(\mathcal{M}_g)$ is defined as

$$P_pH^j(\mathcal{M}_g) := \mathrm{Ker}\left\{H^j(\mathcal{M}_g) \to H^j(h^{-1}(\Lambda^{j-p-1}))\right\} \tag{6.9}$$

The main theorem proven in [CHM] is the celebrated P=W in rank 2.

Theorem 6.5.2. *Under the isomorphism given by the nonabelian Hodge correspondence, we have*

$$W_{2k}H^*(\mathcal{M}_B) = W_{2k+1}H^*(\mathcal{M}_B) = P_kH^*(\mathcal{M}_g)$$

Chapter 7

The Equivariant Localization Formula on $\mathcal{M}_g$

The main result of the present thesis is a new approach to the P=W theorem, which does not rely on the geometric properties of the moduli space $\mathcal{M}_g$, but only on the structure of the equivariant intersection formulas. We obtain partial results towards a new proof for the rank 2 case.

The first idea comes from another description of the perverse filtration of Definition 6.5.1, using the compactification $\overline{\mathcal{M}}_g$ defined in (5.4) and the corresponding infinity divisor (5.3). We refer to [CM] and [CHM] for the general definition of the perverse filtration on an algebraic variety; for us, it is enough to know that there is a well-defined perverse filtration on the cohomology of the compactification $\overline{\mathcal{M}}$ as well. The description we are giving in what follows will be enough for our purposes.

For the sake of notational simplicity, from now on we will drop the index in $\mathcal{M}_g$ and simply write $\mathcal{M}$.

Proposition 7.0.1. *Let* $i : \mathcal{M} \to \overline{\mathcal{M}}$ *be the natural embedding. Then* $x \in P_k(H^j(\mathcal{M}))$ *if and only if there exists a* $y \in P_{k-1}(H^{j-2}(\overline{\mathcal{M}}))$ *such that* $(x + \eta y)\eta^{3g-2-j+k} = 0$.

Proof. From the properties of the perverse truncation functor ${}^p\tau_{\leqslant p}$ [CM], [CHM], which commutes with the restriction to the open $\mathcal{A} \subset \overline{\mathcal{A}}$, we have that the embedding $i : \mathcal{M} \to \overline{\mathcal{M}}$ induces the inclusion

$$i^*(P_k(H^*(\overline{\mathcal{M}})) \subset P_k(H^*(\mathcal{M})). \tag{7.1}$$

On the other hand, from the (η, L)-decomposition[1] of [CM, Corollary 2.1.7], we see that $i^*(P_k(H^*(\overline{\mathcal{M}}))$ induces a filtration on $i^*(H^*(\overline{\mathcal{M}})) \cong H^*(\mathcal{M})$ satisfying a relative Hard Lefschetz theorem of the same type as the perverse filtration P on $H^*(\mathcal{M})$. It follows from this and (7.1) that actually

$$i^*(P_k(H^*(\overline{\mathcal{M}}))) = P_k(H^*(\mathcal{M})). \tag{7.2}$$

[1]Note that we use the notation $\eta = \eta_Z$ for the class L in [CM] while η in [CM] denotes an ample class on $\overline{\mathcal{M}}$.

On the projective $\overline{\mathcal{M}}$ we can use [CM, Proposition 5.2.4.(39)] to deduce[2] that

$$P_k(H^j(\overline{\mathcal{M}})) = \sum_{l=0}^{3g-2-j+k} \operatorname{im}(\eta^l) \cap \ker(\eta^{3g-2-l-j+k}), \tag{7.3}$$

where $\eta : H^j(\overline{\mathcal{M}}) \to H^{j+2}(\overline{\mathcal{M}})$ denotes multiplication by $\eta \in H^2(\overline{\mathcal{M}})$ by an abuse of notation.

Now if $0 \neq x \in H^j(\mathcal{M})$ and $\widetilde{x} \in H^j(\overline{\mathcal{M}})$ is a lift of x, then $\widetilde{x} \notin \operatorname{im}(\eta)$, since $i^*(\eta) = 0$. It follows then from (7.2) and (7.3) that

$$x \in P_k(H^j(\overline{\mathcal{M}})) \Leftrightarrow \eta^{3g-2-j+k}\widetilde{x} = 0 \text{ for some } \widetilde{x} \in H^j(\overline{\mathcal{M}}) \text{ lift of } x.$$

Considering the composition $H^*(\mathcal{M}) \to H^*_{\mathbb{T}}(\mathcal{M})$ (cf. §5.3) with the Kirwan map (5.5), it is clear that any lift of x will have the form $\widetilde{x} = x + \eta y$ for some $y \in H^{j-2}(\overline{\mathcal{M}})$. Finally, again by (7.2) and (7.3), we see that $x \in P_k(H^j(\mathcal{M}))$ if and only if such $y \in P_{k-1}(H^{j-2}(\overline{\mathcal{M}}))$, and this completes the proof. □

We are ready to formulate the enumerative version of P=W.

Theorem 7.0.2 (Enumerative P=W)**.** *The P=W Theorem 6.5.2 holds for* $G = PGL_2$ *if and only if for all* $g \geqslant 2$, $l \geqslant 0$ *and* $m \geqslant 0$ *there is an extension* $\beta^l\gamma^m + \eta q(\alpha, \beta, \gamma, \eta)$ *of* $\beta^l\gamma^m \in H^*(Z)$ *such that*

$$\int_Z \eta^{3g-3-2l-2m}(\beta^l\gamma^m + \eta q(\alpha, \beta, \gamma, u))x = 0 \tag{7.4}$$

for all $x \in H^*(Z)$.

Proof. Notice that for any class $\nu \in H^*(\overline{\mathcal{M}})$, we have $\int_{\overline{\mathcal{M}}} \nu\eta = \int_Z \nu$, and the restriction map $H^*(\overline{\mathcal{M}}) \to H^*(Z)$ is surjective as both rings are generated by the universal classes and η. Therefore, Equation (7.4) is equivalent to

$$\int_{\overline{\mathcal{M}}} \eta^{3g-2-2l-2m}(\beta^l\gamma^m + \eta q(\alpha, \beta, \gamma, \eta))x = 0$$

for all $x \in H^*(\overline{\mathcal{M}})$. By Poincaré duality, this is, in turn, equivalent to

$$\eta^{3g-2-2l-2m}(\beta^l\gamma^m + \eta q(\alpha, \beta, \gamma, \eta)) = 0. \tag{7.5}$$

By Proposition 7.0.1, (7.5) implies that $\alpha^k\beta^l\gamma^m \in P_{2k+2l+4m}(H^{2k+4l+6m}(\mathcal{M}))$ for all k, l and m. Since α, β and γ have weight respectively 2, 2 and 4, Proposition 7.2.1 and Theorem 6.4.3 imply that

$$P_k H^*(\mathcal{M}) \subseteq W_{2k}(\mathcal{M})$$

for all $k \geqslant 0$. Finally, as in [CHM], this, the relative Hard Lefschetz theorem [CM, Theorem 2.1.4] and the curious Hard Lefschetz theorem [HV1, Theorem 1.1.5] imply Theorem 6.5.2 for $G = PGL_2$.

[2]Proposition 5.2.4 is claimed for a smooth total space, but as explained in [CM, Theorem 2.3.1] the results hold for non-smooth varieties for intersection cohomology, and thus for orbifolds with ordinary cohomology.

Conversely, assuming Theorem 6.5.2 holds, then for all $l \geqslant 0$ and $m \geqslant 0$, we have

$$\beta^l\gamma^m \in P_{2l+4m}(H^{4l+6m}(\mathcal{M})).$$

Still by Proposition 7.0.1 this implies that there exists a lift $\widetilde{y} \in H^{2l+4m}(\overline{\mathcal{M}})$ of $\beta^l\gamma^m$ such that $\widetilde{y}\eta^{3g-2-2l-2m} = 0$, and by Proposition 7.2.1 below, we can choose such a lift to be $\mathrm{Sp}(2g, \mathbb{Z})$ invariant; therefore, it is of the form $\widetilde{y} = \beta^l\gamma^m + \eta q(\alpha, \beta, \gamma, \eta)$ for some polynomial q. Then, by Poincaré duality on $\overline{\mathcal{M}}$ we have

$$\int_{\overline{\mathcal{M}}} \eta^{3g-2-2l-2m}(\beta^l\gamma^m + \eta q(\alpha, \beta, \gamma, \eta))x = 0$$

for all $x \in H^*(\overline{\mathcal{M}})$. Finally, since $\int_{\overline{\mathcal{M}}} \eta\nu = \int_Z \nu$ for all ν the proof is complete. □

Remark 7.0.3. There is a straightforward generalization of the result above for PGL_n. We have universal generators for $H^*(\mathcal{M}^d_{\mathrm{Dol}}(\mathrm{PGL}_n))$, whose weights on the character variety side have been computed in [HV1, Sh] and the curious Hard Lefschetz theorem in $H^*(\mathcal{M}^d_B(\mathrm{PGL}_n))$ has been proved in [Me].

7.1 Equivariant integrals on $\mathcal{M}_g$

From Theorem 7.0.2, it becomes clear that in order to approach the P=W theorem from an enumerative standpoint, we need to find useful intersection formulas on Z. Thanks to Lemma 4, this would follow from a formula for the equivariant integrals on $\mathcal{M}_g$.

The first foundational result is a useful intersection formula for the equivariant integrals on $\mathcal{M}_g$, which can be seen as the analogue of the one in Theorem 4.3.4 for Higgs bundles.

Theorem 7.1.1.

$$\oint_{\mathcal{M}} T_{(0)} \exp(Q_{(2)}) =$$

$$= \sum_{r \in \{0, u, -u\}} \operatorname*{Res}_{y=r} \frac{2^{-1}T \cdot (Q'' - 2u/(u^2-y^2))^g}{u^{g-1}\left(e^{Q'\frac{u-y}{u+y}} - e^{-Q'\frac{u+y}{u-y}}\right) y^{2g-2}(u^2-y^2)^{g-1}}\, dy$$

Proof. To calculate the equivariant integral as defined in (5.6), we need to list the components of the fixed point set $\mathcal{M}^{\mathbb{T}}$ of the circle action and identify the corresponding normal bundles. This data may be found in [HT2, §4,5,6], and thus we will only brief it here. There are two sorts of stable $\mathbb{T}$-fixed Higgs bundles (E, Φ) (cf. Proposition 5.1.1):

- E is stable and $\Phi = 0$. This set of Higgs bundles forms a copy $\mathcal{N} \subset \mathcal{M}$ of the moduli of stable bundles, which extends to the embedding $T^*\mathcal{N} \subset \mathcal{M}$.
- $E = L \oplus \Lambda L^{-1}$, $1 \leqslant \deg L \leqslant g-1$, $\Phi|_{\Lambda L^{-1}} = 0$, and $\Phi|_L$ is a nonzero map of line bundles $L \to \Lambda L^{-1}K$. The list of fixed point set of this second type is $F_i \cong S^{2g-2i-1}(C)$, $i = 1, \ldots, g-1$.

We thus have, by (5.6),

$$\oint_{\mathcal{M}} T_{(0)} \exp(Q_{(2)}) = \int_{\mathcal{N}} \frac{T_{(0)} \exp(Q_{(2)})}{e_{\mathbb{T}}(N_{\mathcal{N}})} + \sum_{i=1}^{g-1} \int_{F_i} \frac{T_{(0)} \exp(Q_{(2)})}{e_{\mathbb{T}}(N_{F_i})} \tag{7.6}$$

We start by describing a formula for multiplicative characteristic classes on $\mathcal{N}$. Let $B(t)$ be a formal power series and for a vector bundle V denote by $B(V)$ the corresponding multiplicative characteristic class of the V: denoting the Chern roots of V by t_i, $i = 1, \ldots, \mathrm{rank}(V)$, this class is the product $\prod_i B(t_i)$.

Lemma 6. *Let* $B(t)$ *be a formal power series with* $B(0) \neq 1$. *Then*

$$B(T\mathcal{N}) = [B(0)B(y)B(-y)]_{(0)}^{g-1} \exp\left(\left[\widehat{B}(y) + \widehat{B}(-y)\right]_{(2)}\right), \tag{7.7}$$

where $\widehat{B}(t)$ *satisfies* $\frac{\partial}{\partial t}\widehat{B}(t) = -\log B(t)$.

Proof. Denote by π the projection $C \times \mathcal{N} \to \mathcal{N}$ and by ω the positive integral generator of the second cohomology of C. Then by [Z] we have, for $k > 0$,

$$s_k(T\mathcal{N}) = \pi_*((g-1)(s_k(\mathrm{Ad}(U))\omega - s_{k+1}(\mathrm{Ad}(U))/(k+1)). \tag{7.8}$$

Here s_k is the power sum symmetric polynomial (in any number of variables), i.e. $s_k(b_1, b_2, \ldots) = b_1^k + b_2^k + \ldots$

We write $B(T\mathcal{N})$ as $\exp\left(\sum_i \log(B(t_i))\right)$, and then we apply the equality (7.8) to the sum in the exponential. The result is two terms, the first of which contributes a factor $B(\mathrm{Ad}(U))^{g-1} \cap 1$, while the second gives $\exp(\widehat{B}(\mathrm{Ad}(U)) \cap \omega)$, where 1 and ω are, as usual, the positive integral generators of $H_0(C, \mathbb{Z})$ and $H_2(C, \mathbb{Z})$, respectively. The Chern roots of $\mathrm{Ad}(U)$ are $0, \pm y$, and using the notation introduced in Section 4.3, we obtain (7.7). □

Now we can calculate the first term of the (7.6). Observe that $e_{\mathbb{T}}(T^*\mathcal{N})^{-1}$ is a multiplicative class of $T\mathcal{N}$ corresponding to the function

$$B(t) = \Psi(t) \stackrel{\text{def}}{=} 1/(u-t)$$

Then

$$\widehat{\Psi}(t) = -(u-t)\log(u-t) - t \text{ with } \frac{d}{dt}\widehat{\Psi}(t) = \log(u-t) \text{ and } \frac{d^2}{dt^2}\widehat{\Psi}(t) = \frac{-1}{u-t}$$

and

$$\frac{1}{e_{\mathbb{T}}(T^*\mathcal{N})} = \frac{1}{u^{g-1}} \left[\frac{1}{(u-y)(u+y)}\right]_{(0)}^{g-1} \exp\left(\left[\widehat{\Psi}(y) + \widehat{\Psi}(-y)\right]_{(2)}\right).$$

We have

$$\frac{d}{dy}\left(\Psi(y) + \Psi(-y)\right) = \log(u-y) - \log(u+y)$$

$$\frac{d^2}{dy^2}\left(\Psi(y) + \Psi(-y)\right) = \frac{-1}{u-y} + \frac{-1}{u+y}$$

Now, combining this with (4.9) and (7.7) shows that

$$\int_{\mathcal{N}} \frac{T_{(0)} \exp(Q_{(2)})}{e_{\mathbb{T}}(T^*\mathcal{N})} = \frac{1}{2} \operatorname*{Res}_{y=0} \frac{T \cdot (Q'' - 2u/(u^2-y^2))^g}{u^{g-1}\left(e^{Q'\frac{u-y}{u+y}} - e^{-Q'\frac{u+y}{u-y}}\right) y^{2g-2}(u^2-y^2)^{g-1}} dy. \tag{7.9}$$

This calculates the first summand of (7.6) since, being $\mathcal{N} \subseteq \mathcal{M}$ a Lagrangian subvariety, we have $N_{\mathcal{N}} = T^*\mathcal{N}$.

Now let $F_i \simeq S^{2g-2i-1}(C)$ for $i = 1, \ldots, g$ be the other components of the fixed point set. To evaluate

$$\int_{F_i} \frac{T_{(0)} \exp(Q_{(2)})}{e_{\mathbb{T}}(N_{F_i})} \in \mathbb{C}(u),$$

first we compute $T_{(0)}|_{F_i}$ and $Q_{(2)}|_{F_i}$. For this we define (c.f. [HT2, §5]) universal classes $\eta \in H^2(F_i)$ and $\xi_i \in H^1(F_i)$ for $i = 1 \ldots 2g$ by computing the first Chern class of the universal divisor $\Delta \subset F_i \times C \cong S^{2g-2i-1}(C) \times C$ in Künneth decomposition:

$$c_1(\Delta) = (2g-2i-1)\omega + \sum_{i=1}^{2g} \xi_i e_i + \eta \in H^2(F_i \times C) =$$
$$= H^0(F_i) \times H^2(C) \oplus H^1(F_i) \times H^1(C) \oplus H^2(F_i) \times H^0(C).$$

We also define $\theta_i = \xi_i \xi_{i+g}$ for $i = 1 \ldots g$ and $\theta = \sum_{i=1}^{g} \theta_i$. With these notations we have.

Lemma 7. *We have*

$$T_{(0)}|_{F_i} = T(\eta - u) \in H^*_{\mathbb{T}}(F_i) \cong H^*(F_i)[u] \tag{7.10}$$

and

$$Q_{(2)}|_{F_i} = (1-2i)Q'(\eta - u) - \theta Q''(\eta - u) \in H^*_{\mathbb{T}}(F_i) \cong H^*(F_i)[u] \tag{7.11}$$

Proof. Recall from last displayed line of proof of [HT2, (6.1)] that

$$-c_2(\operatorname{End}(\mathbf{E}))|_{F_i} = ((1-2i)\omega + \eta - u + \sum_{j=1}^{2g} \xi_j e_j)^2.$$

We get (7.10) immidiately and that

$$[y^{2k}]_{(2)}|_{F_i} = (1-2i)(2k)(\eta-u)^{2k-1} + \binom{2k}{2}(-2\theta)(\eta-u)^{2k-2}.$$

This in turn yields (7.11) □

Next we need a formula for the equivariant Euler class $e_{\mathbb{T}}(N_{F_i})$.

Lemma 8. *We have*

$$\begin{aligned} e_{\mathbb{T}}(N_{F_i}) &= (2u-\eta)^{g+2i-2} \exp\left(\frac{\theta}{\eta - 2u}\right)(u)^{g-1}(u-\eta)^{-2i+g} \exp\left(\frac{\theta}{\eta - u}\right) \\ &\quad \cdot (\eta - u)^{2i-2+g} \exp\left(\frac{\theta}{u-\eta}\right) \\ &= u^{g-1}(-1)^g(\eta-u)^{2g-2}(2u-\eta)^{g+2i-2} \exp\left(\frac{\theta}{\eta - 2u}\right). \end{aligned}$$

Proof. We know that the tangent bundle of $\mathcal{M}$ restricted to F_i can be computed as

$$T_{\mathcal{M}}|_{F_i} \cong R^1\pi_* \left(\mathrm{End}_0(\mathbf{E}) \stackrel{Ad(\Phi)}{\rightarrow} \mathrm{End}_0(\mathbf{E}) \otimes K_C \right),$$

where $\pi : F_i \times C \to F_i$ is the projection. To compute this first we have from [HT2, (6.1)]

$$\mathrm{ch}_{\mathbb{T}}(\mathrm{End}_0(\mathbf{E})) =$$

$$= 1 + \exp\left((1-2i)\omega + \eta - u + \sum_{j=1}^{2g} \xi_j e_j \right) + \exp\left((2i-1)\omega - \eta + u - \sum_{j=1}^{2g} \xi_j e_j \right).$$

Thus we can compute

$$\begin{aligned}
\mathrm{ch}_{\mathbb{T}}(T_{\mathcal{M}}|_{F_i}) &= -\mathrm{ch}_{\mathbb{T}}\left(R\pi_* \left(\mathrm{End}_0(\mathbf{E}) \stackrel{Ad(\Phi)}{\rightarrow} \mathrm{End}_0(\mathbf{E}) \otimes K_C \right) \right) \\
&= \pi_* \left(\mathrm{ch}_{\mathbb{T}}(\mathrm{End}_0(\mathbf{E})(-1 + \exp(u)\,\mathrm{ch}(K_C))\mathrm{td}(C) \right) \\
&= \pi_* \left(\mathrm{ch}_{\mathbb{T}}(\mathrm{End}_0(\mathbf{E}))(-1 + \exp(u) + (g-1)(1+\exp(u))\sigma) \right) \\
&= \exp(2u)((g-1)\exp(-\eta) + \exp(-\eta)(2i-1-\theta) \\
&\quad + \exp(u)((g-1)(1+\exp(-\eta)) - \exp(-\eta)(2i-1-\theta) \\
&\quad + ((g-1)(1+\exp(\eta)) - \exp(\eta)(2i-1+\theta) \\
&\quad + \exp(-u)((g-1)\exp(\eta) + \exp(\eta)(2i-1+\theta) \\
&= \exp(2u)((2i-2)\exp(-\eta) + \sum_{i=1}^{g} \exp(-\eta-\theta_i)) \\
&\quad + \exp(u)((g-1) - 2i\exp(-\eta)) - \sum_{i=1}^{g} \exp(-\eta-\theta_i)) \\
&\quad + ((g-1) - 2i\exp(\eta)) - \sum_{i=1}^{g} \exp(\eta+\theta_i)) \\
&\quad + \exp(-u)((2i-2)\exp(\eta) + \sum_{i=1}^{g} \exp(\eta+\theta_i)).
\end{aligned}$$

The four lines give the contributions from the four weight spaces of the $\mathbb{T}$-action on $T_{\mathcal{M}}|_{F_i}$: weight 2, weight 1, weight 0 and weight -1. The 0 weight space corresponds to the tangent space $T_{F_i} < T_{\mathcal{M}}|_{F_i}$. Removing it will yield the normal bundle N_{F_i} of F_i in $\mathcal{M}$. Thus

$$\begin{aligned}
\mathrm{ch}_{\mathbb{T}}(N_{F_i}) &= \exp(2u)((2i-2)\exp(-\eta) + \sum_{i=1}^{g} \exp(-\eta-\theta_i)) \\
&\quad + \exp(u)((g-1) - 2i\exp(-\eta)) - \sum_{i=1}^{g} \exp(-\eta-\theta_i)) \\
&\quad + \exp(-u)((2i-2)\exp(\eta) + \sum_{i=1}^{g} \exp(\eta+\theta_i)).
\end{aligned}$$

Formal computation now gives the Lemma. □

The final ingredient is the intersection theory on symmetric products, examined in §3.4.3. For any power series $A(x) \in \mathbb{C}[[x]]$ and $B(x) \in \mathbb{C}[[x]]$ we have (cfr. Formula (3.7))

$$\int_{F_i} A(\eta)\exp(B(\eta)\theta) = \operatorname*{Res}_{x=0}\left\{\frac{A(x)\,(1+xB(x))^g}{x^{2g-2i}}\right\}dx.$$

So we can proceed to compute with $y = x - u$

$$\int_{F_i}\frac{T_{(0)}\exp(Q_{(2)})}{e_{\mathbb{T}}(N_{F_i})} = \int_{F_i}\frac{T(\eta-u)\exp\left((1-2i)Q'(\eta-u)-\theta Q''(\eta-u)\right)}{u^{g-1}(-1)^g(\eta-u)^{2g-2}(2u-\eta)^{g+2i-2}\exp\left(\frac{\theta}{\eta-2u}\right)}$$

$$= \frac{(-1)^g}{u^{g-1}}\int_{F_i}\frac{T(\eta-u)\exp\left((1-2i)Q'(\eta-u)\right)}{(\eta-u)^{2g-2}(2u-\eta)^{g+2i-2}}\exp\left(\left(-Q''(\eta-u)+\frac{1}{2u-\eta}\right)\theta\right)$$

$$= \frac{(-1)^g}{u^{g-1}}\operatorname*{Res}_{x=0}\frac{T(x-u)\exp\left((1-2i)Q'(x-u)\right)\left(1+x\left(-Q''(x-u)+\frac{1}{2u-x}\right)\right)^g dx}{(x-u)^{2g-2}(2u-x)^{g+2i-2}x^{2g-2i}}$$

$$= u^{1-g}\operatorname*{Res}_{y=-u}\frac{T(y)\exp\left((1-2i)Q'(y)\right)}{y^{2g-2}(u-y)^{g+2i-2}(u+y)^{g-2i}}\left(Q''(y)-\frac{1}{u+y}-\frac{1}{u-y}\right)^g dy$$

We note that the right hand side of this expression has no pole at $y = -u$ for $i \geqslant g$ thus the residue vanishes and so we have

$$\sum_{i=1}^{g-1}\int_{F_i}\frac{T_{(0)}\exp(Q_{(2)})}{e_{\mathbb{T}}(N_{F_i})}$$

$$= \sum_{i=1}^{g-1}u^{1-g}\operatorname*{Res}_{y=-u}\left\{\frac{T(y)\exp\left((1-2i)Q'(y)\right)}{y^{2g-2}(u-y)^{g+2i-2}(y+u)^{g-2i}}\left(Q''(y)-\frac{1}{u+y}-\frac{1}{u-y}\right)^g\right\}dy$$

$$= \sum_{i=1}^{\infty}u^{1-g}\operatorname*{Res}_{y=-u}\left\{\frac{T(y)\exp\left((1-2i)Q'(y)\right)}{y^{2g-2}(u-y)^{g+2i-2}(y+u)^{g-2i}}\left(Q''(y)-\frac{1}{u+y}-\frac{1}{u-y}\right)^g\right\}dy$$

$$= \operatorname*{Res}_{y=-u}\frac{T\cdot(Q''-2u/(u^2-y^2))^g}{u^{g-1}\left(e^{Q'}\frac{u-y}{u+y}-e^{-Q'}\frac{u+y}{u-y}\right)y^{2g-2}(u^2-y^2)^{g-1}}dy.$$

Finally we note that the expression in this residue is odd in y and so will give the same result at $y = u$. Thus we can conclude that

$$\sum_{i=1}^{g-1}\int_{F_i}\frac{T_{(0)}\exp(Q_{(2)})}{e_{\mathbb{T}}(N_{F_i})} =$$

$$= \frac{1}{2}\sum_{r\in\{u,-u\}}\operatorname*{Res}_{y=r}\frac{T\cdot(Q''-2u/(u^2-y^2))^g dy}{u^{g-1}\left(e^{Q'}\frac{u-y}{u+y}-e^{-Q'}\frac{u+y}{u-y}\right)y^{2g-2}(u^2-y^2)^{g-1}}$$

Together with (7.9) this completes the proof of the Theorem. □

7.2 The $\mathrm{Sp}(2g, \mathbb{Z})$ action

Note that the orientation preserving index-2 subgroup of the mapping class group of the Riemann surface underlying C acts on $H^*(C)$ via[3]

$$\Sigma := \mathrm{Sp}(2g, \mathbb{Z})$$

acting on $H^0(C)$ and $H^2(C)$ trivialy and by preserving the symplectic structure on $H^1(C)$ given by the intersection pairing.

We see that Σ acts on $H^3(\mathcal{M}) \cong H^1(C)$, where we identify $H^3(\mathcal{M})$ with $H^1(C)$ by mapping ψ_i to e_i and letting Σ act trivially on $\mathbb{C}[u]$. This will also yield an action of Σ on $H^3_{\mathbb{T}}(\mathcal{M}) \cong H^3(\mathcal{M}) \otimes \mathbb{C}[u]$, where the identification is again done by mapping the equivariant ψ_i to the non-equivariant one. Finally we will let Σ act on $H^2_{\mathbb{T}}(\mathcal{M})$ and $H^4_{\mathbb{T}}(\mathcal{M})$ and so on α and β trivially.

Proposition 7.2.1. *The action defined above induces an action of Σ on $H^*_{\mathbb{T}}(\mathcal{M})$ by $\mathbb{C}[u]$-algebra automorphisms, and on $H^*(Z)$ by ring automorphisms. The perverse filtrations on $H^*(\mathcal{M})$ and on $H^*(Z)$ are invariant under this action.*

Proof. We observe that the equivariant intersection numbers

$$\oint_{\mathcal{M}} \alpha^k \beta^l \prod_{i=1}^{2g} \psi_i^{m_i} \in \mathbb{C}(u)$$

are invariant with respect to this action of Σ. This follows from the fact that in (5.6) the expression for the equivariant Euler class $e_{\mathbb{T}}(T^*\mathcal{N})$ coming from (7.8) is invariant under Σ. Analogously, the formula in Lemma 8 is Σ-invariant.

Thus we deduce that the intersection numbers on $\mathcal{N}$ and F_i are invariant under Σ as the Σ action on $H^*(\mathcal{N})$ and $H^*(F_i)$ comes from the mapping class group of C. It follows that the ideal of relations among the universal generators α, β, ψ_i of $H^*_{\mathbb{T}}(\mathcal{M})$ is invariant under Σ, thus getting an action of Σ on $H^*_{\mathbb{T}}(\mathcal{M})$ by $\mathbb{C}[u]$-algebra automorphisms extending the one on $H^2_{\mathbb{T}}(\mathcal{M}) \oplus H^3_{\mathbb{T}}(\mathcal{M}) \oplus H^4_{\mathbb{T}}(\mathcal{M})$ defined above.

In turn, by (5.7) we get that all intersection numbers on Z will be invariant by Σ, which will yield a Σ-action on $H^*(Z)$ by ring automorphisms. Thus, since $\int_Z x = \int_{\overline{\mathcal{M}}} \eta x$, we see that the perverse filtration on $H^*(\overline{\mathcal{M}})$ will be Σ invariant, and finally also the one on $H^*(\mathcal{M})$ by Proposition 7.0.1. □

7.3 Integrals on the infinity divisor

The formula in Theorem 7.0.2 is an explicit statement, which ought to follow from Theorem 7.1.1 thanks to Lemma 4, but this calculation rather difficult to perform. The main reason is that it involves cancellation of two rather different terms: the residues of a differential form at $y = 0$ and at $y = u$. In this section we mitigate this problem, and rewrite our statement in a completely local form.

Let us consider expression in Theorem 7.1.1. First we observe that the expression in parenthesis may be rewritten as follows

$$e^{Q'}\frac{u-y}{u+y} - e^{-Q'}\frac{u+y}{u-y} = \frac{(u - y\tanh(Q'/2))(u - y\coth(Q'/2))}{u^2 - y^2}.$$

[3]We note that our notation for Γ and Σ are swapped from the notation in [HT2], this is to be more in line with the notation of Γ used in [HT3].

The equivariant integral thus is the sum of the residues of the expression

$$\Omega(u,y) = \frac{2^{-2}T\cdot((u^2-y^2)Q''-2u)^g}{u^{g-1}\sinh(Q')(u-y\tanh(Q'/2))(u-y\coth(Q'/2))y^{2g-2}(u^2-y^2)^{2g-2}} \tag{7.12}$$

at $y=0$ and $y=\pm u$. Let us find all the other poles of this form. The following statement is an easy consequence of the implicit function theorem. Its proof will be omitted.

Proposition 7.3.1. *a. Let us assume that* $T(0)=1$ *and* $Q=Ay^2+P(y)$*, where* A *is a nonzero constant, P is an even polynomial of degree* >2*. Then for* u *sufficiently small, there is a* $c>0$ *such that the form* Ω *in* (7.12) *has the following poles inside the unit disc:*

- *at* $y=0$*, of order* $2g-1$
- *at* $y=\pm u$ *of order* $2g-2$
- *simple poles at* $y=\pm b_0$*, where* $|b_0-\sqrt{u}/A|<cu^2$ *for a constant* c.

b. If $P=0$*, i.e.* $Q=Ay^2$*, then we can describe all remaining poles of* $\Omega(u,y)$*: these are simple poles at* $y=b_n$*,* $n\in\mathbb{Z}\setminus\{0\}$*, where* $|b_n-\pi in|<cu$*.*

Remark 7.3.2. When $P=0$, the numbers $B_u=\{b_n,\ n\in\mathbb{Z}\}$ are the solutions of the Bethe ansatz for the Yang-Yang model [YY]. This was a crucial observation of [MNSh], who also studied a special case of these equivariant integrals in order to find a *regularized volume* of the Higgs moduli. Their formula, obtained using mathematically nonrigorous methods, is an infinite sum over B_u, and can be easily recovered by applying the Residue Theorem to Theorem 7.1.1.

The most important part of Proposition 7.3.1 is that for $Q=Ay^2+P(y)$, and u sufficiently small, the only poles of $\Omega(u,y)$ in the disc $\{y:\ |y|<2u\}$ are the poles $y=0,-u,u$. In this case, we have an integral representation

$$\operatorname*{Res}_{y=0,\pm u}\Omega(u,y)\,dy = \int_{|y|=2\epsilon}\Omega(u,y)\,dy$$

and thus

$$\operatorname*{Res}_{u=0}\operatorname*{Res}_{y=0,\pm u}\Omega(u,y)\,dy\,du = \int_{|u|=\epsilon}\int_{|y|=2\epsilon}\Omega(u,y)\,dy\,du.$$

To perform the last double integral, we must first fix the magnitude of u to be equal to ϵ, and then compute the y-residues inside the circle of the y-plane of radius 2ϵ. However, as the last expression is an integral over a product of circles, we can apply Fubini's theorem and write our integral as

$$\int_{|u|=\epsilon}\int_{|y|=2\epsilon}\Omega(u,y)\,dy\,du = \int_{|y|=2\epsilon}\int_{|u|=\epsilon}\Omega(u,y)\,du\,dy.$$

Finally, we convert this last integral into residues again. Now we fix a value of y of magnitude 2ϵ and look for poles of our form inside the circle of radius ϵ in the u-plane. Again, studying the denominator, we see that we have a pole at $u=0$, but the factor u^2-y^2 now does not contribute. We have a pole, however, at the point $u = y\tanh(Q'/2)$, because this is of order $u \sim 4\epsilon^2 \ll \epsilon$. This leads to the following residue identity:

$$\operatorname*{Res}_{u=0}\operatorname*{Res}_{y=0,\pm u}\Omega(u,y)dy\,du = \operatorname*{Res}_{y=0}\left[\operatorname*{Res}_{u=0} + \operatorname*{Res}_{u=y\tanh(Q'/2)}\right]\Omega(u,y)du\,dy.$$

Combining this with (7.12), we arrive at the following formula for integration on Z:

$$\int_Z T_{(0)}\exp(Q_{(2)}) = -\operatorname*{Res}_{y=0}\left[\operatorname*{Res}_{u=0} + \operatorname*{Res}_{u=y\tanh(Q'/2)}\right]$$
$$\left(\frac{2^{-2}T\cdot((u^2-y^2)Q''-2u)^g\,du\,dy}{u^{g-1}\sinh(Q')(u-y\tanh(Q'/2))(u-y\coth(Q'/2))y^{2g-2}(u^2-y^2)^{2g-2}}\right). \tag{7.13}$$

The second term may be further simplified since the pole of $\Omega(u,y)\,du$ at $u = y\tanh(Q'/2)$ is simple, and thus to calculate the residue, we simply perform the substitution $u = y\tanh(Q')$. Using the identities

$$\sinh(a)(\tanh(a/2)-\coth(a/2)) = -2, \quad \tanh^2(a/2)-1 = \frac{-1}{\cosh^2(a/2)}$$

and $\sinh(a) = 2\sinh(a/2)\cosh(a/2)$, the denominator turns into

$$-2y\cdot y^{g-1}\tanh^{g-1}(Q'/2)\cdot y^{2g-2}\cdot y^{2(2g-2)}\cosh^{-2(2g-2)}(Q'/2),$$

while the numerator is

$$2^{-2}T(u = y\tanh(Q'/2))\cosh^{-2g}(Q'/2)(-y^2Q''-y\sinh(Q'))^g.$$

Canceling the similar factors we arrive at

$$\operatorname*{Res}_{y=0}\frac{2^{-2}T(u=y\tanh(Q'/2))\cdot\cosh^{-2g}(Q'/2)\cdot(-y^2Q''-y\sinh(Q'))^g\,dy}{-2y\cdot y^{g-1}\tanh^{g-1}(Q'/2)\cdot y^{2g-2}\cdot y^{2(2g-2)}\cosh^{-2(2g-2)}} =$$
$$\operatorname*{Res}_{y=0}\frac{-2^{-3}T(u=y\tanh(Q'/2))\cdot\cosh^{2g-4}(Q'/2)\cdot(-yQ''-\sinh(Q'))^g\,dy}{\tanh^{g-1}(Q'/2)\cdot y^{6g-6}}. \tag{7.14}$$

Thus formula (7.13) yields the following result:

Proposition 7.3.3. *We have the following formula for the intersection numbers of Z:*

$$\int_Z T_{(0)}\exp(Q_{(2)}) =$$
$$\operatorname*{Res}_{y=0}\operatorname*{Res}_{u=0}\frac{-2^{-2}T\cdot((u^2-y^2)Q''-2u)^g\,du\,dy}{u^{g-1}\sinh(Q')(u-y\tanh(Q'/2))(u-y\coth(Q'/2))y^{2g-2}(u^2-y^2)^{2g-2}} +$$
$$\operatorname*{Res}_{y=0}\frac{2^{-3}T(u=y\tanh(Q'/2))\cdot\cosh^{2g-4}(Q'/2)\cdot(-yQ''-\sinh(Q'))^g\,dy}{\tanh^{g-1}(Q'/2)\cdot y^{6g-6}}. \tag{7.15}$$

Notice that if the polynomial T is divisible by η^{g-1}, then the first residue at $u = 0$ vanishes. In particular, if $k \leqslant g-1$, we obtain

Corollary 7.3.4. *Define*

$$R^Q_{g,k}(y) \coloneqq (-1)^g 2^{-3} \left(\frac{\sinh(Q'/2)}{y}\right)^{2g-2k-2} \cosh^{2k-2}(Q'/2) \left(Q'' + \frac{\sinh(Q')}{y}\right)^g,$$

then, for $1 \leqslant k \leqslant g-1$,

$$\int_Z \eta^{3g-3-2k} T_{(0)} \exp(Q_{(2)}) = \operatorname*{Res}_{y=0} \frac{dy}{y^{4k-1}} T(u = y \cdot \tanh(Q'/2)) R^Q_{g,k}(y). \tag{7.16}$$

7.4 Defects and the order of the pole

We denote simply by $R_{g,k}(y)$ the function appearing in (7.16) when we set $Q = -Ay^2/2 - Gy^4/4$. Notice that it is a holomorphic function around $y = 0$. Then (5.8) and Corollary 7.3.4 tell us that

$$\int_Z \eta^{3g-3-2k} \alpha^i (4\gamma)^j \beta^m \eta^n \exp(Q_{(2)}) =$$
$$\operatorname*{Res}_{y=0} \frac{dy}{y^{4k-1}} y^{2m} \partial_A^i (\partial_G - y^2 \partial_A)^j (y \tanh(Q'/2))^n R_{g,k}(y).$$

Let $\widetilde{R}_{g,k} = R_{g,k}(y, G = y^{-2}\widetilde{G})$: notice that it is still a holomorphic function in y. Then, since $\partial_G = y^2 \partial_{\widetilde{G}}$, we get

$$\int_Z \eta^{3g-3-2k} \alpha^i (4\gamma)^j \beta^m \eta^n \exp(Q_{(2)}) =$$
$$\operatorname*{Res}_{y=0} \frac{dy}{y^{4k-1}} y^{2m+2n+2j} \partial_A^i (\partial_{\widetilde{G}} - \partial_A)^j (\tanh(Q'/2)/y)^n \widetilde{R}_{g,k}(y)$$

and in this formula, we are taking the residue of a meromorphic function with a pole at $y = 0$ of order $4k - 1 - 2m - 2n - 2j$. Since the order of the pole of the sum two meromorphic functions is at most the maximum between the orders of the two poles, we are brought to make the following definition.

Definition 7.4.1. We define the *defect* of a monomial in α, β, γ and η via the assignment

$$\operatorname{def}(\alpha) = 0, \ \operatorname{def}(\beta) = \operatorname{def}(\gamma) = \operatorname{def}(\eta) = 2$$

and extending it by multiplicativity. We also define the defect of any polynomial in α, β, γ and η to be the *minimum* of the defects of its monomials.

Notice that if x is a class in α, β and γ, so that it can be seen as a class $x \in H^i(\mathcal{M})$, then we immediately verify

$$\operatorname{def}(x) = i - \operatorname{wt}(x)$$

where $\operatorname{wt}(x)$ is the weight of x.

By Theorem 7.0.2 we immediately get the following.

Corollary 7.4.2. *If for every* $x \in H^*(Z)$ *we can find* $y \in H^*(Z)$ *with*

$$\int_Z \eta^{3g-3-\operatorname{def}(x)} (x + \eta y) r = 0, \text{ for all } r \in H^*(Z)$$

then P=W holds for $\mathcal{M}$.

Chapter 8

The matrix problem for the top-defect pairing

Thanks to Corollary 7.4.2, the proof of P=W reduces to a matrix problem for every generator of the cohomology of Z, i.e. monomials in α, β and the ψ_i. We will treat here the case of β^k at top-defect, and solve the matrix problem. Afterwards, we will examine the case of the monomials $\beta^{k-h}\gamma^h$ and prove a criterion for the solvability of the relevant matrix problem. The pairing at lower defects for the monomials will be examined in §9, in which we will show that the problem becomes much more involved even for the "simplest" case β^k.

8.1 The classes β^k

In this section we will prove the following.

Theorem 8.1.1. *Let* $g \geqslant 2$ *and* $1 \leqslant k \leqslant g-1$. *There exists a unique class* $F_k \in H^*(Z)$ *such that, for all* $P \in H^*(Z)$ *with* $\operatorname{def}(P) = 2k-2$, *we have*

$$\int_Z \eta^{3g-3-2k}(\beta^k + \eta F_k)P = 0. \tag{8.1}$$

In the entire section, we will always assume that $1 \leqslant k \leqslant g-1$.

In order to prove Theorem 8.1.1, we need to show that the pairing

$$(F, P) \mapsto \int_Z \eta^{3g-3-2k} FP,$$

$$F \in H^{4k}(Z),\ \operatorname{def}(F) = 2k,\ P \in H^{6g-8}(Z),\ \operatorname{def}(P) = 2k-2$$

where we are taking defect-homogeneous F and P, is degenerate; moreover, we need an element of its kernel to be of the form $\beta^k + \eta F_k$ for some F_k.

To compute the matrix of the pairing, we choose the following basis of the F classes

$$F_{k,a_1,n_1} := \beta^{a_1-n_1}(4\gamma)^{n_1}\eta^{k-a_1}\alpha^{k-a_1-n_1}, \text{ with } \begin{cases} 0 \leqslant n_1 \leqslant a_1, \\ a_1 + n_1 \leqslant k, \end{cases}$$

and the following basis of the P-classes

$$P_{k,a_2,n_2} := \beta^{a_2-n_2}(4\gamma)^{n_2}\eta^{k-1-a_2}\alpha^{3g-3-k-a_2-n_2}, \text{ with } 0 \leqslant n_2 \leqslant a_2 \leqslant k-1.$$

The coefficients have been chosen in order to make computations easier later.

We then perform a column operation and define the matrix M_k of the pairing as follows:

$$(M_k)_{a_1,n_1}^{a_2,n_2} := \frac{(-2)^{k-a_1}}{(k-a_1-n_1)!n_1!} \frac{\int_Z \eta^{3g-3-2k} F_{k,a_1,n_1} P_{k,a_2,n_2}}{\int_Z \eta^{3g-3-2k} F_{k,k,0} P_{k,a_2,n_2}} \tag{8.2}$$

Therefore Theorem 8.1.1 is equivalent to finding a vector in the kernel of M_k^T whose coefficient corresponding to the term β^k (i.e. the row indexed by $(a_1, n_1) = (k, 0)$) is nonzero.

Lemma 9. *We have*

$$(M_k)_{a_1,n_1}^{a_2,n_2} = \binom{3g-3-a_1-n_1-a_2-n_2}{k-a_1-n_1}\binom{g-n_2}{n_1}.$$

Proof. We will use the formula of Corollary 7.3.4 choosing the polynomial $Q = -Ay^2/2 - Gy^4/4$. First of all we perform the computations for

$$\widetilde{F}_{k,a_1,n_1} := \beta^{a_1-n_1}(\alpha\beta+4\gamma)^{n_1}\eta^{k-a_1}\alpha^{k-a_1-n_1},$$

$$\widetilde{P}_{k,a_2,n_2} := \beta^{a_2-n_2}(\alpha\beta+4\gamma)^{n_2}\eta^{k-1-a_2}\alpha^{3g-3-k-a_2-n_2}.$$

Let us write $\widetilde{G} = Gy^2$, so that $\partial_G = y^2\partial_{\widetilde{G}}$. Then by Corollary 7.3.4 we have

$$\int_Z \eta^{3g-3-2k}\widetilde{F}_{k,a_1,n_1}\widetilde{P}_{k,a_2,n_2} =$$

$$\partial_A^{3g-3-a_1-a_2-n_1-n_2}\partial_{\widetilde{G}}^{n_1+n_2} \operatorname*{Res}_{y=0} \frac{dy}{y}\left(\frac{\tanh(-Ay/2-\widetilde{G}y/2)}{y}\right)^{2k-1-a_1-a_2} \widetilde{R}_{g,k}(A,\widetilde{G};y)$$

where $\widetilde{R}_{g,k}(A,\widetilde{G};y) := R_{g,k}(A,\widetilde{G}/y^2;y)$. We see that the form we are taking the residue of has a simple pole, thus its residue is computed by evaluating at $y = 0$. Now

$$\widetilde{R}_{g,k}(A,\widetilde{G};0) = 2^{2k-1-g}(A+\widetilde{G})^{2g-2k-2}(A+2\widetilde{G})^g$$

and the tanh factor gives $(-A/2-\widetilde{G}/2)^{2k-1-a_1-a_2}$. Therefore we have

$$\int_Z \eta^{3g-3-2k}\widetilde{F}_{k,a_1,n_1}\widetilde{P}_{k,a_2,n_2} =$$

$$= (-1)^{a_1+a_2+1}2^{-g+a_1+a_2}\partial_A^{3g-3-a_1-a_2-n_1-n_2}\partial_{\widetilde{G}}^{n_1+n_2}(A+\widetilde{G})^{2g-3-a_1-a_2}(A+2\widetilde{G})^g$$

(notice that the integral does not depend on k). Then using the classes 4γ in the definitions of F_{k,a_1,n_1} and P_{k,a_1,n_1} amounts to change variable $B = A + \widetilde{G}$, so that the formula becomes

$$\int_Z u^{3g-3-2k}F_{k,a_1,n_1}P_{k,a_2,n_2} =$$

$$= (-1)^{a_1+a_2+1} 2^{-g+a_1+a_2} \partial_B^{3g-3-a_1-a_2-n_1-n_2} \partial_G^{n_1+n_2} B^{2g-3-a_1-a_2} (B+G)^g =$$

$$= (-1)^{a_1+a_2+1} 2^{-g+a_1+a_2} (3g-3-a_1-a_2-n_1-n_2)!(n_1+n_2)! \binom{g}{n_1+n_2},$$

dividing out and using the coefficients of (8.2) we obtain

$$(M_k)_{a_1,n_1}^{a_2,n_2} = \binom{3g-3-a_1-n_1-a_2-n_2}{k-a_1-n_1} \binom{g-n_2}{n_1}$$

and the proof is complete. □

Lemma 9 allows us to relate the matrix M_k with a particular evaluation operator on polynomials in two variables.

Corollary 8.1.2. *Let* $v = (v_{a_1,n_1})_{\substack{0 \leqslant n_1 \leqslant a_1 \\ n_1 + a_1 \leqslant k}}$ *be a row vector, and let*

$$p_v(X, Z) := \sum_{\substack{0 \leqslant n_1 \leqslant a_1 \\ a_1 + n_1 \leqslant k}} v_{a_1,n_1} \binom{3g-3-a_1-n_1-Z}{k-a_1-n_1} \binom{g-X}{n_1}$$

Then $vM_k = (p_v(n_2, a_2+n_2))_{0 \leqslant n_2 \leqslant a_2 \leqslant k-1}$.

We are now ready to find a vector in the kernel of M_k^T.

Proposition 8.1.3. *Let*

$$v_k^{a_1,n_1} := \operatorname*{Res}_{t=0} \frac{(-1)^{k-a_1-n_1}(1+t)^{g-k-2}(1+2t)^{g-n_1} dt}{t^{a_1-n_1+1}}, \quad 0 \leqslant n_1 \leqslant a_1, \; a_1+n_1 \leqslant k.$$

Then

$$\sum_{\substack{0 \leqslant n_1 \leqslant a_1 \\ a_1 + n_1 \leqslant k}} v_k^{a_1,n_1} (M_k)_{a_1,n_1}^{a_2,n_2} = 0$$

for all a_2, n_2 *with* $0 \leqslant n_2 \leqslant a_2 \leqslant k-1$. *In particular, the vector* v_k *is in the kernel of* M_k^T.

Proof. Define

$$V_{g,k}(X, Z; v, w) := (1-v)^{-3g+k+2+Z}(1+w)^{g-X},$$

then we see that $\binom{3g-3-a_1-n_1-Z}{k-a_1-n_1}\binom{g-X}{n_1} = \operatorname{Res}_{v,w=0} \frac{V_{g,k}(X,Z;v,w) dv dw}{v^{k-a_1-n_1+1} w^{n_1+1}}$, therefore

$$(M_k)_{a_1,n_1}^{a_2,n_2} = \operatorname*{Res}_{v,w=0} \frac{V_{g,k}(n_2, a_2+n_2; v, w) dv dw}{v^{k-a_1-n_1+1} w^{n_1+1}}. \tag{8.3}$$

Now notice that

$$(1+t)^{g-k-2}(1+2t)^g V_{g,k}(X, Z; -t, \frac{t^2}{1+2t}) = (1+t)^{Z-2X}(1+2t)^X := K(X, Z; t). \tag{8.4}$$

For $0 \leqslant n_2 \leqslant a_2 \leqslant k-1$, the values of $(X, Z) = (n_2, n_2 + a_2)$ belong to

$$\Gamma_k := \{(X, Z) \in \mathbb{Z}^2 \mid X \geqslant 0, \; 2X \leqslant Z \leqslant X + k - 1\}.$$

Notice that for $(X, Z) \in \Gamma_k$, $K(X, Z; t)$ is a polynomial in t of degree $Z - X \leqslant k-1$; therefore, by defining

$$p_k(X, Z) := \operatorname*{Res}_{t=0} \frac{dt}{t^{k+1}} K(X, Z; t) \tag{8.5}$$

we find that $p_k(X, Z) = 0$ on Γ_k. Then, by taking the Taylor series of the left hand side of (8.4), we have

$$\operatorname*{Res}_{t=0} \frac{dt}{t^{k+1}} (1+t)^{g-k-2}(1+2t)^g V_{g,k}(X, Z; -t, \frac{t^2}{1+2t}) =$$

$$= \sum_{0 \leqslant l,n \leqslant \infty} \operatorname*{Res}_{t=0} \frac{(1+t)^{g-k-2}(1+2t)^g}{t^{k+1}} (-t)^l \left(\frac{t^2}{1+2t}\right)^n \operatorname*{Res}_{v,w=0} \frac{V_{g,k}(X, Z; v, w)dv dw}{v^{l+1} w^{n+1}} =$$

$$= \sum_{l+2n \leqslant k} \operatorname*{Res}_{t=0} \frac{(1+t)^{g-k-2}(1+2t)^{g-n}}{t^{k-l-2n+1}} \operatorname*{Res}_{v,w=0} \frac{V_{g,k}(X, Z; v, w)dv dw}{v^{l+1} w^{n+1}} = p_k(X, Z) \tag{8.6}$$

the last equality being formula (8.4). Since $p_k(n_2, a_2 + n_2) = 0$ for $0 \leqslant n_2 \leqslant a_2 \leqslant k-1$, we conclude thanks to Corollary (8.1.2) by substituting $l = k - a_1 - n_1$ in the sum (8.6). □

From this we can find the solution to Equation (8.1), and show that such solution is unique.

Theorem 8.1.4 (Lowest defect)**.** *The solution to Equation* (8.1) *is*

$$\beta^k + uF_k = \frac{1}{p_k(g, 3g-k-2)} \operatorname*{Res}_{t=0} \frac{dt}{t^{k+1}} (1+\beta t)^{g-k-2}(1+2\beta t)^g e^{2\eta t\left(\alpha - \frac{4\gamma t}{1+2\beta t}\right)}$$

Proof. By Proposition 8.1.3 and by (8.2) we have that setting

$$\widetilde{F}_k := \operatorname*{Res}_{t=0} \sum_{l+2n \leqslant k} \frac{(-1)^l (1+t)^{g-k-2}(1+2t)^{g-n}(-2)^{l+n} dt}{t^{k-l-2n+1} n! l!} \beta^{k-l-2n} (4\gamma)^n \eta^{l+n} \alpha^l =$$

$$\operatorname*{Res}_{t=0} \frac{(1+t)^{g-k-2}(1+2t)^g \beta^k dt}{t^{k+1}} \sum_{0 \leqslant l,n \leqslant \infty} \frac{1}{n! l!} \left(\frac{2t\alpha\eta}{\beta}\right)^l \left(\frac{-8t^2\gamma\eta}{\beta^2(1+2t)}\right)^n =$$

$$= \operatorname*{Res}_{t=0} \frac{dt}{t^{k+1}} (1+\beta t)^{g-k-2}(1+2\beta t)^g e^{2\eta t\left(\alpha - \frac{4\gamma t}{1+2\beta t}\right)}$$

then we have $\int_Z u^{3g-3-2k} \widetilde{F}_k P = 0$ for all P with defect $2k-2$. We see that the coefficient of the term β^k in $\widetilde{F}_k$ is

$$p_k(g, 3g-k-2) = \operatorname*{Res}_{t=0} \frac{dt}{t^{k+1}} (1+t)^{g-k-2}(1+2t)^g$$

which is clearly positive for $g \geqslant k+2$ since all exponents are, while for $g = k+1$ we have $p_k(k+1, 2k+1) = 2^{k+1} - 1$, still positive for $k \geqslant 1$. Thus we can in any case divide out and obtain the result. □

Remark 8.1.5. Define D_Z and $D_{X,-1}$ to be respectively the operators such that

$$D_Z f(Z) = f(Z+1) - f(Z), \quad D_{X,-1} f(X) = f(X-1) - f(X)$$

for any function f. Then we immediately see that

$$D_Z K(X,Z;t) = tK(X,Z;t), \quad D_{X,-1} K(X,Z;t) = \frac{t^2}{1+2t} K(X,Z;t).$$

This means that we can also prove Proposition 8.1.3 by using the Newton interpolation formula

$$\sum_{l+2n \leqslant k} (-1)^l D_Z^l D_{X,-1}^n \Big|_{\substack{X=g \\ Z=3g-k-2}} p_k(X,Z) \binom{3g-3+l-k-Z}{l} \binom{g-X}{n} = p_k(X,Z),$$

which is true since p_k is a polynomial of degree k if we set $\deg Z = 1$ and $\deg X = 2$. Thus, we find another expression for the vector in the kernel of M_k^T, namely

$$v_k^{a_1,n_1} = (-1)^{k-a_1-n_1} D_Z^{k-a_1-n_1} D_{X,-1}^{n} \Big|_{\substack{X=g \\ Z=3g-k-2}} p_k(X,Z), \ 0 \leqslant n_1 \leqslant a_1, \ a_1+n_1 \leqslant k.$$

The last formula tells us that the polynomials $p_k(X,Z)$ generate the solution to the equation in Theorem 8.1.1 via subsequent applications of the discrete difference operators D_Z and $D_{X,-1}$ and evaluations at the point $(X,Z) = (g, 3g-k-2)$.

Remark 8.1.6. We immediately verify that

$$D_{Z,-1}^2 K(X,Z;t) = -D_X K(X,Z;t),$$

thus for all k we have

$$D_{Z,-1}^2 p_k(X,Z) = -D_X p_k(X,Z), \tag{8.7}$$

which has the shape of a heat equation in which both X and Z are discrete. Actually, it can be shown that for all $k \geqslant 1$, $p_k(X,Z)$ is the only polynomial solution to Equation (8.7) with initial condition

$$p_k(0,Z) = \binom{Z}{k}.$$

In the following Theorem, we see that the solution found in 8.1.4 is the only one which solves Equation (8.1) at the top defect.

Theorem 8.1.7. *The kernel of M_k^T is one-dimensional. Therefore, the lowest defect part F_k of the solution to the equation in Theorem 8.1.1 is unique.*

Proof. Thanks to Remark 8.1.2, we have to show that by letting

$$\Gamma_k = \{(X,Z) \in \mathbb{Z}^2 \mid X \geqslant 0, \ 2X \leqslant Z \leqslant X+k-1\}$$

there exists only one (up to multiplication by a constant) polynomial of degree at most k in X and Z (recall that $\deg Z = 1$ and $\deg X = 2$) which vanishes on Γ_k.

Letting $p(X,Z)$ be such a polynomial, we can write it as

$$p(X,Z) = \sum_{i=0}^{\lfloor k/2 \rfloor} p_i(Z)\binom{X}{i}$$

where $\deg(p_i) \leqslant k-2i$. Since $p(0,Z) = p_0(Z)$ vanishes in $Z = 0,1,\ldots,k-1$, we see that $p_0(Z) = \lambda_0\binom{Z}{k}$ for some constant λ_0. Then

$$q_1(X,Z) = \frac{1}{X}(p(X,Z) - p(0,Z)) = \sum_{i=1}^{\lfloor k/2 \rfloor} \frac{p_i(Z)}{i}\binom{X-1}{i-1}$$

and $q_1(1,Z) = p_1(Z)$ vanishes in the $k-2$ points $2,3,\ldots,k-1$, therefore $p_i(Z) = \lambda_i\binom{Z-2}{k-2}$. Continuing this way, we see that we can write $p(X,Z)$ as

$$p(X,Z) = \sum_{i=0}^{\lfloor k/2 \rfloor} \lambda_i \binom{Z-2i}{k-2i}\binom{X}{i}$$

for some constant λ_i. Now, for $1 \leqslant j \leqslant \lfloor k/2 \rfloor$, we have $2j \leqslant k \leqslant j+k-1$, therefore

$$0 = p(j,k) = \sum_{i=0}^{\lfloor k/2 \rfloor} \lambda_i \binom{j}{i}, \text{ for } 1 \leqslant j \leqslant \lfloor k/2 \rfloor.$$

These are $\lfloor k/2 \rfloor$ independent conditions on the λ_i's, thus the space of polynomials of degree at most k in Z and X which vanish on Γ_k is at most one-dimensional. By the way, we know that the polynomial $p_k(X,Z) = \mathrm{Res}_{t=0} \frac{dt}{t^{k+1}}(1+t)^X(1+2t)^{Z-2X}$ satisfies this condition, therefore the space is exactly one-dimensional. □

8.1.1 Factorization of M_k

The matrix M_k defined in (8.2) can be conveniently written as a product of two matrices $M_k = Q_k S_k$, where S_k is a matrix with integer entries and Q_k is triangular with ± 1 on the diagonal. This factorization has some striking consequences in itself (see Proposition 8.1.8), and will be of key importance in Section 8.1.2, where it will be used to give the determinantal criterion for P=W at top defect (see Corollary 8.1.11).

Define the following matrix:

$$(Q_k)_{a,n}^{b,m} = (-1)^{k+b} e_{b+m-a-n}(3g-2-k, 3g-1-k, \ldots, 3g-3-a-n)\cdot$$
$$\cdot e_{n-m}(g-n+1, g-n+2, \ldots, g) \tag{8.8}$$

where $e_i(x_1,\ldots,x_j)$ is the i-th elementary symmetric polynomial in $x_1,\ldots,x_j$. Here we adopt the convention that $e_i(x_1,\ldots,x_j) = 0$ if $i < 0$ and $e_i(x_1,\ldots,x_j) = 1$ if $0 \leqslant j < i$. The indexing of Q_k satisfies $0 \leqslant n \leqslant a \leqslant k$, $a+n \leqslant k$ and the same for b and m.

Let also S_k be the matrix defined by

$$(S_k)_{a_1,n_1}^{a_2,n_2} = n_2^{n_1}(a_2+n_2)^{k-n_1-a_1} \tag{8.9}$$

where the rows are indexed by the pairs (a_1, n_1) with $0 \leqslant n_1 \leqslant a_1 \leqslant k$, $n_1 + a_1 \leqslant k$ and the columns by the pairs (a_2, n_2) with $0 \leqslant n_2 \leqslant a_2 \leqslant k-1$. Here we adopt the convention that $0^0 = 1$ (hence the last row of S_k is a row of 1's).

Then a direct computation shows that

$$M_k = Q_k S_k \tag{8.10}$$

and since Q_k is invertible (being upper triangular with ± 1 on the diagonal) we see that $\mathrm{Ker} M_k = \mathrm{Ker} S_k$. In particular, although the entries of M_k are polynomials in g, its kernel is generated by vectors of $\mathbb{Z}^{k(k+1)/2}$, thus independent from g.

It is easy to write the inverse of Q_k as

$$\begin{aligned}(Q_k^{-1})_{a,n}^{b,m} = (-1)^{k+b} h_{b+m-a-n}(3g-2-k, 3g-1-k, \dots, 3g-2-b-m)\cdot \\ \cdot h_{n-m}(g, g-1, \dots, g-m) \end{aligned} \tag{8.11}$$

where $h_r(x_1 \dots, x_n)$ is the *complete symmetric function* of degree r, which is the sum of all monomials of total degree r in the variables $x_1, \dots, x_n$; by convention, $h_r = 0$ for $r < 0$. The fact that this is indeed the inverse of Q_k is a direct consequence of the classical identity

$$\sum_{r=0}^{n} (-1)^r e_r h_{n-r} = \delta_{0,n},$$

which is true for any number of variables.

With this we can show that, in the case of the classes β^k, the perverse filtration is actually a grading.

Proposition 8.1.8. *For all* $k \geqslant 1$, *we have* $\beta^k \notin P_{2k-1}(H^{4k}(\mathcal{M}))$.

Proof. Following Proposition 7.0.1 and Theorem 7.0.2, we write the equation

$$\int_Z \eta^{3g-4-2k}(\beta^k + \eta F)P = 0$$

which must be in particular valid for all P with $\mathrm{def}(P) = 2k$. We thus define $\widetilde{M}_k$ to be the matrix of the pairing $(F, P) \mapsto \int_Z \eta^{3g-4-2k}(\eta F)P$ with $\mathrm{def}(F) = 2k-2$, $\mathrm{def}(P) = 2k$, and with an extra row corresponding to β^k. Then it is easy to see that if we define

$$(\widetilde{S}_k)_{a_1,n_1}^{a_2,n_2} = n_2^{n_1}(a_2+n_2)^{k-n_1-a_1} \tag{8.12}$$

where the rows are indexed by the pairs (a_1, n_1) with $0 \leqslant n_1 \leqslant a_1 \leqslant k$, $n_1 + a_1 \leqslant k$ and the columns by the pairs (a_2, n_2) with $0 \leqslant n_2 \leqslant a_2 \leqslant k$, then

$$Q_k \widetilde{S}_k = \widetilde{M}_k$$

with notations as in (8.8). Since $\widetilde{S}_k$ is obtained by S_k by adding some columns, and since $\mathrm{Ker} S_k^T$ is one-dimensional generated by the coefficients of

$$p_k(X, Z) = \operatorname*{Res}_{t=0} \frac{dt}{t^{k+1}}(1+t)^{Z-2X}(1+2t)^X,$$

then $\mathrm{Ker}\widetilde{S}_k^T = (0)$ if and only if there exist values of $0 \leqslant n_2 \leqslant a_2 \leqslant k$ such that $p_k(n_2, a_2+n_2) \neq 0$. Since $p_k(0, k) = 1$, we can conclude. □

8.1.2 The classes $\beta^{k-h}\gamma^h$

We now consider the general case of the extension problem, namely for $1 \leqslant k \leqslant g-1$ and $0 \leqslant h \leqslant k$,

$$\int_Z \eta^{3g-3-2k}(\beta^{k-h}(4\gamma)^h + \eta F)P = 0 \tag{8.13}$$

where we ask for the existence of some $F \in \mathbb{C}[\alpha, \beta, \gamma, \eta]$ such that (8.13) is satisfied for all $P \in \mathbb{C}[\alpha, \beta, \gamma, \eta]$.

We shall still consider only the top-defect part of the pairing, namely we look for an F with $\mathrm{def}(F) = 2k-2$ such that Equation (8.13) is satisfied for all P with $\mathrm{def}(P) = 2k-2$. Therefore, we see that F is a sum of monomials of the form

$$F_{k,h,a_1,n_1} \coloneqq \beta^{a_1-n_1}(4\gamma)^{n_1}\eta^{k-1-a_1}\alpha^{k+h-a_1-n_1}$$

for $0 \leqslant n_1 \leqslant a_1 \leqslant k-1$, and $a_1 + n_1 \leqslant k+h$. Analogously, P is a sum of monomials of the form

$$P_{k,h,a_2,n_2} \coloneqq \beta^{a_2-n_2}(4\gamma)^{n_2}\eta^{k-1-a_2}\alpha^{3g-3-k-h-a_2-n_2},$$

for $0 \leqslant n_2 \leqslant a_2 \leqslant k-1$ and $a_2 + n_2 \leqslant 3g-3-h-k$. Notice that if we consider the general case $g \geqslant k+1$, the condition on $a_2 + n_2$ is not redundant. However, from $n_2 \leqslant a_2 \leqslant k-1$ we can deduce $a_2 + n_2 \leqslant 2k-2$, so the condition on $a_2 + n_2$ becomes redundant if $2k-2 \leqslant 3g-3-h-k$, that is

$$3g \geqslant 3k + h + 1. \tag{8.14}$$

Fixing values of $0 \leqslant h \leqslant k$, we call the range of values of g which satisfy (8.14) the *redundancy range*. The "smallest" case in which $g \geqslant k+1$ falls out the redundancy range is $k = h = 3$, $g = 4$.

By Proposition 7.0.1, the top-defect part of the statement of the Enumerative P=W Conjecture in the redundancy range is the following.

Conjecture 1. *Let* $k \geqslant 1$ *and* $0 \leqslant h \leqslant k$ *be integers. In the redundancy range, there exists a unique defect-homogeneous* $F \in H^*(Z)$ *with* $\mathrm{def}(F) = 2k-2$ *such that for every defect-homogeneous* $P \in H^*(Z)$ *with* $\mathrm{def}(P) = 2k-2$, *Equation (8.13) is satisfied.*

The case $h = 0$ was proved in last section. We will give an equivalent statement for Conjecture 1 in terms of the non-vanishing of a particular determinant involving the polynomials $p_k(X, Z)$ defined in the previous section.

Let

$$(M_{k,h})^{a_2,n_2}_{a_1,n_1} = \binom{3g-3-a_1-n_1-a_2-n_2}{k+h-a_1-n_1}\binom{g-n_2}{n_1} \tag{8.15}$$

for $0 \leqslant n_1 \leqslant a_1 \leqslant k-1$, $a_1 + n_1 \leqslant k+h$, plus the extra row $(n_1, a_1) = (h, k)$, and $0 \leqslant n_2 \leqslant a_2 \leqslant k-1$.

In the redundancy range, the matrix of the pairing

$$(F, P) \mapsto \int_Z \eta^{3g-3-2k}(\eta F)P$$

for defect-homogeneous classes F and P with $\mathrm{def}(F) = \mathrm{def}(P) = 2k-2$, is $M_{k,h}$ without the last row $(n_1, a_1) = (h, k)$. Such row corresponds to multiplication with

$\beta^{k-h}(4\gamma)^h$, therefore Conjecture 1 is equivalent to stating that in the redundancy range, $\mathrm{Ker}(M_{k,h}^T)$ is one-dimensional, and the entries of its vectors corresponding to the class $\beta^{k-h}\gamma^h$ are nonzero.

Notice that $M_{k,h}$ is a submatrix of M_{k+h} of Definition 8.2. The number of columns of $M_{k,h}$ is

$$c_{k,h} = \frac{k(k+1)}{2}.$$

while the number of rows is

$$r_{k,h} = c_{k,h} - \left\lfloor \frac{(k-h+1)(k-h-3)}{4} \right\rfloor = c_{k+h,0} + \frac{h(h+3)}{2}. \tag{8.16}$$

In particular, if $h = k-3$, $M_{k,h}$ is a square matrix, if $k \geqslant h \geqslant k-2$ then $r_{k,h} = c_{k,h}+1$ and if $h \leqslant k-4$ the number of rows is strictly smaller than the number of columns.

Definition 8.1.9. For $0 \leqslant h \leqslant k$, let $S_{k,h}$ be the matrix defined as

$$(S_{k,h})_{a_1,n_1}^{a_2,n_2} = (a_2+n_2)^{k+h-a_1-n_1} n_2^{n_1}$$

with $0 \leqslant n_2 \leqslant a_2 \leqslant k-1$ and $0 \leqslant n_1 \leqslant a_1 \leqslant k+h$, $a_1+n_1 \leqslant k+h$; here we are using the convention $0^n = 0$ for all n except $0^0 = 1$.

Lemma 10. *The dimension of* $\mathrm{Ker}(S_{k,h}^T)$ *is* $(h+1)(h+2)/2$*. A basis of the kernel is given by the coefficients of the polynomials*

$$Z^j p_{k+h-i}(X,Z), \ \ 0 \leqslant j \leqslant i \leqslant h$$

where p_k *is the polynomial defined in* (8.5)*.*

Proof. Let $v = (v_{a_1,n_1})_{a_1,n_1}$ be a vector in $\mathrm{Ker}(S_{k,h}^T)$. This means that the polynomial

$$p_v(X,Z) = \sum_{a_1,n_1} v_{a_1,n_1} X^{n_1} Z^{k+h-a_1-n_1}$$

vanishes at integers X and Z with $X \geqslant 0$ and $2X \leqslant Z \leqslant X+k-1$. All polynomials of the form

$$Z^j p_{k+h-i}(X,Z), \ \ 0 \leqslant j \leqslant i \leqslant h$$

where p_k is the polynomial defined in (8.5), satisfy this condition. The number of such polynomials is $(h+1)(h+2)/2$, so let us show they are linearly independent. Since the case $h=0$ is the content of Theorem 8.1.7, suppose $k \geqslant h \geqslant 1$.

We will prove the equivalent statement that the polynomials

$$p_{i,j}(X,Z) := \binom{Z-h-k+j}{j} p_{k+h-i}(X,Z), \ \ 0 \leqslant j \leqslant i \leqslant h$$

are linearly independent. To show this, simply form the matrix

$$(T_h)_{0\leqslant j \leqslant i \leqslant h}^{0 \leqslant a \leqslant b \leqslant h} = p_{i,j}(h-b, h+k-b+a)$$

and notice that, up to rearranging rows and columns, T_h is a triangular matrix with powers of two as diagonal entries, thus it has nonzero determinant. We deduce that

$$\mathrm{dimKer} S_{k,h}^T \geqslant (h+1)(h+2)/2.$$

Conversely, assume $p(X,Z)$ is a polynomial of the form

$$p(X,Z) = \sum_{\substack{0\leqslant n\leqslant a\\ n+a\leqslant k+h}} \lambda_{n,a} X^n Z^{k+h-a-n}$$

which vanishes on

$$\Gamma_k = \{(X,Z) \in \mathbb{Z}^2 \mid 0 \leqslant X \leqslant k-1, 2X \leqslant Z \leqslant X+k-1\}.$$

If we define $\deg Z = 1$ and $\deg X = 2$, then $p(X,Z)$ is a polynomial of degree $k+h$. By changing basis, we can write it in the following way

$$p(X,Z) = \sum_{i=0}^{\lfloor (k+h)/2 \rfloor} p_i(Z) \binom{X}{i}$$

where $\deg(p_i) = k+h-2i$. We see that $p_0(Z) = p(0,Z)$ vanishes on $Z = 0,\dots,k-1$, thus we must have

$$p_0(Z) = \sum_{j=0}^{h} \lambda_{0,j} \binom{Z}{k+j}$$

for some constants $\lambda_{0,j}$. Then we consider

$$q_1(X,Z) \coloneqq \frac{1}{X}(p(X,Z) - p(0,Z)) = \sum_{i=1}^{\lfloor (k+h)/2 \rfloor} \frac{p_i(Z)}{i} \binom{X-1}{i-1}$$

and we see that $p_1(Z) = q_1(0,Z)$ vanishes on $Z = 2,\dots,k-1$, so that

$$p_1(Z) = \sum_{j=0}^{h} \lambda_{1,j} \binom{Z-2}{k-2+j}.$$

We can continue this way up to $i = [k/2]$, deducing that

$$p_i(Z) = \sum_{j=0}^{h} \lambda_{i,j} \binom{Z-2i}{k-2i+j}, \quad \text{for } i \leqslant \lfloor k/2 \rfloor.$$

For $\lfloor k/2 \rfloor < i \leqslant \lfloor (k+h)/2 \rfloor$, we do not have any conditions on the vanishing of the polynomials $p_i(Z)$, but we can write anyway

$$p_i(Z) = \sum_{j=2i-k}^{h} \lambda_{i,j} \binom{Z-2i}{k-2i+j}, \quad \text{for } \lfloor k/2 \rfloor \leqslant i \leqslant \lfloor (k+h)/2 \rfloor$$

which is a general polynomial in Z of degree $k+h-2i$. Putting everything together, we can write

$$p(X,Z) = \sum_{i=0}^{\lfloor (k+h)/2 \rfloor} \sum_{j=\max(0,2i-k)}^{h} \lambda_{i,j} \binom{Z-2i}{k-2i+j} \binom{X}{i}$$

and $p(X,Z)$ vanishes on $0 \leqslant X \leqslant \lfloor k/2 \rfloor$, $2X \leqslant Z \leqslant k-1$. The vector space of such polynomials has dimension

$$d_{k,h} = (\lfloor k/2 \rfloor + 1)(h+1) + \sum_{i=\lfloor k/2 \rfloor + 1}^{\lfloor (k+h)/2 \rfloor} (h+k-2i+1).$$

We now impose conditions for $p(X,Z)$ to vanish on other points of Γ_k. Similarly to the proof of Theorem 8.1.7, setting $p(l,k)=0$ for $1 \leqslant l \leqslant \lfloor k/2 \rfloor$, we obtain

$$\sum_{i=0}^{l} \binom{l}{i} \lambda_{i,0} = p(l,k) = 0 \ \text{ for } 1 \leqslant l \leqslant \lfloor k/2 \rfloor,$$

which are $\lfloor k/2 \rfloor$ independent conditions on the $\lambda_{i,0}$ for $1 \leqslant i \leqslant \lfloor k/2 \rfloor$. Analogously, we have $p(l,k+1)=0$ for $2 \leqslant l \leqslant \lfloor (k+1)/2 \rfloor$, which gives

$$\sum_{i=0}^{l} \binom{l}{i} ((k+1)\lambda_{i,0} + \lambda_{i,1}) = 0, \ \text{ for } 2 \leqslant l \leqslant \lfloor (k+1)/2 \rfloor.$$

With this we find $\lfloor (k+1)/2 \rfloor - 1$ independent conditions on the $\lambda_{i,1}$'s for $2 \leqslant i \leqslant \lfloor (k+1)/2 \rfloor$, once the $\lambda_{i,0}$ are chosen. Continuing this way, we obtain $\lfloor (k+j)/2 \rfloor - j$ independent linear conditions on the $\lambda_{i,j}$'s for all $0 \leqslant j \leqslant h$. We conclude that

$$\dim \mathrm{Ker} S_{k,h}^T \leqslant d_{k,h} - \sum_{j=0}^{h} (\lfloor (k+j)/2 \rfloor - j) = (h+1)(h+2)/2$$

where the last equality is tedious but straightforward to verify, thus completing the proof. □

Definition 8.1.10. Let $\{v_1, \dots, v_{(h+1)(h+2)/2}\}$ be independent vectors of $\mathrm{Ker}(S_{k,h}^T)$. We define $\widetilde{Q}_{k,h}$ as the matrix obtained by replacing the last $(h+1)(h+2)/2$ rows of the matrix $Q_{k+h}(g, 3g-3)$ defined in (8.8) by the vectors v_i.

With the same method used to obtain (8.10), we can show that $\widetilde{Q}_{k,h}$ satisfies

$$\widetilde{Q}_{k,h}(g, 3g-3) S_{k,h} = \widetilde{M}_{k,h} \qquad (8.17)$$

where $\widetilde{M}_{k,h}$ is obtained by $M_{k,h}$ by adding $h(h+3)/2$ zero rows, and by replacing the row corresponding to $\beta^{k-h}(4\gamma)^h$ with a zero row. Analogously, if we define $\widehat{Q}_{k,h}$ to be Q_{k+h} with the last rows except the one corresponding to $\beta^{k-h}\gamma^h$ replaced with $h(h+3)/2$ independent vectors of $\mathrm{Ker} S_{k,h}^T$, we get

$$\widehat{Q}_{k,h}(g, 3g-3) S_{k,h} = \widehat{M}_{k,h}$$

where $\widehat{M}_{k,h}$ is obtained by $M_{k,h}$ by adding $h(h+3)/2$ zero rows.

Thus, information on $\widetilde{Q}_{k,h}$ and $\widehat{Q}_{k,h}$ would lead to the solution of the matrix problem. A particularly good result would follow if $\widetilde{Q}_{k,h}$ were invertible.

Lemma 11. *Let $p_k(X,Z)$ be the polynomials defined in* (8.5).

1. *Let*
$$B_{k,h}(X,Z) = (Z^j p_{k+h-i}(X-n, Z-m))_{0\leqslant j\leqslant i\leqslant h}^{0\leqslant n\leqslant m\leqslant h}.$$
Then we have $\det\widetilde{Q}_{k,h}(g,3g-3) = c_{k,h}\det B_{k,h}(g,3g-2-k)$ *for some nonzero constants* $c_{k,h}$.

2. $\det\widehat{Q}_{k,h}(g,3g-3)$ *is the determinant of a first minor of* $B(g,3g-2-k)$.

3. *Define*
$$W_{k,h}(X,Z) := \det(p_k(X-h+i, Z+j))_{0\leqslant i\leqslant h}^{0\leqslant j\leqslant h}. \tag{8.18}$$
Then, up to a nonzero constant,
$$\det(\widetilde{Q}_{k,h}(g,3g-3)) = \prod_{i=0}^{h} W_{k,i}(g,3g-k-i-2).$$

Proof. Let us choose the basis of $\mathrm{Ker}(S_{k,h}^T)$ given by the polynomials $Z^j p_{k+h-i}(X,Z)$ with $0 \leqslant j \leqslant i \leqslant h$ and let us construct $Q_{k,h}$ accordingly.

We consider the matrix $C_{k,h} = \widetilde{Q}_{k,h}(Q_{k+h})^{-1}$, where Q_{k+h} is the matrix defined in (8.8). This is a block upper-triangular matrix with a square block of size $(h+1)(h+2)/2$ at the bottom-right corner and with all other blocks of size 1, each containing a 1; thus our determinant is equal to the determinant of the bottom-right block (up to a sign: it is easy to see that this sign is $+$, since the bottom-right minor of size $(h+1)(h+2)/2$ of Q_{k+h} has determinant 1).

To compute it we use (8.11) along with the identity
$$\sum_{i=0}^{a}(-1)^i \frac{a!}{(a-i)!} h_{b-i}(x,\dots,x-i) = (x-a)^b$$
which is valid for all integers a and b: with this we can take appropriate linear combinations of the last $(h+1)(h+2)/2$ columns of $(Q_{k+h})^{-1}$ to obtain a matrix whose entry $((a,n),(b,m))$ is $(y-b-m+1)^{k+h-a-n}(x-m)^n$ in the last $(h+1)(h+2)/2$ columns. This amounts to performing column operations on $\widetilde{Q}_{k,h}(Q_{k+h})^{-1}$, so this procedure does not change its determinant up to multiplying by nonzero constants.

Recalling that the last $(h+2)(h+1)/2$ rows of $\widetilde{Q}_{k,h}$ consist of the coefficients of $Z^j p_{k+h-i}(X,Z)$, the determinant is then equal to the one in the statement and this proves Point 1. Point 2 is shown similarly when the row which is eliminated corresponds to the polynomial among $(Z^j p_{k+h-i})_{i,j}$ which is not considered, and the column is the index of Q_{k+h} corresponding to the class $\beta^{k-h}(4\gamma)^h$. To show Point 3, use repeatedly the identity $p_k(X,Z+1) - p_k(X,Z) = p_{k-1}(X,Z)$ in the matrix $B_{k,h}$. □

Point 2. of the previous Lemma immediately yields the following.

Corollary 8.1.11. *If* $\widetilde{Q}_{k,h}(g,3g-3)$ *is invertible, then there exists a vector in* $\mathrm{Ker}S_{k,h}^T$ *which can be discarded to give an invertible* $\widehat{Q}_{k,h}(g,3g-3)$.

Theorem 8.1.12. *Assume* $\widetilde{Q}_{k,h}(g,3g-3)$ *is invertible. Then there exists a unique solution to Equation* (8.13).

Proof. If $\widetilde{Q}_{k,h}$ is invertible, then by Corollary 8.1.11 we can choose a $\widehat{Q}_{k,h}$ that is also invertible. This proves that

$$\mathrm{rk}\widetilde{M}_{k,h} = \mathrm{rk}\widehat{M}_{k,h}$$

since both are equal to $\mathrm{rk}S_{k,h}$. Now since $\widehat{M}_{k,h}$ is obtained by $\widetilde{M}_{k,h}$ by replacing a zero row with the row corresponding to $\beta^{k-h}\gamma^h$, we deduce that such row must be a linear combination of the others, thus giving a solution to (8.13).

Now since $\mathrm{Ker}S^T_{k,h}$ has dimension $(h+2)(h+1)/2$, and since $\widetilde{M}_{k,h}$ has exactly $(h+2)(h+1)/2$ zero rows, it follows that $\mathrm{Ker}M^T_{k,h}$ must be one-dimensional, therefore the solution to (8.13) is unique. □

Extensive numerical computations has brought us to state the following.

Conjecture 2. *In the redundancy range,* $W_{k,h}(g, 3g-k-h-2) > 0$.

Conjecture 2 would imply the existence and uniqueness of the lowest defect part of the solution to Equation (8.13) in the redundancy range. Here we provide a proof for $h = 1$.

Proposition 8.1.13. *If* $g \geqslant k+1$, *then* $W_{k,1}(g, 3g-k-3) > 0$.

Proof. With a simple change of variables, we can rewrite the polynomials p_k as

$$p_k(X,Z) = \operatorname*{Res}_{t=0} \frac{dt}{t^{k+1}}(1+t)^X(1-t)^{X-Z+k-1}$$

therefore, we can rewrite the determinants $W_{k,1}(X,Z)$ as

$$\operatorname*{Res}_{t_0,t_1=0} \frac{dt_0 dt_1}{t_0^{k+2}t_1^{k+1}}((1+t_0)(1+t_1))^{X-1}(1-t_0)^{X-Z+k-1}(1-t_1)^{X-Z+k-2}(t_0^2-t_1^2).$$

We decompose it as

$$W_{k,1} = W^0_{k,1} - W^1_{k,1}$$

accordingly to the terms of the Vandermonde factor $t_0^2 - t_1^2$. Then we have the following

Lemma 12. *For all* $k \geqslant 1$, *we have*

$$(k-1)W^0_{k,1}(X,Z) - (k+1)W^1_{k,1}(X,Z) = 2(X-1)W_{k-1,1}(X-1,Z-2)$$

Proof of the lemma. We immediately see that

$$W^0_{k,1}(X,Z) = p_k(X-1,Z)p_{k-1}(X-1,Z-2),$$

$$W^1_{k,1}(X,Z) = p_{k+1}(X-1,Z)p_{k-2}(X-1,Z-2).$$

Moreover we have the easily proven formulas

$$(k+1)p_{k+1}(X,Z) = Xp_k(X-1,Z-2) + (Z-X-k)p_k(X,Z),$$

$$p_k(X-1,Z) = p_{k-1}(X-2,Z-2) + p_k(X-2,Z-1),$$

$$p_k(X,Z) = p_k(X,Z-1) + p_{k-1}(X,Z-1).$$

Applying in order the first, second and third formula to the statement, we get the result. □

Now we can prove the Proposition 8.1.13 by induction on k. Assume that we found the domain where $(X-1)W_{k-1,1}(X-1,Z-2) \geqslant 0$, then

$$\frac{W^1_{k,1}(X,Z)}{W^0_{k,1}(X,Z)} \leqslant \frac{k-1}{k+1} < 1$$

if $W^0_{k,1}(X,Z) > 0$, thus yielding the domain for $W_{k,1}$ by the Lemma.

From the expression of $W^0_{k,1}$ given in the proof of the Lemma, we easily find

$$W^0_{k,1}(X,Z) > 0 \text{ for } \begin{cases} X \geqslant k, \\ Z \geqslant X+k-1. \end{cases}$$

Now since $W_{1,1}(X,Z) = Z$, we find

$$W_{k,1}(X,Z) > 0 \text{ for } \begin{cases} X \geqslant k, \\ Z \geqslant X+k-1. \end{cases}$$

Finally, if $g \geqslant k+1$, we have $3g-k-3 \geqslant g+k-1$ and the proof is complete. □

Remark 8.1.14. We also managed to prove that $W_{k,h}(g, 3g-k-h-2) > 0$ for $g \geqslant k+h+2$. The proof will be provided in a forthcoming work.

Remark 8.1.15. Notice that, in any case, $W_{k,h}(g, 3g-k-h-2)$ is a nonzero polynomial in g with positive leading term. Therefore, existence and uniqueness of the solution to Equation (8.13) is assured for g big enough (depending on k and h).

Remark 8.1.16. If $g \geqslant k+1$ is outside the redundancy range, then the matrix of the pairing $(F,P) \mapsto \int_Z u^{3g-3-2k}(\eta F)P$ can still be defined with the same formula (and the same rows and columns range) as in Definition 8.15: it is indeed easily shown that if $3k+3 \leqslant 3g \leqslant 3k+h$, then the extra columns in $M_{k,h}$ are automatically zero. Thus, the solution to Equation (8.13), even outside the redundancy range, is given by an element of $\mathrm{Ker} M^T_{k,h}$ whose entry corresponding to $\beta^{k-h}(4\gamma)^h$ is nonzero.

However, in this range, computer calculations have shown that, although a solution to Equation (8.13) still exists, we have $\det \widetilde{Q}_{k,h} = 0$ and the solution is never unique at the level of polynomials in α, β, γ and η. By considering the difference of two such solutions, we have noticed that they come from the relations in $H^*(\mathcal{M})$ described in [HT2]. Therefore, we conjecture that the solution to Equation (8.13) is in any case unique in cohomology.

Chapter 9

The matrix problem at lower defects

In order to find a full solution to Equation (8.1), one needs to deal with the case when the pairing

$$(F, P) \mapsto \int_{Z_g} u^{3g-3-2k}(uF)P =: \langle F, P \rangle_{g,k}$$

is not at top-defect. Indeed, assume we have found F^0 with defect $2k-2$ such that, for all defect-homogeneous P of defect $2k-2$, we have $\langle F^0, P \rangle_{g,k} = 0$. The next step would then to find a polynomial F^1 such that for all P^1 of defect $2k-4$, one has

$$\langle F^0 + F^1, P^1 \rangle_{g,k} = 0. \tag{9.1}$$

We can assume the defect of F^1 to be $2k$, so that Equation (9.6) will hold for P of defect $2k-2$ as well. Inductively, the full solution of (8.1) will be a sum $F = F^0 + F^1 + \ldots + F^{k-1}$ such that

$$\langle F^0 + F^1 + \ldots + F^{k-1}, P \rangle_{g,k} = 0$$

for all P, and with defect-homogeneous F^i with defect $2k + 2i - 2$.

For the solution of (9.1), we notice that the pairing $\langle F^1, P \rangle_{g,k}$ for P of homogeneous defect $2k-4$ is still a top-defect pairing. Moreover, recalling that $\deg F^1_{g,k} = 2k-1$ and $\deg P = 3g-4$, $F^1_{g,k}$ and P must be sums of the monomials

$$F^1_{a_1,n_1} = \beta^{a_1-n_1}\gamma^{n_1}u^{k-a_1}\alpha^{k-1-a_1-n_1}, \text{ with } 0 \leqslant n_1 \leqslant a_1,\ a_1 + n_1 \leqslant k-1$$

$$P^1_{a_2,n_2} = \beta^{a_2-n_2}\gamma^{n_2}u^{k-2-a_2}\alpha^{3g-2-a_2-n_2}, \text{ with } 0 \leqslant n_2 \leqslant a_2 \leqslant k-2.$$

Therefore, the pairing at these defects is described exactly by the matrix M_{k-1} and Equation 9.6 is equivalent to the following.

Theorem 9.0.1. *Let F^0 be the solution to Equation* (8.1) *at top-defect, define the vector*

$$w_{a_2,n_2} = \langle F^0, P^1_{a_2,n_2} \rangle_{g,k}, \textit{ with } 0 \leqslant n_2 \leqslant a_2 \leqslant k-2.$$

Then solving Equation (9.6) *is equivalent to finding a vector v^1 such that*

$$v^1 M_{k-1} = w.$$

Notice that the entries of vector w of the statement of Theorem 9.0.1 are the result of pairings of defect two less than the top one. In order to make such a pairing more tractable, we perform a change of variables for F and P.

9.1 Rationalizing the generating function

Recall that in searching for a solution to Equation (8.1), we adopted the intersection formula of Corollary 7.3.4 by choosing the polynomial $Q = -Ay^2/2 - Cy^4/4$. However, the formula is valid for any polynomial multiple of y^2, with any number of formal variables. We will use a polynomial which will yield a *rational* generating function.

In particular, we choose

$$Q^F = 2\int \operatorname{arctanh}(Ay + Cy^3)dy$$

$$Q^P = 2\int \operatorname{arctanh}(Ey + Hy^3)dy.$$

This amounts to a change of basis for the spaces of polynomials F and P.

Definition 9.1.1. Define $\Phi : \mathbb{C}[\alpha, \beta, \gamma, u] \to \mathbb{C}[\alpha, \beta, \gamma, u]$ via the following rules

- Φ is $\mathbb{C}[\beta, u]$-linear;
- $\Phi(\alpha^i \gamma^j) = (-2)^{-i-j+1} \partial_A^i \big|_{A=0} \partial_C^j \big|_{C=0} \int \operatorname{arctanh}(Ay + Cy^3)dy.$

Proposition 9.1.2. *Let $\Omega \in \mathbb{C}[\alpha, \beta, \gamma, u]$ be any class, then*

$$\Phi(\Omega) = \Omega + \Theta$$

with $\operatorname{def}(\Theta) > \operatorname{def}(\Omega)$.

Proof. Write

$$Q = 2\int \operatorname{arctanh}(Ay + Cy^3)dy = Ay^2 + \frac{A^3 + 3C}{6}y^4 + \frac{A^5 + 5A^2C}{15}y^6 + \ldots =$$

$$= \sum_{n=1}^{\infty} P_n(A, C)y^{2n}$$

where the P_n's are polynomials with rational coefficients in A and C. Therefore we have

$$Q_{(2)} = 2\sum_{n=1}^{\infty} n((n-2)\alpha\beta - (n-1)\gamma)\beta^{n-2}P_n = -2\alpha A - 2\gamma C + r$$

with $r = r(\alpha, \beta, \gamma, A, C)$ with defect 4.

Thus we can write $\exp(Q_{(2)}) = \exp(-2(\alpha A + \gamma C))\exp(r)$ so that

$$\partial_A^m \partial_C^n \exp(Q_{(2)})\big|_{\substack{A=0 \\ C=0}} =$$

$$= \sum_{\substack{0\leqslant i\leqslant m\\ 0\leqslant j\leqslant n}} \binom{m}{i}\binom{n}{j} \partial_A^i \partial_C^j \exp(-2(\alpha A + \gamma C)) \partial_A^{m-i} \partial_C^{n-j} \exp(r)\big|_{\substack{A=0\\ C=0}}.$$

Now we have that

$$\operatorname{def} \partial_A^{m-i} \partial_C^{n-j} \exp(r)\big|_{\substack{A=0\\ C=0}} = 2(m+n-i-j)$$

and $\partial_A^i \partial_C^j \exp(-2(\alpha A + \gamma C))\big|_{\substack{A=0\\ C=0}} = (-2)^{i+j}\alpha^i\gamma^j$. This completes the proof for the classes of the form $\alpha^m\gamma^n$, the term $\alpha^m\gamma^n$ corresponding to the term $(i,j) = (m,n)$ in the above sum, and Θ being the remaining terms. The proof for monomials which contain u and β follows immediately, as long as the proof for any class, by $\mathbb{C}[\beta, u]$-linearity. □

By ordering the monomials in $\mathbb{C}[\alpha,\beta,\gamma,u]$ in any way such that $\Omega_1 \prec \Omega_2$ if $\operatorname{def}\Omega_1 < \operatorname{def}\Omega_2$, we have by Proposition 9.1.2 that $\Phi = \mathrm{Id} + T$ where T is a strictly lower triangular endomorphism of $\mathbb{C}[\alpha,\beta,\gamma,u]$ so we have the following.

Corollary 9.1.3. *The endomorphism Φ is a $\mathbb{C}[\beta, u]$-isomorphism.*

So if we find some $F \in \mathbb{C}[\alpha,\beta,\gamma,u]$ such that $\Phi(F)$ satisfies Theorem 8.1.1, the theorem itself would follow. Notice that the lowest defect part of $\Phi(F)$ and the one of F are equal, so the effects of such a change of basis is only seen at higher defects.

After substituting the power series Q^F and Q^P in (7.16), we make the change of variables $B = A + Cy^2$ and $J = E + Hy^2$, to obtain

$$\int_Z u^{3g-3-2k} \exp(Q^F_{(2)} + Q^P_{(2)}) = \tag{9.2}$$

$$= \operatorname*{Res}_{y=0} \frac{(B+J)^{2g-2k-2}}{y^{4k-1}} \frac{(B+J+(C+H)y^2 - (B^2H + J^2C)y^4)^g (1+BJy^2)^{2k-2}}{((1-B^2y^2)(1-J^2y^2))^{2g-2}} :=$$

$$:= \operatorname*{Res}_{y=0} R_{g,k}(B,C,J,H;y)$$

up to some non-zero constant depending only on g. When multiplying the integrand by $T_{(0)}$, the substitution rule becomes

$$\beta \mapsto y^2, \quad u \mapsto \frac{(B+J)y^2}{1+BJy^2}.$$

Notice that in this way, the generating function in (9.2) has become rational.

9.2 The general matrix pairing

We now write the most general matrix pairing, valid for F and P of any defect, in the basis obtained by the automorphism Φ. We let F to be double homogeneous of degree $2k-1$ and defect $2k+2(e-1)$, and P to be double homogeneous of degree $3g-4$ and defect $2k-2(d+1)$.

Then uF must be a linear combination of the terms

$$F^{e}_{k,a_1,n_1} := \beta^{a_1-n_1}\gamma^{n_1}u^{k-a_1+e}\alpha^{k-a_1-n_1-e}, \text{ with } \begin{cases} 0 \leqslant n_1 \leqslant a_1, \\ a_1+n_1 \leqslant k-e, \\ 0 \leqslant e \leqslant k-1 \end{cases}$$

Notice that for $e = k$, the only possible class is u^{2k}, which gives automatically zero when multiplied by $u^{3g-3-2k}$, so we do not consider such case.

Similarly, P must be a linear combination of the terms

$$P^{d}_{k,a_2,n_2} := \beta^{a_2-n_2}\gamma^{n_2}u^{k-1-a_2-d}\alpha^{3g-3-a_2-n_2+d}, \text{ with } \begin{cases} 0 \leqslant n_2 \leqslant a_2 \leqslant k-1-d, \\ 0 \leqslant d \leqslant k-1. \end{cases}$$

Notice that all the monomials in P with perverse degree at least k will give zero after being multiplied by an F of perverse degree k, so they can be safely discarded. Then thanks to Formula (9.2) we have

$$\int_Z u^{3g-3-2k}\Phi(F^{e}_{k,a_1,n_1})\Phi(P^{d}_{k,a_2,n_2}) = \tag{9.3}$$

$$= (-2)^{3-3g-k-d+e+a_1+a_2}\partial_C^{n_1}|_{C=0}\partial_H^{n_2}|_{H=0}\partial_J^{3g-3-a_2-n_2+d}|_{J=0}\partial_B^{k-a_1-n_1-e}|_{B=0}\cdot$$

$$\cdot \operatorname*{Res}_{y=0} y^{2(a_1+a_2-n_1-n_2)}\left(\frac{(B+J)y^2}{1+BJy^2}\right)^{2k-1-a_1-a_2+e-d} R_{g,k}(B,C,J,H;y).$$

We then perform a column operation and ignore the powers of -2 for the moment: we define the relevant matrix $M_{g,k}$ as follows:

$$(M_{g,k})^{d,a_2,n_2}_{e,a_1,n_1} := \frac{1}{n_1!(k-e-n_1-a_1)!}\frac{\int_Z u^{3g-3-2k}\Phi(F^{e}_{k,a_1,n_1})\Phi(P^{d}_{k,a_2,n_2})}{\int_Z u^{3g-3-2k}\Phi(F^{d}_{k,k-d,0})\Phi(P^{d}_{k,a_2,n_2})} \tag{9.4}$$

Therefore Theorem 8.1.1 is equivalent to finding a vector in the kernel of $M^T_{g,k}$ whose coefficient corresponding to the term β^k is nonzero. We now perform a change of variables to simplify the formula further.

Proposition 9.2.1. *Let*

$$V_{g,k}(X,Z;q,v,w) = \frac{(1-v(1-q))^{-g+k-X+Z}(1+w(1-q))^{g-X}}{(1-q)^{2g-2}(1-v)^{2g-X-2}(1+qv)^{g-1}}.$$

Then we have

$$(M_{g,k})^{d,a_2,n_2}_{e,a_1,n_1} = \operatorname*{Res}_{q,v,w=0}\frac{V_k(n_2,a_2+n_2-e)dqdvdw}{q^{d-e+1}v^{k-a_1-n_1-e+1}w^{n_1+1}} \tag{9.5}$$

Proof. We have to perform some algebraic manipulations of (9.3). Notice that

$$\partial_C^{n_1}|_{C=0}\partial_H^{n_2}|_{H=0}\partial_B^{k-a_1-n_1-e}|_{B=0}\operatorname*{Res}_{y=0}\frac{(B+J)^{2k-1-a_1-a_2+e-d}y^{2(2k-1-n_1-n_2+e-d)}}{(1+BJy^2)^{2k-1-a_1-a_2+e-d}}R_{g,k}$$

is a monomial in J of degree $3g-3-a_2-n_2+d$, which is precisely the order of J-derivative we are going to take. Therefore, the effect of the J-derivative is the same as setting $J = 1$ and multiply by $(3g-3-a_2-n_2+d)!$, which is a factor only depending on a_2, n_2 and d, which will be simplified out in the ratio (9.4), so it can be discarded.

Then notice that applying $\partial_H|_{H=0}$ amounts to multiplying by $y^2(1+B^2y^2)$ and lowering by 1 the exponent of the factor $(B+1+(C+H)y^2-(B^2H+C)y^4)$, and eventually setting $H=0$.

After these manipulations we are left with

$$(M_{g,k})^{d,a_2,n_2}_{e,a_1,n_1} = \partial_C^{n_1}|_{C=0}\partial_B^{k-a_1-n_1-e}|_{B=0}\cdot$$

$$\cdot \operatorname*{Res}_{y=0} \frac{(B+1)^{2g-3-a_1-a_2-d+e}(B+1+Cy^2-Cy^4)^{g-n_2}(1+By^2)^{a_1+a_2+d-e-1}}{n_1!(k-a_1-n_1-e)!y^{2(d-e+n_1)+1}(1-B^2y^2)^{2g-n_2-2}(1-y^2)^{2g-2}} =$$

$$= \operatorname*{Res}_{B,C,y=0} \frac{(B+1)^{2g-3-a_1-a_2-d+e}(B+1+Cy-Cy^2)^{g-n_2}(1+By)^{a_1+a_2+d-e-1}}{C^{n_1+1}B^{k-a_1-n_1-e+1}y^{d-e+n_1+1}(1-B^2y)^{2g-n_2-2}(1-y)^{2g-2}}$$

in which we also changed variable $y^2 \mapsto y$ (the ratio in formula (9.4) is implicitly taken here since the corresponding denominator is easily seen to be 1). By rearranging the factors, we find

$$(M_{g,k})^{d,a_2,n_2}_{e,a_1,n_1} =$$

$$= \operatorname*{Res}_{B,C,y=0} \left(\frac{B+1}{B(1+By)}\right)^{k-a_1-n_1-e+1} \left(\frac{B+1}{Cy(1+By)}\right)^{n_1+1} \left(\frac{1+By}{y(B+1)}\right)^{d-e+1} \cdot$$

$$\frac{(B+1)^{2g-k-4-a_2+e}(B+1+Cy-Cy^2)^{g-n_2}(1+By)^{k-e+a_2}y\,dB\,dC\,dy}{(1-B^2y)^{2g-n_2-2}(1-y)^{2g-2}}$$

Thus we perform the change of variable

$$q = \frac{y(B+1)}{1+By}, \quad v = \frac{B(1+By)}{B+1}, \quad w = \frac{Cy(1+By)}{B+1}$$

which after some computations and substituting $X = n_2$, $Z = a_2 + n_2$ exactly gives Formula (9.5). ☐

Remark 9.2.2. Notice that the $V_{g,k}(X,Z;v,w)$ in the proof of Proposition 8.1.3 is just the one of Proposition 9.2.1 when setting $q = 0$.

Motivated by Proposition 9.2.1, we consider the map

$$\mathrm{ev}_k : \mathbb{C}[X,Z] \to \mathbb{C}^{k(k+1)/2}, \quad \mathrm{ev}_k(p) = (p(n, a+n))_{0\leqslant n\leqslant a\leqslant k-1},$$

then the full statement for β^k, at all perversities, is the following.

Conjecture 3. *Let* $k \geqslant 1$. *Then there exist complex numbers* $\lambda^k_{l,n,e}$ *for* $e = 0,\dots,k-1$ *and* $l+2n \leqslant k-e$ *such that, for all* $d = 0,\dots,k-1$, *we have*

$$\sum_{e=0}^{k-1}\sum_{l+2n\leqslant k-e}\lambda^k_{l,n,e}\operatorname*{Res}_{q,v,w=0}\frac{V_{k-e}(X,Z;q,v,w)\,dq\,dv\,dw}{q^{d-e+1}v^{l+1}w^{n+1}} \in \mathrm{Ker}(\mathrm{ev}_{k-d}).$$

We will see how, through this formalism, we can find the next term F^1 of the solution to Equation (8.1).

9.3 The second term

We let

$$H_0(X,Z;t) := (1+t)^{Z-2X}(1+2t)^X$$

so that $p_k(X,Z) = \operatorname{Res}_{t=0} \frac{dt}{t^{k+1}} H_0(X,Z;t)$ are the polynomials of (8.5). Recall that the key relation was

$$(1+t)^{g-k-2}(1+2t)^g V_k\left(X,Z;0,-t,\frac{t^2}{1+2t}\right) = K_0(X,Z;t)$$

so that the coefficients of F^0 are the ones of $(1+t)^{g-k-2}(1+2t)^g$ multiplied by different powers of the factors $(-t)$ and $\frac{t^2}{1+2t}$, namely

$$\lambda^k_{l,n,0} = \operatorname*{Res}_{t=0} \frac{dt}{t^{k+1}}(-t)^l \left(\frac{t^2}{1+2t}\right)^n (1+t)^{g-k-2}(1+2t)^g.$$

Thanks to Proposition 9.2.1, we see that our goal is to find $\lambda^k_{l,n,1}$ such that

$$\sum_{l+2n\leqslant k} \lambda^k_{l,n,0} \operatorname*{Res}_{q,v,w=0} \frac{V_k(X,Z;q,v,w)dqdvdw}{q^2v^{l+1}w^{n+1}} +$$

$$+\sum_{l+2n\leqslant k-1} \lambda^k_{l,n,1} \operatorname*{Res}_{v,w=0} \frac{V_{k-1}(X,Z;0,v,w)dvdw}{v^{l+1}w^{n+1}} \in \operatorname{Ker}(\mathrm{ev}_{k-1}).$$

For the next term, we are interested in $\partial_q V_k$, so we compute (we omit X and Z for notational convenience):

$$\operatorname*{Res}_{q=0} \frac{dq}{q^2} V_k(q,v,w) = \partial_q|_{q=0} V_k(q,v,w) =$$

$$= -v\partial_v V_k(0,v,w) - \left(w - \frac{v(1+w)}{1-v}\right)\partial_w V_k(0,v,w) +$$

$$+\left(\frac{2(g-1)}{1-v} - \frac{v(1+v)}{1-v} + \frac{v^2-(g-1)(2-v)v}{1-v}\right) V_k(0,v,w)$$

and after some straightforward simplifications we have

$$\operatorname*{Res}_{q=0} \frac{dq}{q^2} V_k(X,Z;q,v,w) = (1-v)^{Z-3g+k-1}(1+w)^{g-X}\cdot$$

$$\cdot\left((g-1)(v-1)(v-2) + v(Z-X+k-g) + (g-X)\frac{w(v-1)}{w+1}\right)$$

The following Lemma is proven by an explicit computation and induction on k.

Lemma 13. *For polynomials* p *and* q *in* X *and* Z *and* $k \geqslant 1$, *write* $p \equiv_k q$ *meaning that* $\mathrm{ev}_k(p-q) = 0$. *Then we have*

$$\operatorname*{Res}_{t=0} \frac{dt}{t^k}\frac{\partial}{\partial v} V_k(X,Z;0,-t,\frac{t^2}{1+2t}) \equiv_{k-1} \operatorname*{Res}_{t=0} \frac{dt}{t^k}\frac{3g+Zt-k-2}{1+t} V_k(X,Z;0,-t,\frac{t^2}{1+2t})$$

$$\operatorname*{Res}_{t=0} \frac{dt}{t^k}\frac{\partial}{\partial w} V_{k-1}(X,Z;0,-t,\frac{t^2}{1+2t}) \equiv_k \operatorname*{Res}_{t=0} \frac{dt}{t^{k-2}}\frac{g-X}{(1+t)^2} V_k(X,Z;0,-t,\frac{t^2}{1+2t}).$$

Actually we can get an explicit solution from the expression of $\partial_u|_{u=0}$ and from the equivalences just stated, together with the obvious $V_k \equiv_{\mathrm{ev}_{k-1}} 0$.

Proposition 9.3.1. *Let*

$$H_1(X,Z;t) :=$$
$$:= (1+t)^{Z-2X}(1+2t)^X \left(g-1-\frac{g-2}{t+1}+\frac{3g+Zt-k-2}{1+t}+(g-X)\frac{t^2}{(1+t)^2} \right)$$

and let

$$p_{k,1}(X,Z) := -\operatorname*{Res}_{t=0} \frac{dt}{t^k} H_1(X,Z;t).$$

Then the entries of the vector

$$v_{k,1} := (D_Z^l D_{X,-1}^n \Big|_{\substack{Z=3g-k-1\\X=g}} p_{k,1}(X,Z))_{l+2n\leqslant k-1}$$

are the coefficients of the solution F^1 *to Equation* (9.1).

Proof. Recall from Remark 8.1.5 that letting

$$H_0(X,Z;t) := (1+t)^{Z-2X}(1+2t)^X, \quad p_k(X,Z) := \operatorname*{Res}_{t=0} \frac{dt}{t^{k+1}} H_0(X,Z;t),$$

then if we define

$$v_{k,0} = (D_Z^l D_{X,-1}^n|_{\substack{Z=3g-k-2\\X=g}} p_k(X,Z))_{l+2n\leqslant k}$$

then $v_{k,0}M_{k,0} = 0$. Now, the statement of the Proposition is that

$$v_{k,0}M_{k,1} + v_{k,1}M_{k-1,0} = 0. \tag{9.6}$$

To prove Equation (9.6) we notice that

$$D_Z^l D_{X,-1}^n H_1(X,Z;t) = \frac{t^{l+2n}}{(1+2t)^n}(H_1(X,Z;t)+(n+l)H_0(X,Z;t))$$

and that $\deg p_{k,1} = k-1$ where $\deg Z = 1$ and $\deg X = 2$. Therefore $v_{k,1}M_{k-1,0}$, when writing out the definition, is the (values in X and Z) of the Taylor expansion of

$$\operatorname*{Res}_{t=0} \frac{dt}{t^k}(H_1(g,3g-k-1;t)V_{k-1}(X,Z;0,-t,\frac{t^2}{1+2t})+ \tag{9.7}$$
$$-H_0(g,3g-k-1;t)(t\partial_v V_{k-1}(X,Z;0,-t,\frac{t^2}{1+2t})-\frac{t^2}{1+2t}\partial_w V_{k-1}(X,Z;0,-t,\frac{t^2}{1+2t}))).$$

On the other side, since

$$D_Z^l D_{X,-1}^n H_0(X,Z;t) = \frac{t^{l+2n}}{(1+2t)^n} H_0(X,Z;t)$$

we have that $v_{k,0}M_{k,1}$ is the Taylor expansion of

$$\operatorname*{Res}_{t,u=0} \frac{dt}{t^{k+1}}(H_0(g,3g-k-2;t)\frac{du}{u^2}V_k(X,Z;u,-t,\frac{t^2}{1+2t}))$$

but now we notice that

$$
\begin{aligned}
-\operatorname*{Res}_{u=0}\frac{du}{u^2}H_0(g,3g-k-2;t)V_k(X,Z;u,-t,\frac{t^2}{1+2t})+\\
+tH_1(g,3g-k-1;t)V_{k-1}(X,Z;0,-t,\frac{t^2}{1+2t})+\\
-tH_0(g,3g-k-1;t)(t\partial_v V_{k-1}(X,Z;0,-t,\frac{t^2}{1+2t})+\\
-t\frac{t^2}{1+2t}\partial_w V_{k-1}(X,Z;0,-t,\frac{t^2}{1+2t}))=\\
=(1+t)^{Z-2X}(1+2t)^X(2+g(t-2)+t(Z-X))
\end{aligned}
$$

and it is easy to see that

$$
\operatorname*{Res}_{t=0}\frac{dt}{t^{k+1}}(1+t)^{Z-2X}(1+2t)^X(2+g(t-2)+t(Z-X))=0
$$

for $X \geqslant 0$, $Z \geqslant 2X$ and $Z-X \leqslant k-2$. Noticing that $v_{k,0}M_{k,1}+v_{k,1}M_{k-1,0}$ is just the vector of the values in the region just considered, this completes the proof. □

9.4 Conclusions

In order to find the full solution to Equation (8.1), one needs to find the coefficients $\lambda^k_{l,n,e}$ of Conjecture 3 for all $e \leqslant k-1$. We have not been able to find the higher generating functions H_e, though the key to deal with higher q-derivatives of V_k seems to be the following relation

$$
\partial_q V_k+\frac{v}{1-q}\partial_v V_k+\left(\frac{w}{1-q}-\frac{v(1+w(1-q))}{(1-v)(1-q)^2}\right)\partial_w V_k=
$$

$$
=\left(\frac{2(g-1)}{(1-q)(1-v)(1+qv)}-\frac{v(1+v)}{(1-q)(1-v)(1+qv)}+\frac{v^2-(g-1)(2-v)v}{(1-v)(1+qv)}\right)V_k.
$$

satisfied by V_k. With this, we can hope to find the lower defects of the solution for β^k in a similar way to the one used in the proof of Proposition 9.3.1. For the more general classes $\beta^{k-h}\gamma^h$, already the problem at top-defect seems to be much more complicated, and we have not been able to find any higher-defect solution.

Notice also that the higher-defect part of the solution is *not* unique, since we can simply multiply the lowest-defect part of the solution for $k-1$ by u to get a class which vanishes when multiplied by a class P of defect $k-2$. Eventually, the space of solutions to (8.1) will be an *affine* space of dimension k.

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