Geometric and algorithmic aspects of nilpotent groups

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Geometric and Algorithmic Aspects of Nilpotent Groups

Thèse

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autorise l'impression de la présente thèse, sans exprimer d'opinion sur les propositions qui y sont énoncées.

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La Doyenne

N.B. - La thèse doit porter la déclaration précédente et remplir les conditions énumérées dans les "Informations relatives aux thèses de doctorat à l'Université de Genève".

Résumé

Quand on travaille avec des groupes nilpotents, on pense généralement avoir à faire à des groupes qui se comportent bien. Typiquement, le problème des mots et le problème d'appartenance à des sous-groupes sont décidables efficacement. Leur géométrie est également bien comprise, par exemple leurs fonctions de croissance sont connues de manière bien plus précise que pour d'autres groupes, grâce à des résultats de Pansu et Stoll faisant le lien avec l'étude des groupes de Lie nilpotents.

Dans cette thèse, nous montrerons que ces groupes se comportent de manière surprenante en plus d'un sens, et nous permettent ainsi de résoudre certains problèmes de théorie des groupes via divers exemples aux propriétés exotiques. Nos résultats vont dans différentes directions, parfois purement géométrique comme pour le bord des horofonctions, ou algorithmique pour les problèmes d'appartenance. Cependant, ces résultats restent liés par une même intuition, tirée d'un modèle de chemins pour les groupes nilpotents, spécifiquement les groupes de Heisenberg, d'Engel et de Cartan.

Premièrement, nous exhibons un groupe virtuellement nilpotent pour lequel le nombre de géodésiques de longueur n croit de manière intermédiaire. Ceci réponds à une question restée longtemps ouverte, et vient comme une surprise, les efforts pour construire un tel exemple étant centrés principalement sur des groupes à croissance intermédiaire. En passant, nous prouvons différents résultats sur la géométrie des groupes virtuellement nilpotents, qui seront utiles dans le reste de la thèse.

Par la suite, on étudie les bords des horofonctions de groupes nilpotents, et de manière plus approfondie les groupes de Heisenberg et de Cartan. Dans le premier cas, l'ensemble des points de Busemann est dénombrable, et l'action sur le bord réduit est triviale. Dans le second cas, l'ensemble est indénombrable, et l'action est non-triviale. Ceci réfute des conjectures de Tointon et Yadin, et de Bader et Finkelshtein.

Dans un troisième temps, nous nous intéressons à des problèmes de décision, et plus spécifiquement aux problèmes d'appartenance à un sous-monoïde et à un sous-ensemble rationnel. On montre que le problème d'appartenance à des sous-ensembles rationnels de $H_3(\mathbb{Z})$ est décidable, et en déduit le même résultat pour les sous-monoïdes du groupe de Engel. Par ailleurs, on montre

l'existence d'un groupe (nilpotent !) pour lequel le problème pour les sous-monoïdes est décidable, mais celui pour les sous-ensembles rationnels est indécidable. Ceci confirme une conjecture de Lohrey et Steinberg.

On regarde ensuite à la $\mathbb{N}G$-rationalité des séries de croissance complètes de certains groupes nilpotents, ainsi que certains groupes d'allumeur de réverbères. Dans de nombreux cas, nous montrons que la série n'est pas $\mathbb{N}G$-rationnelle, voire pas $\mathbb{N}G$-algébrique pour $G = H_3(\mathbb{Z})$. Ceci réponds partiellement à deux questions de Grigorchuk, de la Harpe et Nagnibeda.

Dans le dernier chapitre, on étudie la série de Green d'un groupe virtuellement nilpotent introduit par Bishop et Elder et montre qu'elle n'est pas holonomique. Il s'agit du premier exemple de ce type parmi les groupes virtuellement nilpotents. La preuve repose sur un petit miracle permettant de compter certains mots, et deux doses de théorie des nombres.

Abstract

When working with nilpotent groups, we usually think having to deal with a relatively tame class groups. Typically, the Word Problem and the subgroup membership problem are decidable efficiently. Their geometry is also relatively well-understood. For instance their growth functions are known much more precisely than for other groups, thanks to results of Pansu and Stoll making the link with nilpotent Lie groups.

In this thesis, we will highlight that these groups behave surprisingly in many ways, and allow us to answer several problems in group theory, using different examples with exotic properties. Our results go in many directions, some purely geometric with the study of the horofunction boundary, and others more algorithmic with membership problems. However, these results stay connected by a same intuition, coming from a path model for nilpotent groups, specifically the Heisenberg, Engel and Cartan groups.

First, we exhibit a virtually nilpotent group for which the number of geodesics of length n has intermediate growth. This answer a long-standing open question, and comes as a surprise as most efforts to construct such an example were centered around groups of intermediate growth. As a byproduct, we prove several results on the geometry of virtually nilpotent group, that will be useful for the remainder of the thesis.

We follow by studying horofunction boundaries of nilpotent groups, with an in-depth look at the Heisenberg and Cartan groups. For the former, the set of Busemann points is countable and the action on the reduced boundary is trivial. For the latter, the set is uncountable and the action is non-trivial. This disproves conjectures of Tointon–Yadin and Bader–Finkelshtein.

In the third part, we look at decision problems, more specifically the membership problems to submonoids and rational subsets, and the identity problem. We prove that membership to rational subset of $H_3(\mathbb{Z})$ is decidable, and deduce the same result for submonoids of the Engel group. In another direction, we prove the existence of a (nilpotent!) group for which membership to submonoids is decidable, but membership to rational subsets is undecidable. This confirms a conjecture of Lohrey and Steinberg.

Next, we take interest in the $\mathbb{N}G$-rationality of complete growth series of some nilpotent groups, and the lamplighter groups. In many cases, we show that the series is not $\mathbb{N}G$-rational, and not $\mathbb{N}G$-algebraic for $G = H_3(\mathbb{Z})$. This partially answers questions of Grigorchuk, de la Harpe and Nagnibeda.

In the last chapter, we study the Green series of a virtually nilpotent group introduced by Bishop and Elder, and show that it is not D-finite. This is the first example of this type among virtually nilpotent groups. The proof relies on a small miracle allowing to count some words, and a good dose of number theory.

Remerciements

Tout d'abord, j'aimerais remercier Tatiana Nagnibeda pour ses conseils et ses idées, ainsi que la liberté mathématique qu'elle m'a donnée.

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Bien sûr, la liste ne s'arrête pas là, beaucoup de "mathematically grown-up" m'ont expliqué des idées, suggéré des améliorations, ou de nouveaux problèmes. Je remercie entre autre Susan Hermiller, Pierre de la Harpe, Laurent Bartholdi, Christophe Pittet, Slava Grigorchuk, Enrico le Donne, Anna Erschler, Anders Karlsson, Laura Ciobanu, George Metcalfe.

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Contents

Chapter 1

Introduction

This thesis explores different questions of Algorithmic and Geometric Group Theory, with a focus on *nilpotent groups*. Finitely generated torsionfree nilpotent groups coincide with groups of matrices with 1's on the diagonal and 0's below the diagonal. The typical example is the 3-dimensional Heisenberg group. However, instead of considering elements as matrices, we will consider them as classes of paths.

$$\begin{pmatrix} 1 & 8 & 43 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{pmatrix}$$

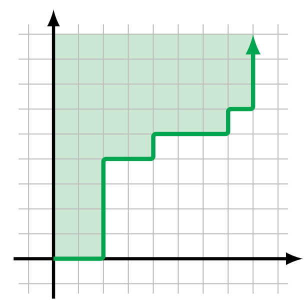


Figure 1.1: A matrix in $H_3(\mathbb{Z})$ and a lattice path representing it.

We will see how this geometric model can be useful to study not only geometric aspects of nilpotent groups, but also algorithmic ones. Specifically, we study classical questions such as membership problems and volume growth of groups, and some more modern twists such as geodesic growth. We will see how these inform each other.

Geometric aspects

A common scheme in Geometric Group Theory is to consider groups as geometric spaces, define some notion of growth and then characterize groups with growth in a given regime by their algebraic properties. The first example is volume growth: given a group G and a generating set S, we associate the *volume growth function*

$$\beta_{G,S}(n) = \#\left\{g \in G \;\middle|\; \|g\|_S \leqslant n\right\},$$

where $\|g\|_S$ is the word metric: the length of the shortest word over S evaluating to g.

Milnor famously asked two questions about the volume growth of groups [Mil68]:

- For which groups do we have $\beta_{G,S}(n) \preceq n^d$ for some constant $d \geqslant 0$? Milnor further suggested the family of virtually nilpotent groups.
- Does the volume growth necessarily satisfy $\beta_{G,S}(n) \preceq n^d$ or $\beta_{G,S}(n) \succeq \exp(n)$?

On the one hand, Gromov confirmed that groups with polynomial volume growth coincide with virtually nilpotent groups [Gro81a]. This is the gold standard of theorems in Geometric Group Theory, as it links a natural geometric condition (polynomial growth) with a purely algebraic one (virtually nilpotent). Doing so, he introduced a large machinery, notably asymptotic cones, to make the link with nilpotent Lie groups.

On the other hand, Grigorchuk constructed a family of groups of intermediate growth (i.e., whose growth is neither bounded above by polynomials, nor below by exponentials). This demonstrates the diversity of groups. This family is now source of groups with many intriguing properties [Gri85]. These results constitute an ideal to aim for.

However, these are not the only questions we can ask about growth, nor the only notions of growth we can consider. For instance, a difficult question is about the regularity and fine asymptotics of growth. Notably, Pansu proved that

$$\beta_{G,S}(n) = c \cdot n^d + o(n^d)$$

for $c = c(G,S) \in \mathbb{R}_{>0}$ and $d = d(G) \in \mathbb{Z}_{\geqslant 0}$ as soon as G is virtually nilpotent [Pan83]. A long standing conjecture is to improve this estimate to $\beta_{G,S}(n) = c \cdot n^d + O(n^{d-1})$. The progress has been slow toward this goal, Stoll proved the result for 2-step nilpotent groups [Sto98], and combined work of Breuillard–Le Donne and Gianella gives

$$\beta_{G,S}(n) = c \cdot n^d + O(n^{d-\frac{1}{s}})$$

for general s-step nilpotent groups [BLD13; Gia17].

We can push further and ask about rationality of the (standard) *growth series*

$$\Sigma_{G,d_S}(t) = \sum_{n=0}^{\infty} \sigma_{G,S}(n) \cdot t^n = \sum_{g \in G} t^{\|g\|_S} \in \mathbb{N}[[t]],$$

where $\sigma(n) = \beta(n) - \beta(n-1)$. (Rationality of $\Sigma(t)$ implies $\beta(n) = c \cdot n^d + O(n^{d-1})$.) Cannon proved that this growth series were rational as soon as G is hyperbolic [Can84], and Benson proved the same result for G virtually abelian [Ben87]. In both cases, the underlying reason are strong language theoretic properties, which can be used to prove more strongly that the *complete growth series*

$$\widehat{\Sigma}_{G,d_S}(t) = \sum_{g \in G} g \cdot t^{\|g\|_S} \in \mathbb{N}G[[t]]$$

is also rational [Lia96; GN97]. Among nilpotent groups, the picture is mixed. Stoll proved that $H_5(\mathbb{Z})$ has a generating set for which the growth series is rational, and another for which the series is transcendental [Sto96]. In contrast Duchin and Shapiro proved that the (standard) growth series of $(H_3(\mathbb{Z}), d_S)$ is always rational [DS19]. This begs the question whether the associated complete growth series are rational, and which are the mechanisms underlying the rationality of growth series in some nilpotent groups.

Another active industry within Geometric Group Theory is to find alternate proof of Gromov's theorem (mentioned earlier). This is not only useful for pedagogical purposes: different proofs give different informations on the group, some arguments can be more effective, or generalize more easily (with the Gap Conjecture in mind). A recent attempt in this direction is the study of the *horofunction boundary* $\partial(G, d)$. This boundary was introduced by Gromov [Gro81b]. It is the adherence of functions

$$\varphi_x(h) = d(x, h) - d(x, e)$$

within the space of functions $G \to \mathbb{Z}$ (with the pointwise convergence topology). This boundary has mostly been studied for hyperbolic, being closely related to the visual boundary [WW06]. Recently, the horofunction boundary has been of a great use to embed hyperbolic groups inside finitely presented simple groups [BBM17; BBMZ23]. Our motivation is somewhat different: G acts on its boundary $\partial(G, d)$ via

$$g \cdot \varphi(h) = \varphi(g^{-1}h) - \varphi(g^{-1}).$$

Something which should be clear from this formula is that $\varphi\colon \mathrm{Stab}_G(\varphi) \to \mathbb{Z}$ is an homomorphism. In particular, finite orbits for the action $G \curvearrowright \partial(G, d)$ give rise to *virtual characters* (that is, epimorphism $\varphi\colon H \twoheadrightarrow \mathbb{Z}$, with $H \leqslant G$ finite-index) which are key ingredients in all known proofs of Gromov's theorem.

This shifts the interest to finding finite orbits under some "small growth" hypothesis. This was achieved by Tointon and Yadin for groups of linear growth [TY16], using the set of *Busemann points*, which are limits of geodesic rays. They conjecture

Conjecture ([TY16])**.** Let (G, d_S) be a polynomially growing group, then the set of Busemann points in $\partial(G, d_S)$ is finite or countable.

Falling short of fixed points, we can also ask for horofunctions which are fixed up to a bounded function. This is the approach explored by Bader and Finkelshtein [BF20]. More precisely, they consider the *reduced boundary*

$$\partial^r(G, d_S) \coloneqq \partial(G, d_S)/O(1),$$

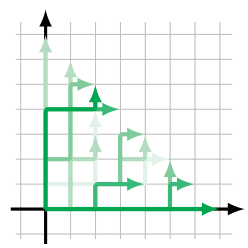
where $O(1)$ is the space of bounded function $G \to \mathbb{Z}$. They proved that $G \curvearrowright \partial^r(G, d_S)$ is trivial for G abelian and $G = H_3(\mathbb{Z})$, and further conjectured

Conjecture. Let (G, d_S) be a nilpotent group, then the action $G \curvearrowright \partial^r(G, d_S)$ is trivial.

Another notion of growth is the *geodesic growth* of group. We consider the function

$$\gamma_{G,S}(n) = \#\big\{w \in S^\star \mid \ell(w) = \|\bar{w}\|_S \leqslant n\big\},$$

which counts the number of geodesic words of length at most n. Most of the work around geodesic growth as been done on the associated series, this is for instance a key part of the argument of Cannon mentioned earlier. Perhaps for this reason, the two analogs of Milnor questions were still open. For a characterization of groups (G, S) with polynomial geodesic growth, we directly realize we are looking at a subclass of all virtually nilpotent groups. Indeed, every element corresponds to at least one geodesic. Moreover, basic groups like $\mathbb{Z}^2$ necessarily have exponential geodesic growth [Sha97].



This gives the impression that pairs (G, S) with polynomial geodesic growth are rare. Despite this, a few families of examples were constructed, containing virtually abelian groups [BBES12; Bis21], and even an example of the form $H_3(\mathbb{Z}) \rtimes C_2$ [BE22]. A few question that remained to be answered is whether polynomial geodesic growth is generic or to the contrary very sparse, and giving a clear criterion. For the existence of a group with intermediate geodesic growth, most of the effort has been concentrated on groups of intermediate volume growths, going all the way back to 1993 [Gri14, p. 755-756]. The results so far have only been negative [Brö16] or inconclusive [Bis17].

At last, we can consider the Word Problem $\mathcal{WP}(G, S) = \{w \in S^\star \mid \bar{w} = e_G\}$, and consider the associated series

$$\Gamma_{G,S}(z) = \sum_{n\geqslant 0} c_n \cdot z^n \in \mathbb{N}[[z]] \quad \text{where } c_n = \#(\mathcal{WP}(G, S) \cap S^n).$$

This series is called the (rescaled) Green series. As for other combinatorial sequences, there is a long tradition of trying to pinpoint $\Gamma_{G,S}(z)$ in the algebraic hierarchy

$$\text{rational} \subset \text{algebraic} \subset \text{diagonal of rational} \subset D\text{-finite} \subset D\text{-algebraic}.$$

This activity also has applications, as computing the Green series gives access to the spectral radius, hence to amenability via Kesten criterion [Kes59], or even to much more precise asymptotics for the return probabilities of random walks [Lal93; Woe12]. Moreover, there are strong links with the complexity of the Word Problem:

- The Green series is rational if and only if the Word Problem is regular, which happens if and only if the group is finite [Ani72; Kou98],

- It is conjectured that the series is algebraic if and only if the Word Problem is context-free, which happens if and only if the group is virtually free [MS83]
- The Green series is a diagonal of rational if G is virtually $F_m \times \mathbb{Z}^n$ [Bis24]. Tthis seems related to Word Problem recognizable by $(F_m \times \mathbb{Z}^n)$-automata (in the sense of [Kam06]), with few exceptions (eg. $F_2 \times F_2$ with standard generating set).

An impressive result is that the Green series of amenable groups of super-polynomial volume growth are not D-finite [BM20]. This left open the question for (non virtually abelian) virtually nilpotent groups. Only very recently some partial (but convincing) results were given by Pak and Soukup [PS22]: if we consider $G = UT_m(\mathbb{Z})$ the group of unitriangular matrices of dimension $m = 9.6 \cdot 10^{85}$ (!) and let the generating set S vary, either some series cannot be represented as a diagonal of rational series, or at least no algorithm can give such a representation (meaning any proof would be ineffective).

In **Chapter 4**, we study the geometry of virtually nilpotent groups, with an emphasis on geodesic growth. Our main result is the construction of a group (virtually 3-step nilpotent!) with intermediate geodesic growth:

Theorem (4.C)**.** *The geodesic growth of the group*

$$v\mathscr{E} = \left\langle a, t \;\middle|\; t^2 = 1;\ [a,[a,a^t]] = [a^t,[a,a^t]] \text{ commutes with } a, a^t \right\rangle$$

with generating set $S = \{a^{\pm 1}, t\}$ *satisfies* $\gamma_{\text{geod}}(n) \asymp \exp\big(n^{3/5} \cdot \log(n)\big)$.

More generally, we provide a criterion saying when the geodesic growth of a pair (G, S) is sub-exponential, when G is virtually nilpotent (Theorem 4.A). For virtually 2-step nilpotent groups, this specializes as a criterion for polynomial geodesic growth.

Along the way, we give estimates on word metrics in virtually nilpotent groups (Proposition 4.1.1), which are used in the subsequent Chapters 5 and 7. Another application is an extension of the Stoll–Breuillard–Le Donne–Gianella estimates on volume growth mentioned earlier to *virtually* nilpotent groups:

Corollary (4.D)**.** *Let G be a virtually s-step nilpotent group, and S a finite symmetric generating set. The volume growth satisfies*

$$\beta_{G,S}(n) = c_{G,S} \cdot n^d + O(n^{d-\delta_s}),$$

where $\delta_s = 1$ *for* $s = 1, 2$ *and* $\delta_s = \frac{1}{s}$ *for* $s \geqslant 3$.

In **Chapter 5**, motivated by the conjecture of Tointon and Yadin, we study the set of Busemann points in $\partial(G, d_S)$ for 2-step and 3-step nilpotent groups.

- We give a characterization of orbits of Busemann points in Heisenberg groups $H_{2n+1}(\mathbb{Z})$ (Theorem 5.3.3), in particular the set of Busemann points is countable.

- In contrast, in $\mathscr{C} = N_{2,3}$ the free nilpotent group of rank 2 and nilpotency class 3, we exhibit uncountably many Busemann points (Theorem 5.5.6).

As a byproduct, we also disprove the conjecture of Bader and Finkelshtein.

In **Chapter 7**, we study when complete growth series are rational, and more specifically when they are $\mathbb{N}G$-rational and $\mathbb{N}G$-algebraic. These conditions translates into the existence of a normal form with strong geometric properties. In turn, we isolate two sufficient conditions to prove non-$\mathbb{N}G$-rationality:

- The first one is purely geometrical, with *dead ends* and *almost saddle elements* appearing. These are notions from "fine geometry", meaning we can replace the original metric d_S by $\tilde{d} = d_S + O(1)$ to study them. This allows us to apply this criterion to lamplighter groups and (non virtually abelian) 2-step nilpotent groups.

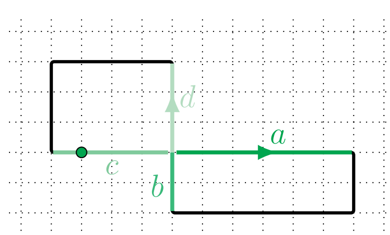
- The second give the existence of a regular normal form consisting of $(1+\varepsilon)$-quasi-geodesics. Here we can replace the original metric by $\tilde{d} = d_S + o(d_S)$. This extra freedom opens the door to use Pansu's theorem comparing the word metrics on nilpotent groups and associated CC-metrics. In particular, we apply the criterion to (non virtually abelian) nilpotent groups of rank 2 and any nilpotency class.

In both cases, we conclude the complete growth series are never $\mathbb{N}G$-rational. For the Heisenberg group $H_3(\mathbb{Z})$, we conclude more strongly that the series are not $\mathbb{N}G$-algebraic. This provides the first examples where the standard growth series is rational, and the complete growth series is not rational, and makes progress toward the conjecture that complete growth series of (non virtually abelian) nilpotent groups is never rational.

Finally, in **Chapter 8**, we prove that the Green series $\Gamma_{v\mathscr{H},S}(z)$ of

$$v\mathscr{H} = H_3(\mathbb{Z}) \rtimes C_2 = \left\langle x, t \;\middle|\; t^2 = [x,[x,txt]] = 1 \right\rangle$$

with respect to the generating multiset $S = \{x, x^{-1}, t, t, t, t, t, t, t, t\}$ is not D-finite. We study the *subword complexity* of the sequence c_n modulo some power of 2. This leads us to count solutions to Diophantine equations, and look at some multiplicative function. We conclude using a lemma of Garrabrant and Pak. This is the first example of virtually nilpotent group whose Green series is not D-finite.



Algorithmic aspects

Membership problems are some of the central motivating questions in algorithmic (semi)group theory. In contrast with Dehn's Word Problem [Deh10] which is easily and efficiently decidable in linear groups [LZ77], even the most basic membership problems already lie at the boundary between decidability and undecidability. The first problem in this family is the Submonoid Membership. For a (semi)group G, we attempt to produce algorithms with the following specifications:

(Uniform) Submonoid Membership ($\mathsf{SMM}(G)$)

Input: Elements g and $g_1, g_2, \ldots, g_n \in G$

Output: Decide whether $g \in \{g_1, g_2, \ldots, g_n\}^*$.

This problem was introduced by Markov [Mar47]. He gave the first undecidability results in semigroup theory: there exist no algorithm deciding if $g \in \{g_1, \ldots, g_n\}^*$, where $g, g_1, \ldots, g_n \in \mathbb{Z}^{6\times 6}$ are 6×6 integers matrices. These undecidability results were extended to the **Subgroup Membership** in $SL_4(\mathbb{Z})$ by Mihailova [Mih58], and to the **Matrix Mortality problem** in $\mathbb{Z}^{3\times 3}$ by Paterson [Pat70].

The next problem in line is the

(Uniform) Rational Subset Membership ($\mathsf{RatM}(G)$)

Input: An element $g \in G$ and a rational subset $R \subseteq G$ (defined by a finite state automaton, labeled by elements in G).

Output: Decide whether $g \in R$.

Rational subsets where first considered inside free monoids, they are then called *regular language*. Deciding membership to a language is one of the core problems in formal language theory, and regular languages are specifically those languages for which a read-only Turing machine can decide membership. This problem was solved in abelian groups by Schützenberger–Eilenberg [ES69], and in free groups by Benois [Ben69]. It's only much more recently this problem was considered in more general groups such as virtually abelian groups [Gru99], wreath products [LSZ15], solvable Baumslag-Solitar groups [CCZ20] and others [KSS07; LS08; LS10; Gra20].

It should be noted that finitely generated submonoids are rational subsets:

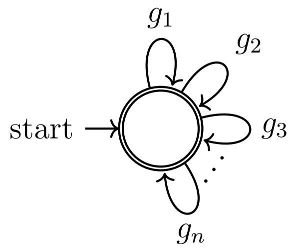


Figure 1.2: An automaton defining the submonoid $R = \{g_1, g_2, \ldots, g_n\}^*$.

Therefore, we have two natural problems, the first being a special case of the second, it is natural to ask whether they are actually equivalent. This was first formally asked by Lohrey and Steinberg [LS08], and they formulated a conjecture in a follow-up paper:

Conjecture ([LS10])**.** There exists a finitely generated group with decidable Submonoid Membership, and undecidable Rational Subset Membership.

In **Chapter 6**, we study these problems within nilpotent groups. Our first result is

Theorem (6.C)**.** *Rational Subset Membership is decidable in* $H_3(\mathbb{Z})$*.*

This is the first nilpotent group for which $\mathsf{RatM}(G)$ is shown to be decidable (except for virtually abelian examples). The proof idea comes from a simple observation:

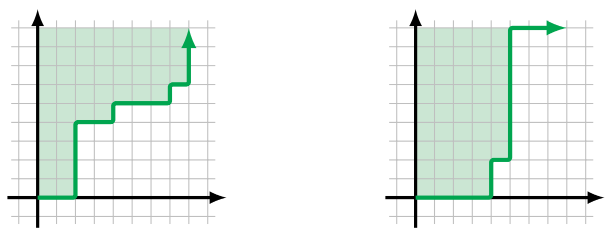


Figure 1.3: Every element in $\{x, y\}^*$ can be written as $x^{n_1}y^{n_2}x^{n_3}y^{n_4}x^{n_5}$ with $n_i \geqslant 0$.

Generalizing this observation, we reduce the general Rational Subset Membership in $H_3(\mathbb{Z})$ to a special case: the Knapsack problem. This is another modern twist on a classical problem from computer science (in $\mathbb{N}$) introduced relatively recently [MNU13]. This concludes as König, Lohrey and Zetzsche proved that the Knapsack problem is decidable in $H_3(\mathbb{Z})$ [KLZ16], using deep results around Hilbert's 10th problem [GS04].

As a corollary, we get that

Corollary (6.4.1)**.** *Submonoid Membership is decidable in the Engel group.*

This extends on previous results of Colcombet, Ouaknine, Semukhin and Worrell solving the Submonoid Membership in $H_3(\mathbb{Z})$ [COSW19].

Using a new reduction and known undecidability results for Rational Subset Membership in some 2-step nilpotent groups [KLZ16; MT17], we also confirm the Lohrey-Steinberg conjecture:

Theorem (6.B)**.** *There exists a finitely generated* 2*-step nilpotent group with decidable Submonoid Membership, and undecidable Rational Subset Membership.*

(It should be noted that Shafrir provided an earlier example, specifically $A \wr \mathbb{Z}^2$ with A finite abelian. However, this work is still unpublished. The proof was nicely written down in [Pot20], and a more general result is now available for $C_p \wr \mathbb{Z}^n$ [Don24].)

Attributions

Section 3.1 and Chapter 4 can be found in

- C. Bodart, *Intermediate geodesic growth in virtually nilpotent groups*, to appear in Groups, Geometry and Dynamics, 2024

Chapter 5 is joint work with Kenshiro Tashiro. Most of the content can be found in

- C. Bodart and Kenshiro Tashiro, *Horofunctions on the Heisenberg and Cartan groups*, `https://arxiv.org/abs/2407.11943`, 2024.

Chapter 6 is extracted from two different articles. Section 6.1 on the Identity problem is taken from a joint paper with Laura Ciobanu and George Metcalfe. The other results on membership problems are taken from a single-authored paper:

- C. Bodart, Laura Ciobanu and George Metcalfe, *Ordering nilpotent groups and the identity problem*, in preparation.
- C. Bodart, *Membership problems in nilpotent groups*, `https://arxiv.org/abs/2401.15504`, 2024.

Sections 7.1, 7.2, 7.4 and Theorem 7.5.3 are joint work with Pierre Bagnoud. Most of it can found in the joint paper

- Pierre Bagnoud and C. Bodart, *Dead ends and rationality of complete growth series*, Advances in Mathematics, August 2024, vol. 452, p. 109821

The proof of Lemma 7.5.12 was suggested by Enrico Le Donne.

Chapter 8 can be found in

- C. Bodart, *A virtually nilpotent group whose Green series is not D-finite*, `https://arxiv.org/abs/2409.13395`, 2024.

Sections 3.2, 7.3, 7.5 and Theorem 5.B are unpublished.

Chapter 2

Background

2.1 Finitely generated groups as metric spaces

Let G be a group, generated as a monoid by a finite set S.[1]

Definition 2.1.1 (Word length)**.** We consider G a group, and S a finite (monoid) generating set. Moreover, we consider $\omega\colon S \to \mathbb{R}_{>0}$ a weight function.

- For every word $w = s_1 s_2 \ldots s_\ell \in S^\star$, we define its *length* as

$$\ell_\omega(w) \coloneqq \omega(s_1) + \omega(s_2) + \ldots + \omega(s_\ell).$$

- There is an *evaluation map* $\mathrm{ev}\colon S^\star \to G$. The image $\mathrm{ev}(w)$ is the element obtained interpreting the word w as a product in G. For short $\mathrm{ev}(w) = \bar{w}$.
- The *word length* of an element $g \in G$ is defined as

$$\|g\|_{S,\omega} \coloneqq \min\{\ell_\omega(w) \mid w \in S^\star \text{ such that } \bar{w} = g\}.$$

Remark 2.1.2. We will often (but not always) make a few additional assumptions:

- Most of the time $\omega \equiv 1$. In this case the subscript ω is dropped from the notations.
- S is *symmetric* if $S = S^{-1}$. In that case, we have $\|g^{-1}\|_S = \|g\|_S$. In particular, the function $d_S(g,h) = \|g^{-1}h\|_S$ is symmetric hence a genuine distance.

The geometrical realization of these length functions are Cayley graphs:

[1]Most of the time, we should picture $S \subseteq G$. However, this is not always the most practical. Often, we should rather think of S as an abstract set of symbols, together with an evaluation map $\mathrm{ev}\colon S^\star \to G$, with possibly several symbols sent to the same element in G. This avoids the use of multisets, for instance when considering decomposition and defining a weight of X in §4.1.1.

Definition 2.1.3 (Cayley graph)**.** Given a group G and a monoid generating set S, the *Cayley graph* is $\mathcal{C}ay(G,S) = (V,E)$ where the vertex and edge sets are given by

$$V = G \quad \text{and} \quad E = \{(g, gs) : g \in G,\ s \in S\}.$$

Edges are considered oriented and labeled (so the edge $g \to gs$ is labeled by s).

The word metric d_S coincide with the graph metric on $\mathcal{C}ay(G,S)$. A natural generalization of Cayley graphs, for coset spaces are Schreier graphs:

Definition 2.1.4 (Schreier graph)**.** Given a group G, a subgroup $H \leqslant G$, and a monoid generating set S, the *Schreier graph* is $\mathcal{S}ch(H\backslash G, S) = (V,E)$ where

$$V = H\backslash G \quad \text{and} \quad E = \{(Hg, Hgs) : Hg \in H\backslash G,\ s \in S\}.$$

Edges are considered oriented and labeled (so the edge $Hg \to Hgs$ is labeled by s).

We can associate many invariants to such graphs.

Definition 2.1.5. To a triplet (G, S, ω), we associate two growth functions:

- The *volume growth* function of (G, S, ω) is
$$\beta_{G,S,\omega}(n) = \#\{g \in G : \|g\|_{S,\omega} \leqslant n\} = \#B_{G,S,\omega}(e, n).$$
- The *spherical growth* function is $\sigma_{G,S,\omega}(n) = \#\{g \in G : \|g\|_{S,\omega} = n\}$.

Volume growth is more robust to change in the metric (eg. when working up to quasi-isometry), while the spherical growth is more practical when working with growth series (eg. when working with geodesic normal forms). Concretely

Definition 2.1.6. Given two increasing functions $f, g\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$, we say that

- g *asymptotically dominates* f (written $f \preceq g$) if there exists a constant $C > 0$ such that $f(n) \leqslant C \cdot g(Cn)$ for all n.
- f and g *are asymptotically equivalent* (written $f \asymp g$) if $f \preceq g$ and $g \preceq f$.

Definition 2.1.7. Let (X, d_X), (Y, d_Y) be metric spaces. A map $f\colon (X, d_X) \to (Y, d_Y)$ is a *quasi-isometry* if there exist constants $\lambda \geqslant 1$, $C_1, C_2 \geqslant 0$ constants such that

(a) For all $x, x' \in X$, we have
$$\frac{1}{\lambda} \cdot d_X(x, x') - C_1 \leqslant d_Y(f(x), f(x')) \leqslant \lambda \cdot d_X(x, x') + C_1.$$

(b) For all $y \in Y$, there exists $x \in X$ such that $d_Y(f(x), y) \leqslant C_2$.

If f only satisfies condition (a), we say that f is a *quasi-isometric embedding.* If $\lambda = 1$, we talk about *roughly isometric embedding* and *rough isometries.*

Fact 2.1.8. *Consider two quasi-isometric groups* (G, S, ω) *and* (G', S', ω') (*for instance* G, G' *equal, or abstractly commensurable*), *then* $\beta_{G,S,\omega}(n) \asymp \beta_{G',S',\omega'}(n)$.

2.2 Reminders on convex geometry

We start by recalling a few notions from convex geometry, and most notably convex polytopes. For a better view on the subject, see the reference book [Zie94].

Definition 2.2.1.

- A subset $C \subseteq \mathbb{R}^d$ is *convex* if $\lambda x + (1-\lambda)y \in C$ for all $x, y \in C$ and $\lambda \in [0,1]$.
- Given a subset $S \subseteq \mathbb{R}^d$, its *convex hull* is the smallest convex set containing S. Equivalently, the convex hull is the set of *convex combinations* of elements of S:

$$\mathrm{ConvHull}(S) := \left\{ \lambda_1 s_1 + \ldots + \lambda_k s_k \;\middle|\; k \in \mathbb{N}, \lambda_i \in \mathbb{R}_{\geqslant 0} \text{ s.t. } \sum_{i=1}^{k} \lambda_i = 1, s_i \in S \right\}.$$

Convex subsets of special interest are convex polytopes. These can be defined in two equivalent ways:

- A *convex polytope* is the convex hull of finitely many points in $\mathbb{R}^d$.
- A *convex polytope* is a bounded subset of $\mathbb{R}^d$ that can be written as the intersection of finitely many half-spaces $\{x \in \mathbb{R}^d \mid f_i(x) \leqslant a_i\}$ where $f_i \colon \mathbb{R}^d \to \mathbb{R}$ are non-trivial linear forms and $a_i \in \mathbb{R}$.

Accordingly, polytopes can be described in two ways:

- A *V-representation* for a polytope P is the data of finitely many points whose convex hull is P.
- A *H-representation* for a polytope P is the data of finitely half-spaces $\{f_i(x) \leqslant a_i\}$ whose intersection is P.

Both descriptions have advantages: V-representations are most practical to *produce* points inside the polytope, while H-representations are useful to *check* whether a point lies in the polytope, or even in its interior. For this reason, it is desirable to find algorithms to produce H-representations from V-representations (and vice versa). This is a classical problem, the *facet enumeration problem* solved in general using Fourier-Motzkin elimination. See for instance [Zie94, §1.2].

Finally, we recall two fundamental theorems:

Theorem 2.2.2 (Caratheodory)**.** *Fix $S \subseteq \mathbb{R}^d$. For each point $x \in \mathrm{ConvHull}(S)$, there exist $d+1$ elements $s_0, s_1, \ldots, s_d \in S$ such that $x \in \mathrm{ConvHull}(\{s_0, s_1, \ldots, s_d\})$.*

Theorem 2.2.3 (Hahn-Banach)**.** *Let $C \subset \mathbb{R}^d$ be a convex set and $p \in \mathbb{R}^d \setminus C$. There exists a non-zero linear form $f \colon \mathbb{R}^d \to \mathbb{R}$ such that $f(x) \geqslant f(p)$ for all $x \in C$.*

2.3 Nilpotent groups

2.3.1 Definition and examples

Let us define the main players of this thesis, namely nilpotent groups:

Definition 2.3.1. The *lower central series* of G is the sequence of subgroups

$$\gamma_1(G) = G \quad \text{and} \quad \gamma_{i+1}(G) = [\gamma_i(G), G] \text{ for all } i \geqslant 1,$$

where $[A, B] = \langle [a, b] \mid a \in A, b \in B \rangle$. For instance $\gamma_2(G) = [G, G]$.

Definition 2.3.2. G is *nilpotent of class* c (or c-step nilpotent) if $\gamma_{c+1}(G) = \{e\}$.

Example 2.3.3. Some examples of nilpotent groups are the following

- Abelian groups are exactly 1-step nilpotent groups. Finitely generated abelian groups are all isomorphic to direct products of cyclic groups:

$$G \simeq \mathbb{Z}^r \times \mathbb{Z}/d_1\mathbb{Z} \times \mathbb{Z}/d_2\mathbb{Z} \times \ldots \times \mathbb{Z}/d_s\mathbb{Z}$$

  (Furthermore, the integers r and d_i are unique if we ask $d_1 \mid d_2 \mid \ldots \mid d_s$.) An important characteristic of f.g. abelian group is their *torsionfree rank* $\mathrm{rk}_{\mathbb{Q}}(G) = r$, which is the rank of the torsionfree part in the previous decomposition.

- The Heisenberg groups are matrix groups

$$H_{2n+1}(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & \mathbf{a} & c \\ & I_n & \mathbf{b}^t \\ & & 1 \end{pmatrix} \,\middle|\, \mathbf{a}, \mathbf{b} \in \mathbb{Z}^n,\ c \in \mathbb{Z} \right\}.$$

  These are alternatively given by the group presentation

$$H_{2n+1}(\mathbb{Z}) = \left\langle x_1, \ldots, x_n, y_1, \ldots, y_n, z \,\middle|\, [x_i, y_i] = z,\ \text{other commutators} = e \right\rangle,$$

  and are all 2-step nilpotent, with $\gamma_2(G) = \langle z \rangle$.

- More generally, groups of unitriangular matrices are nilpotent. For instance

$$UT_4(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & a_{1,2} & a_{1,3} & a_{1,4} \\ & 1 & a_{2,3} & a_{2,4} \\ & & 1 & a_{3,4} \\ & & & 1 \end{pmatrix} \,\middle|\, a_{i,j} \in \mathbb{Z} \right\}.$$

  In general, $UT_n(\mathbb{Z})$ is nilpotent of class $n-1$.

- Nilpotent groups of class c form a variety of groups, defined by the law

$$[g_1, [g_2, [\ \ldots, [g_c, g_{c+1}]]]] = e$$

  for all $g_i \in G$. In particular, we may define free nilpotent groups of any rank r and any nilpotency class c, which we denote by $N_{r,c}$.

Among those, the Heisenberg group $H_3(\mathbb{Z}) = UT_3(\mathbb{Z}) \simeq N_{2,2}$ and the Cartan group $\mathscr{C} = N_{2,3}$ will be particularly important in the following chapters.

2.3.2 Hirsch length

In this paragraph, we recall a notion of dimension for finitely generated nilpotent groups: the Hirsch length. It generalizes the torsionfree rank for finitely generated abelian groups. It should be noted that multiple notions of dimension for nilpotent groups co-exist. For instance, the growth degree (aka homogeneous dimension) is another.

An important property is that f.g. nilpotent groups are *Noetherian*. More explicitly,

Fact 2.3.4. *Subgroups of a finitely generated nilpotent groups are finitely generated.*

In particular, the subgroups $\gamma_i(G)$ are finitely generated, hence the successive quotients

$$\gamma_i(G)/\gamma_{i+1}(G)$$

are finitely generated abelian, hence have well-defined torsionfree ranks. Finally

Definition 2.3.5. The *Hirsch length* of a c-step nilpotent group G is defined as

$$h(G) = \sum_{i=1}^{c} \mathrm{rk}_{\mathbb{Q}}\big(\gamma_i(G)/\gamma_{i+1}(G)\big).$$

Proposition 2.3.6. *Given a finitely generated nilpotent group G, we have*

(a) *For any exact sequence $1 \to N \to G \to Q \to 1$, we have $h(G) = h(N) + h(Q)$.*

(b) *For any subgroup $H \leqslant G$, we have $h(H) \leqslant h(G)$ with equality iff $[G : H] < \infty$.*

2.3.3 Dealing with torsion

We recall some results about torsion in nilpotent groups:

Definition 2.3.7. Given a subgroup $H \leqslant G$, we define the *isolator* of H in G as

$$I_G(H) = \{g \in G : \exists n > 0 \text{ such that } g^n \in H\}.$$

Fact 2.3.8 (See eg. [Khu93, §2.6])**.** *If G is nilpotent, then $I_G(H)$ is a subgroup.*

We add that, if H is normal/characteristic, then $I_G(H)$ is normal/characteristic too. This is obvious as the definition is invariant under automorphisms of G.

The name comes from the following observation: if a subgroup $K \leqslant G$ satisfies $K \geqslant I_G(H)$ and $[K : I_G(H)] < \infty$, then $K = I_G(H)$. Indeed, for all $g \in K$, we have

$$g^{[K:I_G(H)]!} \in I_G(H)$$

by some pigeonhole principle. This means $I_G(H)$ is "isolated" in the lattice of subgroups of G: it cannot be approached by finite-index overgroups.

Isolators are useful to deal with torsion in two ways. First,

Lemma 2.3.9. *Let G be finitely generated, torsion, solvable group. Then G is finite.*

Proof. The proof is by induction on the class.

Base case. For abelian groups, this follows from the classification.

Induction. Suppose that the induction hypothesis holds for class $c-1$ and let G be a solvable group of class c. Observe that

(1) $G/[G,G]$ is finitely generated, abelian and torsion, hence it is finite.

We deduce that $[G,G]$ has finite-index in G, hence is finitely generated too.

(2) By induction, $[G,G]$ is f.g., torsion, solvable of class $c-1$, hence $[G,G]$ is finite.

Combining (1) and (2), we deduce that G is finite. □

Corollary 2.3.10. *If G is finitely generated nilpotent, then*

$$T = \{g \in G : \exists n > 0 \text{ such that } g^n = e\}$$

is a finite characteristic subgroup.

Proof. We have $T = I_G(\{e\})$ therefore T is a subgroup. Fact 2.3.4 tells us T is finitely generated. Combining this with Lemma 2.3.9, we conclude that T is finite. □

Second,

Lemma 2.3.11. *Let G be a finitely generated group. Then $G/I_G([G,G]) \simeq \mathbb{Z}^d$, and every map $G \to G/N$ with G/N torsionfree abelian factors through this first quotient.*

We call $G/I_G([G,G])$ the "torsionfree-abelianization" of G.

Proof. We first note that $I_G([G,G]) = \pi^{-1}(I_A(\{e_A\}))$ where $\pi \colon G \to G/[G,G] =: A$ is the abelianization map, it follows that $I_G([G,G])$ is a normal subgroup of G. Moreover, given $N \trianglelefteq G$, we have that

- G/N is abelian if and only if $N \geqslant [G,G]$,
- G/N is torsionfree if and only if $N = I_G(N)$.

It follows that $G/I_G([G,G])$ is torsionfree abelian, hence $\simeq \mathbb{Z}^d$ if G is finitely generated. Moreover if a quotient G/N is torsionfree abelian, then $N \geqslant I_G([G,G])$. □

2.4 Nilpotent Lie groups

2.4.1 Mal'cev completion

We explain a construction liking finitely generated nilpotent groups and nilpotent Lie groups. This link is key in multiple parts of the thesis.

Theorem 2.4.1 (Mal'cev). *Let H be finitely generated, torsionfree, nilpotent group. Then H embeds as a cocompact lattice inside a simply connected nilpotent Lie group $\bar{H}$.*

Sketch of proof. Consider the descending series of normal subgroups

$$H \trianglerighteq I_H([H,H]) \trianglerighteq I_H(\gamma_3(H)) \trianglerighteq \ldots \trianglerighteq I_H(\gamma_c(H)) \trianglerighteq I_H(\gamma_{c+1}(H)) = \{e\}.$$

(The last equality follows from H being torsionfree.) The successive quotients are finitely generated (Fact 2.3.4), torsionfree and abelian, so we have

$$\forall i = 1, 2, \ldots, c, \qquad I_H(\gamma_i(H))/I_H(\gamma_{i+1}(H)) \simeq \mathbb{Z}^{r_i}.$$

Consider $g_{i,1}, \ldots, g_{i,r_i} \in I_H(\gamma_i(H))$ forming a base modulo $I_H(\gamma_{i+1}(H))$. By construction, every element $g \in H$ can be written uniquely as

$$g = g_{1,1}^{m_{1,1}} \ldots g_{1,r_1}^{m_{1,r_1}} g_{2,1}^{m_{2,1}} \ldots g_{c,r_c}^{m_{c,r_c}}$$

for $m_{i,j} \in \mathbb{Z}$. This allows to identify $H \longleftrightarrow \mathbb{Z}^h$, where $h = \sum_{i=1}^{c} r_i$ is the Hirsch length of H. The $g_{i,j}$ form a *Mal'cev basis*, and the exponents $m_{i,j}$ are the *Mal'cev coordinates*.

Now, the key observation is that multiplication and inversion under this identification are given by polynomials in the coordinates (see [LR04, §2.1]). Therefore, we can extend these operations to $(\mathbb{R}^h, \cdot\,)$. We note that

- The operations are analytical (polynomial).
- The different axioms of groups passes to $(\mathbb{R}^h, \cdot)$ as $\mathbb{Z}^h$ is Zariski dense. Similarly, $(\mathbb{R}^h, \cdot)$ is nilpotent of class c since $(\mathbb{Z}^h, \cdot\,) \simeq H$ is nilpotent of class c.
- $\mathbb{R}^h$ is obviously simply connected, and $\mathbb{Z}^h$ is cocompact.

This proves that any f.g. torsionfree c-step nilpotent group H embeds in a simply connected c-step nilpotent Lie group $\bar{H} = (\mathbb{R}^h, \cdot)$, as a cocompact lattice. □

Remark 2.4.2. Similarly, we can define the rational Mal'cev completion, extending the operations to $\bar{H}_{\mathbb{Q}} \coloneqq (\mathbb{Q}^h, \cdot\,)$. These two constructions are central in the QI classification of nilpotent groups: given two f.g., torsionfree, nilpotent groups G and H,

- It is known that G and H are commensurable if and only if $\bar{G}_{\mathbb{Q}} \simeq \bar{H}_{\mathbb{Q}}$.
- It is conjectured that G and H are quasi-isometric if and only if $\bar{G} \simeq \bar{H}$.

See [Cor18, §19.7] for a discussion.

2.4.2 Lie algebra and exponential coordinates

We recall shortly the standard construction which associate to any real Lie group $(\Gamma, \cdot)$ a real Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$. As Γ is a Lie group, we have smooth *left-multiplication* maps $L_g \colon \Gamma \to \Gamma \colon x \mapsto gx$ defined for each $g \in G$. This allows to define

$$\mathfrak{g} = \left\{ X \in \mathfrak{X}(\Gamma) \;\middle|\; \forall g \in \Gamma,\ L_{g*}X_e = X_g \right\} \simeq T_e\Gamma \simeq \mathbb{R}^h,$$

where $\mathfrak{X}(\Gamma)$ is the set of vector fields on Γ. This vector space is naturally equipped with a Lie bracket $[\cdot, \cdot]_{\mathfrak{g}} \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, defined as $[X, Y]_{\mathfrak{g}} = X \circ Y - Y \circ X$. Some computations show that $[\cdot, \cdot]_{\mathfrak{g}}$ is bilinear, anti-symmetric, and satisfies the Jacobi identity

$$\forall X, Y, Z \in \mathfrak{g}, \quad [X, [Y, Z]_{\mathfrak{g}}]_{\mathfrak{g}} + [Y, [Z, X]_{\mathfrak{g}}]_{\mathfrak{g}} + [Z, [X, Y]_{\mathfrak{g}}]_{\mathfrak{g}} = \mathbf{0}.$$

Definition/Fact 2.4.3 (Exponential map)**.** For each $X \in \mathfrak{g}$, there exists a unique smooth path $\gamma \colon [0, 1] \to \Gamma$ such that

$$\gamma(0) = e \quad \text{and} \quad \forall t \in [0, 1],\ d\gamma|_t = X_{\gamma(t)}.$$

This allows to define the *exponential map* $\exp \colon \mathfrak{g} \to \Gamma$ via $\exp(X) \coloneqq \gamma(1)$. Moreover,

- If Γ is simply connected and nilpotent, then $\exp \colon \mathfrak{g} \to \Gamma$ is a diffeomorphism.
- For every $m \in \mathbb{Z}$, we have $\exp(X)^m = \exp(mX)$.

Let us suppose Γ is simply connected. The first point allows to identify Γ and $\mathfrak{g}$. In particular, given any system of coordinates on the vector space $\mathfrak{g} \simeq \mathbb{R}^h$, we get coordinates on Γ. Such coordinates are called *exponential coordinates*. These have much better properties than Mal'cev coordinates (especially if taken adapted with a gradation of $\mathfrak{g}$). For instance, the second point translates as $g^m = mh$, or in coordinates

$$(v_1, v_2, \dots, v_h)^m = (mv_1, mv_2, \dots, mv_h)$$

for all $m \in \mathbb{Z}$. We take this opportunity to define $g^\mu \coloneqq \mu \cdot g$ for all $g \in \Gamma$ and $\mu \in \mathbb{R}$ (or more formally $g^\mu \coloneqq \exp(\mu \log(g))$, where $\log \colon \Gamma \to \mathfrak{g}$ is the inverse of exp).

More generally, multiplication in Γ is given by the Baker–Campbell–Hausdorff (BCH) formula. However, we will usually try to avoid it, and actually only use it in the case of simply connected 2-step nilpotent Lie groups:

Proposition 2.4.4 (BCH formula, 2-step nilpotent case)**.** *Let $\Gamma \simeq \mathfrak{g}$ be a simply connected 2-step nilpotent Lie group, then for all $g, h \in \Gamma$, we have*

$$g \cdot h = g + h + \frac{1}{2}[g, h]_{\mathfrak{g}}.$$

In particular, we have $[g, h] \coloneqq g \cdot h \cdot g^{-1} \cdot h^{-1} = [g, h]_{\mathfrak{g}}$, so we can drop the subscript.

We can decompose $\Gamma \simeq \mathfrak{g} = V_1 \oplus V_2$, where V_1 is an arbitrary complement of $V_2 = [\mathfrak{g}, \mathfrak{g}]$. Let $\mathrm{Pr}\colon \Gamma \to V_1$ and $A\colon \Gamma \to V_2$ be the associated projection. (A holds for "areas", see Section 3.1.) As Γ is 2-step nilpotent, we observe that $[g,h]$ only depends on $\mathrm{Pr}(g)$ and $\mathrm{Pr}(h)$, so we can consider $[\,\cdot\,,\,\cdot\,]\colon V_1 \times V_1 \to V_2$ instead.

Proposition 2.4.5 (BCH formula, 2-step nilpotent, in gradation)**.** *Let $\Gamma = V_1 \oplus V_2$ be a simply connected 2-step nilpotent Lie group. For all $g, h \in \Gamma$, we have*

$$\begin{aligned} \mathrm{Pr}(g \cdot h) &= \mathrm{Pr}(g) + \mathrm{Pr}(h), \\ A(g \cdot h) &= A(g) + A(h) + \frac{1}{2}[\mathrm{Pr}(g), \mathrm{Pr}(h)]. \end{aligned}$$

More generally, if $g_1, g_2, \ldots, g_n \in \Gamma$, then

$$\begin{aligned} \mathrm{Pr}(g_1 g_2 \ldots g_n) &= \mathrm{Pr}(g_1) + \mathrm{Pr}(g_2) + \ldots + \mathrm{Pr}(g_n), \\ A(g_1 g_2 \ldots g_n) &= \sum_{i=1}^{n} A(g_i) + \frac{1}{2} \sum_{i<j} \big[\mathrm{Pr}(g_i), \mathrm{Pr}(g_j)\big]. \end{aligned}$$

2.4.3 The Stoll metric (aka polytopal sub-Finsler metrics)

In this paragraph, we consider Γ a simply connected nilpotent Lie group.

Definition 2.4.6. A subset $X \subseteq \Gamma$ is a *Lie generating set* if

$$\langle x^{\mu} \mid x \in X, \mu \in \mathbb{R}_{>0} \rangle = \Gamma$$

as an abstract group. If X is symmetric, we can equivalently ask that the set of elements $[x_1, [x_2, \ldots [x_{k-1}, x_k] \ldots]]_{\mathfrak{g}}$ with $x_i \in X$ and $k \geqslant 1$ generates $\mathfrak{g}$ as a vector space.

Typical examples are generating sets of lattices.

Definition 2.4.7 ($\mathbb{R}$-words)**.** Fix a finite Lie generating set X for Γ, and $\omega\colon X \to \mathbb{R}_{>0}$ a weight. An $\mathbb{R}$-*word* is an expression $w = x_1^{\mu_1} \cdot x_2^{\mu_2} \cdot \ldots \cdot x_k^{\mu_k}$ with $x_i \in X$ and $\mu_i \in \mathbb{R}_{>0}$. We denote the set of $\mathbb{R}$-words by $X_{\mathbb{R}}^{\star}$. For each $\mathbb{R}$-word w, we define the followings:

- Its *length*: $\ell_{X,\omega}(w) = \sum_{i=1}^{k} \mu_i \cdot \omega(x_i)$,
- The *total exponent* of a letter $x \in X$ in a word w: $|w|_x = \sum_{i : x_i = x} \mu_i$,
- Its *coarse length*: $k(w) = k$.

Definition 2.4.8 (Stoll metric)**.** Given Γ a simply connected nilpotent Lie group, X a finite Lie generating set and $\omega\colon X \to \mathbb{Z}_{>0}$ a weight function, we define

$$\|h\|_{\mathrm{Stoll},X,\omega} = \inf \left\{ \ell_{\omega}(w) \;\middle|\; w \in X_{\mathbb{R}}^{\star} \text{ and } \overline{w} = h \right\}.$$

Most notably, if X is a generating set of a torsionfree nilpotent group H, then X is a Lie generating set for the Mal'cev completion $\Gamma = \bar{H}$. Since $X^{\star} \subset X_{\mathbb{R}}^{\star}$, we have

$$\forall h \in H, \quad \|h\|_{X,\omega} \geqslant \|h\|_{\mathrm{Stoll},X,\omega}.$$

2.4.4 The Abelian case

We first have a look at the Abelian case and make some useful observations. See [DLM12] for a more thorough treatment. We have $\Gamma = \mathbb{R}^d$. In this case, the Stoll metric has a very geometrical interpretation:

Lemma 2.4.9. *The Stoll metric coincide with a "Minkowski norm": for all $v \in \mathbb{R}^d$,*

$$\|v\|_{\mathrm{Stoll},X,\omega} = \|v\|_{\mathrm{Mink,P}} \coloneqq \min\left\{\lambda \geqslant 0 \mid v \in \lambda \mathrm{P}\right\},$$

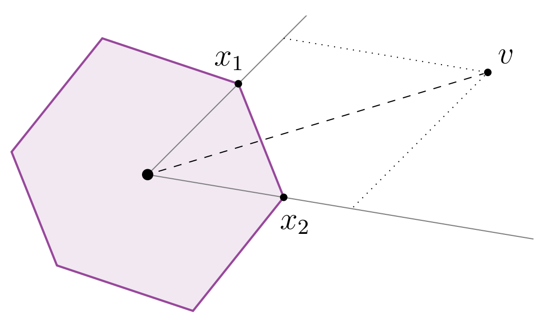
where $\mathrm{P} = \mathrm{ConvHull}\left\{\frac{x}{\omega(x)} \;\middle|\; x \in X\right\} \subset \mathbb{R}^d$. *Moreover, an $\mathbb{R}$-word $x_1^{\mu_1} \cdot \ldots \cdot x_k^{\mu_k}$ is geodesic if and only if all $\frac{x_i}{\omega(x_i)}$ lie on a common facet of* P.

Proof. We can reduce ourselves to the case $\tilde{\omega} \equiv 1$ by setting $\tilde{X} = \left\{\frac{x}{\omega(x)} \mid x \in X\right\} \subset \mathbb{R}^d$. The case $v = 0$ is trivial. Suppose $v \neq 0$ and let $m = \|v\|_{\mathrm{Mink,P}} > 0$.

We first construct an $\mathbb{R}$-word representing v of length m. Consider F the minimal face of P containing $\frac{1}{m} \cdot v$. By the Caratheodory theorem, there exists d vertices of F, say $x_1, \ldots, x_d \in X$, such that $\frac{1}{m} \cdot v \in \mathrm{ConvHull}(x_1, \ldots, x_d)$, i.e.,

$$\exists \nu_1, \ldots, \nu_d \geqslant 0 \quad \text{such that} \quad \nu_1 + \ldots + \nu_d = 1 \quad \text{and} \quad \nu_1 x_1 + \ldots + \nu_d x_d = \frac{1}{m} \cdot v$$

and therefore $v = x_1^{\nu_1 m} \cdot \ldots \cdot x_d^{\nu_d m}$.



Next we show that any $\mathbb{R}$-word $x_1^{\mu_1} \cdot \ldots \cdot x_k^{\mu_k}$ such that all letters x_i lie on a common face F are geodesics. Consider another $\mathbb{R}$-word $y_1^{\lambda_1} \cdot \ldots \cdot y_\ell^{\lambda_\ell} \in X_{\mathbb{R}}^\star$ representing the same element as $x_1^{\mu_1} \cdot \ldots \cdot x_k^{\mu_k}$. As F is a face, there exists a linear form $f \colon \mathbb{R}^d \to \mathbb{R}$ such that $f(p) \leqslant 1$ for all $p \in P$, with equality if and only if $p \in \mathrm{F}$. It follows that

$$\mu_1 + \ldots + \mu_k = f(x_1^{\mu_1} \cdot \ldots \cdot x_k^{\mu_k}) = f(y_1^{\lambda_1} \cdot \ldots \cdot y_\ell^{\lambda_\ell}) = \lambda_1 f(y_1) + \ldots + \lambda_\ell f(y_\ell) \leqslant \lambda_1 + \ldots + \lambda_\ell$$

which means $x_1^{\mu_1} \cdot \ldots \cdot x_k^{\mu_k}$ has indeed minimal length. In particular, this applies to the previously constructed $\mathbb{R}$-word (of length m) for v: we have $\|v\|_{\mathrm{Stoll},X} = m$. □

2.4.5 The 2-step nilpotent case

In the 2-step nilpotent case, Stoll proves more than we could bargain for:

Lemma 2.4.10 ([Sto98, Lemma 3.3]). *There exists a constant K such that, for each $\mathbb{R}$-word $w \in X_{\mathbb{R}}^{\star}$, there exists another $w' \in X_{\mathbb{R}}^{\star}$ satisfying the following conditions:*

- *Both words represent the same elements $\overline{w} = \overline{w'}$ in Γ.*
- *For each letter $x \in X$, we have $|w|_x \geqslant |w'|_x$. In particular, $\ell_\omega(w) \geqslant \ell_\omega(w')$.*
- *w' has uniformly bounded coarse length, precisely $k(w') \leqslant K$.*

If w is geodesic, then so is w', and we have $|w|_x = |w'|_x$ for all $x \in X$.

Corollary 2.4.11. *Let Γ be a simply connected 2-step nilpotent Lie group. Every element $g \in \Gamma$ is represented by a geodesic $w \in X_{\mathbb{R}}^{\star}$ with coarse length $k(w) \leqslant K$.*

Proof. Follows from Lemma 2.4.10 by compactness of

$$\left\{ w \in X_{\mathbb{R}}^{\star} \;\middle|\; \bar{w} = g, \quad \ell(w) \leqslant \|g\|_{\mathrm{Stoll},X,\omega} + 1 \quad \text{and} \quad \ell(w) \leqslant K \right\},$$

and continuity of the evaluation map. □

He deduces the following result, which is central multiple proofs of this thesis:

Proposition 2.4.12 (Rough isometry, [Sto98, Proposition 4.3]). *Let H be a torsion-free 2-step nilpotent group, with a finite generating set X and $\omega\colon X \to \mathbb{Z}_{>0}$ a weight function. Then there exists a constant $C = C(X, \omega)$ such that*

$$\forall h \in H, \quad \|h\|_{\mathrm{Stoll},X,\omega} \leqslant \|h\|_{X,\omega} \leqslant \|h\|_{\mathrm{Stoll},X,\omega} + C.$$

When considering words $w \in X^{\star}$, there is usually a trade-off between having small coarse length $k(w)$, and having length $\ell_\omega(w)$ close to $\|\overline{w}\|_{X,\omega}$. However, in the 2-step nilpotent case, this trade-off does not happen. Analyzing the proofs of Lemma 4.2 and Proposition 4.3 of [Sto98], we see that the word $w \in X^{\star}$ evaluating to h, and witnessing $\|h\|_{X,\omega} \leqslant \|h\|_{\mathrm{Stoll},X,\omega} + C$, has bounded coarse length. More precisely,

Lemma 2.4.13 (The best of both worlds). *There exists $K \geqslant 0$ such that any element $h \in H$ can be written as $h = x_1^{m_1} x_2^{m_2} \dots x_k^{m_k}$ with $x_i \in X$, $m_i \in \mathbb{Z}_{>0}$, $k \leqslant K$ and*

$$\ell_\omega(x_1^{m_1} x_2^{m_2} \dots x_k^{m_k}) \leqslant \|h\|_{\mathrm{Stoll},X,\omega} + C \leqslant \|h\|_{X,\omega} + C.$$

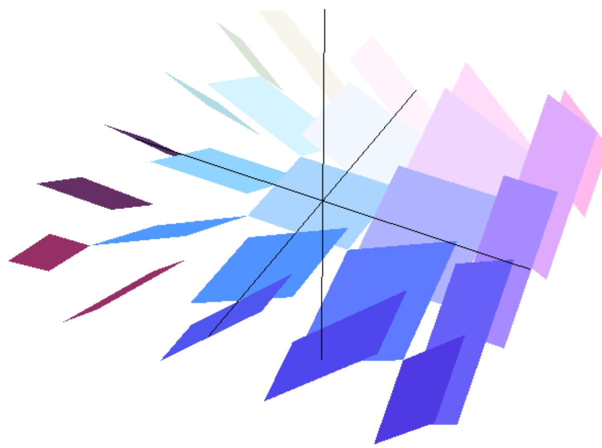
Remark 2.4.14. The Stoll paper only treats the case of symmetric generating set, with weight function $\omega \equiv 1$. However the argument adapts to our more general setting.

2.4.6 Carnot-Caratheory metrics

In this paragraph, we introduce Carnot-Caratheodory metrics. In case the underlying Minkowski metric is defined by a polytope, this is a special case of the Stoll metric. However, we give another point of view, which we will later extend in Chapter 3. Most of this material (and more) can be found in [BLD13; LD17]

Let Γ be a simply connected nilpotent Lie group. As previously, we decompose it as $\Gamma \simeq \mathfrak{g} = V_1 \oplus [\mathfrak{g}, \mathfrak{g}]$, where V_1 is a linear complement of $[\mathfrak{g}, \mathfrak{g}]$, and denote $\mathrm{Pr}\colon \Gamma \to V_1$ the projection on the first coordinate. We consider the left-invariant distribution

$$\Delta = \bigcup_{g \in \Gamma} (L_{g*} V_1)_g \leqslant T\Gamma.$$



Definition 2.4.15. An absolutely continuous curve $\gamma\colon [0, \ell] \to \Gamma$ is *horizontal* if

$$d\gamma|_t \in \Delta_{\gamma(t)} \text{ for almost all } t \in [0, \ell].$$

Fact 2.4.16 (Chow–Rashevskii, see eg. [Mon02, Chapter 2])**.** *For every element $g \in \Gamma$, there exists an horizontal path starting from $\gamma(0) = e$ and ending at $\gamma(\ell) = g$.*

This allows to define a metric on Γ:

Definition 2.4.17 (CC-metric)**.** Let $\Gamma = V_1 \oplus [\Gamma, \Gamma]$ be a simply connected nilpotent Lie group, and $\mathrm{P} \subset V_1$ a full-dimensional, centrally symmetric, compact, convex set. We define

$$\|g\|_{\mathsf{CC},\mathrm{P}} = \inf \left\{ \ell_{\mathrm{P}}(\gamma) \;\middle|\; \gamma \text{ is an horizontal path from } e \text{ to } g \right\},$$

where $\ell_{\mathrm{P}}(\gamma) = \int \|\mathrm{Pr}_*(d\gamma|_t)\|_{\mathrm{Mink},\mathrm{P}} \cdot \mathrm{d}t$. Typically, we consider P the polytope

$$\mathrm{P} = \mathrm{ConvHull} \left\{ \frac{\mathrm{Pr}(x)}{\omega(x)} : x \in X \right\},$$

where (X, ω) is a finite weighted Lie generating set. In that case, we write $\|\cdot\|_{\mathsf{CC},X,\omega}$.

Remark 2.4.18. Observe that $\mathrm{Pr}_*|_{\Delta_g}$ is bijective for all $g \in \Gamma$. In particular, for a given absolutely continuous path $\hat{\gamma}\colon [0,\ell] \to V_1$ such that $\hat{\gamma}(0) = \mathbf{0}$, there exists a unique horizontal path $\gamma\colon [0,\ell] \to \Gamma$ such that $\gamma(0) = e$ and $\hat{\gamma} = \mathrm{Pr} \circ \gamma$. With that point of view, we have the alternative formula

$$\ell_{\mathrm{P}}(\gamma) = \ell_{\mathrm{P}}(\hat{\gamma}) = \int_0^\ell \|\mathrm{d}\hat{\gamma}|_t\|_{\mathrm{Mink,P}} \cdot \mathrm{d}t.$$

Another key take-away is that we could define Γ as the set of absolutely continuous paths in V_1 up to a well-chosen equivalence relation. For instance, comparing with the Definition 2.4.3, elements of $V_1 \subseteq \Gamma$ are represented by straight segments in V_1.

This class of metrics has a nice property, under an additional assumption:

Definition 2.4.19. A Lie algebra is *stratifiable* if there exists $V_1 \leqslant \mathfrak{g}$ such that

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \ldots \oplus V_s$$

where $V_{i+1} = [V_1, V_i]_{\mathfrak{g}}$ and $[V_1, V_s]_{\mathfrak{g}} = \{\mathbf{0}\}$. It is *stratified* if such a decomposition is specified. This extends to simply connected Lie groups.

Definition/Fact 2.4.20. Given a stratified simply connected Lie group Γ decomposed as $V_1 \oplus V_2 \oplus \ldots \oplus V_s$ and $\lambda \in \mathbb{R}$, we defined the *dilation map*

$$\delta_\lambda\colon \begin{pmatrix} \Gamma & \longrightarrow & \Gamma \\ (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s) & \longmapsto & (\lambda\mathbf{v}_1, \lambda^2\mathbf{v}_2, \ldots, \lambda^s\mathbf{v}_s) \end{pmatrix}.$$

(Equivalently, we define $\delta_\lambda|_{V_i} = \lambda^i \cdot \mathrm{id}$.) Then

- If $\lambda \neq 0$, then $\delta_\lambda\colon \Gamma \to \Gamma$ is an automorphism of Lie group.
- In the formalism of Remark 2.4.18, the dilation corresponds to an homothety of factor λ of the curve in V_1. In particular, we have

  $$\|\delta_\lambda g\|_{\mathsf{CC}} = |\lambda| \cdot \|g\|_{\mathsf{CC}}.$$

  (Here, the CC-metric needs to be defined for the same V_1.)

Example 2.4.21. All key examples in the thesis are stratifiable:

- All 2-step nilpotent groups are stratifiable.
- The free nilpotent Lie groups (i.e., the Mal'cev closure of $N_{r,c}$) are stratifiable.
- The Engel group $\bar{\mathscr{E}}$ is stratifiable

In all cases, we can take V_1 any linear complement of $[\mathfrak{g}, \mathfrak{g}]$.

Remark 2.4.22. Stratified nilpotent Lie groups equipped with an (adapted) CC-metric are called *Carnot groups*. These are central in Geometric Group Theory, being the asymptotic cones of finitely generated nilpotent groups.

As for Stoll metrics, one of the main objectives is to estimate word metrics $\|\cdot\|_{X,\omega}$ by the associated CC-metrics $\|\cdot\|_{\mathsf{CC},X,\omega}$. In general, we have the following:

Theorem 2.4.23 ([Pan83])**.** *Let H be finitely generated torsionfree nilpotent group, and (X,ω) a weighted symmetric generating set ($\omega(x^{-1})=\omega(x)$). Then*

$$\|h\|_{X,\omega} \sim \|h\|_{\mathsf{CC},X,\omega} \quad \text{as } \|h\| \to \infty.$$

See also [BLD13, Theorem 4]. We isolate a lemma in their proof:

Lemma 2.4.24. *Let Γ be a simply connected, s-step nilpotent Lie group, and $F \subset \Gamma$ a finite set. We define the "piecewise flattening map"*

$$\mathrm{fl}\colon \begin{pmatrix} F^\star & \longrightarrow & \mathrm{Pr}(F)^\star \\ x & \longmapsto & \mathrm{Pr}(x) \end{pmatrix}.$$

(extended as a morphism). Then, for all $w \in F^\star_{\mathbb{R}}$, we have $d_{\mathsf{CC}}\big(\bar{w}, \overline{\mathrm{fl}(w)}\big) = O\big(\ell(w)^{\frac{s-1}{s}}\big)$.

Sketch of proof. For each $g \in F$, we denote $g = \mathrm{Pr}(g)\cdot c_g$ with $c_g \in [\Gamma,\Gamma]$. We define

$$S_1 = \big\{\mathrm{Pr}(g) \bigm| g \in F\big\},\ S_2 = \big\{c_g \bigm| g \in F\big\} \text{ and } S_i = \big\{[g,h] \bigm| g \in S_j, h \in S_{i-j}, 0<j<i\big\}.$$

Recall that $[\gamma_i(\Gamma),\gamma_j(\Gamma)] \subseteq \gamma_{i+j}(\Gamma)$ hence $S_{s+1} = \{e\}$. Consider $w \in F^\star$ and rewrite it

$$\begin{aligned} \bar{w} &= \big(\mathrm{Pr}(g_1)c_{g_1}\big)\cdots\big(\mathrm{Pr}(g_\ell)c_{g_\ell}\big) \\ &= \mathrm{Pr}(g_1)\cdots\mathrm{Pr}(g_\ell)\cdot\bar{w}_s\cdot\bar{w}_{s-1}\cdot\ldots\cdot\bar{w}_2 \\ &= \overline{\mathrm{fl}(w)}\cdot\bar{w}_s\cdot\bar{w}_{s-1}\cdot\ldots\cdot\bar{w}_2 \end{aligned}$$

where $w_i \in S_i^\star$. For instance $w_2 = c_{g_1}\cdots c_{g_\ell}$, and w_3 contains the commutator $[c_{g_1},\mathrm{Pr}(g_2)]$ which appears when "pushing" c_{g_1} to the right of $\mathrm{Pr}(g_2)$. Every letter of w_i corresponds to a tuple of letters of $\mathrm{Pr}(g_1)c_{g_1}\cdots\mathrm{Pr}(g_\ell)c_{g_\ell}$ with total weight i (where $\mathrm{Pr}(g)$ has weight 1 and c_g has weight 2) with at least one c_g (as we do not permute the $\mathrm{Pr}(g)$ themselves). There are $O(\ell^{i-1})$ such tuples so that $\ell(w_i) = O(\ell^{i-1})$ and finally

$$\|\bar{w}_i\|_{\mathsf{CC}} = O\big(\ell^{\frac{i-1}{i}}\big),$$

using distortion of $\gamma_i(\Gamma)$, giving the desired conclusion. □

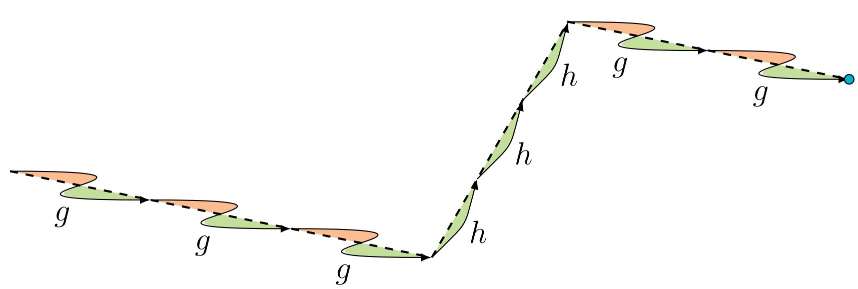


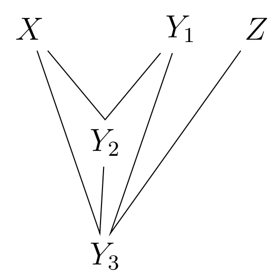
Figure 2.1: Comparison of the path w and $\mathrm{fl}(w)$ in V_1.

These estimates can be improved under an additional assumption:

Definition 2.4.25. A simply connected nilpotent Lie group Γ is *ideal* (or *strictly non-singular*) if, for all $z \in \mathcal{Z}(\Gamma)$ and $g \in \Gamma \setminus \mathcal{Z}(\Gamma)$, there exists $h \in \Gamma$ such that $z = [g, h]$.

Example 2.4.26. This is quite a strong condition.

- Examples of ideal groups are the Heisenberg groups $\Gamma = H_{2n+1}(\mathbb{R})$.
- A non-example is $\Gamma = H_3(\mathbb{R}) \times \mathbb{R}$ (as $z = (0, 0, 0; 1) \in \mathcal{Z}(\Gamma)$ cannot be written as a commutator). This example is important in [Bre14; BLD13]
- All stratifiable ideal groups are nilpotent of class 2. [LDNG18, Theorem A.1]
- A non-stratifiable ideal group of class 3 is given in [Gor95, Example II], see also group $N_{5,2,2}$ in the catalog [LDT22]. This group is given by the diagram:



  Meaning the associated Lie algebra admits a basis $\{X, Y_1, Y_2, Y_3, Z\}$ with only non-zero commutators $[X, Y_1] = Y_2$ and $[X, Y_2] = [Y_1, Z] = Y_3$.

Theorem 2.4.27 ([Kra02; Tas22])**.** *If $(H, d_{X,\omega})$ is a torsionfree, 2-step nilpotent group whose Mal'cev completion $\bar{H}$ is ideal, then there exists K such that*

$$\forall h \in H, \quad \|h\|_{\mathsf{CC},X,\omega} - K \leqslant \|h\|_{X,\omega} \leqslant \|h\|_{\mathsf{CC},X,\omega} + K.$$

As H is cocompact in $\bar{H}$, it follows that $(H, d_{X,\omega}) \hookrightarrow (\bar{H}, d_{\mathsf{CC},X,\omega})$ is a rough isometry.

Whether this generalizes for ideal nilpotent groups of higher nilpotency class remains an open question, see [Fuj16, Question 4].

2.5 Formal languages

Let us start with some basic definitions, and introduce the usual terminology:

Definition 2.5.1.

- An *alphabet* $\mathcal{A}$ is a set (finite or infinite). Elements of $\mathcal{A}$ are *letters*.
- A *word* over $\mathcal{A}$ is a sequence of letters. These sequences can either be finite, or be indexed by $\mathbb{N}$ (infinite) or $\mathbb{Z}$ (bi-infinite). The *empty word* is denoted ε.
- We denote by $\mathcal{A}^\star$ the set of all finite words over $\mathcal{A}$ (including the empty word ε). Equivalently, this is the free monoid over $\mathcal{A}$ (with the *concatenation* operation).
- A (formal) *language* is a subset $\mathcal{L} \subseteq \mathcal{A}^\star$.

Languages are central in computer sciences. For instance, any decision problem can be reformulated as the membership problem to a language. Indeed, the input the decision problem can be encoded as finite words over the alphabet $\mathcal{A} = \{0, 1\}$. We define

$$\mathcal{L} = \big\{w \in \{0,1\}^\star : \text{the answer is YES}\big\}.$$

The decision problem then is equivalent to deciding if a word belong to $\mathcal{L}$. Of course, in this level of generality, not much can be done. For this reason, many classes of languages were defined over time. For the purpose of this thesis, we will only consider the first two levels in Chomsky hierarchy: regular languages and context-free languages.

2.5.1 Regular languages

Regular languages form the lowest class of languages in the Chomsky hierarchy of complexities. Informally, a language is *regular* if its membership problem can be decided by some computer with finite memory. Our model of computer is the following:

Definition 2.5.2. An *automaton* is a 5-uple $M = (V, \mathcal{A}, \delta, v_0, \mathsf{accept})$ where

- V is a set of states / vertices.
- $\mathcal{A}$ is the alphabet.
- $\delta \subseteq V \times \mathcal{A} \times V$ is the transition function. An element $(v_1, a, v_2) \in \delta$ should be seen as an oriented edge from v_1 to v_2, labeled by a.
- $v_0 \in V$ is the initial / start vertex. (Indicated by $\mathsf{start} \rightarrow$.)
- $\mathsf{accept} \subseteq V$ is the set of "accept" / terminal vertices. (Indicated by ◎.)

An automaton is *finite* if both V and δ are finite.

Here are some examples of finite automata:

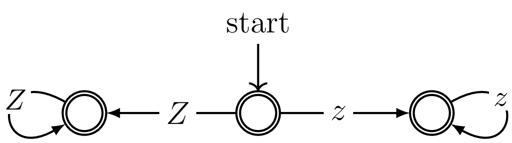


Figure 2.2: M recognizing $\mathcal{L} = \{\varepsilon\} \cup \{z^n, Z^n \mid n \geqslant 1\}$

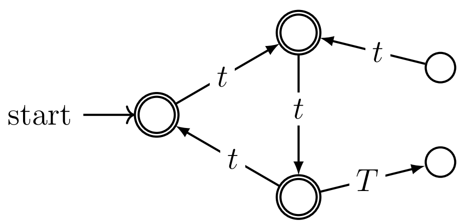


Figure 2.3: M for $\mathcal{L} = \{t^n \mid n \equiv 1, 2 \pmod 3\}$

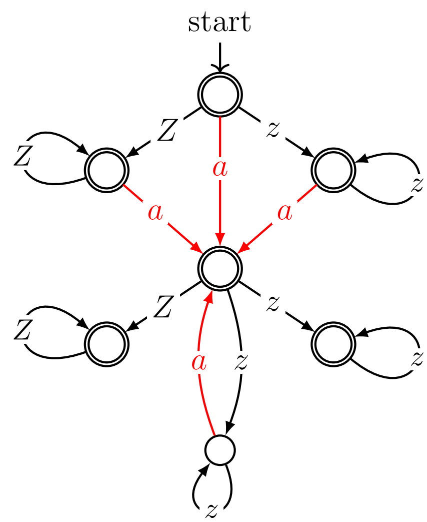


Figure 2.4: M for $\mathcal{L} \underset{\text{ev}}{\longleftrightarrow} C_2 \wr \mathbb{Z}$

Some additional terminology around automata will be needed:

Definition 2.5.3.

- An automaton is *deterministic* whenever, for all $v \in V$ and $a \in \mathcal{A}$, there exists at most one edge exiting v and labeled by a (i.e., $\big|\big(\{v\} \times \{a\} \times V\big) \cap \delta\big| \leqslant 1$).
- An automaton is *saturated* whenever, for all $v \in V$ and $a \in \mathcal{A}$, there exists at least one edge exiting v and labeled by a (i.e., $\big|\big(\{v\} \times \{a\} \times V\big) \cap \delta\big| \geqslant 1$).
- For $v_1, v_2 \in V$, we say that v_2 is *accessible* from v_1 is there exists an oriented path from v_1 to v_2 in M. An automaton is *trim* if, for all $v \in V$, there exists $t \in \mathsf{accept}$ such that t is accessible from v and v is accessible from v_0.
- A *strongly connected component* is a maximal subset $C \subseteq V$ such that, for every $v_1, v_2 \in C$, the state v_2 is accessible from v_1 (and reciprocally).

For instance, automata in Figure 2.2 and 2.3 are deterministic, but not in Figure 2.4. The automaton in Figure 2.3 is not trim.

Definition 2.5.4. A finite automaton M *recognizes* a word $w \in \mathcal{A}^\star$ if w can be read along some oriented path from the initial state v_0 to some terminal state $v \in \mathsf{accept}$. A language $\mathcal{L} \subseteq \mathcal{A}^\star$ is *regular* if it can be written as

$$\mathcal{L} = \{w \in \mathcal{A}^\star \mid w \text{ is recognized by } M\}$$

for some finite automaton $M = (V, \mathcal{A}, \delta, v_0, \mathsf{accept})$.

Remark 2.5.5. The automaton M recognizes the empty word ε if $v_0 \in \mathsf{accept}$.

A fundamental result in the theory is Rabin–Scott theorem:

Theorem 2.5.6 ([RS59]). *Every regular language $\mathcal{L} \subseteq \mathcal{A}^\star$ is recognized by a trim, deterministic automaton. Moreover, such an automaton can be constructed effectively from an automaton recognizing $\mathcal{L}$.*

Proof. Let $M = (V, \mathcal{A}, \delta, v_0, \mathsf{accept})$ be an automaton recognizing $\mathcal{L}$. We define a new deterministic (and saturated) automaton $M' = (V', \mathcal{A}, \delta', v_0', \mathsf{accept}')$ recognizing $\mathcal{L}$. This construction is called the "powerset construction". We define

- $V' = \mathcal{P}(V)$ the powerset of V.
- For every $S \subset V$ (i.e., $S \in V'$) and $a \in \mathcal{A}$, we define

$$\delta'(S, a) \coloneqq \left\{ w \in V \;\middle|\; \exists v \in S \text{ such that } (v, a, w) \in \delta \right\},$$

  and let $\delta' = \{(S, a, \delta'(S, a)) \mid S \subseteq V, a \in \mathcal{A}\}$.
- $v_0' = \{v_0\}$.
- $\mathsf{accept}' = \{S \subseteq V \mid S \cap \mathsf{accept} \neq \emptyset\}$.

The key observation is that the vertex reached in M' when starting from v_0' and following a word $w \in \mathcal{A}^\star$ is exactly the set of vertices we could reach in M when starting from v_0 following the same word w. By definition of accept', it recognizes $\mathcal{L}$.

We can then trim this automaton, removing every vertex which is not accessible from v_0', or from which we cannot reach any vertex in accept'. □

2.5.2 Context-free languages

Definition 2.5.7. A *context-free grammar* is a 4-uple $(\mathcal{A}, \mathcal{N}, P, S)$, where

- $\mathcal{A}$ is a finite set of *terminals*, and $\mathcal{N}$ is a finite set of *non-terminals*.
- P is a finite set of *production rules* of the form $Y \to u$, with $Y \in \mathcal{N}$ and $u \in (\mathcal{A} \sqcup \mathcal{N})^\star$. Formally, P is a finite subset of $\mathcal{N} \times (\mathcal{A} \sqcup \mathcal{N})^\star$.
- $S \in \mathcal{N}$ is the *start symbol*.

We write $v \to w$ if there exist $l, r \in (\mathcal{A} \sqcup \mathcal{N})^\star$ and a production rule $Y \to u$ such that $v = lYr$ and $w = lur$. Moreover we write $u \xrightarrow{*} v$ if there exists a *derivation*

$$u \to v_1 \to v_2 \to \ldots \to v_{n-1} \to v.$$

The language *produced* by the grammar is the set $\mathcal{L} = \{w \in \mathcal{A}^\star \mid S \xrightarrow{*} w\}$. A language is *context-free* if it is produced by a context-free grammar.

Several desirable properties for context-free grammar are the following:

Definition 2.5.8. A context-free grammar is

- *unambiguous* if each word $w \in \mathcal{L}$ admits a unique leftmost derivation, i.e., a unique derivation $Y_0 \to v_1 \to \ldots \to v_{n-1} \to w$ in which each production rule $v_i \to v_{i+1}$ replaces the leftmost non-terminal in v_i. Equivalently, every word $w \in \mathcal{L}$ corresponds to a unique derivation tree.
- *expansive* if there exists a non-terminal Y such that $Y \xrightarrow{*} w_1 Y w_2 Y w_3$. Otherwise, the grammar is *non-expansive* (or *pseudo-linear*)

We recall a fundamental result of Chomsky and Schützenberger on these languages:

Theorem 2.5.9 ([CS63],[Kui70, Theorem 4])**.** *If a language $\mathcal{L}$ is produced by an unambiguous context-free grammar, then its growth series*

$$\Sigma_{\mathcal{L}}(t) = \sum_{n=0}^{\infty} \#\{w \in \mathcal{L} : \ell(w) = n\} \cdot t^n \in \mathbb{N}[[t]]$$

is algebraic. If furthermore the grammar is non-expansive, then $\Sigma_{\mathcal{L}}(t)$ is rational.

Example 2.5.10. We give multiples examples of different flavors:

(a) The set of palindromes over the alphabet $\mathcal{A} = \{a, b\}$ is defined by the grammar

$$S \to \varepsilon \mid a \mid b \mid aSa \mid bSb.$$

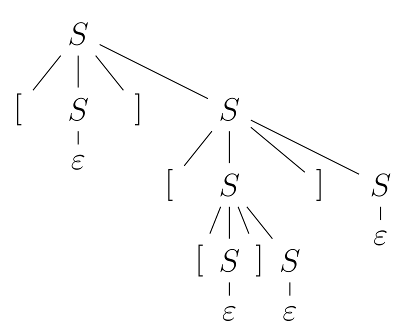
The symbol "|" means "or". This grammar is unambiguous and non-expansive, and indeed its growth series is rational: we have $\Sigma_{\mathcal{L}}(t) = 1 + 2t + 2t^2 \Sigma_{\mathcal{L}}(t)$ hence

$$\Sigma_{\mathcal{L}}(t) = \frac{1 + 2t}{1 - 2t^2} = 1 + 2t + 2t^2 + 4t^3 + 4t^4 + \ldots.$$

(b) The set of "correct string of parenthesis" is defined by the two rules $S \to [S]S$ and $S \to \varepsilon$. An example of derivation is

$$S \to [S]S \to [\,]S \to [\,][S]S \to [\,][[S]S]S \to [\,][[\,]S]S \to [\,][[\,]]S \to [\,][[\,]]$$

corresponding to the derivation tree



This grammar is unambiguous and expansive, and indeed the series is algebraic:

$$\Sigma_{\mathcal{L}}(t) = t \cdot \Sigma_{\mathcal{L}}(t) \cdot t \cdot \Sigma_{\mathcal{L}}(t) + 1 \implies \Sigma_{\mathcal{L}}(t) = \frac{1 - \sqrt{1 - 4t^2}}{2t^2}.$$

(c) Fix $u_i, v_i, w_i \in S^\star$. The grammar

$$\begin{aligned} S &\to w_0 X w_2 Y w_6 \\ X &\to u_1 X v_1 \mid w_1 \\ Y &\to u_2 Y v_2 \mid w_3 Z w_5 \\ Z &\to u_3 Z v_3 \mid w_4 \end{aligned}$$

generates the language

$$\mathcal{L} = \left\{ w_0 u_1^{n_1} w_1 v_1^{n_1} w_2 u_2^{n_2} w_3 u_3^{n_3} w_4 v_3^{n_3} w_5 v_2^{n_2} w_6 \;\middle|\; n_1, n_2, n_3 \geqslant 0 \right\}.$$

This is a typical example of *Dyck loop*, which are central to Section 7.4.

(d) We consider the lamplighter group

$$C_2 \wr \mathbb{Z} = \left\langle a, z \mid a^2 = [a, z^n a z^{-n}] = e \;\text{ for all } n \geqslant 1 \right\rangle.$$

The language of ShortLex representatives for the generating set $\{a, z^\pm\}$ (ordered $a < z^{-1} < z$) is produced by the unambiguous context-free grammar

$$\begin{aligned} S &\to Q_a Q_L Q_R \mid Q_a Q_L z F_R \mid Q_a Q_R z^{-1} F_L; \\ L &\to a \mid z^{-1} L z \mid a z^{-1} L z; \\ R &\to a \mid z R z^{-1} \mid a z R z^{-1}; \\ F_L &\to Q_a Q_L \mid Q_a z^{-1} F_L \\ F_R &\to Q_a Q_R \mid Q_a z F_R \\ Q_a &\to \varepsilon \mid a; \quad Q_L \to \varepsilon \mid z^{-1} L z; \quad Q_R \to \varepsilon \mid z R z^{-1} \end{aligned}$$

We may deduce the associated growth series (recovering a formula of [Joh91])

$$\Sigma_{G,S}(t) = \frac{(1+t)^3 (1-t)^2 (1+t+t^2)}{(1-t^2-t^3)^2 (1-t-t^2)}.$$

Another important class of examples comes from groups:

Theorem 2.5.11 ([MS83]). *Let (G, S) be a marked group. Then the Word Problem*

$$\mathcal{WP}(G, S) \coloneqq \{ w \in S^\star : \bar{w} = e_G \}$$

is context-free if and only if G is virtually free. In that case, $\mathcal{WP}(G, S)$ is unambiguously context-free (even deterministically context-free).

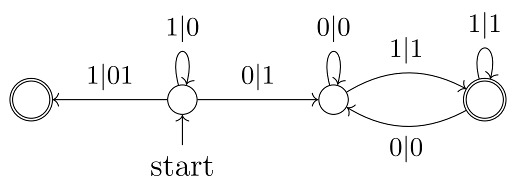
2.5.3 Rational transduction

Definition 2.5.12. A finite-state transducer in a 6-uple $(V, \mathcal{A}, \mathcal{B}, \delta, v_0, \mathsf{accept})$ with

- V is a finite set.
- $\mathcal{A}, \mathcal{B}$ are two alphabets.
- $\delta \subseteq V \times (\mathcal{A} \sqcup \{\varepsilon\}) \times \mathcal{B}^\star \times V$ is a transition function. An element (v_1, a, w, v_2) should be seen as an oriented edge from v_1 to v_2, with label $a \mid w$.
- $v_0 \in V$ is the start vertex.
- $\mathsf{accept} \subseteq V$ is the set of terminal vertices.

A transducer recognizes a *rational relation* $R \subseteq \mathcal{A}^\star \times \mathcal{B}^\star$. (A pair (v, w) belong to R if the words v and w can be read along a common path $v_0 \to \mathsf{accept}$.) Often, this relation is a (partial) function. In this case, its domain is a regular language $\mathcal{L} \subseteq \mathcal{A}^\star$.

Example 2.5.13. The following transducer defines a rational function. The domain is $\{0,1\}^*1$, the set of valid strings of digits of positive integers in binary, read in reverse. On input the string for $n \in \mathbb{Z}_{>0}$, it outputs the string for $n+1$.



Proposition 2.5.14 (See eg. [Niv68])**.** *Consider a language $\mathcal{L} \subseteq \mathcal{A}^\star$. Its "image" by the relation is $R(\mathcal{L}) \coloneqq \{w \in \mathcal{B}^\star \mid \exists v \in \mathcal{L},\ (v,w) \in R\}$. Then*

(a) *If $\mathcal{L}$ is regular, then $R(\mathcal{L})$ is regular.*

(b) *If $\mathcal{L}$ is context-free, then $R(\mathcal{L})$ is context-free.*

(c) *If R is a (partial) function and $\mathcal{L}$ is unambiguously context-free, then $R(\mathcal{L})$ is unambiguously context-free.* [Niv68, Corollaire 1, p.429]

Points (a) and (b) are parts of a more general theorem: "cones of languages" are closed under rational transduction. Examples of cones are the class of regular languages, or the class of context-free languages.

2.6 Rational subsets

Definition 2.6.1. Let G be group (or a monoid).

- A subset $R \subseteq G$ is *rational* (of G) if there exists a regular language $\mathcal{L} \subseteq S^\star$ and a monoid morphism $\mathrm{ev}\colon S^\star \to G$ such that $R = \mathrm{ev}(\mathcal{L})$.
- A subset $R \subseteq G$ is *unambiguously rational* if furthermore $\mathrm{ev}\colon \mathcal{L} \to R$ is one-to-one.

Remark 2.6.2. We denote the alphabet as S as it will usually be identified with $\mathrm{ev}(S)$, which is typically a generating set for G. However, this picture is not always ideal:

- Multiple elements $s, s' \in S$ might be send to the same element $\mathrm{ev}(s) = \mathrm{ev}(s')$, and still we consider two words usv and $us'v$ as distinct.
- The set $\mathrm{ev}(S)$ might not generate G. This is key in Theorem 2.6.3 below. We will also consider rational subsets inside non-finitely-generated groups, eg. $H_3(\mathbb{Q})$.

However, if $\mathrm{ev}(S)$ generates G (as a monoid), then for every rational subset $R \subseteq G$, there exists a regular language $\mathcal{L} \subseteq S^\star$ such that $R = \mathrm{ev}(\mathcal{L})$.

2.6.1 A better language

Given a regular $\mathcal{L} \subseteq S^\star$ recognized by a deterministic finite-state automaton M, and evaluating to $\mathrm{ev}(\mathcal{L}) = R$ inside a *group* G. We construct a new regular language which still evaluates to R and has a few additional properties.

Theorem 2.6.3. *From the automaton M, we can effectively compute*

- *an alphabet $X \sqcup Y$ equipped with an evaluation map* $\mathrm{ev}\colon X \sqcup Y \to G$
- *a rational function* $\mathrm{dec}\colon \mathcal{L} \to X^\star Y : w \mapsto \tilde{w}$ *with domain $\mathcal{L}$ which is into,*
- *a weight function $\omega\colon X \sqcup Y \to \mathbb{Z}_{>0}$*

such that $\mathrm{ev}(\tilde{w}) = \mathrm{ev}(w)$ *and* $\ell_\omega(\tilde{w}) = \ell(w)$ *for all $w \in \mathcal{L}$. As a corollary,*

(a) $\tilde{\mathcal{L}} := \mathrm{dec}(\mathcal{L})$ *satisfies* $\mathrm{ev}(\tilde{\mathcal{L}}) = R$

(b) *If $\mathcal{L}$ is bounded, then $\tilde{\mathcal{L}}$ is bounded.*

(c) *If* $\mathrm{ev}\colon \mathcal{L} \to R$ *is bijective, then* $\mathrm{ev}\colon \tilde{\mathcal{L}} \to R$ *is bijective.*

Moreover, $\tilde{\mathcal{L}}$ satisfies a strong form of the pumping lemma: if $uxv \in \tilde{\mathcal{L}}$ with $x \in X$, then $ux^nv \in \tilde{\mathcal{L}}$ for all $n \geqslant 1$.

(d) *As a corollary, $\langle \mathrm{ev}(X \sqcup Y) \rangle = \langle R \rangle$.*

The construction is inspired from Stallings' "automata theoretic" proof of the Nielsen–Schreier theorem, and a result of Gilman [Gil87, Lemma 5]. Different aspects of the construction are central in Chapters 4, 6 and 7.

Proof. We start with an automaton $M = (V, S, \delta, v_0, \mathsf{accept})$ recognizing R which is both deterministic and trim. Let

$$X = \left\{ tat^{-1} \;\middle|\; \begin{array}{c} t \in S^\star \text{ labels a simple path } v_0 \to p \\ a \in S^\star \text{ labels a simple cycle } p \to p \\ \text{both paths only intersect at } p \end{array} \right\}$$

$$Y = \left\{ y \;\middle|\; y \in S^\star \text{ labels a simple path } v_0 \to q \in \mathsf{accept} \right\}$$

with the obvious evaluation map $\mathrm{ev}(tat^{-1}) := \mathrm{ev}(t)\,\mathrm{ev}(a)\,\mathrm{ev}(t)^{-1}$ and $\mathrm{ev}(y) := \mathrm{ev}(y)$.

▶ We define the *decomposition map*

Any word $w \in \mathcal{L}$ is accepted by a unique path $v_0 \to \mathsf{accept}$ in the automaton M. This path decomposes as a product of conjugates $t_i a_i t_i^{-1}$, via a loop-erasure algorithm:

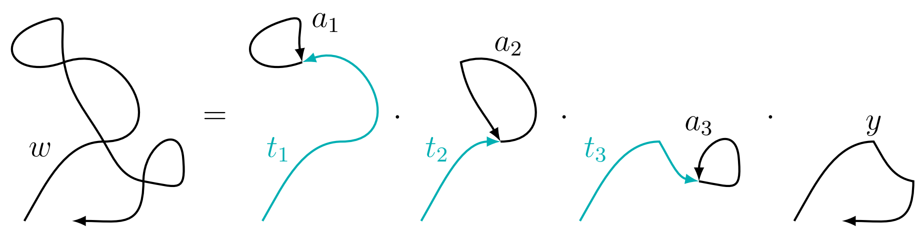


Figure 2.5: A path decomposed as a product of "freeze frames" of the loop-easure algorithm.

The resulting word $\tilde{w} = (t_1 a_1 t_1^{-1}) \dots (t_\ell a_\ell t_\ell^{-1}) \cdot y \in \tilde{\mathcal{L}}$ is the *decomposition* of w. This is indeed a rational function, defined by the transducer $T = (U, S, X \sqcup Y, \delta_T, u_0, \{◎\})$

- $U = \left\{ t \in S^\star \;\middle|\; t \text{ labels a simple path } e \to p \text{ in } M \right\} \sqcup \{◎\}$.
- $u_0 = \varepsilon \in U$ is the initial vertex,
- ◎ is the only of terminal vertex.
- The transition function is

$$\delta_T = \left\{ (t, s, \varepsilon, ts) \;\middle|\; ts \text{ is still a simple path } v_0 \to p \text{ in } M \right\}$$
$$\sqcup \left\{ (t, s, t' a t'^{-1}, t') \;\middle|\; \begin{array}{c} ts \text{ decomposes as } ts = t'a \text{ with} \\ t' \in S^\star \text{ labels a simple path } v_0 \to p \\ a \in S^\star \text{ labels a simple cycle } p \to p \end{array} \right\}$$
$$\sqcup \left\{ (t, \varepsilon, t, ◎) \;\middle|\; t \in S^\star \text{ labels a simple path } v_0 \to p \in \mathsf{accept} \right\}$$

Note that δ_T is almost deterministic: for every $t \in U$ and $s \in S$, there exists at most on element $(t, s, v, t') \in \delta_T$. This follows from M being deterministic. The few ε-transitions do not change the fact that every word $w \in \mathcal{L}$ is accepted by a unique path, hence the resulting transduction is indeed a function.

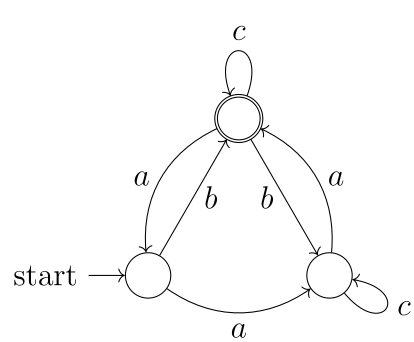


Figure 2.6: An automaton M

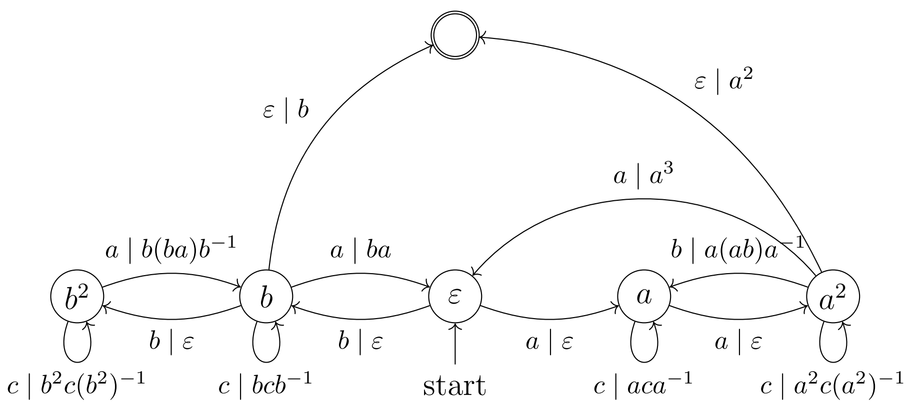


Figure 2.7: The corresponding transducer T.

This proves that $\tilde{\mathcal{L}} \coloneqq \operatorname{dec}(\mathcal{L})$ is a regular language, more explicitely

$$\tilde{\mathcal{L}} = \left\{ x_1x_2\dots x_\ell\, y \in X^\star Y \;\middle|\; \begin{array}{l} t_i \text{ is a prefix of } t_{i+1}a_{i+1} \text{ for } 1 \leqslant i \leqslant \ell-1 \\ t_\ell \text{ is a prefix of } y \end{array} \right\}.$$

Moreover the decomposition map is injective as the sequence of "freeze frames" (this should be called a "movie") is sufficient to reconstruct $w \in \mathcal{L}$.

► We take $\omega(tat^{-1}) = \ell(a)$ and $\omega(y) = \ell(y)$. It should be clear that $\operatorname{ev}(\tilde{w}) = \operatorname{ev}(w)$ (hence point (a), and (c) using injectivity) and $\ell_\omega(\tilde{w}) = \ell(w)$.

(b) We have $\beta_{\mathcal{L}}(n) = \beta_{\tilde{\mathcal{L}},\omega}(n) \succeq \beta_{\tilde{\mathcal{L}}}(n)$. If $\mathcal{L}$ is bounded, then $\mathcal{L}$ and therefore $\tilde{\mathcal{L}}$ have polynomial growth, hence $\tilde{\mathcal{L}}$ is bounded by [Tro81].

(d) We prove that $\langle \operatorname{ev}(X \sqcup Y)\rangle = \langle R\rangle$.

Take $x = tat^{-1} \in X$, where t labels a path $v_0 \to p$. As the automaton M is trim, there exists a path $p \to q \in \mathsf{accept}$ labeled by some word $v \in S^\star$. Consider $tuv \in \mathcal{L}$ and $xx_2\dots x_\ell y \in \tilde{\mathcal{L}}$ its decomposition. Observe that $x_2\dots x_\ell y \in \tilde{\mathcal{L}}$ too, so

$$x = (xx_2\dots x_\ell y)(x_2\dots x_\ell y)^{-1} \in R \cdot R^{-1} \subseteq \langle R\rangle\,.$$

On the other side, we have $Y \subseteq R \subseteq \langle R\rangle$. This proves the inclusion $\operatorname{ev}(X \sqcup Y) \subseteq \langle R\rangle$. The reverse inclusion is clear as $\tilde{\mathcal{L}} \subseteq X^\star Y$ hence $R \subseteq \langle \operatorname{ev}(X \sqcup Y)\rangle$. □

Corollary 2.6.4 ([Gil87, Lemma 5])**.** *Let G be a group and $R \subseteq G$ a (unambiguous) rational subset. Suppose that a subgroup $H \leqslant G$ contains R, then R is a (unambiguous) rational subset of H. Moreover, the result is effective.*

2.6.2 Going to finite-index subgroups

Example 2.6.5. An important example for Theorem 2.6.3 is G group and H a finite-index subgroup. For any $\mathrm{ev}\colon S \to G$, we can consider the language

$$\mathcal{WP}(H\backslash G, S) = \{w \in S^\star \mid \bar{w} \in H\}.$$

This language is regular, recognized by the Schreier graph $\mathcal{S}ch(H\backslash G, S)$ seen as an automaton. We recall the definition of Schreier graphs:

- The vertex set is $V = H\backslash G$,
- The edges/transition function is $\delta = \{(Hg, s, Hg\bar{s})\}$.

In order to make into an automaton, we take $v_0 = H$ and $\mathsf{accept} = \{H\}$. We have

$$X = \left\{ tat^{-1} \;\middle|\; \begin{array}{c} t \in S^\star \text{ labels a simple path } H \to Ht \\ a \in S^\star \text{ labels a simple cycle } Ht \to Ht \end{array} \right\} \xrightarrow{\mathrm{ev}} H$$

and $Y = \{\varepsilon\}$.

The decomposition map w.r.t. this specific automata is particularly useful. We deduce the following Lemma, which complements nicely with Gilman's result:

Lemma 2.6.6. *Let G be a group and $H \leqslant G$ is a finite-index subgroup,*

(a) *If $R \subseteq G$ is rational, then $Hg \cap R$ is rational for each $g \in G$.*

If moreover we are given an automaton $M = (V, S, \delta, v_0, \mathsf{accept})$ for a language $\mathcal{L} \subseteq S^\star$ such that $\mathrm{ev}(\mathcal{L}) = R$, and the Schreier graph $\mathcal{S}ch(H\backslash G, S)$, then

(b) *We can effectively compute an automaton for $\tilde{\mathcal{L}}$ such that $\mathrm{ev}(\tilde{\mathcal{L}}) = Hg \cap R$*

(c) *We can effectively compute $\{Hg : Hg \cap R \neq \emptyset\}$ and compute a subset $\mathcal{K} \subseteq \mathcal{L}$ consisting of a unique representative for each of these cosets.*

Proof. (ab) Consider the decomposition map defined from the automaton $\mathcal{S}ch(H\backslash G, S)$ (with terminal vertex Hg), whose domain is $\{w \in S^\star \mid \bar{w} \in Hg\}$. Then

$$\tilde{\mathcal{L}} = \mathrm{dec}(\mathcal{L})$$

is the desired regular language (over $X \sqcup Y$, with $\mathrm{ev}(X) \subseteq H$).

Alternatively, here is an explicit construction for an automaton (over S). We consider the *tensor product* of $\mathcal{S}ch(H\backslash G, S)$ and M:

- $V' = H\backslash G \times V$ - $v_0' = (H, v_0)$ - $\mathsf{accept}' = \{Hg\} \times \mathsf{accept}$
- $(Hg_1, v_1) \xrightarrow{s} (Hg_2, v_2)$ if and only if $Hg_1 s = Hg_2$ and $v_1 \xrightarrow{s} v_2$.

This new automaton recognizes the language $\mathcal{L} \cap \{w \in S^\star \mid \bar{w} \in Hg\}$.

(c) For each coset, one can compute an automaton for $Hg \cap R$ and then decide if the intersection is non-empty: does there exists a path $v_0' \to \mathsf{accept}'$? If yes, a representative is given by $w \in \mathcal{L}$ labeling the first path found. □

Remark 2.6.7. If a finite presentation $G = \langle S \mid R \rangle$ is given, then the Todd–Coxeter algorithm computes $\mathcal{S}ch(H\backslash G, S)$ from generators for H (given as words over S), so $\mathcal{S}ch(H\backslash G, S)$ does not need to be part of the input.

Chapter 3

A geometrical model

As stated in Section 2.4.6, simply connected nilpotent Lie groups Γ can be endowed with left-invariant distributions $\bigcup_g (L_{g*}V_1)_g$, and every element of Γ can be reached by an horizontal path starting at e. Moreover, absolutely continuous paths in $\Gamma/[\Gamma,\Gamma]$ (starting at $\mathbf{0}$) correspond to horizontal paths in Γ (starting at e) via the lift map. Therefore elements of Γ can be seen as equivalence classes of paths in $\Gamma/[\Gamma,\Gamma]$. In this chapter, we make this equivalence relation explicit for some groups, namely the Heisenberg, Engel and Cartan groups.

This model was already widely used for $H_3(\mathbb{R})$, see for instance [BLD13; DM14]. For the Cartan group, it appears briefly in [Wal11, Section 4] and [AH22].

3.1 The Heisenberg, Engel and Cartan groups

In this section, we define models for some simply connected nilpotent Lie groups. We consider Π the monoid of absolutely continuous paths in $\mathbb{R}^2$ starting from $(0,0)$, with concatenation as operation (see Figure 3.1). Our groups are defined as quotients $\Pi/\!\sim$ for appropriate equivalence relations $\sim$. For any path $g\in\Pi$, we define three parameters:

(1) its second endpoint $\hat g = (x_g, y_g)\in\mathbb{R}^2$.

(•) a distribution of winding numbers. First, we get a closed path g_c by concatenating g with the segment back from $\hat g$ to $(0,0)$. Then the function $W_g\colon \mathbb{R}^2\setminus \operatorname{Im}(g_c)\to\mathbb{Z}$ is defined as $W_g(x,y) =$ the winding number of g_c around (x,y). (See Figure 3.1.)

(2) its total algebraic (or "balayage") area

$$A(g) = \iint_{\mathbb{R}^2} W_g(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int \left(g_x g_y' - g_x' g_y\right)\mathrm{d}t.$$

(3) its (non-normalized) barycenter (or center of gravity)

$$B(g) = \iint_{\mathbb{R}^2} (x, y) \cdot W_g(x, y) \, \mathrm{d}x \, \mathrm{d}y.$$

These parameters behave quite well with concatenation:

Proposition 3.1.1. *Given two paths g, h, their concatenation gh has parameters*

$$\begin{aligned} \widehat{gh} &= \widehat{g} + \widehat{h}, \\ A(gh) &= A(g) + A(h) + \frac{1}{2}\det\big(\hat{g}, \hat{h}\big), \\ B(gh) &= B(g) + B(h) + \hat{g} \cdot A(h) + \frac{1}{3}(2\hat{g} + \hat{h}) \cdot \frac{1}{2}\det\big(\hat{g}, \hat{h}\big). \end{aligned}$$

Proof. The relation $\widehat{gh} = \hat{g} + \hat{h}$ is obvious. The key observation is the decomposition

$$W_{gh} = W_g + W_h \circ \tau_{-\hat{g}} \pm \chi_{\triangle((0,0),\hat{g},\hat{g}+\hat{h})},$$

where

- $\tau_v \colon \mathbb{R}^2 \to \mathbb{R}^2$ is the translation by v,
- $\triangle\big((0,0), \hat{g}, \hat{g} + \hat{h}\big)$ is the convex hull of those three points, $\chi_{\triangle((0,0),\hat{g},\hat{g}+\hat{h})}$ denotes its characteristic function, and the sign $\pm$ depends on the order of $(0,0)$, $\hat{g}$ and $\hat{g} + \hat{h}$ on the boundary of the triangle (-1 if clockwise and $+1$ if anti-clockwise).

Here is a pictorial explanation:

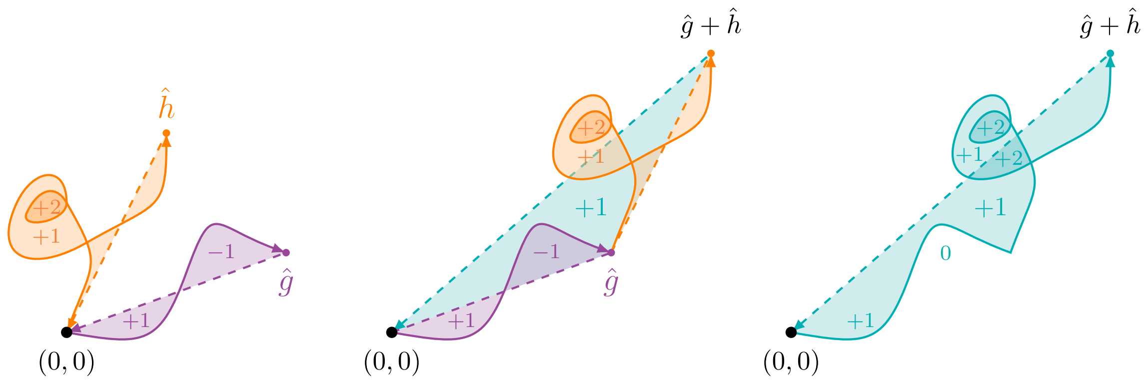


Figure 3.1: Pictures of g, h and gh and some winding numbers.

It only remains to compute $A(gh)$ and $B(gh)$ using the decomposition. Observe that $\frac{1}{2}\det(\hat{g}, \hat{h})$ is the signed area of the triangle $\triangle\big((0,0), \hat{g}, \hat{g} + \hat{h}\big)$, and its (normalized) center of gravity is given by $\frac{1}{3}\big(\mathbf{0} + \hat{g} + (\hat{g} + \hat{h})\big)$. □

As a corollary, we can define different quotients:

- $g \sim h \iff \hat{g} = \hat{h}$, we get as a quotient $\mathbb{R}^2$.
- $g \sim h \iff (\hat{g} = \hat{h}$ and $A(g) = A(h))$, we get as a quotient $H_3(\mathbb{R})$.
- $g \sim h \iff (\hat{g} = \hat{h}$, $A(g) = A(h)$ and $B_y(g) = B_y(h))$ (same "y-coordinate of the barycenter"), we get as a quotient the Engel group $\overline{\mathscr{E}}$.
- $g \sim h \iff (\hat{g} = \hat{h}$, $A(g) = A(h)$ and $B(g) = B(h))$, we get as a quotient the Cartan group $\overline{\mathscr{C}}$, the free 3-step nilpotent Lie group of rank 2.

In each case, the concatenation operation goes to the quotient, and the inverse path gives a multiplicative inverse, so $\Pi/\sim$ defines a group.

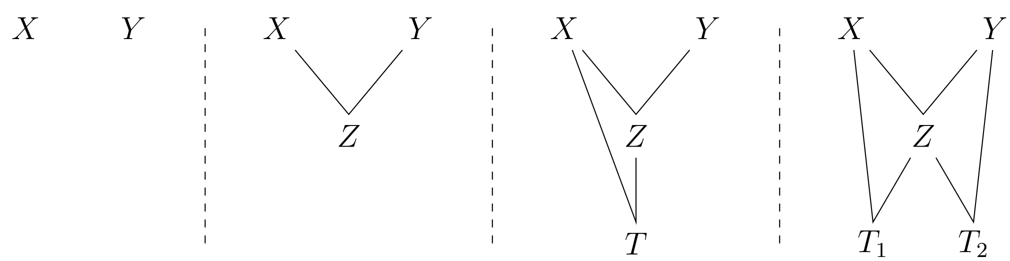


Figure 3.2: The commutator diagrams for the different Lie groups obtained

In each case, the subgroup generated by x and y, the unit segments from $\mathbf{0}$ to $(1,0)$ and $(0,1)$ respectively, is a cocompact lattice. In the second case, we get

$$\langle x, y\rangle \simeq H_3(\mathbb{Z}) = \langle x, y \mid [x,[x,y]] = [y,[x,y]] = e\rangle .$$

3.2 Metabelian nilpotent groups

In this section, we extends the previous models to rank 2 metabelian nilpotent groups. To keep things simple, we formulate the results in the "finitely generated" case. The key motivation is the following model for the free metabelian group of rank 2:

Theorem 3.2.1 ([DV05])**.** *Let $\langle x^{\pm}, y^{\pm}\rangle$ be the monoid of lattices paths in $\mathbb{Z}^2$, starting at $\mathbf{0}$, with concatenation. Consider the equivalence relation*

$$g \sim h \iff (\hat{g} = \hat{h} \text{ and } W_g = W_h)$$

Then $\langle x^{\pm}, y^{\pm}\rangle / \sim$ is a group isomorphic to M_2, the free metabelian group of rank 2.

Remark 3.2.2. We can identify $[M_2, M_2]$ with $\bigoplus_{\mathbb{Z}^2} \mathbb{Z}$, the group of finitely supported functions $f\colon \mathbb{Z}^2 \to \mathbb{Z}$, with addition. Each element $g \in [M_2, M_2]$ corresponds to

$$f_g(a,b) \coloneqq W_g\left(a + \frac{1}{2}, b + \frac{1}{2}\right).$$

For instance $g = [x, y]$ corresponds to $\delta_{\mathbf{0}}$.

As 2-generated metabelian nilpotent groups are quotients of M_2, this indicates that elements can be described in terms of $\hat{g}$ and W_g. It turns out this can be formulated quite cleanly, keeping track of a few parameters defined from W_g.

Definition 3.2.3. The space of polynomial of degree at most n is denoted $\mathbb{R}_n[X,Y]$. For instance $\mathbb{R}_2[X,Y] = \langle 1,\ X, Y,\ X^2, XY, Y^2\rangle$. By convention $\mathbb{R}_{-1}[X,Y] = \{0\}$.

Definition 3.2.4. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function, and $\mathbf{v} \in \mathbb{R}^2$. We define

$$\Delta_{\mathbf{v}} f(\mathbf{x}) \coloneqq f(\mathbf{x}+\mathbf{v}) - f(\mathbf{x}).$$

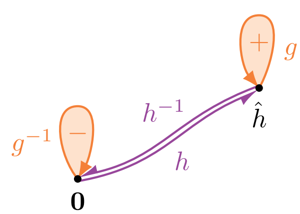
Theorem 3.2.5. *Let $V \leqslant \mathbb{R}_n[X,Y]$ be a linear subspace closed under Δ (i.e., for all $P \in V$ and $\mathbf{v} \in \mathbb{R}^2$, we have $\Delta_{\mathbf{v}} P \in V$). We define*

$$K_V = \left\{ g \in M_2 \;\middle|\; \hat{g} = \mathbf{0} \quad \text{and} \quad \forall P \in V,\ \iint P(\mathbf{x}) \cdot W_g(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} = 0 \right\}.$$

Then $K_V \trianglelefteq M_2$, and the quotient group M_2/K_V is $(n+2)$-step nilpotent.

Proof. We first observe that, if $g \in [M_2, M_2]$ (i.e., if $\hat{g} = 0$) and $h \in M_2$, then

$$W_{[h,g]} = W_g \circ \tau_{-\hat{h}} - W_g = \Delta_{-\hat{h}} W_g \tag{3.2.6}$$



► Let us prove that K_V is normal. We consider $g \in K_V$ and $h \in M_2$. Obviously we have $\widehat{[h,g]} = \mathbf{0}$. Moreover, for all $P \in V$, we have

$$\iint P(\mathbf{x}) \cdot W_{[h,g]}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} = \iint P(\mathbf{x}) \cdot \Delta_{-\hat{h}} W_g(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} = \iint \Delta_{\hat{h}} P(\mathbf{x}) \cdot W_g(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} = 0$$

as $\Delta_{\hat{h}} P \in V$ and $g \in K_V$. This proves that $[h,g] \in K_V$, hence K_V is normal.

► We prove that M_2/K_V is $(n+2)$-step nilpotent. We consider $g \in [M_2, M_2]$ and $h_0, h_1, \ldots, h_n \in M_2$. Using Formula (3.2.6) iteratively, we get

$$W_{[h_0,h_1,\ldots,h_n,g]} = \Delta_{-\hat{h}_0} \Delta_{-\hat{h}_1} \ldots \Delta_{-\hat{h}_n} W_g$$

and therefore

$$\iint P(\mathbf{x}) \cdot W_{[h_0,h_1,\ldots,h_n,g]}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} = \iint \Delta_{\hat{h}_0} \Delta_{\hat{h}_1} \ldots \Delta_{\hat{h}_n} P(\mathbf{x}) \cdot W_g(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} = 0,$$

since $\Delta_{\hat{h}_0} \Delta_{\hat{h}_1} \ldots \Delta_{\hat{h}_n} P(\mathbf{x}) \equiv 0$ for all $P \in \mathbb{R}_n[X,Y]$ (as $\deg(\Delta_{\mathbf{v}} P) \leqslant \deg P - 1$). This proves that $[h_0, h_1, \ldots, h_n, g] \in K_V$, hence M_2/K_V is $(n+2)$-step nilpotent. □

Moreover, the reciprocal holds:

Theorem 3.2.7. *Every 2-generated, metabelian, c-step nilpotent group is a quotient of M_2/K_V with $V = \mathbb{R}_{c-2}[X, Y]$.*

Proof. We consider the relatively free group in the variety of metabelian c-step nilpotent groups, which we denote by $L_{2,c} = M_2/\gamma_{c+1}(M_2)$. We naturally have an epimorphism $\alpha\colon L_{2,c} \twoheadrightarrow M_2/K_V$. The objective is to prove that α is injective.

For every $m, n \geqslant 0$ such that $m + n = c - 1$, we have

$$f_{m,n} \coloneqq \big[\underbrace{x, x, \ldots, x}_{m \text{ copies}}, \underbrace{y, y, \ldots, y}_{n \text{ copies}}, [x, y]\big] \in \gamma_{c+1}(M_2).$$

Using the identification from Remark 3.2.2, we have

$$f_{m,n}(a, b) = \Delta_{\hat{x}}^{m} \Delta_{\hat{y}}^{n} \delta_{\mathbf{0}}(a, b) = (-1)^{m+n-a-b} \binom{m}{a} \binom{n}{b}$$

(which is 0 if $(a, b) \notin [\![0, m]\!] \times [\![0, n]\!]$). For instance, for $c = 6$, we have

$$\begin{array}{cccccc} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \;=\; \begin{array}{cccccc} 1 & -5 & 10 & -10 & 5 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \;=\; \begin{array}{cccccc} \cdot & \cdot & 1 & -3 & 3 & \cdot \\ \cdot & \cdot & -2 & 6 & -6 & 2 \\ \cdot & \cdot & 1 & -3 & 3 & -1 \end{array}$$

modulo $\gamma_7(G)$. Using these relations, we can prove that, for every $f \in [M_2, M_2]$, there exists $\tilde{f}$ such $f - \tilde{f} \in \gamma_{c+1}(M_2)$ and

$$\operatorname{supp}(\tilde{f}) \subseteq T_c \coloneqq \{(a, b) \in \mathbb{Z}^2 \mid a, b \geqslant 0,\ a + b \leqslant c - 1\}.$$

We deduce the existence of an epimorphism $\beta\colon \mathbb{Z}^{|T_c|} \twoheadrightarrow [L_{2,c}, L_{2,c}]$. It follows that

$$h(L_{2,c}) \leqslant 2 + |T_c| = 2 + \frac{(c-1)(c-2)}{2} = 2 + \dim \mathbb{R}_{c-1}[X, Y] = h(M_2/K_V),$$

with equality only if β is an isomorphism. As $h(L_{2,c}) = h(\ker \alpha) + h(M_2/K_V)$, we conclude that $h(\ker \alpha) = 0$ and β is an isomorphism. From that last point, we see that $L_{2,c}$ is torsionfree, hence the only possibility for $h(\ker \alpha) = 0$ is that $\ker \alpha = \{e\}$. □

3.3 Further questions and remarks

An obvious direction of research is

Problem 3.A. Generalize this discussion to groups of higher rank.

There is another model for M_r, which is perhaps more practical for this purpose. We consider $\vec{E}\mathbb{Z}^r$ the set of *oriented edges* of $\mathbb{Z}^r$

$$\vec{E}\mathbb{Z}^r = \{(\mathbf{v}, \mathbf{v} \pm \mathbf{e}_i) \mid \mathbf{v} \in \mathbb{Z}^d\}.$$

For each edge $e = (\mathbf{v}, \mathbf{v} \pm \mathbf{e}_i)$, the reverse edge is $\overleftarrow{e} = (\mathbf{v} \pm \mathbf{e}_i, \mathbf{v})$. For each lattice path $g \in \langle x_1^\pm, \ldots, x_r^\pm \rangle$, we define its "flow function" $F_g \colon \vec{E}\mathbb{Z}^r \to \mathbb{Z}$

$$F_g(e) = \#\text{times } e \text{ is used by } g - \#\text{times } \overleftarrow{e} \text{ is used by } g$$

Theorem 3.3.1 ([DV05])**.** *Let $\langle x_1^\pm, \ldots, x_r^\pm \rangle$ be the monoid of lattices paths in $\mathbb{Z}^r$ starting at $\mathbf{0}$. Consider the equivalence relation*

$$g \sim h \iff F_g = F_h.$$

Then $\langle x_1^\pm, \ldots, x_r^\pm \rangle / \sim$ is a isomorphic to M_r, the free metabelian group of rank r.

Remark 3.3.2. We can recover $\hat{g}$ from F_g, looking at the divergence:

$$d^* F_g(\mathbf{v}) \coloneqq \sum_{e : \omega(e) = \mathbf{v}} F_g(e) \overset{!}{=} \delta_{\hat{g}} - \delta_{\mathbf{0}}.$$

This condition characterizes functions F_g corresponding to paths in $\vec{E}\mathbb{Z}^r$ starting at $\mathbf{0}$ (among finitely supported functions $F \colon \vec{E}\mathbb{Z}^r \to \mathbb{Z}$ with $F(\overleftarrow{e}) = -F(e)$).

The question now becomes

Problem 3.B. Express the equivalence relations $\sim$ on $\langle x_1^\pm, \ldots, x_r^\pm \rangle$ defining metabelian nilpotent groups in terms of F_g.

Problem 3.C. Generalize in the setting of Lie groups.

While integrating winding numbers against polynomials can easily be extended to the continuous setting, the correct continuous analog of F_g is not entirely clear. For $H_3(\mathbb{R})$, what we are looking for is probably the formula

$$A(g) = \int \left(g_x g_y' - g_x' g_y\right) \mathrm{d}t.$$

* * *

Another tempting question is to strengthen Theorem 3.2.7.

Conjecture 3.D. Every *torsionfree* 2-generated metabelian nilpotent group is isomorphic to M_2/K_V for some finite dimensional $V \leqslant \mathbb{R}[X, Y]$ closed under Δ.

Chapter 4

Intermediate geodesic growth

In this chapter, we study the geodesic growth of (marked) groups (G, S). In the spirit of Milnor questions on volume growth, we study if and when it can be polynomial, exponential, or intermediate. A word $w \in S^*$ is a *geodesic* if $\ell(w) = \|\overline{w}\|_S$. The *geodesic growth function* of (G, S) is then defined as

$$\gamma(n) = \gamma_{\text{geod}}^{G,S}(n) = \#\left\{w \in S^\star \;\middle|\; w \text{ is geodesic and } \ell(w) \leqslant n\right\}.$$

We should mention that geodesic growth is sensitive to the choice of a generating set. Indeed, Bridson, Burillo, Elder and Šunić proved that any infinite group G admits a generating set S such that $\gamma_{\text{geod}}^{G,S}(n)$ grows exponentially [BBES12, Example 6].[1]

Figure 4.1: The Cayley graphs of $G = \mathbb{Z}$ w.r.t. $S = \{\pm 1\}$ and $S' = \{\pm 1, \pm 1\}$. The geodesic growth are $\gamma^{\mathbb{Z},S}(n) = 2n + 1$ and $\gamma^{\mathbb{Z},S'}(n) = 2^{n+2} - 3$.

Therefore, Bridson *et al.* ask the following questions:

Question 1. Characterize groups G with polynomial geodesic growth, that is, with $\gamma_{\text{geod}}^{G,S}(n) \preceq n^d$ for some constant $d \geqslant 0$ and for at least one generating set S.

Question 2. Does there exists a pair (G, S) with intermediate geodesic growth?

In the same paper, Bridson *et al.* proposed some partial answers. Their main theorem is a sufficient condition for polynomial geodesic growth:

Theorem ([BBES12, Theorem 1])**.** *Let G be a finitely generated group. If there exists an element $a \in G$ such that $H = \langle\langle a \rangle\rangle_G$ is a finite-index abelian subgroup, then there exists a symmetric generating set S such that (G, S) has polynomial geodesic growth.*

[1] This holds if we allow generating *multisets*. Otherwise, the only virtually nilpotent counter-example is $G = \mathbb{Z}$ (as a corollary of Theorem 4.A), and any other hypothetical counter-example would be of intermediate volume growth (as exponential volume growth implies exponential geodesic growth).

This includes all virtually cyclic groups, and also groups like

$$v\mathscr{Z} = \mathbb{Z}^2 \rtimes C_2 = \left\langle a, t \mid t^2 = [a, a^t] = 1 \right\rangle,$$

where $C_2 = \langle t \mid t^2 = e\rangle$. Subsequently, Bishop and Elder [BE22] proved that the group

$$v\mathscr{H} = H_3(\mathbb{Z}) \rtimes C_2 = \left\langle a, t \mid t^2 = [a, [a, a^t]] = 1 \right\rangle$$

has polynomial geodesic growth w.r.t. the generating set $S = \{a^{\pm}, t\}$.

In the opposite direction, Bridson *et al.* showed that any group factoring onto $\mathbb{Z}^2$ has exponential geodesic growth w.r.t. every generating set. This applies for finitely generated nilpotent groups which are not virtually cyclic [BBES12, Lemma 13]. We generalize most of these results in the following criterion.

Setup. Let G be a virtually s-step-nilpotent group, with S a finite generating set. Consider H a torsion-free, s-step nilpotent, finite-index, normal subgroup of G.[2]

We consider the isolator of the commutator subgroup

$$I_H([H, H]) = \left\{h \in H \mid \exists n > 0,\, h^n \in [H, H]\right\}.$$

We get out of this data a map $\mathrm{Pr}\colon H \twoheadrightarrow H/I_H([H, H]) \simeq \mathbb{Z}^d$, and an action of the finite group $F = G/H$ on $H/I_H([H, H])$ (by conjugation). We define the multiset[3]

$$A = A(S) = \left\{ \frac{\mathrm{Pr}(\bar{a})}{\ell(a)} \in \mathbb{Q}^d \;\middle|\; a \in S^\star \text{ labels a simple cycle in } \mathcal{S}ch(H\backslash G, S) \right\},$$

where $\mathcal{S}ch(H\backslash G, S)$ denotes the Schreier graph. Finally, we define a polytope $\mathrm{P}(S) = \mathrm{ConvHull}(A(S)^F)$, where A^F denotes the orbit of A under conjugation by F.

Theorem 4.A. *If no two elements of $A(S)$ lie on a common facet of $\mathrm{P}(S)$, then the geodesic growth is subexponential. More precisely, we give the following upper bounds:*

- *If $s \leqslant 2$, the geodesic growth is bounded above by a polynomial.*
- *If $s \geqslant 3$, the geodesic growth is bounded above by*

  $$\gamma_{\mathrm{geod}}^{G,S}(n) \preceq \exp\big(n^{\alpha_s} \log(n)\big),$$

  with $0 < \alpha_s < 1$ an explicit constant (eg. $\alpha_3 = 3/5$).

Otherwise the geodesic growth is exponential.

[2] We can always find such an H: G has a finite-index s-step nilpotent group H', which contains a finite-index torsion-free subgroup H'' [Hir46, Thm 3.23]. Finally H'' contains a finite-index subgroup

$$H = \mathrm{core}(H'') = \bigcap_{g \in G} gH''g^{-1} \trianglelefteq G.$$

[3] $A(S)$ is multiset in the following sense: any point $p \in \mathbb{Q}^d$ appears as many times in $A(S)$ as there are simple cycles $a \in S^*$ such that $p = \mathrm{Pr}(\bar{a})/\ell(a)$.

In particular, we reduce the "virtually 2-step nilpotent" case of Question 1 to the characterization of finite subgroups $F \leqslant \mathrm{GL}_d(\mathbb{Z})$ with a certain property, as follows:

Corollary 4.B. *Let G be a finitely generated, virtually 2-step-nilpotent group. Consider $H \trianglelefteq G$ a torsion-free, 2-step nilpotent, finite-index, normal subgroup of G. This defines an action of $F = G/H$ on $\mathbb{Z}^d \simeq H/I_H([H,H])$. The following assertions are equivalent:*

(i) *There exists a generating set S such that (G,S) has polynomial geodesic growth.*

(ii) *There exists a finite set $A \subset \mathbb{Z}^d$ such that $\mathrm{P} = \mathrm{ConvHull}(A^F)$ is a full-dimensional polytope, and no two elements of A lie on the same facet of P.*

Proof. We have to prove (ii) ⇒ (i). We construct a generating set from $A = \{p_1, \dots, p_m\}$ as follows. Consider S_0 a fixed generating set for G, and define

$$S_n = S_0 \cup \{a_1^n, \dots, a_m^n\} \text{ for all } n \in \mathbb{Z}_{>0},$$

where a_i are elements of H satisfying $\mathrm{Pr}(a_i) = p_i$. Observe that the new generators $a_i^n \in H$ only add loops in the Cayley graph of G/H, and therefore

$$A(S_n) = A(S_0) \cup nA.$$

For n large enough, we have $\mathrm{P}(S_n) = \mathrm{ConvHull}(nA^F)$ and $A(S_0) \cap \partial\mathrm{P}(S_n) = \emptyset$. At this point, the hypothesis on A implies that (G, S_n) has polynomial geodesic growth. □

Remark. The statement of this last result can be adapted if we only allow *symmetric* generating sets S. The only other modification needed is

(ii') There exists a *symmetric* finite set $A \subset \mathbb{Z}^d$ such that $\mathrm{P} = \mathrm{ConvHull}(A^F)$ is a full-dimensional polytope, and no two elements of A lie on the same facet of P.

Note that conditions (ii) and (ii') are not equivalent. An example is given by the group $G_2 = \mathbb{Z}^2 \rtimes C_2$ where $C_2 = \langle r \rangle$ acts by 180° rotations (see also [BBES12, Example 16]). If we only look at *symmetric* sets A, we always have $A^{C_2} = A \cup -A = A$, so that both vertices of any facet of P belong to A. In contrast G_2 satisfies condition (ii):

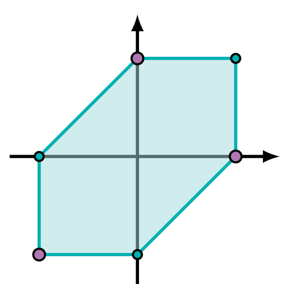


Figure 4.2: $A = \{(1,0),(0,1),(-1,-1)\}$ in purple and $\mathrm{P}(S)$ in green.

This means that the geodesic growth of G_2 is polynomial w.r.t. $S = \{x, y, (xy)^{-1}, r\}$, and exponential w.r.t. any symmetric generating set (as shown in [BBES12]).

Regarding Question 2, we get the following affirmative answer:

Theorem 4.C. *The geodesic growth of the group*

$$v\mathscr{E} = \left\langle a, t \;\middle|\; t^2 = 1;\ [a,[a,a^t]] = [a^t,[a,a^t]] \text{ commutes with } a, a^t \right\rangle$$

with generating set $S = \{a^{\pm 1}, t\}$ *satisfies* $\gamma_{\mathrm{geod}}(n) \asymp \exp\bigl(n^{3/5} \cdot \log(n)\bigr)$.

Note that this group is virtually 3-step nilpotent, more precisely it admits an index-2 subgroup isomorphic to the so-called "Engel group". In some sense, this is the next smallest candidate for intermediate geodesic growth, as virtually abelian have either polynomial or exponential geodesic growth (see [Bis21]), and the same holds true in virtually 2-step nilpotent groups (see Theorem 4.A). The construction relies on the same trick as examples $v\mathscr{Z}$ and $v\mathscr{H}$ of Bridson *and al.* and Bishop-Elder. Our proof re-uses some of their ideas, combined with insights from nilpotent geometry.

As a byproduct, we provide estimates on the volume growth $\beta_{G,S}(n)$ of virtually nilpotent groups G. A celebrated result due to Pansu [Pan83] states that

$$\beta_{G,S}(n) = c_{G,S} \cdot n^d + o(n^d)$$

whenever G is virtually nilpotent. Subsequently the error term was refined whenever G is 2-step nilpotent [Sto98] and more generally s-step nilpotent [BLD13; Gia17]. We extend these results to virtually s-step nilpotent groups:

Theorem 4.D (Corollary 4.1.6)**.** *Let G be a virtually s-step nilpotent group, and S a finite symmetric generating set. The volume growth satisfies*

$$\beta_{G,S}(n) = c_{G,S} \cdot n^d + O(n^{d-\delta_s}),$$

where $\delta_s = 1$ for $s = 1, 2$ and $\delta_s = \frac{1}{s}$ for $s \geqslant 3$.

(The error term coincides with Gianella's error term for nilpotent groups.) Finally, in Corollary 4.3.4, we disprove a conjecture by Breuillard and Le Donne stating that, in any torsion-free nilpotent group H with a symmetric generating set X, we should have

$$\|g\|_X - \|g\|_{\mathrm{Stoll},X} = O_{H,X}(1)$$

[BLD13, Conjecture 6.5]. This conjecture was made in an effort to improve the volume growth estimate to $\beta_{G,S}(n) = cn^d + O(n^{d-1})$.

Organization of the chapter. Section 4.1 gathers some results on word metrics in virtually nilpotent group. We compare word metrics in virtually nilpotent groups and the corresponding nilpotent subgroups, and give some local structure for geodesics. Section 4.2 is devoted to the proof of Theorem 4.A. In Section 4.3, we do a deeper dive into the Engel group. We provide fine lower bounds on word length of some elements, and prove the lower bound on geodesic growth needed for Theorem 4.C. This chapter makes heavy use of Section 2.4.3 from the preliminaries.

4.1 Generalities on virtually nilpotent groups

Let us start with some general observations on virtually nilpotent groups. In this section, G is a group with a finite (non-weighted) generating set S. Let H be a finite-index, torsion-free, s-step nilpotent subgroup. We do not require H to be normal.

4.1.1 A generating set for H

We consider the set X defined in Example 2.6.5, that is,

$$X(S) = \left\{ tat^{-1} \;\middle|\; \begin{array}{c} t \in S^\star \text{ labels a simple path } H \to Ht \\ a \in S^\star \text{ labels a simple cycle } Ht \to Ht \end{array} \right\} \xrightarrow{\text{ev}} H$$

with a weight function $\omega\colon X \to \mathbb{Z}_{>0}$ defined by $\omega(tat^{-1}) = \ell(a)$.

Let us do some reminders and observations:

- We have a *decomposition map*

$$\operatorname{dec}\colon \begin{pmatrix} \{w \in S^\star \mid \bar{w} \in H\} & \longrightarrow & X^\star \\ w & \longmapsto & \tilde{w} \end{pmatrix}$$

  which is injective, and such that $\operatorname{ev}(\tilde{w}) = \operatorname{ev}(w)$ and $\ell_\omega(\tilde{w}) = \ell(w)$. The specific construction of this map will be useful in Sections 4.1.2 and 4.2.

- Since S is a generating set of G (i.e., $\operatorname{ev}(S^\star) = G$), we have $\operatorname{ev}(\mathcal{WP}(H\backslash G, S)) = H$ hence $H = \langle X \rangle$. (This is an easy case of Theorem 2.6.3(d).)
- The equality $\ell_\omega(\tilde{w}) = \ell(w)$ implies that $\|h\|_{X,\omega} \leqslant \|h\|_S$ for all $h \in H$.

4.1.2 Sub-linear control between word metrics

We have $\|h\|_{X,\omega} \leqslant \|h\|_S$ for all $h \in H$. The goal of this paragraph is to prove an inequality in the other direction. In order to state our result, we first define a sequence $(\alpha_s)_{s \geqslant 2} \subset [0,1)$. It starts with $\alpha_2 = 0$ and then

$$\forall s \geqslant 3, \quad \alpha_s = \frac{1 - \frac{1}{s}\alpha_{s-1}}{2 - \alpha_{s-1} - \frac{1}{s}}.$$

An induction shows that $0 \leqslant \alpha_s \leqslant 1 - \frac{1}{s} < 1$. We can now state our main inequality:

Proposition 4.1.1. *Let G be a virtually s-step nilpotent group, and H a finite-index, torsion-free, s-step nilpotent subgroup of G. We consider S a generating set for G and X the associated generating set for H, with the weight function $\omega\colon X \to \mathbb{Z}_{>0}$. Then*

$$\forall h \in H, \quad \|h\|_{X,\omega} \leqslant \|h\|_S \leqslant \|h\|_{X,\omega} + O\big(\|h\|_{X,\omega}^{\alpha_s}\big).$$

We need some preparatory results. First a classical lemma, see eg. [DK18, §14.1.3]

Lemma 4.1.2 (Distortion)**.** *Let H be a finitely generated torsion-free s-step nilpotent group. Consider $\|\cdot\|_E$ an Euclidean norm on $\gamma_s(H) \simeq \mathbb{Z}^c$. Then for $z \in \gamma_s(H)$*

$$\|z\|_{X,\omega} = \Theta\big(\|z\|_E^{1/s}\big) \quad as \;\; \|z\|_E \to \infty.$$

Next we need the following generalization of Lemma 2.4.13:

Lemma 4.1.3 (k versus ℓ_ω)**.** *Any element $h \in H$ with $\|h\|_{X,\omega} = n$ can be represented by a word $v \in X^\star$ with coarse length $k(v) = O(n^{\alpha_s})$ and length $\ell_\omega(v) = n + O(n^{\alpha_s})$.*

Proof. We argue by induction on s. The case $s = 2$ (in which $\alpha_s = 0$) is Lemma 2.4.13.

Suppose the induction hypothesis holds for $s-1 \geqslant 2$. Consider $h \in H$ with $\|h\|_{X,\omega} = n$. Let $u \in X^\star$ be a geodesic word representing h. We can decompose u as a product $u = u_1 u_2 \dots u_m$ with $m = n^\beta + O(1)$ pieces of length $n^{1-\beta} + O(1)$ for $\beta \in (0,1)$ which will be chosen later. By induction hypothesis, there exist words $v_i \in X^\star$ such that

$$\overline{v}_i = \overline{u}_i \bmod \gamma_s(H), \quad k(v_i) = O\big(n^{(1-\beta)\alpha_{s-1}}\big) \;\text{ and }\; \ell_\omega(v_i) = n^{1-\beta} + O\big(n^{(1-\beta)\alpha_{s-1}}\big).$$

Observe that the error $z_i = \bar{u}_i \bar{v}_i^{-1} \in \gamma_s(H)$ has length

$$\|z_i\|_{X,\omega} \leqslant \ell_\omega(u_i) + \ell_\omega(v_i) = O(n^{1-\beta}),$$

hence $\|z_i\|_E = O(n^{(1-\beta)s})$ by Lemma 4.1.2. Therefore, the total error $z = z_1 z_2 \dots z_m$ has size $\|z\|_E = O(n^\beta \cdot n^{(1-\beta)s})$. The same lemma delivers $v_z \in X^\star$ such that $\overline{v}_z = z$ and $\ell_\omega(v_z) = O(n^{1-\frac{s-1}{s}\beta})$. Finally, let $v = v_1 v_2 \dots v_m v_z \in X^\star$. We have $\overline{v} = h$, and

$$k(v) \leqslant \sum_{i=1}^m k(v_i) + k(v_z) = n^\beta \cdot O(n^{(1-\beta)\alpha_{s-1}}) + O(n^{1-\frac{s-1}{s}\beta}),$$
$$\ell_\omega(v) = \sum_{i=1}^m \ell_\omega(v_i) + \ell_\omega(v_z) = n + n^\beta \cdot O(n^{(1-\beta)\alpha_{s-1}}) + O(n^{1-\frac{s-1}{s}\beta}).$$

To conclude, we fine-tune $\beta = \frac{1-\alpha_{s-1}}{2-\alpha_{s-1}-\frac{1}{s}}$ so that $\beta + (1-\beta)\alpha_{s-1} = 1 - \frac{s-1}{s}\beta = \alpha_s$. □

Remark 4.1.4. The first exponents $\alpha_2 = 0$ and $\alpha_3 = \frac{3}{5}$ are optimal (Corollary 4.3.4). Later α_s can probably be improved. For instance, Gianella has proved a result analogous to Lemma 4.1.3 with $\alpha_s = \frac{s}{s+2}$ if we allow v to be an $\mathbb{R}$-word. [Gia17, Lemma 40]

Proof of Proposition 4.1.1. Let us prove the right inequality. Using the Lemma 4.1.3, any element $h \in H$ of length $\|h\|_{X,\omega} = n$ can be represented by a word

$$v = (t_1 a_1 t_1^{-1})^{m_1} \dots (t_k a_k t_k^{-1})^{m_k} \in X^\star$$

with $k(v) = O(n^{\alpha_s})$ and $\ell_\omega(v) = n + O(n^{\alpha_s})$. We convert this word into

$$w = t_1\, a_1^{m_1}\, u_1 \dots t_k\, a_k^{m_k}\, u_k \in S^\star,$$

where $u_i \in S^\star$ is a geodesic representative for t_i^{-1}. This word has length

$$\ell(w) = \ell_\omega(v) + \sum_{i=1}^{k} \big(\ell(t_i) + \ell(u_i)\big) \leqslant \ell_\omega(v) + k(v)\big([G:H] + \max_t \|t^{-1}\|_S\big) = n + O(n^{\alpha_s}),$$

so that $\|h\|_S \leqslant \|h\|_{X,\omega} + O\big(\|h\|_{X,\omega}^{\alpha_s}\big)$ as announced. □

4.1.3 A parte: Volume growth of virtually nilpotent groups

Let us recall the state-of-the-art for growth of finitely generated nilpotent groups:

Theorem 4.1.5 ([Sto98; Gia17])**.** *Let H be a finitely generated s-step nilpotent group, and X be a symmetric generating set. We have*

$$\beta_{H,X}(n) = c_{H,X} \cdot n^d + O(n^{d-\delta_s}),$$

where

- $d = d(H) = \sum_{i=1}^{s} i \cdot \operatorname{rank}_{\mathbb{Q}}\big(\gamma_i(H)/\gamma_{i+1}(H)\big) \in \mathbb{Z}_{\geqslant 0}$ *is the Bass-Guivarc'h exponent,*
- $c_{H,X} \in \mathbb{R}_{>0}$ *is the volume of the unit ball in the asymptotic cone of (H, X) with its associated Carnot-Caratheodory metric.*
- $\delta_s = 1$ *for* $s = 1, 2$ *and* $\delta_s = \frac{1}{s}$ *for* $s \geqslant 3$.

The statement extends if we add a weight function $\omega\colon X \to \mathbb{Z}_{>0}$ into the picture. Our modest contribution is to extend this result to *virtually* nilpotent groups

Corollary 4.1.6. *Let G be a finitely generated, virtually s-step nilpotent group, and S be a symmetric generating set. Let H be a finite-index, torsion-free, s-step nilpotent group, with the associated generating set X and weight function $\omega\colon X \to \mathbb{Z}_{>0}$. We have*

$$\beta_{G,S}(n) = [G:H] \cdot c_{H,X,\omega} \cdot n^d + O(n^{d-\delta_s}).$$

Proof. Let $j = [G:H]$, and decompose $G = \bigsqcup_{i=1}^{j} t_i H$. Picking t_i as short as possible, we may assume that $\|t_i\|_S < j$ for all i. We have

$$\bigsqcup_{i=1}^{j} t_i \cdot \big(B_{G,S}(n-j) \cap H\big) \subseteq B_{G,S}(n) \subseteq \bigsqcup_{i=1}^{j} t_i \cdot \big(B_{G,S}(n+j) \cap H\big)$$

Combining this with Proposition 4.1.1, we get the inclusions

$$\bigsqcup_{i=1}^{j} t_i \cdot B_{H,X,\omega}\big(n - j - O(n^{\alpha_s})\big) \subseteq B_{G,S}(n) \subseteq \bigsqcup_{i=1}^{j} t_i \cdot B_{(H,X,\omega)}(n+j)$$

hence, by Theorem 4.1.5,

$$j \cdot c_{H,X,\omega} \cdot \big(n - j - O(n^{\alpha_s})\big)^d + O(n^{d-\delta_s}) \leqslant \beta_{G,S}(n) \leqslant j \cdot c_{H,X,\omega} \cdot (n+j)^d + O(n^{d-\delta_s}),$$

that is, $\beta_{G,S}(n) = [G:H] \cdot c_{H,X,\omega} \cdot n^d + O(n^{d-\delta_s})$ as $\alpha_s \leqslant 1 - \delta_s < 1$ for all s. □

4.1.4 Structure of almost-geodesics

We give local conditions on almost geodesic $v \in X^\star$. Essentially, most short subwords of v should be geodesics in the abelianization $(\mathrm{Pr}(\bar{H}), \|\cdot\|_{\mathrm{Mink}})$. More precisely

Definition 4.1.7. Let H be a torsion-free s-step nilpotent group, X a finite generating set, and $\omega\colon X \to \mathbb{Z}_{>0}$ a weight function. Consider $\mathrm{Pr}\colon H \twoheadrightarrow H/I_H([H,H]) \simeq \mathbb{Z}^d$ and

$$\mathrm{P} = \mathrm{ConvHull}\left\{ \frac{\mathrm{Pr}(x)}{\omega(x)} \;\middle|\; x \in X \right\} \subseteq \mathbb{R}^d.$$

A word $u \in X \cup X^2$ is called *costly* if

- $u = x$ with $\frac{\mathrm{Pr}(x)}{\omega(x)}$ not on the boundary of P, or
- $u = xy$ with $\frac{\mathrm{Pr}(x)}{\omega(x)}, \frac{\mathrm{Pr}(y)}{\omega(y)}$ not on a common facet of P.

For $v \in X^\star$, we define $N(v)$ as the number of occurrences of costly subwords in v.

Proposition 4.1.8. *There exists a constant $\delta = \delta(H, X, \omega) > 0$ such that*

$$\forall v \in X^\star, \quad \ell_\omega(v) \geqslant \|\overline{v}\|_{X,\omega} + \delta \cdot N(v) - O\big(\|\overline{v}\|_{X,\omega}^{\alpha_s}\big).$$

In some sense, this result is a quantified, discrete analog to the "$(s-1)$-iterated blowup" result of Hakavuori and Le Donne. [HLD23, Corollary 1.4]

Proof. By Lemma 2.4.9, for each costly word u, there exists $u' \in X_{\mathbb{R}}^\star$ such that $\mathrm{Pr}(u) = \mathrm{Pr}(u')$ and that $\ell_\omega(u) - \ell_\omega(u') > 0$. We fix a real number δ such that

$$0 < \delta < \frac{1}{2} \min \big\{ \ell_\omega(u) - \ell_\omega(u') \mid u \text{ is costly} \big\}.$$

We now argue by induction on s.

▶ We first initialize for $s = 2$: Consider a word $v \in X^\star$ with $N = N(v)$ occurrences of costly subwords. Say $M \geqslant \frac{1}{2}N$ of these occurrences are disjoints, we denote them by $u_1, \ldots, u_M$. We replace u_i in v by u_i', thus defining a new $\mathbb{R}$-word v'.

Observe that v' has the same abelianization $\mathrm{Pr}(\overline{w}') = \mathrm{Pr}(\overline{w})$. It only differs in areas:

$$z(\overline{v}') - z(\overline{v}) = \sum_{i=1}^{M} \big(z(\overline{u}_i') - z(\overline{u}_i) \big) \in [\bar{H}, \bar{H}] \simeq \mathbb{R}^c$$

(using Proposition 2.4.5). Recall that $[\bar{H}, \bar{H}]$ is quadratically distorted in $\bar{H}$ (Lemma 4.1.2 but for Lie groups, see rather [Gui73, Lemme II.1] or [Bre14, Theorem 2.7]), hence

$$\|z(\overline{v}') - z(\overline{v})\|_{\mathrm{Stoll},X,\omega} = O\big(M^{\frac{1}{2}}\big).$$

Therefore, there exists an $\mathbb{R}$-word v_z with $\ell_\omega(v_z) = O\big(M^{\frac{1}{2}}\big)$ and $\overline{v} = \overline{v'v_z}$. It follows

$$\begin{aligned}
\ell_\omega(v) = \ell_\omega(v'v_z) \quad &+ \sum_{i=1}^{M} \big(\ell_\omega(u_i) - \ell_\omega(u_i')\big) \quad &- \ \ell_\omega(v_z) \\
\geqslant \|\overline{v}\|_{\mathrm{Stoll},X,\omega} \quad &+ M \cdot \min\big\{\ell_\omega(u) - \ell_\omega(u') \mid u \text{ is costly}\big\} \quad &- O\big(M^{\frac{1}{2}}\big) \\
\geqslant \|\overline{v}\|_{X,\omega} - C &+ \frac{1}{2} N \cdot \min\big\{\ell_\omega(u) - \ell_\omega(u') \mid u \text{ is costly}\big\} &- O\big(N^{\frac{1}{2}}\big) \\
\geqslant \|\overline{v}\|_{X,\omega} \quad &+ \delta \cdot N(v) \ - \ O(1),
\end{aligned}$$

where we have used Proposition 2.4.12 for the second inequality.

► Suppose that the induction hypothesis holds for $s-1 \geqslant 2$. Consider $v \in X^\star$ of length $\ell_\omega(v) = n$. We decompose v as a product $v = v_1 v_2 \dots v_m$ with $m = n^\beta + O(1)$ pieces of length $n^{1-\beta} + O(1)$. Using the hypothesis, there exists $u_i \in X^\star$ such that

$$\overline{u}_i = \overline{v}_i \mod \gamma_s(H) \quad \text{and} \quad \ell_\omega(u_i) \leqslant \ell_\omega(v_i) - \delta \cdot N(v_i) + O\big(n^{(1-\beta)\alpha_{s-1}}\big).$$

We repeat part of the proof of Lemma 4.1.3: the error $z_i = \bar{v}_i \bar{u}_i^{-1} \in \gamma_s(H)$ has length

$$\|z_i\|_{X,\omega} \leqslant \ell_\omega(u_i) + \ell_\omega(v_i) = O(n^{1-\beta})$$

The same argument using distortion gives a word $u_z \in X^\star$ such that $\overline{u}_z = z_1 z_2 \dots z_m$ and $\ell_\omega(u_z) = O(n^{1-\frac{s-1}{s}\beta})$. Finally, let $u = u_1 u_2 \dots u_m u_z \in X^\star$. We have $\overline{u} = \overline{v}$, and

$$\begin{aligned}
\|\overline{v}\|_{X,\omega} \leqslant \ell_\omega(u) &= \sum_{i=1}^{m} \ell_\omega(u_i) + \ell_\omega(u_z) \\
&\leqslant \sum_{i=1}^{m} \Big(\ell_\omega(v_i) - \delta N(v_i) + O\big(n^{(1-\beta)\alpha_{s-1}}\big)\Big) + O(n^{1-\frac{s-1}{s}\beta}) \\
&\leqslant \ell_\omega(v) - \delta\big(N(v) - m\big) + O\big(n^{\beta+(1-\beta)\alpha_{s-1}}\big) + O(n^{1-\frac{s-1}{s}\beta}).
\end{aligned}$$

Fine-tuning $\beta = \frac{1-\alpha_{s-1}}{2-\alpha_{s-1}-\frac{1}{s}}$ gives us the desired result. □

4.2 Criterion for sub-exponential geodesic growth

From now on, we suppose H is a torsion-free, s-step nilpotent, finite index, *normal* subgroup of G. As in Section 4.1.1, we consider the labeled graph $\mathcal{S}ch(H\backslash G, S)$ and

$$X(S) = \left\{ tat^{-1} \;\middle|\; \begin{array}{c} t \in S^\star \text{ labels a simple path } H \ \to \ Ht \\ a \in S^\star \text{ labels a simple cycle } Ht \to Ht \end{array} \right\} \xrightarrow{\text{ev}} H$$

with a weight function $\omega\colon X \to \mathbb{Z}_{>0}$ defined by $\omega(tat^{-1}) = \ell(a)$.

Remark. Observe that $\mathcal{S}ch(H\backslash G, S)$ is transitive since H is normal, hence a word $a \in S^\star$ labels a simple cycle $Ht \to Ht$ if and only if it labels a simple cycle $H \to H$. We can therefore say a word $a \in S^\star$ is a *simple cycle* without ambiguity.

The proof naturally splits into two cases.

▶ First, we suppose that two elements of $A(S)$ lie on the same facet of $\mathrm{P}(S)$, and we conclude (G,S) has exponential geodesic growth. We already provide exponentially many geodesics in the virtually abelian quotient $G/I_H([H,H])$. Observe that

$$\mathrm{P}(S) \stackrel{\text{def}}{=} \operatorname{ConvHull}\left(\left\{\frac{\Pr(\bar a)}{\ell(a)} \;\middle|\; a\in S^\star \text{ simple cycle}\right\}^{G/H}\right) = \operatorname{ConvHull}\left\{\frac{\Pr(x)}{\omega(x)} \;\middle|\; x\in X\right\}.$$

is the polytope that governs the Minkowski norm on $\hat H = H/I_G([H,H]) \simeq \mathbb{Z}^d$ w.r.t. the natural generating multiset $\hat X = \Pr(X)$, with the weight function $\hat\omega(\hat x) = \omega(x)$.

Consider distinct simple cycles $a, b \in S^\star$ (in particular $\bar a, \bar b \in X$) such that

$$\frac{\Pr(\bar a)}{\ell(a)} = \frac{\Pr(\bar a)}{\omega(a)} \quad\text{and}\quad \frac{\Pr(\bar b)}{\ell(b)} = \frac{\Pr(\bar b)}{\omega(b)}$$

lie on a common facet of $\mathrm{P}(S)$. Then, for all $w \in \{a,b\}^\star$, we have

$$\ell(w) = \ell_{\hat\omega}(\hat w) = \|\Pr(\bar w)\|_{\text{Mink,P}} \leqslant \|\Pr(\bar w)\|_{\hat X,\hat\omega} \leqslant \|\bar w\|_{X,\omega} \leqslant \|\bar w\|_S\,.$$

(The second equality follows from Lemma 2.4.9, and the last inequality from the easy part of Proposition 4.1.1.) All these words are geodesics, which concludes. ◀

From now on, we work towards an upper bound. First, we work specifically on geodesics $w \in S^\star$ evaluating in the subgroup H.

▶ Under the hypothesis that no two elements of $A(S)$ lie on the same facet of $\mathrm{P}(S)$, we prove that the coarse length of the decomposition $\tilde w \in X^\star$ (defined in §4.1.1) satisfies

$$k(\tilde w) \leqslant N(\tilde w)\cdot[G:H] + 1 \tag{4.2.1}$$

for any word $w \in S^\star$ evaluating in H. Let us write

$$\tilde w = x_1^{m_1}\, x_2^{m_2}\cdot\ldots\cdot x_k^{m_k} \quad \text{with } x_i \neq x_{i+1} \text{ for all } i.$$

The decomposition process not only gives this sequence of generators $(x_i) \in X$, but also two sequences of simple paths $(t_i) \in S^\star$ and simple cycles $(a_i) \in S^\star$ such that $x_i = \bar t_i \bar a_i \bar t_i^{-1}$. Observe that, for each i, one of t_i and t_{i+1} is a prefix of the other (depending on where the walk re-intersects itself). Consider a time i such that t_{i+1} is a prefix of t_i (including the case $t_{i+1} = t_i$). In particular, we can rewrite $x_{i+1} = \bar t_i \bar b_{i+1} \bar t_i^{-1}$ for some simple cycle $b_{i+1} \in S^\star$. (b_{i+1} is a cyclic permutation of a_{i+1}.)

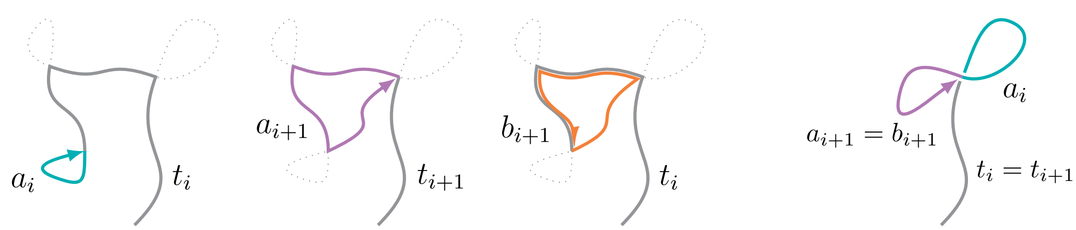


Figure 4.3: $x_i = t_i\, a_i\, t_i^{-1}$ and $x_{i+1} = t_{i+1}\, a_{i+1}\, t_{i+1}^{-1} = t_i\, b_{i+1}\, t_i^{-1}$. The limit case $t_i = t_{i+1}$.

As $x_i \neq x_{i+1}$, we have $a_i \neq b_{i+1}$. Now our hypothesis on $A(S)$ kicks in: both points

$$\frac{\Pr(\bar{a}_i)}{\ell(a_i)} \quad \text{and} \quad \frac{\Pr(\bar{b}_{i+1})}{\ell(b_{i+1})}$$

cannot lie on a common facet of P. We get the same conclusion for

$$\frac{\Pr(x_i)}{\omega(x_i)} \quad \text{and} \quad \frac{\Pr(x_{i+1})}{\omega(x_{i+1})}$$

after conjugation by t_i. We conclude that the subword $x_i x_{i+1}$ is *costly* in the sense of Proposition 4.1.8 (hence counted in $N(\tilde{w})$), as soon as t_{i+1} is a prefix of t_i.

In order to avoid this situation, the path t_i should be a proper prefix of t_{i+1}. This means the t_i's usually get longer and longer, but they have lengths bounded between 0 and $[G:H]-1$. Combining these two observations, we get the desired bound.

▶ We are now able to combine everything. Consider a word $w \in S^\star$ representing an element $h \in H$. Propositions 4.1.1 and 4.1.8 give us

$$\begin{aligned} \|h\|_S &\leqslant \|h\|_{X,\omega} + O\big(\|h\|_{X,\omega}^{\alpha_s}\big) \\ \ell(w) = \ell_\omega(\tilde{w}) &\geqslant \|h\|_{X,\omega} + \delta \cdot N(\tilde{w}) - O\big(\|h\|_{X,\omega}^{\alpha_s}\big) \end{aligned}$$

for some constant $\delta > 0$. If w is a geodesic, then $\ell(w) = \|h\|_S$ and therefore

$$\delta \cdot N(\tilde{w}) = O\big(\|h\|_{X,\omega}^{\alpha_s}\big) = O\big(\ell(\tilde{w})^{\alpha_s}\big).$$

Combined with the inequality (4.2.1), we get a constant $C = C(G,S) > 0$ such that

$$k(\tilde{w}) \leqslant C \cdot \ell(\tilde{w})^{\alpha_s}. \tag{4.2.2}$$

▶ We obtain the desired upper bound. The inequality (4.2.2) gives an injection

$$\big\{w \in S^\star \;\big|\; w \text{ is geodesic with } \overline{w} \in H\big\} \hookrightarrow \big\{v \in X^\star \;\big|\; k(v) \leqslant C \cdot \ell(v)^{\alpha_s}\big\}$$

sending $w \mapsto \tilde{w}$. Observe that $\ell(\tilde{w}) \leqslant \ell_\omega(\tilde{w}) = \ell(w)$, hence

$$\#\big\{w \in S^\star \,\big|\, w \text{ is geodesic, } \ell(w) \leqslant n,\ \overline{w} \in H\big\} \leqslant \#\big\{v \in X^\star \,\big|\, k(v) \leqslant C \cdot n^{\alpha_s},\ \ell(v) \leqslant n\big\}.$$

Note that, in to order to construct a word v satisfying these conditions, it suffices to pick Cn^{α_s} cut points along an interval of length n, pick a letter to fill each of the first Cn^{α_s} "blocks", and leave the last block empty.

$$\big|\, x_3 \;\; x_3 \;\; x_3 \,\big\|\, x_1 \;\; x_1 \;\; x_1 \,\big|\, x_1 \,\big|\, x_4 \;\; x_4 \;\; x_4 \,\big|\, \sqcup \;\; \sqcup$$

This translates into a crude upper bound of

$$\#\big\{v \in X^\star \,\big|\, k(v) \leqslant C \cdot n^{\alpha_s},\ \ell(v) \leqslant n\big\} \leqslant \big((n+1)\,|X|\big)^{Cn^{\alpha_s}} \asymp \exp\big(Cn^{\alpha_s}\log(n)\big).$$

To conclude, each geodesic w in (G,S) can be decomposed as a product

$$w = w_1 s_1 w_2 s_2 \ldots s_{k-1} w_k,$$

where each $w_i \in S^\star$ is a geodesic ending up in H, $s_i \in S$, and $k \leqslant [G:H]$, which gives an upper bound on $\gamma_{\text{geod}}(n)$ of the same type. □

4.3 Geometry of the Engel group

As our main example, we consider a lattice inside the group $\bar{\mathscr{E}}$ defined in Section 3.1. More specifically, we take the lattice generated by the straight segments a and b from $\mathbf{0}$ to $(1,1)$ and $(1,-1)$ respectively. This group is given by the presentation

$$\mathscr{E} = \left\langle a, b \;\middle|\; [a,[a,b]] = [b^{-1},[a,b]] \text{ is central} \right\rangle.$$

Throughout, we fix $X = \{a^{\pm}, b^{\pm}\}$ as a generating set.

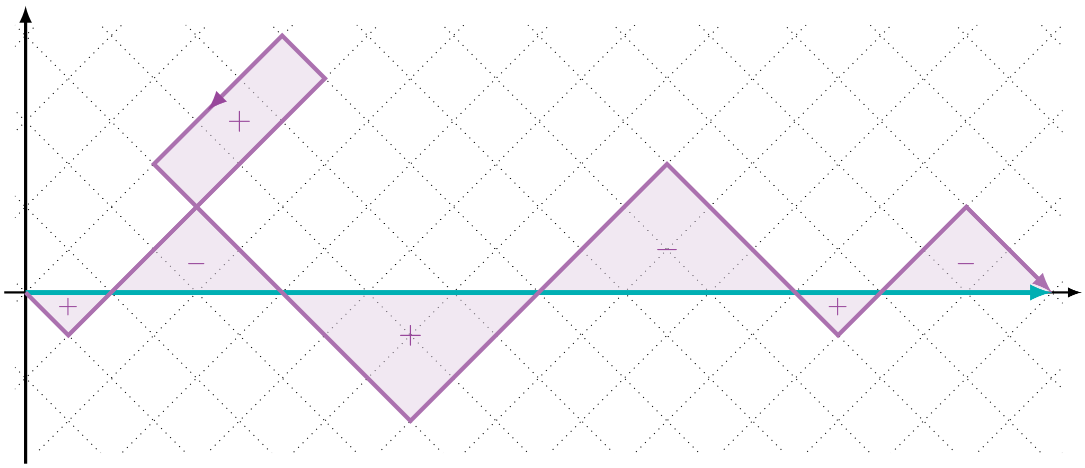


Figure 4.4: A word over X and an equivalent path. Both satisfy $\hat{g} = (24,0)$ and $A(g) = B_y(g) = 0$.

4.3.1 An observation of Stoll and lower bound on word length

In unpublished notes [Sto10], Stoll shows that a key result (Corollary 2.4.11 above) fails in the Engel group[4] $\bar{\mathscr{E}}$. Specifically, he proves that

Proposition ([Sto10]). *The element g_1 represented by a segment from $\mathbf{0}$ to $(1,0)$ has Stoll length $\|g_1\|_{\mathrm{Stoll},X} = 1$, but is not represented by any $\mathbb{R}$-word of length 1.*

By a compactness argument, this proves that Lemma 2.4.13 does not extend either:

Corollary. *Consider $g_n \in \overline{\mathscr{E}}$ the element represented by a segment to $(n,0)$. There does not exist a sequence of $\mathbb{R}$-words w_n representing g_n such that the coarse lengths $k(w_n)$ are uniformly bounded and $\ell(w_n) \leqslant \|g_n\|_{\mathrm{Stoll},X} + o(n) = n + o(n)$.*

Proof of the Corollary. We argue by contradiction, and suppose such a sequence of words does exist. After rescaling by a factor of $\frac{1}{n}$, we get a sequence of $\mathbb{R}$-words of length $1+o(1)$ and coarse length $\leqslant K$, all representing g_1. However, $\mathbb{R}$-words with length $\leqslant 2$ and coarse length $\leqslant K$ form a compact set, so some subsequence converges to an $\mathbb{R}$-word representing g_1 of length 1 and coarse length $\leqslant K$, a contradiction. □

[4]Stoll works with the Cartan group, but his argument works just as well for the Engel group.

In what follows, we quantify the dependence between $k(w_n)$ and $\ell(w_n) - \|g_n\|_{\mathrm{Stoll}}$. We use that the horizontal path is a (particularly bad) abnormal curve in $\overline{\mathscr{E}}$.

Proposition 4.3.1 (B_y back from the depths)**.** *There exists $C > 0$ such that, for any $w \in X_{\mathbb{R}}^\star$ with endpoint $\hat{w} = (n, 0)$ and length $\ell(w) = n + \Delta$, we have*

$$-B_y(\bar{w}) \geqslant \frac{1}{24} \frac{n^3}{\left(k(w) - 1\right)^2} - C \cdot \Delta^2 \cdot \max\left\{\Delta;\ \frac{n}{k(w) - 1}\right\}.$$

Proof. We decompose the path w into a main path (in purple), and some loops and boundary mess (in orange), and estimate the contribution of each part to $-B_y(\bar{w})$.

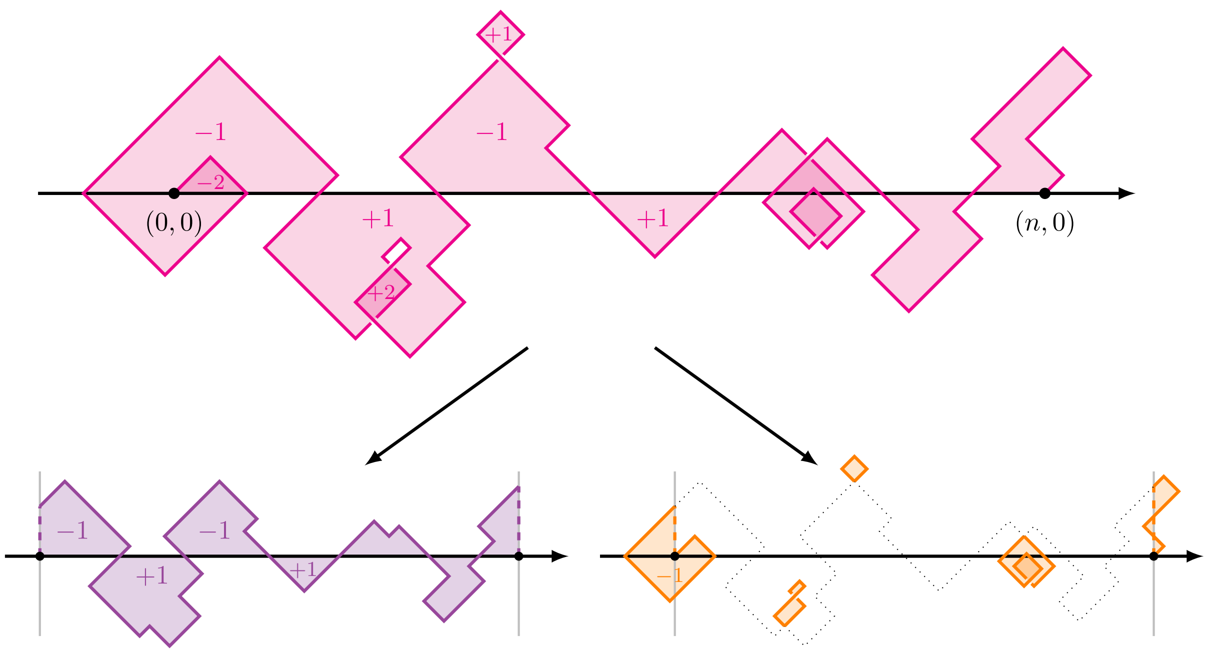


Figure 4.5: The decomposition of a word w. The purple curve is obtained from the interval between the last crossing of the line $x = 0$ and the first crossing of $x = n$, after loop-erasure.

By "contribution", we mean that the winding number distribution of w splits as the sum of the winding numbers of the purple curve and the orange curves (with the added dashed segments). Integrating against the y-coordinate gives two contributions to $B_y(\overline{w})$, which we denote B_y(purple curve) and B_y(orange curves).

► We first estimate the contribution of the purple main path. Note that this path is simple (and we cured out boundary mess), so the winding number of any point is ± 1 if the point lies between the x-axis and the curve - more precisely $+1$ if it lies below the x-axis and -1 if it lies above - and 0 otherwise. In particular, the contribution is non-positive, and bounded by the contribution of the following green area:

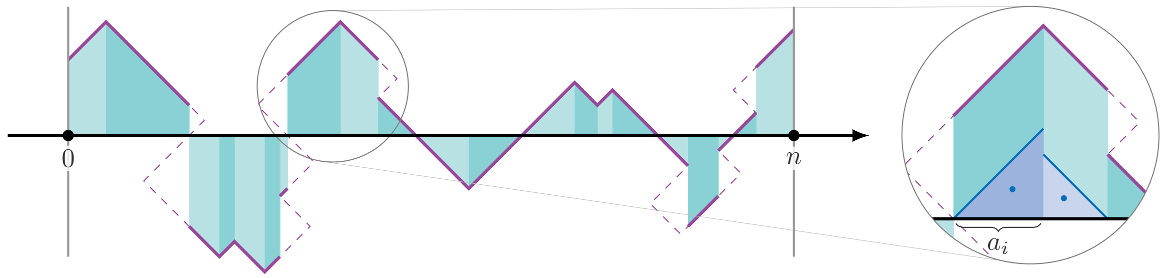


Figure 4.6: The green area decomposed into k' trapezoids, and the blue triangles included inside.

The green area is composed of k' trapezoids/triangles, with $k' \leqslant 2k(w) - 2$ (each segment delimits at most 2 trapezoids if it crosses the x-axis, at most 1 otherwise, and the first and last segments cannot cross the axis). In turn, we bound the contribution of each slice by that of a triangle (with basis and height a_i) included inside it:

$$-B_y(\text{purple curve}) \geqslant -B_y(\text{green zone}) \geqslant -B_y(\text{blue triangles}) = \sum_{i=1}^{k'} \frac{a_i^2}{2} \cdot \frac{a_i}{3}. \quad (4.3.2)$$

Finally, since $\sum_{i=1}^{k'} a_i = n$, the generalized mean inequality gives

$$-B_y(\text{purple curve}) \geqslant \frac{1}{6}\frac{n^3}{k'^2} \geqslant \frac{1}{24}\frac{n^3}{(k(w)-1)^2}.$$

► To control the contribution of orange curves, we need to control both their total area and the y-coordinates. Observe that the purple curve (without the dashed segments) has length at least n as it joins the lines $x = 0$ and $x = n$. It follows that the orange curves have total length at most Δ, hence the two dashed segments too. Therefore

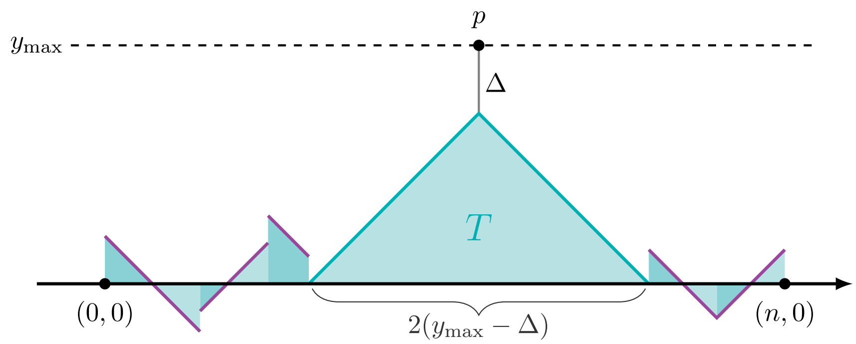
$$\begin{aligned} B_y(\text{orange curves}) &\leqslant \iint_{\mathbb{R}^2} |y| \cdot |W_{\text{orange curves}}(x,y)| \cdot \mathrm{d}x\,\mathrm{d}y \\ &\leqslant y_{\max} \cdot \iint_{\mathbb{R}^2} |W_{\text{orange curves}}(x,y)| \cdot \mathrm{d}x\,\mathrm{d}y \\ &\leqslant y_{\max} \cdot I \cdot (2\Delta)^2, \end{aligned} \quad (4.3.3)$$

where $y_{\max}$ is the largest distance from points of the path w to the x-axis, and I is the isoperimetric constant of $(\mathbb{R}^2, \|\cdot\|_{\text{Mink},X})$ (here $I = \frac{1}{8}$). If $y_{\max}$ is reasonably small, say $y_{\max} \leqslant L \max\left\{\Delta;\ \frac{n}{k(w)-1}\right\}$ for $L = 10\max\{\sqrt{I}, 1\}$, then

$$-B_y(\bar{w}) \geqslant -B_y(\text{purple}) - B_y(\text{orange}) \geqslant \frac{1}{24}\frac{n^3}{(k(w)-1)^2} - 4LI \cdot \Delta^2 \cdot \max\left\{\Delta;\ \frac{n}{k(w)-1}\right\}.$$

► The only remaining case is when $y_{\max}$ is unreasonably large: $y_{\max} \geqslant L\max\left\{\Delta;\ \frac{n}{k(w)-1}\right\}$. In particular, $y_{\max} - \Delta$ is larger than $(1-\frac{1}{L})y_{\max}$, $(L-1)\Delta$ and $(L-1)\frac{n}{k(w)-1}$.

We improve our bound on $-B_y(\text{green zone})$ using that the curve w goes through some point $p = (x_p, \pm y_{\max})$ far away from the x-axis:



In this case, there exists a large triangle T which the $\mathbb{R}$-word w cannot enter. Indeed, for any point q in the interior of T, we have

$$d_{\text{Mink}}\big((0,0),q\big)+d_{\text{Mink}}(q,p)>d_{\text{Mink}}\big((0,0),p\big)+\Delta$$
$$d_{\text{Mink}}(p,q)+d_{\text{Mink}}\big(q,(n,0)\big)>d_{\text{Mink}}\big(p,(n,0)\big)+\Delta$$

so any $\mathbb{R}$-word passing through the four points $(0,0)$, p, q and $\hat{w}=(n,0)$ would have length $>n+\Delta$. This implies that the triangle T must be included inside the green region. Moreover, as previously, we have a lower bound on $-B_y$ for the green regions on both sides of T, composed of at most $2k(w)-2$ trapezoids.

Combining equations (4.3.2), (4.3.3) and the existence of T, we get

$$\begin{aligned}
-B_y(\bar{w}) &\geqslant -B_y(\text{green area})-y_{\max}\cdot 4I\Delta^2\\
&\geqslant \frac{1}{24}\frac{\big(n-2(y_{\max}-\Delta)\big)^3}{(k(w)-1)^2}+\frac{1}{3}(y_{\max}-\Delta)^3-y_{\max}\cdot 4I\Delta^2\\
&\geqslant \frac{1}{24}\frac{n^3}{(k-1)^2}-\frac{1}{4}(y-\Delta)\frac{n^2}{(k-1)^2}+\frac{1}{3}(y-\Delta)^3-4I\cdot y\Delta^2\\
&\geqslant \frac{1}{24}\frac{n^3}{(k-1)^2}+(y-\Delta)^3\left(\frac{1}{3}-\frac{1}{4(L-1)^2}-\frac{4IL}{(L-1)^3}\right)\\
&\geqslant \frac{1}{24}\frac{n^3}{(k(w)-1)^2}.
\end{aligned}$$

(In the third step, we use $(n-h)^3\geqslant n^3-3n^2h$ for all $h\leqslant 3n$ and $y_{\max}\leqslant\frac{1}{2}(n+\Delta)$.) □

We deduce the following quantified version of the main result of [Sto10]:

Corollary 4.3.4. *There exists $C'>0$ s.t., for all $w\in X_{\mathbb{R}}^{\star}$ representing g_n, we have*

$$\ell(w)-\|g_n\|_{\text{Stoll},X}\geqslant C'\cdot k(w)^{-\frac{2}{3}}\cdot\|g_n\|_{\text{Stoll},X}\,.$$

If moreover $n\equiv 0\pmod 6$, then $g_n\in\mathscr{E}$ and $\|g_n\|_X-\|g_n\|_{\text{Stoll},X}=\Omega\left(\|g_n\|_{\text{Stoll},X}^{\frac{1}{3}}\right)$.

Proof. Observe that $\hat{w} = \hat{g}_n = (n, 0)$ and $B_y(\bar{w}) = B_y(g_n) = 0$, therefore

$$\begin{aligned} 0 &\geqslant \frac{1}{24}\frac{n^3}{(k(w)-1)^2} - C\Delta^2 \max\left\{\Delta;\ \frac{n}{k(w)-1}\right\} \\ \iff \max\left\{\Delta^3;\ \frac{n\Delta^2}{k(w)-1}\right\} &\geqslant \frac{1}{24C}\frac{n^3}{(k(w)-1)^2} \\ \iff \Delta &\geqslant \min\left\{\sqrt[3]{\frac{1}{24C}\frac{n^3}{(k(w)-1)^2}};\ \sqrt{\frac{1}{24C}\frac{n^2}{k(w)-1}}\right\}. \end{aligned}$$

Since $\|g_n\|_{\mathrm{Stoll},X} = n$, this implies that

$$\ell(w) - \|g_n\|_{\mathrm{Stoll},X} \geqslant \sqrt{\frac{1}{24C}} \cdot k(w)^{-\frac{2}{3}} \cdot \|g_n\|_{\mathrm{Stoll},X}\,.$$

Finally, as $k(w) \leqslant \ell(w)$ for all $w \in X^\star$, we have $\|g_n\|_X - \|g_n\|_{\mathrm{Stoll},X} \geqslant C' \cdot \|g_n\|_{\mathrm{Stoll},X}^{\frac{1}{3}}$. □

The first part of Corollary 4.3.4 matches the best known upper bound:

Lemma ([Gia17, Lemma 40]). *Let $\bar{H}$ be a simply connected s-step nilpotent Lie group, and X a finite Lie generating set. There exists $C'' > 0$ such that, for every $K \gg 1$ and every $g \in \bar{H}$, there exists an $\mathbb{R}$-word $w \in X_{\mathbb{R}}^\star$ representing g such that*

$$k(w) \leqslant K \quad \text{and} \quad \ell(w) - \|g\|_{\mathrm{Stoll},X} \leqslant C'' \cdot K^{-\frac{2}{s}} \cdot \|g\|_{\mathrm{Stoll},X} + C''.$$

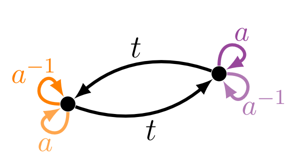
Whether this bound is sharp in s-step nilpotent groups with $s \geqslant 4$ remains open. The second part of Corollary 4.3.4 disproves a conjecture of Breuillard and Le Donne [BLD13, Conjecture 6.5]. (The conjecture states that the difference should be an $O(1)$.)

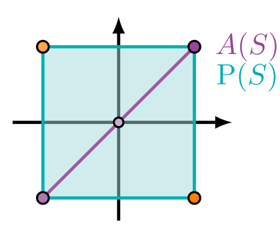
4.3.2 Matching lower bound in a virtually Engel group

In this paragraph, we prove Theorem 4.C, considering the semi-direct product

$$v\mathscr{E} = \mathscr{E} \rtimes C_2 = \left\langle a, t \;\middle|\; t^2 = 1;\ [a,[a,a^t]] = [a^t,[a,a^t]] \text{ commutes with } a, a^t \right\rangle$$

(so $C_2 = \langle t \rangle$ acts by symmetry along the y-axis, and in particular $tat = b^{-1}$), with the generating set $S = \{a^\pm, t\}$. First, we may compute $A(S)$ and $\mathrm{P}(S)$, so that Theorem 1 gives the upper bound $\gamma_{\mathrm{geod}}(n) \preceq \exp\big(n^{3/5} \cdot \log(n)\big)$.





It remains to prove a matching lower bound, i.e., to construct a lot of geodesics. We fix $\kappa > 0$ a small constant (to be determined) and $0 < \varepsilon < \frac{1}{10}$. For any n even integer, let $K \approx \kappa n^{3/5}$ be another even integer, and $m = \frac{n}{2K}$. We show that words of the form

$$w = a^{m_1} t a^{-(m_1+m_2)} t a^{m_2+m_3} t \ldots t a^{-(m_{K-1}+m_K)} t a^{m_K} \in S^\star$$

with $2\sum_i m_i = n$ and $|m_i - m| \leqslant n^\varepsilon$, are all geodesics for n large enough.

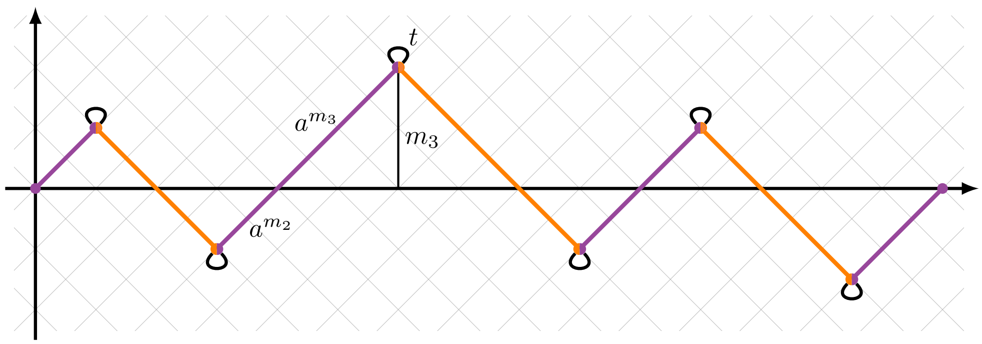


Figure 4.7: The path in $\mathscr{E}$ corresponding to a geodesic in $\upsilon\mathscr{E}$.

► First, we compute B_y for the corresponding element $\overline{w}$. Note that $\overline{w} \in \mathscr{E}$ (as K is even), so this makes sense. Let $\delta_i = m_i - m$ (so that $\sum_i \delta_i = 0$), we have

$$-B_y(\overline{w}) = \sum_{i=1}^{K} \frac{m_i^3}{3} = \sum_{i=1}^{K} \frac{(m+\delta_i)^3}{3} = \frac{Km^3}{3} + \sum_{i=1}^{K}\left(m\delta_i^2 + \frac{\delta_i^3}{3}\right) = \frac{n^3}{24K^2} + O(n^{1+2\varepsilon}). \tag{4.3.5}$$

► Next, we take a shorter word $v \in \{a^\pm, t\}^\star$ ending up in the same coset as w (that is the $\mathscr{E}$ coset), and with same endpoint $\hat{v} = \hat{w} = (n, 0)$, and prove that $B_y(\bar{v}) < B_y(\bar{w})$. This implies that w is a geodesic, as no shorter word represents the same element.

Consider the decomposition $\tilde{v} \in X^\star$. Its coarse length and length are

$$\begin{aligned} k(\tilde{v}) - 1 &\leqslant \text{“number of } t \text{ in } v\text{”} &&= K - d \quad \text{(for some even } d) \\ \ell(\tilde{v}) &= \text{“number of } a^{\pm 1} \text{ in } v\text{”} &&< n + d. \end{aligned}$$

Note that, as you need at least n letters “$a^{\pm 1}$” to reach $\hat{v} = (n, 0)$, the shortening relative to w has to come from the number of “t” in v, so that $2 \leqslant d \leqslant K - 2$.

Combining Proposition 4.3.1 and the equation (4.3.5), we get

$$B_y(\bar{w}) - B_y(\bar{v}) \geqslant \frac{n^3}{24}\left(\frac{1}{(K-d)^2} - \frac{1}{K^2}\right) - C \cdot d^2 \cdot \max\left\{d; \frac{n}{K-d}\right\} - O(n^{1+2\varepsilon}).$$

Now we split into two cases:

- If $d \geqslant K - \frac{n}{K}$, then the term $\frac{n^3}{24(K-d)^2}$ dominates and $B_y(\bar{w}) > B_y(\bar{v})$.

- If $d \leqslant K - \frac{n}{K}$. We have $\frac{1}{(K-d)^2} - \frac{1}{K^2} \geqslant \frac{2d}{K^3}$ by the mean value theorem, hence

$$\begin{aligned} B_y(\bar{w}) - B_y(\bar{v}) &\geqslant \frac{n^3}{12} \cdot \frac{d}{K^3} - C \cdot d \cdot K^2 - O(n^{1+2\varepsilon}) \\ &= d\left(\frac{1}{12\kappa^3} - C\kappa^2\right) \cdot n^{6/5} - o(n^{6/5}) \\ &> 0 \end{aligned}$$

as long as $\kappa < \sqrt[5]{\frac{1}{12C}}$ and n is large enough.

▶ In conclusion, all those words w are geodesics. Moreover there are many degrees of freedom left, leading to plenty of geodesics of length $n+K$. Indeed, the values of the partial sums $\left(\sum_{i=1}^{r} m_i\right)_{r=1,\dots,K-1}$ can be picked independently in $[rm - \frac{1}{2}n^{\varepsilon}, rm + \frac{1}{2}n^{\varepsilon}]$. This shows that, for all n large enough even and $K \approx \kappa n^{3/5}$ even, we have

$$\gamma_{\text{geod}}\big(n+K\big) \geqslant (n^{\varepsilon})^{K-1} \asymp \exp\big(\varepsilon\kappa \cdot n^{3/5} \cdot \log(n)\big).$$

Since the function $\gamma_{\text{geod}}(n)$ is increasing, this bound extends for all n. □

4.4 Further questions and remarks

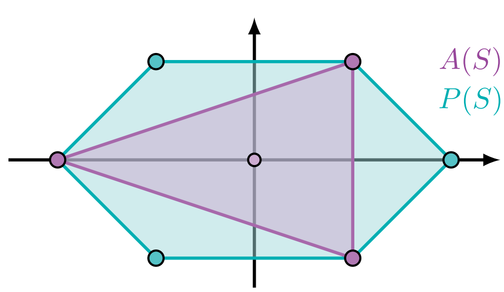
Plenty of questions remain open. The question we would most like to see solved is

Conjecture 4.A. If the geodesic growth of (G, S) is polynomial, with S a *symmetric* generating set, then G is virtually 2-step nilpotent.

Among the possible counter-examples (all virtually nilpotent), treating the virtually 3-step nilpotent cases would be sufficient, as G factors onto $G/\gamma_4(H)$. We emphasize "symmetric" as we have the following intriguing example:

Conjecture 4.B. The geodesic growth of $v\mathscr{E}$ w.r.t. $S = \{a, b, (ab)^{-1}, t\}$ is polynomial.

In contrast, in an $\mathscr{E}$-by-finite group, any *symmetric* generating set S such that $\mathrm{P}(S)$ has vertices on the x-axis will yield exponential geodesic growth (as the x-axis is fixed by automorphisms of $\mathscr{E}$), and any generating set S such that $\mathrm{P}(S)$ has no vertex on the x-axis should yield super-polynomial geodesic growth.



More generally,

Question 4.B'. Can we construct virtually s-step nilpotent groups with polynomial geodesic growth on top of any filiform nilpotent groups of type I

$$\mathscr{F}_s = \left\langle x, y_1, y_2, \ldots, y_s \;\middle|\; [x, y_i] = y_{i+1},\ [x, y_s] = [y_i, y_j] = 1 \right\rangle \ ?$$

These groups have few abnormal curves, covering only the y_1-direction (see [BNV22]).

* * *

If Conjecture 4.A holds, then Theorem 4.B reduces the characterization of groups of polynomial geodesic growth (for symmetric generating set) to the following question:

Problem 4.C. Characterize finite subgroups $F \leqslant \mathrm{GL}_d(\mathbb{Z})$ for which

(ii') there exists a finite *symmetric* set $A \subset \mathbb{Z}^d$ such that $\mathrm{P} = \mathrm{ConvHull}(A^F)$ is full-dimensional and each facet of P contains at most one point of A.

Ideally, the characterization should be algorithmic. Given F as a finite set of integer-valued matrices, one should be able to decide whether or not the condition is satisfied. In contrast, [BBES12, Theorem 1] quoted in the introduction gives the sufficient condition

(iii') There exists $a \in \mathbb{Z}^d$ such that $\mathrm{P} = \mathrm{ConvHull}(\{\pm a\}^F)$ is full-dimensional.

This condition is clearly algorithmic. It is therefore natural to ask

Question 4.D. Are conditions (ii') and (iii') equivalent?

We expect a negative answer, but were not able to find a counter-example.

* * *

Regarding the Breuillard–Le Donne conjecture, we would like to propose the following weakening (already proven for 3-step nilpotent groups, see [Gia17, Proposition 45]):

Conjecture 4.E. Let (H, X) be a f.g., torsionfree, metabelian, nilpotent group. Then

$$\forall h \in H, \quad \|h\|_X - \|h\|_{\mathrm{Stoll},X} = O(\|h\|_X^{1/3}).$$

The idea in the more general setting would be the following: when approximating a geodesic v in the sub-Finsler space to get a genuine word $w \in X^\star$, we create a linear error term in winding number (i.e., $\|W_w - W_v\|_{\ell^1} = O(\ell(v))$). The error in $\gamma_2(G)/\gamma_3(G)$ can be reduced to $O(1)$ for the correct choice of w; this is the idea of Stoll [Sto98]. The higher order terms can be fixed at cost $O(\ell^{\frac{1}{3}})$ as they are (at least) cubically distorted.

Finally, we would like to reiterate

Question 4.F ([Gri14, Problem 13])**.** Does there exist a pair (G, S) of intermediate geodesic growth and intermediate volume growth?

Exponential geodesic growth has been established in several examples of intermediate volume growth by [Brö16], for instance the first Grigorchuk group with standard generating set. The simplest open example is the Fabrikowski–Gupta group. We isolate this question since an answer in either direction would require new insight on these groups.

Chapter 5

Horofunction boundaries

Horofunction boundaries (or horoboundaries) were defined by Gromov in [Gro81b]. The horofunction boundary $\partial(X, d)$ is a topological boundary, which allows to compactify any proper metric spaces (X, d). This comes in contrast with other boundaries which are often ill-defined or trivial: the horofunction boundary $\partial(G, d_S)$ is well-defined and non-trivial for any infinite group G endowed with a word metric d_S.

A key feature of this boundary is that any action by isometry on (X, d) extends as an action by homeomorphisms on $\partial(X, d)$. In particular, when $(X, d) = (G, d_S)$ is a Cayley graph, the action by left-multiplication extends to an action on the boundary.

One of the primary motivations is an observation due to Anders Karlsson [Kar08]:

> If the horoboundary contains a finite orbit $G \cdot \varphi$, then G contains a finite-index subgroup which factors onto $\mathbb{Z}$. ($\varphi\colon \mathrm{Stab}(\varphi) \to \mathbb{Z}$ is a non-trivial homomorphism.)

In particular, if one could prove that all finitely generated groups of polynomial growth admit a finite orbit in their horoboundary, this would provide an alternative proof of Gromov's theorem on groups of polynomial growth [Gro81a]. This was indeed verified for finitely generated nilpotent groups [Wal11], however we would like to prove this statement without using the nilpotent structure, or under weaker growth conditions such as $\beta_{G,S}(n) \preceq n^{\log(n)}$, or even $\beta_{G,S}(n) \preceq \exp(\sqrt{n})$ [RGY24].

This leads us in two different directions:

The first possibility is to find "small" invariant subsets. A nice invariant subset to work with is the set of Busemann points, that is, the set of limits of geodesic rays. For instance, Tointon and Yadin managed to prove that any group growing at most linearly as finitely many Busemann points [TY16]. They formulate a conjecture:

Conjecture (Tointon–Yadin [TY16])**.** If G has polynomial growth, then the set of Busemann points in the horoboundary $\partial(G, d_S)$ is countable.

The second possibility we will consider is to look for "fixed points up to small errors". Horofunctions are real-valued functions on the underlying space X, so one way to formalize this is to look at the *reduced horofunction boundary*

$$\partial^r(X,d) := \partial(X,d)/C_b(X,\mathbb{R}),$$

where $C_b(X,\mathbb{R})$ is the set of bounded continuous functions $f\colon X \to \mathbb{R}$. This is the path explored by Bader and Finkelshtein: they proved that, if G is abelian, or $G = H_3(\mathbb{Z})$, then $G \curvearrowright \partial^r(G,d_S)$ is trivial. They also formulated the following conjecture:

Conjecture (Bader–Finkelshtein [BF20])**.** If G is a finitely generated nilpotent group, then the action of G on its reduced horoboundary $\partial^r(G,d_S)$ is trivial.

We will see that both answers strongly depend on the nilpotency class.

We first provide a complete classification of orbits of Busemann points for a family of 2-step nilpotent groups, including the discrete Heisenberg groups (see Theorem 5.3.3 for a precise statement). This implies the following result:

Theorem 5.A. *Let H be a finitely generated, torsionfree, 2-step nilpotent group with $[H,H] \simeq \mathbb{Z}$, and S any generating set. Then $\partial(H,d_S)$ contains only finitely many orbits of Busemann points. In particular, there are only countably many Busemann points.*

This extends previous results for abelian groups [Rie02; Dev02] and $H_3(\mathbb{Z})$ with its standard generating set [Wal11]. We also prove the second conjecture in class 2:

Theorem 5.B (Theorem 5.4.1)**.** *Let G be a finitely generated 2-step nilpotent group, and S a finite generating set. Then the action $G \curvearrowright \partial^r(G,d_S)$ is trivial.*

However, in class $c \geqslant 3$, both conjectures fail. Precisely, we construct an explicit family of geodesic rays γ_u in $(\mathscr{C},d_S)$ (with S the standard generating set) parametrized by unit vectors $u \in \mathbb{S}^1$. The associated Busemann points satisfy the following property:

Theorem 5.C (Theorem 5.5.6)**.** *The Busemann points $[b_{\gamma_u}]$ are pairwise distinct, and their orbits are infinite in the reduced horoboundary $\partial^r(\mathscr{C},d_S)$.*

This can be extended for free nilpotent groups of class $c \geqslant 3$.

Along the way, we prove some transfer results for quotients, rough isometries, and more generally roughly isometric embeddings (Propositions 5.2.5 and 5.2.1). Combining this with work of Fisher and Nicolussi Golo [FNG21], we get the following result

Theorem 5.D (Corollary 5.4.4)**.** *Let $H_3(\mathbb{Z})$ be the 3-dimensional Heisenberg group, and S any generating set. Then $\partial(H_3(\mathbb{Z}),d_S)$ has the cardinality of the continuum.*

This indicates that "almost all" horofunctions on $H_3(\mathbb{Z})$ are non-Busemann, contrasting with the case of abelian groups. This extends a result of [WW06].

5.1 Preliminaries

5.1.1 Horofunctions and Busemann points

We first define the horofunction boundary of a general proper metric space (X, d). In what follows, we will choose (X, d) to be a finitely generated group with a word metric, or a simply connected nilpotent Lie group with a sub-Finsler metric.

Definition 5.1.1 (Horofunction boundary)**.** Let (X, d) be a proper metric space, together with a base point $e \in X$. Consider the embedding $\iota\colon X \to C(X, \mathbb{R})$ sending

$$x \mapsto \big(\varphi_x(*) = d(x, *) - d(x, e)\big),$$

where $C(X, \mathbb{R})$ is the set of continuous functions $X \to \mathbb{R}$ with the topology of uniform convergence on compact sets. The *horofunction boundary* (or *horoboundary*) is defined as $\partial(X, d) \coloneqq \mathrm{Cl}(\iota(X)) \setminus \iota(X)$, and functions $\varphi \in \partial(X, d)$ are called *horofunctions*.

We observe that, since X is proper (hence second countable), every horofunction can be seen as a limit $\varphi(*) = \lim_{n\to\infty} \varphi_{x_n}(*)$ for some sequence $(x_n) \subseteq X$. An interesting case is when the sequence is a geodesic ray, in which case the sequence φ_{x_n} always converges.

Definition 5.1.2. Let $T \subseteq [0, \infty)$ be an unbounded subset (e.g. $T = \mathbb{N}$), and $\gamma : T \to X$ be a mapping. We say that

- γ is a *geodesic ray* if, for all $s, t \in T$, we have $d(\gamma(s), \gamma(t)) = |t - s|$.
- γ is an *almost geodesic ray* if for all $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that

$$\forall s, t \in T \cap [N, \infty), \quad |d(\gamma(t), \gamma(s)) + d(\gamma(s), \gamma(0)) - t| < \varepsilon.$$

Lemma 5.1.3 ([Rie02, Lemma 4.5])**.** *Let $\gamma\colon T \to X$ be an almost geodesic ray. Then $b_\gamma \coloneqq \lim_{t\to\infty} \varphi_{\gamma(t)}$ exists, such limits are called* Busemann points*.*

Let g be an isometry of (X, d). By [Rie02, Proposition 4.10], g extends uniquely to an homeomorphism of $\mathrm{Cl}(\iota(X))$, sending $\partial(X, d)$ to itself. This defines a (left) group action $\mathrm{Iso}(X) \curvearrowright \partial(X, d)$ by homeomorphisms, given by the explicit formula

$$(g \cdot \phi)(x) = \phi\big(g^{-1}(x)\big) - \phi\big(g^{-1}(e)\big).$$

Moreover, the action $\mathrm{Iso}(X) \curvearrowright (X, d)$ sends almost geodesic rays to almost geodesic rays, hence $\mathrm{Iso}(X) \curvearrowright \partial(X, d)$ sends Busemann points to Busemann points.

5.1.2 Busemann points on Cayley graphs

Let us consider the special case of horofunctions on graphs. The graph structure allows to simplify some definitions and problems. For instance,

- The topology is discrete, hence the topology of uniform convergence on compact sets coincides with the topology of pointwise convergence.
- Every Busemann point is the limit of a geodesic [WW06, Lemma 2.1].

We also have the following folklore lemma:

Lemma 5.1.4 (see eg. [TY16, Lemma 2.4])**.** *In a graph $(\mathcal{G}, d)$, any Busemann point b_γ is represented by a geodesic η starting at $\eta_0 = e$.*

Sketch of proof. The integer-valued sequence $d\big(\gamma_n, e\big) - d\big(\gamma_n, \gamma_0\big)$ is decreasing, so it is constant for $n \geqslant N$, for some N. In particular, we have

$$\forall n \geqslant N, \quad d(\gamma_n, e) = d(\gamma_n, \gamma_N) + d(\gamma_N, e).$$

This means that $\eta = ([\text{geodesic path from } e \text{ to } \gamma_N], \gamma_N, \gamma_{N+1}, \ldots)$ is a geodesic ray, starting at e, which eventually coincides with γ. In particular, $b_\eta = b_\gamma$. □

For the study of Cayley graphs and their horoboundary, we have a rich structure given by graph labeling, left translations and homomorphisms. We consider G a group, with S a finite symmetric generating set, and d_S the induced word metric. Any infinite path $(x_n)_{n\geqslant 0}$ starting at $x_0 = e$ can be parametrized by an infinite word $\gamma = s_1 s_2 \cdots \in S^\infty$, with $s_i = \gamma_i^{-1}\gamma_{i+1}$. We denote prefixes of γ by $\gamma_n \coloneqq s_1 s_2 \cdots s_n \in S^n$. The associated group elements are denoted as $\bar{\gamma}_n \in G$ (so $\bar{\gamma}_n = x_n$).

Definition 5.1.5. The language of infinite geodesics is

$$\operatorname{Geo}(G, S) = \{\gamma \in S^\infty \mid \forall n,\ d_S(e, \bar{\gamma}_n) = n\}.$$

In this formalism, Lemma 5.1.4 easily gives the following:

Lemma 5.1.6. *In a Cayley graph (G, d_S),*

(a) *Every Busemann point is represented by an infinite geodesic word $\gamma \in \operatorname{Geo}(G, S)$.*

(b) *Two Busemann points lie in the same orbit if and only if they are represented by infinite geodesic words which are* cofinal*, i.e., equal up to removing finite prefixes.*

Proof. Given $g \in G$ and $\gamma \in \operatorname{Geo}(G, S)$, the Busemann point $g \cdot b_\gamma$ is represented by a geodesic with the same labeling, starting from g. So the construction in Lemma 5.1.4 gives a geodesic word η representing $g \cdot b_\gamma$ in the same cofinality class as γ.

Reciprocally, given two infinite geodesics $\gamma = uw$ and $\eta = vw$ (with $u, v \in S^\star$ finite prefixes and $w \in S^\infty$ the common infinite suffix) we have $b_\gamma = \bar{u} \cdot b_w = \bar{u}\bar{v}^{-1} \cdot b_\eta$. □

5.1.3 Previous work on Abelian groups

We recall several facts on horofunctions of Abelian groups.

Theorem 5.1.7 ([WW06, Prop 3.5])**.** *Horofunctions on $\partial(\mathbb{Z}^r, S)$ are Busemann points.*

Therefore every horofunction on $\partial(\mathbb{Z}^r, d_S)$ is represented by an infinite geodesic word $\gamma = s_1 s_2 \cdots \in \mathrm{Geo}(\mathbb{Z}^r, S)$. Let $\mathrm{B} = \mathrm{ConvHull}(S) \subset \mathbb{R}^r$ be the convex hull of S. We state a lemma which is implicitly used in the previous work [Dev02; Wal11].

Definition 5.1.8. Given an infinite word $\gamma \in S^\infty$, the set of *directions* of γ is the set of letters of S with infinitely many occurrences in γ. We denote it by D_γ.

Lemma 5.1.9. *Consider an abelian group $(\mathbb{Z}^r, S)$, and $\gamma \in S^\infty$ an infinite word.*

(a) *If γ is a geodesic, the minimal face of* B *containing D_γ is proper. Equivalently, there exists a proper face* F *containing all but finitely many letters of γ.*

(b) *Reciprocally, if a proper face* $\mathrm{F} \subset \mathrm{B}$ *contains* all *letters of γ, then γ is geodesic.*

We will give its proof in a more general setting (see Proposition 5.3.1). The minimal face containing D_γ is the face *associated to* γ. How these associated faces relate to Busemann points might not be clear, until the following result:

Theorem 5.1.10 ([Dev02, Section 4])**.** *Let γ_1, γ_2 be infinite geodesic words in $(\mathbb{Z}^r, S)$.*

(a) *If $b_{\gamma_1} = b_{\gamma_2}$, then γ_1 and γ_2 are associated to the same face* F *of* B.

(b) *Reciprocally, if* all *letters of γ_1 and γ_2 belong to* F, *then $b_{\gamma_1} = b_{\gamma_2}$.*

(c) *$b_{\gamma_1}, b_{\gamma_2}$ lie in the same orbit if and only if γ_1, γ_2 are associated to the same face* F.

Remark 5.1.11. Part (c) follows from the first two parts and Lemma 5.1.6(b).

5.1.4 Reduced horoboundary

In general it is difficult to compute horofunctions since this requires to compute some distances *exactly*. In particular, the horoboundary of a Cayley graph (i.e., its topology) typically depends on the choice of a generating set. However, once we ignore bounded errors, working with horofunctions often become much more manageable. In some cases, the resulting boundary even becomes a quasi-isometry invariant.

Definition 5.1.12. Let $C_b(X, \mathbb{R})$ be the set of bounded continuous functions $X \to \mathbb{R}$. The quotient space $\partial^r(X, d) = \partial(X, d)/C_b(X, \mathbb{R})$ is called the *reduced horoboundary* of (X, d). Elements of the reduced horoboundary are denoted by $[\varphi]$.

Example 5.1.13. The reduced boundary of a proper, complete, Gromov hyperbolic space is homeomorphic to its Gromov boundary [WW05, Proposition 4.4]. It is invariant under quasi-isometries.

We generalize the notion of Busemann points to *roughly Busemann points*. Those are limits of roughly geodesic ray, and are only defined in $\partial^r(X,d)$.

Definition 5.1.14. Let $T \subseteq [0,\infty)$ be an unbounded set. A ray $\gamma\colon T \to X$ is *roughly geodesic* if there exists $C \geqslant 0$ such that

$$\forall s,t \in T, \quad |t-s| - C \leqslant d(\gamma(s),\gamma(t)) \leqslant |t-s| + C.$$

Lemma 5.1.15. *Let γ be a roughly geodesic ray. Then all limits points of γ in $\partial(X,d)$ coincide in $\partial^r(X,d)$. We denote this common limit by $[b_\gamma]$.*

Remark 5.1.16. Despite the notation, $[b_\gamma]$ may not contain any Busemann point. A Busemann point in $\partial^r(X,d)$ is a class containing a Busemann point, and a rough Busemann point is a class containing a rough Busemann point.

Proof. Up to a change of the constant C, we may suppose that $0 \in T$ and $\gamma_0 = e$. Take $x \in X$ and $m,n \in T$ such that $m \geqslant n$. We have

$$\begin{aligned}\varphi_{\gamma_m}(x) - \varphi_{\gamma_n}(x) &= \big(d(x,\gamma_m) - d(e,\gamma_m)\big) - \big(d(x,\gamma_n) - d(e,\gamma_n)\big)\\ &= \big(d(x,\gamma_m) - d(x,\gamma_n)\big) + d(\gamma_0,\gamma_n) - d(\gamma_0,\gamma_m)\\ &\leqslant d(\gamma_m,\gamma_n) + d(\gamma_0,\gamma_n) - d(\gamma_0,\gamma_m)\\ &\leqslant \big(|m-n| + C\big) + (n+C) - (m-C) = 3C\end{aligned}$$

This proves that $\limsup \varphi_{\gamma_m}(x) \leqslant \liminf \varphi_{\gamma_n}(x) + 3C$ hence any two limits points of the sequence $(\varphi_{\gamma_n})_{n\in S}$ (which do exist by compactness of $\mathrm{Cl}(\iota(X))$) are at distance at most $3C$ from each other (in the sup metric) hence coincide in $\partial^r(X,d)$. ☐

5.2 Transfer results

In this section, we give two basic transfer results for rough isometries (and roughly isometric embeddings), and for graph coverings.

5.2.1 Rough isometries and reduced boundary

Since the reduced horoboundary ignores additive errors, it should behave nicely with the coarse setting, meaning it should be stable under rough isometries (Definition 2.1.7).

Proposition 5.2.1. *Let $(X,d_X),(Y,d_Y)$ be two proper metric spaces. Given a roughly isometric embedding $f\colon X \to Y$, we can define a surjective map*

$$f^*\colon \begin{pmatrix} \mathrm{Cl}(\iota(f(X))) \cap \partial^r(Y,d_Y) & \longrightarrow & \partial^r(X,d_X) \\ [\varphi] & \longmapsto & [\varphi\circ f]\end{pmatrix}.$$

Moreover,

- *If $G \curvearrowright X,Y$ by isometries and f is G-equivariant, then f^* is G-equivariant.*
- *If f is a rough isometry, then f^* is a bijection $\partial^r(Y,d_Y) \to \partial^r(X,d_X)$.*

Proof. Fix $e \in X$ and $f(e) \in Y$ two basepoints.

► *Well-defined:* Consider $\varphi \in \mathrm{Cl}(f(X)) \cap \partial(Y, d_Y)$. We fix $(x_n) \subseteq X$ s.t. $f(x_n) \to \varphi$ in $\partial(Y, d_Y)$, and a subsequence (x_{n_k}) such that $x_{n_k} \to \psi \in \partial(X, d_X)$. We have

$$\begin{aligned}&|\varphi(f(g)) - \psi(g)| \\ &\quad = \lim_{k\to\infty} \Big| \Big(d_Y\big(f(g), f(x_{n_k})\big) - d_Y\big(f(e), f(x_{n_k})\big)\Big) - \Big(d_X(g, x_{n_k}) - d_X(e, x_{n_k})\Big)\Big| \\ &\quad \leqslant 2C_1\end{aligned}$$

so that $[\varphi \circ f] = [\psi] \in \partial^r(X, d_X)$. Moreover, given two representatives $\varphi, \varphi' \in [\varphi]$,

$$\|\varphi \circ f - \varphi' \circ f\|_{L^\infty(X)} \leqslant \|\varphi - \varphi'\|_{L^\infty(Y)} < \infty$$

so that $[\varphi \circ f] = [\varphi' \circ f]$.

► *Surjectivity:* Consider $[\psi] \in \partial^r(X, d_X)$, and $(x_n) \subseteq X$ such that $x_n \to \psi$. As $\mathrm{Cl}(f(X)) \cap \partial(Y, d_Y)$ is compact, there exists $\varphi \in \mathrm{Cl}(f(X)) \cap \partial(Y, d_Y)$ and a subsequence (x_{n_k}) such that $f(x_{n_k}) \to \varphi$. Using the first computation, we have

$$f^*([\varphi]) = [\varphi \circ f] = [\psi].$$

► *Equivariance:* For $x \in X$, we have

$$((g \cdot \varphi) \circ f)(x) = \varphi(g \cdot f(x)) - \varphi(g \cdot f(e)) = \varphi(f(g \cdot x)) - \varphi(f(g \cdot e)) = (g \cdot (\varphi \circ f))(x).$$

► *New domain:* We suppose that f is a rough isometry and prove that

$$\mathrm{Cl}(f(X)) \cap \partial^r(Y, d_Y) = \partial^r(Y, d_Y).$$

Fix $[\varphi] \in \partial^r(Y, d_Y)$ and a sequence $(y_n) \subseteq Y$ such that $y_n \to \varphi$. As f is a rough isometry, we can find $(x_n) \subseteq X$ such that $d_Y(y_n, f(x_n)) \leqslant C_2$. As $\partial(Y, d_Y)$ is compact, there is a subsequence x_{n_k} such that $f(x_{n_k}) \to \varphi' \in \partial(Y, d_Y)$. We have

$$\begin{aligned}&|\varphi(g) - \varphi'(g)| \\ &\quad = \lim_{k\to\infty} \Big| \Big(d_X(g, y_{n_k}) - d_X\big(f(e), y_{n_k}\big)\Big) - \Big(d_Y\big(g, f(x_{n_k})\big) - d_Y\big(f(e), f(x_{n_k})\big)\Big)\Big| \\ &\quad \leqslant 2C_2\end{aligned}$$

so that $[\varphi] = [\varphi'] \in \mathrm{Cl}(f(X))$.

► *Inverse:* There exists a rough isometry $g \colon Y \to X$ such that

$$d_X\big(g(f(x)), \mathrm{id}_X(x)\big) \leqslant C_2.$$

It follows that $|\psi \circ g \circ f - \psi| \leqslant C_2$ for all $\psi \in \partial(X, d_X)$, as horofunctions are 1-Lipschitz, hence $f^*(g^*([\psi]) = [\psi \circ g \circ f] = [\psi]$, i.e., $f^* \circ g^* = \mathrm{id}_{\partial^r(X,d_X)}$. Similarly, $g^* \circ f^* = \mathrm{id}_{\partial^r(Y,d_Y)}$. We conclude that f^* is a bijection. □

5.2.2 Graph coverings

In this section, we relate boundaries of graphs and their coverings. Our result mostly apply to Busemann points. In order to state our results, we need a few definitions:

Definition 5.2.2. Let $\mathcal{G}$ and $\mathcal{H}$ be two graphs (possibly with loops and multiple edges). A *graph covering* is a surjection $\pi\colon \mathcal{G} \twoheadrightarrow \mathcal{H}$ which is a local isomorphism, i.e.,

$$\forall v \in V\mathcal{G}, \quad \pi\colon N_v \to N_{\pi(v)}$$

is a bijection, where $N_v = \{e \in E\mathcal{G} \mid v \text{ is incident to } e\}$.

Example 5.2.3. The main examples from come group quotients: any epimorphism $\pi\colon G \twoheadrightarrow H$ induces a graph covering $\pi\colon \mathcal{C}ay(G,S) \twoheadrightarrow \mathcal{C}ay(H,\pi(S))$. More generally, we can consider $\mathcal{C}ay(G,S) \twoheadrightarrow \mathcal{S}ch(K\backslash G,S)$ for any subgroup $K \leqslant G$.

Definition 5.2.4. Let $\mathrm{Geo}(\mathcal{G})$ be the set of geodesics rays $\gamma = (\gamma_n)_{n\geqslant 0}$ in $\mathcal{G}$ starting a $\gamma_0 = e$. For every subset $\Pi \subseteq \mathrm{Geo}(\mathcal{G})$, we define

$$\partial(\Pi) = \{b_\gamma \mid \gamma \in \Pi\} \quad \text{and} \quad \partial^r(\Pi) = \{[b_\gamma] \mid \gamma \in \Pi\}.$$

In particular, $\partial(\mathrm{Geo}(\mathcal{G}))$ is the entire set of Busemann points in $\partial(\mathcal{G},d)$.

The main result of the subsection is the following:

Proposition 5.2.5. *Consider a graph covering $\pi\colon \mathcal{G} \twoheadrightarrow \mathcal{H}$, and let*

$$\Pi = \{\gamma \in \mathrm{Geo}(\mathcal{G}) \mid \pi(\gamma) \textit{ is eventually geodesic in } \mathcal{H}\},$$
$$\Pi^r = \{\gamma \in \mathrm{Geo}(\mathcal{G}) \mid \pi(\gamma) \textit{ is roughly geodesic in } \mathcal{H}\}.$$

Consider $\gamma, \eta \in \mathrm{Geo}(\mathcal{G})$.

(1) *Suppose that $b_\gamma = b_\eta$, then $\gamma \in \Pi \iff \eta \in \Pi$. Moreover, the map*

$$\pi_*\colon \begin{pmatrix} \partial(\Pi) & \longrightarrow & \partial(\mathrm{Geo}(\mathcal{H})) \\ b_\gamma & \longmapsto & b_{\pi(\gamma)} \end{pmatrix}$$

is well-defined and surjective.

(2) *Suppose that $[b_\gamma] = [b_\eta]$, then $\gamma \in \Pi^r \iff \eta \in \Pi^r$. Moreover, the map*

$$\pi_*\colon \begin{pmatrix} \partial^r(\Pi^r) & \longrightarrow & \partial^r(\mathcal{H},d) \\ [b_\gamma] & \longmapsto & [b_{\pi(\gamma)}] \end{pmatrix}$$

is well-defined.

Since $\Pi \subseteq \Pi^r$, we deduce that the restriction $\pi_*\colon \partial^r(\Pi) \to \partial^r(\mathrm{Geo}(\mathcal{H}))$ is well-defined and surjective. Moreover, if $G \curvearrowright \mathcal{G},\mathcal{H}$ by isometries and π is G-equivariant, then the subsets $\partial(\Pi)$, $\partial^r(\Pi)$ and $\partial^r(\Pi^r)$ are G-invariant and π_* is G-equivariant.

Remark 5.2.6. For part (2), we could get away with $\pi\colon X \to Y$ a "rough submetry" (i.e., $d_Y(\pi(x),\pi(x')) \leqslant d_X(x,x') + C$) between general proper metric spaces.

For the proof, we need a lemma due to Walsh, which is useful to distinguish/identify Busemann points in $\partial(\mathcal{G}, d)$, and coarse variation in $\partial^r(X, d)$.

Lemma 5.2.7 ([Wal11, Proposition 2.1])**.** *Consider $\gamma, \eta \in \mathrm{Geo}(\mathcal{G})$. TFAE*

(a) *γ and η converge to the same Busemann point, that is, $b_\gamma = b_\eta$ in $\partial(\mathcal{G}, d)$.*

(b) *For all n, there exists $m \geqslant n$ such that $d(\gamma_m, \eta_n) = d(\eta_m, \gamma_n) = m - n$.*

(c) *There is a geodesic ray sharing infinitely many points with both γ and η.*

Lemma 5.2.8. *Consider γ, η two roughly geodesic rays in (X, d). TFAE*

(a) *$[b_\gamma] = [b_\eta]$ in $\partial^r(X, d)$.*

(b) *There exists $C \geqslant 0$ such that, for all $n \in T_\gamma$ (resp. T_η), there exists $m \in T_\eta$ (resp. T_γ) such that $m \geqslant n$ and $d(\eta_m, \gamma_n) \leqslant m - n + C$ (resp. $d(\gamma_m, \eta_n) \leqslant m - n + C$).*

(c) *There is a roughly geodesic ray sharing infinitely points with γ and η.*

Proof. Suppose γ and η are roughly geodesic with $\gamma_0 = \eta_0 = e$ and parameter $D \geqslant 0$.

(a) $\Rightarrow$ (b): Fix b_γ, b_η, limits points of γ and η respectively. There exists $E \geqslant 0$ such that $|b_\gamma(x) - b_\eta(x)| \leqslant E$ for all $x \in X$. In particular, this holds for $x = \gamma_n$.

- We have $b_\gamma(\gamma_n) \leqslant \limsup_{m \in T_\gamma} d(\gamma_m, \gamma_n) - d(\gamma_m, e) \leqslant -n + 2D$.
- There exists $m \in T_\eta$ with $m \geqslant n$ such that

$$b_\eta(\gamma_n) \geqslant d(\eta_m, \gamma_n) - d(\eta_m, e) - 1 \geqslant d(\eta_m, \gamma_n) - m - D - 1$$

Combining both, we get $-E \leqslant b_\gamma(\gamma_n) - b_\eta(\gamma_n) \leqslant m - n + 3D + 1 - d(\eta_m, \gamma_n)$ hence $d(\eta_m, \gamma_n) \leqslant m - n + (3D + E + 1) =: m - n + C$.

(b) $\Rightarrow$ (c): There exists a strictly increasing sequence $(n_i)_{i \geqslant 0}$ such that, for all $k \geqslant 0$, we have $n_{2k} \in T_\gamma$ and $n_{2k+1} \in T_\eta$ and

$$d(\eta_{n_{2k+1}}, \gamma_{n_{2k}}) \leqslant n_{2k+1} - n_{2k} + C,$$
$$d(\gamma_{n_{2k+2}}, \eta_{n_{2k+1}}) \leqslant n_{2k+2} - n_{2k+1} + C$$

We define $\zeta(n_{2k}) = \gamma(n_{2k})$ and $\zeta(n_{2k+1}) = \eta(n_{2k+1})$ (with domain $T_\zeta = \{n_i\}$). Then ζ is roughly geodesic with parameter $\max\{C, D\}$.

The implication (c) $\Rightarrow$ (a) follows directly from Lemma 5.1.15. □

Proof of Proposition 5.2.5. We consider as basepoints $e \in \mathcal{G}$ and $\pi(e) \in \mathcal{H}$.

(2) Let γ, η be roughly geodesic rays starting at $\gamma_0 = \eta_0 = e$ such that $[b_\gamma] = [b_\eta]$. By Lemma 5.2.8, there exists a roughly geodesic ray ζ interpolating between the two. Suppose that $\pi(\gamma)$ is roughly geodesic with parameter C'.

Consider $m > n$ in T_ζ. Take $l > m$ such that $\zeta_l = \gamma_l$. Then

$$d(\pi(\zeta_m), \pi(\zeta_n)) \leqslant d(\zeta_m, \zeta_n) \leqslant m - n + C$$

and

$$\begin{aligned} d(\pi(\zeta_m), \pi(\zeta_n)) &\geqslant d(\pi(\zeta_l), \pi(\zeta_0)) - d(\pi(\zeta_l), \pi(\zeta_m)) - d(\pi(\zeta_n), \pi(\zeta_0)) \\ &\geqslant (l - 0 - C') - (l - m + C) - (n - 0 + C) \\ &= m - n - (C' + 2C) \end{aligned}$$

as $\pi \colon \mathcal{G} \to \mathcal{H}$ is a submetry.

Using the same argument with $\pi(\zeta)$ and $\pi(\eta)$, we conclude that $\pi(\eta)$ is roughly geodesic too, proving the first part of the statement. Using the other direction of Lemma 5.2.8, we conclude that $[b_{\pi(\gamma)}] = [b_{\pi(\eta)}]$, proving that π_* is well-defined.

(1) Essentially the same with Lemma 5.2.7.

For the surjectivity, we observe that any geodesic (any path) $\gamma \in \mathrm{Geo}(\mathcal{H})$ can be lifted in $\mathcal{G}$ using the graph covering property. This gives us a geodesic ray $\tilde{\gamma} \in \mathrm{Geo}(\mathcal{G})$ such that $\gamma = \pi(\tilde{\gamma})$, in particular $\tilde{\gamma} \in \Pi$ and $\pi_*(b_{\tilde{\gamma}}) = b_\gamma$.

For the invariance/equivariance, take $\gamma \in \Pi$ and $g \in G$. By Lemma 5.1.4, there exists $\eta \in \mathrm{Geo}(\mathcal{G})$ such that η eventually coincides with the translate $g \cdot \gamma$ (up to a shift of indices), in particular $b_\eta = g \cdot b_\gamma$. Observe that $\pi(\eta)$ eventually coincides with $g \cdot \pi(\gamma)$, proving that $\eta \in \Pi$ (hence $g \cdot b_\gamma \in \partial(\Pi)$) and

$$\pi_*(g \cdot b_\gamma) = \pi_*(b_\eta) = g \cdot b_{\pi(\gamma)} = g \cdot \pi_*(b_\gamma).$$

The proof is essentially the same for $\partial^r(\Pi^r)$. □

5.3 Busemann points on Heisenberg groups

Throughout this section, we consider G a torsionfree 2-step nilpotent group, and S a finite symmetric generating set. Note that $\bar{G}$ can be identified with its Lie algebra $\mathfrak{g}$ via the exponential map. Hence the convex hull $\mathrm{B} = \mathrm{ConvHull}(S) \subset \bar{G}$ is well-posed. Moreover, we decompose $\bar{G} = V_1 \oplus V_2$ (with $V_2 = [\bar{G}, \bar{G}]$). Let $\mathrm{Pr} \colon \bar{G} \to V_1$ be the abelianization map. Since Pr is a homomorphism, $\mathrm{Pr}(\mathrm{B})$ is the convex hull of $\mathrm{Pr}(S)$.

5.3.1 Infinite geodesic words

The main goal of this subsection is to extend Lemma 5.1.9 in the setting of finitely generated, torsionfree, 2-step nilpotent groups. Using the previous notations, we prove

Proposition 5.3.1. *Let G be a torsionfree 2-step nilpotent group, and S a finite symmetric generating set. For any infinite geodesic word $\gamma \in \mathrm{Geo}(G, S)$, there exists a proper face $\mathrm{F} \subset \mathrm{Pr}(\mathrm{B})$ such that all but finitely many letters of γ projects to F.*

This argument makes heavy use of the content of Section 2.4.3.

Proof. For every pair $\{s,t\} \subset S$ such that $\Pr(s)$ and $\Pr(t)$ do **not** lie on a common face of Pr(B), there exists an $\mathbb{R}$-word $v = v(s,t)$ such that $\Pr(st) = \Pr(\bar{v})$ and

$$\ell(v) < 2 = \ell(st).$$

We denote by $z(s,t) \in [\bar{G},\bar{G}]$ the element satisfying $st = \bar{v}z$. Let $\delta > 0$ be the minimum value of $2 - \ell(v(s,t))$ over all such pairs.

Consider $\gamma \in S^\infty$ and let us partition $S = D_\gamma \sqcup D_\gamma^c$. We suppose that no proper face of Pr(B) contains $\Pr(D_\gamma)$, and provide a finite subword v of γ such that

$$\|\bar{v}\|_{\text{Stoll}} < \ell(v) - C,$$

hence neither v nor γ can be geodesic w.r.t. the word metric (by Proposition 2.4.12).

By hypothesis, γ can be decomposed as

$$\gamma = w_0 \cdot s_1u_1t_1 \cdot w_1 \cdot s_2u_2t_2 \cdot w_2 \cdot \ldots \cdot w_{n-1} \cdot s_nu_nt_n \cdot w_n \cdot w_\infty$$

where

- $u_i, w_i \in S^\star$ are finite words, with all the (finitely many) instances of D_γ^c in γ appearing in w_0, and $w_\infty \in S^\infty$ is an infinite word,
- s_i, $t_i \in D_\gamma$ are individual letters such that the projections $\Pr(s_i), \Pr(t_i)$ do **not** lie on a common face for each $i \in \{1,2,\ldots,n\}$,
- for each $s \in D_\gamma$, we have $|w_n|_s \geqslant K\,|u_1u_2\ldots u_n|_s + C$,
- the parameter n can be made arbitrarily large.

The subword we are looking for is

$$v = s_1u_1t_1 \cdot w_1 \cdot s_2u_2t_2 \cdot w_2 \cdot \ldots \cdot w_{n-1} \cdot s_nu_nt_n \cdot w_n.$$

As $\bar{G}$ is 2-step nilpotent, the derived subgroup $[\bar{G},\bar{G}]$ is central, we can rewrite

$$\begin{aligned}\overline{v} &= s_1u_1t_1w_1 \cdot \ldots \cdot s_nu_nt_nw_n \\ &= [s_1,u_1]u_1s_1t_1w_1 \cdot \ldots \cdot [s_n,u_n]u_ns_nt_nw_n \\ &= u_1v_1w_1 \cdot \ldots \cdot u_nv_n \cdot w_n \cdot [s_1,u_1]\ldots[s_n,u_n] \cdot z_1\ldots z_n,\end{aligned}$$

where $v_i = v(s_i,t_i)$ and $z_i = z(s_i,t_i)$ defined earlier (the key properties being $s_it_i = v_iz_i$ and $\|v_i\|_{\text{Stoll}} \leqslant 2-\delta$). In particular, we have

$$\|u_1v_1w_1 \cdot \ldots \cdot u_nv_n\|_{\text{Stoll}} \leqslant |s_1u_1t_1w_1 \cdot \ldots \cdot s_nu_nt_n|_S - n\delta,$$

and

$$\|z_1\ldots z_n\|_{\text{Stoll}} = O(\sqrt{n}),$$

so we only need good estimates on $\|w_n \cdot [s_1,u_1]\ldots[s_n,u_n]\|_{\text{Stoll}}$, i.e., a short $\mathbb{R}$-word representing $w_n \cdot [s_1,u_1]\ldots[s_n,u_n]$. Using Lemma 2.4.10, we can find $w_n' \in S_\mathbb{R}^\star$ such that $\bar{w}_n = \bar{w}_n'$ and $k(w_n') \leqslant K$. Two cases present themselves:

- If $|w_n'|_s < |w_n|_s - C$ for some $s \in S$, then $\ell(w_n') < \ell(w_n) - C$ too and

$$\|v\|_{\text{Stoll}} \leqslant |s_1u_1t_1 \cdot w_1 \cdot s_2u_2t_2 \cdot w_2 \cdot \ldots \cdot w_{n-1} \cdot s_nu_nt_n \cdot w_n'|_S < \ell(v) - C,$$

so can conclude without much of the conversation above.

- Otherwise, we have $|w_n'|_s \geqslant |w_n|_s - C \geqslant K\,|u_1u_2\ldots u_n|_s$ for all s. Knowing that $k(w_n') \leqslant K$, the pigeonhole principle implies that the $\mathbb{R}$-word w_n' contains a power s^{λ_s} with $\lambda_s \geqslant |u_1u_2\ldots u_n|_s$ for each $s \in D_\gamma$.

From w_n', we construct a short word w_n'' representing $w_n \cdot [s_1,u_1]\ldots[s_n,u_n]$. For each generator $s \in S_\infty$, we replace in w_n' the previous power s^{λ_s} by the $\mathbb{R}$-word

$$s_n^{-|u_n|_s/\lambda_s} \ldots s_1^{-|u_1|_s/\lambda_s} \cdot s^{\lambda_s} \cdot s_1^{|u_1|_s/\lambda_s} \ldots s_n^{|u_n|_s/\lambda_s}$$

so that $\ell(w_n'') = \ell(w_n') + 2\sum_{s\in S_\infty} \frac{1}{\lambda_s}\,|u_1u_2\ldots u_n|_s \leqslant \ell(w_n) + 2\,|D_\gamma|$. Putting everything together, the triangle inequality gives

$$\begin{aligned}\|\bar v\|_{\text{Stoll}} &\leqslant \|u_1v_1w_1 \cdot \ldots \cdot u_nv_n\|_{\text{Stoll}} + \|w_n \cdot [s_1,u_1]\ldots[s_n,u_n]\|_{\text{Stoll}} + \|z_1\ldots z_n\|_{\text{Stoll}} \\ &\leqslant \Big(\ell(s_1u_1t_1w_1 \cdot \ldots \cdot s_nu_nt_n) - n\delta\Big) + \Big(\ell(w_n) + 2\,|D_\gamma|\Big) + O(\sqrt{n}) \\ &= \ell(v) - n\delta + O(\sqrt{n}) + 2\,|D_\gamma| \\ &< \ell(v) - C\end{aligned}$$

for n large enough. □

5.3.2 Orbits of Busemann points

In this section, we consider H a finitely generated, torsionfree, 2-step nilpotent group, with the additional condition $[H,H] = \langle z\rangle \simeq \mathbb{Z}$. These coincide with lattices inside $H_{2k+1}(\mathbb{R}) \times \mathbb{R}^\ell$ (see e.g. [Sto96, Lemma 7.1]).

The goal is to generalize Theorem 5.1.10 and classify orbits of Busemann points in $\partial(H, d_S)$, for any finite symmetric generating set S. In view of Proposition 5.3.1, we might expect that orbits of Busemann points are once again classified by proper faces of $\mathrm{Pr}(\mathrm{B})$. However, the conclusion is not so straightforward, and we often need more information to chose in which orbit a given Busemann point b_γ fits. Namely, for each geodesic ray $\gamma \in \mathrm{Geo}(H,S)$, we define

- $D_\gamma \subseteq S$, the set of letters that appears infinitely often in γ.
- $\mathrm{E}_\gamma \subseteq \mathrm{B}$, the minimal face of B containing D_γ.
- $\mathrm{F}_\gamma \subseteq \mathrm{Pr}(\mathrm{B})$, the minimal face of $\mathrm{Pr}(\mathrm{B})$ containing $\mathrm{Pr}(D_\gamma)$.

Note that E_γ is a face of $\mathrm{B} \cap \mathrm{Pr}^{-1}(\mathrm{F}_\gamma)$, and F_γ is a *proper* face of $\mathrm{Pr}(\mathrm{B})$.

Definition 5.3.2. A face $\mathrm{F} \subset \mathrm{Pr}(\mathrm{B})$ is *commutative* if $\langle \mathrm{F}\rangle \leqslant \bar H$ is abelian.

Theorem 5.3.3. *Let (H, S) be a 2-step nilpotent group with $[H, H] = \langle z \rangle \simeq \mathbb{Z}$. Consider two geodesic rays $\gamma_1, \gamma_2 \in \mathrm{Geo}(H, S)$.*

(a) *If $b_{\gamma_1} = b_{\gamma_2}$, then $\mathrm{F}_{\gamma_1} = \mathrm{F}_{\gamma_2}$. Moreover, if this face is commutative, then $\mathrm{E}_{\gamma_1} = \mathrm{E}_{\gamma_2}$.*

(b) *Reciprocally, if*

(1) *all the letters of γ_1, γ_2 belong to E with E commutative, or*

(2) *all the letters of γ_1, γ_2 project to F with F non-commutative,*

then $b_{\gamma_1} = b_{\gamma_2}$.

(c) *The Busemann points $b_{\gamma_1}, b_{\gamma_2}$ are in the same orbit if and only if*

(1) *$\mathrm{F}_{\gamma_1} = \mathrm{F}_{\gamma_2}$ is commutative and $\mathrm{E}_{\gamma_1} = \mathrm{E}_{\gamma_2}$, or*

(2) *$\mathrm{F}_{\gamma_1} = \mathrm{F}_{\gamma_2}$ is non-commutative.*

In particular, $\partial(H, d_S)$ contains only finitely many orbits of Busemann points (since B has only finitely many faces), hence countably many Busemann points.

Before coming to a proof of Theorem 5.3.3, let us do some anagrams!

Definition 5.3.4. For $w \in S^\star$, we define

$$Z_w = \{g \in \langle z \rangle : \exists w' \text{ a reodering of } w \text{ such that } \bar{w}' = \bar{w} g\} \subseteq [H, H] \simeq \mathbb{Z}.$$

This definition extends to infinite words $\gamma \in S^\infty$, letting $Z_\gamma = \bigcup_{n \geqslant 0} Z_{\gamma_n}$, where $\gamma_n \in S^\star$ is the prefix of length n of γ.

Lemma 5.3.5. *Let (H, S) be a 2-step nilpotent group with $[H, H] = \langle z \rangle \simeq \mathbb{Z}$. Consider $s, t \in S$ such that $[s, t] \neq e$. Then, for any word $w \in S^*$ containing the letters s, t, s, t in that order, the set $Z_w \subset \mathbb{Z}$ contains both positive and negative numbers.*

Proof. We will show the existence of w' such that $\bar{w}' = \bar{w} z^a$ with $a > 0$. Without lost of generality, we suppose $w = s\, v_1\, t\, v_2\, s\, v_3\, t$. We split into different cases:

- If $[s, v_1 t v_2] \neq e$, then either $w' = s^2 v_1 t v_2 v_3 t$ or $w' = v_1 t v_2 s^2 v_3 t$ satisfies. (More precisely, the first satisfies if $[s, v_1 t v_2] = z^a$ with $a > 0$, and the second otherwise.)
- If $[t, v_2 s v_3] \neq e$, then either $w' = s v_1 t^2 v_2 s v_3$ or $w' = s v_1 v_2 s v_3 t^2$ satisfies.
- Otherwise, we have $\bar{w} = s v_1 t^2 v_2 s v_3$. Moreover

$$[s, v_1 t^2 v_2] = [s, v_1 t v_2] \cdot [s, t] \neq e,$$

hence either $w' = s^2 v_1 t^2 v_2 v_3$ or $w' = v_1 t^2 v_2 s^2 v_3$ satisfies. □

Lemma 5.3.6. *Let (H, S) be a 2-step nilpotent group with $[H, H] = \langle z \rangle \simeq \mathbb{Z}$. Consider an infinite word $\gamma \in S^\infty$ satisfying $\gamma \in D_\gamma^\infty$. Then $Z_\gamma = \langle [s, t] : s, t \in D_\gamma \rangle$.*

Proof. The inclusion $Z_\gamma \subseteq \langle [s,t] : s,t \in D_\gamma \rangle$ is clear. In particular, the statement is true if $[s,t] = e$ for all $s,t \in D_\gamma$. From now on, we suppose the existence of $s,t \in D_\gamma$ such that $[s,t] \neq e$. Let's make some observations on Z_γ:

- Lemma 5.3.5 implies that Z_γ (seen as a subset of $\mathbb{Z}$) is unbounded above and below.
- Consider $\delta = \max\{|a| : \exists s,t \in S,\ [s,t] = z^a\}$. Then $Z_\gamma \subseteq \mathbb{Z}$ is a δ-net, meaning thickening Z_γ gets you $Z_\gamma + [-\delta,\delta] = \mathbb{R}$. Indeed, every reordering of γ_n can be obtained by successively permuting pairs of consecutive letters s,t, hence changing the value by a commutator $[s,t] = z^a$ with $|a| \leqslant \delta$ at each step.
- There exists $u \in D_\gamma^\star$ such that $Z_u \supseteq \langle [s,t] : s,t \in D_\gamma \rangle \cap [-\delta,\delta]$. For instance, if one enumerates $D_\gamma = \{s_1, s_2, \ldots, s_n\}$, then

$$u = (s_1 s_2)^N (s_1 s_3)^M \ldots (s_1 s_n)^M (s_2 s_3)^M \ldots (s_{n-1} s_n)^M$$

  satisfies when M is large enough.

Let $\gamma = \gamma_n \omega_n$ with ω_n the infinite suffix. For n large enough, we can reorder γ_n into a word starting with u, say $\gamma_n' = uv$. Fix $z^a \in \langle [s,t] : s,t \in D_\gamma \rangle$ s.t. $\bar{\gamma}_n' = \bar{\gamma}_n z^a$. Then

$$Z_\gamma = a + Z_{\gamma_n' \omega_n} \supseteq a + Z_u + Z_{v\omega_n} = \langle [s,t] : s,t \in D_\gamma \rangle$$

as $Z_u \supseteq \langle [s,t] : s,t \in D_\gamma \rangle \cap [-\delta,\delta]$ and $Z_{v\omega_n}$ is a δ-net. □

Proof of Theorem 5.3.3. (a) We suppose that $b_{\gamma_1} = b_{\gamma_2}$.

▶ By Proposition 5.3.1, all but finitely many letters of γ_i project to a proper face F_{γ_i}, therefore $\mathrm{Pr}(\gamma_i)$ is eventually geodesic in $(\mathrm{Pr}(H), d_{\mathrm{Pr}(S)})$. We use Proposition 5.2.5(1) and conclude that $b_{\mathrm{Pr}(\gamma_1)} = b_{\mathrm{Pr}(\gamma_2)}$. Finally, Theorem 5.1.10 gives $\mathrm{F}_{\gamma_1} = F_{\gamma_2}$.

▶ Suppose that $\mathrm{F} = \mathrm{F}_{\gamma_1} = \mathrm{F}_{\gamma_2}$ is commutative.

Let $S_{\mathrm{F}} := S \cap \mathrm{Pr}^{-1}(\mathrm{F})$. Then $S_{\mathrm{F}}^{\pm 1}$ generates a discrete abelian subgroup $A \leqslant H$. Let $\bar{A} \leqslant \bar{H}$ be the Mal'cev completion of A, and $\mathrm{ConvHull}_{\bar{A}}(S_{\mathrm{F}}^{\pm 1})$ the convex hull of $S_{\mathrm{F}}^{\pm 1}$ in $\bar{A}$. By Theorem 5.1.10, the orbits of Busemann points of $(A, S_{\mathrm{F}}^{\pm 1})$ are in one-to-one correspondence with the faces of $\mathrm{ConvHull}_{\bar{A}}(S_{\mathrm{F}}^{\pm 1})$. Note that the faces of $\mathrm{ConvHull}_{\bar{A}}(S_{\mathrm{F}}^{\pm 1}) \cap \mathrm{Pr}^{-1}(\mathrm{F})$ coincides with the faces of $\mathrm{B} \cap \mathrm{Pr}^{-1}(\mathrm{F})$.

By Lemma 5.2.7, there exists $\gamma \in \mathrm{Geo}(H,S)$ which intersect γ_1 and γ_2 infinitely often. We have $b_\gamma = b_{\gamma_1} = b_{\gamma_2}$, so all but finitely many letters belong to S_{F}. In particular, these rays are eventually geodesic in $(\mathcal{A}, S_{\mathrm{F}}^{\pm})$. Now we can use the other direction of Lemma 5.2.7: as $\gamma, \gamma_1, \gamma_2$ are eventually geodesic and γ intersects γ_1 and γ_2 infinitely many times, we have $b_{\gamma_1} = b_{\gamma_2}$ in $\partial(\mathcal{A}, S_{\mathrm{F}}^{\pm})$, hence $\mathrm{E}_{\gamma_1} = \mathrm{E}_{\gamma_2}$ by Theorem 5.1.10.

(b1) Let F be a commutative face and E be a face of

$$\mathrm{ConvHull}_{\bar{A}}(S_{\mathrm{F}}^{\pm 1}) \cap \mathrm{Pr}^{-1}(\mathrm{F}) = \mathrm{B} \cap \mathrm{Pr}^{-1}(\mathrm{F}).$$

Consider infinite words $\gamma_1, \gamma_2 \in S^\infty$ such that all the letters belong to E. In particular, we have $\gamma_1, \gamma_2 \in \mathrm{Geo}(A, S_{\mathrm{F}}^{\pm})$. By Theorem 5.1.10, γ_1 and γ_2 yield the same Busemann points in $(A, S_{\mathrm{F}}^{\pm})$. By Lemma 5.2.7, there exists $\gamma \in \mathrm{Geo}(A, S_{\mathrm{F}}^{\pm})$ intersecting with γ_1, γ_2 infinitely many times. Necessarily $\gamma \in S_F^\infty$. Since $\mathrm{Pr}(S_{\mathrm{F}}) \subset \mathrm{F}$, γ_1, γ_2 and γ are also geodesic rays in (H, S). By Lemma 5.2.7 again, we have $b_{\gamma_1} = b_{\gamma_2}$ in $\partial(H, d_S)$.

(b2) Consider $\gamma, \eta \in \mathrm{Geo}(H, S)$ such that $\mathrm{F} = \mathrm{F}_\gamma = \mathrm{F}_\eta$ is a non-commutative face and *all* letters of γ, η belong to S_{F}. We prove that $b_\gamma = b_\eta$. More specifically, we fix a prefix η_m of η, and attempt to extend it into a longer geodesic joining the path γ. Fix n such that all occurrences of letters of $S_{\mathrm{F}} \setminus D_\gamma$ in $\gamma = \gamma_n \omega_n$ belong to γ_n.

Observe that $\langle D_\gamma \rangle$ is a *finite-index* subgroup of $\langle S_{\mathrm{F}} \rangle$. Indeed, we have

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \langle D_\gamma \rangle \cap \langle z \rangle & \hookrightarrow & \langle D_\gamma \rangle & \xrightarrow{\mathrm{Pr}} & \mathrm{Pr}\,\langle D_\gamma \rangle & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle (1)} & & \big\downarrow & & \big\downarrow{\scriptstyle (2)} & & \\
1 & \longrightarrow & \langle S_{\mathrm{F}} \rangle \cap \langle z \rangle & \hookrightarrow & \langle S_{\mathrm{F}} \rangle & \xrightarrow{\mathrm{Pr}} & \mathrm{Pr}\,\langle S_{\mathrm{F}} \rangle & \longrightarrow & 1
\end{array}$$

where the inclusion (1) has finite-index as both are non-trivial subgroup of $\langle z \rangle$ (since the face F is non-commutative), and (2) has finite-index as both are lattices in the hyperplane of $\mathrm{Pr}(H) = V_1$ supporting F. Therefore, there exists $u_1 \in S_{\mathrm{F}}^\star$ such that $\bar{\gamma}_n^{-1} \cdot \overline{\eta_m u_1} \in \langle D_\gamma \rangle$. Using [Wal11, Lemma 4.2], there exist $u_2, v \in D_\gamma^\star$ such that

$$\overline{\eta_m u_1 \cdot u_2} = \overline{\gamma_n \cdot v}.$$

As every letter of D_γ appears infinitely many times in ω_n, there exists a prefix $w_1 \in D_\gamma^\star$ of ω_n (so $\gamma_p = \gamma_n w_1$ for $p > n$) containing all the letters of v. Then w_1 can be reordered as $w_1' = v u_3$ for some $u_3 \in D_\gamma^\star$, and $\bar{w}_1' = \bar{w}_1 g$ with $g \in \langle [s,t] : s, t \in D_\gamma \rangle$.

Using Lemma 5.3.6, we have $g^{-1} \in Z_{\omega_p}$, hence there exists a prefix $w_2 \in D_\gamma^\star$ of ω_p (so $\gamma_q = \gamma_p w_2$ for $q > p$) which can be reordered as w_2' with $\bar{w}_2' = \bar{w}_2 g^{-1}$.

To summarize, we have $\eta_m u_1 u_2 u_3 w_2' \in S_{\mathrm{F}}^\star$ a geodesic extension of η_m such that

$$\overline{\eta_m u_1 u_2 u_3 w_2'} = \overline{\gamma_n v u_3 w_2'} = \overline{\gamma_n w_1' w_2'} = \overline{\gamma_n w_1 w_2} = \bar{\gamma}_q.$$

This can be repeated, extending γ_q to join η, hence $b_\gamma = b_\eta$ by Lemma 5.2.7.

(c) follows from parts (a) and (b), together with Lemma 5.1.6(b). □

5.4 Reduced boundaries of 2-step nilpotent groups

The main objective of this section is Theorem 5.B, which we recall here:

Theorem 5.4.1. *Let G be a finitely generated 2-step nilpotent group, and S a finite generating set. Then the action $G \curvearrowright \partial^r(G, d_S)$ is trivial.*

Let us make an important observation: we have roughly isometric homomorphisms

$$(G, d_S) \twoheadrightarrow (G/T, d_S) \hookrightarrow (\overline{G/T}, d).$$

with $T = I_G(\{e\})$ the (finite!) torsion subgroup of G. Here, d either stands for the Stoll metric $d = d_{\text{Stoll},S}$ (Theorem 2.4.12), or for the Pansu limit metric $d = d_{\mathsf{CC}}$ when $\overline{G/T}$ is ideal (Theorem 2.4.27). Using Proposition 5.2.1, this allows to identify

$$\partial^r(G, d_S) \xleftarrow{\sim} \partial^r(G/T, d_S) \xleftarrow{\sim} \partial^r(\overline{G/T}, d).$$

Since everything is G-equivariant, we only have to prove that $G \curvearrowright \partial^r(\overline{G/T}, d)$ is trivial.

5.4.1 The Heisenberg group

We observe that work of Fisher and Nicolussi Golo on $(H_3(\mathbb{R}), d_{\mathsf{CC}})$ [FNG21] translates easily in the discrete case, where we recover old and new results.

We identify $H_3(\mathbb{R})$ with its Lie algebra $V_1 \oplus V_2$, with $V_1 \simeq \mathbb{R}^2$ and $V_2 \simeq \mathbb{R}$. Recall $\mathrm{Pr}\colon H_3(\mathbb{R}) \to V_1$ is the abelianization map. Moreover the group operation is given by

$$(a, b, c) \cdot (a', b', c') = \left(a + a',\ b + b',\ c + c' + \frac{1}{2}(ab' - a'b)\right).$$

Given a finite symmetric set $S \subset H_3(\mathbb{R})$ generating a lattice $H = \langle S \rangle$, we define

- $\mathsf{s}_1, \ldots, \mathsf{s}_{2N}$ the vertices of $\mathrm{P} \coloneqq \mathrm{ConvHull}(\mathrm{Pr}(S))$, numbered counterclockwise. We take the convention that $\mathsf{s}_0 = \mathsf{s}_{2N}$.
- $e_k \coloneqq \mathsf{s}_k - \mathsf{s}_{k-1}$.
- $\omega\colon V_1 \times V_1 \to \mathbb{R}$ defined by $\omega\big((a, b), (a', b')\big) = a'b - ab'$.
- $\alpha_k(\mathbf{v}) = \frac{\omega(e_k, \mathbf{v})}{\omega(e_k, \mathsf{s}_k)}$ for $\mathbf{v} \in V_1$.

Fisher and Nicolussi Golo characterize all the horofunctions of $(H_3(\mathbb{R}), d_{\mathsf{CC}})$. Up to bounded functions, their result can be re-stated as

Theorem 5.4.2 ([FNG21, Theorem 5.2])**.** *The horofunctions of a polygonal sub-Finsler Heisenberg group $(H_3(\mathbb{R}), d_{\mathsf{CC}})$ are, up to bounded functions, classified as follows:*

(Vertical) $\varphi(\mathbf{v}, c) = -\|\mathbf{v}\|_{\mathrm{Mink},\mathrm{P}} \coloneqq \min\{\lambda \geqslant 0 \mid \mathbf{v} \in \lambda \mathrm{P}\}$,

(Non-vertical) $\varphi(\mathbf{v}, c) = r\alpha_k(\mathbf{v}) + (1 - r)\alpha_{k-1}(\mathbf{v})$ *for some* $r \in [0, 1]$,

(Mixed)

$$\varphi(\mathbf{v},c)=\begin{cases}\alpha_i(\mathbf{v}) & \textit{if}\ \ \omega(\mathsf{s}_i,\mathbf{v})\leqslant 0\ \ (\textit{resp.}\ \ \geqslant 0),\\ r\alpha_i(\mathbf{v})+(1-r)\alpha_{i-1}(\mathbf{v}) & \textit{if}\ \ \omega(\mathsf{s}_i,\mathbf{v})\geqslant 0\ \ (\textit{resp.}\ \ \leqslant 0),\end{cases}$$

or

$$\varphi(\mathbf{v},c)=\begin{cases}\alpha_{i-1}(\mathbf{v}) & \textit{if}\ \ \omega(\mathsf{s}_i,\mathbf{v})\leqslant 0\ \ (\textit{resp.}\ \ \geqslant 0),\\ r\alpha_i(\mathbf{v})+(1-r)\alpha_{i-1}(\mathbf{v}) & \textit{if}\ \ \omega(\mathsf{s}_i,\mathbf{v})\geqslant 0\ \ (\textit{resp.}\ \ \leqslant 0),\end{cases}$$

for some $r\in[0,1]$.

Proposition 5.4.3 ([FNG21, Proposition 6.4])**.** $H_3(\mathbb{R})\curvearrowright\partial^r(H_3(\mathbb{R}),d_{\mathsf{CC}})$ *is trivial.*

Combined with Proposition 5.2.1, we get

Corollary 5.4.4. *Consider a lattice* $H\leqslant H_3(\mathbb{R})$ *with a generating set* S

(a) *The action* $H\curvearrowright\partial^r(H,d_S)$ *is trivial.*

(b) $\partial^r(H,d_S)$ *(hence* $\partial(H,d_S)$*) has the cardinality of the continuum.*

Part (a) is a slight generalization of [BF20, Theorem B].

Part (b) comes in stark contrast with Theorem 5.1.7: on abelian groups, every horofunction is a Busemann point. As proven in Theorem 5.3.3, $\partial(H,d_S)$ contains only countably many Busemann points, hence most horofunctions are non-Busemann.

5.4.2 General case

We prove the following result in the "polytopal sub-Finsler setting". Combined with observations at the beginning of the section, this implies Theorem 5.B.

Theorem 5.4.5. *Let* Γ *be a simply connected* 2*-step nilpotent Lie group, and* S *a finite Lie generating set. Then the action* $\Gamma\curvearrowright\partial^r(\Gamma,d_{\mathrm{Stoll},S})$ *is trivial.*

We start with a lemma from linear algebra

Lemma 5.4.6. *Let* (V,d_E) *be an Euclidean vector space,* $\mathbf{U}\leqslant V$ *a vector subspace, and* $S\subset V$ *a finite (multi)set. Let* $\mathbf{u}\in\mathbf{U}$ *and* $\mathbf{v}\in V$ *satisfying*

$$d_E\big(\mathbf{u},\mathrm{Vect}(S')\big)\leqslant 2d_E\big(\mathbf{v},\mathrm{Vect}(S')\big)$$

for all $S'\subseteq S$ *such that* $\mathrm{Vect}(S')\not\geqslant\mathbf{U}$*. Suppose moreover* $\mathbf{v}=\sum_{\mathbf{s}\in S}\lambda_{\mathbf{s}}\cdot\mathbf{s}$ *with* $\lambda_{\mathbf{s}}\geqslant 0$*. Then there exist* $\alpha_{\mathbf{s}}\in[-C,C]$ *such that*

$$\mathbf{u}=\sum_{\mathbf{s}\in S}\alpha_{\mathbf{s}}\lambda_{\mathbf{s}}\cdot\mathbf{s}.$$

where $C=C(S)>0$ *only depends on* S*.*

Proof. The proof splits in two steps:

(1) We construct inductively a subset $B \subset S$ which freely generates $\mathrm{Vect}(B) \geqslant \mathbf{U}$, such that $\lambda_{\mathbf{s}}$ is "large" for every $\mathbf{s} \in B$.

(2) We estimate the coefficients of $\mathbf{u}$ in the basis B.

(1) We start with $B_0 = \emptyset$. Suppose B_{i-1} is defined, and $\mathrm{Vect}(B_{i-1}) \ngeqslant \mathbf{U}$. We have

$$d_E\big(\mathbf{v}, \mathrm{Vect}(B_{i-1})\big) \geqslant \frac{1}{2} d_E\big(\mathbf{u}, \mathrm{Vect}(B_{i-1})\big)$$

Therefore, there exists $\mathbf{s}_i \in S$ such that

$$d_E\big(\lambda_{\mathbf{s}_i}\mathbf{s}_i, \mathrm{Vect}(B_{i-1})\big) \geqslant \frac{1}{2\,|S|} d_E\big(\mathbf{u}, \mathrm{Vect}(B_{i-1})\big).$$

Let $B_i = B_{i-1} \cup \{\mathbf{s}_i\}$. We iterate until $\mathrm{Vect}(B_d) \geqslant \mathbf{U}$, and declare $B = B_d$.

(2) Let $\mathrm{H}_i = \mathrm{Vect}(B \setminus \{\mathbf{s}_i\})$. We define $\alpha_{\mathbf{s}_i} \in \mathbb{R}$ such that $\mathbf{u} \in \alpha_{\mathbf{s}_i} \cdot \lambda_{\mathbf{s}_i}\mathbf{s}_i + \mathrm{H}_i$ (and $\alpha_{\mathbf{s}} = 0$ if $\mathbf{s} \notin B$). We obviously have $\mathbf{u} = \sum_{\mathbf{s}\in S} \alpha_{\mathbf{s}}\lambda_{\mathbf{s}} \cdot \mathbf{s}$. Moreover,

$$|\alpha_{\mathbf{s}_i}| = \frac{d_E(\mathbf{u}, \mathrm{H}_i)}{d_E(\lambda_{\mathbf{s}_i}\mathbf{s}_i, \mathrm{H}_i)} = \frac{d_E(\mathbf{u}, \mathrm{H}_i)}{d_E(\lambda_{\mathbf{s}_i}\mathbf{s}_i, \mathrm{Vect}(B_{i-1}))} \cdot \frac{d_E(\mathbf{s}_i, \mathrm{Vect}(B_{i-1}))}{d_E(\mathbf{s}_i, \mathrm{H}_i)}.$$

The first factor is bounded by $2\,|S|$ since

$$d_E(\mathbf{u}, \mathrm{H}_i) \leqslant d_E(\mathbf{u}, \mathrm{Vect}(B_{i-1})) \leqslant 2\,|S| \cdot d_E(\lambda_{\mathbf{s}_i}\mathbf{s}_i, \mathrm{Vect}(B_{i-1})).$$

The second factor takes only finitely many values for a fixed S. □

Proof of Theorem 5.4.5. We decompose $\Gamma = V_1 \oplus V_2$, with $\mathrm{Pr}\colon \Gamma \to V_1$ the abelianization map. We consider $d = d_{\mathrm{Stoll},S}$ on Γ, and d_E an Euclidean metric on V_1.

Let $\varphi = \lim \varphi_{g_n}$ an horofunction on (Γ, d). Among vector subspaces $\mathbf{W} \leqslant V_1$ such that

$$\liminf_n d_E(\mathrm{Pr}(g_n), \mathbf{W}) < \infty,$$

we consider $\mathbf{U}$ a minimal subspace satisfying the property. Up to taking a subsequence, we may suppose $d_E(\mathrm{Pr}(g_n), \mathrm{U}) \leqslant D$ for all n. Observe that, if $\mathbf{W} \ngeqslant \mathbf{U}$, then

$$\liminf_n d_E(\mathrm{Pr}(g_n), \mathbf{W}) = \infty$$

(otherwise $\liminf d_E(\mathrm{Pr}(g_n), \mathbf{U} \cap \mathbf{W}) < \infty$, contradicting the minimality of $\mathbf{U}$). We decompose $\mathrm{Pr}(g_n) = \mathbf{u}_n + \boldsymbol{\varepsilon}_n$ with $\mathbf{u}_n \in \mathbf{U}$ and $\boldsymbol{\varepsilon}_n \in \mathrm{U}^\perp$. For $h, x \in \Gamma$, we have

$$\begin{aligned} |\varphi_{g_n}(x) - h \cdot \varphi_{g_n}(x)| &= \Big|\big(\,\|x^{-1}g_n\| - \|g_n\|\,\big) - \big(\,\|x^{-1}hg_n\| - \|hg_n\|\,\big)\Big| \\ &\leqslant \Big|\,\|x^{-1}g_n\| - \|x^{-1}hg_n\|\,\Big| + \Big|\,\|hg_n\| - \|g_n\|\,\Big| \\ &\leqslant \Big|\,\|x^{-1}g_n\| - \|x^{-1}hg_n\|\,\Big| + \|h\|\,. \end{aligned}$$

Since $x^{-1}hg_n = x^{-1}g_n[h,g_n]h = x^{-1}g_n[h,\mathbf{u}_n]\cdot[h,\boldsymbol{\varepsilon}_n]h$, we may continue with

$$\begin{aligned} &\leqslant \Big| \left\|x^{-1}g_n\right\| - \left\|x^{-1}g_n[h,\mathbf{u}_n]\right\| \Big| + \left\|[h,\boldsymbol{\varepsilon}_n]\right\| + 2\left\|h\right\| \\ &\leqslant \Big| \left\|x^{-1}g_n\right\| - \left\|x^{-1}g_n[h,\mathbf{u}_n]\right\| \Big| + 4\left\|h\right\| + 2\left\|\boldsymbol{\varepsilon}_n\right\|_{\mathrm{Mink},\mathrm{Pr}(S)}. \end{aligned}$$

Observe that $\|\boldsymbol{\varepsilon}_n\|_E \leqslant D$, hence $\|\boldsymbol{\varepsilon}_n\|_{\mathrm{Mink}} = O(D)$ as norms on finite-dimensional vector spaces are bi-Lipschitz. It only remains the deal with the first term.

We fix n large enough so that, for each $R \subset S$ such that $\mathrm{Vect}(\mathrm{Pr}(R)) \not\geqslant \mathbf{U}$, we have

$$d_E\big(\mathrm{Pr}(g_n), \mathrm{Vect}(\mathrm{Pr}(R))\big) \geqslant 2d_E\big(\mathrm{Pr}(x), \mathrm{Vect}(\mathrm{Pr}(R))\big) + D. \tag{5.4.7}$$

Using Lemma 2.4.11, there exists a geodesic $\mathbb{R}$-word $w_n = s_1^{\lambda_1}\dots s_k^{\lambda_k} \in S_{\mathbb{R}}^*$ representing $x^{-1}g_n$ with $k \leqslant K$. Using Lemma 5.4.6 for $\mathbf{u} = \mathbf{u}_n$, $\mathbf{v} = \mathbf{u}_n + \boldsymbol{\varepsilon}_n - \mathrm{Pr}(x)$, there exist $\alpha_1,\dots,\alpha_k$ such that $|\alpha_i| \leqslant C = C(KS)$ and $\mathrm{Pr}(s_1^{\alpha_1\lambda_1}\dots s_k^{\alpha_k\lambda_k}) = \mathbf{u}_n$, therefore

$$x^{-1}g_n[h,\mathbf{u}_n] = h^{\alpha_1}s_1^{\lambda_1}h^{-\alpha_1}\dots h^{\alpha_k}s_k^{\lambda_k}h^{-\alpha_k},$$

proving that $\|x^{-1}g_n[h,\mathbf{u}_n]\| \leqslant \|x^{-1}g_n\| + 2CK\left\|\mathrm{Pr}(h)\right\|_{\mathrm{Mink},\mathrm{Pr}(S)}$. The same argument shows that $\|x^{-1}g_n\| \leqslant \|x^{-1}g_n[h,\mathbf{u}_n]\| + 2CK\left\|\mathrm{Pr}(h)\right\|_{\mathrm{Mink},\mathrm{Pr}(S)}$. We conclude

$$|\varphi_{g_n}(x) - h\cdot\varphi_{g_n}(x)| \leqslant (2CK+4)\left\|h\right\| + O_S(D) = O_S(\|h\|) + O_{S,\varphi}(1).$$

□

Remark 5.4.8. The result cannot be extended to virtually nilpotent groups. For instance, for $D_\infty = \langle s,t \mid s^2 = t^2 = e\rangle$, the horoboundary $\partial(D_\infty, d_S)$ consists of exactly two Busemann points $b_{(st)^\infty}$ and $b_{(ts)^\infty}$. We have $t\cdot b_{(st)^\infty} = b_{(ts)^\infty} = -b_{(st)^\infty}$.

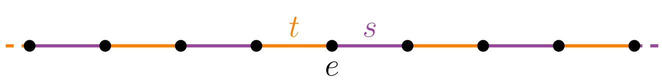


Figure 5.1: The Cayley graph $\mathcal{C}ay(D_\infty, \{s,t\})$.

However, if G contains a finite-index, torsionfree, 2-step nilpotent subgroup H, then we have two roughly isometric homomorphisms (Proposition 4.1.1)

$$(G,d_S) \longleftarrow (H, d_{X,\omega}) \longrightarrow (\bar{H}, d_{\mathrm{Stoll},X,\omega}),$$

so Proposition 5.2.1 allows to identify

$$\partial^r(G,d_S) \xrightarrow{\sim} \partial^r(H,d_{X,\omega}) \xleftarrow{\sim} \partial^r(\bar{H}, d_{\mathrm{Stoll},X,\omega}).$$

As everything is H-equivariant, we conclude that $H \curvearrowright \partial^r(G,d_S)$ is trivial, so the action $G \curvearrowright \partial^r(G,d_S)$ factors through a finite group $G/\langle\!\langle H\rangle\!\rangle_G$.

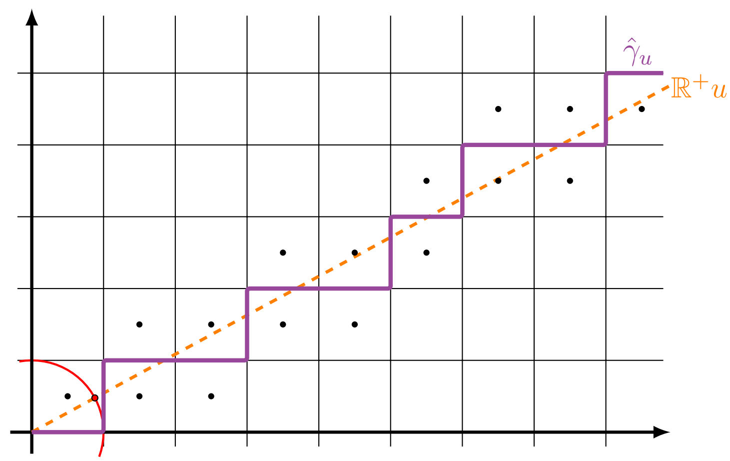
5.5 Busemann points on the Cartan group

In this section, we consider the lattice $\mathscr{C}$ in the Cartan group (see Section 3.1) generated by the segments x and y from $\mathbf{0}$ to $(1,0)$ and $(0,1)$ respectively.

5.5.1 Construction

We are going to define a Busemann point γ_u for each direction $u \in \mathbb{S}^1$ in the plane. Directions can also be parametrized by equivalence classes $[a\!:\!b] \in \big(\mathbb{R}^2 \setminus \{(0,0)\}\big)\big/\mathbb{R}^+$.

Note that, in order to define an infinite path $\gamma = (\gamma_n)_{n\geqslant 0}$, we only need to fix $\gamma_0 \in \mathscr{C}$, and its projection $\hat\gamma = (\hat\gamma_n)_{n\geqslant 0} \in \mathbb{Z}^2 \simeq \mathscr{C}/[\mathscr{C},\mathscr{C}]$. The curve γ is the unique lift. In this case, we take $\gamma_{u,0} = e$, and $(\hat\gamma_{u,n})_{n\geqslant 0}$ which best approximates the ray $\mathbb{R}^+u$. Precisely, unit squares in the grid fall into two categories, depending whether the ray passes above or below their center. $\hat\gamma_u$ is the boundary of the two regions formed.



Whenever $u = [a\!:\!b]$ with a, b odd integers, the ray passes through the center of infinitely many squares. We choose that $\hat\gamma_u$ alternates above and below these squares.

Note that $\hat\gamma_u$ is geodesic in $\mathbb{Z}^2$, so γ_u is an infinite geodesic in $\mathscr{C}$, hence defines a Busemann point in $\partial(\mathscr{C}, d_S)$. We will see these points are distinct in $\partial^r(\mathscr{C}, d_S)$.

5.5.2 Computations of lengths

We first prove a lower bound on the length of certain elements.

Lemma 5.5.1. *If $g \in \mathscr{C}$ such that $\hat g = \hat\gamma_{u,n}$ and $\|g\|_S = n + \Delta$, then*

$$\left\langle B(g); u^\perp \right\rangle \leqslant \left\langle B(\gamma_{u,n}); u^\perp \right\rangle + O(\Delta^3)$$

where $u^\perp$ is the image of u under a rotation of $+90°$. As a corollary, we have that

$$\|h\gamma_{u,n}\|_S = n + \Omega\Big(\sqrt[3]{\langle B(h); u^\perp\rangle}\ \Big)$$

for $h \in [\mathscr{C},\mathscr{C}]$ with $\langle B(h); u^\perp\rangle > 0$.

Remark 5.5.2. If $\langle B(h); u^\perp\rangle \leqslant 0$, we only get the (trivial) inequality $\|h\gamma_{u,n}\| \geqslant n$.

The proof follows the same scheme as Proposition 4.3.1.

Proof. We first explain how the first inequality implies the corollary. Take $g = h\gamma_{u,n}$. As $h \in [\mathscr{C}, \mathscr{C}]$, we indeed have $\hat{g} = \hat{\gamma}_{u,n}$ and

$$B(h\gamma_{u,n}) = B(h) + B(\gamma_{u,n})$$

so, if we define $\Delta = \|g\|_S - n$, we have that

$$\langle B(h); u^\perp\rangle \leqslant O(\Delta^3) \iff \Delta = \Omega\Big(\sqrt[3]{\langle B(h); u^\perp\rangle}\Big).$$

► Now, the main inequality! As $\hat{g} = \hat{\gamma}_{u,n}$, we have $B(g) - B(\gamma_{u,n}) = B(gx) - B(\gamma_{u,n}x)$ for all $x \in \mathscr{C}$, so we are going to estimate $\langle B(g\delta); u^\perp\rangle$ instead of $\langle B(g); u^\perp\rangle$, where δ is the interval of γ_u between $\hat{\gamma}_{u,n}$ and the next intersection with the ray $\mathbb{R}^+u$.

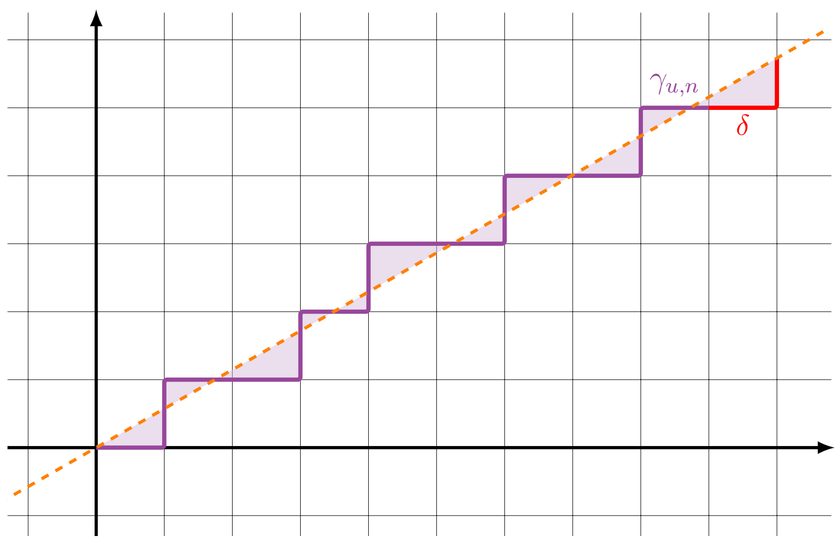


Figure 5.2: The path $\delta_{u,n}\delta$

Fix a geodesic path representing g. We decompose the concatenation $g\delta$ as follows: we first remove all loops, and draw them in blue. We also cut the parts until the last crossing of $\mathbb{R}^-u$, and from the first crossing of "$\mathbb{R}^+u$ after $\widehat{g\delta}$". The remaining is the "main part", drawn in green. The green simple curve and the blue multi-curve both define a winding number distribution (see Figure 5.3), hence both of them have a well-defined (non-normalized) barycenter. Moreover, we have

$$B(g\delta) = B(\text{green area}) + B(\text{blue area}),$$

hence we can estimate both contributions to $\langle B(g\delta); u^\perp\rangle$ separately.

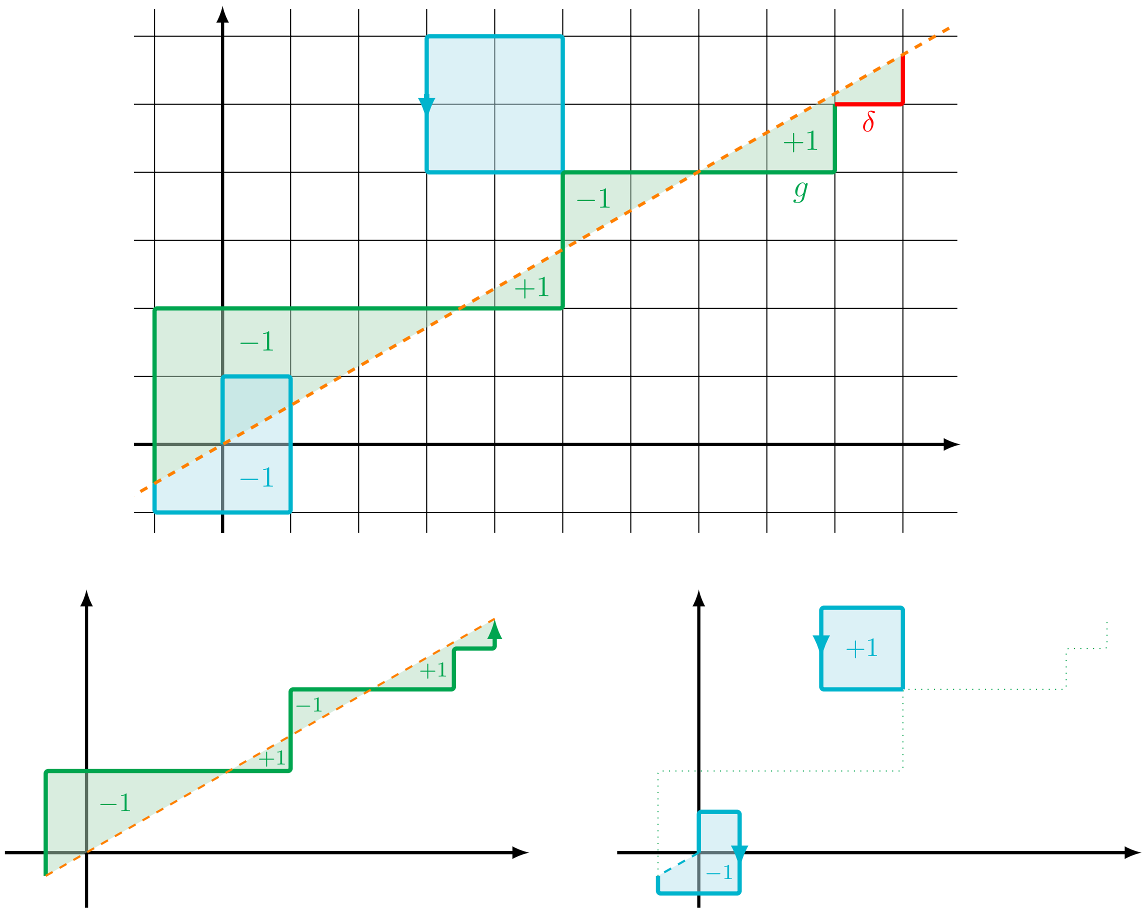


Figure 5.3: Decomposition of geodesic representing g

► Let's first suppose the entire path $g\delta$ stays within a 2Δ-neighborhood of $\mathbb{R}u$.

First observe that, by construction, both $\gamma_{u,n}\delta$ and the green path are simple curves. Their winding numbers (after closing using $\mathbb{R}u$) are ± 1 or 0. More specifically, $+1$'s only appear below $\mathbb{R}u$, and -1's only appear above $\mathbb{R}u$, so the contribution of each region between $g\delta$ (or $\gamma_{u,n}\delta$) and $\mathbb{R}u$ to $\langle B; u^\perp\rangle$ is negative. Moreover, for each square cut in two by $\mathbb{R}u$, one of the two halves has to have non-zero winding number. Now we realize that $\gamma_{u,n}\delta$ is defined so that $\langle B(\gamma_{u,n}\delta); u^\perp\rangle$ is closest to 0 among simple curves reaching the same endpoint, we have

$$\left\langle B(\text{green area}); u^\perp \right\rangle \leqslant \left\langle B(\gamma_{u,n}\delta); u^\perp \right\rangle < 0.$$

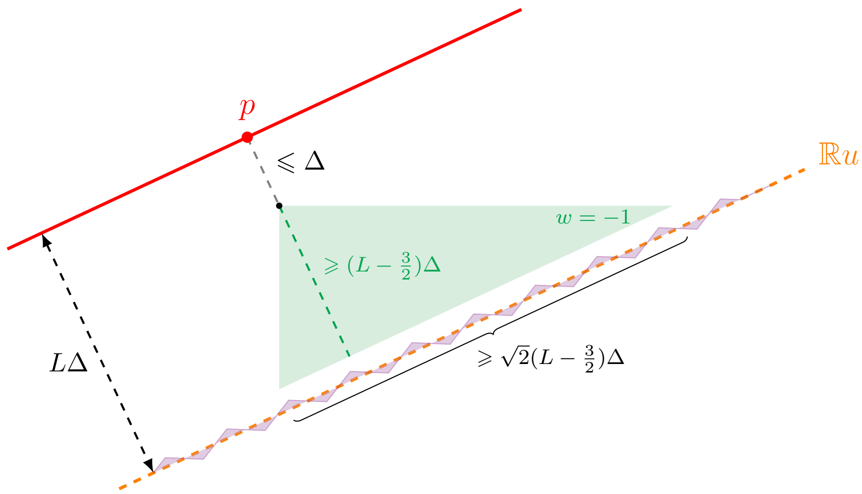
On the other hand, for the blue area, the length of the blue loops is bounded by 2Δ. It follows that the total blue area is bounded by $I(2\Delta)^2$ for some isoperimetric constant $I > 0$ (for the grid, $I = \frac{1}{16}$). As all this area lies within 2Δ from $\mathbb{R}u$, we get that

$$\left\langle B(\text{blue area}); u^\perp \right\rangle \leqslant 8I\Delta^3$$

and finally $\left\langle B(g\delta); u^\perp \right\rangle \leqslant \left\langle B(\gamma_{u,n}\delta); u^\perp \right\rangle + 8I\Delta^3$.

► Let's suppose the furthest $g\delta$ gets from $\mathbb{R}u$ is exactly $L\Delta$, for some $L \geqslant 2$.

As before, the blue contribution is bounded by $\langle B(\text{blue area}); u^\perp \rangle \leqslant 4LI\Delta^3$. On the other hand, the curve $g\delta$ goes through a point p at distance $L\Delta$ from $\mathbb{R}u$. This forces the green area to contain a large triangle, disjoint from the forced half-cut squares.



It follows that

$$\begin{aligned}\left\langle B(\text{green area}); u^\perp) \right\rangle &\leqslant \left\langle B(\gamma_{u,n}\delta); u^\perp \right\rangle + \left\langle B(\text{green triangle}); u^\perp \right\rangle \\ &\leqslant \left\langle B(\gamma_{u,n}\delta); u^\perp \right\rangle - \frac{\sqrt{2}}{6}\left(L - \frac{3}{2}\right)^3 \Delta^3\end{aligned}$$

hence $\langle B(g\delta); u^\perp \rangle \leqslant \langle B(\gamma_{u,n}\delta); u^\perp \rangle + \left(4LI - \frac{\sqrt{2}}{6}\left(L - \frac{3}{2}\right)^3\right)\Delta^3$.

► Finally, we observe that $L \mapsto 4LI - \frac{\sqrt{2}}{6}\left(L - \frac{3}{2}\right)^3$ is eventually decreasing, so

$$\left\langle B(g\delta); u^\perp \right\rangle \leqslant \left\langle B(\gamma_{u,n}\delta); u^\perp \right\rangle + M\Delta^3$$

where $M = \max\left\{8I,\ 4LI - \frac{\sqrt{2}}{6}\left(L - \frac{3}{2}\right)^3 \mid L \geqslant 2\right\}$. □

Next we give an upper bound on the length of specific elements.

Lemma 5.5.3. *For $h \in [\mathscr{C}, \mathscr{C}]$ and n large enough (depending on h), we have*

$$\|h\gamma_{u,n}\|_S = \begin{cases} n + O\left(\sqrt[3]{\langle B(h); u^\perp \rangle + |A(h)|}\right) + O(1) & \text{if } \left\langle B(h); u^\perp \right\rangle \geqslant 0, \\ n + O\left(\sqrt[3]{|A(h)|}\right) + O(1) & \text{if } \left\langle B(h); u^\perp \right\rangle \leqslant 0. \end{cases}$$

*If moreover u is **not** of the form $[a:b]$ for integers a, b such that a xor b is even, then we can improve the estimates and eliminate the dependence on $|A(h)|$:*

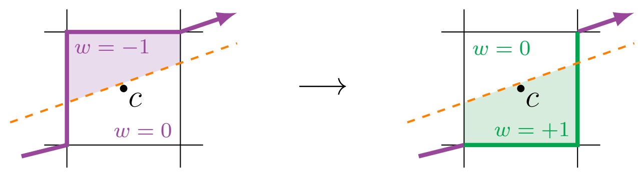
$$\|h\gamma_{u,n}\|_S = \begin{cases} n + O\left(\sqrt[3]{\langle B(h); u^\perp \rangle}\right) + O(1) & \text{if } \left\langle B(h); u^\perp \right\rangle \geqslant 0, \\ n + O(1) & \text{if } \left\langle B(h); u^\perp \right\rangle \leqslant 0. \end{cases}$$

Proof. We are looking for a short path representing $h\gamma_{u,n}$. We modify the path $\gamma_{u,n}$ (with n large) to change the values of A and B from $A(\gamma_{u,n})$ and $B(\gamma_{u,n})$ to

$$A(h\gamma_{u,n}) = A(h) + A(\gamma_{u,n}) \quad \text{and} \quad B(h\gamma_{u,n}) = B(h) + B(\gamma_{u,n}).$$

We fine tune the parameters A, $\langle B; u\rangle$ and $\langle B; u^\perp\rangle$ in three steps at the appropriate cost under the different hypothesis. The different operations are local, hence the endpoint $\hat\gamma_{u,n}$ remains unchanged all along. We control how the length ℓ of the considered path evolves after each step. Initially, $\ell = \|\gamma_{u,n}\|_S = n$.

► First, we modify A by exactly p ($p \in \mathbb{Z}$) at a cost $O(1)$. Let's first suppose $u \neq (\pm1, 0), (0, \pm1)$. Suppose $p > 0$. We find p squares which intersect the ray $\mathbb{R}u$, around which $\hat\gamma_u$ goes clockwise.



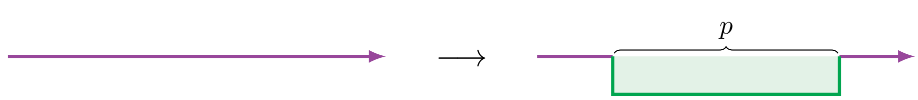
Each time we flip a unit square (from clockwise to counter-clockwise), we add 1 to the winding numbers on the full square, so the parameters change by

$$\begin{aligned} A &\to A + 1, \\ B &\to B + c \end{aligned}$$

The distance from c to $\mathbb{R}u$ being at most $\sqrt{2}/2$, the component $\langle B; u^\perp\rangle$ changes by at most $\sqrt{2}/2$. On the other hand $\langle B; u\rangle$ changes essentially by $\|c\|_E$. After flipping all p squares, the total effect of this operation is

$$\begin{aligned} A &\to A + p, \\ \langle B; u\rangle &\to \langle B; u\rangle + O(p^2) \\ \langle B; u^\perp\rangle &\to \langle B; u^\perp\rangle + O(p), \\ \ell &\to \ell. \end{aligned}$$

(If we take the first p squares along the line around which $\hat\gamma_u$ goes clockwise, that $\|c\|_E = O(p)$ for each square, which explains why $\langle B; u\rangle \to \langle B; u\rangle + O(p^2)$. As we will see later, the size of this change actually doesn't matter.) If $u = (\pm1, 0), (0, \pm1)$, this doesn't quite work, we instead do the following change:



with the same effect except $\ell \to \ell + 2$.

Under the hypothesis that u is not of the form $[a:b]$ with a,b integers such that a xor b is even, we can do better. We modify A by exactly p, at cost 0, and only modifying $\langle B; u^\perp\rangle$ by $O(1)$ (and not $O(p)$ as before).

Thanks to the hypothesis, we can find unit squares with center c either on the ray $\mathbb{R}^+u$ (if $u=[a:b]$ with a,b odd integers), or at distance at most $\frac{1}{|p|}$ from the ray (for irrational slopes). Then, as before, we can select $|p|$ of them to flip from clockwise to counter-clockwise (if $p>0$) or the other way around (if $p<0$). After flipping all p squares, the total effect of this operation is

$$\begin{aligned} A &\to A+p, \\ \langle B;u\rangle &\to \langle B;u\rangle + O_u(p^3) \\ \langle B;u^\perp\rangle &\to \langle B;u^\perp\rangle + O(1), \\ \ell &\to \ell. \end{aligned}$$

(The variation in $\langle B;u\rangle$ may be huge, but this is nothing to worry about.[1])

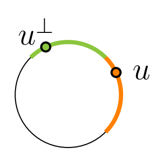
▶ Next we change $\langle B;u^\perp\rangle$ by approximately q at the appropriate cost.

⋆ If $q>0$, we can find $b\in\mathbb{Z}^2$ such that $\langle b;u^\perp\rangle = q+O(1)$ and $\langle b;u\rangle = O(1)$ (hence $\|b\|_E = q+O(1)$), and $z\in[\mathscr{C},[\mathscr{C},\mathscr{C}]]$ such that $B(z)=b$.

Because $[\mathscr{C},[\mathscr{C},\mathscr{C}]]$ is cubically distorted inside $\mathscr{C}$, we have $\|z\|_S = O(\sqrt[3]{q})$. (See for instance [DK18, Corollary 14.16].) So we can find a loop of this length evaluating to z, and then glue it at any point along γ_u. The total effect is

$$\begin{aligned} A &\to A \\ B &\to B+b \\ \langle B;u\rangle &\to \langle B;u\rangle + O(1) \\ \langle B;u^\perp\rangle &\to \langle B;u^\perp\rangle + q + O(1) \\ \ell &\to \ell + O(\sqrt[3]{q}) \end{aligned}$$

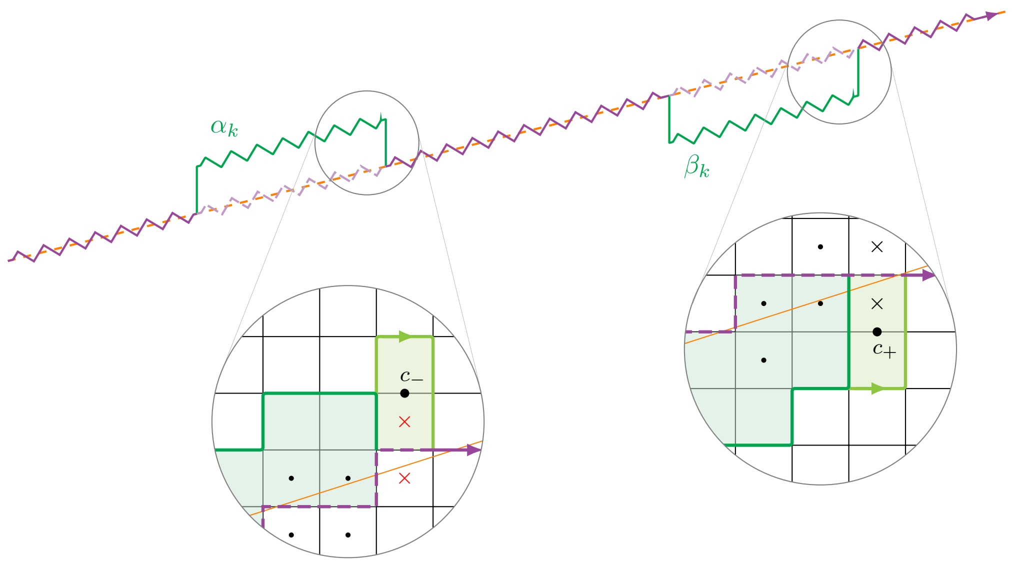
⋆ If $q<0$. There exists $v\in\{(\pm1,0)(0,\pm1)\}$ s.t. $\langle v;u\rangle \geqslant \sqrt{2}/2$, without lost of generality let us assume that $v=(1,0)$. In particular we also have $\langle (0,1);u^\perp\rangle \geqslant \sqrt{2}/2$.



For n large, we can find two disjoint sub-strings α,β in the untouched part of the path $\hat{\gamma}_u$, with $|q|$ occurrences of the letter x each (because $\langle (1,0);u\rangle > 0$). We assume that both α,β start with an x.

[1] The subscript u indicates the implied constants in $O_u(p^3)$ might depend on u.

For $0 \leqslant k \leqslant |q|$, let α_k be the prefix of α which stop right after the k^{th} occurrence of x. By convention $\alpha_0 = \emptyset$. We define β_k similarly. The operation is the following: we replace the string α_k by $y^2\alpha_k y^{-2}$, and the string β_k by $y^{-2}\beta_k y^2$. We prove that, for an appropriate k, the value $\langle B; u^\perp\rangle$ changes by $q + O(1)$.



What happens when we replace k by $k+1$? The winding numbers inside two 1×2 rectangles change by -1 and $+1$ respectively. So the effect is

$$\begin{aligned} A_k &\to A_{k+1} = A_k + 2 - 2 \\ B_k &\to B_{k+1} = B_k + 2(c_+ - c_-) \end{aligned}$$

Recall that, by construction, the ray $\mathbb{R}^+u$ passes in between the two red crosses in the previous picture. In particular,

$$\frac{\sqrt{2}}{4} \leqslant \big\langle (0, 0.5); u^\perp \big\rangle \leqslant \big\langle c_-; u^\perp \big\rangle \leqslant \big\langle (0, 1.5); u^\perp \big\rangle \leqslant \frac{3}{2}$$

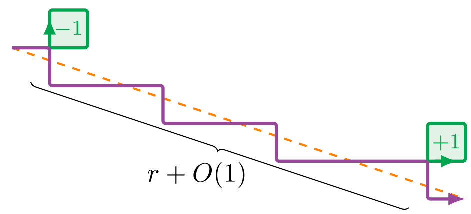
Similarly, we get $-\frac{3}{2} \leqslant \langle c_+; u^\perp\rangle \leqslant -\frac{\sqrt{2}}{4}$. Overall, this gives

$$-6 \leqslant \big\langle B_{k+1}; u^\perp \big\rangle - \big\langle B_k; u^\perp \big\rangle \leqslant -\sqrt{2} < -1$$

It follows that we can find k such that $\langle B_k; u^\perp\rangle - \langle B; u^\perp\rangle = \langle B_k; u^\perp\rangle - \langle B_0; u^\perp\rangle$ falls at distance at most 3 from q. The total effect is

$$\begin{aligned} A &\to A \\ \langle B; u\rangle &\to \langle B; u\rangle + O(q^2) \\ \langle B; u^\perp\rangle &\to \langle B_k; u^\perp\rangle = \langle B; u^\perp\rangle + q + O(1) \\ \ell &\to \ell + 8 \end{aligned}$$

▶ Finally we change $\langle B; u\rangle$ by approximately r at a cost $O(1)$. We glue two loops $[x, y]^{-1}$ and $[x, y]$ on $\hat{\gamma}_u$, at a distance approximately r from each other.



Specifically, if c_+ and c_- are the centers of the two square loops, we ensure that $\|(c_+ - c_-) - qu\|_E = O(1)$. The total effect of this operation is then

$$\begin{aligned} A &\to A, \\ B &\to B + (c_+ - c_-) \\ \langle B; u\rangle &\to \langle B; u\rangle + r + O(1), \\ \langle B; u^\perp\rangle &\to \langle B; u^\perp\rangle + O(1), \\ \ell &\to \ell + 8. \end{aligned}$$

▶ Applying the three steps (in that order), we transform the curve $(\hat\gamma_{u,k})_{0\leqslant k\leqslant n}$ with parameters $A(\gamma_{u,n})$ and $B(\gamma_{u,n})$ into another curve with parameters

$$A(h) + A(\gamma_{u,n}) + O(1) \quad \text{and} \quad B(h) + B(\gamma_{u,n}) + O(1).$$

For the second step, one need to take $q = \langle B(h); u^\perp\rangle + O(|A(h)|)$ in the general case, or $q = \langle B(h); u^\perp\rangle + O(1)$ under the extra hypothesis. This new curve has length $\ell = n + O(1) + O(\max\{\sqrt[3]{q}; 1\}) + O(1)$. Finally, we can fix the last $O(1)$ difference of the endpoint in $\mathscr{C}$ at an extra $O(1)$ cost. Overall, this gives the desired upper bound

$$\|h\gamma_{u,n}\|_S \leqslant \ell = n + O(\max\{\sqrt[3]{q}; 1\}). \qquad \square$$

5.5.3 Back to horofunctions

Theorem 5.5.4. *There exists $C_1, C_2 > 0$ such that, for all $h \in [\mathscr{C}, \mathscr{C}]$, we have*

$$\begin{aligned} C_1\sqrt[3]{\langle -B(h); u^\perp\rangle} \leqslant b_{\gamma_u}(h) &\leqslant C_2\sqrt[3]{\langle -B(h); u^\perp\rangle + |A(h)|} + C_2 && \text{if } \left\langle -B(h); u^\perp\right\rangle \geqslant 0, \\ 0 \leqslant b_{\gamma_u}(h) &\leqslant C_2\sqrt[3]{|A(h)|} + C_2 && \text{if } \left\langle -B(h); u^\perp\right\rangle \leqslant 0. \end{aligned}$$

*If moreover u is **not** of the form $[a:b]$ for integers a, b such that a xor b is even, then we can improve the estimates and remove the dependence on $A(h)$:*

$$b_{\gamma_u}(h) = \begin{cases} \Theta\left(\sqrt[3]{\langle -B(h); u^\perp\rangle}\right) & \text{if } \left\langle -B(h); u^\perp\right\rangle > 0, \\ O(1) & \text{if } \left\langle -B(h); u^\perp\right\rangle \leqslant 0. \end{cases}$$

Proof. This follows directly from Lemma 5.5.1 and 5.5.3, given that

$$b_{\gamma_u}(h) = \lim_{n\to\infty} d(\gamma_n, h) - d(\gamma_n, e) = \lim_{n\to\infty} \left\|h^{-1}\gamma_n\right\|_S - n$$

The only missing piece is $\|h^{-1}\gamma_{u,n}\|_S \geqslant n$, which is true because $h^{-1}\gamma_{u,n}$ has length n in the abelianization (same endpoint as $\gamma_{u,n}$). $\square$

Remark 5.5.5. This should be compared with the discrete Heisenberg group $H_3(\mathbb{Z})$, and more generally pairs (G,S) with "property EH" of [BF20], for which $\varphi(h)=O(1)$ for all horofunction $\varphi\in\partial(G,S)$ and all $h\in[G,G]$.

Finally, we are ready to prove our main theorem

Theorem 5.5.6. *All Busemann points $[b_{\gamma_u}]$ are distinct in the* reduced *boundary $\partial^r(\mathscr{C},d_S)$. Moreover, the stabilizer of $[b_{\gamma_u}]$ for the action $\mathscr{C}\curvearrowright\partial^r(\mathscr{C},d_S)$ satisfies*

$$\mathrm{Stab}_{\mathscr{C}}([b_{\gamma_u}])\leqslant\{g\in\mathscr{C}:\hat{g}\in\mathbb{R}u\}$$

(which is $[\mathscr{C},\mathscr{C}]$ for u irrational).

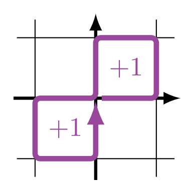
Proof. Consider two *distinct* directions $u,v\in\mathbb{S}^1$. There exists $b\in\mathbb{Z}^2$ such that $\langle -b;u^\perp\rangle>0$ and $\langle -b;v^\perp\rangle\leqslant 0$. Consider $h\in[\mathscr{C},[\mathscr{C},\mathscr{C}]]$ such that $B(h)=b$, then

$$b_{\gamma_u}(h^n)=\Theta(\sqrt[3]{n})$$
$$b_{\gamma_v}(h^n)=O(1).$$

In particular $b_{\gamma_u}-b_{\gamma_v}$ is not a bounded function, that is, $[b_{\gamma_u}]\neq[b_{\gamma_v}]$.

► Now let us have a look at $\mathrm{Stab}_{\mathscr{C}}([b_{\gamma_u}])$. Fix

- $g\in\mathscr{C}$ such that $\hat{g}\notin\mathbb{R}u$ or equivalently $\langle\hat{g};u^\perp\rangle\neq 0$
- $h\in[\mathscr{C},\mathscr{C}]$ such that $A(h)>0$ and $B(h)=0$, for instance $h=[x,y][x^{-1},y^{-1}]$.



Let $C_1,C_2>0$ be the constants in Theorem 5.5.4 such that

$$b_{\gamma_u}(*)\geqslant C_1\cdot\sqrt[3]{\langle -B(*);u^\perp\rangle}\qquad\text{if }\langle -B(*);u^\perp\rangle\geqslant 0,$$
$$b_{\gamma_u}(*)\leqslant C_2\cdot\sqrt[3]{|A(*)|}+C_2\qquad\text{if }\langle -B(*);u^\perp\rangle\leqslant 0.$$

Fix $m\in\mathbb{Z}$ such that $C_1\sqrt[3]{m\langle\hat{g};u^\perp\rangle}>C_2$. (In particular, $m\langle\hat{g};u^\perp\rangle>0$.) We can estimate the values of γ_u and $g^m\cdot\gamma_u$ at h^n:

$$b_{\gamma_u}(h^n)\leqslant C_2\cdot\sqrt[3]{nA(h)}+C_2$$

$$\begin{aligned}(g^m\cdot b_{\gamma_u})(h^n)&:=b_{\gamma_u}(g^{-m}h^n)-b_{\gamma_u}(g^{-m})\\&=b_{\gamma_u}\big(h^n[g^{-m},h^n]g^{-m}\big)+O_{g,u}(1)\\&=b_{\gamma_u}(h^n[g,h]^{-mn})+O_{g,u}(1)&(1)\\&\geqslant C_1\sqrt[3]{m\langle\hat{g};u^\perp\rangle}\cdot\sqrt[3]{nA(h)}+O_{g,u}(1)&(2)\end{aligned}$$

using that

(1) b_{γ_u} is 1-Lipschitz, we can extract the g^{-m} from the right at a cost of $\|g^m\|_S$.

(2) One can compute $B(h^n[g,h]^{-mn}) = -mnB([g,h]) = -mn\big(\hat{g}\cdot A(h)\big)$.

Letting $n \to +\infty$, we conclude that b_{γ_u} and $g^m \cdot b_{\gamma_u}$ are not at bounded distance apart. This means g^m doesn't fix $[b_{\gamma_u}]$, hence neither does g. □

Remark 5.5.7. The same computation shows that $\mathscr{E} \curvearrowright \partial^r(\mathscr{E}, d_S)$ is not trivial, for

$$\mathscr{E} = \big\langle x, y \;\big|\; [x,[x,y]] = [x,[y,[x,y]] = [y,[y,[x,y]] = e \big\rangle$$

the Engel group and $S = \{x^\pm, y^\pm\}$. Specifically, the orbit of $[b_{\gamma_u}]$ where $u = [1:0]$ (that is $[b_{x^\infty}]$) is infinite. However, it is not clear whether $\partial(\mathscr{E}, S)$ contains countably or uncountably many Busemann points.

Looking at larger groups, we get uncountably many Busemann points in many nilpotent groups, combining the previous result and Proposition 5.2.5.

Corollary 5.5.8. *Let $N_{r,c}$ be the free nilpotent group of rank $r \geqslant 2$ and step $c \geqslant 3$ with the standard generating set S. Then $\partial^r(N_{r,c}, d_S)$ contains uncountably many Buseman points, and the action $N_{r,c} \curvearrowright \partial^r(N_{r,c}, d_S)$ is non-trivial.*

Proof. Since any free nilpotent group of rank $r \geqslant 2$ and step $c \geqslant 3$ surjects onto the Cartan group $\mathscr{C}$, the uncountably many Busemann points constructed in $\mathscr{C}$ lift to uncountably many Busemann points in $N_{r,c}$, by Proposition 5.2.5. □

5.6 Further questions and remarks

In the spirit of the Ron-George–Yadin conjecture "$\partial(G, d_S)$ is finite if and only if G virtually cyclic", we propose the following conjecture:

Conjecture 5.A. Let G be a group and S a finite generating set. The horoboundary $\partial(G, d_S)$ is countable if and only if G is virtually abelian.

A motivation to look at these groups is another observation of Karlsson: if $\partial(G, d_S)$ is countable, then $G \curvearrowright \partial(G, d_S)$ admits a finite orbit [Kar08, Corollary 5].

The direction $\Leftarrow$ is part of work in progress with Kenshiro Tashiro. It is not clear how to tackle the direction $\Rightarrow$; even the case of 2-step nilpotent groups is not completely settled. If $\bar{G}$ is ideal, we can reduce the problem to $\partial^r(\bar{G}, d_{\mathsf{CC}})$ (see Section 5.4) and use the description of horofunctions on Carnot groups via Pansu derivatives [FNG21]. For groups whose Mal'cev closure is not ideal, this approach would require a better understanding of horofunctions for subFinsler group, without Pansu derivatives.

We also propose the following weakening of the Tointon–Yadin conjecture:

Conjecture 5.B. If G is 2-step nilpotent, then the set of orbits of Busemann points in $\partial(G, d_S)$ is finite. In particular, there are at most countably many Busemann points.

We do not adventure in a complete characterization of group with countably many Busemann points, as many other groups (or rather marked groups) may enter the picture. For instance, filiform groups (such as the Engel group), or pairs (G, d_S) with sub-exponential geodesic growth (including virtually Cartan groups, see Chapter 4) may only have countably many Busemann points.

Note that Conjecture 5.B reduces to the only case $(N_{2,r}, S_r)$, the free 2-step nilpotent group of r with its standard generating set. Indeed, if G is 2-step nilpotent, generated by S of size $|S| = r$, then there exists an epimorphism $f\colon N_{2,r} \to G$ such that $f(S_r) = S$, hence we can apply Proposition 5.2.5.

* * *

In the computation leading to Theorem 5.5.4, the upper bound presents a non-trivial "number theoretic" dependence in u. We expect that a matching lower bounds can be proven. This behavior should not be present for sub-Finsler metrics.

Conjecture 5.C. Let $(\mathscr{C}, d_S)$ be the Cartan group with the standard generating set.

(a) If $u = [a : b]$ with $a \not\equiv b \pmod 2$, then

$$\forall h \in [\mathscr{C}, \mathscr{C}], \quad b_{\gamma_u}(h) = \Theta\left(\sqrt[3]{|A(h)| + \langle -B(h), u^\perp \rangle}\right).$$

(b) For every subFinsler metric d on $\bar{\mathscr{C}}$. For every $\varphi \in \partial(\bar{\mathscr{C}}, d)$, we have

$$\forall h \in [\bar{\mathscr{C}}, \bar{\mathscr{C}}], \quad B(h) = \mathbf{0} \implies \varphi(h) = O(1).$$

(Or more strongly, $\varphi(h)$ does not depend on $A(h)$, up to a bounded function.)

As an interesting corollary, this would imply that $|d_S - d|$ is not bounded for any choice of subFinsler metric d on $\bar{\mathscr{C}}$, answering a question of Enrico Le Donne.

* * *

Pierre Pansu asked if we could fix the conjecture of Bader–Finkelshtein by a "more reduced" boundary. Specifically, for each $\alpha \in [0, 1)$, let us consider

$$O^\alpha(G, \mathbb{R}) \coloneqq \left\{ f\colon G \to \mathbb{R} \;\middle|\; f(g) = O(\|g\|_S^\alpha) \right\}$$

the space of functions growing slower than some given power of the norm.

Question 5.D. Let (G, d_S) be a finitely generated nilpotent group. Does there exist $\alpha \in [0, 1)$ such that the action $G \curvearrowright \partial(G, d_S)/O^\alpha(G, \mathbb{R})$ is trivial.

For the specific example $\varphi = b_{\gamma_u}$ on $(\mathscr{C}, d_S)$ we have computed, we have only shown that $|g \cdot \varphi(h) - \varphi(h)| = \Omega(\|h\|_S^{2/3})$, so $\alpha = \frac{2}{3}$ seems plausible in this case.

Finally, let us return to our original motivation: the existence of non-trivial virtual characters. For 2-step nilpotent groups, we proved that $G \curvearrowright \partial^r(G, d_S)$ is trivial. This means that, for all $\varphi \in \partial(G, d_S)$ and $g \in G$, we have

$$\forall h \in G, \quad |g \cdot \varphi(h) - \varphi(h)| \leqslant C_{\varphi,g}.$$

For some horofunctions, this can be improved to

$$\forall g, h \in G, \quad |g \cdot \varphi(h) - \varphi(h)| \leqslant C_{\varphi},$$

where the constant C_φ does **not** depend on g, which can be rewritten as

$$\forall g, h \in G, \quad |\varphi(gh) - \varphi(g) - \varphi(h)| \leqslant C_{\varphi},$$

i.e., φ is a quasi-morphism. As $H^2_b(G, \mathbb{R}) = \{\mathbf{0}\}$ for amenable groups, we conclude the existence of a homomorphism $\tilde{\varphi} \colon G \to \mathbb{R}$ such that $|\varphi(h) - \tilde{\varphi}(h)| = O(1)$. As every horofunction is unbounded, we conclude that $\tilde{\varphi}$ is a non-trivial character.

It is natural to ask if this line of argument can be extended for more general groups.

Question 5.E. Let G be a finitely generated nilpotent group. Does there exist $\varphi \in \partial(G, d_S)$ and $\beta \in [0, 1)$ such that

$$\forall g, h \in G, \quad |g \cdot \varphi(h) - \varphi(h)| = O\big(\max\{\|g\|_S\,, \|h\|_S\}^\beta\big)\,?$$

More generally, if we restrict to $g \in H_\varphi$ a finite-index subgroup, does the same hold true for G growing sufficiently slowly (say $\beta_G(n) \preceq n^{\log(n)}$ or $\exp(\sqrt{n})$)?

Question 5.F. Fix $\beta \in (0, 1)$. Consider a map $\varphi \colon G \to \mathbb{R}$ such that

$$\forall g, h \in G, \quad |\varphi(gh) - \varphi(g) - \varphi(h)| = O\big(\max\{\|g\|_S\,, \|h\|_S\}^\beta\big).$$

Suppose G is amenable (or grows sub-exponentially). Can we conclude the existence of a homomorphism $\tilde{\varphi} \colon G \to \mathbb{R}$ such that $|\varphi(g) - \tilde{\varphi}(g)| = O(\|g\|_S^\beta)$?

Chapter 6

Membership problems

This chapter focuses on two decision problems, namely the Submonoid and Rational Subset Membership problems. For a group G (either given as a matrix group, or endowed with a finite generating set S) we look for algorithms with specifications:

Submonoid Membership problem ($\mathsf{SMM}(G)$)

Input: Elements g and $g_1, g_2, \ldots, g_n \in G$ (defined as matrices or words over S).

Output: Decide whether $g \in \{g_1, g_2, \ldots, g_n\}^*$.

Rational Subset Membership problem ($\mathsf{RatM}(G)$)

Input: An element $g \in G$ (given either as a matrix or as a word over S) and a rational subset $R \subseteq G$ (defined by a finite state automaton, labeled by elements in G given either as matrices or as words over S).

Output: Decide whether $g \in R$.

Both problems are known to be decidable for a few classes of groups, starting with free groups [BS86] and virtually abelian groups [Gru99]. We suggest to have a look at surveys [Lo15b; Don23; Lo24] for a better picture.

Finally, we recall the Knapsack Problem which plays a role. This problem is a special case of Rational Subset Membership, introduced more recently in [MNU13].

Knapsack problem ($\mathsf{KS}(G)$)

Input: Elements g and $g_1, g_2, \ldots, g_n \in G$ (defined as matrices or words over S).

Output: Decide whether $g \in \{g_1\}^* \{g_2\}^* \ldots \{g_n\}^*$.

This sub-problem is known to be decidable in a much larger class of group. This includes hyperbolic groups [MNU13] and co-context-free groups [KLZ16].

Finitely generated submonoids are rational, hence the decidability of $\mathsf{RatM}(G)$ implies the decidability of $\mathsf{SMM}(G)$. A natural question is whether the reciprocal holds:

Question (Lohrey-Steinberg)**.** Does there exist a finitely generated group with decidable Submonoid Membership and undecidable Rational Subset Membership?

Lohrey and Steinberg proved both problems are recursively equivalent in RAAGs [LS08] and infinitely-ended groups [LS10]. However, they conjecture a positive answer for more general groups. They also note that the existence of such a group is equivalent to the property "$\mathsf{SMM}(G)$ is decidable" not being closed under free products [LS10, §4].

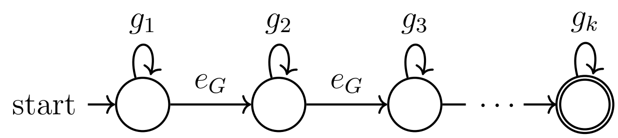


Figure 6.1: Finite state automaton relative to the Knapsack problem

For (non virtually abelian) nilpotent groups, the full picture is not clear yet.

- The Knapsack problem (hence $\mathsf{RatM}(G)$) is undecidable in large nilpotent groups, most notably $G = H_3(\mathbb{Z})^k$ and $N_{k,2}$ for $k \gg 1$. [Lo15a; KLZ16; MT17]
- Submonoid Membership is undecidable in $H_3(\mathbb{Z})^k$ for $k \gg 1$. [Rom23]

Both results rely on the negative solution to Hilbert's 10th problem: there exists no algorithm deciding whether a Diophantine equation (or system of equations) admits an integer solution [Mat93]. On the positive side, the list of results is even shorter:

- The Knapsack problem in decidable in $H_{2m+1}(\mathbb{Z})$ for all $m \geqslant 1$. [KLZ16]
- Colcombet, Ouaknine, Semukhin and Worrell proved that Submonoid Membership is decidable in $H_{2m+1}(\mathbb{Z})$ for all $m \geqslant 1$. [COSW19]

Note that the former relies on deep results on quadratic Diophantine equations [GS04], whereas the latter is elementary. This points to $\mathsf{RatM}(G)$ being harder than $\mathsf{SMM}(G)$, hence the hope to separate both problems within the class of nilpotent groups.

As a first step, we re-interpret and extend Colcombet *and al.*'s result:

Theorem 6.A (Theorem 6.2.1)**.** *There exists an algorithm with specifications*

Input: *A finitely presented nilpotent group G (given by a finite presentation), a finite set $S \subset G$ and an element $g \in G$ (given as words).*

Output: *Finitely many instances $g_i \overset{?}{\in} R_i$ of the Rational Subset Membership in a subgroup $H \leqslant G$ such that $g \in S^*$ if and only if $g_i \in R_i$ for some i.*

Moreover, it solves these instances if $h([H,H]) = h([G,G])$, with h the Hirsch length.

Note that, if $h([G,G]) \leqslant 1$, e.g. for $H_{2m+1}(\mathbb{Z})$, then either $h([H,H]) = h([G,G])$ or H is virtually abelian. In both cases the instances of $\mathsf{RatM}(H)$ are decidable.

In a more conceptual direction, Theorem 6.A can be used to construct examples confirming the conjecture of Lohrey and Steinberg:[1]

Theorem 6.B (Corollary 6.2.3)**.** *There exists a nilpotent group of class* 2 *with decidable Submonoid Membership and undecidable Rational Subset Membership.*

Remark. It is crucial to consider the *uniform* version of the Rational Subset Membership in the previous result. Indeed the instances of $\mathsf{RatM}(H)$ we need to solve *will depend* on $g \in G$, even for a fixed submonoid $S^* \subseteq G$.

Our second main result deals with the discrete Heisenberg group

$$H_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, a, b, c \in \mathbb{Z} \right\} \simeq \left\langle x, y \;\middle|\; [x,[x,y]] = [y,[x,y]] = 1 \right\rangle .$$

The Heisenberg group is the smallest (non virtually-abelian) f.g. nilpotent group, in the sense that it embeds in all such groups. We prove the following:

Theorem 6.C (Theorem 6.3.9)**.** $H_3(\mathbb{Z})$ *has decidable Rational Subset Membership.*

We actually prove a stronger result: it is decidable whether an equation with rational constraints in $H_3(\mathbb{Z})$ admits a solution. This extends on [DLS15].

Along the way, we provide results with application to the following decision problem:

Identity problem ($\mathsf{Id}(G)$)

Input: A finite sets of elements $S \subset G$ (eg. given as matrices or words).

Output: Decide whether the sub-semigroup S^+ contains the neutral element e_G.

This is a natural variation on the **Matrix Mortality problem** introduced by Markov. Instead of asking whether the zero matrix can be written as a product of elements of a subset S given as input, we ask for the identity matrix. This problem was first studied in [CK10], with important contributions [BP09; Do24a; Do24b] for $F_2 \times F_2$, nilpotent groups of class $\leqslant 10$, and metabelian groups respectively.

Building on work of Dong [Do24a], we prove the following result:

Theorem 6.D (Theorem 6.1.3)**.** *The Identity problem is decidable in every finitely generated nilpotent group G.*

The algorithm is uniform in the group (i.e., we could take $G = \langle X \mid R \rangle$ as input). Moreover, for a fixed group G, the algorithm runs in polynomial time.

[1] An earlier example is $A \wr \mathbb{Z}^2$ with $A \neq 1$ finite abelian. See the Bachelor thesis [Pot20], based on an argument of Doron Shafrir [unpublished, 2018]. I thank Markus Lohrey for telling me about it.

6.1 The Identity problem

6.1.1 Characterization of finite-index subgroups

We give a useful result, which can be understood as a discrete version of the Chow-Rashevskii theorem in sub-Riemannian geometry. This allows to detects finite-index subgroups among sub-semigroups. An equivalent result was independently proven by Doron Shafrir [Sha24a, Theorem 2].

Proposition 6.1.1. *Let G be a finitely generated nilpotent group, and consider the map $\mathrm{Pr}\colon G \twoheadrightarrow G/I([G,G]) \simeq \mathbb{Z}^r$. For any set $S \subseteq G$, the following assertions are equivalent*

(a) $\mathrm{ConvHull}(\mathrm{Pr}(S)) \subseteq \mathbb{R}^r$ *contains a ball $B(\mathbf{0},\varepsilon)$ for some $\varepsilon > 0$.*

(b) *For every non-zero linear form $f\colon \mathbb{R}^r \to \mathbb{R}$, there exists $s \in S$ s.t. $f(\mathrm{Pr}(s)) < 0$.*

(c) *For every non-zero homomorphism $f'\colon G \to \mathbb{R}$, there exists $s \in S$ s.t. $f'(s) < 0$.*

(d) *The sub-semigroup S^+ is a finite-index subgroup of G.*

If S is finite, we can restrict to non-zero homomorphism $f'\colon G \to \mathbb{Z}$.

We start with a quick lemma translating that a subgroup $H \leqslant \mathbb{Z}^r$ has infinite index if and only if it is included inside an hyperplane.

Lemma 6.1.2. *A subgroup $H \leqslant \mathbb{Z}^r$ has infinite index if and only if there exists a non-zero morphism $f\colon \mathbb{Z}^r \twoheadrightarrow \mathbb{Z}$ such that $H \leqslant \ker(f)$.*

Proof. First if $H \leqslant \ker(f)$ for some morphism $f\colon \mathbb{Z}^r \to \mathbb{Z}$, then

$$[G : H] \geqslant [G : \ker(f)] = |\mathbb{Z}| = \infty.$$

If H has infinite-index, then $\mathbb{Z}^r/H$ is an infinite, finitely generated, abelian group, hence factors onto $\mathbb{Z}$. The composition $\mathbb{Z}^r \twoheadrightarrow \mathbb{Z}^r/H \twoheadrightarrow \mathbb{Z}$ provides the desired f. □

Proof of Proposition 6.1.1. (a) ⇒ (b) is trivial, and ¬(a) ⇒ ¬(b) is Hahn-Banach for C the interior of $\mathrm{ConvHull}(\mathrm{Pr}(S))$ and $p = \mathbf{0}$. The equivalence (b) ⇔ (c) follows from

$$\mathrm{Pr}^*\colon \begin{pmatrix} \mathrm{Hom}(G/I_G([G,G]),\mathbb{R}) & \longrightarrow & \mathrm{Hom}(G,\mathbb{R}) \\ f & \longmapsto & f \circ \mathrm{Pr} \end{pmatrix}$$

being an isomorphism. The map Pr^* is onto as any homomorphism $f'\colon G \to \mathbb{R}$ factors through $G/I_G([G,G]) \simeq \mathbb{Z}^r$ since $\mathbb{R}$ is torsionfree abelian.

(d) ⇒ (b) follows from Lemma 6.1.2 with $H = \mathrm{Pr}(S^+)$. We prove that (a,b,c) ⇒ (d), arguing by induction on the nilpotency class c. Let $P = \mathrm{ConvHull}(\mathrm{Pr}(S))$.

Base case. We have $c = 1$, i.e., $G \simeq \mathbb{Z}^r \times T$ with T finite abelian. We prove that $-\Pr(s) \in \Pr(S)^+$ for all $s \in S$, hence that $\Pr(S)^+$ is a subgroup (as S is non-empty).

- We first assume that S is finite, it follows that P is a convex polytope. Consider the ray from $\mathbf{0}$ through $-\Pr(s)$. This ray intersects some facet F of P at $-x\Pr(s)$ for some $x > 0$ as $B(\mathbf{0}, \varepsilon) \subset \mathrm{ConvHull}(\Pr(S))$. Using Caratheodory's Theorem, there exist r vertices $\Pr(s_1), \ldots, \Pr(s_r)$ of F such that

  $$\exists y_1, \ldots, y_r \in \mathbb{R}_{\geqslant 0} \text{ summing to 1, such that } \; -x\Pr(s) = y_1 \Pr(s_1) + \ldots + y_r \Pr(s_r).$$

  Moreover we may assume $x \in \mathbb{Q}_{>0}$ and $y_i \in \mathbb{Q}_{\geqslant 0}$ as the coefficients of the underlying system are integers. Finally, we multiply by some well-chosen positive integer N in order to cancel out denominators, and get

  $$-\Pr(s) = (Nx - 1) \cdot \Pr(s) + Ny_1 \cdot \Pr(s_1) + \ldots + Ny_r \cdot \Pr(s_r) \in \Pr(S)^+.$$

- If S is infinite, we can find some finite subset $S_0 \subset S$ such that $\mathbf{0}$ lies in the interior of $P_0 = \mathrm{ConvHull}(\Pr(S_0))$ and $S_0 \ni s$. Now we may repeat the previous argument with S_0 and conclude that $-\Pr(s) \in \Pr(S_0)^+ \subseteq \Pr(S)^+$.

This proves that $\Pr(S)^+$ is a subgroup. This can be extended to S^+. Indeed, for each $s \in S$, there exists $w \in S^+$ such that $sw \in T$ hence $(sw)^{|T|} = e_G$.

Finally $\Pr(S)^+$ is not included inside any hyperplane (condition (b)), hence $\Pr(S)^+$ has finite index in $\mathbb{Z}^r$ by Lemma 6.1.2, and therefore S^+ has finite-index in G.

Induction. Suppose that the induction hypothesis holds for $c - 1 \geqslant 1$. We fix G of nilpotency class c, i.e., $\gamma_{c+1}(G) = \{e_G\}$, and a subset $S \subseteq G$ satisfying condition (abc). We prove that (i.) $S^+ \cap \gamma_c(G)$ is a finite-index subgroup in $\gamma_c(G)$, and deduce that (ii.) S^+ is a finite-index subgroup in G.

i. Proof that $S^+ \cap \gamma_c(G)$ is a finite-index subgroup in $\gamma_c(G)$.

Note that $\gamma_c(G)$ is finitely generated abelian and $S^+ \cap \gamma_c(G) = (S^+ \cap \gamma_c(G))^+$. This is exactly the setup for the base case! We verify condition (c): for each non-zero morphism $f' \colon \gamma_c(G) \to \mathbb{R}$, we provide $t \in S^+ \cap \gamma_c(G)$ such that $f'(t) < 0$. **(Claim 4)**

Claim 1. $H = S^+ \gamma_c(G)$ is a finite-index subgroup of G.

For any non-zero homomorphism $g' \colon G/\gamma_c(G) \to \mathbb{R}$, the composition

$$G \xrightarrow{\pi} G/\gamma_c(G) \xrightarrow{g'} \mathbb{R}$$

is a non-zero morphism hence there exists $s \in S$ s.t. $g'(\pi(s)) < 0$. This means that $\pi(S) \subseteq G/\gamma_c(G)$ satisfies condition (c). However, $G/\gamma_c(G)$ has nilpotency class $c - 1$. By induction hypothesis, it follows that $\pi(S)^+$ is a finite-index subgroup of $G/\gamma_c(G)$.

Claim 2. For each $g \in G$, we have $g^M \in H$ for $M = \operatorname{lcm}\big(1, 2, \ldots, [G : H]\big)$. Equivalently there exists $w_g \in S^+$ and $z_g \in \gamma_c(G) \leqslant \mathcal{Z}(G)$ such that $g^M = w_g z_g$.

By the pigeonhole principle, there exist $0 \leqslant i < j \leqslant [G : H]$ such that $g^i H = g^j H$ hence $g^{j-i} \in H$. It follows that $g^M = (g^{j-i})^{\frac{M}{j-i}} \in H$ which proves the claim.

Claim 3. For each homomorphism $f' \colon \gamma_c(G) \to \mathbb{R}$, either $f' \equiv 0$, or there exist $g \in G$ and $h \in \gamma_{c-1}(G)$ such that $f'([g, h]) < 0$.

Indeed, $\big\{[g, h] \;\big|\; g \in G,\ h \in \gamma_{c-1}(G)\big\}$ is a *symmetric* generating set for $\gamma_c(G)$, hence either $f([g, h]) = 0$ for all g, h and $f' \equiv 0$, or the desired g, h do exist.

Claim 4. There exists $t \in S^+ \cap \gamma_c(G)$ such that $f'(t) < 0$.

Take $g \in G$ and $h \in \gamma_{c-1}(G)$ as in Claim 3. Since $[g, h] \in \gamma_c(G) \leqslant \mathcal{Z}(G)$, one has $[g^p, h^q] = [g, h]^{pq}$ for all $p, q \in \mathbb{Z}$. In particular, for any $p \in \mathbb{Z}_{>0}$, we have

$$[g, h]^{M^2p^2} = [g^{Mp}, h^{Mp}] = w_g^p w_h^p w_{g^{-1}}^p w_{h^{-1}}^p \cdot \big(z_g z_h z_{g^{-1}} z_{h^{-1}}\big)^p$$

for some $w_g, z_g, w_h, z_h, w_{g^{-1}}, z_{g^{-1}}, w_{h^{-1}}, w_{h^{-1}}$ as in Claim 2. It follows that

$$f'\left(w_g^p w_h^p w_{g^{-1}}^p w_{h^{-1}}^p\right) = M^2p^2 \cdot f'\big([g, h]\big) - p \cdot f'\big(z_g z_h z_{g^{-1}} z_{h^{-1}}\big) < 0$$

for p large enough. The element is $t = w_g^p w_h^p w_{g^{-1}}^p w_{h^{-1}}^p \in S^+ \cap \gamma_c(G)$.

ii. Proof that S^+ is a finite-index subgroup inside G.

First we show that S^+ is a subgroup of G. For each $s \in S$, there exists $w \in S^+$ such that $sw \in \gamma_c(G)$: this follows from the fact that s must have an inverse wz in $H = S^+\gamma_c(G)$ (with $w \in S^+$ and $z \in \gamma_c(G)$) by Claim 1. So $sw \in S^+ \cap \gamma_c(G)$ which is a subgroup by (i.), and thus we deduce that $(sw)^{-1} \in S^+ \cap \gamma_c(G)$ which implies $s^{-1} = w(sw)^{-1} \in S^+$.

Finally, we show that S^+ has finite index in G. The index of S^+ in G is given by

$$\begin{aligned}\big[G : S^+\big] &= [G : S^+\gamma_c(G)] \cdot [S^+\gamma_c(G) : S^+] \\ &= \big[G/\gamma_c(G) \;:\; S^+/\big(S^+ \cap \gamma_c(G)\big)\big] \cdot \big[\gamma_c(G) : S^+ \cap \gamma_c(G)\big] < \infty,\end{aligned}$$

using several times the isomorphism theorem $HK/K \simeq H/H \cap K$ for $K \leqslant N_G(H)$. □

6.1.2 The algorithm

Theorem 6.1.3. *The Identity problem is uniformly decidable in f.g. nilpotent groups.*

Input: *A finite presentation $G = \langle X \mid R\rangle$ and a finite set $S \subset (X \cup X^{-1})^*$.*

Output: *Decide whether $g \in S^+$.*

We present our own algorithm which mixes group theory and convex geometry.

Proof. We initiate with $G_0 = G$ and $S_0 = S$.

(1) Compute the "torsionfree-abelian-isation" of G_t, that is, compute the map

$$\mathrm{Pr}\colon G_t \longrightarrow G_t/I_{G_t}([G_t, G_t]) \hookrightarrow \mathbb{Q}^r,$$

where the last map has full rank. Suppose $G_t = \langle X \mid R \rangle$. There is a canonical map $\pi\colon F_X \to \mathbb{Q}^X$. (We write elements of $\mathbb{Q}^X$ as sum of elements of X.)

- Compute a basis of $\mathrm{Vect}(\pi(R))$ and complete it with $x_1, \ldots, x_r \in X$.
- Then $\mathbb{Q}^r = \bigoplus_{i=1}^r \mathbb{Q}x_i$ and Pr is the projection along $\mathrm{Vect}(\pi(R))$.

(2) Write $P = \mathrm{ConvHull}(\mathrm{Pr}(S_t))$ as the intersection of half-spaces $f_i(\mathbf{x}) \geqslant a_i$ with $f_i\colon \mathbb{Q}^r \to \mathbb{Q}$ non-zero linear forms and $a_i \in \mathbb{Q}$. This requires to convert the V-representation of the polytope to an H-representation.

(3) Check the sign of each a_i.

- If $a_i < 0 = f_i(\mathbf{0})$ for all i, this means that $\mathbf{0}$ lies in the interior of P, we may conclude that $e_G \in S_t^+ \subseteq S^+$ using Proposition 6.1.1 (or Lemma 6.1.7).
- If there exists some $a_i > 0$, we conclude that $e_G \notin S^+$. Indeed

$$f_i(\mathrm{Pr}(s_1 s_2 \ldots s_\ell)) = \sum_{j=1}^{\ell} f_i(\mathrm{Pr}(s_j)) \geqslant \ell a_i > 0 = f_i(\mathrm{Pr}(e_G))$$

  for any word $s_1 s_2 \ldots s_\ell \in S_t^+$.
- Otherwise, we go back to step (1) with new inputs

$$G_{t+1} = \bigcap_{i:a_i=0} \ker(f_i \circ \mathrm{Pr}) \quad \text{and} \quad S_{t+1} = S_t \cap G_{t+1}.$$

  This will give the same answer as $e_G \in S_t^+$ if and only if $e_G \in S_{t+1}^+$. Indeed, for any element $\tilde{s} \in S \setminus S_{\mathrm{new}}$, there exists i such that $a_i = 0$ (hence $f_i(\mathrm{Pr}(s)) \geqslant 0$ for all $s \in S$) and $f_i(\mathrm{Pr}(\tilde{s})) > 0$. In particular, for any $w \in S_t^+$ containing $\tilde{s}$,

$$f_i(\mathrm{Pr}(w)) \geqslant f_i(\mathrm{Pr}(\tilde{s})) > 0 = f_i(\mathrm{Pr}(e_G))$$

  so that $w \neq e_G$.

  Note that a presentation for G_{t+1} can be effectively computed combining the solutions for problems (III) and (IV) of [MMNV22, p. 5427].

It remains to justify that the algorithm terminates. The key observation is that the Hirsch length $h(G)$ decreases at each loop. Indeed, there exists some i such that $a_i = 0$. We have $G_{t+1} \leqslant \ker(f_i \circ \mathrm{Pr})$, hence

$$h(G_{t+1}) \leqslant h(\ker(f_i \circ \mathrm{Pr})) = h(G_t) - h(\mathbb{Z}) = h(G_t) - 1.$$

It follows that the algorithm either terminates, or we get to the point where $G_t = \{e_G\}$. In the later case, either $S_t = \emptyset$ or $S_t = \{e_G\}$, we can easily conclude in both cases. □

Remark 6.1.4. This algorithm seems quite wasteful. The presentation, and especially the words representing the elements of S in the successive generating sets gets larger. A better implementation (in P) is given by Dong [Do24a, Algorithm 1]:

(0) Identify $S_0 = S$ as subsets of the Lie algebra $\mathfrak{g}$ of the $\mathbb{Q}$-Mal'cev completion of $\bar{G}_{\mathbb{Q}}$.

(1) Let $S_t = \{s_1, s_2, \ldots, s_\ell\} \subset \mathfrak{g}$. Compute C the cone of solutions $\mathbf{x} \in \mathbb{Q}^\ell_{\geqslant 0}$ to

$$x_1 s_1 + x_2 s_2 + \ldots + x_\ell s_\ell = \mathbf{0} \mod [S_t, S_t]$$

where $[S_t, S_t] = \mathrm{Vect}_{\mathbb{Q}} \left\{ [s_{i_1}, s_{i_2}], [s_{i_1}, s_{i_2}, s_{i_3}], \ldots, [s_{i_1}, \ldots, s_{i_c}] : s_i \in S_t \right\}$.

(2) Let $S_{t+1} = \{s_i \mid \exists \mathbf{x} \in C \text{ with } x_i > 0\}$.

(3) If $S_{t+1} = S_t$, we conclude that $e_G \in S^+$ $(\star)$. If $S_{t+1} = \emptyset$, we conclude $e_G \notin S^+$. Otherwise, go back to step (1) with input S_{t+1}.

The tricky part (only proven for groups of nilpotency class $c \leqslant 10$ in [Do24a]) is the justification of $(\star)$. The condition $S_{t+1} = S_t$ means that, in $\langle S_i \rangle / I([S_i, S_i])$, we can write $\mathbf{0}$ as a positive combination of **all the** s_i, hence $\mathbf{0}$ sits in the interior of the convex hull $\mathrm{ConvHull}(\Pr(S_i))$, and we may apply Proposition 6.1.1.

6.1.3 A parte - Alternate proof via orders

We provide a alternative, weaker lemma which is still sufficient to prove the correctness of Algorithm 6.1.3. The proof uses the theory of left-invariant orders. (This allows for a very short proof, at the cost of being far from self-contained.)

Definition 6.1.5. Let G be a group and $\succ$ a partial order on G

- $\succ$ is *left-invariant* if, for all $g, h, h' \in G$, we have $h \succ h' \implies gh \succ gh'$.
- A total left-invariant order $\succ$ is *Conradian* if, for all $g, h \in G$ such that $g, h \succ e_G$, we have $gh^n \succ h$ for some $n \geqslant 1$.

A fundamental theorem about Conradian order is Conrad's theorem [Con59, Thm 4.1], see also [DNR14, Corollary 3.2.28] for a modern treatment.

Theorem 6.1.6 (Conrad). *Let G be a finitely generated group and $\succ$ a Conradian order on G. There exists an homomorphism $f\colon G \to \mathbb{R}$ such that $g \succ e_G$ implies $f(g) \geqslant 0$.*

Lemma 6.1.7. *Let G be a finitely generated torsion-free nilpotent group, and fix $S \subseteq G$. If $e_G \notin S^+$, then there exists a non-zero morphism $f\colon G \to \mathbb{R}$ s.t. $f(s) \geqslant 0$ for all $s \in S$.*

Proof. As $e_G \notin S^+$, we can define a partial left-invariant order

$$g < h \iff g^{-1}h \in S^+.$$

This order satisfies $s > e_G$ for all $s \in S$. Using [Rhe72, Theorem 4], we can extend this order to a total left-invariant order $\succ$ on G. Moreover the order $\succ$ is Conradian using [Rhe72, Theorem 2]. The existence of f now follows from Conrad's theorem. □

6.2 Dimension gain for Submonoid Membership

6.2.1 Proof of Theorem 6.A

We prove Theorem 6.A, which we re-state for the reader's convenience:

Theorem 6.2.1. *There exists an algorithm with specifications*

Input: *A finitely presented nilpotent group G, a finite set $S \subset G$, and $g \in G$.*

Output: *Finitely many instances $\{g_i \overset{?}{\in} R_i\}$ of the Rational Subset Membership in a subgroup $H \leqslant G$ such that $g \in S^*$ if and only if $g_i \in R_i$ for some i.*

Moreover, the algorithm solves these instances if $h([H,H]) = h([G,G])$.

Proof. First, we may assume $e_G \in S$. The proof splits into three steps

(1) We compute the image of S through $\mathrm{Pr}\colon G \twoheadrightarrow G/I_G([G,G]) \simeq \mathbb{Z}^r$. Using this quotient, we define a subgroup $H \leqslant G$ (depending only on S) and a partition $S = S_0 \sqcup S_+$. For each $s \in S$, we look whether $\mathrm{Pr}(s)$ is invertible in $\mathrm{Pr}(S)^* \leqslant \mathbb{Z}^r$.

(2) We reduce the problem $g \in S^*$ to finitely many instances of $\mathsf{RatM}(H)$.

(3) In the case when $h([H,H]) = h([G,G])$, we solve the previous instances of $\mathsf{RatM}(H)$.

The first step follows [COSW19, Theorem 7] closely, while the last step generalizes "Case II" from the same proof. The observation of the second step seems new.

▶ First we compute a map $\mathrm{Pr}\colon G \to \mathbb{Z}^r$ from a presentation for G. We consider the polytope $\mathrm{ConvHull}(\mathrm{Pr}(S)) \ni \mathbf{0}$ given by a V-representation (namely, $\mathrm{Pr}(S)$). We can compute an H-representation (H for half-space), that is, a finite set of inequalities $\{f_i(\mathbf{v}) \geqslant a_i\}$ with $f_i\colon \mathbb{Z}^r \to \mathbb{Z}$ non-zero linear form and $a_i \in \mathbb{Z}$ such that

$$\mathrm{ConvHull}(\mathrm{Pr}(S)) = \left\{\mathbf{v} \in \mathbb{R}^r \;\middle|\; \forall i \in I,\ f_i(\mathbf{v}) \geqslant a_i\right\}$$

This is the classical *facet enumeration problem.* (See [Zie94, Section 1.2] and references therein for a thorough treatment.) We compute a maximal linearly independent set $\{f_1, f_2, \ldots, f_s\}$ of linear forms among all inequalities $f_i(\mathbf{v}) \geqslant 0$ in the H-representation, and define an homomorphism $f\colon G \to \mathbb{Z}^s$ via

$$f(h) = \big(f_1(\mathrm{Pr}(h)),\ f_2(\mathrm{Pr}(h)), \ldots,\ f_s(\mathrm{Pr}(h))\big).$$

The image of $f\colon G \to \mathbb{Z}^s$ has finite-index (by linear independence), and $f(S) \subset \mathbb{Z}^s_{\geqslant 0}$. We define $H = \ker f$ and partition S into $S_0 = S \cap H$ and $S_+ = S \setminus H$.

Geometric intermezzo: We are looking at the minimal face F of $\mathrm{ConvHull}(\mathrm{Pr}(S))$ containing $\mathbf{0}$ (i.e., no proper sub-face of F contains $\mathbf{0}$). This face is given by

$$F = \mathrm{ConvHull}(\mathrm{Pr}(S)) \cap K = \mathrm{ConvHull}(\mathrm{Pr}(S_0))$$

where $K = \left\{\mathbf{v} \in \mathbb{R}^r \;\middle|\; f_i(\mathbf{v}) = 0 \text{ for } i = 1, 2, \ldots, s\right\}$. The minimality of F can be stated as "there exists $\varepsilon > 0$ such that $B(\mathbf{0}, \varepsilon) \cap K \subset F$".

▶ There exists only finitely many words $w = u_1 u_2 \dots u_k \in S_+^\star$ such that

$$f(\bar{w}) = \sum_{i=1}^{k} f(u_i) = f(g)$$

(Indeed, k is bounded by the sum of the components of $f(g)$.) For each word w, we need to decide whether $g \in S_0^* u_1 S_0^* u_2 S_0^* \dots S_0^* u_k S_0^*$. Observe that

$$S_0^* \cdot u_1 S_0^* \cdot u_2 S_0^* \cdot \ldots \cdot u_k S_0^* = S_0^* \cdot \left(v_1 S_0 v_1^{-1}\right)^* \cdot \left(v_2 S_0 v_2^{-1}\right)^* \cdot \ldots \cdot \left(v_k S_0 v_k^{-1}\right)^* \cdot v_k$$

where $v_i = u_1 u_2 \dots u_i$ hence the problem can be restated as

$$g v_k^{-1} \overset{?}{\in} R \coloneqq S_0^* \cdot \left(v_1 S_0 v_1^{-1}\right)^* \cdot \left(v_2 S_0 v_2^{-1}\right)^* \cdot \ldots \cdot \left(v_k S_0 v_k^{-1}\right)^* \subseteq H$$

which is an instance of the Rational Subset Membership Problem in H.

We note that algorithms presented in [MMNV22] allow to compute a presentation for H, and then rewrite all elements $g v_k^{-1} \in H$ and $v_i s v_i^{-1} \in H$ as words over the corresponding generating set. (Problems (III), (IV) and (II) in the article.)

▶ We solve the Membership problem under the hypothesis $h([G,G]) = h([H,H])$.

Observe that $I_G([G,G]) = I_H([H,H])$. Indeed, we have $I_G([G,G]) \leqslant H$ as H is the kernel of $f\colon G \to \mathbb{Z}^s$ with $\mathbb{Z}^s$ torsionfree abelian, and

$$m = \Big[I_G([G,G]) : [H,H]\Big] = \Big[I_G([G,G]) : [G,G]\Big]\Big[[G,G] : [H,H]\Big] < \infty.$$

For every $g \in I_G([G,G])$, we have $g^{m!} \in [H,H]$ hence $g \in I_H([H,H])$, proving the observation. In particular, we can identify $H/I_H([H,H]) = H/I_G([G,G]) = \Pr(H)$.

Condition (a) of Proposition 6.1.1 (for H and S_0) now reads "$F = \operatorname{ConvHull}(\Pr(S_0))$ contains a ball $B(\mathbf{0},\varepsilon) \cap K$", which holds by minimality of F. We conclude that S_0^* is a finite-index subgroup of H. Finally, the problem can be restated as

$$\langle S_0 \rangle\, g v_k^{-1} \ \cap\ \left(v_1 S_0 v_1^{-1}\right)^* \cdot \left(v_2 S_0 v_2^{-1}\right)^* \cdot \ldots \cdot \left(v_k S_0 v_k^{-1}\right)^* \overset{?}{=} \emptyset$$

which is easily decided using Lemma 2.6.6(c). □

6.2.2 Proof of Theorem 6.B

We give a lemma on abstract commensurability classes of subgroups of $N_{2,m} \times \mathbb{Z}^n$. Some ideas can already be found in [GG55, Theorem 7].

Lemma 6.2.2. *Any subgroup $H \leqslant N_{2,m} \times \mathbb{Z}^n$ admits a finite-index subgroup $H' \trianglelefteq H$ isomorphic to $N_{2,k} \times \mathbb{Z}^\ell$ for some $k \leqslant m$ and $\ell \leqslant \binom{m}{2} - \binom{k}{2} + n$.*

Proof. Let $G = N_{2,m} \times \mathbb{Z}^n$ and fix a subgroup $H \leqslant G$. Consider the map

$$\alpha\colon N_{2,m} \times \mathbb{Z}^n \longrightarrow N_{2,m} \longrightarrow \mathbb{Z}^m$$

The image $\alpha(H) \leqslant \mathbb{Z}^m$ is isomorphic to $\mathbb{Z}^k$ for some $k \leqslant m$. Fix $g_1, \ldots, g_k \in H$ such that $\{\alpha(g_i)\}$ is a basis for $\alpha(H)$.

Claim 1. The morphism $N_{2,k} \twoheadrightarrow \langle g_i \rangle$ sending $x_i \mapsto g_i$ is an isomorphism.

As $N_{2,k}$ is torsionfree, any non-trivial subgroup has positive Hirsch length, hence any proper quotient satisfies $h(Q) < h(N_{2,k})$. On the other side, the abelianization of $\langle g_i \rangle$ is $\mathbb{Z}^k$ by construction, and its derived subgroup is generated by the $\binom{k}{2}$ linearly independent commutators $[g_i, g_j]$ with $i < j$ (as the $\alpha(g_i)$ form a linearly independent subset of $\mathbb{Z}^m$). This implies that $\langle g_i \rangle$ cannot be a proper quotient of $N_{2,k}$.

Observe that $K = \ker \alpha$ is the center of G, in particular is abelian. We have

$$K \longrightarrow\!\!\!\!\!\rightarrow K/\langle g_i \rangle \cap K \xrightarrow{\ \sim\ } T \times \mathbb{Z}^{\binom{m}{2}-\binom{k}{2}+n} \longrightarrow\!\!\!\!\!\rightarrow \mathbb{Z}^{\binom{m}{2}-\binom{k}{2}+n}$$

where T is some finite abelian group. Indeed $K \simeq \mathbb{Z}^{\binom{m}{2}+n}$ and $\langle g_i \rangle \cap K \simeq \mathbb{Z}^{\binom{k}{2}}$. We denote the whole composition by β, and the composition of the first two arrows by γ.

The image $\beta(H \cap K) \leqslant \mathbb{Z}^{\binom{m}{2}-\binom{k}{2}+n}$ is isomorphic to $\mathbb{Z}^\ell$ for some $\ell \leqslant \binom{m}{2} - \binom{k}{2} + n$. Fix $h_1, \ldots, h_\ell \in H \cap K$ such that $\{\beta(h_j)\}$ is a basis of $\beta(H \cap K)$.

Claim 2. As K is abelian, $\beta\colon \langle h_j \rangle \twoheadrightarrow \beta(H \cap K) \simeq \mathbb{Z}^\ell$ is an isomorphism.

Claim 3. We have $H' \coloneqq \langle g_i, h_j \rangle \simeq N_{2,k} \times \mathbb{Z}^\ell$. It suffices to check that

- $\langle g_i \rangle$ and $\langle h_j \rangle$ commutes, which is true as $\langle h_j \rangle \leqslant K = \mathcal{Z}(G)$.
- $\langle g_i \rangle \cap \langle h_j \rangle = 1$, which is true as $\langle h_j \rangle \subset K$, $\beta|_{\langle h_j \rangle}$ is injective and $\beta|_{\langle g_i \rangle \cap K}$ is zero.

Claim 4. $H' \trianglelefteq H$ and H/H' is finite.

As $\alpha(H') = \alpha(H)$, we have $[H', H'] = [H, H]$ and therefore $H' \trianglelefteq H$. Moreover,

$$H/H' \simeq H \cap K/H' \cap K \simeq \gamma(H \cap K)/\gamma(H' \cap K) \simeq \gamma(H \cap K) \cap T$$

using the Nine lemma, the third isomorphism theorem, and the Nine lemma again.

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & H' \cap K & \rightarrow & H' & \rightarrow & \alpha(H') & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow\wr & & \\
1 & \rightarrow & H \cap K & \rightarrow & H & \rightarrow & \alpha(H) & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & H \cap K/H' \cap K & \overset{\sim}{\dashrightarrow} & H/H' & \rightarrow & 1 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 1 & \rightarrow & \gamma(H' \cap K) & \xrightarrow{\sim} & \beta(H' \cap K) & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow\wr & & \\
1 & \rightarrow & \gamma(H \cap K) \cap T & \rightarrow & \gamma(H \cap K) & \rightarrow & \beta(H \cap K) & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \gamma(H \cap K) \cap T & \overset{\sim}{\dashrightarrow} & \gamma(H \cap K)/\gamma(H' \cap K) & \rightarrow & 1 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

It should be noted that everything is effectively computable from generators for H. $\square$

Finally, Theorem 6.B follows easily:

Corollary 6.2.3. *There exist $m, n \geqslant 0$ such that $N_{2,m} \times \mathbb{Z}^n$ has decidable Submonoid Membership Problem and undecidable Rational Subset Membership Problem.*

Proof. Consider a group $G = N_{2,m} \times \mathbb{Z}^n$ with undecidable $\mathsf{RatM}(G)$ and m minimal. Such a group does exist by [KLZ16], and [MT17] even proves $m \leqslant 26$.

We prove that $\mathsf{SMM}(G)$ is decidable. Using Theorem 6.2.1, the problem reduces to finitely many instances of $\mathsf{RatM}(H)$ for subgroups $H \leqslant G$ with $h([H,H]) < h([G,G])$. However Lemma 6.2.2 tells us such subgroups admit finite-index subgroups $H' \simeq N_{2,k} \times \mathbb{Z}^\ell$ with $k < m$ and $\ell \leqslant \binom{m}{2} - \binom{k}{2} + n$. In turn, $\mathsf{RatM}(H)$ reduces to $\mathsf{RatM}(N_{2,k} \times \mathbb{Z}^\ell)$ using Lemma 2.6.6(a), and the latter is decidable as $k < m$. □

Remark 6.2.4. It is important that subgroups of $N_{2,m} \times \mathbb{Z}^n$ fall into finitely many abstract commensurability classes, otherwise $\mathsf{RatM}(H)$ could be decidable for each group H without any uniform algorithm working for all H. Here, we can compute the values of k and ℓ, and decide which of the finitely many algorithms to apply.

6.3 Rational subsets are bounded

The goal of this section is to prove the following technical proposition

Proposition 6.3.1. *Let $R \subseteq H_3(\mathbb{Z})$ be a rational subset, i.e., $R = \mathrm{ev}(\mathcal{L})$ for some regular language $\mathcal{L} \subseteq S^\star$. There exists a* bounded *regular language $\mathcal{L}'$ such that $R = \mathrm{ev}(\mathcal{L}')$. Moreover, an automaton for $\mathcal{L}'$ can be effectively computed.*

Recall a regular language is *bounded* if it satisfies the following equivalent conditions:

Theorem ([Tro81])**.** *Fix Σ an alphabet and $\mathcal{K} \subseteq \Sigma^\star$ a regular language. TFAE*

(a) *$\mathcal{K}$ has polynomial growth.*

(b) *$\mathcal{K}$ is* bounded*, i.e., $\mathcal{K} \subseteq \{w_1\}^*\{w_2\}^* \ldots \{w_r\}^*$ for some $w_i \in \Sigma^\star$.*

(c) *$\mathcal{K}$ is a finite union of languages $t_0\,\{u_1\}^*\,t_1\,\{u_2\}^*\,t_2 \ldots \{u_s\}^*\,t_s$ with $t_i, u_i \in \Sigma^\star$.*

6.3.1 The Abelian case

We first prove the analogous result for (virtually) abelian groups, using a classical result due to Eilenberg and Schützenberger.

Theorem 6.3.2 ([ES69])**.** *Let G be an abelian group and R be a rational subset, then R is unambiguously rational. Moreover, given a language $\mathcal{L}$ such that $\mathrm{ev}(\mathcal{L}) = R$, we can effectively compute a regular normal form $\mathcal{L}'$ for R.*

Corollary 6.3.3. *Any rational subset of a virtually abelian group can be represented by a bounded regular language $\mathcal{L}'$. Moreover, $\mathcal{L}'$ can be effectively computed.*

Proof. Take $\mathcal{L}'$ the normal form from Theorem 6.3.2, and let S be the (finite) set of elements appearing as labels on an automaton recognizing $\mathcal{L}$. Note that $\ell(w) \geqslant \|\bar{w}\|_S$ for all $w \in S^*$ hence $\beta_{\mathcal{L}'}(n) \leqslant \beta_{(\langle S\rangle,S)}(n)$, but $\langle S\rangle$ is f.g. virtually abelian hence has polynomial growth, so $\mathcal{L}'$ has polynomial growth hence is bounded. □

6.3.2 Reduction to automata with $\mathsf{start} = \mathsf{accept} = \{v\}$

This is a technical subsection to reduce to the case $\mathsf{start} = \mathsf{accept} = \{v\}$ (i.e., to the case of rational submonoids). In this subsection, G can be an arbitrary group.

We consider a general rational subset $R = \mathrm{ev}(\mathcal{L})$, where $\mathcal{L}$ is accepted by a, automaton $M = (V, S, \delta, v_0, \mathsf{accept})$.

▶ Using a loop-erasure algorithm, we decompose each word $w \in \mathcal{L}$ as

$$w = w_0 s_1 w_1 s_2 \dots s_\ell w_\ell$$

where $s_1, s_2, \dots, s_\ell \in G$ label a simple path $v_0 \xrightarrow{s_1} v_1 \xrightarrow{s_2} \dots \xrightarrow{s_\ell} v_\ell$ with $v_\ell \in \mathsf{accept}$, and each $w_i \in S^\star$ labels a cycle $v_i \to v_i$. We proceed as follows:

Start at the vertex v_0 and skip directly to the last visit of v_0, hence bypassing a (possibly empty) cycle w_0 from v_0 to v_0, then go to the next vertex. Each time you enter a new vertex v_i, skip directly to the last visit of v_i (bypassing another cycle), then keep going.

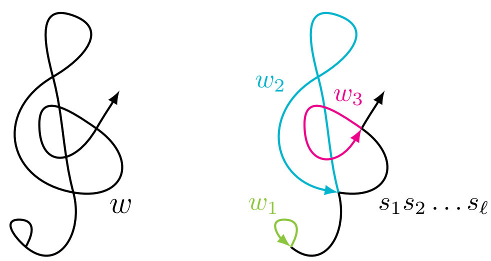


Figure 6.2: A path in the automaton and its decomposition

▶ It follows that

$$\mathcal{L} = \bigcup \mathcal{L}_{v_0\to v_0} \cdot s_1 \cdot \mathcal{L}_{v_1\to v_1} \cdot s_2 \cdot \dots \cdot s_\ell \cdot \mathcal{L}_{v_\ell\to v_\ell}$$

where the union is taken over simple paths $v_0 \xrightarrow{s_1} v_1 \xrightarrow{s_2} \dots \xrightarrow{s_\ell} v_\ell$ with $v_\ell \in \mathsf{accept}$, and $\mathcal{L}_{v\to v}$ is the language of words labeling cycles $v \to v$. The union is finite, and each language $\mathcal{L}_{v\to v}$ is regular (accepted by the G-automaton $M_{v\to v} = (V, \delta, v, \{v\})$).

If we managed to find *bounded* regular languages $\mathcal{L}'_{v\to v}$ such that $\mathrm{ev}(\mathcal{L}'_{v\to v}) = \mathrm{ev}(\mathcal{L}_{v\to v})$, we would be done with the language

$$\mathcal{L}' = \bigcup_{s_1 s_2 \dots s_\ell} \mathcal{L}'_{v_0\to v_0} \cdot s_1 \cdot \mathcal{L}'_{v_1\to v_1} \cdot s_2 \cdot \dots \cdot s_\ell \cdot \mathcal{L}'_{v_\ell\to v_\ell}$$

which is bounded, regular, and evaluates to R. This is the subject of the next subsection.

6.3.3 Main discussion

We now fix $G = H_3(\mathbb{Z})$. It is often useful to think of it as the lattice generated by x, y in $H_3(\mathbb{R})$ (see Section 3.1). Recall that $[x, y] = z$, and more generally $[g, h] = z^{\det(\hat{g},\hat{h})}$.

We consider an automaton $M = (V, S, \delta, v, \{v\})$ recognizing a language $\mathcal{L}$, evaluating to a rational subset R. In particular $\mathcal{L}$ is a submonoid of $S^\star$, and R is a submonoid of $H_3(\mathbb{Z})$. As in Theorem 2.6.3, let

$$X = \left\{ \bar{t}\,\bar{u}\,\bar{t}^{-1} \;\middle|\; \begin{array}{c} t \in S^\star \text{ labels a simple path } v \to p \\ u \in S^\star \text{ labels a simple cycle } p \to p \\ \text{both paths only intersect at } p \end{array} \right\}.$$

We will work with both $\mathcal{L}$ and $\tilde{\mathcal{L}}$ defined in Theorem 2.6.3. (Note that $Y = \{\varepsilon\}$.)

Let $\mathrm{Pr}\colon H_3(\mathbb{Z}) \twoheadrightarrow \mathbb{Z}^2 \colon g \mapsto \hat{g}$ be the abelianization map. We discuss depending on the subset positively spanned by $\mathrm{Pr}(X)$, i.e., depending on

$$\{\lambda_1 y_1 + \ldots + \lambda_r y_r \mid \lambda_i \in \mathbb{R}_{\geqslant 0},\ y_i \in \mathrm{Pr}(X)\}.$$

(1) If $\mathrm{Pr}(X)$ is included in a line.

(2) If $\mathrm{Pr}(X)$ spans the whole plane (i.e., $\mathbf{0}$ belong to the interior of $\mathrm{ConvHull}(\mathrm{Pr}(X))$).

(3) If $\mathrm{Pr}(X)$ spans a half-plane.

(4) If $\mathrm{Pr}(X)$ spans a cone.

In each case, we provide an (effectively computable) bounded regular language $\mathcal{L}'$ such that $\mathrm{ev}(\mathcal{L}') = \mathrm{ev}(\mathcal{L})$. Case (4) will take most of our time.

(1) $\mathrm{Pr}(X)$ spans $\{\mathbf{0}\}$, a ray or a line

In this case, the subgroup $\langle X\rangle$ is abelian (isomorphic to $\{e\}$, $\mathbb{Z}$ or $\mathbb{Z}^2$, but one might as well think we work in $\mathbb{Z}^X$). As R is rational in $\langle X\rangle$ (Corollary 2.6.4), we can compute a bounded regular language $\mathcal{L}' \subset \langle X\rangle^\star$ representing R (Corollary 6.3.3).

(2) $\mathrm{Pr}(X)$ spans the whole plane

For each $x \in X$, the set R contains $w_{v\to p}\, x^m\, w_{p\to v}$, where $w_{p\to q}$ are fixed words labeling paths $p \to q$, and m is arbitrarily large. Applying Proposition 6.1.1 for $S = R$, we get that R is a subgroup, hence $R = \langle X\rangle$ by Theorem 2.6.3(d). A Mal'cev basis is given by

- $a, b \in \langle X\rangle$ such that $\mathrm{Pr}(a), \mathrm{Pr}(b)$ form a basis of $\mathrm{Pr}(\langle X\rangle) \leqslant \mathbb{Z}^2$. These can be found using Gaussian elimination on the matrix with vectors $\mathrm{Pr}(x)$ $(x \in X)$ as rows.
- $z^d \in \langle X\rangle$ with $d > 0$ minimum. (This d exists as $z^{\det(\mathrm{Pr}(a),\mathrm{Pr}(b))} = [a; b] \in \langle X\rangle$, and can be found as the Subgroup Membership is decidable in $H_3(\mathbb{Z})$ [Mal58].)

We deduce a regular normal form $\mathcal{L}' = \{a^p b^q (z^d)^r \mid p, q, r \in \mathbb{Z}\} \subset G^\star$ for R.

(3) $\mathrm{Pr}(X)$ spans a half-plane

By hypothesis, we can find

- $\bar{s}\,\bar{a}\,\bar{s}^{-1}$ and $\bar{t}\,\bar{c}\,\bar{t}^{-1} \in X$ such that $\hat{a}, \mathbf{0}$ and $\hat{c} \in \mathbb{Z}^2$ are aligned in that order. We also fix $\tilde{s} \in S^\star$ labeling a path from the endpoint of s to v. Define $\tilde{t} \in S^\star$ similarly.
- $b \in \mathcal{L}$ such that $\hat{b}$ doesn't lie on the same line. (Take $\bar{r}\,\bar{u}\bar{r}^{-1} \in X$ such that $\hat{u}$ doesn't lie on the line. Fix $\tilde{r} \in S^\star$ from the endpoint of r to v, and take $b = ru\tilde{r}$.)

Each vector $\mathbf{v} \in \mathbb{Z}^2$ can be decomposed uniquely as $\mathbf{v} = \alpha(\mathbf{v}) \cdot \hat{a} + \beta(\mathbf{v}) \cdot \hat{b}$.

Lemma 6.3.4. *There exists a computable $K \geqslant 0$ such that, if $g \in R$ satisfies $\beta(\hat{g}) \geqslant K$, then g can be written as*

$$g = \mathrm{ev}\big(sa^{n_1}\tilde{s} \cdot b^m \cdot tc^{n_2}\tilde{t} \cdot b^m \cdot sa^{n_3}\tilde{s} \cdot b^{n_4} \cdot w\big)$$

where m is fixed, $n_1, n_2, n_3, n_4 \in \mathbb{Z}_{\geqslant 0}$, and w varies in a fixed finite subset of $\mathcal{L}$.

Proof. Let $h(n_1, n_2, n_3, n_4) = \mathrm{ev}\left(sa^{n_1}\tilde{s} \cdot b^m \cdot tc^{n_2}\tilde{t} \cdot b^m \cdot sa^{n_3}\tilde{s} \cdot b^{n_4}\right)$. We fix $p, q \in \mathbb{Z}_{>0}$ such that $p\hat{a} + q\hat{c} = \mathbf{0}$. In particular, we have

$$\mathrm{Pr}\left(h(n_1+p, n_2+q, n_3, n_4)\right) = \mathrm{Pr}\left(h(n_1, n_2, n_3, n_4)\right) = \mathrm{Pr}\left(h(n_1, n_2+q, n_3+p, n_4)\right)$$

Let us see how the area changes under the same transformations:

$$A\big(h(n_1+p, n_2+q, n_3, n_4)\big) - A\big(h(n_1, n_2, n_3, n_4)\big) = \underbrace{pA(a) + qA(c) + [p\hat{a}; m\hat{b} + \mathrm{Pr}(\tilde{s}t)]}_{d_+(m)}$$

$$A\big(h(n_1, n_2+q, n_3+p, n_4)\big) - A\big(h(n_1, n_2, n_3, n_4)\big) = \underbrace{pA(a) + qA(c) - [p\hat{a}; m\hat{b} + \mathrm{Pr}(\tilde{t}s)]}_{d_-(m)}$$

We fix m large enough so that $d_+(m) \cdot d_-(m) < 0$ and fix $d = \gcd(d_+, d_-)$.

Consider the finite-index normal subgroup $N = \langle a^d, b^d, c^d, z^d \rangle$. The quotient Cayley graph is easily constructed. (To know if $g, h \in H_3(\mathbb{Z})$ lie in the same coset, compare $\mathrm{Pr}(g), \mathrm{Pr}(h)$ in $\mathbb{Z}^2/d\langle \mathrm{Pr}(a), \mathrm{Pr}(b), \mathrm{Pr}(c)\rangle$ and $A(g), A(h)$ in $\mathbb{Z}/d\mathbb{Z}$.) Using Lemma 2.6.6, we construct a finite subset $\mathcal{K} \subset \mathcal{L}$ representing each coset of N intersecting R.

We take $K = \beta(\mathrm{Pr}(h(0,0,0,0))) + \max_{w \in \mathcal{K}} \beta(\hat{w})$. Recall that R is a submonoid in $H_3(\mathbb{Z})$, hence its image in the finite quotient $H_3(\mathbb{Z})/N$ is a subgroup. It follows that, for any $g \in R$, we can find $w \in \mathcal{K}$ such that

$$N \cdot \bar{w} = N \cdot h(0,0,0,0)^{-1} g$$

(as g and $h(0,0,0,0) \in R$). If furthermore $\beta(g) \geqslant K$, there exists $m_1, m_2, m_4 \in \mathbb{Z}_{\geqslant 0}$ s.t.

$$\begin{aligned}\mathrm{Pr}(g) &= \mathrm{Pr}\left(h(0,0,0,0)\right) + d\big(m_1\,\mathrm{Pr}(a) + m_2\,\mathrm{Pr}(c) + m_4\,\mathrm{Pr}(b)\big) + \mathrm{Pr}(\bar{w}) \\ &= \mathrm{Pr}\left(h(dm_1, dm_2, 0, dm_4)\bar{w}\right)\end{aligned}$$

(We can ensure $m_1, m_2 \geqslant 0$ as $\Pr(a), \Pr(c)$ are colinear, in opposite direction. Moreover, $dm_4 = \beta(\hat{g}) - \beta(\Pr(h(0,0,0,0))) - \beta(\hat{w}) \geqslant \beta(\hat{g}) - K \geqslant 0$.) Finally

$$A(g) \equiv A\big(h(0,0,0,0)\bar{w}\big) \equiv A\big(h(dm_1, dm_2, 0, dm_4)\bar{w}\big) \pmod{d}$$

hence, using the two transformations described above, we can find $n_1, n_2, n_3, n_4 \in \mathbb{Z}_{\geqslant 0}$ such that $g = h(n_1, n_2, n_3, n_4)\bar{w}$. □

Finally, we can decompose into two (effectively computable) regular languages

$$\mathcal{L} = \{w \in \mathcal{L} \mid \beta(\hat{w}) \geqslant K\} \sqcup \{w \in \mathcal{L} \mid \beta(\hat{w}) < K\} =: \mathcal{L}_{\text{reg}} \sqcup \mathcal{L}_{\text{abn}}.$$

- The first term can be replaced by a bounded regular language

$$\mathcal{L}'_{\text{reg}} = \{sa^{n_1}\tilde{s} \cdot b^m \cdot tc^{n_2}\tilde{t} \cdot b^m \cdot sa^{n_3}\tilde{s} \cdot b^{n_4} \cdot w \mid w \in \mathcal{K}\} \subseteq \mathcal{L}.$$

  Using Lemma 6.3.4, we have $\text{ev}(\mathcal{L}_{\text{reg}}) \subseteq \text{ev}(\mathcal{L}'_{\text{reg}}) \subseteq \text{ev}(\mathcal{L}) = R$.

- For the second term, we compute a trim automaton for $\mathcal{L}_{\text{abn}}$ and compute the set X associated to each strongly connected component. For each component, $\Pr(X)$ is contained in the line through $\Pr(a)$. We can apply the arguments of §6.3.2 and §6.3.3 to get a bounded regular language $\mathcal{L}'_{\text{abn}}$ such that $\text{ev}(\mathcal{L}_{\text{abn}}) = \text{ev}(\mathcal{L}'_{\text{abn}})$.

The language we are looking for is $\mathcal{L}' = \mathcal{L}'_{\text{reg}} \cup \mathcal{L}'_{\text{abn}}$.

(4) $\Pr(X)$ **spans a cone**

We construct a bounded regular language $\mathcal{L}'_+$ such that

$$\{g \in \text{ev}(\mathcal{L}) \mid A(g) \geqslant 0\} \subseteq \text{ev}(\mathcal{L}'_+) \subseteq \text{ev}(\mathcal{L}).$$

Obviously, we can do the same thing for elements of negative area, hence taking the union of both languages gives the desired bounded regular language $\mathcal{L}'$.

Consider

- tat^{-1} with t a simple path from v to some p, and a a simple loop from p to p such that $\Pr(\bar{a})$ belong on the lower side of the cone, and $\frac{A(tat^{-1})}{\|\Pr(a)\|}$ is maximized. We also fix $\tilde{t} \in S^\star$ labeling a path back from p to v.
- $s^{-1}bs$ with s a simple path from some q to v, and b a simple loop from q to q such that $\Pr(\bar{b})$ belong on the upper side of the cone, and $\frac{A(s^{-1}bs)}{\|\Pr(b)\|}$ is maximized. We also fix $\tilde{s} \in S^\star$ labeling a path from v to q.

Each $\mathbf{v} \in \mathbb{Z}^2$ can be decomposed uniquely as $\mathbf{v} = \alpha(\mathbf{v}) \cdot \hat{a} + \beta(\mathbf{v}) \cdot \hat{b}$. Moreover, $\mathbf{v}$ belongs to the cone if and only if $\alpha(\mathbf{v}) \geqslant 0$ and $\beta(\mathbf{v}) \geqslant 0$.

(4a) There does not exist $x \in X$ such that $\Pr(x) = \mathbf{0}$ and $A(x) > 0$.

This is the difficult case. If $\hat{g}$ is far from the border of the cone, and $A(g)$ is far from its maximum (for a fixed value of $\hat{g}$), then we can found a word representing g is a bounded sub-language of $\mathcal{L}$.

Lemma 6.3.5. *There exist computable $K, m \geqslant 0$ such that, if $g \in R$ satisfies*

$$0 \leqslant A(g) \leqslant \tfrac{1}{2}[a;b] \cdot \alpha(\hat{g})\beta(\hat{g}) + A(tat^{-1}) \cdot \alpha(\hat{g}) + A(s^{-1}bs) \cdot \beta(\hat{g}) - K,$$
$$\alpha(\hat{g}) \geqslant K \quad \text{and} \quad \beta(\hat{g}) \geqslant K,$$

then g can be written as

$$g = \operatorname{ev}\big(ta^{n_1}\tilde{t} \cdot \tilde{s}b^{n_2}s \cdot ta^{n_3}\tilde{t} \cdot w \cdot \tilde{s}b^{p_1}s \cdot ta\tilde{t} \cdot \tilde{s}b^{p_2}s \cdot \ldots \cdot ta\tilde{t} \cdot \tilde{s}b^{p_m}s\big)$$

where $n_1, n_2, n_3, p_1, p_2, \ldots, p_m$ varies in $\mathbb{Z}_{\geqslant 0}$ and w varies in a fixed finite subset of $\mathcal{L}$.

Proof. Let

$$h(n_1, n_2, n_3; w; p_1, \ldots, p_m) = \operatorname{ev}\big(ta^{n_1}\tilde{t} \cdot \tilde{s}b^{n_2}s \cdot ta^{n_3}\tilde{t} \cdot w \cdot \tilde{s}b^{p_1}s \cdot ta\tilde{t} \cdot \tilde{s}b^{p_2}s \cdot \ldots \cdot ta\tilde{t} \cdot \tilde{s}b^{p_m}s\big)$$

The operation $(p_i, p_{i+1}) \to (p_i + 1, p_{i+1} - 1)$ preserves $\hat{h}$, and decreases the area by

$$A\big(h(\ldots, p_i, p_{i+1}, \ldots)\big) - A\big(h(\ldots, p_i + 1, p_{i+1} - 1, \ldots)\big) = [s \cdot ta\tilde{t} \cdot \tilde{s}; b].$$

Note that $st\tilde{t}\tilde{s}$ labels a cycle in the automaton, so $\alpha(\Pr(st\tilde{t}\tilde{s})) \geqslant 0$. Let

$$d = [s \cdot ta\tilde{t} \cdot \tilde{s}; b] = \big(1 + \alpha(\Pr(st\tilde{t}\tilde{s}))\big) \cdot [a;b] \geqslant [a;b] > 0.$$

We fix $m = 2d + 1$ and consider $N = \langle a^d, b^d, z^d \rangle$. Using Lemma 2.6.6(c), we compute a finite set $\mathcal{K} \subset \mathcal{L}$ containing a representative for each coset of N intersecting R.

For each $g \in R$, we find parameters so that $h(\ldots) = g$. We split this into four steps

(1) We find $w \in \mathcal{K}$ such that $g = h(0, 0, 0; w; 0, \ldots, 0)$ in G/N.

(2) For K (computably) large enough, we find $N_1, P_m \geqslant 0$ such that

$$g = h(N_1, 0, 0; w; 0, \ldots, 0, P_m)$$

in G/N and $G/[G, G] = \mathbb{Z}^2$. Moreover $A(g) \leqslant A(h)$.

(3) For K large enough, we find $n_1, n_2, n_3, p_m \geqslant 0$ with $n_2 \leqslant \frac{3}{5}\beta(\hat{g})$ and

$$g = h(n_1, n_2, n_3; w; 0, \ldots, 0, p_m)$$

in G/N and $G/[G, G] = \mathbb{Z}^2$. Moreover

$$A(h) \geqslant A(g) \geqslant A(h) - d\left(\frac{3}{5}\beta(\hat{g}) + \beta(\Pr(\tilde{t}\tilde{s}st))\right)[a;b].$$

(4) For K large enough, we find $n_1, n_2, n_3, p_1, \ldots, p_m \geqslant 0$ such that $g = h(\ldots)$.

(1) Take $w \in \mathcal{K}$ such that

$$w = (t\tilde{t} \cdot \tilde{s}s \cdot t\tilde{t})^{-1} \cdot g \cdot \left(\tilde{s}s \cdot ta\tilde{t} \cdot \tilde{s}s \cdot \ldots \cdot ta\tilde{t} \cdot \tilde{s}s\right)^{-1} \qquad \text{in } G/N.$$

Such a w exists as the image of the monoid R in G/N is a subgroup.

(2) Take

$$N_1 = \alpha(\hat{g}) - \alpha(\hat{h}(0,0,0;w;0,\ldots,0))$$
$$P_m = \beta(\hat{g}) - \beta(\hat{h}(0,0,0;w;0,\ldots,0))$$

For K computably large enough, we have $N_1, P_m \geqslant 0$. By construction, we have

$$\hat{g} = \hat{h}(N_1,0,0;w;0,\ldots,0,P_m).$$

Moreover $d \mid N_1, P_m$, hence g and h still coincide in G/N. The area of h is given by

$$A(ta^{N_1} \cdot v \cdot b^{P_m}s) = \frac{1}{2}[a;b] \cdot \alpha(\hat{g})\beta(\hat{g}) + A(tat^{-1}) \cdot \alpha(\hat{g}) + A(s^{-1}bs) \cdot \beta(\hat{g}) - C(w)$$

with $v = \tilde{t}\tilde{s}st\tilde{t}w(\tilde{s}sta\tilde{t})^{m-1}\tilde{s}$, and $C(w)$ computable. For K computably large enough, we may therefore assume $A(g) \leqslant A(h(N_1,0,0;w;0,\ldots,0,P_m))$.

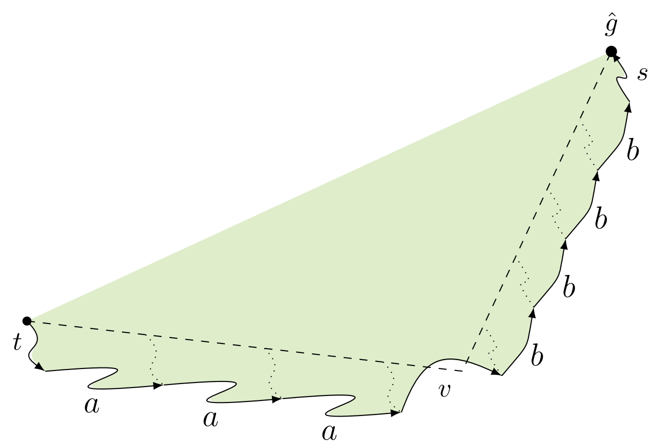


Figure 6.3: The path corresponding to $h(N_1,0,0;w;0,\ldots,0,P_m)$.

Indeed, the big triangle has area $\frac{1}{2}[a;b] \cdot \alpha(\hat{g})\beta(\hat{g})$. The smaller regions bordered by a and b have area $A(tat^{-1})$ and $A(s^{-1}bs)$ respectively, and there are N_1 and P_m of those.

(3) Next, we apply two transformations in order

- $(n_2, p_m) \to (n_2 + d, p_m - d)$. This doesn't change $\hat{h}$, and decreases the area by

$$A\big(g(N_1,n_2,0;w;0,\ldots,0,p_m)\big) - A\big(g(N_1,n_2+d,0;w;0,\ldots,0,p_m-d)\big)$$
$$= [s \cdot t\tilde{t} \cdot w \cdot (\tilde{s}s \cdot ta\tilde{t})^{m-1} \cdot \tilde{s}; b^d] > 0$$

  Repeat as long as $n_2 \leqslant \frac{3}{5}\beta(\hat{g})$ and $A(h) \geqslant A(g)$.

- $(n_1,n_3)\to(n_1-d,n_3+d)$. This doesn't change $\hat{h}$, and decreases the area by

$$\begin{aligned}&A\big(g(n_1,n_2,n_3;w;\ldots)\big)-A\big(g(n_1-d,n_2,n_3+d;w;\ldots)\big)\\&=[a^d;\tilde{t}\cdot\tilde{s}b^{n_2}s\cdot t]=d\big(n_2+\beta(\Pr(\tilde{t}\tilde{s}st))\big)[a;b]\end{aligned}$$

  Repeat as long as $n_1\geqslant 0$ and $A(h)\geqslant A(g)$.

At the end of this process, we have found an element $h(n_1,n_2,n_3;w;0,\ldots,0,p_m)$ with $n_1,n_2,n_3,p_m\geqslant 0$ and $n_2\leqslant\frac{3}{5}\beta(\hat{g})$, which coincides with g in G/N and $G/[G,G]$. Moreover, as we cannot apply the second step anymore, we have

$$0\leqslant A(g)-A(h)\leqslant d\big(n_2+\beta(\Pr(\tilde{t}\tilde{s}st))\big)[a;b]\leqslant d\left(\frac{3}{5}\beta(\hat{g})+\beta(\Pr(\tilde{t}\tilde{s}st))\right)[a;b]$$

Here we use that $A(g)\geqslant 0$. If we had continued until the conditions $n_2\leqslant\frac{3}{5}\beta(\hat{g})$ and $n_1\geqslant 0$ were the limiting conditions, we would have reached an area of

$$A(h)\approx A\left(0,\frac{3}{5}P_m,N_1;w;0,\ldots,0,\frac{2}{5}P_m\right)\approx-\frac{1}{10}[a;b]\cdot\alpha(\hat{g})\beta(\hat{g})<0\leqslant A(g)$$

for K computably large enough. So the condition $A(h)\geqslant A(g)$ stopped us.

(4) Finally, we use the operation $(p_i,p_{i+1})\to(p_i+1,p_{i+1}-1)$. We can use this operation up to $(m-1)\cdot(P_m-n_2)$ times, decreasing the area by exactly d each time. The total variation obtainable is therefore larger than

$$2d\cdot\left(\frac{2}{5}\beta(\hat{g})-\beta(\hat{h}(0,0,0;w;0,\ldots,0))\right)\cdot d\geqslant d\left(\frac{3}{5}\beta(\hat{g})+\beta(\Pr(\tilde{t}\tilde{s}st))\right)[a;b]$$

as $d\geqslant[a;b]$ and $\beta(\hat{g})\geqslant K$. In conclusion, we have $A(h)=A(g)$ at some point. □

As in the previous case, we decompose $\mathcal{L}$ into two regular languages, with all elements in the first set treated by Lemma 6.3.5, and all elements in the second case following the border of the cone (hence being easier to treat). We first need a lemma to take care of the "area condition" with a finite state automaton.

Lemma 6.3.6. *Consider a word $w\in\mathcal{L}$ and $x_1x_2\ldots x_\ell\in X^*$ its decomposition. Let*

$$\begin{aligned}M(w)&=\#\left\{i\;\middle|\;1\leqslant i\leqslant\ell\text{ and }\alpha(x_i),\beta(x_i)>0\right\}\\N(w)&=\#\left\{(i,j)\;\middle|\;1\leqslant i<j\leqslant\ell\text{ and }\alpha(x_j),\beta(x_i)>0\right\}\end{aligned}$$

There exists a computable constant L such that, if

$$A(\bar{w})\geqslant\frac{1}{2}[a;b]\cdot\alpha(\hat{w})\beta(\hat{w})+A(tat^{-1})\cdot\alpha(\hat{w})+B(s^{-1}bs)\cdot\beta(\hat{w})-K,$$

then $M(w),N(w)\leqslant L$.

Proof. Let

$$\Delta = \max\Big(\big\{A(x) \mid x \in X\big\} \cup \{0\}\Big)$$
$$\delta = \min\Big(\big\{\alpha(x), \beta(x) \mid x \in X\big\} \setminus \{0\}\Big)$$

We can bound the area of $\bar{w}$ by

$$\begin{aligned}
A(\bar{w}) &= \sum_{i<j} \frac{1}{2}[x_i; x_j] + \sum_i A(x_i) \\
&= \frac{1}{2}[a;b]\left(\sum_{i<j} \alpha(x_i)\beta(x_j) - \alpha(x_j)\beta(x_i)\right) + \sum_i A(x_i) \\
&\leqslant \frac{1}{2}[a;b]\left(\sum_i \alpha(x_i) \cdot \sum_j \beta(x_j) - N(w)\cdot\delta^2\right) + \\
&\quad + \sum_{i:\beta(x_i)=0} \alpha(x_i)A(tat^{-1}) + \sum_{i:\alpha(x_i)=0} \beta(x_i)A(s^{-1}bs) + M(w)\cdot\Delta \\
&\leqslant \frac{1}{2}[a;b]\cdot\alpha(\hat{w})\beta(\hat{w}) + A(tat^{-1})\cdot\alpha(\hat{w}) + B(s^{-1}bs)\cdot\beta(\hat{w}) + \\
&\quad + M(w)\cdot\Delta - N(w)\cdot\delta^2
\end{aligned}$$

hence $M(w)\cdot\Delta - N(w)\cdot\delta^2 \geqslant -K$. Combining with $\binom{M(w)}{2} \leqslant N(w)$ we get

$$K \geqslant \frac{\delta^2}{2}\cdot M(w)\big(M(w)-1\big) - \Delta\cdot M(w)$$

hence $M(w) \leqslant L$ for some computable L. Finally $N(w) \leqslant \frac{1}{\delta^2}(M(w)\cdot\Delta + k)$. □

We decompose the language $\tilde{\mathcal{L}}$ given by Theorem 2.6.3 as

$$\tilde{\mathcal{L}} = \left\{\tilde{w} \in \tilde{\mathcal{L}} \;\middle|\; \alpha(\hat{w}), \beta(\hat{w}) \geqslant K \text{ and } N(w) \geqslant L\right\} \sqcup \big\{\text{complement}\big\} =: \mathcal{L}_{\text{reg}} \sqcup \mathcal{L}_{\text{abn}}$$

- We replace $\mathcal{L}_{\text{reg}}$ by the bounded regular language

  $$\mathcal{L}'_{\text{reg}} = \big\{ta^*\tilde{t}\cdot\tilde{s}b^*s\cdot ta^*\tilde{t}\cdot w\cdot(\tilde{s}b^*s\cdot ta\tilde{t})^{m-1}\cdot\tilde{s}b^*s \;\big|\; w\in\mathcal{K}\big\}$$

  so that $\{g \in \mathrm{ev}(\mathcal{L}_{\text{reg}}) \mid A(g) \geqslant 0\} \subseteq \mathrm{ev}(\mathcal{L}'_{\text{reg}}) \subseteq R$.

- For the second term, we compute a trim automaton for $\mathcal{L}_{\text{abn}}$ and compute the set X associated to each strongly connected component.

  For each component, $\mathrm{Pr}(X)$ is contained in the line through $\mathrm{Pr}(a)$ or through $\mathrm{Pr}(b)$. We can apply §6.3.2 and case (1) of the proof to get a bounded regular language $\mathcal{L}'_{\text{abn}}$ such that $\mathrm{ev}(\mathcal{L}'_{\text{abn}}) = \mathrm{ev}(\mathcal{L}_{\text{abn}})$.

The language we are looking for is $\mathcal{L}'_+ = \mathcal{L}'_{\text{reg}} \cup \mathcal{L}'_{\text{abn}}$.

(4b) There does exist $x \in X$ such that $\Pr(x) = \mathbf{0}$ and $A(x) > 0$.

This case is similar to case (3). We fix $r, c, \tilde{r} \in S^\star$ labeling a path $v \to p$, a cycle $p \to p$ and a path $p \to v$ respectively, such that $\hat{c} = \mathbf{0}$ and $A(\bar{c}) > 0$.

Lemma 6.3.7. *There exists a computable $K \geqslant 0$ such that, if $g \in R$ satisfies*

$$A(g) \geqslant 0, \quad \alpha(\hat{g}) \geqslant K \quad \text{and} \quad \beta(\hat{g}) \geqslant K,$$

then g can be written as $g = \mathrm{ev}\big(\tilde{s}b^{n_1}s \cdot ta^{n_2}\tilde{t} \cdot rc^{n_3}\tilde{r} \cdot w\big)$ where n_1, n_2, n_3 varies in $\mathbb{Z}_{\geqslant 0}$ and w varies in a fixed finite subset of $\mathcal{L}$.

Proof. We introduce the notation

$$h(n_1, n_2, n_3) = \mathrm{ev}\big(\tilde{s}b^{n_1}s \cdot ta^{n_2}\tilde{t} \cdot rc^{n_3}\tilde{r}\big).$$

Say $c = z^d$. We consider $N = \langle a^d, b^d, z^d \rangle$. Using Lemma 2.6.6(c), we construct a finite subset $\mathcal{K} \subset \mathcal{L}$ of representatives for each coset of N intersecting R. For $g \in R$, we take

- $w \in \mathcal{K}$ such that $\bar{w} = h(0,0,0)^{-1}g$ in G/N.
- $n_1 = \alpha(\hat{g}) - \alpha(\hat{h}(0,0,0)) - \alpha(\hat{w})$ and $n_2 = \beta(\hat{g}) - \beta(\hat{h}(0,0,0)) - \beta(\hat{w})$
- $n_3 = \frac{1}{d}\big(A(g) - A\big(h(n_1,n_2,0) \cdot \bar{w}\big)\big)$.

We should note that n_1, n_2, n_3 are integers with $d \mid n_1, n_2$. Moreover, if we suppose $\alpha(\hat{g}), \beta(\hat{g}) \geqslant K$ for some large K, we have $n_1, n_2 \geqslant K - C_1(w) \geqslant 0$ and

$$A\big(h(n_1,n_2,0) \cdot \bar{w}\big) = -\frac{1}{2}[a;b] \cdot n_1 n_2 + C_2(w) \cdot n_1 + C_3(w) \cdot n_2 + A(w) \leqslant 0$$

where C_1, C_2, C_3 are computable, implying $n_1, n_2, n_3 \geqslant 0$. Exactly how large K needs to be can be computed. Finally $g = h(n_1, n_2, n_3) \cdot \bar{w}$. □

Finally, we decompose $\mathcal{L}$ into two regular languages

$$\mathcal{L} = \{w \in \mathcal{L} \mid \alpha(\hat{w}), \beta(\hat{w}) \geqslant K\} \sqcup \{w \in \mathcal{L} \mid \alpha(\hat{w}) < K \text{ or } \beta(\hat{w}) < K\} =: \mathcal{L}_{\mathrm{reg}} \sqcup \mathcal{L}_{\mathrm{abn}}.$$

- The first term can be replaced by a bounded regular language

  $$\mathcal{L}'_{\mathrm{reg}} = \big\{\tilde{s}b^{n_1}s \cdot ta^{n_2}\tilde{t} \cdot rc^{n_3}\tilde{r} \cdot w \;\big|\; n_1, n_2, n_3 \geqslant 0, w \in \mathcal{K}\big\} \subseteq \mathcal{L}$$

  Using Lemma 6.3.7, we have $\{g \in \mathrm{ev}(\mathcal{L}_{\mathrm{reg}}) \mid A(g) \geqslant 0\} \subseteq \mathrm{ev}(\mathcal{L}'_{\mathrm{reg}}) \subseteq \mathrm{ev}(\mathcal{L})$.
- For the second term, we compute a trim automaton for $\mathcal{L}_{\mathrm{abn}}$ and compute the set X associated to each strongly connected component. For each component, $\Pr(X)$ is contained in the line through $\Pr(a)$ or through $\Pr(b)$. We can apply §6.3.2 and §6.3.3 to get a bounded regular language $\mathcal{L}'_{\mathrm{abn}}$ such that $\mathrm{ev}(\mathcal{L}'_{\mathrm{abn}}) = \mathrm{ev}(\mathcal{L}_{\mathrm{abn}})$.

The language we are looking for is $\mathcal{L}'_+ = \mathcal{L}'_{\mathrm{reg}} \cup \mathcal{L}'_{\mathrm{abn}}$.

6.3.4 Proof of Theorem 6.C

Finally, we prove that $\mathsf{RatM}(H_3(\mathbb{Z}))$ is decidable, using Proposition 6.3.1. We prove a stronger result about solving equations under rational constraints.

Definition 6.3.8. An equation with rational constraints in G is the data of

- an element $w \in F(x_1, \ldots, x_n) * G$ and
- n rational subsets $R_1, \ldots, R_n \subseteq G$.

This equation admits a solution if there exists $(g_1, \ldots, g_n) \in R_1 \times \ldots \times R_n$ such that $w(g_1, \ldots, g_n) := f(w) = e_G$, where f is the homomorphism

$$f\colon \begin{pmatrix} F(x_1, \ldots, x_n) * G & \longrightarrow & G \\ g & \longmapsto & g \\ x_i & \longmapsto & g_i \end{pmatrix}.$$

This generalizes the Rational Subset Membership $g \overset{?}{\in} R$, as we may consider the equation $x_1 = g$ under the rational constraint $x_1 \in R$.

Theorem 6.3.9. *There exists an algorithm which takes as input an equation with rational constraints in $H_3(\mathbb{Z})$, and decides whether it admits a solution.*

This extends the analogous result without rational constraints, due to Duchin, Liang and Shapiro. The proof is a straightforward adaptation of [DLS15, Theorem 3] and [KLZ16, Theorem 6.8], with Proposition 6.3.1 as a starting point.

Proof. We first prove the statement under the extra assumption that each rational constraint is given as $R_i = h_{i,0}\, \{k_{i,1}\}^*\, h_{i,1}\, \{k_{i,2}\}^*\, h_{i,2} \ldots h_{i,\ell_i-1}\, \{k_{i,\ell_i}\}^*\, h_{i,\ell_i}$. (Such a set will be called "Knapsack-like".) In particular, a generic element of R_i is given as

$$g_i = g_i(n_1, \ldots, n_{\ell_i}) = h_{i,0}\, k_{i,1}^{n_1}\, h_{i,1}\, \{k_{i,2}\}^*\, h_{i,2} \ldots h_{i,\ell_i-1}\, \{k_{i,\ell_i}\}^*\, h_{i,\ell_i}$$

with $n_j \in \mathbb{N}$. In coordinates, the existence of a solution reduces to the system

$$\begin{cases} \hat{w}(g_1, \ldots, g_n) &= \mathbf{0} \\ A(w(g_1, \ldots, g_n)) &= 0 \end{cases}$$

This system consists of two linear and one quadratic equations, with coefficients in $\frac{1}{2}\mathbb{Z}$ and unknowns $n_{i,j} \in \mathbb{N}$, hence we can decide if it admits a solution using [GS04].

In general, we can write each rational constraint as $R_i = \bigcup_{j=1}^{m_i} R_{ij}$ where each R_{ij} is Knapsack-like using Proposition 6.3.1. Therefore, we only have to check whether one of $m_1 \ldots m_n$ system with Knapsack-like constraints admits a solution. The final answer is "Yes" if any of these answers is yes, "No" otherwise. □

Remark 6.3.10. It should be noted that the result about Rational Subset Membership can be extended to $H_3(\mathbb{Q})$ as every finitely generated subgroup of $H_3(\mathbb{Q})$ is (effectively) isomorphic to a subgroup of $H_3(\mathbb{Z})$. Indeed, given a finite set of matrices

$$M_i = \begin{pmatrix} 1 & a_i & c_i \\ 0 & 1 & b_i \\ 0 & 0 & 1 \end{pmatrix} \in H_3(\mathbb{Q}) \quad (i = 1, 2, \ldots, r),$$

find $N \neq 0$ such that $Na_i, Nb_i, N^2c_i \in \mathbb{Z}$. The dilation $\delta_N : (a, b, c) \mapsto (Na, Nb, N^2c)$ is an automorphism $H_3(\mathbb{Q}) \to H_3(\mathbb{Q})$ such that $\delta_N\big(\langle M_i \rangle\big) \leqslant H_3(\mathbb{Z})$.

6.4 Further questions and remarks

Note that the decidability of $\mathsf{RatM}(H_3(\mathbb{Z}))$ proven in the chapter, combined with the reduction Theorem 6.A, allows to conclude

Corollary 6.4.1. *The Engel group*

$$\mathscr{E} = \big\langle x, y_1, y_2, y_3 \;\big|\; [x, y_i] = y_{i+1} \textit{ for } i = 1, 2,\; [x, y_3] = [y_i, y_j] = 1 \big\rangle$$

has decidable Submonoid Membership.

Indeed, this nilpotent group has Hirsch length $h(\mathscr{E}) = 4$, hence any infinite-index subgroup admits a finite-index subgroup isomorphic to 1, $\mathbb{Z}$, $\mathbb{Z}^2$, $\mathbb{Z}^3$ or $H_3(\mathbb{Z})$. It can be seen as a subgroup of the unitriangular matrices $UT_4(\mathbb{Z})$:

$$\mathscr{E} \simeq \mathbb{Z}^3 \rtimes_X \mathbb{Z} = \left\{ \begin{pmatrix} X^n & \mathbf{y} \\ 0 & 1 \end{pmatrix} \;\middle|\; n \in \mathbb{Z},\ \mathbf{y} \in \mathbb{Z}^3 \right\} \quad \text{where } X = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

For this reason, extending Theorem 6.C to higher Heisenberg group is desirable.

Conjecture 6.A. Higher $H_{2m+1}(\mathbb{Z})$ have decidable Rational Subset Membership.

In turn, this would imply that Submonoid Membership is decidable in f.g. nilpotent groups if $h([G, G]) \leqslant 2$, or if $G = N_{2,3} \times \mathbb{Z}^n, N_{3,2} \times \mathbb{Z}^n$.[2] Indeed, all of their subgroups satisfy $h([H, H]) = 0, 1, 3$. For $G = N_{2,3} \times \mathbb{Z}^n$, this follows from Lemma 6.2.2.

Most algorithms presented rely on results on quadratic Diophantine equations which are completely ineffective [Sie72; GS04]. There is one exception though:

Problem 6.B. Find effective bounds on the complexity of the Submonoid Membership in $G = H_{2m+1}(\mathbb{Z}) \times \mathbb{Z}^n$. Is this problem in P (for fixed m, n)?

[2] A few months after this article was first made public, Doron Shafrir proved a case of this conjecture sufficient to conclude that these groups have decidable Submonoid Membership [Sha24b].

One may wonder for which class of groups can the Rational Subset Membership be reduced to the Knapsack problem (as in Proposition 6.3.1):

Question 6.C. Characterize groups G such that every rational subset $R \subseteq G$ can be represented by a *bounded* regular language $\mathcal{L}' \subset G^\star$. In particular, does this hold if

1. G is 2-step nilpotent with cyclic derived subgroup?
2. G is 2-step nilpotent?
3. $G = \mathscr{E}$ is the Engel group?
4. Does any group of super-polynomial growth have this property?

▶ Using Theorem 2.6.3(b) and Lemma 2.6.6, we can prove that this property passes to quotients, subgroups and finite-index overgroups.

▶ Recall that 2-step nilpotent groups with infinite cyclic derived subgroup contain copies of $H_{2m+1}(\mathbb{Z}) \times \mathbb{Z}^n$ as finite-index subgroups, see [Sto96, Lemma 7.1]. In particular, we only have to consider $G = H_{2m+1}(\mathbb{Z}) \times \mathbb{Z}^n$ to answer Question 6.C.1.

A positive answer to Question 6.C.1 would be particularly interesting as it would imply decidability for the Rational Subset Membership for those groups. Indeed, the Knapsack problem is also decidable for those groups (adapting the proof of [KLZ16, Theorem 6.8] from $H_3(\mathbb{Z}) \times \mathbb{Z}^n$ to $H_{2m+1}(\mathbb{Z}) \times \mathbb{Z}^n$, then using [KLZ16, Theorem 7.3]).

▶ An instance of Question 6.C.2 of special interest is

$$G = N_{2,r} = \left\langle x_1, x_2, \ldots, x_r \;\middle|\; \text{2-step nilpotent} \right\rangle$$

with $R = \{x_1, x_2, \ldots, x_r\}^*$. A positive answer (effective in r) would allow to reduce the Submonoid Membership Problem to the Knapsack Problem in any 2-step nilpotent groups. In turn, this would provide an example of group with decidable Submonoid Membership and undecidable Knapsack problem (adapting the proof of Corollary 6.2.3). Note that the answer is positive for $r = 2$, as

$$\{x, y\}^* = \{x\}^*\{y\}^*\, x\, \{y\}^*\{x\}^* \sqcup \{y\}^*$$

in $N_{2,2} = H_3(\mathbb{Z})$. This equality is one of the key motivations behind Proposition 6.3.1.

▶ The property fails for the free 3-step nilpotent group of rank 2

$$G = N_{3,2} = \left\langle x, y \;\middle|\; \text{3-step nilpotent} \right\rangle.$$

For instance $R = \{x, y\}^*$ cannot be represented by a bounded regular language. Let $\mathcal{L}' \subset G^\star$ be a bounded regular language such that $\operatorname{ev}(\mathcal{L}') \subseteq R$, say

$$\mathcal{L}' = \bigcup_{i=1}^{I} v_{i,0}\, \{w_{i,1}\}^*\, v_{i,1}\, \{w_{i,2}\}^*\, v_{i,2} \ldots v_{i,\ell_i-1} \{w_{i,\ell_i}\}^*\, v_{i,\ell_i}.$$

Observe that $\hat{w}_{i,j} \in \mathbb{Z}^2_{\geqslant 0}$, otherwise the word

$$w = v_{i,0}\, v_{i,1} \ldots v_{i,j-1}\, w_{i,j}^N\, v_{i,j} \ldots v_{i,\ell_i} \in \mathcal{L}'$$

will project to $\hat{w} = N \cdot \hat{w}_{i,j} + O(1) \notin \mathbb{Z}^2_{\geqslant 0} = \mathrm{Pr}(R)$ for N large enough.

Take a direction $u \in \mathbb{R}^2_{\geqslant 0}$ which is not proportional to any of the $\hat{w}_{i,j}$ and take an element $\gamma_{u,n} \in \{x, y\}^*$ of length n which best follows the ray $\mathbb{R}^+ u$ (see Section 5.5). Adapting computations from Sections 4.3 and 5.5, we have

$$\left\langle B(\gamma_{u,n}) - B(\bar{w}); u^\perp \right\rangle = \Theta_u(n^3)$$

for all elements $w \in \mathcal{L}'$ such that $\hat{w} = \hat{\gamma}_{u,n}$. In particular, $\gamma_{u,n} \notin \mathrm{ev}(\mathcal{L}')$ for n large.

► We expect that no group of super-polynomial growth has the property. We confirm this conjecture for solvable groups and linear groups.

Lemma 6.4.2. *Suppose that there exists an epimorphism $\pi\colon G \twoheadrightarrow \mathbb{Z}$ and two elements $x, y \in G$ such that $\pi(x), \pi(y) > 0$ and $R = \{x, y\}^* \subset G$ is a free submonoid. Then G doesn't have the property.*

Proof. We first observe that $\#\{g \in R \mid \pi(g) \leqslant n\} \asymp 2^n$. (In particular, this set is very much finite.) On the other side, let us consider a language

$$\mathcal{L}' := h_0 \cdot g_1^* \cdot h_1 \cdot g_1^* \cdot \ldots \cdot g_d^* \cdot h_d \subset G^\star$$

such that $\mathrm{ev}(\mathcal{L}') \subseteq R$. Then $\pi(g_i) \geqslant 0$, with equality if and only if g_i has finite order. Consider $T = \{i \in [\![1, d]\!] \mid g_i \text{ is torsion}\}$. We compute

$$\#\{g \in \mathrm{ev}(\mathcal{L}') \mid \pi(g) \leqslant n_0 + n\} \leqslant \prod_{i \in T} |\langle g_i \rangle| \cdot \binom{n + d - |T|}{d - |T|} \preceq n^{d - |T|},$$

where $n_0 = \pi(h_0 h_1 \ldots h_d)$. As bounded regular languages are a finite union of languages of the previous type, this proves that no bounded regular language may represent R. □

This lemma obviously applies to non-abelian free groups. In [Bre07, Theorem 1.4], Breuillard proves that every f.g. virtually solvable group admits a finite-index subgroup for which the lemma applies. The Tits alternative concludes for linear groups.

Chapter 7

Complete growth series

We are interested in fine properties of growth functions of groups. Recall that, for a metric group (G, d), its spherical growth function is

$$\sigma_{G,d}(n) = \#\left\{g \in G \;\middle|\; \|g\| \coloneqq d(e,g) = n\right\}.$$

When studying the “regularity” of a sequence, it is natural to consider the associated series, and indeed a large literature looks at the *spherical growth series* of a group (G, d)

$$\Sigma_{G,d}(t) = \sum_{n=0}^{\infty} \sigma(n) \cdot t^n = \sum_{g \in G} z^{\|g\|} \in \mathbb{N}[[t]].$$

The key questions are to prove or disprove rationality (or algebraicity) of this series, or at least to compute its radius of convergence (the growth rate).

In this chapter, we will look at an even richer sequence, namely $S_n = \sum_{\|g\|=n} g$, seen as elements of the group semiring $\mathbb{N}G$. Just as in the standard case, the sequence $(S_n)_n$ gives rise to a growth series: the *complete* (*spherical*) *growth series* of G

$$\widehat{\Sigma}_{G,d}(t) = \sum_{n=0}^{\infty} S_n \cdot t^n = \sum_{g \in G} g \cdot t^{\|g\|} \in \mathbb{N}G[[t]].$$

Once again, the main question is whether the series are rational or not. Here there are two different notions of rationality: being $\mathbb{N}G$-rational or $\mathbb{Z}G$-rational (see Section 7.1 for the necessary definitions). The interest of rationality for standard growth series is that their coefficients are relatively easy to compute: they satisfy a linear recurrence equation. Moreover precise asymptotics are known. Similarly, if the complete growth series is $\mathbb{Z}G$-rational, then the list of elements of length n satisfies some linear recurrence relations. In some sense, being $\mathbb{N}G$-rational is even better (see Remark 7.1.12).

Proving complete growth series are $\mathbb{N}G$-rational (and sometimes computing them) has been central since their introduction in the two thesis of Liardet and Nagnibeda. They both prove strong positive results: Liardet proved that $\widehat{\Sigma}_{G,d_S}(t)$ is $\mathbb{N}G$-rational for any virtually abelian group G [Lia96], and Grigorchuk and Nagnibeda proved the same result for hyperbolic groups [GN97]. Combined with the following known (and easy) implications, this recovers more classical results due to Benson and Cannon respectively.

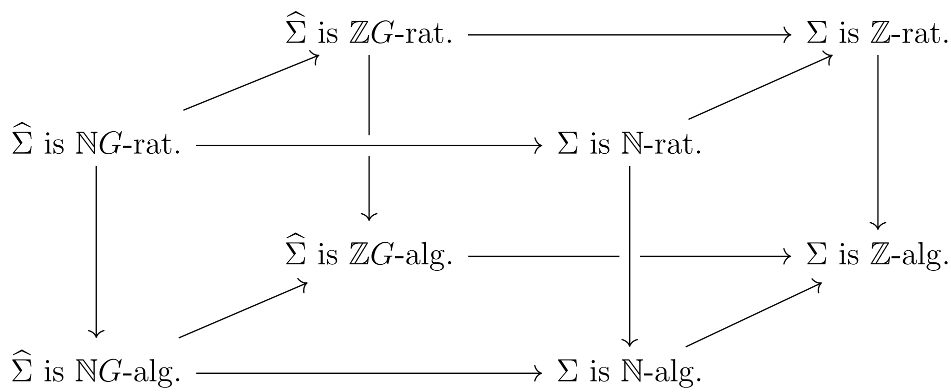


Figure 7.1: Implications between different properties (see Remark 7.1.6).

On the side of negative results, much less was known. This is illustrated by the following question of Grigorchuk and de la Harpe, and Grigorchuk and Nagnibeda:

Question 1 ([GH97]). Does there exists (G, d_S) such that the complete growth series $\widehat{\Sigma}_{G,d_S}(t)$ is *not* rational, while the standard growth series $\Sigma_{G,d_S}(t)$ *is* rational?

They suggest $G = H_3(\mathbb{Z})$ with its standard generating set. More generally,

Conjecture 2 ([GN97]). If G is nilpotent and the complete growth series $\widehat{\Sigma}_{G,d_S}(t)$ is rational. Does this implies that G is virtually abelian?

We answer positively to the first question and make progress toward the second, when understanding "rational" as "$\mathbb{N}G$-rational". We isolate two necessary conditions:

Theorem 7.A (Theorem 7.2.5). *Let (G, d) be a metric group. Suppose that*

- *(G, d_S) contains dead ends of arbitrarily large depth (and G is infinite), or*
- *(G, d_S) contains almost saddle elements of arbitrarily large depth and fixed margin.*

Then $\widehat{\Sigma}_{G,d}(t)$ is not $\mathbb{N}G$-rational. (See Section 7.2 for definitions.)

Theorem 7.B (Theorem 7.5.3). *Suppose that $\widehat{\Sigma}_{G,d_S}(t)$ is $\mathbb{N}G$-rational, then G admits a $(1 + \varepsilon, 0)$-quasi-geodesic regular normal formal, for any $\varepsilon > 0$*

Both criteria have advantages. The first is purely geometrical, "local", and robust to change to roughly isometric metrics. The second still contains elements of formal language theory, but is robust to change to asymptotically equivalent metrics.

We start by testing both criteria on the lamplighter group. We prove

Theorem 7.C (Theorem 7.2.8, Proposition 7.5.11)**.** *Consider $G = L \wr \mathbb{Z}$ with $L \not\simeq \{e\}$, and $S = S_L \sqcup \{z^{\pm}\}$ a standard generating set. Then $\widehat{\Sigma}_{G,d_S}(t)$ is not $\mathbb{N}G$-rational.*

Whenever L is finite, this follows from the existence of dead ends of arbitrarily large depths, as proven in [CT03]. (This was the first example of this type, answering another question of [GH97].) For more general groups L, we need the new notion of saddle elements. It should be noted that these results can be extended to other base groups using recent results of Silva [Sil23]. This contrast with results of Bartholdi proving that $\widehat{\Sigma}_{G,d_S}(t)$ is $\mathbb{N}G$-algebraic as soon as $\widehat{\Sigma}_{L,d_{S_L}}(t)$ is $\mathbb{N}L$-algebraic [Bar17, Theorem C.3].

Next, using the machinery of sub-Finsler geometry, we construct almost saddle elements in any (non virtually abelian) 2-step nilpotent group, and deduce that

Theorem 7.D (Theorem 7.3.6)**.** *Let (G, d_S) be a virtually 2-step nilpotent group such that $\widehat{\Sigma}_{G,d_S}(t)$ is $\mathbb{N}G$-rational, then G is virtually abelian.*

This extends results of Stoll, that (non virtually abelian) 2-step nilpotent groups do not admit geodesic regular normal form [Sto95]. For $G = H_3(\mathbb{Z})$, almost saddle elements can be replaced by dead ends of arbitrarily large depths, whose existence was proven by Warshall [War07]. We give an alternate proof of this fact.

Using a more ad-hoc argument which mixes CC-geometry and the characterization of context-free languages of polynomial growth via Dyck loops, we prove

Theorem 7.E (Theorem 7.4.5)**.** *Let (G, d_S) be a group containing a finite-index subgroup isomorphic to $H_3(\mathbb{Z})$, then $\widehat{\Sigma}_{G,d_S}(t)$ is not $\mathbb{N}G$-algebraic.*

This is quite striking as Duchin and Shapiro proved in a *tour de force* that $H_3(\mathbb{Z})$ has rational (standard) growth series with respect to any generating set [DS19]. This highlights that the "robust" mechanisms underlying the rationality in $H_3(\mathbb{Z})$ are completely different from what is going on in virtually abelian and hyperbolic groups.

Finally, using some more CC-geometry

Theorem 7.F (Theorem 7.5.14)**.** *Let (G, d_S) be a group containing a finite-index 2-generated nilpotent group. If $\widehat{\Sigma}_{G,d_S}(t)$ is $\mathbb{N}G$-rational, then G is virtually abelian.*

The geometry "up to rough isometry" of groups of nilpotency class $c \geqslant 3$ is not well-understood. However, the metric is asymptotically equivalent to the CC-metric by Pansu's theorem, which is the reason the second criterion can still be useful.

To summarize, we have the following results:

	$\mathbb{N}$-rational	$\mathbb{N}$-algebraic	$\mathbb{N}G$-rat.	$\mathbb{N}G$-alg.
Virtually abelian	✓	✓	✓	✓
Hyperbolic	✓	✓	✓	✓
Heisenberg $H_3(\mathbb{Z})$	✓	✓	✗	✗
Virtually $H_3(\mathbb{Z})$	depends?	depends?	✗	✗
Virt. nilpotent of step 2	depends	depends	✗	never?
Virt. nilpotent of rank 2	depends?	depends?	✗	never?
$H_5(\mathbb{Z})$ (standard gen.)	✗	✗	✗	✗
$C_2 \wr \mathbb{Z}$ (standard gen.)	✓	✓	✗	✓
$C_2 \wr F_2$ (standard gen.)	✗	✓	✗	✓

Table 7.1: Summary of our results (in red) together with some relevant known results (in black). The first six lines hold for any finite symmetric generating set.

This proves that the lattice of properties forming the front face of Figure 7.1 doesn't collapse: knowing that the standard growth series $\Sigma_{G,d_S}(t)$ is $\mathbb{N}$-rational does not imply that complete growth series $\widehat{\Sigma}_{G,d_S}(t)$ is $\mathbb{N}G$-rational or even $\mathbb{N}G$-algebraic.

7.1 Preliminaries

Definition 7.1.1. A metric monoid G is a monoid with a norm $\|\cdot\|$ satisfying

- $\|g\| \geqslant 0$ with equality if and only if $g = e$.
- $\|gh\| \leqslant \|g\| + \|h\|$.

If G is a group, we also suppose $\|g^{-1}\| = \|g\|$. In this case, norms are in one-to-one correspondence with distances d satisfying

- For all $g, x, y \in G$, we have $d(gx, gy) = d(x, y)$. **(Left-invariant)**

To go back and forth, we define $d(x, y) = \|x^{-1}y\|$ and $\|g\| = d(e, g)$. When working with growth series, we will additionally require that

- For all $x, y \in G$, we have $d(x, y) \in \mathbb{Z}$. **(Integer-valued)**
- For all $r \geqslant 0$, the ball $B(e, r) = \{g \in G \mid d(e, g) \leqslant r\}$ is finite. **(Proper)**

For short, we say that d is a LIP.

Our main interest lies on word metrics d_S. However, in order to prove results for that particular setting, we will often need to consider more general metrics, such as weighted word metrics, their restriction to subgroups (which may no longer be coarsely geodesic), or sub-Finsler metrics. Another example is the length ℓ on $S^\star$.

In this chapter, we will suppose generating sets are symmetric, i.e., $S = S^{-1}$.

7.1.1 Rational and algebraic series

In this section, we recall the definition of R-rational and R-algebraic formal series, where R is a fixed semiring. In what follows, we will mostly consider $R = \mathbb{N}G$, other relevant cases being $R = \mathbb{N}, \mathbb{N}S^\star, \mathbb{Z}$ and $\mathbb{Z}G$. We then proceed with alternate characterizations. See [DKV09; Sal+78] for a complete treatment.

Definition 7.1.2. An *algebraic system* over $R[s]$ is a system of equations

$$X_i = P_i(X_1, X_2, \dots, X_n) \quad \text{for } i = 1, 2, \dots, n$$

where $P_i \in R[t]\langle X_1, X_2, \dots, X_n\rangle$ are polynomials with (a priori) non-commutative coefficients and variables, i.e., finite sums of monomials $r_0 X_{i_1} r_1 X_{i_2} r_2 \dots X_{i_d} r_d \cdot t^k$ with $d, k \geqslant 0$ integers, $1 \leqslant i_j \leqslant n$ and $r_j \in R$. A system is *proper* if it contains

- no constant term (i.e., no monomial with $d = k = 0$);
- no monomial with $d = 1$ and $k = 0$.

A system is *linear* if furthermore all monomials satisfy $d = 0$, or $d = 1$ and $r_1 = 1_R$.

Definition 7.1.3. A formal series $S = \sum_{n=0}^{\infty} a_n \cdot t^n \in R[[t]]$ is *proper* (or *quasi-regular*) if its constant coefficient is zero, that is $a_0 = 0$.

The main motivation for those two definitions is the following result:

Theorem 7.1.4 (See [Sal+78, Theorem IV.1.1])**.** *Every proper algebraic system admits a unique solution $(S_1, S_2, \dots, S_n) \in R[[t]]^n$ consisting of proper formal series.*

Finally, we say a proper series is R-rational (resp. R-algebraic) if it is solution to a proper linear (resp. algebraic system). More generally

Definition 7.1.5 (R-rational and R-algebraic series)**.** A formal series $S \in R[[t]]$ is *R-rational* (resp. *R-algebraic*) if $S - a_0$ is the first component of a proper solution $(S_1, S_2, \dots, S_n) \in R[[t]]$ to a proper linear (resp. algebraic) system over $R[t]$.

Remark 7.1.6. Any morphism $f\colon R \to R'$ extends to a morphism $f\colon R[[t]] \to R'[[t]]$, and sends R-rational (resp. R-algebraic) series to R'-rational (resp. R'-algebraic) series. A few important examples are the following:

- The augmentation map $\epsilon\colon \mathbb{N}G \to \mathbb{N}$, sending $\sum a_g g \mapsto \sum a_g$. This proves that, if $\widehat{\Sigma}_{G,d}$ is $\mathbb{N}G$-rational, then $\Sigma_{G,d}$ is $\mathbb{N}$-rational.
- More generally, given a monoid morphism $\pi\colon G \to H$, we deduce a morphism $\pi\colon \mathbb{N}G \to \mathbb{N}H$. The case $H = \{e\}$ recovers the augmentation map. Another example which will be relevant in Section 7.5 is the evaluation map $\mathrm{ev}\colon S^\star \to G$.
- The inclusion map $\iota\colon \mathbb{N}G \hookrightarrow \mathbb{Z}G$.

Let us give another less technical definition of rational series over non-commutative semirings. We first need the notion of quasi-inverse:

Definition 7.1.7 (Quasi-inverse and star height)**.**

- Given a *proper* series $S(t) \in R[[t]]$, its *quasi-inverse* is defined as

$$S(t)^* = \sum_{n=0}^{\infty} S(t)^n,$$

  where $S(t)^n$ is the multiplication of $S(t)$ with itself n times and $S(t)^0 = 1$.

- We define inductively a sequence of semirings $\mathrm{Rat}_n \subset R[[t]]$. First $\mathrm{Rat}_0 = R[t]$, then Rat_{n+1} is the semiring generated by $\mathrm{Rat}_n \cup \{S^* \mid S \in \mathrm{Rat}_n \text{ proper}\}$.

- Given $S \in R[[t]]^{\mathrm{rat}} := \bigcup_n \mathrm{Rat}_n$, its *star height* is the minimal n such that $S \in \mathrm{Rat}_n$.

The name quasi-inverse comes from the informal equality $(1 - S)S^* = 1$. This linear equation shows that, if S is rational, then so is S^*. Reciprocally, all rational series can be obtained from polynomials using addition, multiplication and quasi-inversion:

Theorem 7.1.8 ([Sal+78, II.1.2 and II.1.4])**.** *The set of R-rational series is $R[[t]]^{\mathrm{rat}}$.*

7.1.2 Link with formal languages

Rational and algebraic series can be characterized in terms of automata and languages.

Definition 7.1.9. An $R[t]$-*automaton* is a finite oriented graph $A = (V, E)$ with an edge labeling $p\colon E \to R[t]$. A series $S(t) \in R[[t]]$ is *recognized by the automaton A* if there exist two initial and terminal vertices $I, T \in V$ such that

$$S(t) = \sum_{\gamma \in P(I,T)} p(\gamma),$$

where $P(I, T)$ is the set of all oriented paths from I to T, and $p(e_1 \ldots e_\ell) := p(e_1) \cdots p(e_\ell)$.

Theorem 7.1.10 (Kleene-Schützenberger [Sch61], see also [DKV09, Theorem 3.2.5])**.** *A formal series is R-rational if and only if it is recognized by an $R[t]$-automaton.*

Remark 7.1.11. In the case $R = \mathbb{N}G$, we may *and will* assume that labels are of the form $p(e) = w(e) \cdot t^{\ell(e)}$ for $w(e) \in G$ and $\ell(e) \in \mathbb{N}$.

Remark 7.1.12. The computational advantages of $\mathbb{N}G$-rational series over $\mathbb{Z}G$-rational series are made clearer through the formalism of $R[t]$-automata. Given an $\mathbb{N}G[t]$-automaton for a complete growth series, we can pick a path at random, compute the associated term $g \cdot t^\ell$, and get a certificate that the element g has length ℓ. If the Word Problem is efficiently solvable, this proves that computing the length of an element (given a word) is in NP. In contrast, if the automaton had labels in $\mathbb{Z}G[t]$, the term $g \cdot t^\ell$ might cancel out with other terms, so that we need to compute all terms with the same exponent t^ℓ before having a certificate g has indeed length ℓ.

Corollary 7.1.13. *Let (G,d) be a group, and $H \leqslant G$ a subgroup.*

(a) *If $\widehat{\Sigma}_{G,d}(t)$ is $\mathbb{N}G$-rational and $[G:H]<\infty$, then $\widehat{\Sigma}_{H,d|_H}(s)$ is $\mathbb{N}G$-rational.*

(b) *If $\widehat{\Sigma}_{H,d|_H}(t)$ is $\mathbb{N}G$-rational, then it is $\mathbb{N}H$-rational.*

Proof. The proof is identical to Lemma 2.6.6 and Corollary 2.6.4 respectively. □

Theorem 7.1.10 admits a generalization linking R-algebraic series and pushdown automata [DKV09, §7.5.3]. For complete growth series, this can be reformulated as

Theorem 7.1.14. *Let (G,d) be a group. The complete growth series $\widehat{\Sigma}_{G,d}(t)$ is $\mathbb{N}G$-algebraic if and only if there exists an unambiguous context-free language $\mathcal{L}\subset(G\times\mathbb{N})^\star$ such that the evaluation map* $\mathrm{ev}\colon \mathcal{L}\to G\times\mathbb{N}$ *is injective and its image is given by*

$$\mathrm{ev}(\mathcal{L})=\big\{(g,\|g\|):g\in G\big\}.$$

Corollary 7.1.15. *Let (G,d) be a group, and $H\leqslant G$ a finite-index subgroup. If $\widehat{\Sigma}_{G,d}(t)$ is $\mathbb{N}G$-algebraic, then $\widehat{\Sigma}_{H,d|_H}(t)$ is $\mathbb{N}H$-algebraic.*

Proof. There are only finite many rules in a context-free grammar, so there exists a finite set $S\subseteq G$ such that $\mathcal{L}\subseteq(S\times\mathbb{N})^\star$. We consider the Schreier graph $\mathcal{S}ch(H\backslash G,S)$, and rational function $w\mapsto\tilde{w}$ defined from it, as in Example 2.6.5. Then

$$\tilde{\mathcal{L}}\subseteq(X\times\mathbb{N})^\star\subseteq(H\times\mathbb{N})^\star$$

is unambiguously CF by Proposition 2.5.14(c), and evaluates to $\{(h,\|h\|):h\in H\}$. □

7.2 Dead ends and saddle elements

We recall the notion of dead ends due to [Bog97], and introduce some related notions:

Definition 7.2.1. Consider (G,d) a metric group, and let $D>0$ and $M\geqslant 0$ be real numbers. An element $g\in G$ is

- a *dead end of depth* at least D if
$$\forall h\in G \text{ such that } \|h\|\leqslant D,\quad \|gh\|\leqslant\|g\|.$$
- an *almost dead end of depth* at least D and *margin* M if
$$\forall h\in G \text{ such that } \|h\|\leqslant D,\quad \|gh\|\leqslant\|g\|+M.$$
- a *saddle element of depth* at least D if
$$\forall h\in G \text{ such that } \|h\|\leqslant D,\quad \|gh\|=\|gh^{-1}\|.$$

- an *almost saddle element* of depth $\geqslant D$ and margin M if

$$\forall h \in G \text{ such that } \|h\| \leqslant D, \ \Big| \|gh\| - \|gh^{-1}\| \Big| \leqslant M.$$

We say that (G, d) has *deep pockets* if it contains dead ends of arbitrarily large depths.

Remark 7.2.2. Finite groups have deep pockets: element of maximal length are dead ends of infinite depth. The notion of deep pockets gets way more interesting when looking at non-bounded groups, as all elements have finite depth.

The notions of (almost) saddle element is new. The notion of almost dead ends appears implicitly in Warshall's work [War10; War07]. Their main interest is that the existence of almost dead ends of large depth is preserved by rough isometries:

Lemma 7.2.3 (Compare with [War10, Proposition 7])**.** *Let (G, d_G) and (H, d_H) be two roughly isometric groups. Suppose that G contains almost dead ends of arbitrarily large depth for some fixed margin $M \geqslant 0$, then the same is true in H.*

Proof. Let $f : G \to H$ be a rough isometry. Up to translation we may suppose that $f(e_G) = e_H$. Let $g \in G$ be an almost dead end of margin M and depth D. We prove that $f(g)$ is an almost dead end. For any $h \in H$ such that $\|h\| \leqslant D - 2K$, there exists $x \in G$ such that $d\big(f(x), f(g)h\big) \leqslant K$ hence

$$d(g, x) \leqslant d\big(f(g), f(x)\big) + K \leqslant d\big(f(g), f(g)h\big) + d\big(f(g)h, f(x)\big) + K \leqslant D.$$

By hypothesis g is an almost dead end of depth D so that $\|x\| \leqslant \|g\| + M$ hence

$$\|f(g)h\| \leqslant \|f(x)\| + d\big(f(x), f(g)h\big) \leqslant \big(\|x\| + K\big) + K \leqslant \|g\| + M + 2K \leqslant \|f(g)\| + M + 3K.$$

This means that $f(g)$ is an almost dead end of margin $M + 3K$ (which is fixed) and depth $D - 2K$ (which can be made arbitrarily large for well chosen $g \in G$). □

We should think of the situation where the depth D is much larger than the margin M. In case of integer-valued metric and $D \gg M$, the following lemma ensures that we can "promote" almost dead ends into genuine dead ends of comparatively large depth.

Lemma 7.2.4 ([War10, Proposition 6])**.** *Let X be a metric space, $f : X \to \mathbb{N}$ a function, and $D, M \in \mathbb{N}$. Suppose that there exists $x \in X$ such that*

$$\forall x' \in B(x, D), \quad f(x') < f(x) + M.$$

Then there exists $x' \in B_D(x)$ such that f reaches a maximum on $B(x', \frac{D}{M})$ at x'.

7.2.1 Criterion

Let us now relate dead ends with the rationality of complete growth series

Theorem 7.2.5. *Let (G, d) be a group with LIP metric, and $M \geqslant 0$ a fixed constant. Suppose that (G, d) contains either*

- *an infinite set of (almost) dead ends of arbitrarily large depths, or*
- *an infinite set of almost saddle elements of arbitrarily large depths and margin M.*

Then the complete growth series of (G, d) is not $\mathbb{N}G$-rational.

Proof. By contraposition, suppose that the growth series is $\mathbb{N}G$-rational, recognized by some finite state automaton (V, E). We prove that, for any $M > 0$, there exist only finitely many dead ends (resp. almost saddle elements of margin M) and depth

$$D \geqslant 2M \cdot |V| \cdot \max_{e \in E} \|w(e)\| .$$

Each element $g \in G$ corresponds to a path γ_g with $w(\gamma_g) = g$ and $\ell(\gamma_g) = \|g\|$. Except for finitely many elements, this path has length $\geqslant M\,|V|$. By the pigeonhole principle some vertex $v \in V$ appears at least $M + 1$ times by γ_g among the last $M\,|V| + 1$ last visited vertices. Therefore, we can decompose

$$\gamma_g = \alpha\beta_1 \dots \beta_M \delta$$

with β_i non-empty paths going from v to v, and $|\beta_1 \dots \beta_M \delta| \leqslant M\,|V|$. Note that $\ell(\beta_i) \neq 0$, otherwise all the elements $g_n = w(\alpha\beta_1 \dots \beta_i^n \dots \beta_M\delta)$ would have the same length. We deduce that $\ell(\beta_1 \dots \beta_M) \geqslant M$. Finally we consider

$$h = w(\delta)^{-1} w(\beta_1 \dots \beta_M\, \delta).$$

We have $\|h\| \leqslant \|w(\delta)\| + \|w(\beta_1 \dots \beta_M\, \delta)\| \leqslant 2M\,|V| \cdot \max_{e \in E} \|w(e)\| \leqslant D$. Moreover

$$\|gh\| - \|g\| = \ell(\alpha\beta_1 \dots \beta_M\beta_1 \dots \beta_M\delta) - \ell(\alpha\beta_1 \dots \beta_M\delta) = \ell(\beta_1 \dots \beta_M) \geqslant M > 0$$

proving that is not a dead end of the required depth. Similarly, we have

$$\|gh\| - \|gh^{-1}\| = \ell(\alpha\beta_1 \dots \beta_M\beta_1 \dots \beta_M\delta) - \ell(\alpha\delta) = 2\ell(\beta_1 \dots \beta_M) \geqslant 2M > M$$

proving that g is not an almost saddle element of margin M and the required depth. □

7.2.2 Application: Lamplighter groups

In this paragraph, we consider (restricted) wreath products

$$L \wr Q \coloneqq \Big(\bigoplus_{q \in Q} L \Big) \rtimes Q,$$

where Q acts by permuting the entries of $\bigoplus_{q \in Q} L$. The group L is the *lamp group*, while Q is the *base group*. Elements are pairs (Φ, q) with $\Phi\colon Q \to L$ finitely supported and $q \in Q$. We fix an embedding $L \hookrightarrow L \wr Q$, mapping $g \mapsto (\Phi, e_Q)$ where Φ is defined as $\Phi(p) = g$ is $p = e_Q$, and $\Phi(p) = e_L$ if $p \neq e_Q$.

Let us start with a classical result for lengths in wreath products.

Proposition 7.2.6. *Consider $G = L \wr Q$, endowed with the standard generating set $S = S_L \cup S_Q$. Elements $g \in L \wr Q$ can be identified with pairs (Φ, q) with $\Phi\colon Q \to L$ a finitely supported function, and $q \in Q$. Moreover, the length of g is given by the formula*

$$\|g\|_{\text{std}} = \sum_{p \in Q} \|\Phi(p)\|_{S_L} + TS(e_Q;\, \text{supp}(\Phi);\, q)$$

where $TS(x; S; y)$ is the length of the shortest path in the Cayley graph $Cay(Q, S_Q)$ starting at x, going through S in some order, and ending at y.

Using this formula, it was shown in [CT03] that many of those groups have deep pockets:

Theorem 7.2.7 ([CT03, Theorem 6.1])**.** *Consider $G = L \wr \mathbb{Z}$ with L non-trivial, with the standard generating set $S = S_L \cup \{t^{\pm}\}$. Suppose that L has dead ends of arbitrary depth w.r.t. S_L (for instance if L is finite), then so does G with respect to S.*

Note that the condition on (L, S_L) is indeed important, as wreath products like $\mathbb{Z} \wr \mathbb{Z}$ do not have dead ends (w.r.t. the standard generating set). However, the question of $\mathbb{N}G$-rationality of their complete growth series is still settled by the following proposition:

Theorem 7.2.8. *Let $G = L \wr \mathbb{Z}$ with L non-trivial. Consider the standard generating set $S = S_L \cup \{z^{\pm}\}$ with S_L symmetric. There exist infinitely many elements (g_d) satisfying*

$$\forall h \in G \text{ such that } \|h\|_S \leqslant d, \quad \|g_d h\|_S = \|g_d h^{-1}\|_S.$$

As a corollary, the associated complete growth series in not $\mathbb{N}G$-rational.

Proof. Let $\ell \in L \setminus \{e_L\}$ and define

$$\Psi_d(q) = \begin{cases} \ell & \text{if } q = \pm d \\ e_L & \text{otherwise.} \end{cases}$$

We consider $g_d = (\Psi_d, 0)$ i.e., the element with only lamps in a non-trivial state on site $\pm d$, and the lamplighter guy back at 0. Consider $h = (\Phi, q) \in L \wr \mathbb{Z}$ with $\|h\|_S \leqslant d$. We have $h^{-1} = (\Phi(-q + \,\cdot\,)^{-1}, -q)$. Note that $\|h^{-1}\|_S = \|h\|_S \leqslant d$ so that

$$\text{supp}\,\Phi \cap \text{supp}\,\Psi_d = \emptyset = \text{supp}\,\Phi(-q + \,\cdot\,)^{-1} \cap \text{supp}\,\Psi_d.$$

It follows that $\|g_d h\|_S$ can be easily computed:

$$\begin{aligned} \|g_d h\|_S &= \sum_{p \in \mathbb{Z}} \|\Psi_d(p)\|_{S_L} + \sum_{p \in \mathbb{Z}} \|\Phi(p)\|_{S_L} + TS(0;\, B(0,d);\, q) \\ &= 2\,\|\ell\|_{S_L} + \sum_{p \in \mathbb{Z}} \|\Phi(p)\|_{S_L} + 2d - |q| \end{aligned}$$

(We use that any path going through $\pm d$ must go through the entire interval.) Similarly

$$\begin{aligned}\left\|g_d h^{-1}\right\|_S &= \sum_{p\in\mathbb{Z}} \|\Psi_d(p)\|_{S_L} + \sum_{p\in\mathbb{Z}} \left\|\Phi(-q+p)^{-1}\right\|_{S_L} + TS(0;\, B(0,d);\, -q) \\ &= 2\,\|\ell\|_{S_L} + \sum_{p\in\mathbb{Z}} \left\|\Phi(p)^{-1}\right\|_{S_L} + 2d - |q|\end{aligned}$$

Recall S_L was symmetric and compare both formula: we have $\|g_d h\|_S = \|g_d h^{-1}\|_S$. $\square$

Remark 7.2.9. Both Theorems 7.2.7 and 7.2.8 can be generalized when $(\mathbb{Z}, \{t^{\pm}\})$ is replaced by (Q, S_Q) such that the associated Cayley graph is an infinite tree (implying $Q \simeq \mathbb{Z}^{*m} * C_2^{*n}$). The element g_d can be taken as $g_d = (\Psi_d, e_Q)$ with

$$\Psi_d(q) = \begin{cases} \ell & \text{if } \|q\|_{S_Q} = d, \\ e_L & \text{otherwise.} \end{cases}$$

However, in those cases, standard growth series are already non $\mathbb{Z}$-rational [Par92].

7.3 Almost saddle elements

The goal of this section is to prove that complete growth series of a virtually 2-step nilpotent group (G, d_S) is $\mathbb{N}G$-rational only if G is virtually abelian. To achieve this, we exhibit almost saddle elements in 2-step nilpotent Lie groups with polytopal sub-Finsler metrics, then use various transfer results and Criterion 7.2.5 to get the conclusion.

7.3.1 Lipschitz control away from a pathological set

Let Γ be a simply connected 2-step nilpotent Lie group, and $X \subset \Gamma$ a finite Lie generating set. We consider the metrics $d_{\mathrm{Stoll},X}$ and d_E (any Euclidean metric) on $\Gamma = V_1 \oplus V_2$.

We prove a result allowing us to add small areas $z \in V_2$ to large elements $g \in \Gamma$ at a small marginal cost. This result holds for elements g far away from the following pathological set P: for each subset $T \subseteq X$, we define $P_T = V_1 \oplus \mathrm{Vect}[T, \Gamma]$, and

$$P = \bigcup_{T : P_T \neq \Gamma} P_T.$$

Lemma 7.3.1. *There exist two constants $C, \varepsilon > 0$ (depending only on Γ, X) such that, for all $\Delta > 0$ and $g \in \Gamma$ satisfying $d_E(g, P) \geqslant \Delta^2$ and $d_{\mathrm{Stoll}}(e, g) \leqslant \varepsilon\Delta^2$. Then,*

$$\forall z \in V_2, \quad d_{\mathrm{Stoll}}(e, gz) \leqslant d_{\mathrm{Stoll}}(e, g) + \frac{C}{\Delta} \cdot d_E(\mathbf{0}, z).$$

Proof. Fix a geodesic $\mathbb{R}$-word $g = x_1^{\mu_1} x_2^{\mu_2} \cdots x_k^{\mu_k}$ with $k \leqslant K$ (Corollary 2.4.11).

► We first prove lower bounds for sufficiently many μ_i, precisely if

$$T_g \coloneqq \left\{x_i : |\mu_i| \geqslant C_1\Delta\right\}$$

(for some constant $C_1 = C_1(X)$ which will be fixed later) then $\mathrm{Vect}[T_g, \Gamma] = V_2$.

Suppose $P_{T_g} \neq \Gamma$, then $P_{T_g} \subseteq P$. It follows that

$$\begin{aligned}
\Delta^2 &\leqslant d_E(g, P) \leqslant d_E(g, P_{T_g}) \\
&= d_E\left(\sum_{i=1}^k \mu_i x_i + \frac{1}{2}\sum_{i<j} \mu_i\mu_j[x_i, x_j],\ P_{T_g}\right) \\
&= d_E\left(\sum_{i=1}^k \mu_i A(x_i) + \frac{1}{2}\sum_{i<j:x_i,x_j\notin T_g} \mu_i\mu_j[x_i, x_j],\ P_{T_g}\right) \\
&\leqslant d_E\left(\sum_{i=1}^k \mu_i A(x_i) + \frac{1}{2}\sum_{i<j:x_i,x_j\notin T_g} \mu_i\mu_j[x_i, x_j],\ \mathbf{0}\right) \\
&\leqslant d_{\mathrm{Stoll}}(e, g) \cdot \max_i \|A(x_i)\|_E + K^2 \cdot (C_1\Delta)^2 \cdot \max_{i,j} \|[x_i, x_j]\|_E \\
&\leqslant \left(C_1^2 \cdot K^2 \max_{i,j} \|[x_i, x_j]\|_E + \varepsilon \cdot \max_i \|A(x_i)\|_E\right) \cdot \Delta^2
\end{aligned}$$

which is wrong for $C_1 < \left(K\sqrt{\max_{i,j}\|[x_i, x_j]\|_E}\right)^{-1}$ and $\varepsilon > 0$ sufficiently small.

► The above geodesic $\mathbb{R}$-word contains long segments in each directions of T_g, which "generates" V_2. Using these segments, we prove the inequality.

Consider $r > 0$ the largest number such that

$$B_E(\mathbf{0}, r) \subseteq \sum_{x\in T}[x, B_{\mathrm{Stoll}}(e, 1)]$$

for all $T \subseteq X$ satisfying $\mathrm{Vect}[T, \Gamma] = V_2$. Here $B_E(\mathbf{0}, r)$ is the ball of radius r in (V_2, d_E), and $B_{\mathrm{Stoll}}(e, 1)$ is the ball of radius 1 in $(\Gamma, d_{\mathrm{Stoll}})$. Note that this (Minkowski) sum is a full-dimensional, centrally symmetric, convex set of V_2 containing $\mathbf{0}$ for each T satisfying the condition $\mathrm{Vect}[T, \Gamma] = V_2$, so indeed $r > 0$.

It follows that

$$\frac{r}{d_E(\mathbf{0}, z)} \cdot z \in \sum_{x\in T_g}[x, B_{\mathrm{Stoll}}(e, 1)] \subseteq \sum_{i=1}^k \frac{\mu_i}{C_1\Delta} \cdot [x_i, B_{\mathrm{Stoll}}(e, 1)].$$

Finally we can pick $t_i \in B_{\mathrm{Stoll}}\left(e, \frac{d_E(1,z)}{rC_1\Delta}\right)$ such that $z = \sum_{i=1}^k \mu_i[x_i, t_i]$ hence

$$gz = t_1^{-1}x_1^{\mu_1}t_1 \cdot t_2^{-1}x_2^{\mu_2}t_2 \cdot \ldots \cdot t_k^{-1}x_k^{\mu_k}t_k$$

has length $d_{\mathrm{Stoll}}(1, gz) \leqslant d_{\mathrm{Stoll}}(1, g) + 2K\frac{d_E(\mathbf{0},z)}{rC_1\Delta}$. We take $C = \frac{2K}{rC_1}$. □

Remark 7.3.2. This lemma was isolated (and the presentation much improved) during conversations with Enrico Le Donne and Luca Nalon. It should be compared to results of [LDNG18; LDN24]. The set P_{T_g} should be compared with the set I_γ in [BNV22]. Variations on the lemma can be obtained by taking

$$P_T = \operatorname{Vect}(T + [T,\Gamma]) \qquad \text{or} \qquad P_T = \operatorname{Vect}(\Pr(T) + [T,\Gamma]).$$

Some variation might give meaningful information on a set of *full* asymptotic density.

7.3.2 Almost saddle elements in $(\bar{H}, d_{\mathrm{Stoll}})$

Theorem 7.3.3. *Let Γ be a simply connected 2-step nilpotent Lie group with $[\Gamma,\Gamma]$ infinite, X a finite symmetric Lie generating set. Then $(\Gamma, d_{\mathrm{Stoll}})$ has almost saddle elements g of arbitrarily large depth D and fixed margin M, that is*

$$\forall h \in \Gamma \text{ such that } \|h\|_{\mathrm{Stoll}} \leqslant D, \quad \Big| \|gh\|_{\mathrm{Stoll}} - \|gh^{-1}\|_{\mathrm{Stoll}} \Big| \leqslant M.$$

Moreover, if $H \leqslant \Gamma$ is a lattice, we may suppose $g \in [H,H]$.

Proof. Fix $g_0 \in [\Gamma,\Gamma] \setminus P$, say $d_E(g_0, P) = \delta$. If $H \leqslant \Gamma$ is a lattice, we take $g_0 \in [H,H]$. We prove that $g = g_0^n$ is an almost saddle element of depth D (and some fixed margin M to be determined) for n large enough. We are aiming to prove

$$\forall h \in G \text{ such that } \|h\|_{\mathrm{Stoll}} \leqslant D, \quad \|gh^{-1}\|_{\mathrm{Stoll}} \leqslant \|gh\|_{\mathrm{Stoll}} + M.$$

By Corollary 2.4.11, there exists a geodesic $\mathbb{R}$-word $gh = x_1^{\mu_1} \cdot x_2^{\mu_2} \cdot \ldots \cdot x_k^{\mu_k}$ with $k \leqslant K$. Note that the element

$$f = x_1^{-\mu_1} \cdot x_2^{-\mu_2} \cdot \ldots \cdot x_k^{-\mu_k}$$

isn't too far away from gh^{-1}: we have $\Pr(f) = -\Pr(gh) = \Pr(gh^{-1})$ as $g \in [\Gamma,\Gamma]$, and

$$A(f) = A(gh) - 2\sum_{i=1}^{n} \mu_i\, A(x_i) = A(gh^{-1}) - \Big(2\sum_{i=1}^{n} \mu_i\, A(x_i) - 2A(h)\Big) =: A(gh^{-1}) - z$$

by Proposition 2.4.5. We just have to fix this error z in areas, with a budget of M. This is done using Lemma 7.3.1. We observe that

$$\begin{aligned}
\|f\|_{\mathrm{Stoll}} &\leqslant \|gh\|_{\mathrm{Stoll}} = O(\sqrt{n}) + O(D) \\
d_E(f,P) &\geqslant d_E(g,P) - \|A(h)\|_E - 2\sum_{i=1}^{n} \mu_i \,\|A(x_i)\|_E = n\delta - O(\sqrt{n}) - O(D^2) \\
\|z\|_E &\leqslant O(\sqrt{n}) + O(D)
\end{aligned}$$

(as $\sum_i \mu_i = \|gh\|_{\mathrm{Stoll}}$) where the hidden constants only depends on Γ, X and g_0. Finally, we can use Lemma 7.3.1 (with $\Delta = \sqrt{d_E(f,P)}$) and conclude that

$$\begin{aligned}
\|gh^{-1}\|_{\mathrm{Stoll}} = \|fz\|_{\mathrm{Stoll}} &\leqslant \|f\|_{\mathrm{Stoll}} + C\frac{\|z\|_E}{\sqrt{d_E(f,P)}} \\
&\leqslant \|gh\|_{\mathrm{Stoll}} + O_{n\to\infty}(1).
\end{aligned}$$

□

Remark 7.3.4. For $\Gamma = H_3(\mathbb{R})$, and more generally for Γ ideal (see Definition 2.4.25), we have $P \subseteq V_1$, so every large enough element $g \in [\Gamma, \Gamma]$ is an almost saddle element of large depth and fixed margin M

Remark 7.3.5. The motivation comes from dilations. In nilpotency class 2, we have

$$\Pr(\delta_{-1}(x)) = -\Pr(x) \quad \text{and} \quad A(\delta_{-1}x) = A(x).$$

If g is a large commutator and h a short element, we have $\Pr(gh^{-1}) = -\Pr(gh)$ and $A(gh^{-1}) \approx A(g) \approx A(gh)$. It follows that $gh^{-1} \approx \delta_{-1}(gh)$ and $\|gh^{-1}\| \approx \|gh\|$.

7.3.3 Conclusion

The main theorem follows easily.

Theorem 7.3.6. *Let (G, d_S) be a virtually 2-step nilpotent group such that the complete growth series $\widehat{\Sigma}_{G,d_S}(t)$ is $\mathbb{N}G$-rational, then G is virtually abelian.*

Proof. We argue by contraposition. Let $H \leqslant G$ be a finite-index, torsionfree, 2-step nilpotent subgroup, and (X, ω) the weighted generating set of H defined in §4.1.1.

Since G is not virtually abelian, neither is H, hence $[H, H]$ is infinite. Proposition 7.3.3 implies that $(H, d_{\mathrm{Stoll},X,\omega})$ has almost saddle elements of arbitrarily large depths and fixed margin. Propositions 2.4.12 and 4.1.1 combined imply that the isomorphism

$$\mathrm{id}\colon (H, d_S|_H) \longrightarrow (H, d_{X,\omega}) \longrightarrow (H, d_{\mathrm{Stoll},X,\omega})$$

is a rough isometry. It follows that $(H, d_S|_H)$ has almost saddle elements of arbitrarily large depths and fixed margin. Using Theorem 7.2.5, we conclude that $\widehat{\Sigma}_{H,d_S|_H}(t)$ is not $\mathbb{N}H$-rational, and finally $\widehat{\Sigma}_{G,d_S}(t)$ is not $\mathbb{N}G$-rational by Corollary 7.1.13. □

7.4 Non-algebraicity for $H_3(\mathbb{Z})$

The proof splits into two part. We first refine Warshall result about the existence of dead ends of arbitrary depths in $H_3(\mathbb{Z})$ [War07], proving that *large commutators* are almost dead ends. Combining this with the classifications of CF-language of polynomial growth, we prove that complete growth series of $G = H_3(\mathbb{Z})$ are not $\mathbb{N}G$-algebraic.

7.4.1 CC-metrics on $H_3(\mathbb{R})$

Let us recall some known results on the CC-geometry of $H_3(\mathbb{R})$. We fix $\mathrm{P} \subset \mathbb{R}^2$ centrally symmetric convex polygon, recall

$$\|g\|_{\mathsf{CC}} = \min \left\{ \ell_{\mathrm{P}}(\gamma) \;\middle|\; \gamma \text{ is a path representing } g \right\}.$$

Here $\gamma\colon [0, \ell] \to V_1 = \mathbb{R}^2$ represents g if it has the correct endpoint $\gamma(\ell) = \Pr(g)$ and the correct area $A(\gamma) = A(g)$, therefore computing $\|g\|_{\mathsf{CC}}$ reduces to solving the isoperimetric problem, where the perimeter is computed w.r.t. the norm $\|\cdot\|_{\mathrm{Mink,P}}$.

CC-geodesics on $H_3(\mathbb{R})$ can be described precisely. First a classical result of Busemann on the isoperimetric problem in the plane with Minkowski metric:

Theorem 7.4.1 ([Bus49])**.** *Fix* $\mathrm{P} \subset \mathbb{R}^2$ *a centrally symmetric convex polygon. For any absolutely continuous closed curve* γ*, we have*

$$\frac{\ell_{\mathrm{P}}(\gamma)}{A(\gamma)} \geqslant \frac{\ell_{\mathrm{P}}(I)}{A(I)},$$

where I *is the isoperimetrix, i.e., the boundary of the rotation by* $\pm\frac{\pi}{2}$ *of the polar dual*

$$\mathrm{P}^* = \left\{ x \in \mathbb{R}^2 \;\middle|\; \forall y \in \mathrm{P},\ \langle x; y\rangle \leqslant 1 \right\}.$$

Moreover, the equality is reached uniquely if γ *is homothetic to* I*.*

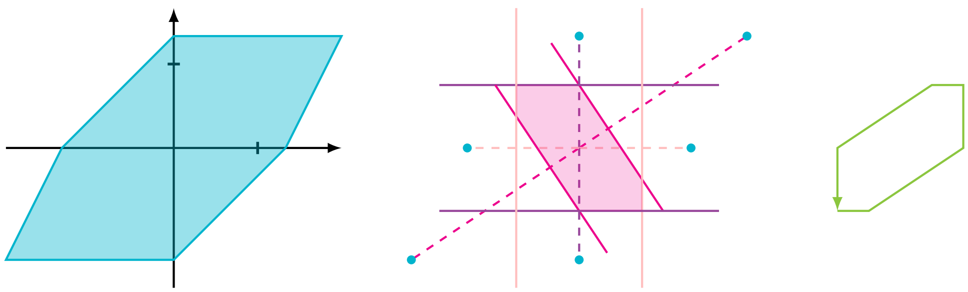


Figure 7.2: Examples of P, P^* and I.

Note that I is a polygon with sides parallel to vertices of P.

This was generalized for all CC-geodesics in a beautiful paper of Duchin and Mooney:

Theorem 7.4.2 ([DM14, Structure Theorem])**.** CC-*geodesics split into two classes:*

- Regular geodesics *which follows a portion of a dilate of the isoperimetrix.*
- Unstable geodesics *for which all tangent directions* $\gamma'(t)$ *lie in a common positive cone spanned by two consecutive vertices of* P*, i.e.,* γ *is a geodesic in* $(\mathbb{R}^2, \|\cdot\|_{\mathrm{Mink}})$*.*

Moreover all such paths are CC*-geodesics.*

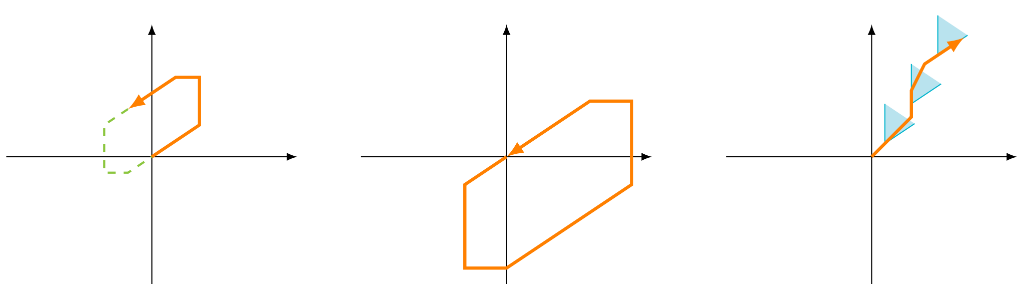


Figure 7.3: A few CC-geodesics of both types for the previous choice of P.

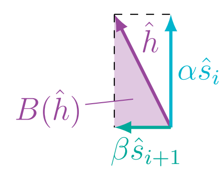
7.4.2 Almost dead ends in $H_3(\mathbb{R})$

We prove the following result:

Proposition 7.4.3. *Fix* $\mathrm{P} \subset \mathbb{R}^2$ *centrally symmetric convex polygon. There exist constants* C, M *depending on* P *such that, for any* $D \gg 1$ *large enough and* $n \geqslant C \cdot D^4$, *the element* $g = z^n = [x,y]^n$ *satisfies*

$$\forall h \in H_3(\mathbb{R}) \text{ s.t. } \|h\|_{\mathsf{CC}} \leqslant D, \quad \|gh\|_{\mathsf{CC}} \leqslant \|g\|_{\mathsf{CC}} - \|\hat{h}\|_{\mathrm{P}} + M.$$

Proof. Fix $g, h \in H_3(\mathbb{R})$ as in the statement. There exist consecutive vertices $\hat{s}_i, \hat{s}_{i+1}$ of P such that $\hat{h}$ lives in the positive cone spanned by $\hat{s}_i, \hat{s}_{i+1}$, say $\hat{h} = \alpha\hat{s}_i + \beta\hat{s}_{i+1}$ with $0 \leqslant \alpha, \beta \leqslant D$. We define $B(\hat{h})$ as the area of the triangle with sides $\hat{h}$, $-\alpha\hat{s}_i$ and $-\beta\hat{s}_{i+1}$ (in that order).



Note that the area $A(h) + B(\hat{h})$ is enclosed by a curve of length $\|h\|_{\mathsf{CC}} + \|\hat{h}\|_{\mathrm{P}} \leqslant 2D$ (specifically h concatenated with the two segments $-\alpha\hat{s}_i$ and $-\beta\hat{s}_{i+1}$), so that

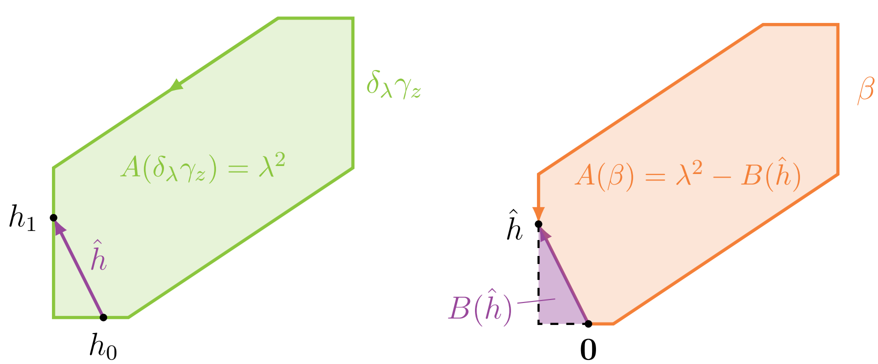
$$\left|A(h) + B(\hat{h})\right| \leqslant \left(\frac{2D}{\|z\|_{\mathsf{CC}}}\right)^2$$

by Busemann's isoperimetric inequality.

Consider γ_z any geodesic representing z and consider the dilation $\delta_\lambda\gamma_z$ with

$$\lambda = \sqrt{A(g) + A(h) + B(\hat{h})} \geqslant \sqrt{C \cdot D^4 - O(D^2)} = \sqrt{C} \cdot D^2 - O(1).$$

We consider $D \gg 1$ so that all sides of $\delta_\lambda\gamma_z$ have Minkowski-length $\geqslant D$. In particular we can find two points h_0 and h_1 on the curve $\delta_\lambda\gamma_z$, more precisely on the sides with direction $-\hat{s}_{i+1}$ and $-\hat{s}_i$ respectively, differing by a vector $\hat{h}$. Up to picking a different starting point on the geodesic $\delta_\lambda\gamma_z$, we may assume $h_0 = \mathbf{0}$ and $h_1 = \hat{h}$.



We consider the curve β following $\delta_\lambda\gamma_z$ from $h_0 = \mathbf{0}$ to $h_1 = \hat{h}$. The curve β has endpoint $h_1 = \hat{h}$ (just as gh) and area $\lambda^2 - B(\hat{h}) = A(g) + A(h) = A(gh)$, hence represents gh.

It remains to bound the length of β to estimate $\|gh\|_{\mathsf{CC}}$. We have

$$\lambda = \sqrt{n + A(h) + B(\hat{h})} \leqslant \sqrt{n} + \frac{1}{2}\frac{A(h) + B(\hat{h})}{\sqrt{n}} \leqslant \sqrt{n} + \frac{2D^2}{\sqrt{n}\,\|z\|_{\mathsf{CC}}^2}$$

and therefore

$$\begin{aligned}\|gh\|_{\mathsf{CC}} &\leqslant \ell_{\mathrm{P}}(\beta) = \ell_{\mathrm{P}}(\delta_\lambda\gamma_z) - \|\hat{h}\|_{\mathrm{P}} = \lambda\,\|z\|_{\mathsf{CC}} - \|\hat{h}\|_{\mathrm{P}} \\ &\leqslant \|g\|_{\mathsf{CC}} - \|\hat{h}\|_{\mathrm{P}} + \frac{2D^2}{\sqrt{n}\,\|z\|_{\mathsf{CC}}} \leqslant \|g\|_{\mathsf{CC}} - \|\hat{h}\|_{\mathrm{P}} + \frac{2}{\sqrt{C}\,\|z\|_{\mathsf{CC}}}.\end{aligned}$$

as $\|g\|_{\mathsf{CC}} = \sqrt{n}\,\|z\|_{\mathsf{CC}}$ and $n \geqslant CD^4$. We conclude by setting $C = 4/M^2\,\|z\|_{\mathsf{CC}}^2$. □

7.4.3 CF languages of polynomial growth and Dyck loops

A k-Dyck word is a word over the set of symbols $[_1,]_1, [_2,]_2, \dots, [_k,]_k$ satisfying

- each symbol appears exactly once,
- $[_i$ appears before $]_i$,
- and if $[_j$ appears in between $[_i$ and $]_i$, then so does $]_j$.

For instance, $[_1[_2]_2[_3[_4]_4]_3]_1[_5]_5$ is a Dyck word while $[_1[_2]_1]_2$ and $]_1$ are not.

We fix $\mathcal{A}$ an alphabet and a k-Dyck word z. A k-Dyck loop with underlying word z is the set of words obtained by placing fixed words $w_0, \dots, w_{2k} \in \mathcal{A}^*$ in between the parenthesis, and replacing parenthesis $[_i$ and $]_i$ by powers $u_i^{n_i}$ and $v_i^{n_i}$ respectively, with $u_i, v_i \in \mathcal{A}^*$ fixed, and n_i any positive integer. For instance,

$$\left\{ab(a)^{n_1}bc(ac)^{n_2}ac(c)^{n_2}(da)^{n_3}(b)^{n_3}abc : n_1, n_2, n_3 \in \mathbb{N}\right\}$$

is a 3-Dyck loop with underlying word $[_1[_2]_2]_1[_3]_3$ obtained with $u_1 = a$, $u_2 = ac$, $v_2 = c$, $v_1 = \varepsilon$, $u_3 = da$, $v_3 = b$ and $w_0 = ab$, $w_1 = bc$, $w_2 = ac$, $w_3 = w_4 = w_5 = \varepsilon$, $w_6 = abc$.

We have the following structural result due to Ilie, Rozenberg and Salomaa:

Theorem 7.4.4 ([IRS00]). *A* CF *language* $\mathcal{L} \subseteq \mathcal{A}^*$ *satisfies*

$$\beta_{\mathcal{L}}(n) := \#\{w \in \mathcal{L} : \ell(w) \leqslant n\} = O(n^k)$$

if and only if it is a finite union of k-Dyck loops.

7.4.4 Main proof

Theorem 7.4.5. *Let (G, d_S) be a group containing a finite-index subgroup $H \simeq H_3(\mathbb{Z})$, with S a finite symmetric generating set. Then $\widehat{\Sigma}_{G,d_S}(t)$ is not $\mathbb{N}G$-algebraic.*

Proof. The proof goes by contradiction. We suppose that the complete growth series of (G, S) is $\mathbb{N}G$-algebraic. We first observe that $\widehat{\Sigma}_{H,d_S|_H}(t)$ is $\mathbb{N}H$-algebraic by Corollary 7.1.15. In particular Theorem 7.1.14 gives a CF language $\mathcal{L} \subset (H \times \mathbb{N})^\star$ evaluating to

$$\big\{(g, \|g\|_S) : g \in H\big\}.$$

The growth $\beta_{\mathcal{L}}(n)$ is a lower bound on the volume growth of (H, d_S), so grows polynomially. It follows from Theorem 7.4.4 that $\mathcal{L}$ is a finite union of Dyck loops.

Consider one of the Dyck loops forming $\mathcal{L}$. When evaluating this language, we get pairs $(g, \ell) \in H \times \mathbb{N}$. Informally, the language "predicts" g has length $\|g\|_S = \ell$, and we'd like to prove this cannot be done coherently. More precisely, we get elements

$$g(n_1, n_2, \ldots, n_k) \in H \quad \text{of length} \quad \alpha + n_1\tau_1 + n_2\tau_2 + \ldots + n_k\tau_k$$

for some fixed $\alpha, \tau_1, \tau_2, \ldots, \tau_k$ (depending only on the chosen Dyck loop). Just to make the notation clear(er), if the word underlying the Dyck loop is $[_1[_2[_3]_3[_4]_4]_2]_1$, then

$$g(n_1, n_2, n_3, n_4) = w_0\, u_1^{n_1}\, w_1\, u_2^{n_2}\, w_2\, u_3^{n_3}\, w_3\, v_3^{n_3}\, w_4\, u_4^{n_4}\, w_5\, v_4^{n_4}\, w_6\, v_2^{n_2}\, w_7\, v_1^{n_1}\, w_8$$

for some fixed $u_i, v_i, w_i \in H$. We may always suppose $(u_i, v_i) \neq (e, e)$ (otherwise just forget about those, fuse some w_j's and lower k) hence $\tau_i > 0$ (otherwise varying n_i gives infinitely many words evaluating in a given sphere in H). We may also suppose $w_0 = w_1 = \ldots = w_{2k-1} = e$ (in which case we will also drop the index for w_{2k}). For instance the previous example can be rewritten

$$(\underbrace{w_0\, u_1\, w_0^{-1}}_{\tilde{u}_1})^{n_1}\ (\underbrace{w_0 w_1\, u_2\, (w_0 w_1)^{-1}}_{\tilde{u}_2})^{n_2} \ldots (\underbrace{w_0 \ldots w_7\, v_1\, (w_0 \ldots w_7)^{-1}}_{\tilde{v}_1})^{n_1} \cdot (\underbrace{w_0 w_1 \ldots w_8}_{\tilde{w}}).$$

Note that each element $g \in H$ appears as a $g(n_1, n_2, \ldots, n_k)$ when evaluating one of the Dyck loop. Using the pigeonhole principle we know infinitely many elements of $[H, H]$ appear in a given Dyck loop. From now on we only consider this specific Dyck loop.

The remainder of the proof goes as follows:

(a) We show that $\mathbf{0} \in \operatorname{ConvHull}\{\hat{u}_i + \hat{v}_i\}$ (where $\hat{g} = \operatorname{Pr}(g) \in V_1$).

In particular there exist weights $\lambda_i \geqslant 0$ such that

$$\sum_{i=1}^{k} \lambda_i(\hat{u}_i + \hat{v}_i) = \mathbf{0}.$$

As $\hat{u}_i + \hat{v}_i \in \mathbb{Z}^2$, we may even suppose $\lambda_i \in \mathbb{N}$.

(b) We treat (i.e., find a contradiction) the case when the underlying word has the form $[_1 \cdots]_1$. This is done using *central* almost dead ends of large depth.

(c) We treat the other case i.e., when the underlying word contains at least two disjoints pairs of brackets $[_1 \cdots]_1 [_j \cdots]_j \cdots$. This is done using CC-geometry.

(i) First we deduce two elements $h_1, h_2 \in H_3(\mathbb{R}) \setminus \{e\}$ such that $\hat{h}_1 + \hat{h}_2 = \mathbf{0}$ and

$$\forall m, n \in \mathbb{R}^+, \quad \|\delta_m h_1 \cdot \delta_n h_2\|_{\mathsf{CC}} = \|\delta_m h_1\|_{\mathsf{CC}} + \|\delta_n h_2\|_{\mathsf{CC}} .$$

Here $\|\cdot\|_{\mathsf{CC}} = \|\cdot\|_{\mathsf{CC},X,\omega}$, where (X, ω) is as defined in §4.1.1.

(ii) Using Duchin–Mooney's Structure Theorem (Theorem 7.4.2 above), we prove that no such elements exist in $H_3(\mathbb{R})$.

(a) Suppose on the contrary that $\mathbf{0} \notin \operatorname{ConvHull}\{\hat{u}_i + \hat{v}_i\}$, then there exists a linear form $h\colon \mathbb{R}^2 \to \mathbb{R}$ such that $h(\hat{u}_i + \hat{v}_i) > 0$ for all i. Let $m > 0$ be the minimum of those k values. We have

$$h\Big(\Pr\big(g(n_1, n_2, \ldots, n_k)\big)\Big) \geqslant (n_1 + n_2 + \ldots + n_k) \cdot m + h(\hat{w})$$

so that $h(\hat{g}) > 0$ except for finitely many choices of $n_1, n_2, \ldots, n_k$. It follows that $\hat{g} = \mathbf{0}$ (i.e., $g \in [H, H]$) for only finitely many values of the parameters, a contradiction.

(b) Consider $g_0 = g(n_1, n_2, \ldots, n_k)$ any commutator appearing in our Dyck loop. Define

$$g_n = g(n_1 + n\lambda_1, n_2 + n\lambda_2, \ldots, n_k + n\lambda_k)$$

Note that all those g_n are distinct commutators. As $(H, d_S|_H)$ and $(H, d_{\mathsf{CC},X,\omega})$ are roughly isometric (Proposition 4.1.1 and Theorem 2.4.27), hence Theorem 7.4.3 tells us that the commutator g_n is an almost dead end of depth $\geqslant D$ and margin M (in $(H, d_S|_H)$) for all n large enough. Now consider for any n

$$\tilde{g}_n = g(n_1 + n\lambda_1 + M + 1, n_2 + n\lambda_2, \ldots, n_k + n\lambda_k).$$

Recall that the underlying word starts and ends by $[_1$ and $]_1$ respectively, and that g_n is a commutator (hence is central in H). It follows that we can rewrite

$$\tilde{g}_n = u_1^{M+1} \cdot g_n \cdot w^{-1}\, v_1^{M+1}\, w = g_n \cdot u_1^{M+1} \cdot w^{-1}\, v_1^{M+1}\, w$$

Let us take $h = u_1^{M+1} \cdot w^{-1}\, v_1^{M+1}\, w$, fix a depth $D = \|h\|_S$ and n large enough for that D. We have g_n and a nearby element $\tilde{g}_n = g_n h$. We compute their lengths in order to find a contradiction. As both g_n and $\tilde{g}_n$ appears in the Dyck loop, we know

$$\begin{aligned} \|g_n\|_S &= \alpha + (n_1 + n\lambda_1)\tau_1 + \ldots + (n_k + n\lambda_k)\tau_k \\ \|\tilde{g}_n\|_S &= \alpha + (n_1 + n\lambda_1 + M + 1)\tau_1 + \ldots + (n_k + n\lambda_k)\tau_k = \|g_n\|_S + (M+1)\lambda_1 \end{aligned}$$

We have $\|g_n h\|_S > \|g_n\|_S + M$ which is a contradiction with g_n being an almost dead end of depth D and margin M.

(c-i) Suppose that the underlying word factors into (non-empty) disjoints Dyck words $z = [_1 \ldots]_1 \cdot [_j \ldots]_j \ldots$. Write

$$g(n_1, n_2, \ldots, n_k) = g_1(n_1, n_2, \ldots, n_{j-1}) \cdot g_2(n_j, \ldots, n_k) \cdot w.$$

Fix $m, n \in \mathbb{N}$. We are going to consider $g_1(m\lambda_1, \ldots, m\lambda_{j-1})$ and $g_2(n\lambda_j, \ldots, n\lambda_k)$ and approximate them by better-behaved $\delta_m h_1$ and $\delta_n h_2$. We take

$$F = \{u_i, v_i \mid i = 1, \ldots, k\},$$

and consider the map $\mathrm{fl}\colon F^\star \to \mathrm{Pr}(F)^\star$ defined in Lemma 2.4.24. The elements h_1 and h_2 are defined as follows:

$$h_1 = \mathrm{ev}\big(\mathrm{fl}\big(g_1(\lambda_1, \lambda_2, \ldots \lambda_{j-1})\big)\big) \quad \text{and} \quad h_2 = \mathrm{ev}\big(\mathrm{fl}\big(g_2(\lambda_j, \ldots, \lambda_k)\big)\big).$$

For instance, if $g_1(n_1, n_2) = u_1^{n_1} u_2^{n_2} v_2^{n_2} v_1^{n_1}$, then $h_1 = \mathrm{Pr}(u_1)^{\lambda_1} \mathrm{Pr}(u_2)^{\lambda_2} \mathrm{Pr}(v_2)^{\lambda_2} \mathrm{Pr}(v_1)^{\lambda_1}$.

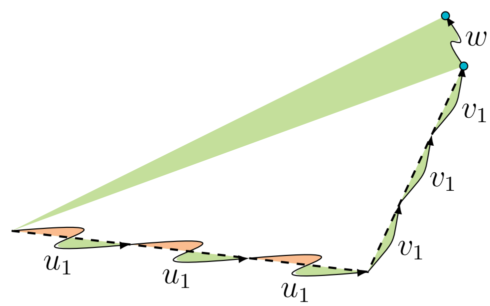


Figure 7.4: $g_1(p\lambda_1)w$ and $\delta_p h_1$. (Underlying word starts with $[_1]_1$ and $\lambda_1 = 1$)

Lemma 2.4.24 gives $\|\delta_{mp} h_1\|_{\mathsf{CC}} = \|g_1(mp\lambda_1, \ldots, mp\lambda_{j-1}) \cdot w\|_{\mathsf{CC}} + O(\sqrt{p})$. It follows

$$\begin{aligned} p\,\|\delta_m h_1\|_{\mathsf{CC}} = \|\delta_{mp} h_1\|_{\mathsf{CC}} &\sim \|g_1(mp\lambda_1, \ldots, mp\lambda_{j-1}) \cdot g_2(0, \ldots, 0)\|_{\mathsf{CC},X,\omega} \\ &\sim \|g_1(mp\lambda_1, \ldots, mp\lambda_{j-1}) \cdot g_2(0, \ldots, 0)\|_S \\ &= \alpha + mp(\lambda_1\tau_1 + \ldots + \lambda_{j-1}\tau_{j-1}) \end{aligned}$$

(where the second $\sim$ follows from Theorem 2.4.27 and Proposition 4.1.1). Similarly

$$\begin{aligned} \|\delta_{np} h_2\|_{\mathsf{CC}} &= \|g_2(np\lambda_j, \ldots, np\lambda_k) \cdot w\|_{\mathsf{CC}} + O(\sqrt{p}) \\ \|\delta_{mp} h_1 \cdot \delta_{np} h_2\|_{\mathsf{CC}} &= \|g_1(mp\lambda_1, \ldots, mp\lambda_{j-1}) \cdot g_2(np\lambda_j, \ldots, np\lambda_k)\|_{\mathsf{CC}} + O(\sqrt{p}) \end{aligned}$$

so that

$$\begin{aligned} p\,\|\delta_n h_2\|_{\mathsf{CC}} &\sim \alpha + np(\lambda_j\tau_j + \ldots + \lambda_k\tau_k) \\ p\,\|\delta_m h_1 \cdot \delta_n h_2\|_{\mathsf{CC}} &\sim \alpha + p\big(m(\lambda_1\tau_1 + \ldots + \lambda_{j-1}\tau_{j-1}) + n(\lambda_j\tau_j + \ldots + \lambda_k\tau_k)\big) \end{aligned}$$

and finally $\|\delta_m h_1 \cdot \delta_n h_2\|_{\mathsf{CC}} = \|\delta_m h_1\|_{\mathsf{CC}} + \|\delta_n h_2\|_{\mathsf{CC}}$ for all $m, n \in \mathbb{N}$.

(c-ii) We have two elements $h_1, h_2 \neq e$ such that $\hat{h}_1 + \hat{h}_2 = \mathbf{0}$ and

$$\|\delta_m h_1 \cdot \delta_n h_2\|_{\mathsf{CC}} = \|\delta_m h_1\|_{\mathsf{CC}} + \|\delta_n h_2\|_{\mathsf{CC}}$$

for all $m, n \geqslant 0$. This means that, for any CC-geodesics γ_1, γ_2 representing h_1 and h_2, the concatenation $\delta_m \gamma_1 \cdot \delta_n \gamma_2$ is a CC-geodesic for $\delta_m h_1 \cdot \delta_n h_2$.

Let us first consider $m = n = 1$. We know that $h_1 h_2$ is a commutator, so any geodesic has to follow a rescaled isoperimetrix. Suppose w.l.o.g. that $\|h_1\|_{\mathsf{CC}} \geqslant \|h_2\|_{\mathsf{CC}}$, then any geodesic γ_1 for h_1 has to cover at least half the perimeter of the isoperimetrix.

- If either the isoperimetrix has $\geqslant 6$ sides or $\|h_1\|_{\mathsf{CC}} > \|h_2\|_{\mathsf{CC}}$, then γ_1 has to cover two corners of the isoperimetrix, hence the scale of the isoperimetrix followed by any geodesic continuation of γ_1 is fixed, and $\gamma_1\gamma_2$ is the maximal geodesic continuation. In particular the longer curve $\gamma_1 \cdot \delta_2\gamma_2$ cannot be a geodesic continuation.

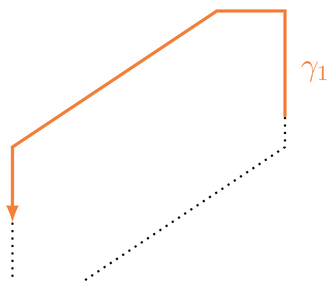


Figure 7.5: All geodesic continuations of γ_1 have to follow the dotted path.

- The only remaining case appears when the isoperimetrix has 4 sides, $\|h_1\|_{\mathsf{CC}} = \|h_2\|_{\mathsf{CC}}$ and the geodesics γ_1 and γ_2 meet at corners of the isoperimetrix. However we still have the same contradiction as $\gamma_1 \cdot \delta_2\gamma_2$ is not a CC-geodesic. □

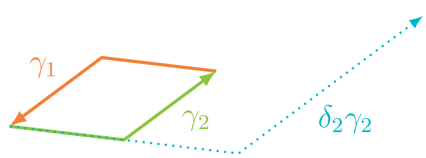


Figure 7.6: The path $\gamma_1 \cdot \delta_2\gamma_2$ is not quite geodesic.

Remark. This extends for higher Heisenberg groups with cubical-like generating sets.

7.5 Quasi-geodesic regular normal forms

Definition 7.5.1. Let (G, d_S) be a group with a word metric. A normal form $\mathcal{L} \subseteq S^\star$ (i.e. a language such that $\mathrm{ev}\colon \mathcal{L} \to G$ is one-to-one) is (λ, C)-quasi-geodesic if

$$\forall w \in \mathcal{L}, \quad \ell(w) \leqslant \lambda \|\bar{w}\|_S + C.$$

If $\lambda = 1$ and $C = 0$, we say the normal form is *geodesic*.

Remark 7.5.2. In the literature, (λ, C)-quasi-geodesics usually satisfy

$$\ell(v) \leqslant \lambda \|\bar{v}\|_S + C$$

for all subwords v of w (i.e. $w = avb$). If the normal form is recognized by an ergodic finite state automaton, both conditions are equivalent up to a change of constants.

The starting point is the observation that having a geodesic regular normal form implies that $\widehat{\Sigma}_{G,d_S}(t)$ is $\mathbb{N}G$-rational, and a partial converse due to Laurent Bartholdi:

Proposition ([Bar17, Exercice B.3])**.** *Suppose that $\widehat{\Sigma}_{G,d_S}(t)$ is $\mathbb{N}G$-rational, then (G, d_S) admits a $(\lambda, 0)$-quasi-geodesic normal form $\mathcal{L} \subseteq S^\star$ for some $\lambda \geqslant 1$.*

We give the following strengthening

Theorem 7.5.3. *Suppose that $\widehat{\Sigma}_{G,d_S}(t)$ is $\mathbb{N}G$-rational, then (G, d_S) admits a $(1+\varepsilon, 0)$-quasi-geodesic normal form $\mathcal{L} \subseteq S^\star$ for any $\varepsilon > 0$.*

7.5.1 Distortion of series

For the proof, we consider more general series.

Definition 7.5.4. Let $(M, \|\cdot\|)$ be a metric monoid, and $\Gamma \in \mathbb{N}M[[t]]$ (with $\Gamma \neq 0$).

- A monomial $m \cdot t^\alpha$ (with $m \in M$) *appears in* $\Gamma(t)$ if $\Gamma(t) - m \cdot t^\alpha \in \mathbb{N}M[[t]]$.
- The *distortion* of Γ is

$$\operatorname{dist}(\Gamma) = \sup\left\{ \frac{\|m\|}{\alpha} \;\middle|\; mt^\alpha \text{ appears in } \Gamma \right\}.$$

- The *asymptotic distortion* of Γ is

$$\operatorname{adist}(\Gamma) = \limsup_{\alpha\to\infty}\left\{ \frac{\|m\|}{\alpha} \;\middle|\; mt^\alpha \text{ appears in } \Gamma \right\}.$$

We prove the following Proposition, which implies Theorem 7.5.3.

Proposition 7.5.5. *Consider a $\mathbb{N}G$-rational series $\Gamma \in \mathbb{N}G[[t]]$ and $\varepsilon > 0$. There exists a $\mathbb{N}S^\star$-rational series $\Lambda \in \mathbb{N}S^\star[[t]]$ such that $\operatorname{ev}(\Lambda) = \Gamma$ and $\operatorname{dist}(\Lambda) \leqslant \operatorname{dist}(\Gamma) + \varepsilon$.*

Take $\Gamma = \widehat{\Sigma}_{G,d_S}$. The *support* of Λ - the set of words $w \in S^\star$ such that wt^α appears in Λ for some α - is rational (using Theorem 7.1.10). Take $\mathcal{L}$ the support, we have

$$\forall w \in \mathcal{L}, \quad \ell(w) \leqslant \operatorname{dist}(\Lambda)\cdot\alpha \leqslant (\operatorname{dist}(\widehat{\Sigma}_{G,d_S}) + \varepsilon)\cdot\alpha = (1+\varepsilon)\cdot\|\bar{w}\|_S\,,$$

hence Theorem 7.5.3.

Lemma 7.5.6. *Let* $\Gamma_1, \Gamma_2 \in \mathbb{N}M[[t]]$ *(with* $\Gamma_1, \Gamma_2 \neq 0$*).*

(a) $\operatorname{dist}(\Gamma_1 + \Gamma_2) = \max\{\operatorname{dist}(\Gamma_1), \operatorname{dist}(\Gamma_2)\}$ *(idem for* adist*).*

(b) $\operatorname{dist}(\Gamma_1 \cdot \Gamma_2) \leqslant \max\{\operatorname{dist}(\Gamma_1), \operatorname{dist}(\Gamma_2)\}$ *(idem for* adist*).*

(c) *If* Γ *is proper, then* $\operatorname{dist}(\Gamma^*) = \operatorname{dist}(\Gamma)$.

(d) *If* Γ *is proper, then* $\limsup_{n\to\infty} \operatorname{dist}(\Gamma^n) \leqslant \operatorname{adist}(\Gamma^*)$.

(e) $\operatorname{adist}(\Gamma_1) \leqslant \operatorname{adist}(\Gamma_1\Gamma_2)$*, with equality if* Γ_2 *is polynomial.*

Proof. Nothing more than $\|mn\| \leqslant \|m\| + \|n\|$. □

Lemma 7.5.7. *If* $\Gamma(t) \in \mathbb{N}M[[t]]$ *is* $\mathbb{N}M$*-rational, and the monomial* mt^α *appears in* $\Gamma(t)$*, then* $\Gamma(t) - mt^\alpha$ *is also* $\mathbb{N}M$*-rational and has the same star height as* $\Gamma(t)$*.*

Proof. The proof is by induction on the star height. First, the statement is trivial if $\Gamma(t) \in \mathrm{Rat}_0$ is a polynomial. Suppose the statement is true for $\Gamma \in \mathrm{Rat}_h$.

- Suppose that mt^α appears in $\Lambda = \Gamma^*$ with $\Gamma \in \mathrm{Rat}_h$, then it appears in Γ^n for some n, which has star height h. We can rewrite

$$\Lambda - mt^\alpha = 1 + \Gamma + \ldots + \Gamma^{n-1} + \big(\Gamma^n - mt^\alpha\big) + \Gamma^{n+1} \cdot \Gamma^* \in \mathrm{Rat}_{h+1}.$$

- Suppose that mt^α appears in $\Lambda = n_0 t^{\beta_0} \cdot \Gamma_1^* \cdot n_1 t^{\beta_1} \cdot \ldots \cdot \Gamma_k^* \cdot n_k t^{\beta_k}$ with $\Gamma_i \in \mathrm{Rat}_h$. There exists monomials $m_i t_i^\alpha$ appearing in Γ_i such that

$$mt^\alpha = n_0 m_1 n_1 \ldots m_k n_k \cdot t^{\beta_0 + \alpha_1 + \beta_1 + \ldots + \alpha_k + \beta_k}.$$

  We write and expend

$$\Lambda = n_0 t^{\beta_0} \cdot \big((\Gamma_1^* - m_1 t^{\alpha_1}) + m_1 t^{\alpha_1}\big) \cdot n_1 t^{\beta_1} \cdot \ldots \cdot \big((\Gamma_k^* - m_k t^{\alpha_k}) + m_k t^{\alpha_k}\big) \cdot n_k t^{\beta_k},$$

  After removing the term mt^α which appears in the RHS, we get an expression for $\Lambda - mt^\alpha$ proving that $\Lambda - mt^\alpha \in \mathrm{Rat}_{h+1}$.

- In full generality, elements $\Lambda \in \mathrm{Rat}_{h+1}$ are finite sums of products of the previous type, and mt^α appears in one of the products. □

Corollary 7.5.8. *Given a* $\mathbb{N}S^\star$*-rational series* $\Lambda \in \mathbb{N}S^\star[[t]]$ *and* $\varepsilon > 0$*, there exists a rational series* $\Lambda' \in \mathbb{N}S^\star[[t]]$ *with the same star height s.t.* $\operatorname{ev}(\Lambda) = \operatorname{ev}(\Lambda') \in \mathbb{N}G[[t]]$ *and*

$$\operatorname{dist}(\Lambda') \leqslant \max\{\operatorname{adist}(\Lambda) + \varepsilon,\ \operatorname{dist}(\operatorname{ev}(\Lambda))\}.$$

Proof. In the series Λ, there are finitely many monomials $w_1 s^{\alpha_1}, \ldots, w_n s^{\alpha_n}$ with

$$\frac{\ell(w_i)}{\alpha_i} > \operatorname{adist}(\Lambda) + \varepsilon.$$

For each of them, pick word $w_i' \in S^\star$ such that $\mathrm{ev}(w_i') = \mathrm{ev}(w_i)$ and $\ell(w_i') = \|\mathrm{ev}(w_i)\|_S$. In particular $\frac{\ell(w_i')}{\alpha_i} = \frac{\|\mathrm{ev}(w_i)\|_S}{\alpha_i} \leqslant \mathrm{dist}(\mathrm{ev}(\Lambda))$. We consider

$$\Lambda' = \Lambda - \sum_{i=1}^{n} w_i s^{\alpha_i} + \sum_{i=1}^{n} w_i' s^{\alpha_i}.$$

By Lemma 7.5.7, this series is $\mathbb{N}S^\star$-rational of the same star height, and

$$\mathrm{dist}(\Lambda') \leqslant \max\{\mathrm{adist}(\Lambda) + \varepsilon, \mathrm{dist}(\mathrm{ev}(\Lambda))\}.$$

□

Proof of Proposition 7.5.5. We prove by induction on the star height of Γ that, for any $\varepsilon > 0$, we can find $\mathbb{N}S^\star$-rational series $\tilde{\Gamma}, \hat{\Gamma} \in \mathbb{N}S^\star[[t]]$ evaluating to Γ such that

$$\mathrm{adist}(\tilde{\Gamma}) \leqslant \mathrm{adist}(\Gamma) + \varepsilon$$
$$\mathrm{dist}(\hat{\Gamma}) \leqslant \mathrm{dist}(\Gamma) + \varepsilon$$

Base case. If $\Gamma \in \mathrm{Rat}_0$ is a polynomial, this is trivial : replace each g by a geodesic word over S. We get $\mathrm{dist}(\Lambda) = \mathrm{dist}(\Gamma)$ (and asymptotic distortions are equal to $-\infty$).

Induction. Let $\Gamma \in \mathrm{Rat}_{h+1}$. It can be written as a finite sum of terms of the form

$$\begin{aligned} &h_0(S_1)^* h_1 (S_2)^* \cdots (S_k)^* \cdot h_k t^\beta \\ &\quad = (h_0 S_1 h_0^{-1})^* \big((h_0h_1) S_2 (h_0h_1)^{-1}\big) \cdots \big((h_0h_1\dots h_{k-1}) S_1 (h_0 h_1 \dots h_{k-1})^{-1}\big)^* h_0 h_1 \dots h_k t^\beta \\ &\quad =: (T_1)^* \cdots (T_k)^* \cdot h t^\beta, \end{aligned}$$

where $S_i, T_i \in \mathrm{Rat}_h$. Using Lemma 7.5.6(d,e,a), we get exponents n_i such that

$$\mathrm{dist}(T_i^{n_i}) \leqslant \mathrm{adist}(T_i^*) \leqslant \mathrm{adist}(\Gamma) + \varepsilon$$

so we can rewrite this term as

$$(\underbrace{1 + T_1 + \ldots + T_1^{n_1 - 1}}_{\Sigma_1})(\underbrace{T_1^{n_1}}_{\Pi_1})^* \cdots (\underbrace{1 + T_m + \ldots + T_m^{n_m - 1}}_{\Sigma_m})(\underbrace{T_m^{n_m}}_{\Pi_m})^*\, h t^\beta$$

As $1 + T_i + \ldots + T_i^{n_i - 1} \in \mathrm{Rat}_h$, our induction hypothesis gives $\tilde{\Sigma}_i \in \mathbb{N}S^\star[[t]]^{\mathrm{rat}}$ s.t.

$$\mathrm{adist}(\tilde{\Sigma}_i) \leqslant \mathrm{adist}(1 + T_i + \ldots + T_i^{n_i-1}) + \varepsilon \leqslant \mathrm{adist}(\Gamma) + \varepsilon$$

where the second inequality comes from Lemma 7.5.6(e,a). Similarly as $T_i^{n_i} \in \mathrm{Rat}_h$ we get a series $\hat{\Pi}_i \in \mathbb{N}S^\star[[t]]^{\mathrm{rat}}$ evaluating to $T_i^{n_i}$ such that

$$\mathrm{dist}(\hat{\Pi}_i) \leqslant \mathrm{dist}(T_i^{n_i}) + \varepsilon \leqslant \mathrm{adist}(\Gamma) + 2\varepsilon.$$

Finally we pick a word $w_h \in S^\star$ evaluating to h. Putting everything together we get

$$\mathrm{adist}\big(\tilde{\Sigma}_1(\hat{\Pi}_1)^* \cdots \tilde{\Sigma}_m(\hat{\Pi}_m)^* \cdot w_h s^\beta\big) = \mathrm{adist}\big(\tilde{\Sigma}_1(\hat{\Pi}_1)^* \cdots \tilde{\Sigma}_m(\hat{\Pi}_m)^*\big) \leqslant \mathrm{adist}(\Gamma) + 2\varepsilon$$

Summing finitely many terms, we get $\tilde{\Gamma} \in \mathbb{N}S^\star[[t]]^{\mathrm{rat}}$ such that $\mathrm{adist}(\tilde{\Gamma}) \leqslant \mathrm{adist}(\Gamma) + 2\varepsilon$ and $\mathrm{ev}(\tilde{\Gamma}) = \Gamma$. Corollary 7.5.8 gives $\hat{\Gamma} \in \mathbb{N}S^\star[[t]]^{\mathrm{rat}}$ with $\mathrm{dist}(\hat{\Gamma}) \leqslant \mathrm{dist}(\Gamma) + 3\varepsilon$. □

7.5.2 Optimality of Theorem 7.5.3

One may hope that $\mathbb{N}G$-rationality of complete growth series of a pair (G, d_S) implies the existence of geodesic regular normal form. Such hopes are washed away by the results of Liardet, combined with an example due to Neumann and Shapiro:

Theorem ([NS97])**.** *There exists a virtually abelian group G generated by a finite set S such that no regular language $\mathcal{L} \subseteq S^\star$ that surjects onto G is geodesic. Specifically*

$$G_{\text{NS}} = \left\langle x, y, a, b \;\middle|\; \begin{array}{c} x^\alpha, x^\beta, y^\alpha, y^\beta \text{ commutes for all } \alpha, \beta \in \langle a, b\rangle \\ a^2 = b^4 = [a,b] = e, \quad x \cdot x^a = y \cdot y^{b^2} \end{array} \right\rangle$$

generated by $S_{\text{NS}} = \{x^\pm, y^\pm, a, b^\pm\}$ is an example (using the notation $g^h := hgh^{-1}$).

This group is virtually abelian, hence has $\mathbb{N}G$-rational complete growth series [Lia96], but admits no geodesic regular normal form. The next best thing to hope for is the existence of a $(1, C)$-quasi-geodesic normal form, that is, to replace the multiplicative error with an additive error. This holds for virtually abelian groups:

Theorem 7.5.9. *Let G be virtually abelian group, and S a finite generating set. Then G admits a regular normal form $\mathcal{L} \subset S^\star$ which is $(1, C)$-quasi-geodesic, for some $C \geqslant 0$.*

Proof. Consider a finite-index normal abelian group $(H, d_{X,\omega})$ as defined in §4.1.1. The language of ShortLex representative is regular (eg. [Eps+92, Theorem 4.3.1]): if

$$X = \left\{t_1 a_1 t_1^{-1}, t_2 a_2 t_2^{-1}, \ldots, t_k a_k t_k^{-1}\right\}$$

in increasing order, then the ShortLex language consist of all words

$$w = (t_1 a_1 t_1^{-1})^{n_1} (t_2 a_2 t_2^{-1})^{n_2} \ldots (t_k a_k t_k^{-1})^{n_k}$$

with $\mathbf{n} = (n_1, n_2, \ldots, n_k) \in \mathbf{U} \subseteq \mathbb{N}^k$, with $\mathbf{U}^c$ an ideal of $(\mathbb{N}^k, +)$ (i.e., something that a finite-state automaton can deal with: check for finitely many vectors $\mathbf{v} \in \mathbb{N}^k$ if $\mathbf{n} \geqslant \mathbf{v}$ component-wise). The language we are looking for is

$$\mathcal{L} = \left\{ t_1 a_1^{n_1} u_1 \cdot t_2 a_2^{n_2} u_2 \cdots t_k a_k^{n_k} u_k \;\middle|\; (n_1, n_2, \ldots, n_k) \in \mathbf{U} \right\},$$

where $u_i \in S^\star$ are fixed words such that $\bar{u}_i = t_i^{-1}$. Proposition 4.1.1 concludes. □

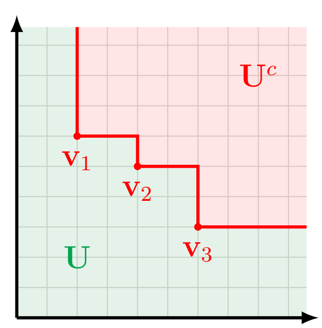


Figure 7.7: An ideal $\mathbf{U}^c$ in $(\mathbb{N}^2, +)$, its complement $\mathbf{U}$, and finitely many generators of $\mathbf{U}^c$.

However, this improvement does not hold for more general pairs (G, d_S).

Proposition 7.5.10. *There exists a group G generated by a finite set S such that*

(a) *The complete growth series of (G, d_S) is $\mathbb{N}G$-rational, and*

(b) *For any regular language $\mathcal{L} \subseteq S^\star$ that surjects onto G, there exists $\varepsilon > 0$ and infinitely many words $w \in \mathcal{L}$ such that $\ell(w) > (1+\varepsilon) \cdot \|\bar{w}\|_S$.*

Specifically, an example is given by $G = G_{\mathrm{NS}} * \mathbb{Z}$, generated by $S = S_{\mathrm{NS}} \sqcup \{z^\pm\}$.

Proof. (a) $\widehat{\Sigma}_{G,d_S}(t)$ is $\mathbb{N}G$-rational. Indeed, [All+11, Corollary 3.5] gives

$$\widehat{\Sigma}_{G,d_S}(t) = \widehat{\Sigma}_{\mathrm{NS}}(t)\left((\widehat{\Sigma}_{\mathbb{Z}}(t)-1)(\widehat{\Sigma}_{\mathrm{NS}}(t)-1)\right)^* \widehat{\Sigma}_{\mathbb{Z}}(t)$$

(Note that removing a term "1" preserves $\mathbb{N}G$-rationality by Lemma 7.5.7.)

(b) Let $\mathcal{L} \subset S^*$ be a regular that surjects onto G. Consider a trim deterministic automaton $M = (V, S, \delta, v_0, \mathsf{accept})$ accepting $\mathcal{L}$, with $|V| = m$ states. Let $g = (zh)^m$ for some $h \in G_{\mathrm{NS}}$. Consider $w \in \mathcal{L}$ a representative for g. Using a loop-erasing algorithm, one can decompose w as the union of a simple path of length $\leqslant m-1$, and at most m cycles (possibly empty or non-simple)

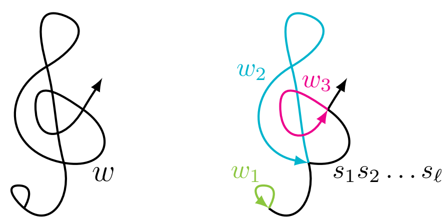


Figure 7.8: Start at the base point of w, then skip all the way to the last visit of this vertex, effectively skipping over a cycle w_1. Then go to the next vertex and repeat.

On the other hand, using cut vertices in $\mathcal{C}ay(G, S)$, we can decompose w as $w = v_0\, sv_1\, sv_2 \cdots sv_m$ such that $\mathrm{ev}(v_0\, zv_1 \cdots zv_i) = (zh)^i$ for all $i = 1, 2, \ldots, m$. It follows that at least one of the $v_1, v_2, \ldots, v_m$ (which all evaluates to h) lie completely inside one of the cycles. Therefore

$$\mathcal{L}' = \{v \in S^\star \mid v \text{ is a subword of (a word labeling) a cycle in } A \text{ and } \bar{v} \in G_{\mathrm{NS}}\}$$

surjects onto G_{NS}.

- Either some word $v \in \mathcal{L}'$ contains a letter $z^\pm$, hence is not geodesic.
- Or $\mathcal{L}' \subseteq S_{\mathrm{NS}}^\star$. In particular, we can rewrite

$$\mathcal{L}' = \{v \in S_{\mathrm{NS}}^\star \mid v \text{ is a subword of a cycle in } A\}.$$

  It follows that $\mathcal{L}'$ is regular. (Recognized by the automaton A, where we consider all states as initial and terminal, and we remove all edges labeled $z^\pm$, and all edges going from a strongly connected component to another.)

In both cases, it follows that $\mathcal{L}'$ contains some word v which is not geodesic. Consider β a cycle in A (say from $p \in V$ to itself) containing v as a subword, and α, γ two paths in A going from a starting vertex to p, and from p to an accept vertex respectively. Observe that the word labeling β is not geodesic, pick $\varepsilon > 0$ such that $\ell(\beta) > (1+\varepsilon)\left\|\bar{\beta}\right\|_S$. The words w_n labeling $\alpha\beta^n\gamma$ satisfy

$$\ell(w_n) = \ell(\alpha) + n\ell(\beta) + \ell(\gamma) > (1+\varepsilon)\big(\|\bar{\alpha}\| + n\left\|\bar{\beta}\right\| + \|\bar{\gamma}\|\big) \geqslant (1+\varepsilon)\|\bar{w}_n\|$$

for all n large enough, as wanted. □

7.5.3 Application: Lamplighter groups

In this section, we apply the previous criterion to lamplighter groups. In some sense, this completes the proof suggested in [Bar17, Exercice C.1].

Proposition 7.5.11. *Consider $G = F \wr \mathbb{Z}$ with F finite and non-trivial, and with the standard generating set $S = S_F \sqcup \{z^{\pm}\}$. Let $D = \operatorname{diam}(F, d_{S_F})$. Then*

(a) *G admits a 3-quasi-geodesic regular normal form $\mathcal{L} \subset S^\star$*

(b) *G admits no λ-quasi-geodesic regular normal form, for $1 \leqslant \lambda < \frac{D+3}{D+1}$.*

In particular, $\widehat{\Sigma}_{G,d_S}(t)$ is not $\mathbb{N}G$-rational.

Proof. (a) Let $\mathcal{F} \subset S_F^\star$ be a set of geodesic representatives for $F \setminus \{e_F\}$, and let $\mathcal{Z} = \{z\}^+ \sqcup \{\varepsilon\} \sqcup \{z^{-1}\}^+$. We consider the "left-first" normal form [CT03]

$$\mathcal{L} = \mathcal{Z} \sqcup \mathcal{Z}\mathcal{F}(z^+\mathcal{F})^*\mathcal{Z}.$$

We denote $q\colon G \twoheadrightarrow \mathbb{Z}$ the "endpoint map". Consider $g = (\Phi, q) \in F \wr \mathbb{Z}$.

- If $\operatorname{supp}(\Phi) = \emptyset$, the element g is represented by a word in $\mathcal{Z}$, which is geodesic.
- If $\operatorname{supp}(\Phi) \neq \emptyset$, let $m = \min\operatorname{supp}(\Phi)$ and $M = \max\operatorname{supp}(\Phi)$. The word $w \in \mathcal{L}$ representing g first goes m, then switch all the lamps needed from left to right until M, and finally goes to its endpoint q. In particular,

$$\begin{aligned}\ell(w) &= d(\mathbf{0}, m) + d(m, M) + d(M, q) + \sum_{i\in\mathbb{Z}} \|\Phi(i)\|_{S_F} \\ &\leqslant 3 \cdot TS(\mathbf{0}; \{m, M\}; q\}) + \sum_{i\in\mathbb{Z}} \|\Phi(i)\|_{S_F} \\ &< 3\|g\|_S\end{aligned}$$

  Indeed, every edge of $\mathbb{Z}$ we use (each at most three times) has to be used at least once in the optimal solution for the traveling salesman problem.

(b) Informally, we prove that regular normal form cannot avoid "going left-first at large scale" for "right-first elements", or the opposite. Pictorially,

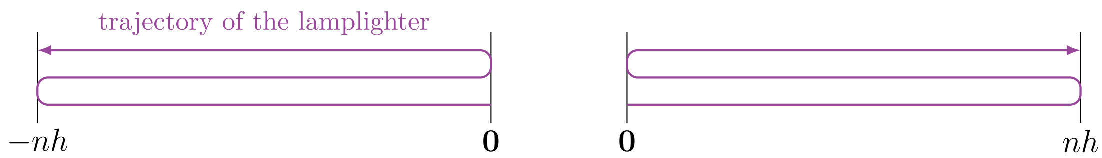


Figure 7.9: Regular normal forms of $L \wr \mathbb{Z}$ do at least one of these mistakes.

Let $\mathcal{L} \subseteq S^\star$ be a regular language such that $\mathrm{ev}(\mathcal{L}) = G$. Consider $f \in F \setminus \{e\}$, and let

$$g = z^{-N} f z^N \cdot z^N f z^{-N},$$

an element with lamps at position $\pm N$ in a non-trivial state and the lamplighter back at the origin, with N larger than the size of the automaton recognizing $\mathcal{L}$. Fix $w = s_1 s_2 \dots s_\ell \in \mathcal{L}$ such that $\mathrm{ev}(w) = g$. There exist i, j (w.l.o.g. $0 < i < j < \ell$) such that

$$q(\mathrm{ev}(s_1 \dots s_i)) = -N \quad \text{and} \quad q(\mathrm{ev}(s_1 \dots s_j)) = N.$$

As N is large, we can decompose $w = u_0 v_1 u_1 v_2 u_2 v_3 u_3$ such that

$$\forall n_1, n_2, n_3 \geqslant 0, \quad w(n_1, n_2, n_3) \coloneqq u_0\, v_1^{n_1}\, u_1\, v_2^{n_2}\, u_2\, v_3^{n_3}\, v_3 \in \mathcal{L}$$

and $q(\bar{v}_1) < 0$, $q(\bar{v}_2) > 0$ and $q(\bar{v}_3) < 0$. (In particular, we have a control on the number of instances of $z^\pm$ in v_1, v_2, v_3.) There exist $\mu_1, \mu_2, \mu_3 > 0$ such that

$$-\mu_1 q(\bar{v}_1) = \mu_2 q(\bar{v}_2) = -\mu_3 q(\bar{v}_3) =: h$$

Let $w_n \coloneqq w(\mu_1 n, \mu_2 n, \mu_3 n)$ and $(\Phi_n, q_n) = \bar{w}_n$. On the one side, we have

$$\begin{aligned} \mathrm{supp}(\Phi_n) &\subseteq [-nh + O_{n\to\infty}(1),\ O_{n\to\infty}(1)] \\ q_n &= -nh + O_{n\to\infty}(1) \end{aligned}$$

so that geodesic for $\bar{w}_n$ should be "right-first". By Proposition 7.2.6

$$\|\bar{w}_n\|_S = nh + \sum_{i\in\mathbb{Z}} \|\Phi_n(i)\|_{S_F} + O_{n\to\infty}(1),$$

with $\sum_{i\in\mathbb{Z}} \|\Phi_n(i)\|_{S_F} \leqslant D \cdot nh + O_{n\to\infty}(1)$. On the other side, we have

$$\ell(w_n) \geqslant 3nh + \sum_{i\in\mathbb{Z}} \|\Phi_n(i)\|_{S_F}.$$

(The trajectory w_n is "left-first".) It follows that $\frac{\ell(w_n)}{\|\bar{w}_n\|_S} \geqslant \frac{D+3}{D+1} + O_{n\to\infty}\left(\frac{1}{n}\right)$. □

7.5.4 Application: Nilpotent groups of rank 2

We give a second application using CC-geometry and Pansu's theorem.

Lemma 7.5.12. *Let Γ be a simply connected stratified nilpotent Lie group with $V_1 \simeq \mathbb{R}^2$. Consider $\mathrm{P} \subset V_1$ a centrally symmetric convex polygon, and $\|\cdot\|_{\mathsf{CC},\mathrm{P}}$ the associated Carnot-Caratheodory metric on Γ. There exists $\lambda > 1$ such that*

$$\ell_{\mathrm{P}}(\gamma) \geqslant \lambda \, \|\bar{\gamma}\|_{\mathsf{CC},\mathrm{P}}$$

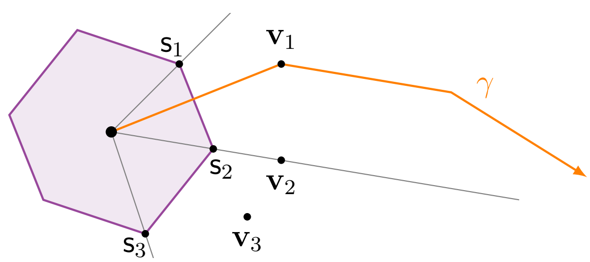
for every triangular $\gamma\colon [0, \ell] \to V_1$. (Recall that $\bar{\gamma}$ is the endpoint of the horizontal lift.)

Proof. We prove that no triangle is CC-geodesic, by contraposition. Let $\gamma\colon [0, \ell] \to \mathbb{R}^2$ be a piecewise-C^1 geodesic. In this setting, [HLD23, Corollary 1.4] tells us that, for all t, the directions $\gamma'(t^-)$ and $\gamma'(t^+)$ belong to a common cone $\mathbb{R}^+\mathrm{F}$, with F a facet of P.

Let $\mathsf{s}_1, \mathsf{s}_2 \ldots, \mathsf{s}_{2N}$ be the vertices of P in order. As P is centrally symmetric, we have $N \geqslant 2$ and $\mathsf{s}_{N+i} = -\mathsf{s}_i$. If we moreover suppose γ is made of three segments, this implies the three directions $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ belong two at most two consecutive cones, say

$$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^+\mathsf{s}_1 + \mathbb{R}^+\mathsf{s}_2 + \mathbb{R}^+\mathsf{s}_3 \subseteq \mathbb{R}\mathsf{s}_1 + \mathbb{R}^+\mathsf{s}_2.$$

Everything lives in a half-plane, the only way that the curve is closed is that γ is reduced to a point. (Unless we "truly use" s_2, we cannot use both $\mathsf{s}_1, \mathsf{s}_3$.)



This proves that, for any triangle γ, we have $\|\bar{\gamma}\|_{\mathsf{CC},\mathrm{P}} < \ell_P(\gamma)$. We get the stronger conclusion by compactness: the set of triangles γ with $\ell_{\mathrm{P}}(\gamma) = 1$ is compact, and the function $\gamma \mapsto \|\bar{\gamma}\|_{\mathsf{CC},\mathrm{P}}$ is continuous, so it reaches its maximum $M < 1$. Finally

$$\ell_{\mathrm{P}}(\gamma) \geqslant \frac{1}{M} \, \|\bar{\gamma}\|_{\mathsf{CC},\mathrm{P}}$$

for all triangles γ (using dilations to extend the result when $\ell_{\mathrm{P}}(\gamma) \neq 1$). □

Remark 7.5.13. This argument does not extend to $(r+1)$-gons in higher rank $r \geqslant 3$. For instance, if we endow $\Gamma = H_3(\mathbb{R}) \times \mathbb{R}$ with the cubical-like Lie generating set $S = \{(x^\pm, \pm 1), (y^\pm, \pm 1)\}$ (hence $\mathrm{P} = \operatorname{ConvHull}\operatorname{Pr}(S)$ is a cube), the path

$$\gamma = (x, 1)^\mu \cdot (y, -1)^\mu \cdot (x^{-1}, 1)^\mu \cdot (y^{-1}, -1)^\mu$$

is a geodesic representative for z^{μ^2}. This corresponds to the equality

$$(1,1,1)+(1,-1,-1)+(-1,1,-1)+(-1,-1,1)=\mathbf{0},$$

where each pair of terms lie on common facet of P (i.e., an entry stays unchanged). However, we still get some partial results in higher rank when the "minimal combinatorial distance between orthogonal points on the surface of P" is large enough.

Theorem 7.5.14. *Consider (G, d_S) such that G contains a finite-index, 2-generated, nilpotent subgroup H. If $\widehat{\Sigma}_{G,d_S}(t)$ is $\mathbb{N}G$-rational, then G is virtually abelian.*

Proof. We argue by contraposition, and suppose that G is not virtually abelian.

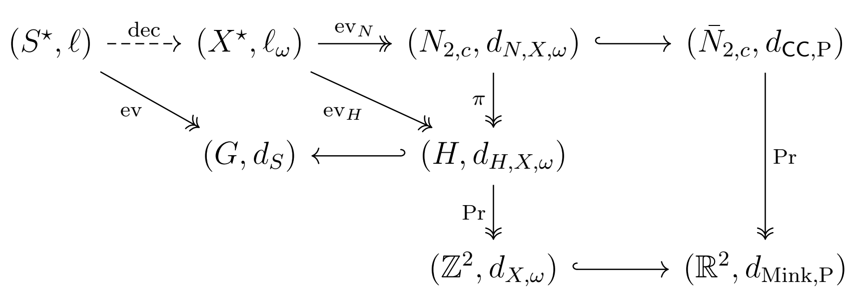
We consider H a finite-index, 2-generated, c-step nilpotent subgroup. In particular, we have an epimorphism $\pi\colon N_{2,c} \twoheadrightarrow H$. We consider (X,ω) the weighted generating set defined in §4.1.1. We define a morphism $\mathrm{ev}_N\colon X^\star \to N_{2,c}$ s.t. $\pi(\mathrm{ev}_N(w)) = \mathrm{ev}_H(w)$. Observe that $\mathrm{ev}_N(X)$ is generating set of $N_{2,c}$, indeed we have

$$\langle \mathrm{ev}_N(X)\rangle\,[N_{2,c}, N_{2,c}] = N_{2,c} \implies \langle \mathrm{ev}_N(X)\rangle = N_{2,c}$$

[Khu93, Theorem 2.2.3(d)]. Let $\mathcal{L} \subseteq S^\star$ be a regular surjecting onto G. We consider

$$\tilde{\mathcal{L}} = \left\{\tilde{w} \;\middle|\; w \in \mathcal{L} \text{ and } \mathrm{ev}(w) \in H\right\} \subseteq X^\star$$

where $\tilde{w}$ is the decomposition map defined in §2.6.5. Observe that $\tilde{\mathcal{L}}$ is also regular and satisfies $\mathrm{ev}_H(\tilde{\mathcal{L}}) = H$. As H is not virtually abelian, its abelianization is $\mathbb{Z}^2$ and it derived subgroup $[H,H]$ is infinite. It particular, there exists infinitely many $\tilde{w} \in \tilde{\mathcal{L}}$ such that $\Pr(\mathrm{ev}_H(\tilde{w})) = \mathbf{0}$.



Pick $\tilde{w} \in \tilde{\mathcal{L}}$ a sufficiently long word such that $\Pr(\mathrm{ev}_H(\tilde{w})) = \mathbf{0}$. We can write it as a product $\tilde{w} = u_0v_1u_1v_2u_2v_3u_3$ where $v_1, v_2, v_3 \neq \varepsilon$, such that $\tilde{w}(n_1,n_2,n_3) = u_0v_1^{n_1}u_1v_2^{n_2}u_2v_3^{n_3}u_3$ belong to $\tilde{\mathcal{L}}$ for all $n_1,n_2,n_3 \geqslant 0$, and crucially

$$\mathbf{0} \in \mathrm{ConvHull}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}, \quad \text{where } \mathbf{v}_i \coloneqq \Pr(\mathrm{ev}_H(v_i)).$$

In particular, there exist $\mu_1,\mu_2,\mu_3 \in \mathbb{Z}_{>0}$ such that $\tilde{w}_n \coloneqq \tilde{w}(n\mu_1,\, n\mu_2,\, n\mu_3)$ satisfies $\Pr(\mathrm{ev}_H(\tilde{w}_n)) = \mathbf{0}$ for all n. We prove that these words are at best λ-quasi-geodesic.

If $\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{v}_3 = \mathbf{0}$, this is trivial as $[H, H]$ is quadratically distorted, hence

$$\|\mathrm{ev}(w_n)\|_S \sim \|\mathrm{ev}(w_n)\|_{X,\omega} \preceq \sqrt{n},$$

by Proposition 4.1.1, while $\ell(w_n) = \ell_\omega(\tilde{w}_n) \asymp n$ (as $v_i \neq \varepsilon$). Let us suppose at least one of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is non-zero. On the one side, we have

$$\begin{aligned}\ell(w_n) = \ell_\omega(\tilde{w}_n) &\geqslant n\mu_1 \cdot \ell_\omega(v_1) + n\mu_2 \cdot \ell_\omega(v_2) + n\mu_3 \cdot \ell_\omega(v_3) \\ &\geqslant n\mu_1 \|\mathbf{v}_1\|_{\mathrm{Mink,P}} + n\mu_2 \|\mathbf{v}_2\|_{\mathrm{Mink,P}} + n\mu_3 \|\mathbf{v}_3\|_{\mathrm{Mink,P}} \\ &= \ell_{\mathrm{P}}(\gamma_n) \geqslant \lambda \cdot \|\bar{\gamma}_n\|_{\mathsf{CC},\mathrm{P}}\end{aligned}$$

where γ_n is the triangular path with sides $n\lambda_1\mathbf{v}_1$, $n\lambda_2\mathbf{v}_2$ and $n\lambda_3\mathbf{v}_3$, and λ is the constant given in Lemma 7.5.12. (Note that $\bar{N}_{2,c}$ is stratified.) On the other side,

$$\begin{aligned}\|\mathrm{ev}(w_n)\|_S &\sim \|\mathrm{ev}_H(\tilde{w}_n)\|_{H,X,\omega} && (1)\\ &\leqslant \|\mathrm{ev}_N(\tilde{w}_n)\|_{N,X,\omega} && (2)\\ &= \Big\|(\mathrm{ev}_N(u_0)\,\mathrm{ev}_N(v_1)\,\mathrm{ev}_N(u_0)^{-1})^{n\lambda_1}(\mathrm{ev}_N(u_0u_1)\,\mathrm{ev}_N(v_2)\,\mathrm{ev}_N(u_0u_1)^{-1})^{n\lambda_2}\ldots \\ &\qquad \ldots(\mathrm{ev}_N(u_0u_1u_2)\,\mathrm{ev}_N(v_3)\,\mathrm{ev}_N(u_0u_1u_2)^{-1})^{n\lambda_3}\,\mathrm{ev}_N(u_0u_1u_2u_3)\Big\|_{N,X,\omega} \\ &= \Big\|(\mathrm{ev}_N(u_0)\,\mathrm{ev}_N(v_1)\,\mathrm{ev}_N(u_0)^{-1})^{n\lambda_1}(\mathrm{ev}_N(u_0u_1)\,\mathrm{ev}_N(v_2)\,\mathrm{ev}_N(u_0u_1)^{-1})^{n\lambda_2}\ldots \\ &\qquad \ldots(\mathrm{ev}_N(u_0u_1u_2)\,\mathrm{ev}_N(v_3)\,\mathrm{ev}_N(u_0u_1u_2)^{-1})^{n\lambda_3}\Big\|_{N,X,\omega} + O(1) \\ &\sim \Big\|(\mathrm{ev}_N(u_0)\,\mathrm{ev}_N(v_1)\,\mathrm{ev}_N(u_0)^{-1})^{n\lambda_1}(\mathrm{ev}_N(u_0u_1)\,\mathrm{ev}_N(v_2)\,\mathrm{ev}_N(u_0u_1)^{-1})^{n\lambda_2}\ldots \\ &\qquad \ldots(\mathrm{ev}_N(u_0u_1u_2)\,\mathrm{ev}_N(v_3)\,\mathrm{ev}_N(u_0u_1u_2)^{-1})^{n\lambda_3}\Big\|_{\mathsf{CC},\mathrm{P}} && (3)\\ &\sim \|\bar{\gamma}_n\|_{\mathsf{CC},\mathrm{P}} && (4)\end{aligned}$$

which concludes. Here we use (1) Proposition 4.1.1, (2) π is a submetry, (3) Pansu's Theorem 2.4.23 and (4) Lemma 2.4.24 with

$$\begin{aligned}F = \big\{&\mathrm{ev}_N(u_0)\,\mathrm{ev}_N(v_1)\,\mathrm{ev}_N(u_0)^{-1},\ \mathrm{ev}_N(u_0u_1)\,\mathrm{ev}_N(v_2)\,\mathrm{ev}_N(u_0u_1)^{-1}, \\ &\mathrm{ev}_N(u_0u_1u_2)\,\mathrm{ev}_N(v_3)\,\mathrm{ev}_N(u_0u_1u_2)^{-1}\big\}.\end{aligned}$$ □

Remark 7.5.15. We could prove Lemma 7.5.12 without the "stratified" assumption, using the same trick of lifting to a stratified cover. This would lighten the notations for the second part of the proof. Anyway, at least we have a fun little diagram.

Chapter 8

Non-D-finite Green series

We consider the (re-scaled[1]) Green series of groups. Given a group G and a finite generating (multi)set S, we associate a series $\Gamma_{G,S}(z)$ defined as

$$\Gamma_{G,S}(z) = \sum_{\ell=0}^{\infty} c_\ell \cdot z^\ell \in \mathbb{Z}[[z]]$$

where $c_\ell = \#\{(s_1, \ldots, s_\ell) \in S^\ell \mid s_1 \ldots s_\ell = e_G\}$ is the number of closed paths $e_G \to e_G$ of length ℓ in $\mathcal{C}ay(G, S)$. In the spirit of the study of other combinatorial sequences, we would like to pin down these series inside the following algebraic hierarchy:

$$\text{rational} \Rightarrow \text{algebraic} \Rightarrow \text{diagonal of rational} \Rightarrow D\text{-finite} \Rightarrow D\text{-algebraic}.$$

Lots of work has been done in this direction. For instance,

- Kouskov proved that this series is rational if and only if G is finite [Kou98].
- $\Gamma_{G,S}(z)$ is algebraic as soon as G is virtually free. This follows from the Muller–Schupp theorem [MS83], as the Word Problem is unambiguously context-free in this case. It is an open problem whether the converse holds.
- Bishop recently proved that $\Gamma_{G,S}(z)$ is the diagonal of a rational series as soon as G is virtually $F_m \times \mathbb{Z}^n$ [Bis24]. This improves upon previous results of [Eld+14].

We also have a few results in the negative direction:

- Most proofs use the asymptotics. For instance, if $\sum_{n\geqslant 0} a_n \cdot z^n$ is D-finite and has positive radius of convergence, then there exist $\alpha_i \in \mathbb{Q}, \beta_i \in \mathbb{N}, \lambda_i \in \overline{\mathbb{Q}}$ such that

$$a_n \sim \sum_{i=1}^{m} A_i \cdot \lambda_i^n \cdot n^{\alpha_i} \cdot (\log n)^{\beta_i}.$$

[1] The Green series is usually defined as $G(z) = \sum_{n\geqslant 0} \mathbb{P}[X_n = e] \cdot z^n$, where $(X_n)_{n\geqslant 0}$ is the simple random walk on $\mathcal{C}ay(G, S)$, starting at $X_0 = e$. Therefore $\Gamma_{G,S}(z) = G(|S| \cdot z)$.

This is the approach used by Kouskov for the result quoted earlier. Perhaps the most notable result in this direction is the proof by Bell and Mishna that $\Gamma_{G,S}(z)$ is not D-finite for amenable groups of super-polynomial growth [BM20].

- Garrabrant and Pak proved that $F_2 \times F_2$ had a specific generating multiset for which $\Gamma_{G,S}(t)$ is not D-finite [GP17]. Their approach is quite different as the asymptotics for this group are not too wild (combining [Cha17, Theorem 1.3] and [CHR13]). Instead, they manufacture a generating set so that the sequence ($c_\ell \bmod 4$) has large *subword complexity*, and prove this cannot happen for D-finite series.

For virtually nilpotent groups, the consensus was not clear[2]. The known asymptotics [Ale02, Corollary 1.17] perfectly match what is possible among D-finite series. For instance, for $G = H_3(\mathbb{Z})$ with generating set $S = \{x^\pm, y^\pm, e\}$, Diaconis and Hough proved that $c_\ell = \left(\frac{25}{16}\ell^{-2} + O(\ell^{-5/2})\right) 5^\ell$ [DH21]. The first conclusive evidence was given by Pak and Soukup. They proved the undecidability of some decision problem related to Green series. As a corollary, they obtain the following result:

Theorem ([PS22])**.** *There exists a nilpotent group G such that*

- *Either there exists a finite generating multiset S such that the Green series $\Gamma_{G,S}(z)$ is not the diagonal of rational series,*
- *Or at least, there is no algorithm which, given a generating set S, computes a representation of the Green series as the diagonal of rational series.*

Specifically, they consider the group of $m \times m$ unitriangular matrices $G = UT_m(\mathbb{Z})$, with $m = 9.6 \cdot 10^{85}$. We improve on this result, proving that the first conclusion holds, at the cost of allowing for virtually nilpotent groups. The group in consideration is

$$v\mathscr{H} = H_3(\mathbb{Z}) \rtimes C_2 = \left\langle x, t \;\middle|\; [x,[x,x^t]] = [x^t,[x,x^t]] = t^2 = 1 \right\rangle .$$

This group presentation was introduced in [BE22] as the first example of group with polynomial geodesic growth which is not virtually abelian.

Theorem 8.A. *The Green series of the virtually nilpotent group $v\mathscr{H}$ with respect to the generating multiset $S = \{x, x^{-1}, t, t, t, t, t, t, t, t\}$ is not D-finite.*

Our proof is most similar to Pak–Garrabrant argument. We also consider the values of some derived sequence modulo a large power of 2, and compute its subword complexity.

Remark. We can promote this result to generating *sets* using $G' = v\mathscr{H} \times D_8$ and $S' = \{(x^\pm, 0)\} \cup \{(t,g) \mid g \in D_8\}$. This follows from the previous result as

$$\Gamma_{v\mathscr{H},S}(z) = 8 \cdot \Gamma_{G',S'}(z) - 7 \cdot \Gamma_{\langle x\rangle,\{x^\pm\}}(z),$$

and $\Gamma_{\langle x\rangle,\{x^\pm\}}(t)$ is an algebraic series.

[2]The reader can compare the different versions of [GP17, §6]

8.1 Preliminaries

8.1.1 D-finite series

Definition 8.1.1. A series $\sum_{n\geqslant 0} a_n \cdot z^n$ is *D-finite* (or *holonomic*) if its coefficients are P-recursive, i.e., if there exist polynomials $p_0, \ldots, p_k \in \mathbb{Z}[X]$ with $p_0 \neq 0$ such that

$$p_0(n) \cdot a_n + p_1(n) \cdot a_{n-1} + \ldots + p_k(n) \cdot a_{n-k} = 0 \quad \text{for all } n \geqslant k.$$

D-finite series are closed under many operations, as shown by Stanley:

Proposition 8.1.2 ([Sta80, Theorems 2.1, 2.3, 2.7])**.** *Let $A, \Gamma, \tilde{\Gamma} \in \mathbb{Z}[[z]]$ be series.*

(a) *If $A(z)$ is an algebraic series, then $A(z)$ is D-finite.*

(b) *If $\Gamma(z), \tilde{\Gamma}(z)$ are D-finite, then $c \cdot \Gamma(z) + \tilde{c} \cdot \tilde{\Gamma}(z)$ and $\Gamma(z) \cdot \tilde{\Gamma}(z)$ are D-finite.*

(c) *If $\Gamma(z)$ is D-finite and $A(z)$ is algebraic with $A(0) = 0$, then $\Gamma(A(z))$ is D-finite.*

We add another operation to the list:

Proposition 8.1.3. *If $\sum_{n\geqslant 0} a_n \cdot z^n$ is D-finite, then the extracted series*

$$\sum_{n=0}^{\infty} a_{2n} \cdot z^n \quad \text{and} \quad \sum_{n=0}^{\infty} a_{2n+1} \cdot z^n$$

are D-finite.

Proof. Let $\Gamma(z) = \sum_{n\geqslant 0} a_n \cdot z^n$. Using Proposition 8.1.2, the series

$$\sum_{n\geqslant 0} a_{2n} \cdot z^{2n} = \frac{1}{2}\big(\Gamma(z) + \Gamma(-z)\big)$$

is D-finite. Therefore, there exist polynomials $p_0, \ldots, p_{2k} \in \mathbb{Z}[X]$ such that

$$p_0(2n) \cdot a_{2n} + p_2(2n) \cdot a_{2n-2} + \ldots + p_{2k}(2n) \cdot a_{2n-2k} = 0 \quad \text{for all } 2n \geqslant 2k.$$

Taking $q_i(n) = p_{2i}(2n)$ concludes. The proof for the other series is analogous. □

8.1.2 Subword complexity

Definition 8.1.4. Given a sequence $(a_n)_{n\geqslant 0} \in \mathcal{A}^{\mathbb{Z}_{\geqslant 0}}$, its *subword complexity* (or *block complexity*) is the function $p_a \colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$.

$$p_a(n) = \#\big\{(u_1, \ldots, u_n) \in \mathcal{A}^n \mid \exists x \geqslant 0,\ a_{x+i} = u_i \text{ for } i = 1, \ldots, n\big\}.$$

Many "algebraically nice" sequences have low complexity. For instance,

- Eventually periodic sequences (eg. coefficients of rational series in $\mathbb{F}_q[[X]]$) are characterized by the property $p_a(n) = O(1)$. Otherwise $p_a(n) \geqslant n+1$.
- Automatic sequences (eg. coefficients of algebraic series in $\mathbb{F}_q[[X]]$, or equivalently diagonal of rational series in $\mathbb{F}_q[[X]]$) satisfy $p_a(n) = O(n)$.

We recall another result in that direction, which will be key in our argument:

Theorem 8.1.5 (Garrabrant-Pak, [GP17, Lemma 4])**.** *Let $\sum_{n\geqslant 0} a_n \cdot z^n \in \mathbb{Z}[[X]]$ be a D-finite series. Then the sequence $(a_n \bmod 2)_{n\geqslant 0}$ has subword complexity $p_a(n) = o(2^n)$.*

8.2 Complexity of multiplicative sequences

In contrast with coefficients of D-finite series, we prove that many *multiplicative* functions have maximal subword complexity. Some proof ideas appear in [Li20, §4].

Definition 8.2.1. A function $f\colon \mathbb{Z}_{>0} \to \mathbb{C}$ is *multiplicative* if

$$\forall m, n \in \mathbb{Z}_{>0} \text{ such that } \gcd(m,n) = 1, \qquad f(mn) = f(m)f(n).$$

Theorem 8.2.2. *Let $f\colon \mathbb{Z}_{>0} \to \{\pm 1\}$ be a multiplicative function. Suppose that*

- *the set $P_f = \{p \text{ prime} \mid \exists q = p^m, f(q) = -1\}$ is infinite, and*
- *the set $Q_f = \{q \text{ prime power} \mid f(q) = -1\}$ is sparse in the sense $\sum_{q\in Q} \frac{1}{q} < \infty$.*

Then the subword complexity of $\big(f(n)\big)_{n>0}$ is maximal, that is, $p_f(n) = 2^n$.

Remark. Neither assumption can be fully dropped. For instance, if Q_f (hence P_f) is finite, then the function is periodic and $p_f(n) = O(1)$. If we drop the "sparseness" condition, some automatic sequences enter the picture, such as

$$f(n) = \frac{n}{2^{\nu_2(n)}} \pmod 4.$$

The Liouville function $\lambda(n)$ satisfies the first condition, and $p_\lambda(n) \geqslant (1+\varepsilon)^n$ is a long-standing open problem related to Sarnak's conjecture on Möbius disjointness.

Proof. Fix $(u_1, \ldots, u_n) \in \{\pm 1\}^n$, we find x such that $f(x+i) = u_i$ for all $1 \leqslant i \leqslant n$.

Let $I = \{i \in [\![1, n]\!] \mid f(i) \neq u_i\}$ (the "failure set" for $x = 0$). For each $i \in I$, we pick a prime power $p_i^{m_i}$ such that $f(p_i^{m_i}) = -1$. The first hypothesis ensures we can take all primes $p_i > n$ and distincts. If $i \notin I$, we take as convention $p_i = 1$. The Chinese remainder theorem gives infinitely many x satisfying the conditions

- For each prime $p \leqslant n$, we take $x \equiv 0 \pmod{p^{m+1}}$, where $m = \lfloor \log_p(n) \rfloor$.

- For each index $i \in I$, we take $x \equiv p_i^{m_i} - i \pmod{p_i^{m_i+1}}$.

They are all of the form $x = kM + R$ where

$$M = \prod_{p \leqslant n} p^{m+1} \cdot \prod_{i \in I} p_i^{m_i+1}$$

and $0 \leqslant R < M$. By construction, we have $x + i = i \cdot p_i^{m_i} \cdot (kM_i + R_i)$ where $kM_i + R_i$ doesn't contain any extra factor $p \leqslant n$ or p_j (with $1 \leqslant j \leqslant n$). In particular,

$$f(x+i) = f(i) \cdot f(p_i^{m_i}) \cdot f(kM_i + R_i) = u_i \cdot f(kM_i + R_i).$$

We prove that a positive proportion of all k satisfy $q \nmid kM_i + R_i$ for all $q \in Q_f$ and $1 \leqslant i \leqslant n$. The only prime factors that still matter come from

$$\tilde{P}_f = \left\{ p \in P_f \;\middle|\; p > n \text{ and } p \neq p_i \right\}.$$

For each rank N, we partition $\tilde{P}_f = \tilde{P}_{f,\leqslant N} \sqcup \tilde{P}_{f,>N}$. For each $p \in P_f$, let m be the smallest integer such that $f(p^m) = -1$

$$\frac{1}{X} \# \left\{ k \leqslant X \;\middle|\; \begin{array}{l} \forall p \in \tilde{P}_{f,\leqslant N}, \\ \forall i \in [\![1, n]\!], \end{array} \quad p^m \nmid kM_i + R_i \right\} = \prod_{p \in \tilde{P}_{f,\leqslant N}} \left(1 - \frac{n}{p^m} \right) + O_N\left(\frac{1}{X} \right)$$

(Indeed, the count is exact each time X is a common multiple of the p^m for $p \in \tilde{P}_{f,\leqslant N}$. The variation in between is accounted by $O_N(\frac{1}{X})$.)

$$\begin{aligned} \frac{1}{X} \cdot \# \left\{ k \leqslant X \;\middle|\; \exists p \in \tilde{P}_{f,>N}, \exists i \in [\![1, n]\!], \quad p^m \mid kM_i + R_i \right\} \\ \leqslant \frac{n}{X} \cdot \# \left\{ k \leqslant (M+1)X \;\middle|\; \exists p \in \tilde{P}_{f,>N}, \quad p^m \mid k \right\} \\ \leqslant (M+1)\, n \cdot \sum_{p \in \tilde{P}_{f,>N}} \frac{1}{p^m} \end{aligned}$$

Using the hypothesis $\sum_{p \in P_f} \frac{1}{p^m} < \infty$, we have

$$\mathbb{P}_N \coloneqq \prod_{p \in \tilde{P}_{f,\leqslant N}} \left(1 - \frac{n}{p^m} \right) - (M+1)\, n \cdot \sum_{p \in \tilde{P}_{f,>N}} \frac{1}{p^m} \longrightarrow \prod_{p \in \tilde{P}_f} \left(1 - \frac{n}{p^m} \right) > 0.$$

Fixing N large enough and letting $X \to \infty$, we get

$$\liminf_{X \to \infty} \frac{1}{X} \left\{ k \leqslant X \;\middle|\; \forall p \in \tilde{P}_f, \forall i \in [\![1, n]\!], \quad p^m \nmid kM_i + R_i \right\} \geqslant \mathbb{P}_N > 0$$

which concludes. $\square$

8.3 Main argument

Reduction to paths without backtracking.

Let us consider the following language (the "reduced Word Problem").

$$\mathcal{R} = \left\{ w \in S^* \;\middle|\; \bar{w} = e, \text{ no subword } xx^{-1} \text{ or } x^{-1}x \right\},$$

and $R(z) = \sum_{\ell \geqslant 0} r(\ell) \cdot z^\ell$ the associated growth series. Adapting the proof of the Bartholdi–Grigorchuk cogrowth formula [Bar99, Corollary 2.6], we get

$$\frac{R(z)}{1 - z^2} = \frac{\Gamma\left(\frac{z}{1+z^2}\right)}{1 + z^2}.$$

(As we only remove "bumps" xx^{-1} and $x^{-1}x$, we should take $d = 2$ in the formula.) It follows from Proposition 8.1.2(c) that $R(z)$ is D-finite if and only if $\Gamma_{\mathscr{vH},S}(z)$ is D-finite.

Counting paths with few t's. We decompose $\mathcal{R}$ into three disjoint sets:

$$\begin{aligned}
\mathcal{R}_1 &= \left\{ w \in \mathcal{R} \;\middle|\; \text{at most four } t, \text{ or six } t \text{ including two consecutive} \right\}, \\
\mathcal{R}_2 &= \left\{ w \in \mathcal{R} \;\middle|\; \text{exactly six } t \text{ and no subword } tt \right\}, \\
\mathcal{R}_3 &= \left\{ w \in \mathcal{R} \;\middle|\; \text{at least eight } t \right\}.
\end{aligned}$$

(1) Observe that $\mathcal{R}_1 = WP\big(F_2 \rtimes C_2, \{x, x^{-1}, 8 \cdot t\}\big) \cap \mathscr{C}$ where $\mathscr{C}$ is a rational language encoding the fact that our words are reduced and the condition on the t's. Indeed, if

$$x^{n_0}\, t\, x^{n_1}\, t\, x^{n_2}\, t\, x^{n_3}\, t\, x^{n_4} = x^{n_0} y^{n_1} x^{n_2} y^{n_3} x^{n_4}$$

is trivial in $\mathscr{vH}$, then it is also trivial in $F_2 \rtimes C_2$. Using the easy direction of Müller–Schupp's theorem 2.5.11, we conclude that $\mathcal{R}_1$ is unambiguously context-free.

(2) Paths of length $2\ell + 6$ in $\mathcal{R}_2$ come in two types and four orientations:

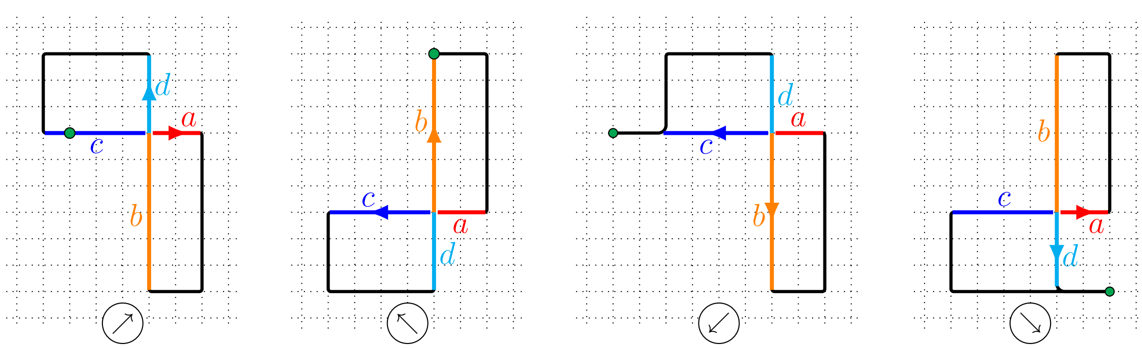


Figure 8.1: Two paths of each type, and all four orientations. For the first type, the perimeter should be 2ℓ. For the second type, the perimeter of the "main shape" (i.e., without the "tail") should be $2k < 2\ell$. The first picture corresponds to the word $w = x^5tx^{-6}tx^{-2}tx^9tx^{-4}tx^{-3}tx$.

Shapes of perimeter $2n$ are parametrized by solutions $(a,b,c,d) \in \mathbb{Z}^4_{>0}$ to the system

$$\begin{cases} a+b+c+d = n, \\ ab = cd. \end{cases}$$

We denote the set of solutions by S_n.

For paths of the first type, we have $2(a+c)+3$ starting points (anywhere along an horizontal segment). For the second type, we have exactly 6 starting points (the "tail" can be attached at any corner, its length $\ell - k$ is fixed). This leads to

$$r_2(2\ell+6) = 8^6 \cdot \left(\sum_{(a,b,c,d)\in S_\ell} 4 \cdot \big(2(a+c)+3\big) + \sum_{k<\ell} 4 \cdot 6 \cdot |S_k| \right)$$

(3) Finally $8^8 \mid r_3(\ell)$ as we can choose any of the 8 copies for each instance of t.

Counting solutions to a Diophantine equation. We compute $|S_n| \pmod 8$. A key observation is that, in addition to the four orientations used earlier, there is an extra symmetry. Specifically $D_8 \curvearrowright S_n$ generated by the involutions

$$\sigma(a,b,c,d) = (a,b,d,c) \quad \text{and} \quad \tau(a,b,c,d) = (c,d,a,b).$$

Using the Orbit-Stabilizer formula, we get

$$\begin{aligned} |S_n| = {} & \quad 1 \cdot \#\{(a,a,a,a)\} && + \; 4 \cdot \#\{(a,b,a,b) : a < b\} \\ & + \; 4 \cdot \#\{(a,b,c,c) : a < b\} && + \; 8 \cdot \#\{(a,b,c,d) : a < c < d < b\} \\ = {} & \quad \mathbb{1}_{\{4|n\}} && + \; 4 \cdot \mathbb{1}_{\{2|n\}} \cdot \left\lfloor \tfrac{n-1}{4} \right\rfloor \\ & + \; 4 \cdot \#\{(a,b,c,c) : a < b\} && + \; 8 \cdot \text{an integer} \end{aligned}$$

For each $n \in \mathbb{Z}_{>0}$, let's compute the number of solutions $0 < a < c < b$ to the system

$$\begin{cases} a+b+2c = n, \\ \quad ab \quad\;\; = c^2. \end{cases}$$

Let $d = \gcd(a,b)$. We can write $a = dX$ and $b = dY$ with $X < Y$ coprime integers. As $d^2XY = ab = c^2$, we conclude that both X and Y are perfect squares, more precisely $a = dx^2$, $b = dy^2$ and $c = dxy$. Now the first equation becomes

$$d(x+y)^2 = n.$$

Reciprocally, for each integer $z \geqslant 3$ such that $z^2 \mid n$, we have $\frac{1}{2}\varphi(z)$ choices for $x < y$ such that $x + y = z$ and $\gcd(x,y) = 1$, where φ is the Euler's totient function. Using Gauss formula $\sum_{d|m} \varphi(d) = m$, we conclude that

$$\#\{(a,b,c,c) \in S_n : a < b\} = \frac{1}{2} \sum_{\substack{z^2|n \\ z \geqslant 3}} \varphi(z) = \frac{1}{2} \sum_{\substack{z|m \\ z \geqslant 3}} \varphi(z) = \frac{m(n) - 1 - \mathbb{1}_{\{4|n\}}}{2},$$

where $m(n) = \prod_p p^{\lfloor v_p(n)/2 \rfloor}$ is the largest integer such that $m^2 \mid n$.

End game. Let us put everything together for $\ell = 2j+1$ odd:

$$
\begin{aligned}
r(4j+8) &\equiv r_1(4j+8) + 8^6 \cdot 4 \cdot \left(4 \cdot \frac{m(2j+1)-1}{2} + \sum_{k<2j+1} 6 \cdot \mathbb{1}_{\{4|k\}}\right) \pmod{8^6 \cdot 4 \cdot 8} \\
&\equiv r_1(2\ell+6) + 2^{21} \cdot \left(m(2j+1) - 1 + 3 \cdot \left\lfloor \frac{j}{2} \right\rfloor\right)
\end{aligned}
$$

hence

$$
\frac{1}{2^{22}}\left(r(4j+8) - r_1(4j+8) - 2^{21} \cdot 3 \cdot \left\lfloor \frac{j}{2} \right\rfloor\right) \equiv \frac{m(2j+1)-1}{2} \pmod{2}.
$$

Let

$$
S(z) := \sum_{j=0}^{\infty} \frac{1}{2^{22}}\left(r(4j+8) - r_1(4j+8) - 2^{21} \cdot 3 \cdot \left\lfloor \frac{j}{2} \right\rfloor\right) \cdot z^j.
$$

Observe that

$$
\begin{aligned}
\frac{m(2j+1)-1}{2} &\equiv 0 \pmod{2} \iff f(2j+1) \equiv 1 \pmod{4}, \\
\frac{m(2j+1)-1}{2} &\equiv 1 \pmod{2} \iff f(2j+1) \equiv -1 \pmod{4},
\end{aligned}
$$

where $f\colon \mathbb{Z}_{>0} \to \{\pm 1\}$ is the multiplicative function defined as

$$
f(n) = m\left(\frac{n}{2^{\nu_2(n)}}\right) \pmod{4},
$$

which satisfies the hypothesis of Theorem 8.2.2. We conclude that the subword complexity of the coefficients of $S(z)$ modulo 2 is $p(n) = 2^n$, hence $S(z)$ cannot be D-finite by Theorem 8.1.5. As the generating series of $r_1(4j+8)$ and $\lfloor \frac{j}{2} \rfloor$ are algebraic, it follows that $R(z)$ and $\Gamma_{\upsilon\mathscr{H},S}(z)$ are not D-finite (using Propositions 8.1.2 and 8.1.3). □

Remark 8.3.1. It is possible to bypass Section 8.2 and get the weaker conclusion that the Green series cannot be written as a diagonal of rational series.

We first repeat the argument of Section 8.3: if $\Gamma(z)$ is the diagonal of rational series, then $S(z)$ is too. Using [DL87, Theorem 5.2] and [Chr+80, Théorème 1], we get that the sequence $\big(m(n) \bmod 4\big)_{n\geqslant 1}$ is 2-automatic, and therefore its multiplicative cousin

$$
f(n) = m\left(\frac{n}{2^{\nu_2(n)}}\right) \pmod{4}
$$

(with values in $\{\pm 1\}$) is 2-automatic too. However, sequences that are both automatic and multiplicative are classified [KLM22], and f does not appear on the list.

Another tempting argument was to use asymptotic frequencies. Unfortunately, this reduces to a well-known open problem. Recall that squarefree numbers have asymptotic density $\frac{6}{\pi^2}$. It follows that naturals such that $m(n) = k$ have density $\frac{1}{k^2} \cdot \frac{6}{\pi^2}$, hence

$$\lim_{X\to\infty} \frac{\#\{n \leqslant X \mid m(n) \equiv 1 \bmod 4\}}{X} = \frac{6}{\pi^2} \sum_{k\geqslant 0} \frac{1}{(4k+1)^2} = \frac{3}{\pi^2}\left(\frac{\pi^2}{8} + G\right) = \frac{3}{8} + \frac{3G}{\pi^2}$$

where $\frac{\pi^2}{8} = \sum_{\ell\geqslant 0} \frac{1}{(2\ell+1)^2}$ and $G = L(2, \chi_{-4}) = \sum_{\ell\geqslant 0} \frac{(-1)^\ell}{(2\ell+1)^2}$ is the Catalan constant. It is widely believed that π^2 and G are $\mathbb{Q}$-linearly independent ([CDT24] for recent progress), hence the sequence $(m(n) \bmod 4)_{n\geqslant 1}$ cannot be 2-automatic [Cob72, Theorem 6].

8.4 Further questions and remarks

The most tempting question is to extend the result to the discrete Heisenberg group.

Problem 8.A. Consider $H_3(\mathbb{Z}) = \langle x, y \mid [x,[x,y]] = [y,[x,y]] = e\rangle$. Prove that the Green series $\Gamma_{H_3(\mathbb{Z}),S}(z)$ is not D-finite for some generating (multi)set S.

Part of the motivation is that any proof looking at the coefficients (mod p^m) would then pass to any group containing $H_3(\mathbb{Z})$, for instance any virtually nilpotent group which is not virtually abelian (for a well-chosen generating multiset).

However, the small trick of adding multiple copies of a generator doesn't seem to work. For instance, if we take $S = \{x, y, y, z\}^{\pm}$ as our generating set, and look modulo 2^{2K}, we have to count closed paths that stay within K-neighborhood of the abelian subgroup $\langle x, z\rangle$, hence the associated series should be the diagonal of rational series.

This means that we need to find extra symmetries on the entire set of closed paths (and not just $\mathcal{R}_2$ in our argument), for instance find some 2-group action. Hopefully, once filtering by increasing orbit size, the first terms would be provably good (i.e. D-finite), and we can find a provably bad term before getting stuck on unprovably bad ones.

Another interesting group is the lattice $G = \langle a, b, c\rangle$ in $H_3(\mathbb{R})$, with a, b, c given by

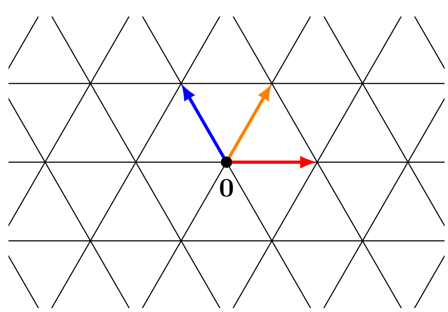


Figure 8.2: A triangular lattice and the generators a, b and c in red, orange and blue respectively.

The point is that $D_{12} \curvearrowright G$ by isometries, giving a few symmetries to start with. This group is isomorphic to $\langle x^2,\ xyx,\ y\rangle \leqslant H_3(\mathbb{Z})$.

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