Wall-crossings for moduli spaces of parabolic bundles on curves

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Résumé

La formule de Verlinde est une expression pour la caractéristique d'Euler des fibrés linéaires sur les espaces de modules des fibrés stables sur une courbe. Cette formule motivée par la physique quantique est un de plus bean résultat de la géométrie énumérative. Au fil des années, de nombreuses preuves différentes de cette formule ont été proposées.

Dans le chapitre 2 de cette thèse, nous donnons une nouvelle preuve de la variante parabolique plus difficile de cette formule basée sur une comparaison des croisements de murs en théorie géométrique des invariants et de certains calculs de résidus itérés. En cours de route, nous développons une variante tautologique des correspondances de Hecke, calculons les polynômes de Hilbert des espaces de modules et présentons une nouvelle approche transparente du problème de ρ-shift de la théorie.

Dans le chapitre 3, nous montrons que les méthodes de résidu/croisement de mur du chapitre 2 peuvent être utilisées pour décrire les applications poussées en avant dans la K-théorie des espaces de modules et présenter de nouvelles formules explicites pour la caractéristique d'Euler d'une classe plus large de fibrés vectoriels sur l'espace des modules des fibrés paraboliques stables.

Notre travail a été motivé par les résultats de Teleman et Woodward sur l'indice des classes K-théorieque des champ de modules.

Summary

The Verlinde formula, an expression for the Euler characteristic of line bundles on the moduli spaces of stable bundles on a curve, is a strikingly beautiful statement in enumerative geometry motivated by quantum physics. It has attracted a lot of attention over the years, and has a number of different proofs.

In Chapter 2 of this thesis, we give a new proof of the more difficult parabolic variant of this formula based on a comparison of wall-crossings in Geometric Invariant Theory and certain iterated residues calculus. On the way, we develop a tautological variant of Hecke correspondences, calculate the Hilbert polynomials of the moduli spaces, and present a new, transparent approach to the ρ-shift problem of the theory.

In Chapter 3 we show that the residue/wall-crossing methods of Chapter 2 may be successfully employed to describe the pushforward maps in the K-theory of moduli spaces and present new, explicit formulas for the Euler characteristic of a wider class of vector bundles on the moduli space of stable parabolic bundles.

Our work was motivated by the results of Teleman and Woodward on the index of K-theory classes of moduli stacks.

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Contents

Chapter 1

Introduction

1.1. The parabolic Verlinde formula

The Verlinde formula is a strikingly beautiful statement in Enumerative Geometry motivated by quantum physics [29]. Our focus in the first part of this thesis will be the more difficult, parabolic variant of this formula, which we briefly describe below.

Let C be a smooth, complex projective curve of genus $g \geqslant 2$, and fix a point $p \in C$. Denote by Δ the set of vectors $c = (c_1 > c_2 > ... > c_r) \in \mathbb{R}^r$ such that $\sum c_i = 0$ and $c_1 - c_r < 1$. We will call a vector $c \in \Delta$ *regular* if no nontrivial subset of its coordinates sums to an integer. For such a $c \in \Delta$, there exists a smooth projective moduli space $P_0(c)$ of dimension $(r^2-1)(g-1)+\binom{r}{2}$ ([19, 13, 4]), whose points are in one-to-one correspondence with equivalence classes of pairs (W, F_*), where W is a vector bundle of rank r on C with trivial determinant, F_* is a full flag in the fiber W_p, and the pair satisfies a certain parabolic stability condition depending on a regular $c \in \Delta$ (cf. §2.1.1). This condition roughly states that for a proper subbundle $W' \subset W$, the degree $\deg(W')$ is strictly smaller than the sum of a subset of the coordinates of c depending on the position of W'_p with respect to F_*.

There is a natural way to associate to a positive integer k and an integer vector $\lambda \in \mathbb{Z}^r$ satisfying $\lambda_1 + \cdots + \lambda_r = 0$ a line bundle $\mathcal{L}(k;\lambda)$ on $P_0(c)$, in such a way that if $c = \lambda/k$, then $\mathcal{L}(k;\lambda)$ is ample. The *parabolic Verlinde formula* is the following expression for the Euler characteristic of the ample line bundle $\mathcal{L}(k;\lambda)$: assume $c = \lambda/k$ is regular; then

$$\chi(P_0(c), \mathcal{L}(k;\lambda)) = N_{r,k} \cdot \sum \frac{(-i)^{\binom{r}{2}} \exp(2\pi i \widehat{\lambda} \cdot x)}{\prod_{i<j} \left(2 \sin \pi(x_i - x_j)\right)^{2g-1}}, \tag{1.1}$$

where $N_{r,k} = r(r(k+r)^{r-1})^{g-1}$, $\widehat{\lambda} = \lambda + \frac{1}{2}(r-1, r-3, \ldots, 1-r)$, and the sum is taken over the finite set of those points in the interior of the parallelopiped

$$\{x = (x_1, x_2, \ldots, x_r = 0) |\ 0 < x_i - x_{i+1} < 1 \text{ for } i = 1, \ldots, r-1\}$$

which satisfy the conditions $(k+r)x \in \mathbb{Z}^r$ and $x_i - x_j \notin \mathbb{Z}$ for $1 \leqslant i < j < r$.

Remark 1.1.1. This finite set is a set of lattice points in the interior of $(r-1)!$ identical simplices. (These are the orange-colored points in the rhombus on Figure 1.1). By symmetrizing with respect to the group of permutations of the r coordinates, one obtains the same function on each of these simplices. Using the Weyl character formula, this allows one to rewrite (1.1) in a more familiar form as

$$\chi(P_0(c), \mathcal{L}(k;\lambda)) = (r(k+r)^{r-1})^{g-1} \cdot \sum \frac{\chi_\lambda(x)}{\prod_{i<j}\left(2\sin\pi(x_i - x_j)\right)^{2g-2}}, \tag{1.2}$$

where χ_λ is the character of the irreducible $SU(r)$-representation of highest weight λ, and the sum is now taken over the lattice points of the form $(k+r)x \in \mathbb{Z}^r$ in the interior of a single simplex $\{x = (1 > x_1 > x_2 > \cdots > x_{r-1} > x_r = 0)\}$.

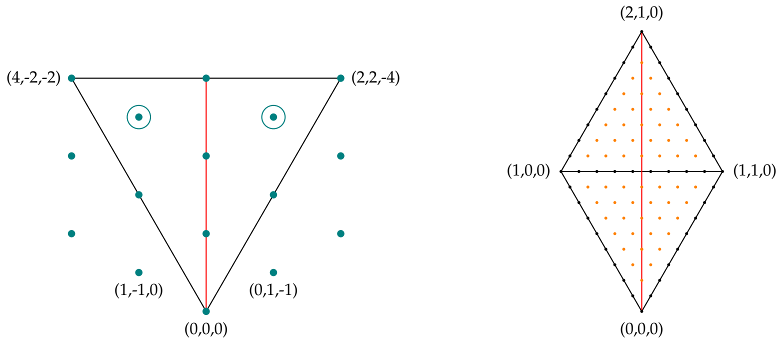

Figure 1.1 – The set of λs (left), and the finite set from (1.1) (right) for $k = 6$, $r = 3$.

Remark 1.1.2. Equality (1.1) remains valid in greater generality, for certain cases when λ/k is non-regular. This slightly more technical statement will be given in Theorems 2.3.7 and 3.2.3.

Equality (1.1), the parabolic Verlinde formula, has attracted a lot of attention over the years, and there is a number of different proofs (cf. e.g. [1, 12, 24]). In Chapter 2 of this thesis, we give a novel proof of this result, which stands out with its technical simplicity. Below, we give a quick sketch of the strategy of our proof.

Strategy of the proof

Our proof is based on three ideas. We start with the study of the right-hand side of equation (1.1). As observed in [20], this finite sum can be written as a piecewise polynomial function in (k, λ). We will briefly explain the idea in the simplest case $r = 2$.

We fix k and $(\lambda, -\lambda) \in \mathbb{Z}^2$; in rank-2 case, the sum on the right-hand side of (1.1) may be written in the following simplified form

$$2(2k+4)^{g-1} \sum_{j=1}^{k-1} \frac{-i \cdot \exp(2\pi i(\lambda + \frac{1}{2})j/(k+2))}{(2\sin(\pi j/(k+2)))^{2g-1}}.$$

Note that this sum is periodic in $\lambda + \frac{1}{2}$ modulo $k+2$. We introduce the notation $\{q\}$ for the fractional part of $q \in \mathbb{R}$, and using the residue theorem, we evaluate the sum as

$$(-1)^{g-1}(2k+4)^g \operatorname*{Res}_{z=1} \frac{z^{\{(\lambda+\frac{1}{2})/(k+2)\}(k+2)}}{(z^{1/2} - z^{-1/2})^{2g-1}(1 - z^{k+2})} \frac{dz}{z} \overset{z=e^u}{=}$$

$$(-1)^{g-1}(2k+4)^g \operatorname*{Res}_{u=0} \frac{e^{\{(\lambda+\frac{1}{2})/(k+2)\}(k+2)u}}{(e^{u/2} - e^{-u/2})^{2g-1}(1 - e^{u(k+2)})} du. \tag{1.3}$$

A simple calculation shows that we obtained a periodic, piecewise polynomial function in the pair (λ, k), which is polynomial on the cones bounded by the lines $(\lambda + \frac{1}{2})/(k+2) \in \mathbb{Z}$. When $r > 2$, one can still present the sums on the right-hand side of (1.1) as *iterated residues* of certain rational differential forms (cf. Theorem 2.3.7). In this case the combinatorics of the residue formulas is considerably more complicated, and is best treated using the notion of diagonal bases and of hyperplane arrangements (cf. §2.2).

We observe that the simplex Δ (cf. page 1) of *parabolic weights* c parametrizing stability conditions contains a finite number of hyperplanes (*walls*) on whose complement (the set of regular elements in Δ) the stability condition is locally constant. This induces a *chamber* structure on Δ, such that the left-hand side (via the Grothendieck-Riemann-Roch theorem) and the right-hand side (written as residue formula) of (1.1) are manifestly polynomial in the variables $(k; \lambda)$ on each chamber. We introduce the notation $l_c(k; \lambda)$ and $r_c(k; \lambda)$ for these polynomials, where c is any element of the corresponding chamber.

In §2.4 we derive a simple formula (cf. Theorem 2.4.7) for the *wall-crossing* difference in geometric invariant theory; using this formula, in §2.5 we show that the differences between the two polynomials associated to neighbouring chambers (specified by c_+ and c_-) for the left-hand side and the right-hand side coincide:

$$l_{c_+} - l_{c_-} = r_{c_+} - r_{c_-}. \tag{1.4}$$

The next step of our proof relies on the *Hecke correspondence* between moduli spaces of bundles of different degrees, which was introduced in [17]. In §2.6 of this thesis we describe a "tautological" variant of this construction, which identifies the same space with moduli spaces of parabolic bundles of different degrees and weights. We choose a pair of chambers adjacent to two special vertices of the simplex Δ, and consider the corresponding pairs of polynomials

$$l_{c_>}(k; \lambda),\ l_{c_<}(k; \lambda) \quad \text{and} \quad r_{c_>}(k; \lambda),\ r_{c_<}(k; \lambda) \tag{1.5}$$

from the left-hand side and the right-hand side of (1.1), respectively. Using the tautological Hecke correspondence and Serre duality, in §2.7 we derive certain symmetry properties of $l_{c_>}(k; \lambda)$ and $l_{c_<}(k; \lambda)$, and then we prove that $r_{c_>}(k; \lambda)$ and $r_{c_<}(k; \lambda)$ satisfy the same symmetries.

Finally, in §2.7.4 we show that a set of polynomials parametrized by the chambers in Δ is uniquely determined by the wall-crossing terms (1.4) and our symmetry properties for the polynomials (1.5), and thus we obtain that $l_c(k; \lambda)$ and $r_c(k; \lambda)$ coincide.

Historical remarks

There is a long list of proofs of the Verlinde formula. Below, we give reference to the works that are closest in spirit to what we do.

The proofs of the Verlinde formula fall in two categories: proofs of the fusion rules and proofs that find some interpretation of the "Fourier transformed" discrete sum on the right hand side of (1.1); as explained above, the present work belongs to this second group. Another line of division concerns the model, which one uses for the moduli spaces: via the Narasimhan-Seshadri correspondence, the moduli spaces of parabolic vector bundles may equally be presented as symplectic manifolds of certain types of flat connections on punctured Riemann surfaces, and this opens the way of using the methods of symplectic geometry.

A paper closely related to our work is that of Jeffrey and Kirwan [12], which approaches the problem from a symplectic/cohomological point of view (cf. [11]), and has a somewhat different angle form ours. This paper also uses the residue calculus introduced in [20, 21], but not quite as consistently as our work, and the parabolic case was not resolved from this point of view (cf. [10]).

The idea of proving the Verlinde formula via wall-crossings appeared in the seminal paper of Michael Thaddeus [25]. He used a geometric approach and managed to prove the Verlinde formula in rank 2 by crossing walls in the moduli of stable pairs. The *master space construction*, which plays a central role in our paper, first appeared in his works as well [26]. In a sense, the present work may be thought of as the completion of his program.

1.2. Euler characteristics of tautological bundles

In the second part of this thesis, we apply the wall-crossing/residue technique of Chapter 2 to obtain formulas for the Euler characteristics of a wider class of vector bundles on the moduli space $P_0(c)$: we associate to a dominant weight ν of GL_r a tautological vector bundle U_ν on $P_0(c)\times C$ and calculate

$$\chi(P_0(c), \mathcal{L}(k;\lambda)\otimes\pi_!(U_\nu\otimes\mathcal{K}^{\frac{1}{2}})), \tag{1.6}$$

where $\pi : P_0(c)\times C\to P_0(c)$ is the projection, and $\mathcal{K}$ is the canonical bundle on C (cf. Theorem 3.2.3 for the result).

As in the case of Euler characteristics of line bundles (cf. (1.3)), the formulas for (1.6) we obtain have the form of iterated residues of rational differential forms. For example, in the simplest rank $r=2$ case, the answer may be described as follows.

We fix a dominant weight $\nu=(\nu_1,\nu_2)\in\mathbb{Z}^2$ of GL_2; denote by ρ_ν the irreducible representation of GL_2 with highest weight ν, and by $\bar\rho_\nu$ its restriction to $SU_2\subset GL_2$. Let $U_\nu\to P_0(c)\times C$ be the bundle associated to the representation ρ_ν. We introduce the notation

$$\phi(x)=\frac{\sinh((\nu_1-\nu_2+1)x/2)}{\sinh(x/2)}$$

for the character function of $\bar\rho_\nu$ on the Lie algebra of the maximal torus of SU_2. Let

$$\dot\phi(x)=2\frac{d}{dx}\phi(x)\quad\text{and}\quad\ddot\phi(x)=2\frac{d}{dx}\dot\phi(x),$$

then

$$\chi(P_0(c), \mathcal{L}(k;\lambda)\otimes\pi_!(U_\nu\otimes\mathcal{K}^{\frac{1}{2}}))=$$
$$(-(2k+4))^g\operatorname*{Res}_{u=0}\frac{\exp(u(\lambda+\frac{1}{2}+\frac{\nu_1+\nu_2}{2}))}{(2\sinh\left(\frac{u}{2}\right))^{2g-1}(1-e^{u(k+2)})}\left(\frac{g\,\ddot\phi(u)}{2k+4}+\frac{e^{(k+2)u}\dot\phi(u)}{(1-e^{u(k+2)})}\right)du. \tag{1.7}$$

In Chapter 3 of this thesis we will follow the ideas described in §1.1 above, where the case of the line bundles on the moduli spaces was treated. Let us highlight some of the new phenomena that we encountend in this higher rank case.

The symmetry of Euler characteristics (1.6) on the moduli spaces $P_0(c_>)$ and $P_0(c_<)$ (cf. (1.5)) is only true after an affine transformation; in fact, they need to be shifted by a linear combination of Euler characteristics of line bundles, which then can be calculated using the results of Chapter 2 (cf. Propositions 3.4.2, 3.4.5 and 3.4.6). The appearance of Hessians (higher rank variants of the function $\ddot{\varphi}(u)$ in the formula (1.7) above) in our framework in the formulas for Euler characteristics is remarkably simply explained by the relations in the cohomology ring of the curve (cf. page 53). The directional derivatives (higher rank variants of the function $\dot{\varphi}(u)$ above), on the other hand, appear from a comparison of the Chern characters of the corresponding vector bundles under the Hecke isomorphism (cf. Proposition 3.3.9).

Remarks

The idea of the formulas for push-forwards in the cohomological setting, in particular, the Hessian, first appeared in the seminal paper of Witten [30]. Mathematically sound approaches in this cohomological/symplectic setting were employed by Jeffrey and Kirwan [12] and Meinrenken [14]. In particular, the wall-crossing ideas, which play a major role in our work already appeared in [12].

The results of Chapter 3 of this thesis own a lot to the paper of Teleman and Woodward [24], where a similar formula is derived for Euler characteristics of vector bundles on stacks. In the present thesis, we demonstrate, in particular, that the sophisticated tools employed in [24], at least in this instance, may be replaced by a simple combinatorial device. There are also subtle differences in the final formulas, which are manifest, in particular, in the appearance of certain determinantal factors in our formalism.

The formulas we find, even though they are similar to the results of [12] and [24], are new, and, in fact, are the first explicit formulas for these quantities.

Chapter 2

The parabolic Verlinde formula

This chapter is based on the work [22] and gives a new proof for the parabolic Verlinde formula in all ranks via a comparison of wall-crossings in Geometric Invariant Theory and certain iterated residue calculus.

2.1. Parabolic bundles

In this section, we briefly review the definition of parabolic bundles, recall the basic facts about their moduli spaces and describe the chamber structure on the space of the relevant parameters, known as *parabolic weights*.

2.1.1. Definitions

Let C be a smooth complex projective curve of genus $g \geqslant 2$, and fix a point $p \in C$.

- A *parabolic bundle* on C is a vector bundle W of rank r with a full flag F_* in the fiber over p:

$$W_p = F_r \supsetneq \dots \supsetneq F_1 \supsetneq F_0 = 0$$

  and *parabolic weights* $c = (c_1, \dots, c_r)$ assigned to $F_r, F_{r-1}, \dots, F_1$, satisfying the conditions

$$c_1 > c_2 > \dots > c_r \text{ and } c_1 - c_r < 1.$$

- The *parabolic degree*[1] and the *parabolic slope* of W are defined as

$$\mathit{pardeg}(W) = \deg(W) - \sum_{i=1}^{r} c_i; \quad \mathit{parslope}(W) = \frac{\mathit{pardeg}(W)}{\operatorname{rank}(W)}.$$

- A *morphism* $f : W \to W'$ of parabolic bundles is a morphism of vector bundles satisfying $f_p(F_i) \subset F'_{j-1}$ if $c_{r-i+1} < c'_{r-j+1}$. In particular, an *endomorphism* of a parabolic bundle W is a vector bundle endomorphism preserving the flag F_*.

[1]For technical reasons, we have chosen a sign convention opposite to that in the majority of treatments in the literature.

- Denote by $\mathrm{ParHom}(W, W')$ the sheaf of parabolic morphisms from W to W'. Then there is a short exact sequence of sheaves

$$0 \to \mathrm{ParHom}(W, W') \to \mathrm{Hom}(W, W') \to T_p \to 0, \tag{2.1}$$

where T_p is a torsion sheaf supported at p. The rank of T_p is the number of pairs (i, j), s.t. $c_i < c'_j$ (cf. [5]).

If $W' \subset W$ is a subbundle of W, then both W' and the quotient W/W' inherit a parabolic structure from W in a natural way (cf. [13], definition 1.7).

- A parabolic bundle W is *stable of weight* c, if any proper subbundle $W' \subset W$ satisfies *parslope*$(W') <$ *parslope*(W); and W is *semistable of weight* c, if the inequality is not strict.

Remark 2.1.1. Note that the parabolic stability condition depends on the parabolic weights only up to adding the same constant to all weights c_i.

2.1.2. *Construction of the moduli spaces*

We start with a quick review of the construction of Mehta and Seshadri [13] of the moduli space of stable parabolic bundles. It follows from Remark 2.1.1 that, without loss of generality, we can assume that the parabolic weights of a rank-r degree-d bundle belong to the simplex

$$\Delta_d = \left\{ (c_1, c_2, ..., c_r) \,|\, c_1 > c_2 > ... > c_r,\ c_1 - c_r < 1,\ \sum_i c_i = d \right\}.$$

Definition 2.1.2. We will call a vector $c = (c_1, \dots, c_r) \in \mathbb{R}^r$ such that $\sum_i c_i \in \mathbb{Z}$ *regular* if for any nontrivial subset $\Psi \subset \{1, 2, \dots, r\}$, we have $\sum_{i \in \Psi} c_i \notin \mathbb{Z}$.

Now choose an integer $d \gg 0$ such that $H^1(W) = 0$ and W is generated by global sections for any rank-r degree-d semistable parabolic bundle W of parabolic degree 0. Put $N = r(1 - g) + d$ and consider the

- Groethendieck quot scheme $\mathrm{Quot}(N, r)$ ([8]) parametrizing quotients $\mathcal{O}^N \twoheadrightarrow W$, where W is a coherent sheaf of degree d and rank r.

- This space is endowed with a universal bundle UQ, and a generically free action of the group $G = PSL(N)$, which does not, however, lift to UQ.

- Let LFQuot $\subset \mathrm{Quot}(N, r)$ be the open subscheme consisting of locally free quotients W, such that the induced map $H^0(\mathcal{O}^N) \to H^0(W)$ is an isomorphism.

- Denote by XQ the total space of the flag bundle $\mathrm{Flag}(UQ_p)$ on LFQuot $\times\, p$. This space is endowed with the flag of vector bundles $Fl_1 \subset \cdots \subset Fl_{r-1} \subset Fl_r = UQ_p$.

- Let $k \in \mathbb{Z}$ and $(\lambda_1, ..., \lambda_r) \in \mathbb{Z}^r$, such that $\sum_{i=1}^r \lambda_i = kd$, and consider the line bundle

$$L(k; \lambda) = \det(UQ_p)^{k(1-g)} \otimes \det(\pi_* UQ)^{-k} \otimes (Fl_r/Fl_{r-1})^{\lambda_1} \otimes ... \otimes (Fl_1)^{\lambda_r}$$

on XQ, which does carry a G-linearization (lift of the G-action from XQ).

- Finally, assume $c \in \Delta_d$ is regular (cf. Definition 2.1.2 above) and define $\widetilde{P}_d(c)$, the moduli space of stable parabolic weight-c vector bundles on C as the GIT quotient $XQ /\!\!/^c G$ of XQ with respect to any linearization $L(k;\lambda)$, such that $\lambda/k = c$.

Theorem 2.1.3 ([19]). *Assume that $c \in \Delta_d$ is a regular weight vector. Then the moduli space $\widetilde{P}_d(c)$ is a smooth projective variety of dimension $r^2(g-1) + \binom{r}{2} + 1$, whose points are in one-to-one correspondence with the set of isomorphism classes of stable parabolic bundles of weight c (cf. §2.1.1).*

Remark 2.1.4. Via the determinant map, the moduli space $\widetilde{P}_d(c)$ fibers over the Jacobian of degree-d line bundles with isomorphic fibers, and in this thesis, we will focus on the moduli space

$$P_d(c) = \{W \in \widetilde{P}_d(c) |\ \det W \simeq \mathcal{O}(dp)\},$$

which is smooth, projective and has dimension $(r^2-1)(g-1) + \binom{r}{2}$.

Remark 2.1.5. Note that tensoring with the line bundle $\mathcal{O}(mp)$ induces an isomorphism: $\otimes\mathcal{O}(mp) : P_d(c) \to P_{d+rm}(c)$, so the moduli spaces $P_d(c)$, essentially, depend only on d modulo r.

2.1.3. *The Picard group of $P_d(c)$*

For a regular $c \in \Delta_d$, there exist universal bundles $\mathcal{U}$ over $P_d(c) \times C$ endowed with a flag $\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{r-1} \subset \mathcal{F}_r = \mathcal{U}_p$, and satisfying the obvious tautological properties. In general, such universal bundles $\mathcal{U}$, and hence the flag line bundles $\mathcal{F}_{i+1}/\mathcal{F}_i$ are unique only up to tensoring by the pull-back of a line bundle from $P_d(c)$. Nevertheless, we have the following statement.

Lemma 2.1.6. *For $k \in \mathbb{Z}$ and $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathbb{Z}^r$, such that $\sum_{i=1}^r \lambda_i = kd$, the line bundle*

$$\mathcal{L}_d(k;\lambda) = \det(\mathcal{U}_p)^{k(1-g)} \otimes \det(\pi_*\mathcal{U})^{-k} \otimes (\mathcal{F}_r/\mathcal{F}_{r-1})^{\lambda_1} \otimes \dots \otimes (\mathcal{F}_1)^{\lambda_r} \tag{2.2}$$

on $P_d(c)$ is independent of the choice of the universal bundle $\mathcal{U}$.

Proof. Note that tensoring $\mathcal{U}$ with a pullback $\pi^*\mathcal{L}$ of a line bundle $\mathcal{L}$ on $P_0(c)$ changes $\det(\mathcal{U}_p)$ by $\mathcal{L}^r$ and $\det(\pi_*\mathcal{U})$ by $\mathcal{L}^{d-r(g-1)}$. □

Remark 2.1.7. The line bundle $L(k;\lambda)$ defined in §2.1.2 descends to the line bundle $\mathcal{L}_d(k;\lambda)$ on the GIT quotient $P_d(c)$.

Notation: We will say that $\mathcal{U}$ is *normalized* if the line subbundle $\mathcal{F}_1 \subset \mathcal{U}_p$ is trivial. The parameter k is often called the *level*.

Let $\omega \in H^2(C)$ be the fundamental class of our curve C, and $e_1, \dots, e_{2g}$ a basis of $H^1(C)$, such that $e_i e_{i+g} = \omega$ for $1 \leqslant i \leqslant g$, and all other intersection numbers $e_i e_j$ equal 0. For a class $\delta \in H^*(P \times C)$ of a product, we introduce the following notation for its Künneth components (cf. [30]):

$$\delta = \delta_{(0)} \otimes 1 + \sum_i \delta_{(e_i)} \otimes e_i + \delta_{(2)} \otimes \omega \in \bigoplus_{i=0}^{2} H^{*-i}(P) \otimes H^i(C). \tag{2.3}$$

We will need the following formula.

Lemma 2.1.8. *The equality* $2c_1(\mathcal{L}_d(r; d, ..., d)) = c_2(\mathrm{End}_0(U_d))_{(2)}$ *holds, where* End_0 *stands for traceless endomorphisms.*

Proof. Taking the first Chern class on both sides of (2.2), we obtain

$$c_1(\mathcal{L}_d(r; d, ..., d)) = r(1-g)c_1(U_d)_{(0)} - rc_1(\pi_*(U_d)) + dc_1(U_d)_{(0)},$$

where we can evaluate the middle term using the Groethendieck-Riemann-Roch theorem, and $c_1(U_d)_{(2)} = d$:

$$\begin{aligned} c_1(\pi_*(U_d)) = ch_1(\pi_!(U_d)) &= \pi_* ch_2(U_d) - (g-1)c_1(U_d)_{(0)} \\ &= c_1(U_d)_{(0)} d - c_2(U_d)_{(2)} - (g-1)c_1(U_d)_{(0)}. \end{aligned}$$

This leads to the formula

$$c_1(\mathcal{L}_d(r; d, ..., d)) = -d(r-1)c_1(U_d)_{(0)} + rc_2(U_d)_{(2)},$$

which is easily seen to equal $\frac{1}{2}c_2(\mathrm{End}_0(U_d))_{(2)}$. □

2.1.4. *Walls and chambers*

The central question we address in this thesis is how the moduli space of stable parabolic bundles depends on the choice of parabolic weights. Let W be a vector bundle of degree d with a fixed full flag F_* of the fiber W_p, and let us try to determine the structure of the set of parabolic weights $c \in \Delta_d$ for which W is stable. Clearly, for this we need to study the set of parabolic weights $c = (c_1, c_2, \ldots c_r)$ for which one can find a proper subbundle $W' \subset W$ such that

$$parslope(W') = parslope(W) = 0. \tag{2.4}$$

A subbundle $W' \subset W$ determines a short exact sequence of parabolic bundles

$$0 \to W' \to W \to W'' \to 0$$

and the position of W'_p with respect to F_* gives rise to a nontrivial partition of the set $\{1, 2, \ldots, r\}$ into two sets, Π' and Π'' (cf. [13], definition 1.7); the parabolic weights of W' and W'' are then $c' = (c_i)_{i\in\Pi'}$ and $c'' = (c_i)_{i\in\Pi''}$, correspondingly. The slope condition (2.4) translates into a pair of equivalent equalities:

$$d' = \sum_{i\in\Pi'} c_i, \quad d'' = \sum_{i\in\Pi''} c_i, \tag{2.5}$$

where d', $d'' = d - d'$ are the degrees of W' and W'', respectively. This means that the critical values of $c \in \Delta_d$ for which (2.4) is possible lie on the union of affine hyperplanes (or *walls*) defined by the equations

$$\sum_{i\in\Pi'} c_i = l, \text{ where } l \in \mathbb{Z}, \text{ and } \Pi' \subset \{1, 2, \ldots, r\} \text{ nontrivial.}$$

As only finitely many of these walls intersect the simplex Δ_d, their complement is a finite union of open polyhedral *chambers*. It is easy to verify that as we vary c inside one of these chambers, the stability condition, and thus the moduli space $P_d(c)$ does not change.

Example 1. Consider the case of rank-3 degree-0 stable parabolic bundles with parabolic weights $c = (c_1, c_2, c_3) \in \Delta_0$. The set Δ_0 is an open triangle with vertices $(0,0,0), (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ and $(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$ (cf. Figure 2.1), and there exist only two essentially different stability conditions. The wall separating the two regimes is given by the condition $c_2 = 0$. We write $P_0(>)$ for the moduli space $P_0(c_1, c_2, c_3)$ with $c_2 > 0$, and $P_0(<)$ for $P_0(c_1, c_2, c_3)$ with $c_2 < 0$.

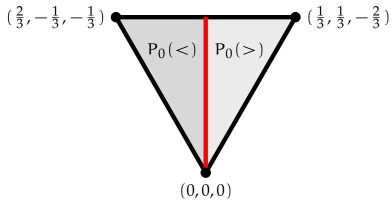

Figure 2.1 – The space of admissible parabolic weights for rank $r = 3$.

2.2. Wall-crossing in the Verlinde formula

A key component of our approach is the notion of *diagonal basis* and the associated generalized Bernoulli polynomials introduced for general hyperplane arrangements in [20]. Using this formalism, we will be able to formulate our main result, Theorem 2.3.8.

2.2.1. *Notation*

We begin by setting up some extra notation for the space of parabolic weights introduced in §2.1.1.

- Let $V = \mathbb{R}^r/\mathbb{R}(1,1,\dots,1)$ be the $r-1$-dimensional vector space, obtained as the quotient of $\mathbb{R}^r$. The dual space V^* is then naturally represented as

  $$V^* = \{a = (a_1, \dots, a_r) \in \mathbb{R}^r |\ a_1 + \dots + a_r = 0\}.$$

  Let $x_1, x_2, \dots, x_r$ be the coordinates on $\mathbb{R}^r$; given $a \in V^*$, we will write $\langle a, x\rangle$ for the linear function $\sum_i a_i x_i$ on V. We will sometimes identify this linear function with the vector a itself.

- The vector space V^* is endowed with a lattice Λ of full rank:

  $$\Lambda = \{\lambda = (\lambda_1, \dots, \lambda_r) \in \mathbb{Z}^r |\ \lambda_1 + \dots + \lambda_r = 0\}.$$

  In particular, for $1 \leqslant i \neq j \leqslant r$, we can define the element $\alpha^{ij} = x_i - x_j$ in Λ.

- Our arrangement is the set of hyperplanes $\{x_i = x_j\} \subset V$, $1 \leqslant i < j \leqslant r$. It will be convenient for us to think about this set as the set of roots of the A_{r-1} root system with the opposite roots identified:

  $$\Phi = \{\ \pm\alpha^{ij} | 1 \leqslant i < j \leqslant r\}.$$

  Note that V^* carries a natural action of the permutation group Σ_r, permuting the coordinates x_j, $j = 1, \dots, r$, and this action restricts to an action on Φ as well.

- The basic object of the theory is an *ordered* linear basis **B** of V^* consisting of the elements of Φ. Let us denote the set of these objects by $\mathcal{B}$:

$$\mathcal{B} = \left\{ \mathbf{B} = \left(\beta^{[1]}, \dots, \beta^{[r-1]} \right) \in \Phi^{r-1} | \ \mathbf{B} - \text{basis of } V^* \right\}$$

- For $\mathbf{B} \in \mathcal{B}$, we will write $\mathrm{Fl}(\mathbf{B})$ for the full flag

$$\left[V^* = \langle \beta^{[1]}, \beta^{[2]}, \dots, \beta^{[r-1]} \rangle_{\text{lin}}, \dots, \langle \beta^{[r-1]}, \beta^{[r-2]} \rangle_{\text{lin}}, \langle \beta^{[r-1]} \rangle_{\text{lin}} \right],$$

where $\langle \cdot \rangle_{\text{lin}}$ stands for linear span.

2.2.2. *Diagonal bases*

Definition 2.2.1. • For $\tau \in \Sigma_{r-1}$ and $\mathbf{B} \in \mathcal{B}$, we will write $\mathbf{B} \circlearrowleft \tau$ for the permuted sequence $(\beta^{[\tau(1)]}, \beta^{[\tau(2)]}, \dots, \beta^{[\tau(r-1)]})$.

- For two elements $\mathbf{B}, \mathbf{C} \in \mathcal{B}$ we will write $\mathbf{B} \dashv \mathbf{C}$ if for any $\tau \in \Sigma_{r-1}$, we have $\mathrm{Fl}(\mathbf{B} \circlearrowleft \tau) \neq \mathrm{Fl}(\mathbf{C})$.

- A subset $\mathcal{D} \subset \mathcal{B}$ of $(r-1)!$ elements is called a *diagonal basis* if for any two different elements $\mathbf{B}, \mathbf{C} \in \mathcal{D}$, we have $\mathbf{B} \dashv \mathbf{C}$.

Remark 2.2.2. This definition is motivated by a construction [20], which associates to each diagonal basis $\mathcal{D}$ a pair of dual bases of the middle homology and the cohomology of the complexified hyperplane arrangement on $V \otimes_{\mathbb{R}} \mathbb{C}$ defined by Φ. The dimension of these (co)homology spaces is $(r-1)!$.

2.2.3. *Combinatorial interpretation*

This notion has the following purely combinatorial form.

- We can think of Φ as the edges of the complete graph on r vertices.

- Then the set $\mathcal{B}$ may be thought of as the set of spanning trees of this graph with edges enumerated from 1 to $r-1$. We will introduce the notation

$$\mathbf{B} \mapsto \mathrm{Tree}(\mathbf{B})$$

for this ordered tree.

- In this language, the flag $\mathrm{Fl}(\mathbf{B})$ corresponds to a sequence of r nested partitions of the vertices (starting with the total partition into 1-element sets and ending with the trivial partition) associated to $\mathrm{Tree}(\mathbf{B})$, the jth partition being the one induced by the first $j-1$ edges. For example, the ordered tree $[(2,4)(1,3),(1,2)]$ induces the same sequence of partitions as $[(1,4),(2,3),(1,2)]$ (see Figure 2.2)

- A diagonal basis $\mathcal{D}$ is then a set of $(r-1)!$ ordered trees such that the $(r-1)!$ partition sequences obtained by reordering the edges of any one of the ordered trees are different from $(r-1)!-1$ sequences of partitions obtained from the remaining elements of $\mathcal{D}$.

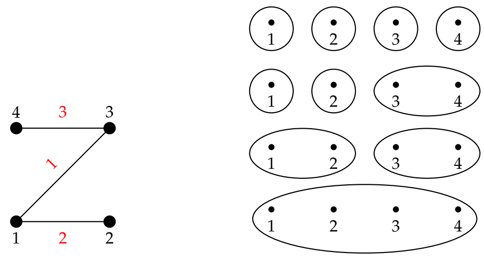

Figure 2.2 – $\mathbf{B} = (\alpha^{1,3}, \alpha^{1,2}, \alpha^{3,4})$

2.2.4. Examples

There are essentially 2 known constructions of diagonal bases [20].

I. **The Hamiltonian basis.** For each permutation $\sigma \in \Sigma_r$, we can define

$$\sigma(\mathbf{B}) = (\alpha^{\sigma(r-1),\sigma(r)}, \alpha^{\sigma(r-2),\sigma(r-1)}, \dots, \alpha^{\sigma(1),\sigma(2)}) \in \mathcal{B}. \tag{2.6}$$

The set $\mathcal{H}_m = \{\sigma(\mathbf{B})| \ \sigma \in \Sigma_r, \sigma(1) = m\}$ is then a diagonal basis. In the combinatorial description, this diagonal basis corresponds to the set of Hamiltonian paths starting at vertex m, and endowed with the reversed natural ordering of edges.

Example 2. Here are some examples of Hamiltonian bases:

- for $r = 3$: $\mathcal{H}_1 = \{(\alpha^{2,3}, \alpha^{1,2}), (\alpha^{3,2}, \alpha^{1,3})\}$,
- and for $r = 4$:

$$\mathcal{H}_1 = \{(\alpha^{3,4}, \alpha^{2,3}, \alpha^{1,2}), (\alpha^{2,4}, \alpha^{3,2}, \alpha^{1,3}), (\alpha^{4,3}, \alpha^{2,4}, \alpha^{1,2}), \\ (\alpha^{3,2}, \alpha^{4,3}, \alpha^{1,4}), (\alpha^{4,2}, \alpha^{3,4}, \alpha^{1,3}), (\alpha^{2,3}, \alpha^{4,2}, \alpha^{1,4})\}.$$

II. **The no-broken-circuit bases.** Let $\upsilon : \{1, \dots, r(r-1)/2\} \to \Phi$ be a total ordering, which we will represent as an order relation $\overset{\upsilon}{<}$ on Φ. To this ordering, one can associate the following, so called *noncommutative no-broken-circuit diagonal basis* [20]:

$$\mathcal{D}[\upsilon] = \left\{ \left(\beta^{[1]}, \dots, \beta^{[r-1]}\right) \in \mathcal{B} \,\middle|\, \beta^{[1]} \overset{\upsilon}{<} \dots \overset{\upsilon}{<} \beta^{[r-1]}, \text{ and} \\ \alpha^{ij} \overset{\upsilon}{<} \beta^{[m]} \Rightarrow (\alpha^{ij}, \beta^{[m]}, \dots, \beta^{[r-1]}) \text{ linearly independent} \right\}.$$

Example 3. Let $\alpha^{1,3} \overset{\upsilon}{<} \alpha^{1,4} \overset{\upsilon}{<} \alpha^{2,3} \overset{\upsilon}{<} \alpha^{2,4} \overset{\upsilon}{<} \alpha^{1,2} \overset{\upsilon}{<} \alpha^{3,4}$ be the ordering of the positive roots for rank $r = 4$. Then

$$\mathcal{D}[\upsilon] = \{(\alpha^{1,3}, \alpha^{1,2}, \alpha^{3,4}), (\alpha^{1,3}, \alpha^{1,4}, \alpha^{2,3}), (\alpha^{1,3}, \alpha^{1,4}, \alpha^{2,4}), \\ (\alpha^{1,3}, \alpha^{1,4}, \alpha^{1,2}), (\alpha^{1,3}, \alpha^{2,3}, \alpha^{3,4}), (\alpha^{1,3}, \alpha^{2,3}, \alpha^{2,4})\}$$

is the corresponding no-broken-circuit diagonal basis.

Remark 2.2.3. The hyperplane arrangement induced by Φ is invariant under the natural action of Σ_r on the vector space V. It follows easily from the definition that if $\mathcal{D}$ is a diagonal basis and $\sigma \in \Sigma_r$ is a permutation, then $\sigma(\mathcal{D})$ is also a diagonal basis.

2.3. The residue formula and the main result

In this section, we recall the residue formula from [20] for $\mathrm{Ver}(k,\lambda)$, the discrete Verlinde sum on the right hand side of (1.1). The key feature of this formula is that it exposes the piecewise polynomial nature of $\mathrm{Ver}(k,\lambda)$, which is key for our wall-crossing analysis. While the objects are relatively simple, the formalism is heavy with notation, so we begin by describing the 1-dimensional case.

2.3.1. *The residue formula in dimension 1*

The story begins with the Fourier series

$$\frac{1}{(2\pi i)^m}\sum_{n\in\mathbb{Z}\setminus 0}\frac{\exp(2\pi i a n)}{n^m} \tag{2.7}$$

for $m \geqslant 2$, which is a periodic, piecewise polynomial function given by the formula

$$\operatorname*{Res}_{x=0}\frac{\exp(\{a\}x)}{1-\exp(x)}\frac{dx}{x^m},$$

where $\{a\}$ is the fractional part of the real number a. The polynomial functions thus obtained on the interval $[0,1]$ are called *Bernoulli polynomials*. The polynomial on the interval containing the real number $c \in \mathbb{R}\setminus\mathbb{Z}$ is given by

$$\operatorname*{Res}_{x=0}\frac{\exp((a-[c])x)}{1-\exp(x)}\frac{dx}{x^m},$$

where $[c]$ is the integer part of c.

Now we pass to a trigonometric version of this formula, calculating finite sums of values of rational trigonometric functions over rational points with denominators equal to an integer k.

We replace thus the rational function x^{-m} by the (hyperbolic) trigonometric function $f(x) = (2\sinh(x/2))^{-2m}$, and introduce an integer parameter λ related to a via $ka = \lambda$. We consider the sum of values of the function f over a finite set of rational points in analogy with (2.7):

$$\sum_{n=1}^{k-1}\frac{\exp(2\pi i\lambda n/k)}{(2\sin(\pi n/k))^{2m}},$$

where $\lambda, k \in \mathbb{Z}$. This sum is again periodic in $\lambda \mod k$, and for $m \geqslant 2$ we can evaluate it via the residue theorem as

$$(-1)^m\operatorname*{Res}_{z=1}\frac{z^{k\{\lambda/k\}}}{(z^{1/2}-z^{-1/2})^{2m}}\cdot\frac{k\,dz}{z(1-z^k)} \overset{z=\exp(x/k)}{=} (-1)^m\operatorname*{Res}_{x=0}\frac{\exp(\{\lambda/k\}\cdot x)}{1-\exp(x)}\cdot f(x/k)\,dx.$$

Again, this is a piecewise polynomial function in the pair (k,λ), which is polynomial in the cones bounded by the lines $\lambda = qk$, $q \in \mathbb{Z}$.

Note that in these calculations, a key role is played by the Bernoulli operator:

$$f \mapsto \mathrm{Ber}[f](a) = \frac{f(x)\exp(ax)\,dx}{1-\exp(x)}, \tag{2.8}$$

which transforms meromorphic functions in the variable x into polynomials in a, and plays the role of a generalized Fourier operator.

2.3.2. *The multidimensional case*

Now we return to the setup of §2.2 with the vector space V endowed with the hyperplane arrangement Φ. We introduce the notation $\mathcal{F}_\Phi$ *for the space of meromorphic functions defined in a neighborhood of* 0 *in* $V \otimes_{\mathbb{R}} \mathbb{C}$ *with poles on the union of hyperplanes*

$$\bigcup_{1\leqslant i<j\leqslant r} \{x|\, \langle \alpha^{ij}, x\rangle = 0\}.$$

In particular, the inverse of the function

$$w_\Phi = \prod_{i<j} \left(2\sinh(\pi(x_i - x_j))\right)$$

is an element of $\mathcal{F}_\Phi$.

To write down our residue formula, we need a multidimensional generalization of the notions of integer and fractional parts. Given a basis $\mathbf{B} = (\beta^{[1]}, \ldots, \beta^{[r-1]}) \in \mathcal{B}$ of V^*, and an element $a \in V^*$, we define $[a]_{\mathbf{B}}$ and $\{a\}_{\mathbf{B}}$ to be the unique elements of V^* satisfying

- $[a]_{\mathbf{B}} = a - \{a\}_{\mathbf{B}} \in \Lambda$, and
- $\{a\}_{\mathbf{B}} \in \sum_{j=1}^{r-1}[0,1)\beta^{[j]}$.

This notion naturally induces a chamber structure on V^*: we will call $a \in V^*$ *regular* if a is a point of continuity for the functions $a \mapsto [a]_{\mathbf{B}}, \{a\}_{\mathbf{B}}$ for all $\mathbf{B} \in \mathcal{B}$, i.e. when $\{a\}_{\mathbf{B}} \in \sum_{j=1}^{r-1}(0,1)\beta^{[j]}$. Now, for regular a and b we define the equivalence relation

$$a \sim b \text{ when } [a]_{\mathbf{B}} = [b]_{\mathbf{B}} \quad \forall \mathbf{B} \in \mathcal{B}. \tag{2.9}$$

The equivalence classes for this relation form a Λ-periodic system of *chambers* in V^*.

Convention: We will think of a partition Π of $\{1, 2, \ldots, r\}$ into two nonempty sets as an ordered partition $\Pi = (\Pi', \Pi'')$ such that $r \in \Pi''$, and we will call these objects *nontrivial partitions* for short.

The following statement is straightforward.

Lemma 2.3.1. *The equivalence classes of the relation $\sim$ are precisely the chambers in V^* created by the walls parameterized by a nontrivial partition $\Pi = (\Pi', \Pi'')$ of the first r positive integers, and an integer l:*

$$S_{\Pi,l} = \{c \in V^*|\, \sum_{j\in\Pi'} c_j = l\} \tag{2.10}$$

Remark 2.3.2. Note that the walls given in (2.10) are precisely the same as the ones given in (2.5) for the case $d = 0$, where they play the role of walls separating the chambers of parabolic weights c in which the parabolic moduli spaces $P_0(c)$ are naturally the same. This "coincidence" is precisely what we need for our comparative wall-crossing strategy. There is a small terminological issue here: the "chambers" in §2.1.4 are the intersections of the equivalence classes of $\sim$ defined above with the open simplex $\Delta_0 \stackrel{\text{def}}{=} \Delta$, where the parabolic weights live (cf. Figures 2.1 and 2.3). We will use the term "chamber" in both cases if this causes no confusion.

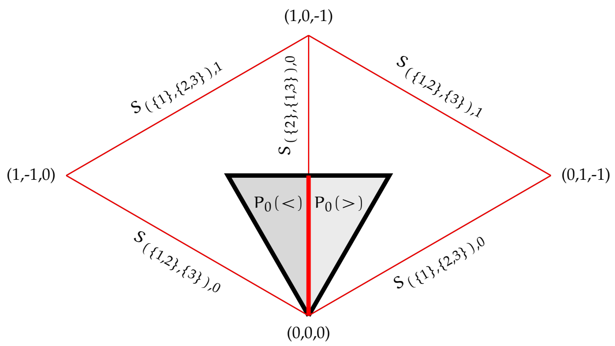

Figure 2.3 – Chambers for rank $r = 3$.

Each element $\mathbf{B} = (\beta^{[1]}, \ldots, \beta^{[r-1]}) \in \mathcal{B}$ defines an *iterated* version of the Bernoulli operator (2.8) on the space of functions $\mathcal{F}_\Phi$: interpreting the elements $a, \beta^{[j]} \in V^*$ as linear functions on V, we define

$$\operatorname*{iBer}_{\mathbf{B}}[f(x)](a) = \frac{1}{(2\pi i)^{r-1}} \int_{Z(\mathbf{B})} \frac{f(x)\exp\langle a, x\rangle \, d\langle \beta^{[1]}, x\rangle \wedge \cdots \wedge d\langle \beta^{[r-1]}, x\rangle}{(1 - \exp(\langle \beta^{[1]}, x\rangle)) \ldots (1 - \exp(\langle \beta^{[r-1]}, x\rangle))}, \tag{2.11}$$

where the naturally oriented cycle $Z_{\mathbf{B}}$ is defined by

$$Z_{\mathbf{B}} = \{v \in V \otimes_{\mathbb{R}} \mathbb{C} : |\langle \beta^{[j]}, x\rangle| = \varepsilon_j, \ j = \ldots, r-1\} \subset V \otimes_{\mathbb{R}} \mathbb{C} \backslash \{w_\Phi(x) = 0\},$$

with real constants ε_j satisfying $0 \leqslant \varepsilon_{r-1} \ll \cdots \ll \varepsilon_1$. Thus again, $\mathrm{iBer}_{\mathbf{B}}$ is a linear operator associating to a function in $\mathcal{F}_\Phi$ a polynomial on V^*.

Remark 2.3.3. Let us make a small remark about the computational aspects of the operator $\mathrm{iBer}_{\mathbf{B}}$. Denoting the coordinate $\langle \beta^{[j]}, x\rangle$ by y_j, $j = 1, \ldots, r-1$, and writing f and a in these coordinates: $f(x) = \hat{f}(y)$, $\langle a, x\rangle = \langle \hat{a}, y\rangle$, we can rewrite (2.11) as

$$\operatorname*{iBer}_{\mathbf{B}}[f(x)](a) = \operatorname*{Res}_{y_1=0} \ldots \operatorname*{Res}_{y_{r-1}=0} \frac{\hat{f}(y)\exp\langle \hat{a}, y\rangle \, dy_1 \wedge \cdots \wedge dy_{r-1}}{(1 - \exp(y_1)) \ldots (1 - \exp(y_{r-1}))},$$

where *iterating* the residues here means that we keep the variables with lower indices as unknown constants, and then use geometric series expansions of the type

$$\frac{1}{1 - \exp(y_1 - y_2)} = \frac{y_1 - y_2}{1 - \exp(y_1 - y_2)} \cdot \frac{1}{y_1 - y_2} = \frac{y_1 - y_2}{1 - \exp(y_1 - y_2)} \cdot \sum_{n=0}^{\infty} \frac{y_2^n}{y_1^{n+1}}.$$

2.3.3. *Invariance of diagonal bases and the main results*

Diagonal bases have the following key invariance property.

Theorem 2.3.4 ([20]). *Let $f \in \mathcal{F}_\Phi$, and $c \in V^*$ be regular; let $\mathcal{D}$ be a diagonal basis of Φ. Then the functional (cf. (2.11) above)*

$$f \mapsto \sum_{\mathbf{B} \in \mathcal{D}} \operatorname*{iBer}_{\mathbf{B}}[f(x)](a - [c]_{\mathbf{B}})$$

transforming a meromorphic function $f \in \mathcal{F}_\Phi$ into a polynomial in the variable $a \in V^*$ is independent of the choice of the diagonal basis $\mathcal{D}$. In particular, for regular $a \in V^*$, the functional

$$f \mapsto \sum_{\mathbf{B} \in \mathcal{D}} \underset{\mathbf{B}}{\mathrm{iBer}}[f(x)](\{a\}_{\mathbf{B}}) \tag{2.12}$$

transforms f into a well-defined piecewise polynomial function on V^*, which is polynomial in each chamber.

As this functional is invariantly defined, it is not surprising that it is equivariant with respect to the symmetries of our hyperplane arrangement. For $\sigma \in \Sigma_r$, we define, as usual

$$\sigma \cdot f(x) = f(\sigma^{-1}x). \tag{2.13}$$

This convention is consistent with (2.6).

Lemma 2.3.5. *Let $f \in \mathcal{F}_\Phi$, and $\sigma \in \Sigma_r$, and pick any diagonal basis $\mathcal{D}$. Then*

$$\sum_{\mathbf{B} \in \mathcal{D}} \underset{\mathbf{B}}{\mathrm{iBer}}[f(x)](\sigma \cdot a - [\sigma \cdot c]_{\mathbf{B}}) = \sum_{\mathbf{B} \in \mathcal{D}} \underset{\mathbf{B}}{\mathrm{iBer}}[\sigma^{-1} \cdot f(x)](a - [c]_{\mathbf{B}})$$

Proof. Indeed, recall that $\sigma \in \Sigma_r$ takes a diagonal basis to another diagonal basis (cf. Remark 2.2.3), and thus we have

$$\sum_{\mathbf{B} \in \mathcal{D}} \underset{\mathbf{B}}{\mathrm{iBer}}[f(x)](\sigma \cdot a - [\sigma \cdot c]_{\mathbf{B}}) = \sum_{\mathbf{B} \in \mathcal{D}} \underset{\sigma\mathbf{B}}{\mathrm{iBer}}[f(x)](\sigma \cdot a - [\sigma \cdot c]_{\sigma\mathbf{B}}).$$

Now we perform the linear substitution $x = \sigma(y)$, and obtain

$$\sum_{\mathbf{B} \in \mathcal{D}} \underset{\sigma\mathbf{B}}{\mathrm{iBer}}[f(x)](\sigma \cdot a - [\sigma \cdot c]_{\sigma\mathbf{B}}) = \sum_{\mathbf{B} \in \mathcal{D}} \underset{\mathbf{B}}{\mathrm{iBer}}[\sigma^{-1} \cdot f(y)](a - [c]_{\mathbf{B}}).$$

□

Remark 2.3.6. By picking the Hamiltonian diagonal basis $\mathcal{H}_1 = \{\sigma \cdot \mathbf{B}_0 |\ \sigma \in \mathrm{Stab}(1, \Sigma_r)\}$, we can turn the argument in the proof above around, and obtain the following formula:

$$\sum_{\mathbf{B} \in \mathcal{H}_1} \underset{\mathbf{B}}{\mathrm{iBer}}[f(x)](a - [c]_{\mathbf{B}}) = \sum_{\sigma \in \mathrm{Stab}(1,\Sigma_r)} \underset{\mathbf{B}_0}{\mathrm{iBer}}[\sigma \cdot f(x)](\sigma \cdot a - [\sigma \cdot c]_{\mathbf{B}}) =$$
$$\underset{y_1=0}{\mathrm{Res}} \dots \underset{y_{r-1}=0}{\mathrm{Res}} \sum_{\sigma \in \mathrm{Stab}(1,\Sigma_r)} \frac{\sigma \cdot f(y) \exp\langle \sigma \cdot a - [\sigma \cdot c]_{\mathbf{B}}, y \rangle \, dy_1 \wedge \dots \wedge dy_{r-1}}{(1 - \exp(y_1)) \ \dots (1 - \exp(y_{r-1}))},$$

where

$$\mathbf{B}_0 = (y_1 = x_{r-1} - x_r, \dots, y_{r-2} = x_2 - x_3, y_{r-1} = x_1 - x_2) \in \mathcal{B}.$$

Now we are ready to write down the residue formula for the Verlinde sums proved in [21, Theorem 4.2]. Recall that we denoted by $\mathrm{Ver}(k, \lambda)$ the finite sum on the right hand side of (1.1).

Theorem 2.3.7. *Let $g \geqslant 1$, $k \in \mathbb{Z}^{>0}$, $\lambda \in \Lambda$, and let $\mathcal{D}$ be any diagonal basis of Φ. Introducing the notation $\widehat{k} = k + r$, and $\widehat{\lambda} = \lambda + \rho$, we have*

$$\mathrm{Ver}(k,\lambda) = \tilde{N}_{r,k} \sum_{\mathbf{B}\in\mathcal{D}} \underset{\mathbf{B}}{\mathrm{iBer}} \left[w_{\Phi}^{1-2g}(x/\widehat{k}) \right] \left(\widehat{\lambda}/\widehat{k} - [\widehat{\varsigma}]_{\mathbf{B}} \right), \tag{2.14}$$

where $\tilde{N}_{r,k} = (-1)^{\binom{r}{2}(g-1)} N_{r,k}$ (cf. (1.1)) and $\widehat{\varsigma} \in V^*$ is a regular point in a chamber that contains $\widehat{\lambda}/\widehat{k}$ in its closure.

Now, if we look at our main goal (1.1): proving the equality

$$\mathrm{Ver}(k,\lambda) = \chi(P_0(\lambda/k), \mathcal{L}_0(k;\lambda)), \tag{2.15}$$

then we discover a rather embarrassing mismatch. Both sides are piecewise polynomial functions, however,

- according to the HRR theorem, $\chi(P(\lambda/k), \mathcal{L}_0(k;\lambda))$ is polynomial on the cones over the the equivalence classes (cf. (2.9)) of λ/k, while
- according to (2.14), $\mathrm{Ver}(k,\lambda)$ is polynomial on the cones over the equivalence classes of $\widehat{\lambda}/\widehat{k}$,

and these conic partitions of $\{(k,\lambda)|\ \lambda/k \in \Delta\}$ could clearly be different (cf. Figure 2.4 for a sketch of this problem).

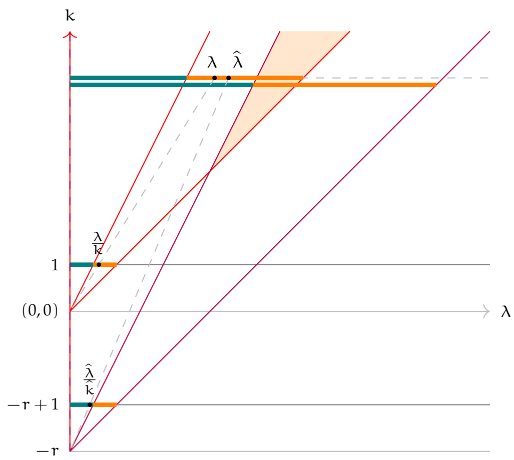

Figure 2.4 – λ/k is in the orange chamber, while $\widehat{\lambda}/\widehat{k}$ is in the green chamber.

Thus for (1.1) to be true, some miracle needs to occur, and these miracles are well-known in the area of "quantization commutes with reduction" [15, 28, 23]. We will return to this problem in §2.9, but for now, we will be satisfied to use (2.14) to write down a (conjectural for the moment) formula for $\chi(P_0(\lambda/k), \mathcal{L}_0(k;\lambda))$, which is manifestly polynomial on the cones where λ/k is in a fixed equivalence class.

Let us fix a regular $c \in \Delta$ marking a particular chamber in Δ. The two cones $\{(k;\lambda)|\ \lambda/k \sim c\}$ and $\{(k;\lambda)|\ \widehat{\lambda}/\widehat{k} \sim c\}$ intersect along an open cone (this cone is shaded in orange on Figure

2.4), and on this intersection, the expression

$$\sum_{\mathbf{B}\in\mathcal{D}} \underset{\mathbf{B}}{\mathrm{iBer}}\left[w_\Phi^{1-2g}(x/\hat{k})\right]\left(\hat{\lambda}/\hat{k} - [\lambda/k]_{\mathbf{B}}\right) \tag{2.16}$$

coincides with the right hand side of (2.14). As (2.16) is manifestly polynomial on each cone where λ/k is in a particular chamber in Δ, this expression will be then our main candidate for $\chi(P_0(\lambda/k), \mathcal{L}_0(k;\lambda))$.

Our plan is thus to split the proof of (2.15) into three parts: the first is equality (2.14), and the other two are given in our main theorem below. We formulated all our statements in a manner that allows us to treat the cases when λ/k or $\hat{\lambda}/\hat{k}$ are on a boundary separating two of our chambers in Δ.

Theorem 2.3.8. *Let $\lambda \in \Lambda$ and $k \in \mathbb{Z}^{>0}$ be such that $\lambda/k \in \Delta$. Let ς and $\hat{\varsigma} \in \Delta$ be regular elements, specifying two chambers in Δ, which contain λ/k and $\hat{\lambda}/\hat{k}$ in their closures, correspondingly. Then for any diagonal basis $\mathcal{D}$, the following two equalities hold:*

$$\chi(P_0(\varsigma), \mathcal{L}(k;\lambda)) = \tilde{N}_{r,k} \sum_{\mathbf{B}\in\mathcal{D}} \underset{\mathbf{B}}{\mathrm{iBer}}\left[w_\Phi^{1-2g}(x/\hat{k})\right]\left(\hat{\lambda}/\hat{k} - [\varsigma]_{\mathbf{B}}\right), \tag{I.}$$

and

$$\sum_{\mathbf{B}\in\mathcal{D}} \underset{\mathbf{B}}{\mathrm{iBer}}\left[w_\Phi^{1-2g}(x/\hat{k})\right]\left(\hat{\lambda}/\hat{k} - [\varsigma]_{\mathbf{B}}\right) = \sum_{\mathbf{B}\in\mathcal{D}} \underset{\mathbf{B}}{\mathrm{iBer}}\left[w_\Phi^{1-2g}(x/\hat{k})\right]\left(\hat{\lambda}/\hat{k} - [\hat{\varsigma}]_{\mathbf{B}}\right). \tag{II.}$$

Remark 2.3.9. Part I. of the theorem implies that if $\lambda/k \in \Delta$ is not regular, then

$$\chi(P_0(c^+), \mathcal{L}(k;\lambda)) = \chi(P_0(c^-), \mathcal{L}(k;\lambda)),$$

for regular $c^\pm \in \Delta$ in two neighboring chambers that contain λ/k in their closure (cf. Proposition 2.9.1 and Remark 2.9.4).

Before we proceed, we formulate a mild generalization of part I. of our theorem. As observed above, if we fix a generic $c \in \Delta$, and vary (λ, k) in such a way that $\lambda/k \sim c$, then both sides of the equality (I.) are manifestly polynomial, and thus we can extend the validity of this equality as follows.

Corollary 2.3.10. *Let $c \in \Delta$ be a regular element, which thus specifies a chamber in Δ and a parabolic moduli space $P_0(c)$ as well. Then for a diagonal basis $\mathcal{D}$, an arbitrary weight $\lambda \in \Lambda$, and a positive integer k, we have*

$$\chi(P_0(c), \mathcal{L}(k;\lambda)) = \tilde{N}_{r,k} \sum_{\mathbf{B}\in\mathcal{D}} \underset{\mathbf{B}}{\mathrm{iBer}}[w_\Phi^{1-2g}(x/\hat{k})](\hat{\lambda}/\hat{k} - [c]_{\mathbf{B}}). \tag{2.17}$$

Example 4. Let us write down these formulas in case of $r = 3$ explicitly. Let $\mathcal{D}$ be the diagonal basis from Example 2; then using Remark 2.3.6, we obtain

$$\chi(P_0(<), \mathcal{L}(k;\lambda)) = (-1)^{g-1}(3(k+3)^2)^g \operatorname*{Res}_{y=0}\operatorname*{Res}_{x=0} \frac{e^{\lambda_1 x+(\lambda_1+\lambda_2)y+x+y} - e^{\lambda_1 x+(\lambda_1+\lambda_3)y+x}}{(1-e^{x(k+3)})(1-e^{y(k+3)})w_\Phi(x,y)^{2g-1}} dx dy$$

and

$$\chi(P_0(>),\mathcal{L}(k;\lambda)) = (-1)^{g-1}(3(k+3)^2)^g \operatorname*{Res}_{y=0}\operatorname*{Res}_{x=0} \frac{e^{\lambda_1 x+(\lambda_1+\lambda_2)y+x+y} - e^{\lambda_1 x+(\lambda_1+\lambda_3)y+x+y(k+3)}}{(1-e^{x(k+3)})(1-e^{y(k+3)})w_\Phi(x,y)^{2g-1}}\, dxdy,$$

where $w_\Phi(x,y) = 2\sinh(\frac{x}{2})2\sinh(\frac{y}{2})2\sinh(\frac{x+y}{2})$.

2.3.4. *The walls*

Our first step is to identify the wall-crossing terms of the residue formula (2.17), which originate in the discontinuities of the function $c \mapsto \{c\}_{\mathbf{B}}$. These discontinuities occur on "walls": the affine hyperplanes (2.10). The following is straightforward:

Lemma 2.3.11. *Let $S_{\Pi,l}$ be the wall defined by (2.10), and $\mathbf{B} = (\beta^{[1]},\dots,\beta^{[r-1]}) \in \mathcal{B}$ an ordered basis of V^*. Then, as a function of c, the fractional part function $\{c\}_{\mathbf{B}}$ has a discontinuity at a generic point of the wall $S_{\Pi,l}$ exactly when* $\mathrm{Tree}(\mathbf{B})$ *(cf. page 11) is a union of a tree on Π', a tree on Π'' (the enumeration of the edges is irrelevant here) and a single edge (which we will call the* ***link****) connecting Π' and Π''.*

Notation: We will denote the element of $\mathbf{B}$ corresponding to this edge by β_{link}; this vector thus depends on $\mathbf{B}$ and the partition Π.

Proof. We fix $\mathbf{B}$, and note that for our purposes, $c \in S_{\Pi,l}$ will be considered *generic* if it belongs to only this one wall in Δ; this is equivalent to the condition that the only nontrivial subsets of coordinates of c which sum up to an integer are Π' and Π''.

Note that an element

$$c = \sum_{j=1}^{r-1} b_j \beta^{[j]} \in \Delta$$

is a point of discontinuity of the fractional part function $\{\cdot\}_{\mathbf{B}}$ if and only if $b_j \in \mathbb{Z}$ for some $1 \leqslant j \leqslant r-1$. Next, we express the coefficient b_j via the coordinates of c: we show that for all $1 \leqslant j \leqslant r-1$ we have

$$b_j = \sum_{i\in\Psi_j} c_i \quad \text{for some subset} \quad \Psi_j \subset \{1,...,r\}. \tag{2.18}$$

Now we orient the edges of $\mathrm{Tree}(\mathbf{B})$ in such way that they are all directed "away" from the root vertex r, and, without loss of generality (recall that $\sum c_i = 0$), we can assume that this orientation agrees with the signs of the elements $\beta^{[j]} \in \mathbf{B}$. It is easy to verify then that the subset

$$\Psi_j = \{k \in \{1,...,r\} |\ \text{the unique directed path in } \mathrm{Tree}(\mathbf{B}) \text{ from } r \text{ to } k \text{ contains the edge corresponding to } \beta^{[j]}\},$$

satisfies (2.18).

Hence we can conclude that if $c \in S_{\Pi,l}$ is generic and the coefficient b_j is an integer, then necessarily $\Psi_j = \Pi'$, and thus $\Pi'' = \{1,...,r\}\backslash\Psi_j$, and cutting the edge corresponding to $\beta^{[j]}$ from $\mathrm{Tree}(\mathbf{B})$ results in two disjoint trees, on Π' and on Π'', respectively. □

Now choose two regular elements $c^+, c^- \in V^*$ in two neighboring chambers separated by the wall $S_{\Pi,l}$, in such a way that

$$[c^+_{\Pi'}] = l \quad \text{and} \quad [c^-_{\Pi'}] = l - 1, \tag{2.19}$$

where

$$c_{\Pi'} \stackrel{\text{def}}{=} \sum_{i \in \Pi'} c_i,$$

and, as usual, $[q]$ stands for the integer part of the real number q. Now introduce the notation

$$p_\pm(k;\lambda) = \tilde{N}_{r,k} \sum_{\mathbf{B} \in \mathcal{D}} \operatorname*{iBer}_{\mathbf{B}}[w_\Phi^{1-2g}(x/\hat{k})](\hat{\lambda}/\hat{k} - [c^\pm]_{\mathbf{B}})$$

for the two polynomial functions in (k, λ) corresponding to c^+ and c^-, respectively. We define the *wall-crossing term* in our residue formula (2.17) as the difference between these two polynomials:

$$p_+(k;\lambda) - p_-(k;\lambda).$$

Using Lemma 2.3.1 and (2.19), we obtain the following simple residue formula for this difference.

Lemma 2.3.12. *Let (Π, l), c^+ and c^- be as above, and let us fix a diagonal basis $\mathcal{D} \subset \mathcal{B}$. Denote by $\mathcal{D}|\Pi$ the subset of those elements of $\mathcal{D}$, which satisfy the condition described in Lemma 2.3.11. Then*

$$p_+(k,\lambda) - p_-(k,\lambda) = \tilde{N}_{r,k} \sum_{\mathbf{B} \in \mathcal{D}|\Pi} \operatorname*{iBer}_{\mathbf{B}} \left[(1 - \exp(\beta_{\text{link}}(x))) w_\Phi^{1-2g}(x/\hat{k}) \right] \left(\lambda/\hat{k} - [c^+]_{\mathbf{B}} \right), \tag{2.20}$$

where β_{link} is the "link" element of $\mathbf{B}$ (depending on Π and $\mathbf{B}$) defined after Lemma 2.3.11.

Remark 2.3.13. Note that the multiplication by $1 - \exp(\beta_{\text{link}}(x))$ in (2.20) has the effect of canceling one of the factors in the denominator in the definition (2.11) of the operation iBer.

Example 5. Calculating the difference of two polynomials from Example 4, we obtain the wall-crossing term for rank 3 case:

$$p_-(k;\lambda) - p_+(k;\lambda) = (-3(k+3)^2)^g \operatorname*{Res}_{y=0} \operatorname*{Res}_{x=0} \frac{e^{\lambda_1 x + (\lambda_1+\lambda_3)y + x}}{(1 - e^{x(k+3)}) w_\Phi(x,y)^{2g-1}} dx dy.$$

2.3.5. Wall-crossing and diagonal bases

Now we pass to the study of the combinatorial object $\mathcal{D}|\Pi$ defined in Lemma 2.3.12. One thing we will discover is that even though each diagonal basis consists of $(r-1)!$ elements and the right hand side of (2.20) does not depend on the choice of $\mathcal{D}$, the number of elements in $\mathcal{D}|\Pi$ might vary with $\mathcal{D}$.

First we look at the case of the Hamiltonian basis $\mathcal{H}_1$. Form now on, we will use the notation $|\Pi'| = r'$ and $|\Pi''| = r''$ for a nontrivial partition $\Pi = (\Pi', \Pi'')$, (recall the convention $r \in \Pi''$). The following statement is easy to verify.

Lemma 2.3.14. *Let $\Pi = (\Pi', \Pi'')$ be a nontrivial partition, such that $1 \in \Pi'$ (the other case is analogous). Then*

$$\mathcal{H}_1|\Pi = \{\sigma(\mathbf{B})|\ \sigma(1) = 1, \text{ and } \sigma(\Pi') \in \Pi'\}.$$

In particular, $|\mathcal{H}_1|\Pi| = (r'-1)! \cdot r''!$.

It turns out that for our geometric applications, instead of $\mathcal{H}_1$, we will need to choose a particular nbc-basis, where the ordering is chosen to be consistent with Π.

To simplify our terminology, we will use the language of graphs and edges introduced in §2.2.3, and we will think of $\alpha^{ij} \in \Phi$ as an edge in the complete graph on r vertices. To define the ordering υ, we need to choose an edge between Π', Π''; the choice is immaterial, but for simplicity we settle for $m \stackrel{\text{def}}{=} \max\{i \in \Pi'\}$ and $r \in \Pi''$, and set $\beta_{\text{link}} = \alpha^{m,r}$ to be the smallest element according to υ.

The υ-ordered list of edges thus starts with β_{link}, and then continues with the remaining $r' \cdot r'' - 1$ edges connecting Π' and Π''. Next we list the $r'(r'-1)/2$ edges connecting vertices in Π' in any order, and finally, we list the remaining edges, those connecting vertices in Π''.

Notation: We introduce the natural notation Φ' and Φ'' for the $A_{r'}$ and $A_{r''}$ root systems corresponding to Π' and Π'', and we denote by $\mathcal{D}[\upsilon]$, $\mathcal{D}'[\upsilon]$ and $\mathcal{D}''[\upsilon]$, the diagonal nbc-bases induced by the ordering υ on Φ, Φ' and Φ'', respectively.

The following is easy to verify.

Lemma 2.3.15. *Given elements $\mathbf{B}' \in \mathcal{D}'[\upsilon]$ and $\mathbf{B}'' \in \mathcal{D}''[\upsilon]$, we can define an element of $\mathcal{D}[\upsilon]$ as follows: we start with β_{link}, then append $\mathbf{B}'$, and then continue with $\mathbf{B}''$. This construction creates a one-to-one correspondence*

$$\mathcal{D}'[\upsilon] \times \mathcal{D}''[\upsilon] \to \mathcal{D}[\upsilon]|\Pi; \tag{2.21}$$

in particular, $|\mathcal{D}[\upsilon]|\Pi| = (r'-1)! \cdot (r''-1)!$.

Finally, putting Lemmas 2.3.12 and 2.3.15 together, we arrive at the following elegant statement:

Proposition 2.3.16. *Let (Π, l), c^+ and c^- be as in Lemma 2.3.12, and let $\mathcal{D}'$ and $\mathcal{D}''$ be diagonal bases of Φ' and Φ'' correspondingly. Then*

$$p_+(k;\lambda) - p_-(k;\lambda) = (k+r)\tilde{N}_{r,k} \cdot \sum_{\mathbf{B}' \in \mathcal{D}'} \sum_{\mathbf{B}'' \in \mathcal{D}''} \operatorname*{Res}_{\beta_{\text{link}}=0} \operatorname*{iBer}_{\mathbf{B}'} \operatorname*{iBer}_{\mathbf{B}''} \left[w_\Phi^{1-2g}(x/\hat{k}) \right] \left(\hat{\lambda}/\hat{k} - [c^+]_{\mathbf{B}} \right) d\beta_{\text{link}}, \tag{2.22}$$

where $\operatorname{Res}_{\beta_{\text{link}}=0} \operatorname{iBer}_{\mathbf{B}'} \operatorname{iBer}_{\mathbf{B}''} d\beta_{\text{link}}$ is simply $\operatorname{iBer}_{\mathbf{B}}$ (cf (2.11)) with $\mathbf{B}$ obtained by appending $\mathbf{B}'$, and then $\mathbf{B}''$ to β_{link}, and with the factor $(1 - \exp\langle \beta_{\text{link}}, x \rangle)$ removed from the denominator.

Remark 2.3.17. The expression

$$\operatorname*{Res}_{\beta_{\text{link}}=0} \operatorname*{iBer}_{\mathbf{B}'} \operatorname*{iBer}_{\mathbf{B}''} \left[w_\Phi^{1-2g}(x/\hat{k}) \right] \left(\hat{\lambda}/\hat{k} - [c^+]_{\mathbf{B}} \right) d\beta_{\text{link}}$$

may equally be interpreted as follows. We write

$$\hat{\lambda}/\hat{k} - [c^+]_{\mathbf{B}} = m_{\text{link}}\beta_{\text{link}} + n' + n''$$

according to the splitting of **B**, think of $w(x/\widehat{k})$ as a function in $\mathcal{F}_{\Phi''}$ with some fixed values of the parameters from $\mathbf{B}'$ and β_{link}, and then calculate

$$\operatorname*{iBer}_{\mathbf{B}''}[w_{\Phi}^{1-2g}(x/\widehat{k})](n'').$$

The result will be a rational function Q in the variables from $\mathbf{B}'$ and β_{link}, and we proceed to calculate $\mathrm{iBer}_{\mathbf{B}'}[Q](n')$ to obtain a function F in the variable β_{link}, and finally the answer is $\mathrm{Res}_{\beta_{\text{link}}=0} \exp(m_{\text{link}}\beta_{\text{link}})F(\beta_{\text{link}})d\beta_{\text{link}}$.

We observe that since the trees $\mathrm{Tree}(B')$ and $\mathrm{Tree}(B'')$ are disjoint, the order of the application of the operations $\mathrm{iBer}_{B'}$ and $\mathrm{iBer}_{B''}$ is immaterial.

2.4. Wall-crossing in master space

Master spaces were introduced by Thaddeus in [26] in order to understand the dependence of GIT quotients on their linearizations. Following his footsteps, in this section, we describe a simple but very effective method to control the changes in the Euler characteristics of line bundles when crossing a wall in the space of linearizations. (Similar results appeared in [7]).

2.4.1. Wall-crossing and holomorphic Euler characteristics

We begin by recalling the basic notions of Geometric Invariant Theory.

Let X be a smooth projective variety over $\mathbb{C}$, and G a reductive group acting on X. A *linearization* of this action is a line bundle L on X with a lifting of the G-action to a linear action on L. An ample linearization is G-effective, if L^n has a nonzero G-invariant section for some $n > 0$; the space of such linearizations $\mathrm{Cone}_G(X)$ is called the G-*effective ample cone*.

For $L \in \mathrm{Cone}_G(X)$, we define the invariant-theoretic quotient $M_L = X /\!\!/^L G$ as the Proj of the graded ring of invariant sections of the powers of L:

$$M_L = \mathrm{Proj} \bigoplus_n H^0(X, L^n)^G.$$

According to Mumford's Geometric Invariant Theory [16], there is a partition of X (depending on L)

$$X = X^s[L] \cup X^{sss}[L] \cup X^{us}[L] \tag{2.23}$$

into the set of stable, strictly semistable, and unstable points, such that there is a surjective map $(X^s[L] \cup X^{sss}[L])/G \to M_L$; when $X^{sss}[L]$ is empty, this map is a bijection, and the quotient $M_L = X^s[L]/G$ is a smooth orbifold.

In [6], Dolgachev and Hu studied the dependence of the GIT quotient $M_L = X /\!\!/^L G$ on L. They showed that $\mathrm{Cone}_G(X)$ is divided by hyperplanes, called walls, into finitely many convex chambers, such that when L varies within a chamber, the partition (2.23) and thus the GIT quotient M_L remains unchanged. Moreover, an ample effective linearization lies on a wall precisely when it possesses a strictly semistable point.

Now let us consider two neighboring chambers, with smooth GIT quotients M_+ and M_-. We pick an arbitrary linearization $\mathcal{L}$ of the G-action on X, which descends to M_+ and M_-.

This last condition means that if $S \subset G$ is the stabilizer of a generic point in X, then S acts trivially on the fibers of $\mathcal{L}$. We will call such linearizations *descending*.

Thus, given such a descending linearization $\mathcal{L}$ of the G-action on X, we obtained two line bundles: one on M_+ and one on M_-, which, by abuse of notation, we will denote by the same letter $\mathcal{L}$. Via taking Chern classes, this construction creates a correspondence between classes in $H^2(M_+, \mathbb{Z})$ and $H^2(M_-, \mathbb{Z})$, which we will assume to be an isomorphism of free $\mathbb{Z}$-modules. We will thus identify these lattices, and introduce the notation Γ for them:

$$\Gamma = H^2(M_+, \mathbb{Z}) \simeq H^2(M_-, \mathbb{Z}).$$

The walls mentioned above can be thought of as hyperplanes in $\Gamma_{\mathbb{R}} = \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$.

Our goal in this section is to compare the holomorphic Euler characteristics $\chi(M_+, \mathcal{L})$ and $\chi(M_-, \mathcal{L})$, which are given by the Hirzebruch-Riemann-Roch theorem:

$$\chi(M_\pm, \mathcal{L}) = \int_{M_\pm} \exp(c_1(\mathcal{L}))\mathrm{Todd}(M_\pm).$$

As this expression is manifestly polynomial in $c_1(\mathcal{L})$, we obtain thus two polynomials on Γ, and our goal is to calculate their difference, the *wall-crossing term*

$$\chi(M_+, \mathcal{L}) - \chi(M_-, \mathcal{L}). \tag{2.24}$$

2.4.2. *The master space construction*

To simplify our setup, we will make some additional assumptions.

Assumptions 2.4.1. 1. The generic stabilizer of X is trivial.

2. Let L_+ and L_- be two ample linearizations of the G-action on X from the adjacent chambers corresponding to the quotients M_+ and M_-. Without loss of generality, we can assume that the linearization $L_0 = L_+ \otimes L_-$ lies on the single wall separating the two chambers, and that the interval connecting $c_1(L_+)$ and $c_1(L_-)$ in $\Gamma_{\mathbb{R}} = \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ does not intersect any other walls.

3. Let X^0 be the set of those semistable points $x \in X^{ss}[L_0]$ which are not stable for $L_\pm$:

$$X^0 := X^{ss}[L_0] \backslash (X^s[L_+] \cup X^s[L_-])$$

   We assume that X^0 is smooth, and that for $x \in X^0$ the stabilizer subgroup $G_x \subset G$ is isomorphic to $\mathbb{C}^*$.

4. Assume that there is a linearization $\vec{L}$ of the G-action on X such that $L_+ = L_- \otimes \vec{L}^n$ for some positive integer n, and such that for each $x \in X^0$, the stabilizer subgroup G_x acts freely on $\vec{L}_x \backslash 0$.

Now we introduce the *master space* construction of Thaddeus [26]. Consider the variety $Y = \mathbb{P}(\mathcal{O} \oplus \vec{L})$, which is a $\mathbb{P}^1$-bundle over X endowed with the additional $\mathbb{C}^*$-action $(1, t^{-1})$. As Y is a projectivization of a vector bundle on X, it comes equipped with $\mathcal{O}(1)$, which is the standard $G \times \mathbb{C}^*$-equivariant line bundle. To simplify our notation, we will denote the same way the linearizations of the G-action on X and their pull-backs (with tautological G-action) to Y.

The *master space* Z then is the GIT quotient of Y with respect to the linearization $L_-(n) = L_- \otimes \mathcal{O}(n)$:

$$Z = Y /\!\!/^{L_-(n)} G,$$

which inherits a $\mathbb{C}^*$-action from Y. Some additional notation:

- We will denote this copy of $\mathbb{C}^*$ by T,
- the projection $Y \to X$ by π, and the quotient map $Y^s \to Z$ by ψ.
- Introduce the notation $Y(0 : \cdot)$ and $Y(\cdot : 0)$ for the two copies of X in Y, corresponding to the two poles of the projective line; then Y is partitioned into 3 sets:

  $$Y = Y(0 : \cdot) \sqcup Y(\cdot : 0) \sqcup \vec{L}^\circ,$$

  where $\vec{L}^\circ$ is the line bundle $\vec{L}$ with the zero-section removed. We will write $\pi_\circ$ for the restriction of π to $\vec{L}^\circ$. We collect our maps on the following diagram.

$$\begin{tikzcd} \vec{L}^\circ \arrow[r, hook] \arrow[drr, "\pi_\circ"] & Y = \mathbb{P}(\mathcal{O} \oplus \vec{L}) \supset Y^s \arrow[d, "\pi"] \arrow[r, "\psi"] & Z \\ & X & \end{tikzcd} \tag{2.25}$$

Proposition 2.4.2. *1. There are embeddings*

$$\iota_- : M_- \to Z \quad \textit{and} \quad \iota_+ : M_+ \to Z$$

obtained as the quotients $Y^s \cap Y(\cdot : 0)/G$ *and* $Y^s \cap Y(0 : \cdot)/G$*, correspondingly.*

2. *The strictly semistable locus of* Y *with respect to the linearization* $L_-(n)$ *is empty, and the GIT quotient* $Z = Y^s/G$ *is smooth.*

3. *There is an embedding* $\iota_0 : X^0/G \to Z$*, obtained via* $\psi({\pi_\circ}^{-1}(X^0))$*. We denote the image of* ι_0 *by* Z^0*.*

4. *The fixed point locus* Z^T *is the disjoint union of* $\iota_+(M_+)$*,* $\iota_-(M_-)$*, and* Z^0*.*

Proof. (1)-(3) follow from [26, 4.2, 4.3]. To prove (4), first note that $Y(\cdot : 0)$ and $Y(0 : \cdot)$ are fixed by T, so we immediately obtain that $M_\pm \subset Z$ are fixed components. Also the G-action on Y commutes with the T-action, so a point $\psi(y) \in \psi(\pi_\circ^{-1}(X))$ is fixed by T if and only if the T-orbit $T \cdot y \subset \pi_\circ^{-1}(X)$ is contained in the G-orbit $G \cdot y \subset \pi_\circ^{-1}(X)$. Since $T \cdot y \subset \pi_\circ^{-1}(x)$ for some $x \in X$, we need $y \in \pi_\circ^{-1}(X^0)$. Moreover, for any $y \in \pi_\circ^{-1}(x) \subset \pi_\circ^{-1}(X^0)$, $T \cdot y = \pi_\circ^{-1}(x) = G_x \cdot y$, so a point $\psi(y) \in \psi(\pi_\circ^{-1}(X))$ is fixed by T if and only if $\psi(y) \in \psi(\pi_\circ^{-1}(X^0)) = Z^0$. □

Construction: Given a G-equivariant vector bundle E on X, we can construct a T-equivariant vector bundle $\zeta(E) \to Z$ on Z by first pulling E back from X to Y, and endowing the resulting bundle π^*E with the trivial action of T, and the action of G pulled back from X. We then obtain $\zeta(E) \to Z$ by descending π^*E to Z.

Before we formulate our wall-crossing formula, we need one more ingredient: the identification of the normal bundles of the fixed point components of Z.

Lemma 2.4.3. *1. The normal bundle on the component M_+ of Z^T is $\zeta(\vec{L}^{-1})|_{M_+}$, and the normal bundle of M_- is $\zeta(\vec{L})|_{M_-}$.*

2. Let $x \in X^0$, denote by G_x the stabilizer of x in the group G, and consider the point $\iota_0(x) \in Z^0$ (cf. Proposition 2.4.2 (3)). Then the normal vector space of $Z^0 \subset Z$ at the point $\iota_0(x)$ is canonically T-equivariantly isomorphic to the T-vector space $\vec{L}_x^\circ \times_{G_x} N_xX^0$, where N_xX^0 is the vector space normal to $X^0 \subset X$ at x, and the $T \simeq \mathbb{C}^*$-action is induced by left multiplication by t^{-1} on $\vec{L}_x$.

Proof. Part (1) immediately follows from the formula for the tangent space of the projective line: $T\mathbb{P}(V) \simeq \mathrm{Hom}(S, Q)$, where $S \to V \to Q$ is the tautological sequence on $\mathbb{P}(V)$, the projectivization of the vector space V.

For part (2), consider diagram 2.25; our goal is to identify the descent to Z_0 of the normal bundle $N_{\pi_\circ^{-1}X^0}$ to $\pi_\circ^{-1}X^0$ in $\vec{L}^\circ$. We only need to observe that this bundle may be identified with the pull-back $\pi_\circ^*NX^0$ of the normal bundle to X^0 in X, endowed with the natural G-action and a T-action, which is trivial on the fibers. □

Remark 2.4.4. Note that restricting the operator ζ to X^0, we can construct a T-equivariant vector bundle on Z^0 from a G-equivariant vector bundle on X^0. Then the normal bundle N_{Z^0} of $Z^0 \subset Z$ may be also described as $\zeta|_{X^0}(NX^0)$. The T-weights of the action may be computed by fixing $x \in X^0$, identifying the stabilizer subgroup $G_x \subset G$ with T via its action on the fiber $\vec{L}_x$, and then considering the action of G_x on N_xX^0.

Lemma 2.4.5. *The restriction of the line bundle $\zeta(\vec{L})$ to Z^0 is trivial with T-weight 1.*

Proof. Note that $\pi_\circ^*\vec{L}$ admits a G-equivariant tautological non-vanishing section. For calculating the weight, we observe that while T acts on $\vec{L}_x$ with weight -1, the T-weight of $\vec{L}_x^\circ \times_{G_x} \vec{L}$ is $+1$. □

Definition 2.4.6. Given a T-vector bundle V on a manifold on which T acts trivially, the T-equivariant *K-theoretical Euler class* of V^*, which we denote by $E_t(V)$, may be described as follows: let $x_1, \dots, x_n$ be the Chern roots of V, and $l_1, \dots l_n \in \mathbb{Z}$ be the corresponding T-weights. Then

$$E_t(V) = \prod_{j=1}^{n} \left(1 - t^{-l_j} \exp(-x_j)\right).$$

Now we are ready to write down our wall-crossing formula for (2.24). A key role will be played by the following notion: given a rational differential 1-form on the Riemann sphere, let us denote taking the sum of residues at 0 and at infinity by $\mu \mapsto \mathrm{Res}_{t=0,\infty} \mu$:

$$\operatorname*{Res}_{t=0,\infty} \stackrel{\text{def}}{=} \operatorname*{Res}_{t=0} + \operatorname*{Res}_{t=\infty}.$$

Theorem 2.4.7. *Let $\mathcal{L}$ be a linearization of the G-action on X, and denote, as above, by $\zeta(\mathcal{L})$ the T-equivariant line bundle on Z obtained by pull-back to Y and descent to Z. If Assumptions 2.4.1 hold, then*

$$\chi(M_+,\mathcal{L}) - \chi(M_-,\mathcal{L}) = \operatorname*{Res}_{t=0,\infty} \int_{Z^0} \frac{\mathrm{ch}_t(\zeta(\mathcal{L})|_{Z^0})}{E_t(N_{Z^0})} \mathrm{Todd}(Z^0)\frac{dt}{t}, \tag{2.26}$$

where N_{Z^0} is the T-equivariant bundle on Z^0 described in Lemma 2.4.3, ch_t is the T-equivariant Chern character, and $E_t(N_{Z^0})$ is the K-theoretical Euler class of $N^*_{Z^0}$.

Proof of Theorem 2.4.7. The Atiyah-Bott fixed-point formula [2] applied to the line bundle $\zeta(\mathcal{L})$ on our master space Z yields

$$\chi_t(Z,\zeta(\mathcal{L})) = \sum_{F\subset Z^T} \int_F \frac{\mathrm{ch}_t(\zeta(\mathcal{L})|_F)}{E_t(N_F)} \mathrm{Todd}(F), \tag{2.27}$$

where the sum is taken over the connected components of the fixed point locus Z^T.

In Proposition 2.4.2, we identified these components as M_+, M_- and Z^0. Lemma 2.4.3 identifies the equivariant normal bundles of M_+ and M_-, and thus the corresponding contributions are

$$\int_{M_+} \frac{\mathrm{ch}(\mathcal{L})\mathrm{Todd}(M_+)}{1-t^{-1}\exp(c_1(\vec{L}))} \quad \text{and} \quad \int_{M_-} \frac{\mathrm{ch}(\mathcal{L})\mathrm{Todd}(M_-)}{1-t\exp(-c_1(\vec{L}))}.$$

We observe that $\chi_t(Z,\zeta(\mathcal{L}))$ is a Laurent polynomial in t since it is the alternating sum of T-characters of finite dimensional vector spaces. Thus, as a function of t, $\chi_t(Z,\zeta(\mathcal{L}))$ has poles only at $t = 0,\infty$, and by the Residue Theorem, we have

$$\operatorname*{Res}_{t=0,\infty} \chi_t(Z,\zeta(\mathcal{L}))\frac{dt}{t} = 0.$$

On the other hand, since

$$\operatorname*{Res}_{t=0,\infty} \frac{A}{1-t^{-1}B}\frac{dt}{t} = -A \quad \text{and} \quad \operatorname*{Res}_{t=0,\infty} \frac{A}{1-tB}\frac{dt}{t} = A,$$

we have

$$\operatorname*{Res}_{t=0,\infty} \int_{M_+} \frac{\mathrm{ch}(\mathcal{L})\mathrm{Todd}(M_+)}{1-t^{-1}\exp(c_1(\vec{L}))}\frac{dt}{t} = -\chi(M_+,\mathcal{L}) \text{ and}$$

$$\operatorname*{Res}_{t=0,\infty} \int_{M_-} \frac{\mathrm{ch}(\mathcal{L})\mathrm{Todd}(M_-)}{1-t\exp(-c_1(\vec{L}))}\frac{dt}{t} = \chi(M_-,\mathcal{L}).$$

Now, applying the functional $\mathrm{Res}_{t=0,\infty}$ to the two sides of (2.27) multiplied by dt/t we obtain the desired result (2.26). □

2.5. Wall-crossings in parabolic moduli spaces

In this section, we apply Theorem 2.4.7 to wall-crossings in the moduli space of parabolic bundles.

From now on, we assume that $d = 0$, and we write Δ for the corresponding set of admissible parabolic weights Δ_0. Recall from Section 2.1.2 that for regular $c \in \Delta$, the moduli space of stable parabolic bundles $P_0(c)$ is the GIT quotient $XQ /\!\!/^c PSL(\chi)$, where XQ is a subspace of the total space of a flag bundle over the Quot scheme. Let us fix a partition $\Pi = (\Pi', \Pi'')$ and an integer l, and introduce the notation Δ'_l and Δ''_{-l} for the simplices of parabolic weights of Π' and Π''. Let $\phi \in \Sigma_r$ be the unique permutation which sends $\{1, ..., r'\}$ to Π' preserving the order of first r' and the last r'' elements. We choose $c^0 = (c^0_1, ..., c^0_r) \in S_{\Pi,l}$ and two regular elements $c^+, c^- \in \Delta$ in two neighboring chambers separated by the wall $S_{\Pi,l}$, such that

$$c^{\pm} = c^0 \pm \epsilon(..., 0, 1, 0, ..., 0, -1)$$

for some positive $\epsilon \in \mathbb{Q}$, where 1 and -1 are on the $\phi(r')^{\text{th}}$ and r^{th} places, respectively. Let

$$c' = \sum_{i \in \Pi'} c^0_i x_i \in \Delta'_l \quad \text{and} \quad c'' = \sum_{i \in \Pi''} c^0_i x_i \in \Delta''_{-l}.$$

For $(k, \lambda) \in \mathbb{Z} \times \Lambda$, consider the polynomials

$$q_{\pm}(k, \lambda) = \chi(P_0(c^{\pm}), \mathcal{L}_0(k; \lambda)).$$

Our goal is to calculate the difference of these two polynomials.
Notation: To simplify our notation, from now on, we omit the index t from the symbols for equivariant characteristic classes.

2.5.1. *The master space construction*

We construct the master space Z from §2.4.2 using the following data:

- a smooth variety $X = XQ$ (cf. §2.1.2);
- linearizations $L^{\pm} = L(k; \lambda^{\pm})$ of the G-action on X (cf. §2.1.2), such that $\lambda^{\pm}/k = c^{\pm}$;
- the linearization $\vec{L} = L(0; x_{\phi(r')} - x_r)$ of the G-action on X.

The following statement is easy to verify.

Lemma 2.5.1. *([5, §3.2]) The subset $X^0 \subset X$ is the set of points representing vector bundles W on C, such that W splits as a direct sum $W' \oplus W''$, where W' and W'' are, respectively, c' and c''-stable parabolic bundles. Therefore, we have the following description of the locus Z^0:*

$$Z^0 = \{W = W' \oplus W'' \mid W' \in \widetilde{P}_l(c');\ W'' \in \widetilde{P}_{-l}(c'');\ \det(W) \simeq \mathcal{O}\}.$$

Remark 2.5.2. Note that Z^0 is fibered over Jac^l with fibre $P_l(c') \times P_{-l}(c'')$ by the determinant map $\widetilde{P}_l(c') \to Jac^l$ and

$$H^*(Z^0, \mathbb{Q}) \simeq H^*(P_l(c') \times P_{-l}(c''), \mathbb{Q}) \otimes H^*(Jac^l, \mathbb{Q}). \tag{2.28}$$

Remark 2.5.3. If the rank of the vector bundle $W \in \widetilde{P}_l(c)$ is 1, then $c = l$ and $\widetilde{P}_l(l)$ is isomorphic to Jac^l, while $P_l(l)$ is a point.

Now we need to verify the hypotheses of Theorem 2.4.7. Note that in our present construction X is not projective, however, it contains all semisimple points of the flag bundle over the open subscheme of the Quot scheme parametrizing locally free quotients (cf. §2.1.2) for all possible polarizations, and hence the missing points of the Quot scheme have no effect on any of our constructions (a similar argument appeared in [26]).

Assumptions 2.4.1 (1)-(2) are trivially satisfied, so we study the action of the stabilizer $G_x \subset PSL(N)$ of point $x \in X$ on the fiber $\vec{L}_x\backslash 0$.

- For a general point $x \in X$ the stabilizer of x is the center $\mathbb{Z}_N \subset SL(N)$, which acts trivially on the fiber $\vec{L}_x\backslash 0$.
- For $x \in X^0$, any element of the stabilizer of x induces an automorphism of the corresponding vector bundle $W = W' \oplus W''$, so the stabilizer of x in $GL(N)$ is isomorphic to $\mathbb{C}^* \times \mathbb{C}^* \subset GL(N)$. An element $(t_1, t_2) \in \mathbb{C}^* \times \mathbb{C}^*$ is in $SL(N)$ if and only if $t_1^{N'} t_2^{N''} = 1$, where $N' = \chi(W')$ and $N'' = \chi(W'')$. Note that (t_1, t_2) acts on $\vec{L}_x$ as $t_1 t_2^{-1}$, and we need $t_1 = t_2$ (hence $t_1^N = 1$) for this action to be trivial, so the stabilizer of any point in $\vec{L}_x\backslash 0$ is the center $\mathbb{Z}_N \subset SL(N)$.

Then the action of $G = PSL(N)$ is free on $Y\backslash(Y(0:\cdot) \cup Y(\cdot:0))$, and the action of $G_x \subset PSL(N)$ on $\vec{L}_x\backslash 0$ induces an isomorphism $G_x \simeq \mathbb{C}^* \simeq T$.

Now by Theorem 2.4.7, the wall-crossing polynomial $q_-(k;\lambda) - q_+(k;\lambda)$ is equal to

$$\operatorname*{Res}_{t=0,\infty} \int_{Z^0} \frac{ch(\mathcal{L}_0(k;\lambda)|_{Z^0})}{E(N_{Z^0})} Todd(Z^0)\, \frac{dt}{t}. \tag{2.29}$$

Note that in our case, the T-action on Z is free outside the fixed locus Z^T, so as a function in $t \in T$, the integral in (2.29) may have poles only at $t = 0, 1, \infty$. Then, using the Residue Theorem and substituting $t = e^u$, we conclude that (2.29) equals

$$-\operatorname*{Res}_{u=0} \int_{Z^0} \frac{ch(\mathcal{L}_0(k;\lambda)|_{Z^0})}{E(N_{Z^0})} Todd(Z^0)\, du, \tag{2.30}$$

and thus our goal is to calculate this integral.

Our first step is to identify the characteristic classes under the integral sign (cf. Proposition 2.5.11 for the result).

We start with the study of the restriction of the line bundle $\mathcal{L}_0(k;\lambda)$ to the fixed locus $Z^0 \subset Z$. First, we describe a parametrization of the factor $H^*(Jac^l, \mathbb{Q})$ in (2.28). Let $\mathcal{J}$ be the Poincare bundle over $Jac \times C$, such that $c_1(\mathcal{J})_{(0)} = 0$; define $\eta \in H^2(Jac)$ by $(\sum_i c_1(\mathcal{J})_{(e_i)} \otimes e_i)^2 = -2\eta \otimes \omega$ (cf. §2.1.3), then (cf. [31]) for any $m \in \mathbb{Z}$

$$\int_{Jac} e^{\eta m} = m^g. \tag{2.31}$$

As Z^0 is a connected component of the fixed locus of the T- action on Z, its equivariant cohomology factors: $H^*_T(Z^0) \simeq H^*(Z^0) \otimes \mathbb{C}[u]$. In particular, there are canonical embeddings $H^*(Z^0) \hookrightarrow H^*_T(Z^0)$ and $\mathbb{C}[u] \hookrightarrow H^*_T(Z^0)$.

Remark 2.5.4. It follows from Lemma 2.4.5 that $c_1(\zeta(\vec{L})|_{Z^0}) = u$.

Recall that for a parabolic weight $c = (c_1, ..., c_r) \in \Delta$, we have set $c_{\Pi'} = \sum_{i\in\Pi'} c_i$.

Lemma 2.5.5. *Let $\lambda = (\lambda_1, ..., \lambda_r) \in \Lambda$, $k \in \mathbb{Z}^{>0}$ and let $\Pi = (\Pi', \Pi'')$ be a nontrivial partition with $r \in \Pi''$. Let*

$$\lambda' = \sum_{i\in\Pi'} \lambda_i x_i \quad \textit{and} \quad \lambda'' = \sum_{i\in\Pi''} \lambda_i x_i,$$

and define δ by $(\lambda/k)_{\Pi'} = l + \delta$. Then

$$\begin{aligned} \mathrm{ch}(\mathcal{L}_0(k;\lambda)\big|_{Z^0}) = e^{k\delta u} \exp\left(\frac{\eta k}{r'} + \frac{\eta k}{r''}\right) \cdot \\ \mathrm{ch}(\mathcal{L}_l(k;\lambda'_1, ..., \lambda'_{r'} - k\delta) \boxtimes \mathcal{L}_{-l}(k;\lambda''_1, ..., \lambda''_{r''} + k\delta)), \end{aligned}$$

where $\boxtimes$ denotes the external tensor product of line bundles on $P_l(c') \times P_{-l}(c'')$.

Proof. First, note that

$$\mathrm{ch}(\mathcal{L}_0(0;\lambda)\big|_{Z^0}) \;=\; e^{k(l+\delta)u}\mathrm{ch}(\mathcal{L}_l(0;\lambda'_1, ..., \lambda'_{r'} - kl - k\delta) \boxtimes \mathcal{L}_{-l}(0;\lambda''_1, ..., \lambda''_{r''} + kl + k\delta)),$$

and thus it will be sufficient to identify the restriction of $\mathcal{L}(k;0)$. It follows from Lemma 2.1.8, that

$$c_1(\mathcal{L}_0(k;0)) = \frac{k}{2r} c_2(\mathrm{End}_0(U))_{(2)}.$$

Note that

$$c_2(\mathrm{End}_0(U))_{(2)} = -2r\,\mathrm{ch}_2(U)_{(2)} + c_1^2(U)_{(2)} = -r\,\mathrm{ch}_2(U)_{(2)},$$

and thus

$$c_1(\mathcal{L}_0(k;0)) = -k\,\mathrm{ch}_2(U)_{(2)}.$$

Denote by $\tilde{U}'$ and $\tilde{U}''$ the normalized (cf. §2.1.3) universal bundles over $\tilde{P}_l(c') \times C$ and $\tilde{P}_{-l}(c'') \times C$, respectively. Since

$$\mathrm{ch}_2(U\big|_{Z^0})_{(2)} = \mathrm{ch}_2(\tilde{U}' \otimes \zeta(\vec{L})\big|_{Z^0})_{(2)} + \mathrm{ch}_2(\tilde{U}''\big|_{Z^0})_{(2)},$$

we have (cf. Remark 2.5.4)

$$\begin{aligned} c_1(\mathcal{L}_0(k;0)\big|_{Z^0}) = -k\,\mathrm{ch}_2(\tilde{U}')_{(2)} - k\,u\,c_1(\tilde{U}')_{(2)} - k\,\mathrm{ch}_2(\tilde{U}'')_{(2)} = \\ \frac{k}{2r'}c_2(\tilde{U}')_{(2)} - \frac{k}{2r'}c_1^2(\tilde{U}')_{(2)} + \frac{k}{2r''}c_2(\tilde{U}'')_{(2)} - \frac{k}{2r''}c_1^2(\tilde{U}'')_{(2)} - kl\,u. \end{aligned}$$

Now, since

$$c_1^2(\tilde{U}')_{(2)} = 2l\,c_1(U')_{(0)} - 2\eta \quad \text{and} \quad c_1^2(\tilde{U}'')_{(2)} = -2l\,c_1(U'')_{(0)} - 2\eta,$$

by Lemma 2.1.8, we have

$$\begin{aligned} c_1(\mathcal{L}_0(k;0)\big|_{Z^0}) = \frac{k}{r'}c_1(\mathcal{L}_l(r';l, ..., l)) - \frac{kl}{r'}c_1(U')_{(0)} + \eta\frac{k}{r'} + \\ \frac{k}{r''}c_1(\mathcal{L}_{-l}(r';-l, ..., -l)) + \frac{kl}{r''}c_1(U'')_{(0)} + \eta\frac{k}{r''} - kl\,u = \\ c_1(\mathcal{L}_l(k;(0, ..., 0, kl))) + c_1(\mathcal{L}_{-l}(k;(0, ..., 0, -kl))) + \eta\left(\frac{k}{r'} + \frac{k}{r''}\right) - kl\,u, \end{aligned}$$

and this completes the proof. ∎

Lemma 2.5.6. *Denote by $\widetilde{\mathcal{U}}'$ and $\widetilde{\mathcal{U}}''$ the normalized (cf. §2.1.3) universal bundles over $\widetilde{P}_\iota(c') \times C$ and $\widetilde{P}_{-\iota}(c'') \times C$, and denote by π the projections along C. Then the T-equivariant normal bundle to the fixed locus $Z^0 \subset Z$ is*

$$N_{Z^0} = R^1_T\pi_*(\mathcal{P}ar\mathcal{H}om(\widetilde{\mathcal{U}}', \widetilde{\mathcal{U}}'')) \oplus R^1_T\pi_*(\mathcal{P}ar\mathcal{H}om(\widetilde{\mathcal{U}}'', \widetilde{\mathcal{U}}')), \tag{2.32}$$

where the $T \simeq \mathbb{C}^*$-action has weights -1 and and $+1$ on the two summands, respectively.

Remark 2.5.7. As we are working with fixed determinant moduli spaces, the push-forwards in (2.32) are to be taken along the curve C in the part of $\widetilde{P}_\iota(c') \times \widetilde{P}_{-\iota}(c'') \times C$ where $\det(W') \cdot \det(W'') \simeq \mathcal{O}$ (cf. Lemma 2.5.1).

Proof. According to Lemma 2.4.3, for any point $x \in X^0$, the normal bundle N_{Z^0} at the point $\iota_0(x) \in Z^0$ may be identified with the T-vector space $\vec{L}^\circ_x \times_{G_x} N_x X^0$, where $N_x X^0$ is the normal bundle to $X^0 \subset X$ at x, with the T-action induced by left multiplication by t^{-1} on $\vec{L}_x$.

Denote by $\mathcal{U}Q$ the universal bundle over X, which descends to the normalized universal bundles on $P_0(c^\pm)$. Recall that any point $x \in X^0$ represents a vector bundle which splits as a direct sum of two subbundles, hence we have $\mathcal{U}Q_x = \mathcal{U}^+_x \oplus \mathcal{U}^-_x$, and

$$N_x X^0 = H^1(C, \mathcal{P}ar\mathcal{H}om(\mathcal{U}^+_x, \mathcal{U}^-_x)) \oplus H^1(C, \mathcal{P}ar\mathcal{H}om(\mathcal{U}^-_x, \mathcal{U}^+_x))$$

(c.f. [18, Proposition 1.13] for the description of the deformation space of parabolic bundles). A simple calculation (cf. Remark 2.4.4 and Lemma 2.4.5) shows we have a a T-module isomorphism

$$\vec{L}^\circ_x \times_{G_x} H^1(C, \mathcal{P}ar\mathcal{H}om(\mathcal{U}^+_x, \mathcal{U}^-_x)) \simeq \vec{L}_x \otimes H^1(C, \mathcal{P}ar\mathcal{H}om(\mathcal{U}^+_x, \mathcal{U}^-_x))$$

with T-weight -1 induced by multiplication on $\vec{L}_x$ and trivial action on $\mathcal{U}^+_x$ and $\mathcal{U}^-_x$; applying the projection formula we obtain that

$$\vec{L}_x \otimes H^1(C, \mathcal{P}ar\mathcal{H}om(\mathcal{U}^+_x, \mathcal{U}^-_x)) \simeq H^1_T(C, \mathcal{P}ar\mathcal{H}om(\mathcal{U}^+_x \otimes \vec{L}^{-1}_x, \mathcal{U}^-_x)).$$

Similarly, we have

$$\vec{L}^\circ_x \times_{G_x} H^1(C, \mathcal{P}ar\mathcal{H}om(\mathcal{U}^-_x, \mathcal{U}^+_x)) \simeq \vec{L}^{-1}_x \otimes H^1(C, \mathcal{P}ar\mathcal{H}om(\mathcal{U}^-_x, \mathcal{U}^+_x)) \simeq \\ H^1_T(C, \mathcal{P}ar\mathcal{H}om(\mathcal{U}^-_x, \mathcal{U}^+_x \otimes \vec{L}^{-1}_x))$$

with T-action of weight 1.

Finally, we observe that according to our normalizations, the bundles $\mathcal{U}^+ \otimes \vec{L}^{-1}$ and $\mathcal{U}^-$ descend to the normalized universal bundles $\widetilde{\mathcal{U}}'$ and $\widetilde{\mathcal{U}}''$ over $\widetilde{P}_\iota(c') \times C$ and $\widetilde{P}_{-\iota}(c'') \times C$, respectively, and this completes the proof. □

2.5.2. *Calculation of the characteristic classes of N_{Z^0}*

Before we calculate the equivariant K-theoretical Euler class of the conormal bundle $N^*_{Z^0}$, we need to introduce some notations. Recall that for $1 \leqslant i, j \leqslant r$, the differences $x_i - x_j \in V^*$ are linear functions on V, and the function $x_i - x_j$ corresponds to the linearization $L_0(0; x_i - x_j)$ on X, which descends to the line bundle $\mathcal{L}_0(0; x_i - x_j)$ on the moduli space $P_0(c)$ (cf. §2.1.2).

As in §2.4.2, we denote by $\zeta(L_0(0; x_i - x_j))$ the line bundle on Z obtained by the pullback and then descent. This way, we obtain a correspondence between the linear functions $x_i - x_j$ and the T-equivariant line bundles on Z.

Recall the definition of the permutation $\varphi \in \Sigma_r$ given at the beginning of this chapter: φ takes the first r' numbers to Π', preserving the order of the first r' and the last r'' elements. We introduce the symbols

$$\begin{aligned} &z'_i - z'_j = c_1(\zeta(L_0(0; x_{\varphi(i)} - x_{\varphi(j)}))\big|_{Z^0}),\ (1 \leqslant i, j \leqslant r') \\ &z''_i - z''_j = c_1(\zeta(L_0(0; x_{\varphi(r'+i)} - x_{\varphi(r'+j)}))\big|_{Z^0}),\ (1 \leqslant i, j \leqslant r'') \\ &u = (z'_{r'} - z''_r) = c_1(\zeta(L_0(0; x_{\varphi(r')} - x_r))\big|_{Z^0}) \end{aligned} \tag{2.33}$$

for the equivariant cohomology classes in $H^2_T(Z^0)$. The last equalities are consistent with Lemma 2.4.5.

Remark 2.5.8. Note that (cf. Remark 2.5.2)

$$z'_i - z'_j = c_1(\mathcal{F}'_{r-i+1}/\mathcal{F}'_{r-i} \otimes (\mathcal{F}'_{r-j+1}/\mathcal{F}'_{r-j})^*) \in H^2(P_l(c')),$$

$$z''_i - z''_j = c_1(\mathcal{F}''_{r-i+1}/\mathcal{F}''_{r-i} \otimes (\mathcal{F}''_{r-j+1}/\mathcal{F}''_{r-j})^*) \in H^2(P_{-l}(c'')),$$

where $\mathcal{F}'_i$ and $\mathcal{F}''_i$ are the flag bundles (cf. §2.1.3) on $P_0(c')$ and $P_0(c'')$, correspondingly.

Taking into account these identifications, functions on V give rise to equivariant cohomology classes on Z^0. To make the splitting $H^*_T(Z^0) \simeq H^*(Z^0) \otimes \mathbb{C}[u]$, explicit, however, we will write these classes in the form $f_u(z', z'')$, thinking of them as functions of the differences of the z'_is and the differences of the z''_is, depending on the parameter u. With this convention, we introduce

$$w^\times_u(z', z'') = \prod_{\substack{i,j \\ \varphi(i) < \varphi(r'+j)}} 2\sinh(z'_i - z''_j) \prod_{\substack{i,j \\ \varphi(r'+j) < \varphi(i)}} 2\sinh(z''_j - z'_i),$$

$$\rho^\times_u(z', z'') = \frac{1}{2} \sum_{\substack{i,j \\ \varphi(i) < \varphi(r'+j)}} (z'_i - z''_j) + \frac{1}{2} \sum_{\substack{i,j \\ \varphi(r'+j) < \varphi(i)}} (z''_j - z'_i),$$

where according to (2.33),

$$z'_i - z''_j = (z'_i - z'_{r'}) + u - (z''_j - z''_r) = c_1(\zeta(\mathcal{L}_0(0; x_{\varphi(i)} - x_{\varphi(r'+j)}))\big|_{Z^0}) \in H^2_T(Z^0).$$

Now we are ready to write down our formula for the K-theoretical Euler class $E(N_{Z^0})$ (cf. definition 2.4.6 with $t = e^u$).

Proposition 2.5.9.

$$\begin{aligned} E(N_{Z^0})^{-1} = &(-1)^{lr + r'r''(g-1)} e^{-rlu} \exp\left(\frac{\eta r}{r'} + \frac{\eta r}{r''}\right) w^\times_u(z', z'')^{1-2g} \exp(\rho^\times_u(z', z'')) \\ &\mathrm{ch}(\mathcal{L}_l(r''; -l, \ldots, -l, -l + rl) \boxtimes \mathcal{L}_{-l}(r'; l, \ldots, l, l - rl)). \end{aligned}$$

Proof. It follows from the short exact sequence (2.1) for parabolic morphisms that

$$\mathrm{ch}(-\pi_!(\mathrm{ParHom}(\tilde{U}'',\tilde{U}')) = -\mathrm{ch}(\pi_!(\mathrm{Hom}(\tilde{U}'',\tilde{U}'))) + \sum_{\substack{i,j\\ \phi(i)<\phi(r'+j)}} e^{z'_i - z''_j}$$

and

$$\mathrm{ch}(-\pi_!(\mathrm{ParHom}(\tilde{U}',\tilde{U}'')) = -\mathrm{ch}(\pi_!(\mathrm{Hom}(\tilde{U}',\tilde{U}'')) + \sum_{\substack{i,j\\ \phi(r'+j)<\phi(i)}} e^{z''_j - z'_i},$$

so by Lemma 2.5.6

$$\begin{aligned} \mathrm{ch}(N_{Z^0}) &= \mathrm{ch}(-\pi_!(\mathrm{Hom}(\tilde{U}'',\tilde{U}')) \oplus -\pi_!(\mathrm{Hom}(\tilde{U}'',\tilde{U}')^*)) \\ &+ \sum_{\substack{i,j\\ \phi(i)<\phi(r'+j)}} e^{z'_i - z''_j} + \sum_{\substack{i,j\\ \phi(r'+j)<\phi(i)}} e^{z''_j - z'_i}. \end{aligned} \tag{2.34}$$

Let $f(x)$ be a power series in one variable, and W a vector bundle of rank r with (equivariant) Chern roots $y_1, \dots, y_r$. Then we denote by $[f(x)]^W$ the multiplicative (equivariant) characteristic class of W given by the function $f(x)$ in Chern roots of W:

$$[f(x)]^W = \prod_{j=1}^{r} f(y_j).$$

Lemma 2.5.10. *Let P be a smooth variety, and let S be a T-vector bundle on $P \times C$ with T-weight 1; pick a point $p \in C$ and denote by $\pi : P \times C \to P$ the projection along the curve. Then*

$$E(-\pi_! S \oplus -\pi_! S^*)^{-1} = (-1)^{\mathrm{rk}(-\pi_! S)} \frac{\exp(-\mathrm{ch}_2(S)_{(2)})}{[(2\sinh(x/2))^{2g-2}]^{S_p}}.$$

Proof. Note that

$$E(-\pi_! S)^{-1} = \left[\frac{1}{1 - t^{-1}e^{-x}}\right]^{-\pi_! S} = \left[\frac{-te^x}{1 - te^x}\right]^{-\pi_! S}$$

and

$$E(-\pi_! S^*)^{-1} = \left[\frac{1}{1 - te^{-x}}\right]^{-\pi_! S^*} = \left[\frac{1}{1 - te^x}\right]^{(-\pi_! S^*)^*}.$$

Applying Serre duality and the Grothendieck-Riemann-Roch Theorem we obtain

$$\begin{aligned} \mathrm{ch}(-\pi_! S) + \mathrm{ch}((-\pi_! S^*)^*) &= \mathrm{ch}(-\pi_! S) + \mathrm{ch}(\pi_!(S \otimes K_C)) = \\ &\mathrm{ch}(-\pi_! S) + \pi_*(\mathrm{ch}(S \otimes K_C)\mathrm{Todd}(C)) = \\ &\mathrm{ch}(-\pi_! S) + \mathrm{ch}(\pi_! S) + (2g-2)\mathrm{ch}(S_p) = (2g-2)\mathrm{ch}(S_p), \end{aligned}$$

where K_C is the canonical sheaf on the curve C, hence

$$\left[\frac{1}{1 - te^x}\right]^{-\pi_! S \oplus (-\pi_! S^*)^*} = \left[\frac{1}{(1 - te^x)^{2g-2}}\right]^{S_p} = \frac{\exp(-c_1(S_p)(g-1))}{[(2\sinh(x/2))^{2g-2}]^{S_p}}.$$

Since

$$[-te^x]^{-\pi_! S} = (-1)^{\mathrm{rk}(-\pi_! S)} \exp(c_1(-\pi_! S))$$

and by the Grothendieck-Riemann-Roch theorem

$$\mathrm{ch}_1(-\pi_! S) = \mathrm{ch}_1(S_p)(g-1) - \mathrm{ch}_2(S)_{(2)},$$

we conclude that

$$[-te^x]^{-\pi_! S} = (-1)^{\mathrm{rk}(-\pi_! S)} \exp(c_1(S_p)(g-1)) \exp(-\mathrm{ch}_2(S)_{(2)}),$$

which finishes the proof of Lemma 2.5.10. □

Note that the last two terms in (2.34) are the sums of Chern characters of line bundles, so they contribute the multiplicative factor

$$\frac{\exp(\rho_u^\times(z', z''))}{w_u^\times(z', z'')}$$

to the equivariant class $E(N_{Z^0})^{-1}$; and using Lemma 2.5.10 with $S = \mathrm{Hom}(\tilde{U}'', \tilde{U}')$, we obtain that the inverse of the K-theoretical Euler class of the first term in (2.34) is

$$(-1)^{lr + r'r''(g-1)} w_u^\times(z', z'')^{2-2g} \exp(-\mathrm{ch}_2(\mathrm{Hom}(\tilde{U}'', \tilde{U}'))_{(2)}).$$

Note that

$$\begin{aligned} -\mathrm{ch}_2(\mathrm{Hom}(\tilde{U}'', \tilde{U}'))_{(2)} &= \frac{1}{2} c_2(\mathrm{End}_0(\tilde{U}' \oplus \tilde{U}''))_{(2)} - \frac{1}{2} c_2(\mathrm{End}_0(\tilde{U}'))_{(2)} - \frac{1}{2} c_2(\mathrm{End}_0(\tilde{U}''))_{(2)} \\ &= c_1\left(\mathcal{L}(r;0)\big|_{Z^0} \otimes \mathcal{L}_l(-r'; -l, ..., -l) \boxtimes \mathcal{L}_{-l}(-r''; l, ..., l)\right). \end{aligned}$$

The latter equality follows from Lemma 2.1.8. Finally, using Lemma 2.5.5 to calculate the Chern character of $\mathcal{L}(r;0)\big|_{Z^0}$, we obtain the formula for the class $E(N_{Z^0})^{-1}$, and the proof of the Lemma is complete. □

2.5.3. *The wall-crossing formula*

Putting Lemma 2.5.5 and Proposition 2.5.9 together, we obtain the following.

Proposition 2.5.11. *The wall-crossing term (2.30) is equal to*

$$\begin{aligned} K \operatorname*{Res}_{u=0} e^{(k\delta - rl)u} \int_{P_l(c') \times P_{-l}(c'')} & \big[(w_u^\times(z', z''))^{1-2g} \exp(\rho_u^\times(z', z''))\cdot \\ & \mathrm{ch}(\mathcal{L}_l(k + r''; \lambda'_1 - l, ..., \lambda'_{r'-1} - l, \lambda'_{r'} - l - k\delta + rl) \boxtimes \\ & \mathcal{L}_{-l}(k + r'; \lambda''_1 + l, ..., \lambda''_{r''-1} + l, \lambda''_{r''} + l + k\delta - rl)) \mathrm{Todd}(P_l(c') \times P_{-l}(c''))\big]\, du, \end{aligned}$$

where δ is a parameter depending on λ and the wall $S_{\Pi, l}$ (cf. Lemma 2.5.5) and K is the constant $(-1)^{lr + r'r''(g-1)} \frac{(r(k+r))^g}{(r'r'')^g}$.

Now all that is left to do is to perform the integral, using an induction on the rank based on Corollary 2.3.10. We will begin with the case $l=0$, as it is simpler. For $l=0$, the integral from Proposition 2.5.11 has the form

$$\int_{P_0(c')\times P_0(c'')} \big[w_u^\times(z',z'')^{1-2g} e^{\rho_u^\times(z',z'')}\mathrm{Todd}(P_0(c'))\mathrm{Todd}(P_0(c''))$$
$$\mathrm{ch}(\mathcal{L}_0(k+r'';\lambda_1',\dots,\lambda_{r'-1}',\lambda_{r'}'-k\delta)\boxtimes\mathcal{L}_0(k+r';\lambda_1'',\dots,\lambda_{r''-1}'',\lambda_{r''}''+k\delta))\big]. \quad (2.35)$$

The inductive hypothesis (2.17) maybe cast in the following form

$$\int_{P_0(c)} \mathrm{ch}(\mathcal{L}_0(k;\lambda))\mathrm{Todd}(P_0(c)) = \tilde{N}_{r,k}\sum_{\mathbf{B}\in\mathcal{D}} \operatorname*{iBer}_{\mathbf{B}}[\exp\langle\lambda,x/\hat{k}\rangle\cdot w_\Phi(x/\hat{k})^{1-2g}](\rho/\hat{k}-[c]_{\mathbf{B}}). \quad (2.36)$$

Now let us fix k, and allow to vary λ. We can extend this equality by linearity to arbitrary linear combinations of Chern characters of line bundles of the form

$$\sum_i \mathrm{ch}(\mathcal{L}_0(k;\lambda^i)) = \mathrm{ch}(\mathcal{L}_0(k;0))\cdot\sum_i \mathrm{ch}(\mathcal{L}_0(0;\lambda^i)).$$

Since any polynomial on V, up to a fixed degree may be represented as a linear combination of exponential functions of the form $\exp\langle\lambda,x/\hat{k}\rangle$, formula (2.36) may be generalized in the following way.

Lemma 2.5.12. *Let $G(x)$ be a formal power series on V, and denote by $G(z)$ the characteristic class in $H^*(P_0(c))$ obtained by the identification of functions on V and cohomology classes of $P_0(c)$, described before the equation (2.33). Then we have*

$$\int_{P_0(c)} \mathrm{ch}(\mathcal{L}_0(k;0))G(z)\mathrm{Todd}(P_0(c)) = \tilde{N}_{r,k}\cdot\sum_{\mathbf{B}\in\mathcal{D}} \operatorname*{iBer}_{\mathbf{B}}[G(x/\hat{k})\cdot w_\Phi^{1-2g}(x/\hat{k})](\rho/\hat{k}-[c]_{\mathbf{B}}). \quad (2.37)$$

Finally, let $\mathcal{D}'$ and $\mathcal{D}''$ be Hamiltonian bases (cf. §2.3.5). Since

$$w_{\Phi'}(x/\hat{k})w_{\Phi''}(x/\hat{k})w_u^\times(x/\hat{k}) = w_\Phi(x/\hat{k}),$$
$$\rho'(x/\hat{k})\rho''(x/\hat{k})\rho_u^\times(x/\hat{k}) = \rho(x/\hat{k}),$$

where $w_{\Phi'}, w_{\Phi''}$ and ρ',ρ'' are naturally defined for the root systems Φ' and Φ'' (cf. §2.3.5), the integral (2.35) is equal to

$$\tilde{N}_{r',k+r''}\tilde{N}_{r'',k+r'}\sum_{\mathbf{B}'\in\mathcal{D}'}\sum_{\mathbf{B}''\in\mathcal{D}''}\operatorname*{iBer}_{\mathbf{B}'}\operatorname*{iBer}_{\mathbf{B}''}[w_\Phi(x/\hat{k})^{1-2g}e^{\rho(x/\hat{k})}]$$
$$((\lambda_1',\dots,\lambda_{r'-1}',\lambda_{r'}'-k\delta)/\hat{k}-[c']_{\mathbf{B}'}+(\lambda_1'',\dots,\lambda_{r''-1}'',\lambda_{r''}''+k\delta)/\hat{k}-[c'']_{\mathbf{B}''}).$$

Identifying u (cf. (2.33)) with the "link" element of the diagonal basis $\mathcal{D}=(\alpha^{\phi(r'),r}\,\mathcal{D}'\,\mathcal{D}'')$ (cf. §2.3.5), and moving the factor $e^{k\delta u}$ from Proposition 2.5.11 inside the argument of iBer, we obtain the proof of the following theorem for $l=0$.

Theorem 2.5.13. *Let $c^{\pm} \in \Delta$ be in the neighbouring chambers; then the wall-crossing term*

$$\chi(P_0(c^+), \mathcal{L}_0(k;\lambda)) - \chi(P_0(c^-), \mathcal{L}_0(k;\lambda))$$

is equal to

$$(k+r)\tilde{N}_{r,k} \sum_{B' \in \mathcal{D}'} \sum_{B'' \in \mathcal{D}''} \operatorname*{Res}_{\alpha^{\phi(r'),r}=0} \operatorname*{iBer}_{B'} \operatorname*{iBer}_{B''} [w_{\Phi}(x/\widehat{k})^{1-2g}](\widehat{\lambda}/\widehat{k} - [c^+]_B)\, d\alpha^{\phi(r'),r},$$

where $\mathcal{D}'$ and $\mathcal{D}''$ are the diagonal bases of Φ' and Φ'' (cf. §2.3.5) correspondingly.

Remark 2.5.14. Note that this wall-crossing term coincides with the one from Proposition 2.3.16.

Example 6. It follows from Example 1 that in case of rank 3, the permutation $\phi \in \Sigma_3$ sends $(1,2,3)$ to $(1,3,2)$. Then $u = c_1(\mathcal{F}_1' \otimes \mathcal{F}_1''^*)$ and let $z = z_1'' - z_2'' = c_1(\mathcal{F}_2''/\mathcal{F}_1'' \otimes \mathcal{F}_1''^*)$. Then the inverse of the K-theoretical Euler class of the conormal bundle is (cf. Proposition 2.5.9)

$$\operatorname{ch}(\mathcal{L})e^{\frac{9\eta}{2}}e^{\frac{z}{2}}\left(2\sinh\left(\frac{u}{2}\right)2\sinh\left(\frac{z-u}{2}\right)\right)^{1-2g},$$

where $\mathcal{L} = \mathcal{L}_0(2;0,0)$ is a line bundle on the moduli space P_0 of rank-2 degree-0 stable parabolic bundles. The Chern character of the restriction of the line bundle $\mathcal{L}_0(k;\lambda_1,\lambda_2,\lambda_3)$ to Σ is

$$e^{\frac{3k\eta}{2}}\operatorname{ch}(\mathcal{L}_0^k)e^{\lambda_1 z+\lambda_2 u}.$$

Hence the wall-crossing term

$$\chi(P_0(<), \mathcal{L}_0(k,\lambda)) - \chi(P_0(>), \mathcal{L}_0(k,\lambda))$$

is equal to

$$-\left(\frac{3(k+3)}{2}\right)^g \operatorname*{Res}_{u=0}\frac{e^{\lambda_2 u}}{(2\sinh(\frac{u}{2}))^{2g-1}} \cdot \int_{P_0} \frac{\operatorname{ch}(\mathcal{L}_0(k+1;\lambda_1+\frac{1}{2},-\lambda_1-\frac{1}{2}))}{(2\sinh(\frac{z-u}{2}))^{2g-1}}\operatorname{Todd}(P_0)du.$$

The integral is the Euler charactersitics of a line bundle on a moduli space of degree-0 rank-2 stable parabolic bundles, so we can calculate it using the induction by rank. It is equal to

$$(-1)^{g-1}(2(k+3))^g \operatorname*{Res}_{z=0}\frac{e^{(\lambda_1+1)z}}{(2\sinh(\frac{z-u}{2})2\sinh(\frac{z}{2}))^{2g-1}(1-e^{(k+3)z})}dz,$$

so the wall-crossing term is

$$(-3(k+3)^2)^g \operatorname*{Res}_{u=0}\operatorname*{Res}_{z=0}\frac{e^{\lambda_1 z+\lambda_2 u+z}}{\tilde{w}_\phi(z,u)^{2g-1}(1-e^{(k+3)z})}dzdu,$$

where $\tilde{w}_\phi(z,u) = 2\sinh(\frac{z-u}{2})2\sinh(\frac{u}{2})2\sinh(\frac{z}{2})$. Note that this is exactly the same polynomial as in Example 5 after changing (z,u) to $(x,-y)$.

2.6. Tautological Hecke correspondences

If $l \neq 0$, then we need one more step in our proof, which uses the Hecke correspondence to calculate the wall-crossing term (2.30).

2.6.1. *The Hecke correspondence*

Given a rank-r degree-d vector bundle W with a full flag $0 \subsetneq F_1 \subsetneq \ldots \subsetneq F_r = W_p$ at p, one can obtain a rank-r degree-$d-1$ vector bundle W' with a full flag $0 \subsetneq G_1 \subsetneq \ldots \subsetneq G_r = W'_p$ using the tautological Hecke correspondence construction as follows.

The evaluation map $W \to W_p$ induces the short exact sequence of the associated sheaves of sections

$$0 \to \mathcal{W}' \xrightarrow{\tilde{\alpha}} \mathcal{W} \to W_p/F_{r-1} \to 0 \tag{2.38}$$

on curve C. Since $\mathcal{W}'$ is a kernel of $\tilde{\alpha}$, it is a locally free sheaf, thus gives a rank-r vector bundle W' over C with $\det(W') \simeq \det(W) \otimes \mathcal{O}(-p)$. The image of the associated morphism of vector bundles α at the point p is $F_{r-1} \subset W_p$, so $\alpha_p : W'_p \to W_p$ has a one-dimensional kernel $G_1 \subset W'_p$. Moreover, compositions of α_p with the quotient morphisms $F_{r-1} \to F_{r-1}/F_i$ induce a full flag of the corresponding kernels $G_1 \subsetneq \ldots \subsetneq G_{r-1} \subsetneq G_r = W'_p$ in W'_p.

Denote this operator between the sets of isomorphism classes of degree-d and $d-1$ vector bundles with a flag at p by

$$\mathcal{H} : (W, F_*) \mapsto (W', G_*).$$

Similarly, for any $m \geqslant 0$, one can define the operator $\mathcal{H}^m$ between the sets of isomorphism classes of degree-d and $d-m$ vector bundles with a flag at the point p by iterating the above construction m times. Clearly, these maps are independent of the parabolic weights.

Proposition 2.6.1. *Let $c \in \Delta$ be a regular (cf. page 7) point. Then the operator $\mathcal{H}$ induces an isomorphism between the moduli spaces $P_d(c_1, \ldots, c_r)$ and $P_{d-1}(c_2, \ldots, c_r, c_1 - 1)$.*

Proof. First, we need to show that if $W \in P_d(c_1, \ldots, c_r)$ is a parabolic stable bundle with parabolic weights $(c_1, \ldots, c_r)$, then W', its image under the Hecke operator $\mathcal{H}$, is parabolic stable with respect to parabolic weights $(c_2, \ldots, c_r, c_1 - 1)$. For this, consider the subbundle $V' \subset W'$ and let $\alpha(V') = V \subset W$ (cf. (2.38)) be its image. Since W is parabolic stable,

$$\mathit{parslope}(V) < \mathit{parslope}(W) = \mathit{parslope}(W').$$

We need to prove that $\mathit{parslope}(V') < \mathit{parslope}(W')$. There are two possible cases:

- If α maps V' to V isomorphically, then $\deg(V') = \deg(V)$ and $V_p \subset F_{r-1}$, hence $\mathit{parslope}(V') = \mathit{parslope}(V) < \mathit{parslope}(W')$.
- Otherwise, $\deg(V') = \deg(V) - 1$, and V_p is not contained in F_{r-1}, so one of the parabolic weights of V' is $c_1 - 1$. Then, as in the previous case, $\mathit{parslope}(V') = \mathit{parslope}(V)$, and the result follows.

To show that the map $\mathcal{H}$ is an isomorphism, note that $\mathcal{H}^r$ maps

$$P_d(c_1, c_2, \ldots, c_r) \to P_{d-r}(c_1 - 1, c_2 - 1, , \ldots, c_r - 1). \tag{2.39}$$

It is easy to check that given $\mathcal{W}$ and iterating the associated morphism of locally free sheaves of sections (2.38) r times, we obtain a subsheaf $\mathcal{W}' \subset \mathcal{W}$ of sections of W which vanishes at the point p. So the map (2.39) is just tensoring by $\mathcal{O}(-p)$, and hence it is an isomorphism. □

Now we can define an operator $\mathcal{H}^m$ for any $m \in \mathbb{Z}$, taking the inverse map if necessary. We will need the following statement, which follows from Proposition 2.6.1 and the construction of $\mathcal{H}^m$.

Corollary 2.6.2. *Let $m \geqslant 0$. Then under the isomorphism $\mathcal{H}^m$ the line bundle $\mathcal{L}_d(k;\lambda_1,\dots,\lambda_r)$ corresponds to the line bundle $\mathcal{L}_{d-m}(k;\lambda_{r-m+1},\dots,\lambda_r,\lambda_1-k,\dots,\lambda_{r-m}-k)$.*

2.6.2. *The effect of the Hecke correspondence on the integral*

Recall that our goal is to calculate the wall-crossing term from Proposition 2.5.11. For simplicity, we assume that l is positive (the other case is analogous). We apply the Hecke operators $\mathcal{H}^l$ and $\mathcal{H}^{-l}$ to the moduli spaces $P_l(c')$ and $P_{-l}(c'')$ to obtain

$$P'_0 = P_0(c'_{l+1},\dots,c'_{r'},c'_1-1,\dots,c'_l-1) \simeq P_l(c') \text{ and}$$

$$P''_0 = P_0(c''_{r''-l+1}+1,\dots,c''_{r''}+1,c''_1,\dots,c''_{r''-l}) \simeq P_{-l}(c'').$$

Recall (cf. page 10) that there is a natural action of the group Σ_r on V^*, and hence (cf. page 31) on $H^2(P_l(c')\times P_{-l}(c''))$. Let $\tau' \in \Sigma_{r'}$ and $\tau'' \in \Sigma_{r''}$ be the cyclic permutations defined by

$$\tau'\cdot(c'_1-1,\dots,c'_l-1,c'_{l+1},\dots,c'_{r'}) = (c'_{l+1},\dots,c'_{r'},c'_1-1,\dots,c'_l-1)$$

and

$$\tau''\cdot(c''_1,\dots,c''_{r''-l},c''_{r''-l+1}+1,\dots,c''_r+1) = (c''_{r''-l+1}+1,\dots,c''_{r''}+1,c''_1,\dots,c''_{r''-l}).$$

And set $\tau = (\tau',\tau'') \in \Sigma_{r'}\times\Sigma_{r''} \subset \Sigma_r$. Note that

$$\tau'\cdot(-l+r',\dots,-l+r',-l,\dots,-l) = \tau'\cdot\rho'-\rho'$$

and

$$\tau''\cdot(l,\dots,l,l-r'',\dots,l-r'') = \tau''\cdot\rho''-\rho'',$$

so applying the Hecke operator $\mathcal{H}^l\times\mathcal{H}^{-l}$ to the wall-crossing term from Proposition 2.5.11 and using Corollary 2.6.2, we obtain that the wall-crossing term (2.30) is equal to

$$\begin{aligned} K \operatorname*{Res}_{u=0} e^{(k\delta-rl)u} \int_{P'_0\times P''_0} & (\tau\cdot w_u^{\times}(z',z''))^{1-2g} e^{\tau\cdot\rho_u^{\times}(z',z'')} \\ & \mathrm{ch}(\mathcal{L}_0(k+r'';\tau'\cdot(\lambda'_1-\hat{k},\dots,\lambda'_l-\hat{k},\lambda'_{l+1},\dots,\lambda'_{r'-1},\lambda'_{r'}-k\delta+rl)) \\ & \mathrm{ch}(\mathcal{L}_0(k+r';\tau''\cdot(\lambda''_1,\dots,\lambda''_{r'-l},\lambda''_{r''-l+1}+\hat{k},\dots,\lambda''_{r''}+\hat{k}+k\delta-rl)) \\ & \quad e^{\tau''\cdot\rho''(z',z'')-\rho''(z',z'')} e^{\tau'\cdot\rho'(z',z'')-\rho'(z',z'')} \mathrm{Todd}(P'_0)\mathrm{Todd}(P''_0))\, du. \end{aligned} \tag{2.40}$$

As in §2.5.3, according to Lemma 2.5.12, we can calculate this integral using the induction on rank. Let $\mathcal{D}'$ and $\mathcal{D}''$ be two Hamiltonian diagonal bases. Then $\tau'(\mathcal{D}')$ and $\tau''(\mathcal{D}'')$ are also

Hamiltonian diagonal bases (cf. Remark 2.2.3) and the integral in (2.40) is equal to

$$(-1)^{lr}\tilde{N}_{r',k+r''}\tilde{N}_{r'',k+r'}\sum_{\mathbf{B}'\in\tau'(\mathcal{D}')}\sum_{\mathbf{B}''\in\tau''(\mathcal{D}'')}\operatorname*{iBer}_{\mathbf{B}'}\operatorname*{iBer}_{\mathbf{B}''}$$
$$[\tau\cdot w_u^{\times}(x/\hat{k})^{1-2g}(w_{\Phi'}(x/\hat{k})w_{\Phi''}(x/\hat{k}))^{1-2g}e^{\tau\cdot\rho(x/\hat{k})}]$$
$$(\tau'\cdot(\lambda'_1-\hat{k},...,\lambda'_l-\hat{k},\lambda'_{l+1},...,\lambda'_{r'-1},\lambda'_{r'}-k\delta+rl)/\hat{k}-[\tau'\cdot(c'_1-1,...,c'_l-1,c'_{l+1},...,c'_{r'})]_{\mathbf{B}'}+$$
$$\tau''\cdot(\lambda''_1,...,\lambda''_{r'-l},\lambda''_{r''-l+1}+\hat{k},...,\lambda''_{r''-1}+\hat{k},\lambda''_{r''}+\hat{k}+k\delta-rl)/\hat{k}-$$
$$[\tau''\cdot(c''_1,...,c''_{r'-l},c''_{r''-l+1}+1,...,c''_{r''}+1)]_{\mathbf{B}''}). \quad (2.41)$$

To arrive at Theorem 2.5.13, we need to make additional transformations of formula (2.41): first, we shift λ' and λ'', and then we apply Lemma 2.3.5 to eliminate the cyclic permutation τ.

Note that given an ordered basis $\mathbf{B}\in\mathcal{B}$ and an element $\nu\in V^*$ such that $\{\nu\}_{\mathbf{B}}=0$, for any weight $\lambda\in\Lambda$ and positive integer k one have

$$(\lambda+\hat{k}\nu)/\hat{k}-[c+\nu]_{\mathbf{B}}=\lambda/\hat{k}-[c]_{\mathbf{B}}. \quad (2.42)$$

In particular, to perform the shift of λ' in (2.41), we use the following equality for any $\mathbf{B}'\in\mathcal{D}'$:

$$(\lambda'_1-\hat{k},...,\lambda'_l-\hat{k},\lambda'_{l+1},...,\lambda'_{r'-1},\lambda'_{r'}-k\delta+rl)/\hat{k}-[(c'_1,...,c'_{r'-1},c'_{r'}-l)-(1,...,1,0,...0,-l)]_{\mathbf{B}'}=$$
$$(\lambda'_1,...,\lambda'_{r'-1},\lambda'_{r'}-k\delta+rl-l\hat{k})/\hat{k}-[(c'_1,...,c'_{r'-1},c'_{r'}-l)]_{\mathbf{B}'}, \quad (2.43)$$

which clearly remains true after changing $\mathcal{D}'$ to $\tau'(\mathcal{D}')$ and applying τ' to both sides of the equation. Similarly, shifting the last terms of (2.41) by $\tau''(0,...,0,-1,...-1,-1+l)$, we can rewrite (2.41) as

$$(-1)^{lr}\tilde{N}_{r',k+r''}\tilde{N}_{r'',k+r'}\sum_{\mathbf{B}'\in\tau'(\mathcal{D}')}\sum_{\mathbf{B}''\in\tau''(\mathcal{D}'')}\operatorname*{iBer}_{\mathbf{B}'}\operatorname*{iBer}_{\mathbf{B}''}$$
$$[\tau\cdot w_u^{\times}(x/\hat{k})^{1-2g}(w_{\Phi'}(x/\hat{k})w_{\Phi''}(x/\hat{k}))^{1-2g}e^{\tau\cdot\rho(x/\hat{k})}]$$
$$(\tau'\cdot(\lambda'_1,...,\lambda'_{r'-1},\lambda'_{r'}-k\delta+rl-l\hat{k})/\hat{k}-[\tau'\cdot(c'_1,...,c'_{r'-1},c'_{r'}-l)]_{\mathbf{B}'}+$$
$$\tau''\cdot(\lambda''_1,...,\lambda''_{r''-1},\lambda''_{r''}+k\delta-rl+l\hat{k})/\hat{k}-[\tau''\cdot(c''_1,...,c''_{r''-1},c''_{r''}+l)]_{\mathbf{B}''}).$$

Finally, identifying u (cf. (2.33)) with the "link" element of the diagonal basis $\tau(\mathcal{D})=(\alpha^{\tau\phi(r'),\tau(r)}\,\tau'(\mathcal{D}')\,\tau''(\mathcal{D}''))$ (cf. §2.3.5) and

- moving the factor $e^{(k\delta-rl)u}$ from (2.40) inside the argument of $\mathrm{iBer}_{\mathbf{B}}$, where $\mathbf{B}=(\alpha^{\tau\phi(r'),\tau(r)}\,\mathbf{B}'\,\mathbf{B}'')$,
- applying (2.42) with $\mathbf{B}=(\alpha^{\tau\phi(r'),\tau(r)}\,\mathbf{B}'\,\mathbf{B}'')$ and $\nu=l\alpha^{\tau\phi(r'),\tau(r)}$,
- applying Lemma 2.3.5,
- and using the fact that

$$\tau^{-1}\cdot(w_{\Phi'}(x/\hat{k})w_{\Phi''}(x/\hat{k}))=(-1)^{lr}w_{\Phi'}(x/\hat{k})w_{\Phi''}(x/\hat{k}),$$

we obtain the formula of Theorem 2.5.13 for arbitrary $l\in\mathbb{Z}$.

2.7. Affine Weyl symmetry and the proof of part I of Theorem 2.3.8

In this section, we prove certain symmetry properties of our Hilbert polynomials on the left hand side of (1.1), and we finish the proof of part I of Theorem 2.3.8. We start with the basic instance of symmetry of Hilbert polynomials: relative Serre duality.

2.7.1. Serre duality

Proposition 2.7.1. *Let $\mathcal{E} \to X$ be a rank 2 vector bundle over a smooth variety X, $\pi : Y = \mathbb{P}(\mathcal{E}) \to X$ its projectivization and $\omega_{X/Y}$ the relative cotangent line bundle. Then*

$$\chi(Y, \pi^*\mathcal{L} \otimes \omega_{X/Y}^m) = -\chi(Y, \pi^*\mathcal{L} \otimes \omega_{X/Y}^{-m+1})$$

for any line bundle $\mathcal{L} \in \mathrm{Pic}(X)$.

Proof. By Serre duality for families of curves [9, Ch III, §7-8] for any integer n

$$\chi(Y, \pi^*\mathcal{L} \otimes \mathcal{O}(n)) = -\chi(Y, \pi^*(\mathcal{L} \otimes (\wedge^2\mathcal{E})^{n+1}) \otimes \mathcal{O}(-n-2)). \tag{2.44}$$

Denote by $\Omega_{X/Y}$ the sheaf of relative differentials on Y; it follows from the short exact sequence

$$0 \to \Omega_{X/Y} \otimes \mathcal{O}_X \to \pi^*\mathcal{E}(-1) \to \mathcal{O}_X \to 0,$$

that

$$\omega_{X/Y} = \wedge^2(\pi^*\mathcal{E}(-1)) = \pi^*(\wedge^2\mathcal{E}) \otimes \mathcal{O}(-2).$$

Then the statement follows from (2.44) by substituting $n = -2m$. □

Now we can generalize this statement to the case of flag bundles.

Proposition 2.7.2. *Let $\pi : Y = \mathrm{Flag}(\mathcal{E}) \to X$ be a rank-r flag bundle over X. Let $\mathcal{L}$ be a line bundle on X, and $\mathcal{F}_1$, $\mathcal{F}_2/\mathcal{F}_1, ..., \mathcal{F}_r/\mathcal{F}_{r-1}$ the standard flag line bundles on Y. For $k \in \mathbb{Z}$ and $\lambda = (\lambda_1, ..., \lambda_r) \in \Lambda$ denote by*

$$\mathcal{L}(k;\lambda) = (\pi^*\mathcal{L})^k \otimes (\mathcal{F}_r/\mathcal{F}_{r-1})^{\lambda_1} \otimes (\mathcal{F}_{r-1}/\mathcal{F}_{r-2})^{\lambda_2} \otimes ... \otimes \mathcal{F}_1^{\lambda_r}.$$

Consider the polynomial

$$q(k;\lambda_1, \lambda_2, ..., \lambda_r) = \chi(Y, \mathcal{L}(k;\lambda_1, \lambda_2, ..., \lambda_r))$$

in $(k, \lambda) \in \mathbb{Z} \times \Lambda$ and extend this definition to $\mathbb{R} \times V^*$. Then $q(k;\lambda - \rho)$ is anti-invariant under the permutations of $\lambda_1, \lambda_2, ..., \lambda_r$.

Proof. For $1 \leqslant k < r$, let $\mathrm{Flag}_{\hat{k}}(\mathcal{E}) \to X$ be the flag bundle over X obtained from Y by forgetting the k-dimensional subspace. Then $Y \simeq \mathbb{P}(\mathcal{F}_{k+1}/\mathcal{F}_{k-1}) \to \mathrm{Flag}_{\hat{k}}(\mathcal{E})$ is a $\mathbb{P}^1$-bundle over $\mathrm{Flag}_{\hat{k}}(\mathcal{E})$, and thus applying Proposition 2.7.1 we obtain

$$\chi(Y, \mathcal{L}(k;\lambda_1, ..., \lambda_{r-k}, \lambda_{r-k+1}, ..., \lambda_r)) = -\chi(Y, \mathcal{L}(k;\lambda_1, ..., \lambda_{r-k+1} - 1, \lambda_{r-k} + 1, ..., \lambda_r)),$$

and the result follows. □

2.7.2. *The Weyl anti-symmetry of the functions* q_1 *and* q_{-1}

Armed with this statement, we are ready to take on the symmetries of the Hilbert polynomial of our parabolic moduli spaces. We note that the two sets $\Delta_{\pm 1}$ of weights for degree-± 1 stable parabolic bundles are simplices with one of their vertices at $(\frac{1}{r}, ..., \frac{1}{r})$ and $(\frac{-1}{r}, ..., \frac{-1}{r})$, correspondingly (cf. §2.1.2).

Denote by $N_{\pm 1}$ the moduli spaces of rank-r degree-± 1 stable vector bundles and by UN any universal bundle over $N_{\pm 1} \times C$ (cf. e.g [3]).

Lemma 2.7.3. *Let* $c = (c_1, ..., c_r)$ *be a parabolic weight from the chamber in* Δ_1*, which has as one of its vertices the (regular) point* $(\frac{1}{r}, ..., \frac{1}{r})$*. Then the moduli space* $P_1(c)$ *of rank-*r *degree-*1 *stable parabolic bundles is isomorphic to the flag bundle* $\mathrm{Flag}(UN_p)$ *over* N_1*. An analogous statement holds in the case of degree* -1 *and the point* $(\frac{-1}{r}, ..., \frac{-1}{r}) \in \Delta_{-1}$*.*

Proof. A simple calculation shows that the point $(c_1, ..., c_r) \in \Delta_1$, such that all $c_i > 0$, lies inside the chamber in Δ_1 with the vertex $(\frac{1}{r}, ..., \frac{1}{r})$. Hence it is enough to prove the first statement for the moduli space $P_1(c_1, ..., c_r)$ with positive parabolic weights.

Moreover, it is sufficient to show that if (W, F_*) is a parabolic stable vector bundle which represents a point in $P_1(c_1, ..., c_r)$, then W is stable as an ordinary bundle. Assume that W admits a proper subbundle W' with $\mathit{slope}(W') \geqslant \mathit{slope}(W) = \frac{1}{r}$, then $\deg(W') \geqslant 1$. Since all parabolic weights of W are positive, this implies that $\mathit{parslope}(W') > 0 = \mathit{parslope}(W)$, and therefore W is parabolic unstable. The proof for degree-(-1) bundles is analogous. □

Denote the moduli spaces described above by $P_1(>)$ and $P_{-1}(<)$, correspondingly, and their images under the Hecke isomorphisms $\mathcal{H}$ and $\mathcal{H}^{-1}$ by $P_0(>)$ and $P_0(<)$.

The following statement is straightforward (cf. Lemma 2.1.8).

Lemma 2.7.4. *The line bundles* $\mathcal{L}_1(r; 1, ..., 1)$ *and* $\mathcal{L}_{-1}(r; -1, ..., -1)$ *on* $P_1(>)$ *and* $P_{-1}(<)$ *defined in Lemma 2.1.6 may be obtained as pullbacks of the ample generators of the Picard groups* $\mathrm{Pic}(N_{\pm 1})$*.*

Example 7. In case of rank-3 parabolic bundles the moduli space $P_1(c_1, c_2, c_3)$ with $2c_3 > c_1 + c_2 - 1$ is a flag bundle over N_1 and it is isomorphic to the moduli space $P_0(>)$ from Example 1, while the moduli space $P_{-1}(c_1, c_2, c_3)$ with $2c_1 < c_2 + c_3 + 1$ is a flag bundle over N_{-1} and it is isomorphic to $P_0(<)$.

Now we establish the Weyl anti-symmetry of the polynomials

$$q_{-1}(k; \lambda_1, ..., \lambda_r) = \chi(P_0(<), \mathcal{L}_0(k; \lambda_1, ..., \lambda_r))$$

and

$$q_1(k; \lambda_1, ..., \lambda_r) = \chi(P_0(>), \mathcal{L}_0(k; \lambda_1, ..., \lambda_r))$$

defined on $\mathbb{R} \times \Lambda$, as in Proposition 2.7.2. Let $\tau \in \Sigma_r$ be the cyclic permutation, such that $\tau \cdot (c_1, ..., c_r) = (c_2, ..., c_r, c_1)$, and consider two points in V^*:

$$\theta_1[k] = \frac{k+r}{r} \cdot (1, 1, \ldots, 1) - (k+r)x_r - \rho = \tau \cdot (\frac{k}{r} - k, \frac{k}{r}, ..., \frac{k}{r}) - \tau \cdot (\rho) =$$
$$\left(\frac{k}{r} - \frac{r-1}{2} + 1, \frac{k}{r} - \frac{r-1}{2} + 2, ..., \frac{k}{r} - \frac{r-1}{2} + r - 1, -k + \frac{k}{r} - \frac{r-1}{2}\right)$$

and

$$\theta_{-1}[k] = -\frac{k+r}{r}\cdot(1,1,\dots,1) + (k+r)x_1 - \rho = \tau^{-1}\cdot(-\frac{k}{r},\dots,-\frac{k}{r},-\frac{k}{r}+k) - \tau^{-1}\cdot(\rho) = \left(k-\frac{k}{r}+\frac{r-1}{2},-\frac{k}{r}-\frac{r-1}{2},-\frac{k}{r}-\frac{r-1}{2}+1,\dots,-\frac{k}{r}-\frac{r-1}{2}+r-2\right).$$

Proposition 2.7.5. *The polynomials $q_1(k;\lambda+\theta_1[k])$ and $q_{-1}(k;\lambda+\theta_{-1}[k])$ are anti-invariant under the action of the Weyl group by permutations of $\lambda_1,\dots,\lambda_r$.*

Proof. Recall that the moduli space $P_0(>)$ is isomorphic to the flag bundle $P_1(>)$ over N_1 under the Hecke isomorphism $\mathcal{H}^{-1}$. Then using Corollary 2.6.2, Proposition 2.7.2 and Lemma 2.7.4, for any permutation $\sigma\in\Sigma_r$ we obtain

$$\begin{aligned} q_1(k;\sigma\cdot\lambda+\theta_1[k]) &\overset{\text{def}}{=} \chi(P_0(>),\mathcal{L}_0(k;\sigma\cdot\lambda+\theta_1[k])) \overset{2.6.2}{=} \\ &\chi(P_1(>),\mathcal{L}_1(k;\tau^{-1}\cdot\sigma\cdot\lambda+(\frac{k}{r},\dots,\frac{k}{r})-\rho)) \overset{2.7.2\,2.7.4}{=} \\ &(-1)^{\sigma}\chi(P_1(>),\mathcal{L}_1(k;\tau^{-1}\cdot\lambda+(\frac{k}{r},\dots,\frac{k}{r})-\rho)) \overset{2.6.2}{=} \\ &(-1)^{\sigma}\chi(P_0(>),\mathcal{L}_0(k;\lambda+\theta_1[k])) \overset{\text{def}}{=} (-1)^{\sigma}q_1(k;\lambda+\theta_1[k]). \end{aligned}$$

The proof for q_{-1} is similar. □

The two group actions in Proposition 2.7.5 may be combined in the following manner. For $k\geqslant 0$, we define an action of the *affine Weyl group* $\Sigma\rtimes\Lambda$ on $\Lambda\times\mathbb{Z}$, which acts trivially on the second factor, the level, and the action at level k is given by setting

$$\sigma.\lambda = \sigma\cdot(\lambda+\rho)-\rho \quad\text{and}\quad \gamma.\lambda = \lambda+(k+r)\gamma \text{ for } \sigma\in\Sigma,\ \gamma\in\Lambda.$$

We denote the resulting group of affine-linear transformations of V^* by $\widetilde{\Sigma}[k]$, and note that the action is defined in such a way that

$$\sigma.\lambda+\rho = \sigma\cdot(\lambda+\rho) \text{ and } (\gamma.\lambda+\rho)/\hat{k} = \gamma+(\lambda+\rho)/\hat{k}. \tag{2.45}$$

It is easy to verify that the stabilizer subgroup

$$\Sigma_r^+ \overset{\text{def}}{=} \mathrm{Stab}(\theta_1[k],\widetilde{\Sigma}[k]) \subset \widetilde{\Sigma}[k]$$

is generated by the transpositions $s_{i,i+1}$, $1\leqslant i\leqslant r-2$ and the reflection $\alpha^{r-1,r}\circ s_{r-1,r}$; similarly,

$$\Sigma_r^- \overset{\text{def}}{=} \mathrm{Stab}(\theta_{-1}[k],\widetilde{\Sigma}[k]) \subset \widetilde{\Sigma}[k]$$

is generated by $s_{i,i+1}$, $2\leqslant i\leqslant r-1$ and the reflection $\alpha^{1,2}\circ s_{1,2}$.

Then Proposition 2.7.5 maybe recast in the following form: the polynomial $q_1(k;\lambda)$ is anti-invariant with respect to the copy Σ_r^+ of the symmetric group Σ_r, while $q_{-1}(k;\lambda)$ is anti-invariant with respect to the copy Σ_r^- of the symmetric group Σ_r.

The following statement is straightforward:

Lemma 2.7.6. *Both subgroups $\Sigma_r^{\pm}$ are isomorphic to Σ_r and for $r>2$, the two subgroups generate the affine Weyl group $\widetilde{\Sigma}[k]$.*

2.7.3. *The Weyl anti-symmetry of the polynomials* p_1 *and* p_{-1}

Following (2.17), we define the two polynomials

$$p_{\pm 1}(k;\lambda) = \sum_{\mathbf{B}\in\mathcal{D}} \operatorname{iBer}_{\mathbf{B}}[w_\Phi^{1-2g}(x/\hat{k})](\hat{\lambda}/\hat{k} - [\theta_{\pm 1}]_{\mathbf{B}}),$$

where $\theta_1 = \frac{1}{r}\cdot(1,1,\dots,1) - x_r$, and $\theta_{-1} = -\frac{1}{r}\cdot(1,1,\dots,1) + x_1$.

Proposition 2.7.7. *The polynomial* $p_1(k;\lambda)$ *is anti-invariant with respect to* Σ_r^+*, and* $p_{-1}(k;\lambda)$ *is anti-invariant with respect to* Σ_r^-*.*

Proof. We recall that the points $\theta_{\pm 1}[k]$ are the fixed points of the actions of $\Sigma^\pm$, and clearly $\lim_{k\to\infty}\theta_{\pm 1}[k]/k = \theta_{\pm 1}$. This means, that we can fix a small open ball $D \subset V^*$ centered at θ_1 such that

$$\lambda/k \in D \Longrightarrow \forall \sigma \in \Sigma^+ : \ (\sigma.\lambda + \rho)/\hat{k} \sim \theta_1. \tag{2.46}$$

Then for $\lambda/k \in D$ we have

$$p_1(k;\lambda) = \sum_{\mathbf{B}\in\mathcal{D}} \operatorname{iBer}_{\mathbf{B}}[w_\Phi^{1-2g}(x/\hat{k})](\{\hat{\lambda}/\hat{k}\}_{\mathbf{B}}).$$

Now, let us consider a generator of Σ^+ of the type $\sigma = s_{i,i+1}$, $1 \leqslant i \leqslant r-2$. Using (2.45), and Lemma 2.3.5, and the fact that $\sigma \cdot w_\Phi = -w_\Phi$ we obtain

$$p_1(k;\sigma.\lambda) = \sum_{\mathbf{B}\in\mathcal{D}} \operatorname{iBer}_{\mathbf{B}}[w_\Phi^{1-2g}(x/\hat{k})](\sigma\cdot\{\hat{\lambda}/\hat{k}\}_{\mathbf{B}}) =$$
$$\sum_{\mathbf{B}\in\mathcal{D}} \operatorname{iBer}_{\mathbf{B}}[(-w_\Phi)^{1-2g}(x/\hat{k})](\{\hat{\lambda}/\hat{k}\}_{\mathbf{B}}) = -p_1(k;\lambda)$$

The case of the last generator $\alpha^{r-1,r} \circ s_{r-1,r}$ is similar, but after the substitution, we need to use the equality $\{\alpha^{r-1,r} + \hat{\lambda}/\hat{k}\}_{\mathbf{B}} = \{\hat{\lambda}/\hat{k}\}_{\mathbf{B}}$ to obtain $p_1(k;\hat{k}\alpha^{r-1,r} + s_{r-1,r}.\lambda) = -p_1(k;\lambda)$. □

2.7.4. *Proof of part I. of Theorem 2.3.8*

Recall that in Lemma 2.3.1, we introduced a chamber structure on $\Delta \subset V^*$ created by the walls $S_{\Pi,l}$, where $\Pi = (\Pi',\Pi'')$ is a nontrivial partition, and $l \in \mathbb{Z}$. Before we proceed, we introduce some extra notation. Denote by

$$\check{\Delta} = \{(k;a) |\, a/k \in \Delta\} \subset \mathbb{R}^{>0} \times V^*$$

the cone over $\Delta \subset V^*$, and let

$$\check{\Delta}^{\mathrm{reg}} = \{(k;a) |\, a/k \in \Delta \text{ is regular}\} \subset \check{\Delta}$$

be the set of its regular points. Denote by $\check{S}_{\Pi,l} \subset \check{\Delta}$ the cone over the wall $S_{\Pi,l} \subset \Delta$; then $\check{\Delta}^{\mathrm{reg}}$ is the complement of the union of walls $\check{S}_{\Pi,l}$ in $\check{\Delta}$. Finally, denote by $\check{\Delta}^{\mathrm{reg}}_{\Lambda}$ the intersection of the lattice $\mathbb{Z}^{>0} \times \Lambda$ with $\check{\Delta}^{\mathrm{reg}}$.

By substituting $\varsigma = \lambda/k$, we can consider the left-hand side and the right-hand side of formula I. of Theorem 2.3.8 as functions in $(k, \lambda) \in \check{\Delta}_{\Lambda}^{\mathrm{reg}}$. We denote by $q(k;\lambda)$ and $p(k;\lambda)$ the left-hand side and the right-hand side, correspondingly.

We showed that $q(k;\lambda)$ and $p(k;\lambda)$ are *polynomials* on the cone over each chamber in Δ (cf. Theorem 2.3.4, §2.1.4). We proved that the wall-crossing terms, i.e. the differences between polynomials on neighbouring chambers, for $q(k;\lambda)$ (cf. Theorem 2.5.13) and for $p(k;\lambda)$ (cf. Proposition 2.3.16) coincide, hence there exists a polynomial $\Theta(k;\lambda)$ on $\mathbb{Z}^{>0} \times \Lambda$, such that the restriction of $\Theta(k;\lambda)$ to $\check{\Delta}_{\Lambda}^{\mathrm{reg}}$ is equal to the difference $p(k;\lambda) - q(k;\lambda)$.

Now for $r > 2$, we can conclude that

$$\Theta(k;\lambda) = p_1(k;\lambda) - q_1(k;\lambda) = p_{-1}(k;\lambda) - q_{-1}(k;\lambda),$$

where $p_{\pm 1}(k;\lambda)$ and $q_{\pm 1}(k;\lambda)$ are the restrictions of $p(k;\lambda)$ and $q(k;\lambda)$ to two specific chambers defined in §2.7.3 and §2.7.2. Then, according to Propositions 2.7.5 and 2.7.7, the polynomial $\Theta(k;\lambda)$ is anti-invariant with respect to the action of the subgroups $\Sigma_r^{\pm}$, and hence by Lemma 2.7.6, it is anti-invariant under the action of the entire affine Weyl group $\widetilde{\Sigma}[k]$. It is easy to see that any such polynomial function has to vanish, and thus $p(k;\lambda) = q(k;\lambda)$, and this completes the proof of part I. of Theorem 2.3.8 for the case when $\lambda/k \in \Delta$ is regular.

As in Corollary 2.3.10, we can extend $p(k;\lambda)$ from the interior of each chamber to its boundary by polynomiality. Clearly, to prove part I. of Theorem 2.3.8 for the cases when λ/k is not regular, it is sufficient to show, that these extensions from the chambers containing λ/k in their closure give the same value on $(k;\lambda)$. It follows from Remark 2.9.4, that this is the case, and this completes the proof of part I. of Theorem 2.3.8 (cf. Remark 2.3.9).

2.8. Rank 2, two points

Unfortunately, the argument above does not work for $r = 2$, because, in this case, $\theta_1[k] = \theta_{-1}[k]$, the groups Σ_r^- and Σ_r^+ coincide, and thus they do not generate the entire affine Weyl group. The way out is to pass to the 2-punctured case.

2.8.1. Wall-crossing

We will thus fix two points: $p, s \in C$, and study the moduli space of rank-2, stable parabolic bundles W with fixed determinant isomorphic to $\mathcal{O}(pd)$, with parabolic structure given by a line $F_1 \subset W_p$ with weight $(c, -c)$, and a line $G_1 \subset W_s$ with weight $(a, -a)$.

Now we need to repeat the analysis of our work so far in this somewhat simpler case; some details thus will be omitted.

Set $d = 0$; then the space of admissible weights (cf. Figure 2.5) is a square

$$\square = \{(c, a) \,|\, 1 > 2c > 0,\, 1 > 2a > 0,\, \},$$

which has two adjacent chambers defined by the conditions

$$c > a \quad \text{and} \quad c < a.$$

Denote the corresponding moduli spaces by $P_0(c > a)$ and $P_0(c < a)$.

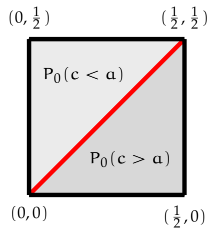

Figure 2.5 – The space of admissible weights in the case of rank $r=2$, two points.

Again, we have universal bundles over $P_0(c>a)\times C$ and $P_0(c<a)\times C$, which we will denote by the same symbol U; this bundle is endowed with two flags, $\mathcal{F}_1\subset\mathcal{F}_2=U_p$ and $\mathcal{G}_1\subset\mathcal{G}_2=U_s$. For $\mu,\lambda\in\mathbb{Z}$, we introduce the line bundle

$$\mathcal{L}(k;\lambda,\mu)=\det(U_p)^{k(1-g)}\otimes\det(\pi_*(U))^{-k}\otimes(\mathcal{F}_2/\mathcal{F}_1)^{\lambda}\otimes(\mathcal{F}_1)^{-\lambda}\otimes(\mathcal{G}_2/\mathcal{G}_1)^{\mu}\otimes(\mathcal{G}_1)^{-\mu}.$$

We repeat the construction of the master space from Section 5.1, choosing a point (c^0,c^0) on the wall and two points

$$(c,a)^{\pm}=(c^0,c^0)\pm\epsilon(1,0)\in\square,\quad \epsilon\in\mathbb{Q}_{>0}$$

from the adjacent chambers. We can identify the fixed point set Z^0 as follows.

Lemma 2.8.1. *The locus Z^0 defined in Proposition 2.4.2 is*

$$Z^0\simeq \mathrm{Jac}^0\simeq\{V=L\oplus L^{-1}\,|\,L_s=F_1,L_p^{-1}=G_1\}.$$

As in §2.5.1, denote by $\mathcal{J}$ the universal bundle over $\mathrm{Jac}^0\times C$ normalized in such a way that $c_1(\mathcal{J})_{(0)}=0$ (cf. (2.3)). Define

$$\eta\in H^2(\mathrm{Jac})\quad\text{by}\quad\left(\sum_i c_1(\mathcal{J})_{(e_i)}\otimes e_i\right)^2=-2\eta\otimes\omega;$$

we have then $\int_{\mathrm{Jac}}e^{\eta m}=m^g$ for $m\in\mathbb{Z}$.

Let $\pi:\mathrm{Jac}^0\times C\to\mathrm{Jac}^0$ be the projection and N_{Z^0} be the equivariant normal bundle to Z^0 in Z. Then, as in Lemma 2.5.6, Proposition 2.5.9 and Lemma 2.5.5, we obtain the identifications:

- $N_{Z^0}=R^1_T\pi_*(\mathrm{ParHom}(\mathcal{J},\mathcal{J}^{-1}))\oplus R^1_T\pi_*(\mathrm{ParHom}(\mathcal{J}^{-1},\mathcal{J}))$, where $T\simeq\mathbb{C}^*$-action has weights (-1,1);
- $E(N_{Z^0})^{-1}=(-1)^g(2\sinh(\frac{u}{2}))^{-2g}\exp(4\eta)$;
- $\mathrm{ch}_T(\mathcal{L}(k;\lambda,\mu)|_{Z^0})=\exp(2k\eta)\exp(u(\lambda-\mu))$.

Now we define the polynomials:

$$h_>(k;\lambda,\mu)\stackrel{\text{def}}{=}\chi(P_0(c>a),\mathcal{L}(k;\lambda,\mu)),\ h_<(k;\lambda,\mu)\stackrel{\text{def}}{=}\chi(P_0(c<a),\mathcal{L}(k;\lambda,\mu)).$$

and, applying Theorem 2.4.7, we obtain the following expression for their difference.

Lemma 2.8.2. *The wall-crossing term equals*

$$h_>(k;\lambda,\mu)-h_<(k;\lambda,\mu)=\ (-1)^g(2k+4)^g\operatorname*{Res}_{u=0}\frac{\exp(u(\lambda-\mu))}{(2\sinh(\frac{u}{2}))^{2g}}\,du.$$

2.8.2. Symmetry

Denote by $P_{-1}(c > a)$ the image of the moduli space $P_0(c > a)$ under the Hecke isomorphism $\mathcal{H}$ (cf. §2.6) at the point p and by $P_{-1}(c < a)$ the image of the moduli space $P_0(c < a)$ under the Hecke isomorphism $\mathcal{H}$ at the point s.

We have the following analogue of Lemma 2.7.3.

Lemma 2.8.3. *Denote by N_{-1} the moduli space of rank-2 degree-(-1) stable bundles on C and by $\mathcal{U}N$ any universal bundle over $N_{-1} \times C$. Then the moduli spaces $P_{-1}(c > a)$ and $P_{-1}(c < a)$ are isomorphic to the bundle $\mathbb{P}(\mathcal{U}N_p) \times \mathbb{P}(\mathcal{U}N_s)$ over N_{-1}.*

Denote by $\mathcal{T}[p]$ and $\mathcal{T}[s]$ the vertical tangent lines of $\mathbb{P}(\mathcal{U}N_p)$ and $\mathbb{P}(\mathcal{U}N_s)$, respectively, and by $\mathcal{L}_{-1}$ the pullback of the ample generator of the Picard group of N_{-1} to $\mathbb{P}(\mathcal{U}N_p) \times \mathbb{P}(\mathcal{U}N_s)$ (cf. Lemma 2.7.4). Then a simple calculation shows the following.

Lemma 2.8.4. *Under the Hecke isomorphism $\mathcal{H}$ at p, the line bundle $\mathcal{L}(2k; \lambda, \mu)$ on $P_0(c > a)$ corresponds to the line bundle $\mathcal{L}_{-1}^k \otimes \mathcal{T}[p]^{-\lambda+k} \otimes \mathcal{T}[s]^{\mu}$ on $P_{-1}(c > a)$.*
Under the Hecke isomorphism $\mathcal{H}$ at the point s, $\mathcal{L}(2k; \lambda, \mu)$ on $P_0(c < a)$ corresponds to the line bundle $\mathcal{L}_{-1}^k \otimes \mathcal{T}[p]^{\lambda} \otimes \mathcal{T}[s]^{-\mu+k}$ on $P_{-1}(c < a)$.

As in §2.7.2, applying Serre duality for families of curves (cf. Proposition 2.7.2) to the line bundles on the two $\mathbb{P}^1 \times \mathbb{P}^1$ bundles over N_{-1}, we obtain that the polynomials $h_>(k; \lambda, \mu)$ and $h_<(k; \lambda, \mu)$ are anti-invariant under the action of the Weyl group $\Sigma_2 \times \Sigma_2$ with the center at $\theta_1[k] = (\frac{k+1}{2}, \frac{-1}{2})$ and $\theta_2[k] = (\frac{-1}{2}, \frac{k+1}{2})$, correspondingly (cf. Figure 2.6). In other words, we obtain the following 4 identities.

Lemma 2.8.5.

$$h_>(k; \lambda, \mu) = -h_>(k; \lambda, -\mu - 1) = -h_>(k; -\lambda + k + 1, \mu);$$
$$h_<(k; \lambda, \mu) = -h_<(k; -\lambda - 1, \mu) = -h_<(k; \lambda, -\mu + k + 1).$$

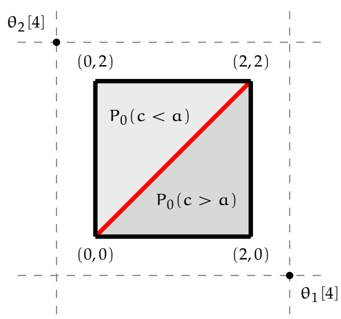

Figure 2.6 – $k = 4$, $r = 2$, two points.

Now, define the polynomials

$$\widetilde{h}_>(k; \lambda, \mu) = (-1)^{g-1}(2k+4)^g \operatorname*{Res}_{u=0} \frac{\exp(u(\lambda+\mu+1)) - \exp(u(\lambda-\mu))}{(2\sinh\left(\frac{u}{2}\right))^{2g}(1 - e^{u(k+2)})} du;$$

$$\widetilde{h}_<(k; \lambda, \mu) = (-1)^{g-1}(2k+4)^g \operatorname*{Res}_{u=0} \frac{\exp(u(\lambda+\mu+1)) - \exp(u(\lambda-\mu+k+2))}{(2\sinh\left(\frac{u}{2}\right))^{2g}(1 - e^{u(k+2)})} du,$$

and from here we can follow the logic of the proof of part I of Theorem 2.3.8.

Proposition 2.8.6. *The polynomials introduced above, in fact, coincide:*

$$h_>(k;\lambda,\mu) = \widetilde{h}_>(k;\lambda,\mu) \quad \textit{and} \quad h_<(k;\lambda,\mu) = \widetilde{h}_<(k;\lambda,\mu).$$

Proof. It is a simple exercise to show that $\widetilde{h}_>(k;\lambda,\mu)$ and $\widetilde{h}_<(k;\lambda,\mu)$ satisfy the identities appearing in Lemmas 2.8.2 and 2.8.5, and hence the polynomial

$$\Theta(k;\lambda,\mu) = h_>(k;\lambda,\mu) - \widetilde{h}_>(k;\lambda,\mu) = h_<(k;\lambda,\mu) - \widetilde{h}_<(k;\lambda,\mu)$$

satisfies all four Σ_2-symmetries listed in Lemma 2.8.5. These groups together generate a double action of the affine Weyl group $\widetilde{\Sigma}$ in λ and μ separately, and this implies the vanishing of Θ. □

As $P_0(c > a)$ is a $\mathbb{P}^1$-bundle over the moduli space of rank-2 degree-0 stable parabolic bundles $P_0(c,-c)$, substituting $\mu = 0$ in $\widetilde{h}_>$, we obtain the Verlinde formula for rank 2.

Corollary 2.8.7.

$$\chi(P_0(c,-c),\mathcal{L}_0(k;\lambda)) = (-1)^{g-1}(2k+4)^g \operatorname*{Res}_{u=0} \frac{\exp(u(\lambda+\frac{1}{2}))}{(2\sinh\left(\frac{u}{2}\right))^{2g-1}(1-e^{u(k+2)})} du.$$

2.9. The combinatorics of the $[Q,R]=0$

In this section, we give a proof of the second part of Theorem 2.3.8. Let $\lambda/k \in \Delta$, and fix a regular element $\varsigma \in \Delta$ in a chamber containing λ/k in its closure, and another regular element $\widehat{\varsigma} \in \Delta$ containing $\widehat{\lambda}/\widehat{k}$ in its closure. Our goal is to prove the the equality $p_\varsigma(k;\lambda) = p_{\widehat{\varsigma}}(k;\lambda)$, where we define

$$p_c(k;\lambda) = \tilde{N}_{r,k} \sum_{\mathbf{B}\in\mathcal{D}} \operatorname{iBer}_{\mathbf{B}}[w_\Phi^{1-2g}(x/\widehat{k})](\widehat{\lambda}/\widehat{k} - [c]_{\mathbf{B}}) \tag{2.47}$$

for a regular $c \in \Delta$ and diagonal basis $\mathcal{D}$. This is a subtle statement, which is a combinatorial-geometric projection of the idea of quantization commutes with reduction (or $[Q,R]=0$ for short, cf. [15, 23]).

If $\lambda/k \sim \widehat{\lambda}/\widehat{k}$, i.e. when λ/k and $\widehat{\lambda}/\widehat{k}$ are regular elements in the same chamber in Δ, then $p_\varsigma(k;\lambda) = p_{\widehat{\varsigma}}(k;\lambda)$ is a tautology. We assume thus that this is not the case, and denote by $\mathcal{S}(k,\lambda)$ the set of walls separating ς and $\widehat{\varsigma}$, or containing either λ/k or $\widehat{\lambda}/\widehat{k}$ or both. Equivalently, the wall $S_{\Pi,l}$ belongs to $\mathcal{S}(k,\lambda)$ if

$$(\lambda/k)_{\Pi'} \geqslant l \geqslant (\widehat{\lambda}/\widehat{k})_{\Pi'} \text{ or } (\lambda/k)_{\Pi'} \leqslant l \leqslant (\widehat{\lambda}/\widehat{k})_{\Pi'},$$

where $c_{\Pi'} = \sum_{i\in\Pi'} c_i$ for an element $c = (c_1,\dots,c_r) \in V^*$. Clearly, there is a path in Δ connecting ς and $\widehat{\varsigma}$, which intersects only walls from $\mathcal{S}(k,\lambda)$ in a generic points. Then to prove the equality $p_\varsigma(k;\lambda) = p_{\widehat{\varsigma}}(k;\lambda)$, it is enough to show the following, at first sight somewhat surprising fact.

Proposition 2.9.1. *Assume $g \geqslant 1$, $\lambda/k \in \Delta$, $S_{\Pi,l} \in \mathcal{S}(k,\lambda)$ and let $c^\pm \in \Delta$ be two points in two neighboring chambers separated by the wall $S_{\Pi,l}$. Then*

$$p_{c^+}(k;\lambda) = p_{c^-}(k;\lambda). \tag{2.48}$$

Proof. The difference of the two sides of (2.48) is expressed as a residue in (2.29). The integral in (2.29) is a rational expression in the variable t, and our plan is to show by degree count in t and t^{-1} that its residues at zero and at ∞ vanish. We define the degree of the quotient of two polynomials $R = P/Q$ of the variable t as $\deg_t(R) = \deg_t(P) - \deg_t(Q)$, and we set $\deg_{t^{-1}}(R) = \deg_t(R(t^{-1}))$. Then, clearly,

$$\deg_t(R) < 0 \Longrightarrow \operatorname*{Res}_{t=\infty} R\,\frac{dt}{t} = 0 \quad \text{and} \quad \deg_{t^{-1}}(R) < 0 \Longrightarrow \operatorname*{Res}_{t=0} R\,\frac{dt}{t} = 0.$$

A convenient expression for (2.29) will be (2.40), where we change variables via $t = e^u$. In what follows, we will always tacitly assume this substitution, and we will write, for example, $\deg_{t^{\pm 1}}(1/(e^u - e^{-u})) = -1$. We thus obtain a formula of the form $\mathrm{Res}_{t=0,\infty} f(t)\,dt/t$, and to show that this is zero, it is sufficient to show that $\deg_t(f) < 0$ and $\deg_{t^{-1}}(f) < 0$.

Now we observe that the variable u occurs only in the first line of (2.40), and thus, calculating the degrees in t and t^{-1} separately, we obtain the following formula:

$$\deg_{t^{\pm 1}}(f) = \pm(k\delta - rl) + (1-2g)\deg_{t^{\pm 1}}(\tau \cdot w_u^\times) + \deg_{t^{\pm 1}}(\exp(\tau \cdot \rho_u^\times)). \tag{2.49}$$

Recall that here δ represents the distance of λ/k from the wall $S_{\Pi,l}$, while $w_u^\times$ and $\rho_u^\times$, represent the parts of the Weyl denominator and the ρ-shift corresponding to roots connecting Π' and Π'', respectively.

We begin the study of this expression with some simple remarks. We recall that the permutation τ preserves the partition $\Pi = (\Pi', \Pi'')$, and thus we have

$$\deg_{t^{\pm 1}}(\tau \cdot w_u^\times) = \deg_{t^{\pm 1}}(w_u^\times) = \frac{r'r''}{2}.$$

Using, in addition, that $\rho_u^\times$ is linear in u, we obtain

$$\deg_t(\exp(\tau \cdot \rho_u^\times)) = -\deg_{t^{-1}}(\exp(\tau \cdot \rho_u^\times)) = \deg_t(\exp(\rho_u^\times)).$$

Combining these equalities, and assuming $g \geqslant 1$, we arrive at the following conclusion.

Lemma 2.9.2. *The inequality*

$$\left|(k\delta - rl) + \deg_t(\exp(\rho_u^\times))\right| < \frac{r'r''}{2} \tag{2.50}$$

implies the vanishing of the wall-crossing term: equality (2.48).

Before we proceed, we introduce some notation. Denote by

$$\mathrm{Inv}(\Pi) = \{(i,j) \,|\, \Pi' \ni i > j \in \Pi''\}$$

the set of "inverted" pairs of elements of the partition Π. The number of these pairs $|\mathrm{Inv}(\Pi)|$ coincides with the standard notion of length of the shuffle permutation $\phi \in \Sigma_r$ introduced in §2.5.

Each pair (i,j) which is not inverted contributes $+u/2$ to $\rho_u^\times$, while each inverted pair contributes $-u/2$, and thus we have

$$\deg_t(\exp(\rho_u^\times)) = \frac{r'r''}{2} - |\mathrm{Inv}(\Pi)|. \tag{2.51}$$

Also, recall the notation $c_{\Pi'} = \sum_{i\in\Pi'} c_i$ for an element $c = (c_1, ..., c_r) \in V^*$; in particular, we have $(\lambda/k)_{\Pi'} = l + \delta$ and

$$\rho_{\Pi'} = \sum_{i\in\Pi'} \frac{r+1}{2} - i.$$

The following is a simple exercise, whose proof will be omitted:

$$\deg_t(\exp(\rho_u^\times)) = \rho_{\Pi'}. \tag{2.52}$$

Now we come to a key point of our argument.

Lemma 2.9.3. *If the intersection of the wall $S_{\Pi,l}$ with Δ is non-empty, then*

$$-\frac{r'r''}{2} < lr - \rho_{\Pi'} < \frac{r'r''}{2}. \tag{2.53}$$

Proof. Pick a point $c = (c_1, ..., c_r)$ in the intersection $S_{\Pi,l} \cap \Delta$, and recall that for any $1 \leqslant i < j \leqslant r$, we have $0 < c_i - c_j < 1$, and

$$\sum_{i\in\Pi'} c_i = -\sum_{i\in\Pi''} c_i = l.$$

Then

$$-|\mathrm{Inv}(\Pi)| < \sum_{(i,j)\in\mathrm{Inv}(\Pi)} (c_i - c_j) \leqslant \sum_{i\in\Pi'} \sum_{j\in\Pi''} (c_i - c_j),$$

and, similarly,

$$\sum_{i\in\Pi'} \sum_{j\in\Pi''} (c_i - c_j) < r'r'' - |\mathrm{Inv}(\Pi)|.$$

Now, since

$$\sum_{i\in\Pi'} \sum_{j\in\Pi''} (c_i - c_j) = r'' \sum_{i\in\Pi'} c_i - r' \sum_{j\in\Pi''} c_j = lr,$$

we can conclude

$$-|\mathrm{Inv}(\Pi)| < lr < r'r'' - |\mathrm{Inv}(\Pi)|.$$

In view of (2.51) and (2.52), these inequalities are equivalent to (2.53), and this completes the proof. □

Now we are ready to prove (2.48). The condition $S_{\Pi,l} \in \mathcal{S}(k, \lambda)$, i.e. that $S_{\Pi,l}$ separates λ/k and $\widehat{\lambda}/\widehat{k}$ or contains λ/k or $\widehat{\lambda}/\widehat{k}$, may occur in two ways.

- $(\lambda/k)_{\Pi'} \geqslant l \geqslant (\widehat{\lambda}/\widehat{k})_{\Pi'}$, which is equivalent to the two inequalities: $\delta \geqslant 0$ and $lk + lr \geqslant \lambda_{\Pi'} + \rho_{\Pi'}$. After canceling lk and reordering the terms, we can rewrite these as

$$0 \geqslant k\delta - lr + \rho_{\Pi'} \geqslant \rho_{\Pi'} - lr. \tag{2.54}$$

Using Lemma 2.9.3 then we can conclude that

$$0 \geqslant k\delta - lr + \rho_{\Pi'} > -\frac{r'r''}{2},$$

which, in view of the equality (2.52), implies the necessary estimate (2.50).

- The second case is similar: $(\lambda/k)_{\Pi'} \leqslant l \leqslant (\widehat{\lambda}/\widehat{k})_{\Pi'}$ is equivalent to $\delta \leqslant 0$ and $lk + lr \leqslant \lambda_{\Pi'} + \rho_{\Pi'}$. This leads to
$$0 \leqslant k\delta - lr + \rho_{\Pi'} \leqslant \rho_{\Pi'} - lr, \tag{2.55}$$
which, in turn, implies
$$0 \leqslant k\delta - lr + \rho_{\Pi'} < \frac{r'r''}{2},$$
and hence (2.50).

This completes the proof of Proposition 2.9.1: indeed, a simple calculation shows that if $\lambda/k \in \Delta$ then $\widehat{\lambda}/\widehat{k} \in \Delta$, so the conditions of Lemma 2.9.3 hold. We have just shown that this implies (2.50), and according to Lemma 2.9.2, we can conclude the vanishing of the wall-crossing term (2.48). □

Remark 2.9.4. Note that if $\lambda/k \in \Delta$ is non-regular, then it belongs to some wall from the set $\mathcal{S}(k, \lambda)$. Hence proposition 2.9.1 implies that the right-hand side of formula (I.) of Theorem 2.3.8 is a well-defined function on the cone over Δ:
$$\{(k, \lambda) \in \mathbb{Z}^{>0} \times \Lambda | \, \lambda/k \in \Delta\}.$$

Chapter 3

Euler characteristics of tautological bundles

In this chapter we show that the residue/wall-crossing method of Chapter 2 may be successfully employed to describe the pushforward maps in the K-theory of moduli spaces of stable parabolic bundles on smooth curves. This chapter is based on the work [27] and presents formulas for the Euler characteristic of associated bundles over the moduli spaces.

3.1. Rank 2 case

We start with presenting our arguments for the case of moduli spaces of rank-2 parabolic bundles, when the formula for the Euler characteristic has a simple form (cf. (3.6)).

3.1.1. *The residue formula for rank 2*

As in §2.8, in this case we need to consider the moduli space of vector bundles with parabolic structures at *two* points to calculate our wall-crossing terms.

Recall (cf. §2.8.1) that there are two such moduli spaces of rank-2 degree-0 parabolic bundles, which we denoted by $P_0(c > a)$ and $P_0(c < a)$; recall that for $\lambda, \mu \in \mathbb{Z}$, we introduced the line bundle

$$\mathcal{L}(k;\lambda,\mu) = \det(U_p)^{k(1-g)} \otimes \det(\pi_*(U))^{-k} \otimes (\mathcal{F}_2/\mathcal{F}_1)^{\lambda} \otimes (\mathcal{F}_1)^{-\lambda} \otimes (\mathcal{G}_2/\mathcal{G}_1)^{\mu} \otimes (\mathcal{G}_1)^{-\mu}$$

on moduli spaces $P_0(c > a) \times C$ and $P_0(c < a) \times C$.

Let $\nu = (\nu_1 \geqslant \nu_2) \in \mathbb{Z}^2$ be a dominant weight of GL_2, denote by ρ_ν the irreducible representation of GL_2 with highest weight ν, and by $\bar{\rho}_\nu$ its restriction to $SU_2 \subset GL_2$. We denote by $U_\nu \to P_0(c,a) \times C$ the bundle associated to the representation ρ_ν, and by $\mathcal{K}$ the canonical sheaf on C.

Our goal is to calculate Euler characteristics

$$\begin{aligned} \chi^{\nu}_{>}(k;\lambda,\mu) &\stackrel{\text{def}}{=} \chi(P_0(c > a), \mathcal{L}(k;\lambda,\mu) \otimes \pi_!(U_\nu \otimes \mathcal{K}^{\frac{1}{2}})) \quad \text{and} \\ \chi^{\nu}_{<}(k;\lambda,\mu) &\stackrel{\text{def}}{=} \chi(P_0(c < a), \mathcal{L}(k;\lambda,\mu) \otimes \pi_!(U_\nu \otimes \mathcal{K}^{\frac{1}{2}})). \end{aligned} \tag{3.1}$$

Let $\mathrm{Exp} : \mathrm{Lie}(SU_2) \to SU_2$ be the exponential map and let

$$\phi(x) = \mathrm{trace}(\bar{\rho}_\nu \circ \mathrm{Exp}(x/2)) = \frac{\sinh((\nu_1 - \nu_2 + 1)x/2)}{\sinh(x/2)}$$

be the character function on the Lie algebra of a maximal torus of SU_2.

We introduce the notation

$$\dot{\phi}(x) = 2\frac{d}{dx}\phi(x) \quad \text{and} \quad \ddot{\phi}(x) = 2\frac{d}{dx}\dot{\phi}(x),$$

(where the factors of 2 are introduced for convenience) and define two polynomials in k, λ, μ which as we will show, equal to (3.1):

$$R^\nu_>(k;\lambda,\mu) \overset{\text{def}}{=} (-1)^g \frac{\partial}{\partial\delta}\Big|_{\delta=0} \frac{1}{(2\pi i)} \int\limits_{|u|=\varepsilon} \frac{(e^{u(\lambda+\mu+1)} - e^{u(\lambda-\mu)})e^{u(\nu_1+\nu_2)/2}(2k+4+\delta\ddot{\phi}(u))^g}{(2\sinh\left(\frac{u}{2}\right))^{2g}(1-e^{u(k+2)+\delta\dot{\phi}(u)})} du$$

and

$$R^\nu_<(k;\lambda,\mu) \overset{\text{def}}{=} (-1)^g \frac{\partial}{\partial\delta}\Big|_{\delta=0}$$
$$\frac{1}{(2\pi i)} \int\limits_{|u|=\varepsilon} \frac{(e^{u(\lambda+\mu+1)} - e^{u(\lambda-\mu+k+2)+\delta\dot{\phi}(u)})e^{u(\nu_1+\nu_2)/2}(2k+4+\delta\ddot{\phi}(u))^g}{(2\sinh\left(\frac{u}{2}\right))^{2g}(1-e^{u(k+2)+\delta\dot{\phi}(u)})} du,$$

where ε is a real constant and $\delta \ll \varepsilon$. We note two facts about this pair of polynomials:

Fact 1. The difference of these polynomials has the form:

$$R^\nu_>(k;\lambda,\mu) - R^\nu_<(k;\lambda,\mu) = g(-(2k+4))^{g-1} \operatorname*{Res}_{u=0} \frac{e^{u(\lambda-\mu)}e^{u(\nu_1+\nu_2)/2}\ddot{\phi}(u)}{(2\sinh\left(\frac{u}{2}\right))^{2g}} du.$$

Fact 2. An easy calculation via substitutions shows the following:

$$R^\nu_>(k;\lambda,\mu) = -R^\nu_>(k;\lambda,-\mu-1) = -R^\nu_>(k;-\lambda+k+1-(\nu_1+\nu_2),\mu) -$$
$$(-(2k+4))^g \operatorname*{Res}_{u=0} \frac{(e^{u(\lambda+\mu+1)} - e^{u(\lambda-\mu)})e^{u(\nu_1+\nu_2)/2}\dot{\phi}(u)}{(2\sinh\left(\frac{u}{2}\right))^{2g}(1-e^{u(k+2)})} du$$

and

$$R^\nu_<(k;\lambda,\mu) = -R^\nu_<(k;-\lambda-1-(\nu_1+\nu_2),\mu) =$$
$$-R^\nu_<(k;\lambda,-\mu+k+1) - (-(2k+4))^g \operatorname*{Res}_{u=0} \frac{(e^{u(\lambda+\mu+1)} - e^{u(\lambda-\mu+k+2)})e^{u(\nu_1+\nu_2)/2}\dot{\phi}(u)}{(2\sinh\left(\frac{u}{2}\right))^{2g}(1-e^{u(k+2)})} du.$$

3.1.2. *Hecke correspondences, Serre duality and the symmetry argument*

In this section, we prove that the polynomials $\chi^{\nu}_{>}(k;\lambda,\mu)$ and $\chi^{\nu}_{<}(k;\lambda,\mu)$ (cf. (3.1)) satisfy the same antisymmetries as the polynomials $R^{\nu}_{>}$ and $R^{\nu}_{<}$ (cf. Fact 2).

In §2.6.1 (c.f. also §3.3.4) we describe the tautological variant of the Hecke correspondence which identifies the moduli spaces of parabolic bundles with different degrees and weights. Applying the Hecke correspondence at the point p and q to $P_0(c>a)$ and $P_0(c<a)$ respectively, we can identify these spaces as $\mathbb{P}^1\times\mathbb{P}^1$-bundles over the moduli space N_{-1} of stable bundles of degree -1 (cf. Lemma 2.8.3):

$$\mathbb{P}^1\times\mathbb{P}^1\to P_0(c>a)\to N_{-1}\leftarrow P_0(c<a)\leftarrow\mathbb{P}^1\times\mathbb{P}^1.$$

For each copy of $\mathbb{P}^1$, the moduli space $P_0(c>a)$ can be considered as a $\mathbb{P}^1$-bundle; applying Serre duality for families of curves as in §3.4.1, we obtain the following two equalities:

$$\chi^{\nu}_{>}(k;\lambda,\mu)=-\chi^{\nu}_{>}(k;\lambda,-\mu-1)$$

and

$$\chi^{\nu}_{>}(k;\lambda,\mu)=-\chi^{\nu}_{>}(k;-\lambda+k+1-(\nu_1+\nu_2),\mu)+\sum_{i=0}^{\nu_1-\nu_2}(\nu_1-\nu_2-2i)\chi(P_0(c>a),\mathcal{L}(k;\lambda+\nu_1-i,\mu)).$$

Similarly, for $P_0(c<a)$ we obtain that

$$\chi^{\nu}_{<}(k;\lambda,\mu)=-\chi^{\nu}_{<}(k;-\lambda-(\nu_1+\nu_2)-1,\mu)=-\chi^{\nu}_{<}(k;\lambda,-\mu+k+1)+\sum_{i=0}^{\nu_1-\nu_2}(\nu_1-\nu_2-2i)\chi(P_0(c>a),\mathcal{L}(k;\lambda+\nu_1-i,\mu)).$$

We showed in Proposition 2.8.6 that

$$\sum_{i=0}^{\nu_1-\nu_2}(\nu_1-\nu_2-2i)\chi(P_0(c>a),\mathcal{L}(k;\lambda+\nu_1-i,\mu))=(-1)^{g-1}(2k+4)^g\operatorname*{Res}_{u=0}\frac{(e^{u(\lambda+\mu+1)}-e^{u(\lambda-\mu)})e^{u(\nu_1+\nu_2)/2}\phi(u)}{(2\sinh\left(\frac{u}{2}\right))^{2g}(1-e^{u(k+2)})}du$$

and

$$\sum_{i=0}^{\nu_1-\nu_2}(\nu_1-\nu_2-2i)\chi(P_0(c<a),\mathcal{L}(k;\lambda+\nu_1-i,\mu))=(-1)^{g-1}(2k+4)^g\operatorname*{Res}_{u=0}\frac{(e^{u(\lambda+\mu+1)}-e^{u(\lambda-\mu+k+2)})e^{u(\nu_1+\nu_2)/2}\phi(u)}{(2\sinh\left(\frac{u}{2}\right))^{2g}(1-e^{u(k+2)})}du,$$

hence the polynomials $\chi^{\nu}_{>}$ and $\chi^{\nu}_{<}$ satisfy the same antisymmetries as $R^{\nu}_{>}$ and $R^{\nu}_{<}$ (cf. Fact 2 on page 51).

3.1.3. Wall-crossing in moduli spaces

Our next step is to compare the difference $\chi^{\nu}_{>} - \chi^{\nu}_{<}$ with the difference $R^{\nu}_{>} - R^{\nu}_{<}$ from Fact 1 on page 51.

In §2.4 we presented a simple formula for the wall-crossing difference in Geometric Invariant Theory. The formula has the form of a residue of an equivariant integral, taken with respect to an equivariant parameter. In the rank-2 case (cf. Lemma 2.8.1), the space Z^0 over which we integrate is isomorphic to the Jacobian of degree-0 line bundles on C:

$$Z^0 \simeq \{V = L \oplus L^{-1} \mid L \in \mathrm{Jac}^0,\, L_p = F_1, L_q^{-1} = G_1\}.$$

We thus obtain the following expression for the wall-crossing difference:

$$\chi^{\nu}_{>}(k;\lambda,\mu) - \chi^{\nu}_{<}(k;\lambda,\mu) = (-1)^g \operatorname*{Res}_{u=0} \frac{\exp(u(\lambda-\mu))}{(2\sinh(u/2))^{2g}} \int_{\mathrm{Jac}} e^{\eta(2k+4)} \mathrm{ch}(\pi_!(U_\nu \otimes \mathcal{K}^{\frac{1}{2}})|_{\mathrm{Jac}})\, du, \tag{3.2}$$

where u plays the role of the equivariant parameter, the generator of $H^*_{\mathbb{C}^*}(pt)$; let (cf. page 28) $\mathcal{J}$ be the Poincare bundle over $\mathrm{Jac} \times C$, satisfying $c_1(\mathcal{J}_p) = 0$, then the class $\eta \in H^2(\mathrm{Jac})$ is defined through the Künneth decomposition of $c_1(\mathcal{J})^2$.

It follows from the Groethendieck-Riemann-Roch theorem that

$$\mathrm{ch}(\pi_!(U_\nu \otimes \mathcal{K}^{\frac{1}{2}})) = \pi_* \mathrm{ch}(U_\nu).$$

A simple calculation shows that the restriction $U|_{Z^0} = \mathcal{J} \oplus \mathcal{J}^{-1}$ has $\mathbb{C}^*$-weight 1, hence we have

$$\mathrm{ch}(U_\nu|_{Z^0}) = \bigoplus_{i=0}^{\nu_1-\nu_2} \mathrm{ch}(\mathcal{J}^{\nu_1-\nu_2-2i}) \exp((\nu_1 - i)u).$$

Note that that $\pi_*(\mathrm{ch}(\mathcal{J}^n)) = -n^2\eta$, and thus

$$\pi_*(\mathrm{ch}(U_\nu|_{Z^0})) = -\eta \sum_{i=0}^{\nu_1-\nu_2} (\nu_1 - \nu_2 - 2i)^2 \exp((\nu_1 - i)u) = -\eta \exp((\nu_1+\nu_2)u/2)\ddot{\phi}(u).$$

Using (2.31), we obtain that the wall-crossing difference (3.2) is equal to

$$g(-(2k+4))^{g-1} \operatorname*{Res}_{u=0} \frac{\exp(u(\lambda-\mu)) e^{u(\nu_1+\nu_2)/2} \ddot{\phi}(u)}{(2\sinh(u/2))^{2g}}\, du, \tag{3.3}$$

and thus we have (cf. Fact 1 on page 51)

$$R^{\nu}_{>} - R^{\nu}_{<} = \chi^{\nu}_{>} - \chi^{\nu}_{<}. \tag{3.4}$$

Now we are ready for the final argument: we can rearrange equation (3.4) to describe the equality of wall-crossings as

$$R^{\nu}_{>}(k;\lambda,\mu) - \chi^{\nu}_{>}(k;\lambda,\mu) = R^{\nu}_{<}(k;\lambda,\mu) - \chi^{\nu}_{<}(k;\lambda,\mu); \tag{3.5}$$

we introduce the notation $\Theta(k;\lambda,\mu)$ for this polynomial. Then $\Theta(k;\lambda,\mu)$ satisfies 4 antisymmetries:

$$\Theta(k;\lambda,\mu) = -\Theta(k;\lambda,-\mu-1) = -\Theta(k;-\lambda+k+1-(\nu_1+\nu_2),\mu) = \\ -\Theta(k;-\lambda-1-(\nu_1+\nu_2),\mu) = -\Theta(k;\lambda,-\mu+k+1),$$

hence it is anti-invariant with respect to the affine Weyl group action on λ and μ separately, and this implies $\Theta = 0$.

As $P_0(c > a)$ is a $\mathbb{P}^1$-bundle over the moduli space of rank-2 degree-0 stable parabolic bundles $P_0(c)$, substituting $\mu = 0$ in $R^{\nu}_{>}$ and taking the derivative with respect to δ, we obtain the formula for rank 2:

$$\chi(P_0(c),\mathcal{L}(k;\lambda)\otimes\pi_!(U_\nu\otimes\mathcal{K}^{\frac{1}{2}})) = \\ (-(2k+4))^g \operatorname*{Res}_{u=0} \frac{\exp(u(\lambda+\frac{1}{2}+\frac{\nu_1+\nu_2}{2}))}{(2\sinh\left(\frac{u}{2}\right))^{2g-1}(1-e^{u(k+2)})}\left(\frac{g\,\ddot{\phi}(u)}{2k+4}+\frac{e^{(k+2)u}\dot{\phi}(u)}{(1-e^{u(k+2)})}\right)du. \quad (3.6)$$

3.2. Main result and wall-crossing in residue formulas

In this section we describe tautological vector bundles on parabolic moduli spaces we are considering and present the main result of this chapter, generalised variant of formula (3.6) above for all ranks (cf. Theorem 3.2.3).

3.2.1. *Vector bundles on the moduli space of parabolic bundles*

Let $\nu = (\nu_1, ..., \nu_r)$ be a dominant weight of GL_r, consider the irreducible representation ρ_ν with highest weight ν, and denote by $\bar{\rho}_\nu$ its restriction to the subgroup $SU_r \subset GL_r$. We denote by ϕ^ν the character $\phi^\nu = \mathrm{trace}(\bar{\rho}_\nu \circ \mathrm{Exp})$ on the Lie algebra V of a maximal torus $T \subset SU_r$. We collect our maps on the following diagram.

$$\begin{tikzcd} & GL_r \arrow[r, "\rho_\nu"] & GL(V_\nu) \\ V \arrow[r, "\mathrm{Exp}"] & T \subset SU_r \arrow[u, hook] \arrow[ur, "\bar{\rho}_\nu"] & \end{tikzcd}$$

Given a representation ρ_ν of GL_r, we denote by U_ν the vector bundle over $P_0(c) \times C$ associated to the principal GL_r-bundle.

The vector bundle U_ν has the following explicit construction. Let U be the normalized universal bundle on $P_0(c) \times C$ (cf. §2.1.3), and consider the full flag bundle $\mathrm{Flag}(U) \xrightarrow{f} P_0(c) \times C$. Denote by $L_1, ..., L_r$ the standard quotient line bundles on $\mathrm{Flag}(U)$. Then

$$U_\nu = f_*(L_1^{\nu_1}\otimes L_2^{\nu_2}\otimes ...\otimes L_r^{\nu_r}). \quad (3.7)$$

Remark 3.2.1. Note that the vector bundles $\mathcal{F}_r, ..., \mathcal{F}_1$ (cf. §2.1.3) on the moduli space $P_0(c)$ define a section of the flag bundle $\mathrm{Flag}(U_p) \to P_0(c) \times \{p\}$.

3.2.2. Main result

Before we formulate the main result of this chapter, Theorem 3.2.3, we introduce a generalized version of the iterated Bernoulli operator defined in §2.3. In this section, we will use the notation of §§2.2.1-2.3.

Recall the notation $\mathcal{F}_\Phi$ for the space of meromorphic functions defined in a neighborhood of 0 in $V\otimes_{\mathbb{R}}\mathbb{C}$ with poles on the union of hyperplanes

$$\bigcup_{1\leqslant i<j\leqslant r}\{x|\,\langle\alpha^{ij},x\rangle=0\};$$

then the inverse of

$$w_\Phi \stackrel{\text{def}}{=} \prod_{i<j}\left(2\sinh(x_i-x_j)\right)$$

is a function in $\mathcal{F}_\Phi$. Any ordered linear basis $\mathbf{B}\in\mathcal{B}$ of V^* (cf. §2.2.1) induces an isomorphism $V^*\simeq V$, and we will write $\check{\alpha}_{\mathbf{B}}$ for the image of $\alpha\in V^*$ under this isomorphism. We will sometimes omit the index $\mathbf{B}$ to simplify the notation. For a function Q on V and $\alpha\in V^*$, we introduce the directional derivative $Q_{\check{\alpha}_{\mathbf{B}}}$. Then, given Q and $\mathbf{B}\in\mathcal{B}$, we fix a homomorphism

$$\alpha\mapsto\exp(Q_{\check{\alpha}_{\mathbf{B}}})$$

from the additive group V^* to the multiplicative group of non-vanishing holomorphic functions on $V\otimes_{\mathbb{R}}\mathbb{C}$.

Let $K=\frac{1}{2r}\sum_{i<j}\alpha_{ij}^2$ be the normalized Killing form of SU_r and let $\delta\in\mathbb{R}$ be a small parameter; given a basis $\mathbf{B}=\left(\beta^{[1]},\dots,\beta^{[r-1]}\right)\in\mathcal{B}$ of V^*, a function $f\in\mathcal{F}_\Phi$ and a holomorphic function $Q=\mathrm{const}\cdot K-\delta\phi$, defined in a neighborhood of 0 in $V\otimes_{\mathbb{R}}\mathbb{C}$, we define

$$\underset{\mathbf{B},Q}{\mathrm{iBer}}\,[f(x)]\,(a)\stackrel{\text{def}}{=}\frac{1}{(2\pi i)^{r-1}}\int_{Z_{\mathbf{B}}}\frac{f(x)\exp(Q_{\check{a}})\,dQ_{\check{\beta}^{[1]}}\wedge\dots\wedge dQ_{\check{\beta}^{[r-1]}}}{(1-\exp(Q_{\check{\beta}^{[1]}}))\,\dots(1-\exp(Q_{\check{\beta}^{[r-1]}}))},\tag{3.8}$$

where the naturally oriented cycle $Z_{\mathbf{B}}$ is given by

$$Z_{\mathbf{B}}=\{x\in V\otimes_{\mathbb{R}}\mathbb{C}:\,|\langle\beta^{[j]},x\rangle|\;=\varepsilon_j,\,j=1,\dots,r-1\}\subset V\otimes_{\mathbb{R}}\mathbb{C}\backslash\{w_\Phi(x)=0\}$$

with sufficiently small fixed real constants ε_j satisfying $0\leqslant\varepsilon_{r-1}\ll\dots\ll\varepsilon_1$. Thus $\mathrm{iBer}_{\mathbf{B},Q}$ is a linear operator associating to a meromorphic function $f\in\mathcal{F}_\Phi$ a polynomial on V^*.

We introduce the notation $\mathcal{H}^\Phi$ for the space of holomorphic functions of the form $Q=\mathrm{const}\cdot K-\delta\phi$, defined in a neighborhood of 0 in $V\otimes_{\mathbb{R}}\mathbb{C}$. We will always assume that our parameter δ is small enough, so that the cycle given by $\{x\in V\otimes_{\mathbb{R}}\mathbb{C}:\,|Q_{\check{\beta}^{[j]}}(x)|\;=\varepsilon_j,\,j=1,\dots,r-1\}\subset V\otimes_{\mathbb{R}}\mathbb{C}\backslash\{w_\Phi(x)=0\}$ is homotopic to the cycle $Z_{\mathbf{B}}$.

Notation: We will write $\mathrm{iBer}_{\mathbf{B}}$ for $\mathrm{iBer}_{\mathbf{B},K}$ to simplify the notation. Note that this agrees with (2.11).

We will need the following property of the operator $\mathrm{iBer}_{\mathbf{B},Q}$.

Lemma 3.2.2. *Let $Q=(k+r)K-\delta\varphi\in\mathcal{H}^\Phi$, then for any vector $w\in\Lambda$ and a function $f\in\mathcal{F}_\Phi$, which depends on δ, we have*

$$\frac{\partial}{\partial\delta}\Big|_{\delta=0}\underset{\mathbf{B},Q}{\mathrm{iBer}}\,[f(x)]\,(a+w)=$$

$$\frac{\partial}{\partial\delta}\Big|_{\delta=0}\underset{\mathbf{B},Q}{\mathrm{iBer}}\,[f(x)\exp((k+r)w)]\,(a)-\underset{\mathbf{B},(k+r)K}{\mathrm{iBer}}\left[f(x)\big|_{\delta=0}\exp((k+r)w)\phi_{\check{w}}(x)\right](a).\tag{3.9}$$

Proof. Note that

$$\operatorname*{iBer}_{\mathbf{B},Q}[f(x)](a+w) = \operatorname*{iBer}_{\mathbf{B},Q}[f(x)\exp(Q_{\check{w}})](a) = \operatorname*{iBer}_{\mathbf{B},Q}[f(x)\exp((k+r)w-\delta\phi_{\check{w}}(x))](a);$$

then taking the derivative with respect to δ at zero, we obtain the result. □

We are now ready to present the formula for the Euler characteristic of associated vector bundles on the moduli spaces.

Theorem 3.2.3. *Let $\mathcal{K}$ be the canonical class of the curve C, $\lambda \in \Lambda$, $k \in \mathbb{Z}_{>0}$, $\nu = (\nu_1 \geqslant \nu_2 \ldots \geqslant \nu_r) \in \mathbb{Z}^r$, $\widehat{\lambda} = \lambda + \rho$, $\nu_{\det} = (1,\ldots,1,1-r)\frac{\sum \nu_i}{r}$, $Q = (k+r)K - \delta\varphi^{\nu} \in \mathcal{H}^{\Phi}$ and let $c \in \Delta$ be a regular element (cf. page 7). Then for any diagonal basis $\mathcal{D} \in \mathcal{B}$, the following equality holds:*

$$\chi(P_0(c), \mathcal{L}(k;\lambda) \otimes \pi_!(U_\nu \otimes \mathcal{K}^{\frac{1}{2}})) = \\ N_r \cdot \frac{\partial}{\partial \delta}\Big|_{\delta=0} \sum_{\mathbf{B}\in\mathcal{D}} \operatorname*{iBer}_{\mathbf{B},Q}\left[\det(\operatorname{Hess}(Q))^{g-1}(x) w_{\Phi}^{1-2g}(x) \exp(\langle \widehat{\lambda} + \nu_{\det}, x\rangle)\right](-[c]_{\mathbf{B}}), \quad (3.10)$$

where $N_r = (-1)^{\binom{r}{2}(g-1)} r^g$ and by $\operatorname{Hess}(Q)$ we denote the Hessian matrix of Q.[1]

Taking the derivative with respect to δ, we obtain the following explicit formulas.

Corollary 3.2.4. *Let λ, k, ν, Q, c be as above, and $N_{r,k} = (-1)^{\binom{r}{2}(g-1)} r(r(k+r)^{r-1})^{g-1}$, then*

$$\chi(P_0(c), \mathcal{L}(k;\lambda) \otimes \pi_!(U_\nu \otimes \mathcal{K}^{\frac{1}{2}})) = \\ N_{r,k} \sum_{\mathbf{B}\in\mathcal{D}} \operatorname*{iBer}_{\mathbf{B}}\Bigg[w_{\Phi}^{1-2g}(x/\widehat{k}) \exp(\langle \widehat{\lambda} + \nu_{\det}, x/\widehat{k}\rangle)\Bigg(\frac{-g}{k+r}\operatorname{tr}(\operatorname{Hess}(\varphi^{\nu})(x/\widehat{k})) - \\ \sum_i \frac{\varphi^{\nu}_{\check{\beta}^{[i]}}(x/\widehat{k})\exp(\langle \beta^{[i]}, x\rangle)}{1-\exp(\langle \beta^{[i]}, x\rangle)} + \sum_i \langle [c], \check{\beta}^{[i]}\rangle \varphi^{\nu}_{\check{\beta}^{[i]}}(x/\widehat{k})\Bigg)\Bigg](-[c]_{\mathbf{B}}).$$

Example 8. Denote by U the normalized universal bundle on the moduli spaces of rank-3 parabolic bundles $P_0(>)$ and $P_0(<)$ defined in Example 1. We have $U \simeq U_\nu$ for $\nu = (1,0,0)$ and

$$\phi(\alpha^{12},\alpha^{23}) = e^{\frac{2\alpha^{12}+\alpha^{23}}{3}} + e^{\frac{\alpha^{23}-\alpha^{12}}{3}} + e^{\frac{-\alpha^{12}-2\alpha^{23}}{3}}; \;\; \phi_{\check{\alpha}^{12}}(\alpha^{12},\alpha^{23}) = e^{\frac{2\alpha^{12}+\alpha^{23}}{3}} - e^{\frac{\alpha^{23}-\alpha^{12}}{3}}; \\ \phi_{\check{\alpha}^{23}}(\alpha^{12},\alpha^{23}) = e^{\frac{\alpha^{23}-\alpha^{12}}{3}} - e^{\frac{-\alpha^{12}-2\alpha^{23}}{3}}; \;\; \operatorname{tr}(\operatorname{Hess}(\phi(\alpha^{12},\alpha^{23}))) = \frac{2}{3}\phi(\alpha^{12},\alpha^{23}).$$

Let $\mathcal{D}$ be the diagonal basis from Example 2; writing the operator $\operatorname{iBer}_{\mathbf{B}}$ for $\mathbf{B} \in \mathcal{D}$ in the variables (x,y) as explained in Remark 2.3.3 and using Remark 2.3.6, we obtain

$$\chi(P_0(<), \mathcal{L}(k;\lambda) \otimes \pi_!(U \otimes \mathcal{K}^{\frac{1}{2}}) = \\ N \cdot \operatorname*{Res}_{y=0}\operatorname*{Res}_{x=0} \frac{(e^{\lambda_1 x + (\lambda_1+\lambda_2)y + x + y + \frac{x+2y}{3}} - e^{\lambda_1 x + (\lambda_1+\lambda_3)y + x + \frac{x-y}{3}})}{(1-e^{x(k+3)})(1-e^{y(k+3)}) w_{\Phi}(x,y)^{2g-1}} \cdot \\ \left(\frac{2g}{3(k+3)}\phi(x,y) + \frac{e^{(k+3)x}\phi_{\check{x}}(x,y)}{(1-e^{(k+3)x})} + \frac{e^{(k+3)y}\phi_{\check{y}}(x,y)}{(1-e^{(k+3)y})}\right) dx dy$$

[1] For $\mathbf{B} = (\beta^{[1]}, \ldots, \beta^{[r-1]})$, we set $\operatorname{Hess}(Q)_{ij} = \frac{\partial}{\partial \beta^{[j]}} Q_{\check{\beta}^{[i]}}$.

and

$$\chi(P_0(>), \mathcal{L}(k;\lambda)\otimes\pi_!(\mathcal{U}\otimes\mathcal{K}^{\frac{1}{2}}) =$$

$$N\cdot \operatorname*{Res}_{y=0}\operatorname*{Res}_{x=0} \frac{\left(e^{\lambda_1 x+(\lambda_1+\lambda_2)y+x+y+\frac{x+2y}{3}} - e^{\lambda_1 x+(\lambda_1+\lambda_3)y+x+(k+3)y+\frac{x-y}{3}}\right)}{(1-e^{x(k+3)})(1-e^{y(k+3)})w_\Phi(x,y)^{2g-1}}\cdot$$
$$\left(\frac{2g}{3(k+3)}\phi(x,y)+\frac{e^{(k+3)x}\phi_{\check{x}}(x,y)}{(1-e^{(k+3)x})}+\frac{e^{(k+3)y}\phi_{\check{y}}(x,y)}{(1-e^{(k+3)y})}\right)dxdy-$$
$$N\cdot \operatorname*{Res}_{y=0}\operatorname*{Res}_{x=0}\frac{e^{\lambda_1 x+(\lambda_1+\lambda_3)y+x+(k+3)y+\frac{x-y}{3}}\phi_{\check{y}}(x,y)}{(1-e^{x(k+3)})(1-e^{y(k+3)})w_\Phi(x,y)^{2g-1}}dxdy,$$

where $w_\Phi(x,y) = 2\sinh(\frac{x}{2})2\sinh(\frac{y}{2})2\sinh(\frac{x+y}{2})$ and $N = (-1)^g(3(k+3)^2)^g$. One can compare these formulas with the ones from Example 4.

3.2.3. *Wall-crossing in residue formulas*

We start the proof of Theorem 3.2.3 following the strategy of Chapter 2. Our first step is to calculate the wall-crossing terms of the residue expressions from Theorem 3.2.3. We choose two regular elements $c^+, c^- \in \Delta$ in two neighbouring chambers separated by the wall $S_{\Pi,l}$ (cf. (2.10)) such that

$$[c^+_{\Pi'}] = l \text{ and } [c^-_{\Pi'}] = l-1,$$

where we use the notation $c_{\Pi'} = \sum_{i\in\Pi'} c_i$ for $c\in\Delta$. We denote by

$$R^\gamma_\pm(k,\lambda) = N_r\cdot\frac{\partial}{\partial\delta}|_{\delta=0}\sum_{\mathbf{B}\in\mathcal{D}}\operatorname*{iBer}_{\mathbf{B},Q}\left[\det(\mathrm{Hess}(Q))^{g-1}(x)w_\Phi^{1-2g}(x)\exp(\langle\hat{\lambda}+\nu_{\det},x\rangle)\right](-[c^\pm]_{\mathbf{B}})$$

the two polynomials in $(k,\lambda)\in\mathbb{Z}_{>0}\times\Lambda$. Then the wall-crossing term in the residue formula is the difference

$$R^\gamma_+(k,\lambda) - R^\gamma_-(k,\lambda).$$

Using Lemma 3.2.5, we obtain the following expression for this difference.

Lemma 3.2.5. *Let* (Π, l) *and* c^+, c^- *be as above, and fix a diagonal basis* $\mathcal{D}\subset\mathcal{B}$. *Denote by* $\mathcal{D}|\Pi$ *the subset of those elements* $\mathbf{B}$ *of* $\mathcal{D}$ *for which* $\mathrm{Tree}(\mathbf{B})$ *(cf. §2.1.3) is a union of a tree on* Π'*, a tree on* Π'' *and a single edge* β_{link} *(which we will call the link) connecting* Π' *and* Π''*. Then*

$$R^\gamma_+(k,\lambda) - R^\gamma_-(k,\lambda) = N_r\cdot\frac{\partial}{\partial\delta}|_{\delta=0}$$
$$\sum_{\mathbf{B}\in\mathcal{D}|\Pi}\operatorname*{iBer}_{\mathbf{B},Q}\left[(1-\exp(Q_{\check{\beta}_{\mathrm{link}}}(x)))\det(\mathrm{Hess}(Q))^{g-1}w_\Phi^{1-2g}(x)\exp(\langle\hat{\lambda}+\nu_{\det},x\rangle)\right](-[c^+]_{\mathbf{B}})$$

Remark 3.2.6. Note that the multiplication by $(1-\exp(Q_{\check{\beta}_{\mathrm{link}}}(x)))$ in Lemma 3.2.5 has the effect of canceling one of the factors in the denominator in the definition (3.8) of the operation iBer.

As observed in Chapter 2, even though this difference does not depend on the choice of $\mathcal{D}$, it is convenient to choose a particular diagonal basis (cf. page 19). Recall the notation Φ' and Φ'' for the $A_{r'}$ and $A_{r''}$ root systems corresponding to Π' and Π''; using Lemma 2.3.16 and taking the derivative with respect to δ at $\delta = 0$, we arrive at the following statement.

Corollary 3.2.7. *Let $\mathcal{D}'$ and $\mathcal{D}''$ be diagonal bases of Φ' and Φ'' correspondingly. Then*

$$R^{\nu}_{+}(k,\lambda) - R^{\nu}_{-}(k,\lambda) = (k+r)N_{r,k} \sum_{\mathbf{B}'\in\mathcal{D}'} \sum_{\mathbf{B}''\in\mathcal{D}''} \operatorname*{Res}_{\beta_{\text{link}}=0} \operatorname*{iBer}_{\mathbf{B}'} \operatorname*{iBer}_{\mathbf{B}''}$$
$$\Bigg[w_{\Phi}^{1-2g}(x/\widehat{k}) \exp(\langle \widehat{\lambda} + \nu_{\det}, x/\widehat{k}\rangle) \Bigg(\frac{-g}{k+r} \operatorname{tr}(\operatorname{Hess}(\varphi^{\nu}))(x/\widehat{k}) + l\varphi^{\nu}_{\check{\beta}_{\text{link}}}(x/\widehat{k}) -$$
$$\sum_{i\neq \text{link}} \frac{\varphi^{\nu}_{\check{\beta}^{[i]}}(x/\widehat{k}) \exp(\langle \beta^{[i]}, x\rangle)}{1-\exp(\langle \beta^{[i]}, x\rangle)} + \sum_{i\neq \text{link}} \langle [c^{+}], \check{\beta}^{[i]}\rangle \varphi^{\nu}_{\check{\beta}^{[i]}}(x/\widehat{k}) \Bigg) \Bigg] \left(-[c^{+}]_{\mathbf{B}}\right) \, d\beta_{\text{link}}, \quad (3.11)$$

where $\operatorname{Res}_{\beta_{\text{link}}=0} \operatorname{iBer}_{B'} \operatorname{iBer}_{B''} \, d\beta_{\text{link}}$ is simply $\operatorname{iBer}_{\mathbf{B}}$ (cf (3.8)) with $\mathbf{B}$ obtained by appending $\mathbf{B}'$, and then $\mathbf{B}''$ to β_{link}, and the factor $(1-\exp(\langle \beta_{\text{link}}, x\rangle))$ removed from the denominator.

Example 9. Calculating the difference of the two polynomials from Example 8, we obtain the wall-crossing term:

$$- \quad N \quad \cdot \quad \operatorname*{Res}_{y=0} \operatorname*{Res}_{x=0} \frac{e^{\lambda_1 x + (\lambda_1+\lambda_3)y + x + \frac{x-y}{3}}}{(1-e^{x(k+3)}) w_{\Phi}(x,y)^{2g-1}} \left(\frac{2g}{3(k+3)} \phi(x,y) + \frac{e^{(k+3)x}\phi_{\check{x}}(x,y)}{(1-e^{(k+3)x})} \right) dxdy.$$

3.3. Wall-crossing in Euler characteristics

In this section, we calculate the changes in Euler characteristics of vector bundles when varying the moduli spaces of parabolic bundles. The main result is Proposition 3.3.6, where we present explicit formulas for the wall-crossing terms for the left-hand side of (3.10).

3.3.1. *Wall-crossing in master space*

Fix the wall $S_{\Pi,l}$ given by an ordered partition $\Pi = (\Pi', \Pi'')$ of the first r integers and an integer l, and two regular elements $c^{+}, c^{-} \in \Delta$ in two neighbouring chambers separated by the wall $S_{\Pi,l}$. Let

$$c' = \sum_{i\in\Pi'} x_i \text{ and } c'' = \sum_{i\in\Pi''} x_i.$$

In §2.5.1 we constructed the "master space" Z whose quotients, under different linearizations, by a fixed $\mathbb{C}^*$-action, are the moduli spaces of $c^{\pm}$-stable parabolic bundles. We showed that the elements $c^{\pm}$ may be chosen within their chambers so that Z is a smooth, projective variety with a $\mathbb{C}^*$-action, and identified the connected components of the fixed locus:

$$Z^{\mathbb{C}^*} \simeq P_0(c^{+}) \sqcup P_0(c^{-}) \sqcup Z^0,$$

where Z^0 is the set of points representing rank-r vector bundles W on C, such that W splits as a direct sum $W' \oplus W''$, where W' and W'' are, respectively, c' and c''-stable parabolic bundles of degree l and $-l$, rank $r' = |\Pi'|$ and $r'' = |\Pi''|$ (cf. Lemma 2.5.1):

$$Z^0 = \{ W = W' \oplus W'' \,|\, W' \in \widetilde{P}_l(c');\ W'' \in \widetilde{P}_{-l}(c'');\ \det(W) \simeq \mathcal{O} \}.$$

Recall that we also have (cf. Remark 2.5.2)

$$H^*(Z^0, \mathbb{Q}) \simeq H^*(P_l(c'), \mathbb{Q}) \otimes H^*(P_{-l}(c''), \mathbb{Q}) \otimes H^*(Jac^l, \mathbb{Q}). \tag{3.12}$$

Consider the polynomials

$$\chi^\nu_\pm(k;\lambda) = \chi(P_0(c^\pm), \mathcal{L}(k;\lambda) \otimes \pi_!(\mathcal{U}_\nu \otimes \mathcal{K}^{\frac{1}{2}})).$$

Our goal is to calculate the difference $\chi^\nu_+(k;\lambda) - \chi^\nu_-(k;\lambda)$.

Applying the Atiyah-Bott fixed-point formula to the master space Z with the $\mathbb{C}^*$-action, we showed (cf. Theorem 2.4.7) that the wall-crossing polynomial $\chi^\nu_+(k;\lambda) - \chi^\nu_-(k;\lambda)$ is equal to

$$\operatorname*{Res}_{u=0} \int_{Z^0} \frac{ch((\mathcal{L}(k;\lambda) \otimes \pi_!(\mathcal{U}_\nu \otimes \mathcal{K}^{\frac{1}{2}}))|_{Z^0})}{E(N_{Z^0})} \mathrm{Todd}(Z^0)\, du, \tag{3.13}$$

where $E(N_{Z^0})$ is the K-theoretical Euler class (cf. definition 2.4.6) of the conormal bundle of Z^0 in Z and u is an equivariant parameter.

Before we calculate this integral, we need to introduce some extra notations.

3.3.2. *Restriction. Representations*

For any weight $\nu = (\nu_1, ..., \nu_r)$ of GL_r, we define

$$|\nu| \stackrel{\text{def}}{=} \sum_i \nu_i;$$

the irreducible representation ρ_ν of $GL_r \simeq (SL_r \times \mathbb{C}^*)/\mathbb{Z}_r$ with highest weight ν can be decomposed by restriction as a product of the irreducible representation $\bar{\rho}_\nu$ of SU_r and the one-dimensional representation $\rho[|\nu|] : t \mapsto t^{|\nu|}$ of the center $Z(GL_r) \simeq \mathbb{C}^*$.

Let $GL_{r'} \times GL_{r''}$ be the subgroup of GL_r induced by an ordered partition (Π', Π'') of the first r positive integers. The restriction of the irreducible representation ρ_ν of GL_r decomposes as a direct sum of irreducible representations of $GL_{r'} \times GL_{r''}$:

$$\rho_\nu = \sum_{(\nu', \nu'')} \rho_{\nu'} \otimes \rho_{\nu''}.$$

Similarly, the restriction of the representation $\bar{\rho}_\nu$ to $SU_r \cap (GL_{r'} \times GL_{r''}) \subset GL_r$ can be decomposed as a direct sum

$$\bar{\rho}_\nu = \sum_{(\nu', \nu'')} \bar{\rho}_{\nu'} \otimes \bar{\rho}_{\nu''} \otimes \rho[rs]$$

of products of irreducible representations of $SU_{r'}$, $SU_{r''}$ and the one-dimensional torus $\mathbb{C}^* \simeq (Z(GL_{r'}) \times Z(GL_{r''})) \cap SU_r$, where $s = \sum_{i \in \Pi'} (\nu'_i - |\nu|/r)$. Let

$$w \stackrel{\text{def}}{=} \frac{s}{r'} \sum_{i \in \Pi'} x_i - \frac{s}{r''} \sum_{i \in \Pi''} x_i \in V^*,$$

then the corresponding decomposition of character functions (cf. end of §3.2.1) has the form

$$\phi^{\nu} = \sum_{(\nu',\nu'')} \phi^{\nu'}\phi^{\nu''}\exp(w). \tag{3.14}$$

Recall (cf. Lemma 3.2.5) that the a key role in our wall-crossing terms is played by the bases $\mathbf{B}$ of V^*, obtained by appending $\mathbf{B}'$, and then $\mathbf{B}''$ to β_{link}. Using expression (3.14), we arrive at the following equalities for the directional derivatives of ϕ^{ν}.

Lemma 3.3.1. *Let $\mathbf{B} = \beta_{\text{link}}\ \mathbf{B}'\ \mathbf{B}''$ be a basis of V^* described above. Then:*

1. *For any $\alpha \in \mathbf{B}'$ and any $\beta \in \mathbf{B}''$ we have*

$$\phi^{\nu}_{\check{\alpha}} = \sum_{(\nu',\nu'')} \phi^{\nu'}_{\check{\alpha}}\phi^{\nu''}\exp(w) \quad \textit{and} \quad \phi^{\nu}_{\check{\beta}} = \sum_{(\nu',\nu'')} \phi^{\nu''}_{\check{\beta}}\phi^{\nu'}\exp(w);$$

2. $\check{\beta}_{\text{link}} = \frac{r''}{r}\sum_{i\in\Pi'} x_i - \frac{r'}{r}\sum_{i\in\Pi''} x_i$*, and thus*

$$\phi^{\nu}_{\check{\beta}_{\text{link}}} = \sum_{(\nu',\nu'')} \frac{sr}{r'r''}\phi^{\nu'}\phi^{\nu''}\exp(w) + (\phi^{\nu'}\phi^{\nu''})_{\check{\beta}_{\text{link}}}\exp(w);$$

3. $\operatorname{tr}(\operatorname{Hess}(\phi^{\nu})) =$

$$\sum_{(\nu',\nu'')} (\operatorname{tr}(\operatorname{Hess}(\phi^{\nu'}))\phi^{\nu''} + \operatorname{tr}(\operatorname{Hess}(\phi^{\nu''}))\phi^{\nu'} + s^2\left(\frac{r}{r'r''}\right)\phi^{\nu'}\phi^{\nu''})\exp(w).$$

3.3.3. *Restriction. Bundles*

Recall that our goal is to calculate the integral (3.13); our first step is to identify the characteristic classes under this integral. We showed in Theorem 2.5.13, that

$$\operatorname*{Res}_{u=0}\int_{Z^0}\frac{\operatorname{ch}(\mathcal{L}(k;\lambda)|_{Z^0})}{E(N_{Z^0})}\operatorname{Todd}(Z^0)\,du = (k+r)N_{r,k}\sum_{\mathbf{B}'\in\mathcal{D}'}\sum_{\mathbf{B}''\in\mathcal{D}''}$$
$$\operatorname*{Res}_{\beta_{\text{link}}=0}\operatorname*{iBer}_{\mathbf{B}'}\operatorname*{iBer}_{\mathbf{B}''}[w_{\Phi}(x/\widehat{k})^{1-2g}\exp(\langle\widehat{\lambda}, x/\widehat{k}\rangle)](-[c^+]_{\mathbf{B}})\,d\beta_{\text{link}}, \tag{3.15}$$

where $\phi \in \Sigma_r$ is the unique permutation which sends $\{1,\ldots,r'\}$ to Π' preserving the order of the first r' and the last r'' elements. Now we study the restriction of the class $\operatorname{ch}(\pi_!(\mathcal{U}_{\nu}\otimes\mathcal{K}^{\frac{1}{2}}))$ to Z^0.

Recall that in §2.5.1 we denoted by $\omega \in H^2(C)$ the fundamental class of our curve C, and by $e_1,\ldots,e_{2g}$ a basis of $H^1(C)$, such that $e_ie_{i+g} = \omega$ for $1 \leqslant i \leqslant g$, and all other intersection numbers e_ie_j equal 0. For a class $\gamma \in H^*(P\times C)$ of a product, we recall the following notation for its Künneth components:

$$\gamma = \gamma_{(0)}\otimes 1 + \sum_i \gamma_{(e_i)}\otimes e_i + \gamma_{(2)}\otimes\omega \in \bigoplus_{i=0}^{2} H^{*-i}(P)\otimes H^i(C). \tag{3.16}$$

It follows from the Groethendieck-Riemann-Roch theorem that

$$\mathrm{ch}(\pi_!(\mathcal{U}_\nu \otimes \mathcal{K}^{\frac{1}{2}})) = \mathrm{ch}(\mathcal{U}_\nu)_{(2)}. \tag{3.17}$$

Recall our notation $\mathcal{J}$ for the Poincare bundle over $\mathrm{Jac} \times C$, such that $c_1(\mathcal{J})_{(0)} = 0$ and an element $\eta \in H^2(\mathrm{Jac})$ defined by $(\sum_i c_1(\mathcal{J})_{(e_i)} \otimes e_i)^2 = -2\eta \otimes \omega$. The following statement is straightforward.

Lemma 3.3.2. *Denote by* $\mathcal{U}[l]'$ *and by* $\mathcal{U}[-l]''$ *the normalized universal bundles on the moduli spaces* $P_l(c') \times C$ *and* $P_{-l}(c'') \times C$*, correspondingly (cf. beginning of §3.2.1). Let* $\mathcal{U}[l]_{\nu'}$ *and* $\mathcal{U}[-l]_{\nu''}$ *be the the associated vector bundles on* $P_l(c') \times C$ *and* $P_{-l}(c'') \times C$ *(cf. (3.7)). Then*

$$\mathrm{ch}(\mathcal{U}_\nu)_{(2)}\big|_{Z^0} = \sum_{(\nu',\nu'')} \exp(u \sum_i \nu'_i)\Big(\mathrm{ch}(\mathcal{U}[l]_{\nu'}) \boxtimes \mathrm{ch}(\mathcal{U}[-l]_{\nu''}) \boxtimes \\ \Big(1 + \Big(l\frac{sr}{r'r''} - \Big(\frac{sr}{r'r''}\Big)^2 \eta\Big) \otimes \omega\Big)\Big)_{(2)}.$$

Applying the Künneth decomposition (cf. (3.12)), one can write any class $\gamma \in H^{i+j+k}(Z_0)$ as a sum

$$\gamma = \sum_{(i,j,k)} \gamma_i \otimes \gamma_j \otimes \gamma_k,$$

where $\gamma_i \in H^i(P_0(c'))$, $\gamma_j \in H^j(P_0(c''))$ and $\gamma_k \in H^k(\mathrm{Jac})$. We will say that the summand $\gamma_i \otimes \gamma_j \otimes \gamma_k$ is of *odd type*, if at least one number from $\{i, j, k\}$ is odd. Note that the integral over Z^0 of any class of odd type is zero.

Now putting Lemma 3.3.2 and equation (3.17) together, we obtain the following statement.

Lemma 3.3.3. *In the notation of Lemma 3.3.2,*

$$\mathrm{ch}(\pi_!(\mathcal{U}_\nu \otimes \mathcal{K}^{\frac{1}{2}})\big|_{Z^0}) = \sum_{(\nu',\nu'')} \exp(u \sum_i \nu'_i)\Big(\mathrm{ch}(\mathcal{U}[l]_{\nu'})_{(2)} \boxtimes \mathrm{ch}(\mathcal{U}[-l]_{\nu''})_{(0)} + \\ \mathrm{ch}(\mathcal{U}[l]_{\nu'})_{(0)} \boxtimes \mathrm{ch}(\mathcal{U}[-l]_{\nu''})_{(2)} + \mathrm{ch}(\mathcal{U}[l]_{\nu'})_{(0)} \boxtimes \mathrm{ch}(\mathcal{U}[-l]_{\nu''})_{(0)} \boxtimes \Big(l\frac{sr}{r'r''} - \Big(\frac{sr}{r'r''}\Big)^2 \eta\Big)\Big) \\ + \textit{classes of odd type}.$$

Example 10. It follows from Example 1 that in rank 3 case $\Pi' = \{2\}$, $\Pi'' = \{1, 3\}$ and the fixed locus Z^0 is the set of vector bundles that split as a direct sum of rank-2 degree-0 stable parabolic bundle and a line bundle of degree 0. We denote by $\mathcal{U}''$ the normalized universal bundle on the moduli space P_0 of rank-2 stable parabolic bundles with trivial determinant. Then for the universal bundle $\mathcal{U}$ from Example 8, the Chern character $\mathrm{ch}(\mathcal{U})_{(2)}|_{Z^0}$ has two summands:

- for $\nu' = (0), \nu'' = (1, 0)$ we have $\mathrm{ch}(\mathcal{U}'')_{(2)} - \frac{1}{4}\eta\, \mathrm{ch}(\mathcal{U}'')_{(0)}$;
- for $\nu' = (1), \nu'' = (0, 0)$ we have $e^u \eta$.

Remark 3.3.4. Recall that in §2.5.2 we identified the functions on V with cohomology classes on $P_0(c)$ and an equivariant cohomology classes on Z^0. Under these identifications, $\mathrm{ch}(U_{\nu'})_{(0)}$ corresponds to the function $\phi^{\nu'}(x)\exp(\langle \nu'_{\det}, x\rangle)$ and $\mathrm{ch}(U_{\nu''})_{(0)}$ corresponds to the function $\phi^{\nu''}(x)\exp(\langle \nu''_{\det}, x\rangle)$.

Now our goal is to calculate the wall-crossing integral (3.13) applying induction by rank based on Theorem 2.3.8. Using (3.17), we can write the inductive hypothesis in the following form:

$$\int_{P_0(c)} \mathrm{ch}(\mathcal{L}(k;\lambda))\mathrm{ch}(U_\nu)_{(2)}\mathrm{Todd}(P_0(c)) =$$

$$N_r \cdot \frac{\partial}{\partial\delta}\Big|_{\delta=0} \sum_{\mathbf{B}\in\mathcal{D}} \underset{\mathbf{B},Q}{\mathrm{iBer}} \left[\det(\mathrm{Hess}(Q))^{g-1}(x) w_\Phi^{1-2g}(x)\exp(\langle\widehat{\lambda}+\nu_{\det},x\rangle)\right](-[c]_{\mathbf{B}}). \quad (3.18)$$

Fixing k and varying λ, we can extend this hypothesis by linearity to the following linear combinations of Chern characters of line bundles

$$\sum_i \mathrm{ch}(\mathcal{L}(k;\lambda^i)) = \mathrm{ch}(\mathcal{L}(k;0))\cdot\sum_i \mathrm{ch}(\mathcal{L}(0;\lambda^i)).$$

Since any polynomial on V, up to a fixed degree may be represented as a linear combination of exponential functions of the form $\exp(\langle\lambda, x\rangle)$, formula (3.18) may be generalized in the following way.

Lemma 3.3.5. *Let $G(x)$ be a formal power series on V, and denote by $G(z)$ the characteristic class in $H^*(P_0(c))$ obtained by the identification of functions on V and cohomology classes on $P_0(c)$ (cf. Remark 3.3.4). Then*

$$\int_{P_0(c)} \mathrm{ch}(\mathcal{L}_0(k;0))G(z)\mathrm{ch}(U_\nu)_{(2)}\mathrm{Todd}(P_0(c)) = N_r\cdot\frac{\partial}{\partial\delta}\Big|_{\delta=0}$$

$$\sum_{\mathbf{B}\in\mathcal{D}} \underset{\mathbf{B},Q}{\mathrm{iBer}} \left[\det(\mathrm{Hess}(Q))^{g-1}(x)G(x) w_\Phi^{1-2g}(x)\exp(\langle\widehat{\lambda}+\nu_{\det},x\rangle)\right](-[c]_{\mathbf{B}}). \quad (3.19)$$

Armed with this statement and equality (3.15), we are ready to calculate the integral (3.13). We start with the case $l = 0$.

- Note that for $l = 0$, $[c^+] = [c'] + [c'']$.
- Then using the induction hypothesis (3.19) and Remark 3.3.4, we conclude that the first summand in Lemma 3.3.3 contributes

$$(k+r)N_{r,k}\sum_{(\nu',\nu'')} \underset{u=0}{\mathrm{Res}}\exp(u\sum\nu'_i)\sum_{\mathbf{B}'\in\mathcal{D}'}\sum_{\mathbf{B}''\in\mathcal{D}''}\underset{\mathbf{B}'}{\mathrm{iBer}}\,\underset{\mathbf{B}''}{\mathrm{iBer}}\Big[w_\Phi^{1-2g}(x/\widehat{k})$$

$$\exp(\langle\widehat{\lambda}+\nu'_{\det}+\nu''_{\det},x/\widehat{k}\rangle)\Big(\frac{-g}{k+r}\mathrm{tr}(\mathrm{Hess}(\varphi^{\nu'})(x/\widehat{k}))\varphi^{\nu''}(x/\widehat{k}) - \sum_{\beta^{[i]}\in\mathbf{B}'}\varphi^{\nu''}(x/\widehat{k})\cdot$$

$$\frac{\varphi^{\nu'}_{\check{\beta}^{[i]}}(x/\widehat{k})\exp(\langle\beta^{[i]},x\rangle)}{1-\exp(\langle\beta^{[i]},x\rangle)} + \sum_{\beta^{[i]}\in\mathbf{B}'}\langle[\c{c}],\check{\beta}^{[i]}\rangle\varphi^{\nu'}_{\check{\beta}^{[i]}}(x/\widehat{k})\varphi^{\nu''}(x/\widehat{k})\Big)\Big](-[c^+]_{\mathbf{B}})\,du \quad (3.20)$$

to the wall-crossing integral (3.13).

- Note that
$$\langle v'_{\mathrm{det}} + v''_{\mathrm{det}}, x\rangle + (x_{\varphi(r')} - x_r)\sum v_i = v_{\mathrm{det}} + w, \tag{3.21}$$
hence after the identification of u with $\beta_{\mathrm{link}} = x_{\varphi(r')} - x_r$ justified in (3.3.7) (see also Remark 3.3.4), we can replace the factors
$$\exp(\langle v'_{\mathrm{det}} + v''_{\mathrm{det}}, x/\hat{k}\rangle)\exp(u\sum v'_i) = \exp(\langle v_{\mathrm{det}} + w, x/\hat{k}\rangle)$$
in (3.20).

- The second summand in Lemma 3.3.3 has the same form as (3.20) with exchanged ν' and ν'', $\mathbf{B}'$ and $\mathbf{B}''$.

- Since
$$\int_{\mathrm{Jac}} \left(-\left(\frac{sr}{r'r''}\right)^2 \eta \cdot \exp\left(\eta \frac{(k+r)r}{r'r''}\right)\right) = \frac{-g}{(k+r)} \frac{s^2 r}{r'r''} \left(\frac{(k+r)r}{r'r''}\right)^g,$$
where the second factor comes from the restriction of $\mathcal{L}(k;\lambda)/E(N_{Z^0})$ to Z^0 (cf. Lemma 2.5.5 and Proposition 2.5.9), the third summand in Lemma 3.3.3 for $l = 0$ contributes
$$-g N_{r,k} \frac{s^2 r}{r'r''} \sum_{(\nu',\nu'')} \operatorname*{Res}_{\beta_{\mathrm{link}}=0} \sum_{\mathbf{B}'\in\mathcal{D}'} \sum_{\mathbf{B}''\in\mathcal{D}''} \operatorname*{iBer}_{\mathbf{B}'} \operatorname*{iBer}_{\mathbf{B}''} [w_\Phi^{1-2g}(x/\hat{k}) \exp(\langle\hat{\lambda}, x/\hat{k}\rangle)$$
$$\exp(\langle v_{\mathrm{det}} + w, x/\hat{k}\rangle)\varphi^{\nu'}(x/\hat{k})\varphi^{\nu''}(x/\hat{k})] \left(-[c^+]_{\mathbf{B}}\right) d\beta_{\mathrm{link}}.$$
to the wall-crossing integral (3.13).

- Finally, using Lemma3.3.1, we arrive at the following statement for $l = 0$.

Proposition 3.3.6. *Let $\mathcal{D}'$ and $\mathcal{D}''$ be diagonal bases of Φ' and Φ'' and let β_{link} be the link edge (cf. page 58). Then*

$$\chi^\nu_+(k;\lambda) - \chi^\nu_-(k;\lambda) = (k+r)N_{r,k} \sum_{\mathbf{B}'\in\mathcal{D}'} \sum_{\mathbf{B}''\in\mathcal{D}''} \operatorname*{Res}_{\beta_{\mathrm{link}}=0} \operatorname*{iBer}_{\mathbf{B}'} \operatorname*{iBer}_{\mathbf{B}''}$$
$$\Bigg[w_\Phi^{1-2g}(x/\hat{k}) \exp(\langle\hat{\lambda} + v_{\mathrm{det}}, x/\hat{k}\rangle)\Bigg(\frac{-g}{k+r}\mathrm{tr}(\mathrm{Hess}(\varphi^\nu))(x/\hat{k}) + l\varphi^\nu_{\check{\beta}_{\mathrm{link}}}(x/\hat{k}) -$$
$$\sum_{i\neq \mathrm{link}} \frac{\varphi^\nu_{\check{\beta}^{[i]}}(x/\hat{k})\exp(\langle\beta^{[i]}, x\rangle)}{1-\exp(\langle\beta^{[i]}, x\rangle)} + \sum_{i\neq\mathrm{link}} \langle [c^+], \check{\beta}^{[i]}\rangle \varphi^\nu_{\check{\beta}^{[i]}}(x/\hat{k})\Bigg)\Bigg] \left(-[c^+]_{\mathbf{B}}\right) d\beta_{\mathrm{link}}. \tag{3.22}$$

Remark 3.3.7. Note that this wall-crossing term coincides with the one from Corollary 3.2.7, and hence with the one from Lemma 3.2.5.

Example 11. Let $z = c_1(\mathcal{F}''_2/\mathcal{F}''_1 \otimes \mathcal{F}''^*_1) \in H^2(P_0)$, where $\mathcal{F}''_i$ are flag bundles on P_0 (cf. Example 10). In particular, we have $\mathrm{ch}(U'')_{(0)} = e^z + 1$.

We saw in Example 6 that in rank-3 case the Chern character of the restriction of the line bundle $\mathcal{L}(k;\lambda)$ multiplied by the inverse of the K-theoretical Euler class of the conormal bundle of Z^0 is equal to

$$\exp\left(\frac{3(k+3)\eta}{2}\right) e^{\lambda_2 u}\mathrm{ch}\left(\mathcal{L}''(k+1;\lambda_1)\right) e^{\frac{z}{2}} \left(2\sinh(u/2)2\sinh((z-u)/2)\right)^{2g-1},$$

where $\mathcal{L}''(k;\lambda)$ is a line bundle $\mathcal{L}(k,(\lambda,-\lambda))$ on P_0. Using Example 10 and Theorem 2.3.8, we conclude (cf. (3.13)) the the wall-crossing term

$$\chi(P_0(<),\mathcal{L}_0(k,\lambda)\otimes\pi_!(U\otimes\mathcal{K}^{\frac{1}{2}}))-\chi(P_0(>),\mathcal{L}_0(k,\lambda)\otimes\pi_!(U\otimes\mathcal{K}^{\frac{1}{2}})) \tag{3.23}$$

is equal to

$$-\left(\frac{3(k+3)}{2}\right)^g \operatorname*{Res}_{u=0} e^{\lambda_2 u}\int_{P_0}\frac{\mathrm{ch}(\mathcal{L}''(k+1;\lambda_1)\otimes\pi_!(U''\otimes\mathcal{K}^{\frac{1}{2}}))e^{\frac{z}{2}}}{(2\sinh(\frac{u}{2})2\sinh(\frac{z-u}{2}))^{2g-1}}\mathrm{Todd}(P_0)du$$
$$-\frac{2g}{3(k+3)}N\cdot\operatorname*{Res}_{u=0}\operatorname*{Res}_{z=0}\frac{e^{\lambda_1 z+\lambda_2 u+z}(e^u+\frac{1+e^z}{4})}{\tilde{w}_\phi(z,u)^{2g-1}(1-e^{(k+3)z})}dzdu,$$

where $\tilde{w}_\phi(z,u)=2\sinh(\frac{z-u}{2})2\sinh(\frac{u}{2})2\sinh(\frac{z}{2})$ and $N=(-1)^g(3(k+3)^2)^g$. This integral is the Euler charactersitic of a vector bundle on the moduli space of degree-0 rank-2 stable parabolic bundles, so we can calculate it using the induction by rank (cf. formula (3.6)). A simple calculation shows that the wall-crossing term (3.23) is equal to

$$-\frac{2g}{3(k+3)}N\cdot\operatorname*{Res}_{u=0}\operatorname*{Res}_{z=0}\frac{e^{\lambda_1 x+\lambda_2 u+z}(1+e^u+e^z)}{(1-e^{z(k+3)})\tilde{w}_\phi(z,u)^{2g-1}}dzdu$$
$$-N\cdot\operatorname*{Res}_{u=0}\operatorname*{Res}_{z=0}\frac{e^{\lambda_1 x+\lambda_2 u+z+(k+3)z}(1-e^z)}{(1-e^{z(k+3)})^2\tilde{w}_\phi(z,u)^{2g-1}}dzdu.$$

Note that this is exactly the same polynomial as in Example 9 after changing (z,u) to $(x,-y)$.

If $l\neq 0$, we will need one more step to calculate the wall-crossing term (3.13), which uses the tautological Hecke correspondences.

3.3.4. *Hecke correspondence*

In §2.6.1 we defined the tautological Hecke operators between the moduli spaces of parabolic bundles with different degrees and parabolic weights as follows: given a vector bundle W on C with a full flag F_* in the fibre W_p at $p\in C$, we considered the associated sheaf of sections $\mathcal{W}$ and defined the subsheaf

$$\mathcal{W}[-1]=\{\gamma\in H^0(C,\mathcal{W})\,|\,\gamma(p)\subset F_{r-1}\}\subset\mathcal{W}.$$

Then $\mathcal{W}[-1]$ is locally free, and thus defines a vector bundle, which we denote by $W[-1]$. Considering the associated morphism of vector bundles $W[-1]\to W$, we defined the full flag G_* in the fibre $W[-1]_p$ and denoted this operator by $\mathcal{H}:(W,F_*)\mapsto(W[-1],G_*)$. We proved that $\mathcal{H}$ induces an isomorphism of the moduli spaces

$$\mathcal{H}:P_d(c_1,c_2,...,c_r)\simeq P_{d-1}(c_2,...,c_r,c_1-1).$$

Applying $\mathcal{H}$ to the normalized universal bundle U on the moduli space $P_0(c)\times C$ we obtain a short exact sequence for the corresponding sheaves of sections:

$$0\to\mathcal{U}[-1]\to\mathcal{U}\to\mathcal{F}_r/\mathcal{F}_{r-1}\to 0.$$

Considering the associated vector bundles, we arrive at the following equality

$$\mathrm{ch}(\mathcal{U}) = \mathrm{ch}(\mathcal{U}[-1]) + \omega \cdot \mathrm{ch}(\mathcal{L}(0;(1,0,..,0,-1))), \tag{3.24}$$

where $\omega \in H^2(C)$ is the fundamental class of the curve (cf. the beginning of §3.3.3).

Remark 3.3.8. Note that under the Hecke isomorphism $\mathcal{H}$, the normalized (cf. §3.2.1) universal bundle $\mathcal{U}$ on the moduli space $P_0(c_1, c_2, ..., c_r) \times C$ corresponds to the universal bundle $\mathcal{U}[-1]$ on the moduli space $P_{-1}(c_2, ..., c_r, c_1 - 1) \times C$ such that the line bundle $\mathcal{F}_2/\mathcal{F}_1$ is trivial.

Similarly, applying the Hecke operator $\mathcal{H}^l$ to the normalized universal bundle $\mathcal{U}$, we obtain the universal bundle $\mathcal{U}[-l]$ on $P_{-l}(c_{l+1}, ..., c_r, c_1 - 1, ..., c_l - 1) \times C$.
Notation: Given an irreducible representation $\rho_\nu : GL_r \to GL(V_\nu)$ of highest weight ν, we consider its weight decomposition

$$V_\nu = \bigoplus_{\mu \in \mathbb{Z}^r} V[\mu],$$

where $V[\mu]$ is the weight space of the weight μ, and we denote by $m_\mu = \dim(V[\mu])$.

Proposition 3.3.9. *Let $\mathcal{U}[-l]_\nu$ be the vector bundle on $P_{-l}(c) \times C$ associated to the irreducible representation ρ_ν of GL_r with highest weight ν and the normalized universal bundle $\mathcal{U}[-l]$ (cf. §3.2.1). Then*

$$\mathrm{ch}(\mathcal{U}_\nu) = \mathrm{ch}(\mathcal{U}[-l]_\nu) + \omega \sum_\mu m_\mu (\mu_1 + ... + \mu_l)\mathrm{ch}(\mathcal{L}(0;(\mu_1, ..., \mu_{r-1}, \mu_r - |\nu|))),$$

where $|\nu| = \sum_i \nu_i$ and the sum runs over the weights μ of n ρ_ν with highest weight ν.

Proof. Given a rank-r vector bundle V on $P_0(c) \times C$ and a symmetric polynomial $f \in \mathbb{C}[y_1, ..., y_r]^{\Sigma_r}$, denote by $f(V) \in H^*(P_0(c) \times C)$ the cohomology class obtained by evaluating f at the Chern roots of V. The flag $\mathcal{F}_1 \subset \mathcal{F}_2 \subset ... \subset \mathcal{F}_r = \mathcal{U}_p$ defines the cohomology classes

$$\xi_i = c_1(\mathcal{F}_{r-i+1}/\mathcal{F}_{r-i} \otimes \mathcal{F}_1^*) \in H^2(P_0(c)),$$

and thus we have

$$\mathrm{ch}(\mathcal{U}_p) = e^{\xi_1} + ... + e^{\xi_{r-1}} + 1;$$

it follows from Remark 3.2.1 that the Chern character of an associated bundle $\mathcal{U}_\nu$ is given by

$$\mathrm{ch}((\mathcal{U}_\nu)_p) = \sum_\mu m_\mu \exp(\mu_1\xi_1 + ... + \mu_{r-1}\xi_{r-1}).$$

We note that the cohomology class $f(\mathcal{U}_p)$ in $H^*(P_0(c) \times C)$ is well-defined for any (not necessarily symmetric) polynomial $f \in \mathbb{C}[y_1, ..., y_r]$.

We introduce the notation $f_i(y_1, ..., y_r) = \frac{1}{i!}(y_1^i + ... + y_r^i)$; in particular, for any vector bundle V on $P_0(c) \times C$, we have $f_i(V) = \mathrm{ch}_i(V)$. It follows from (3.24) that

$$f_i(\mathcal{U}) = f_i(\mathcal{U}[-1]) + \omega\, \partial_{y_1} f_i(\mathcal{U}_p),$$

and thus

$$f_i(\mathcal{U})f_j(\mathcal{U}) = f_i(\mathcal{U}[-1])f_j(\mathcal{U}[-1]) + \omega\,(\partial_{y_1} f_i(\mathcal{U}_p)f_j(\mathcal{U}[-1]) + \partial_{y_1} f_j(\mathcal{U}[-1]_p)f_i(\mathcal{U})) =$$
$$f_i(\mathcal{U}[-1])f_j(\mathcal{U}[-1]) + \omega\, \partial_{y_1}(f_i f_j)(\mathcal{U}_p). \tag{3.25}$$

For the last equality, we used the facts that $\omega\,\mathrm{ch}(U) = \omega\,\mathrm{ch}(U_p)$ and that according to (3.24), $\mathrm{ch}(U_p) = \mathrm{ch}(U[-1]_p)$.

Since any symmetric polynomial $f \in \mathbb{C}[y_1, ..., y_r]^{\Sigma_r}$ may be written as a polynomial in f_i's, (3.25) implies that for any symmetric polynomial f we have:

$$f(U) = f(U[-1]) + \omega\,\partial_{y_1} f(U_p).$$

Let

$$g_\nu(y_1, ..., y_r) = \sum_\mu m_\mu \exp(\mu_1 y_1 + ... + \mu_r y_r);$$

since $g_\nu(U) = \mathrm{ch}(U_\nu)$, we have

$$\mathrm{ch}(U_\nu) = \mathrm{ch}(U[-1]_\nu) + \omega\,\partial_{y_1} g_\nu(U_p),$$

and thus

$$\mathrm{ch}(U_\nu) = \mathrm{ch}(U[-1]_\nu) + \omega \sum_\mu m_\mu \mu_1 \exp(\mu_1 \xi_1 + ... + \mu_{r-1}\xi_{r-1}).$$

Finally, note that

$$\exp(\mu_1 \xi_1 + ... + \mu_{r-1}\xi_{r-1}) = \mathrm{ch}(\mathcal{L}(0; (\mu_1, ..., \mu_{r-1}, \mu_r - |\nu|))),$$

hence we obtain the proof for $l = 1$. Iterating this argument, we obtain the proof for the general case. □

3.3.5. *Wall-crossing for $l \neq 0$*

Recall that our goal is to calculate the wall-crossing integral (3.13) for non-zero l, or, more precisely, to prove Proposition 3.3.6 for the case when $l \neq 0$. The treatment of this case follows the logic of §2.6.2 (cf. page 38), hence, in this section, we will only highlight the differences which arise in our, more general, situation. For simplicity, we assume that l is positive (the other case is analogous).

- We first apply the Hecke operators $\mathcal{H}^l$ and $\mathcal{H}^{-l}$ to the moduli spaces $P_l(c')$ and $P_{-l}(c'')$ to obtain
$$P_0' = P_0(c_{l+1}', ..., c_{r'}', c_1' - 1, ..., c_l' - 1) \simeq P_l(c') \text{ and}$$
$$P_0'' = P_0(c_{r''-l+1}'' + 1, ..., c_{r''}'' + 1, c_1'', ..., c_{r''-l}'') \simeq P_{-l}(c'').$$

- Next, applying the Hecke operator $\mathcal{H}^l \times \mathcal{H}^{-l}$ to the wall-crossing term (3.13), we recast it as an integral over the moduli spaces of degree-0 parabolic bundles $P_0' \times P_0''$, and thus we can calculate this integral using the induction by rank as in §3.3.3.

- As in Chapter 2, to arrive at Proposition 3.3.6 we will need to make additional transformations of the formulas we obtained. We perform this transformation by applying Lemma 3.2.2 with $\mathbf{B} = (\alpha^{\tau\phi(r'),\tau(r)}\,\mathbf{B}'\,\mathbf{B}'')$ and

$$w = \sum_{i=1}^{l} (x_{\phi(r'-l+i)} - x_{\phi(r'+i)}) \in \Lambda,$$

where $\phi \in \Sigma_r$ is the permutation which sends $\{1, ..., r'\}$ to Π' preserving the order of the first r' and the last r'' elements.

The first summand on the right-hand side of (3.9) coincides with the shift of λ we treated in §2.6.2 (cf. equation (2.43)). An easy calculation shows that the second summand on the right-hand side of (3.9) eliminates the changes (cf. Proposition 3.3.9 and equation (3.17)) of the Chern character of $\pi_!(U_\gamma \otimes \mathcal{K}^{\frac{1}{2}})|_{Z^0}$ under the Hecke transformations $\mathcal{H}^l$ and $\mathcal{H}^{-l}$.

- Finally, applying Lemma 3.2.2 with $\mathbf{B} = (\alpha^{\tau\phi(r'),\tau(r)}\mathbf{B}'\mathbf{B}'')$ and $\nu = l\alpha^{\tau\phi(r'),\tau(r)}$, we perform the shift of λ we treated on page 38 and obtain l times the second summand in $\phi^\gamma_{\check{\beta}_{\text{link}}}$ (cf. Lemma 3.3.1). The first summand in $\phi^\gamma_{\check{\beta}_{\text{link}}}$ is obtained from Lemma 3.3.2.

This completes the proof of Proposition 3.3.6 for arbitrary $l \in \mathbb{Z}$.

3.4. Symmetry

The main result of this section is Proposition 3.4.2, where we prove certain symmetry for the Euler characteristics of our vector bundles on the moduli spaces of parabolic bundles.

3.4.1. *Symmetries through Serre duality*

Recall that in §2.7.2 we denoted by $N_{\pm 1}$ the moduli spaces of rank-r degree-± 1 stable vector bundles and by $UN^{\pm}$ the universal bundle over $N_{\pm 1} \times C$, normalized in such a way that $\det(UN^-_p) \simeq \mathcal{L}_{-1}(-r; (1, ..., 1))$ and $\det(UN^+_p) \simeq \mathcal{L}_1(r; (1, ..., 1))$.

In Lemma 2.7.3 we identified the moduli spaces $P_1(>)$ and $P_{-1}(<)$, which are isomorphic to the flag bundles

$$P_1(>) \simeq \mathrm{Flag}(UN^+_p) \xrightarrow{p} N_1 \quad \text{and} \quad P_{-1}(<) \simeq \mathrm{Flag}(UN^-_p) \xrightarrow{p} N_{-1}.$$

The following is easy to verify.

Lemma 3.4.1. *Under the normalization described above, the line bundles $\mathcal{F}_1 \subset p^*(UN^{\pm}_p)$ are isomorphic to $\mathcal{L}_{-1}(-1; (0, ..., 0, 1))$ and $\mathcal{L}_1(1; (0, ..., 0, 1))$, respectively (cf. §3.2.1).*

Applying the Hecke operators $\mathcal{H}^{-1}$ and $\mathcal{H}$ (cf. §3.3.4) to the moduli spaces $P_{-1}(<)$ and $P_1(>)$ we obtain

$$P_0(<) \simeq P_{-1}(<) \quad \text{and} \quad P_0(>) \simeq P_1(>).$$

Let $\tau \in \Sigma_r$ be the cyclic permutation $\tau \cdot (c_1, ..., c_r) = (c_2, ..., c_r, c_1)$, and consider two points in V^*:

$$\theta_1[k] = \frac{k+r}{r} \cdot (1, 1, \ldots, 1) - (k+r)x_r - \rho = \tau \cdot \left(\frac{k}{r} - k, \frac{k}{r}, ..., \frac{k}{r}\right) - \tau \cdot \rho,$$
$$-\frac{k+r}{r} \cdot (1, 1, \ldots, 1) + (k+r)x_1 - \rho = \tau^{-1} \cdot \left(-\frac{k}{r}, ..., -\frac{k}{r}, -\frac{k}{r} + k\right) - \tau^{-1} \cdot \rho.$$

Now we define two polynomials

$$\chi^\gamma_<(k; \lambda) = \chi(P_0(<), \mathcal{L}(k; \lambda) \otimes \pi_!(U_\gamma \otimes \mathcal{K}^{\frac{1}{2}})),$$

$$\chi^{\nu}_{>}(k;\lambda) = \chi(P_0(>), \mathcal{L}(k;\lambda) \otimes \pi_!(U_\nu \otimes \mathcal{K}^{\frac{1}{2}}))$$

and establish the Weyl antisymmetry for the modified polynomials

$$f^{\nu}_{<}(k;\lambda) = \chi^{\nu}_{<}(k;\lambda) - \sum_{\mu} m_\mu\, \mu_1 \chi(P_0(<), \mathcal{L}(k;\lambda + (\mu_1, \ldots, \mu_{r-1}, \mu_r - |\nu|)))$$

and

$$f^{\nu}_{>}(k;\lambda) = \chi^{\nu}_{>}(k;\lambda) + \sum_{\mu} m_\mu\, \mu_r \chi(P_0(>), \mathcal{L}(k;\lambda + (\mu_1, \ldots, \mu_{r-1}, \mu_r - |\nu|))),$$

where we sum over all weights μ of the irreducible representation ρ_ν and $|\nu| = \sum_i \nu_i$ (cf. notation on page 65).

Example 12. In case of rank-3 parabolic bundles and $\nu = (1,0,0)$ (cf. Example 8), we have

$$f^{\nu}_{<}(k,\lambda) = \chi(P_0(<), \mathcal{L}(k;\lambda) \otimes \pi_!(U \otimes \mathcal{K}^{\frac{1}{2}})) - \chi(P_0(<), \mathcal{L}(k;(\lambda_1+1, \lambda_2, \lambda_3 - 1));$$

$$f^{\nu}_{>}(k,\lambda) = \chi(P_0(>), \mathcal{L}(k;\lambda) \otimes \pi_!(U \otimes \mathcal{K}^{\frac{1}{2}})) + \chi(P_0(>), \mathcal{L}(k;(\lambda_1, \lambda_2, \lambda_3)).$$

Proposition 3.4.2. *Let* $\nu_{\det} = \frac{\sum \nu_i}{r}(1, \ldots, 1, 1-r)$*; then the polynomials*

$$f^{\nu}_{<}(k;\lambda + \theta_{-1}[k] - \nu_{\det}) \quad \text{and} \quad f^{\nu}_{>}(k;\lambda + \theta_1[k] - \nu_{\det}))$$

are anti-invariant under the action of the group of permutations of $\lambda_1, \ldots, \lambda_r$.

Proof. First, we will show the anti-invariance of the Euler characteristics of vector bundles on the moduli spaces of degree ± 1 parabolic bundles $P_1(>)$ and $P_{-1}(<)$, as it is simpler. Let $U[1]$ and $U[-1]$ be the universal bundles on $P_1(>) \times C$ and $P_{-1}(<) \times C$ that correspond to the normalized (cf. §3.2.1) universal bundles on $P_0(>)$ and $P_0(<)$, respectively, and let

$$\tilde{\theta}_{-1} = -\tau \cdot \nu_{\det} - \rho \quad \text{and} \quad \tilde{\theta}_1 = -\tau^{-1} \cdot \nu_{\det} - \rho.$$

Applying Serre duality for family of curves to the associated vector bundles $U[\pm 1]_\nu$ (cf. (3.7)) on the moduli spaces $P_{-1}(<)$ and $P_1(>)$, we obtain the following.

Lemma 3.4.3. *The Euler characteristics* $\chi(P_{-1}(<), \mathcal{L}_{-1}(k;\lambda + \tilde{\theta}_{-1}) \otimes \pi_!(U[-1]_\nu \otimes \mathcal{K}^{\frac{1}{2}}))$ *and* $\chi(P_1(>), \mathcal{L}_{-1}(k;\lambda + \tilde{\theta}_1) \otimes \pi_!(U[1]_\nu \otimes \mathcal{K}^{\frac{1}{2}}))$ *are anti-invariant under the permutations of* $\lambda_1, \ldots, \lambda_r$.

Proof. Note that $U[-1] \simeq p^*(UN^-) \otimes (\mathcal{F}_2/\mathcal{F}_1)^*$ (cf. Remark 3.3.8), hence

$$U[-1]_\nu \simeq p^*(UN^-_\nu) \otimes (\mathcal{F}_2/\mathcal{F}_1)^{-\sum \nu_i},$$

where UN^-_ν is a vector bundle on $N_{-1} \times C$ obtained by (3.7) from the universal bundle UN^-. Then

$$\pi_!(U[-1]_\nu \otimes \mathcal{K}^{\frac{1}{2}}) \simeq \pi_!(p^*(UN^-_\nu) \otimes \mathcal{K}^{\frac{1}{2}}) \otimes \mathcal{L}_{-1}(1;(0, \ldots, 0, -1, 0))^{\sum \nu_i}$$

by Lemma 3.4.1, and thus

$$\chi(P_{-1}(<), \mathcal{L}_{-1}(k;\lambda + \tilde{\theta}_{-1}) \otimes \pi_!(U[-1]_\nu \otimes \mathcal{K}^{\frac{1}{2}}) =$$
$$\chi(P_{-1}(<), \mathcal{L}_{-1}(k + \sum \nu_i; \lambda - \frac{\sum \nu_i}{r}(1, \ldots, 1) - \rho) \otimes \pi_!(p^*(UN^-_\nu) \otimes \mathcal{K}^{\frac{1}{2}})).$$

Since the line bundle $\mathcal{L}_{-1}(r;(-1, \ldots, -1))$ is a pullback of the ample generator of $\mathrm{Pic}(N_{-1})$ (cf. Lemma 2.7.4), the statement follows from Serre duality for families of curves (cf. Proposition 2.7.1). The proof for the Euler characteristic on the moduli space $P_1(>)$ is similar. □

Recall that our goal is to show certain antisymmetries for the polynomials $f^{\nu}_{\lessgtr}(k;\lambda)$, which are the linear combinations of the Euler characteristics of vector bundles on the moduli spaces $P_0(\lessgtr)$. We will follow the argument for the polynomial $f^{\nu}_{<}$ (the proof for $f^{\nu}_{>}$ is analogous).

Under the isomorphism $\mathcal{H} : P_0(<) \xrightarrow{\sim} P_{-1}(<)$, vector bundles on $P_0(<)$ correspond to vector bundles on $P_{-1}(<)$. Below, we will write this correspondence explicitly and then will apply Lemma 3.4.3 to the vector bundles on $P_{-1}(<)$ to obtain antisymmetries for the Euler characteristics.

Note that trivially

$$-\nu_{\det} + (\mu_1, \ldots, \mu_{r-1}, \mu_r - \sum \nu_i) = -\frac{\sum \nu_i}{r}(1, \ldots, 1) + \mu, \tag{3.26}$$

and thus it follows from Proposition 3.3.9 that

$$\begin{aligned}
&\chi(P_0(<), \mathcal{L}(k; \lambda + \theta_{-1}[k] - \nu_{\det}) \otimes \pi_!(U_\nu \otimes \mathcal{K}^{\frac{1}{2}})) = \\
&\qquad \chi(P_{-1}(<), \mathcal{L}_{-1}(k; \tau \cdot \lambda - \frac{k}{r}(1, \ldots, 1) + \tilde{\theta}_{-1}) \otimes \pi_!(U[-1]_\nu \otimes \mathcal{K}^{\frac{1}{2}})) + \\
&\qquad\qquad \sum_\mu m_\mu\, \mu_1 \chi(P_0(>), \mathcal{L}(k; \lambda + \theta_{-1}[k] - \frac{\sum \nu_i}{r}(1, \ldots, 1) + \mu)).
\end{aligned} \tag{3.27}$$

Using Lemma 3.4.3 and equations (3.26) and (3.27), for any permutation $\sigma \in \Sigma_r$ we obtain

$$\begin{aligned}
&f^{\nu}_{<}(k; \sigma \cdot \lambda + \theta_{-1}[k] - \nu_{\det}) \overset{(3.26)(3.27)}{=} \\
&\quad \chi(P_{-1}(<), \mathcal{L}_{-1}(k; \tau \cdot \sigma \cdot \lambda - \frac{k}{r}(1, \ldots, 1) + \tilde{\theta}_{-1}) \otimes \pi_!(U[-1]_\nu \otimes \mathcal{K}^{\frac{1}{2}})) + \\
&\quad \sum_\mu m_\mu\, \mu_1 \chi(P_0(<), \mathcal{L}(k; \sigma \cdot \lambda + \theta_{-1}[k] - \frac{\sum \nu_i}{r}(1, \ldots, 1) + \mu)) - \\
&\quad \sum_\mu m_\mu\, \mu_1 \chi(P_0(<), \mathcal{L}(k; \sigma \cdot \lambda + \theta_{-1}[k] - \frac{\sum \nu_i}{r}(1, \ldots, 1) + \mu)) \overset{3.4.3}{=} \\
&= (-1)^\sigma \chi(P_{-1}(<), \mathcal{L}_{-1}(k; \tau \cdot \lambda - \frac{k}{r}(1, \ldots, 1) + \tilde{\theta}_{-1}) \otimes \pi_!(U[-1]_\nu \otimes \mathcal{K}^{\frac{1}{2}})) \overset{(3.27)}{=} \\
&\quad (-1)^\sigma \chi(P_0(<), \mathcal{L}_0(k; \lambda + \theta_{-1}[k] - \nu_{\det}) \otimes \pi_!(U[-1]_\nu \otimes \mathcal{K}^{\frac{1}{2}})) - \\
&\quad (-1)^\sigma \sum_\mu m_\mu\, \mu_1 \chi(P_0(>), \mathcal{L}(k; \lambda + \theta_{-1}[k] - \frac{\sum \nu_i}{r}(1, \ldots, 1) + \mu)) \overset{\text{def}}{=} \\
&\qquad\qquad (-1)^\sigma f^{\nu}_{<}(k; \lambda + \theta_{-1}[k] - \nu_{\det}),
\end{aligned} \tag{3.28}$$

which completes the proof of Proposition 3.4.2 for $f^{\nu}_{<}$. The proof for $f^{\nu}_{>}$ is similar. □

3.4.2. *The Affine Weyl group*

We define an action of the *affine Weyl group* $\Sigma_\nu \ltimes \Lambda$ on $\Lambda \times \mathbb{Z}_{>0}$, which acts trivially on the second factor, the level, and the action at level $k > 0$ is given by

$$\sigma.\lambda = \sigma \cdot (\lambda + \rho + \nu_{\det}) - \rho - \nu_{\det}$$

and

$$\gamma.\lambda = \lambda + (k+r)\gamma \quad \text{for} \quad \sigma \in \Sigma,\ \gamma \in \Lambda.$$[2]

We denote the resulting group of affine-linear transformations of V^* by $\widetilde{\Sigma}[k]$. It is easy to verify that the stabilizer subgroup

$$\Sigma^+_{r,\nu} \stackrel{\text{def}}{=} \mathrm{Stab}(\theta_1[k] - \nu_{\det}, \widetilde{\Sigma}[k]) \subset \widetilde{\Sigma}[k]$$

is generated by the transpositions $s_{i,i+1}$, $1 \leqslant i \leqslant r-2$ and the reflection $\alpha^{r-1,r} \circ s_{r-1,r}$; similarly,

$$\Sigma^-_{r,\nu} \stackrel{\text{def}}{=} \mathrm{Stab}(\theta_{-1}[k] - \nu_{\det}, \widetilde{\Sigma}[k]) \subset \widetilde{\Sigma}[k]$$

is generated by $s_{i,i+1}$, $2 \leqslant i \leqslant r-1$ and the reflection $\alpha^{1,2} \circ s_{1,2}$.

Then Proposition 3.4.2 maybe recast in the following form: the polynomial $f^\nu_>(k;\lambda)$ is anti-invariant with respect to the copy $\Sigma^+_{r,\nu}$ of the symmetric group Σ_r, while $f^\nu_<(k;\lambda)$ is anti-invariant with respect to the copy $\Sigma^-_{r,\nu}$ of the symmetric group Σ_r.

The following statement is straightforward:

Lemma 3.4.4. *Both subgroups $\Sigma^\pm_{r,\nu}$ are isomorphic to Σ_r and for $r > 2$ the two subgroups generate the affine Weyl group $\widetilde{\Sigma}[k]$.*

3.4.3. *Symmetries in residue formulas*

The main result of this section is Proposition 3.4.5, where we show the antisymmetries for the residues formulas on the right-hand side of (3.10).

Recall that in §3.4.1 we defined a pair of polynomials $\chi^\nu_{\lessgtr}$ corresponding to the Euler characteristics from the left-hand side of (3.10) and proved the Weyl antisymmetry for the modified polynomials $f^\nu_{\lessgtr}$. Now we define the two polynomials corresponding to the residue expressions from the right-hand side of (3.10):

$$R^\nu_>(k;\lambda) = N_r \cdot \frac{\partial}{\partial\delta}\Big|_{\delta=0} \sum_{\mathbf{B}\in\mathfrak{D}} \underset{\mathbf{B},Q}{\mathrm{iBer}} \left[\det(\mathrm{Hess}(Q))^{g-1}(x) w_\Phi^{1-2g}(x) \exp(\langle\widehat{\lambda} + \nu_{\det}, x\rangle)\right] (-[\theta_1]_{\mathbf{B}})$$

and

$$R^\nu_<(k;\lambda) = N_r \cdot \frac{\partial}{\partial\delta}\Big|_{\delta=0} \sum_{\mathbf{B}\in\mathfrak{D}} \underset{\mathbf{B},Q}{\mathrm{iBer}} \left[\det(\mathrm{Hess}(Q))^{g-1}(x) w_\Phi^{1-2g}(x) \exp(\langle\widehat{\lambda} + \nu_{\det}, x\rangle)\right] (-[\theta_{-1}]_{\mathbf{B}}),$$

where $\theta_1 = \frac{1}{r} \cdot (1,1,\ldots,1) - x_r$, and $\theta_{-1} = -\frac{1}{r} \cdot (1,1,\ldots,1) + x_1$, and establish the Weyl antisymmetry for the modified pair of polynomials:

$$F^\nu_>(k;\lambda) = R^\nu_>(k;\lambda) + N_{r,k} \cdot \\ \sum_\mu m_\mu\, \mu_r \sum_{\mathbf{B}\in\mathfrak{D}} \underset{\mathbf{B},(k+r)K}{\mathrm{iBer}} \left[w_\Phi^{1-2g}(x) \exp(\langle\widehat{\lambda} + (\mu_1, \ldots, \mu_{r-1}, \mu_r - |\nu|), x\rangle)\right] (-[\theta_1]_{\mathbf{B}})$$

[2]Note that this action differs from the one defined in §2.7.2 by a shift by $\nu_{\det}$.

and

$$F^{\nu}_{<}(k;\lambda) = R^{\nu}_{<}(k;\lambda) - N_{r,k}\cdot \sum_{\mu} m_{\mu}\,\mu_1 \sum_{\mathbf{B}\in\mathcal{D}} \underset{\mathbf{B},(k+r)K}{\mathrm{iBer}} \left[w_{\Phi}^{1-2g}(x)\exp(\langle\widehat{\lambda}+(\mu_1,\dots,\mu_{r-1},\mu_r-|\nu|),x\rangle)\right](-[\theta_{-1}]_{\mathbf{B}}),$$

where, as usual, the sum runs over all weights μ of the irreducible representation ρ_ν and $|\nu| = \sum_i \nu_i$ (cf. notation on page 65).

Proposition 3.4.5. *The polynomial $F^{\nu}_{>}(k;\lambda)$ is anti-invariant with respect to $\Sigma^{+}_{r,\nu}$, and $F^{\nu}_{<}(k;\lambda)$ is anti-invariant with respect to $\Sigma^{-}_{r,\nu}$.*

Proof. We first consider a generator of $\Sigma^{-}_{r,\nu}$ of the type $\sigma = s_{i,i+1}$, $2 \leqslant i \leqslant r-1$. Note that

$$\sigma.\lambda+\rho+\nu_{\det} = \sigma(\lambda+\rho+\nu_{\det}) \quad\text{and}\quad \sigma.\lambda+\rho+\mu-|\nu|x_r = \sigma(\lambda+\rho-|\nu|x_r)+\mu.$$

Using Lemma 2.3.5 and the facts that

$$\sigma\cdot\det(\mathrm{Hess}(Q))(x) = \det(\mathrm{Hess}(Q))(x) \quad\text{and}\quad \sigma\cdot w_{\Phi}^{1-2g}(x) = -w_{\Phi}^{1-2g}(x),$$

we obtain

$$\begin{aligned}
&F^{\nu}_{<}(k;\sigma.\lambda) = \\
&N_r\frac{\partial}{\partial\delta}\Big|_{\delta=0}\sum_{\mathbf{B}\in\mathcal{D}} \underset{\mathbf{B},Q}{\mathrm{iBer}}[-\det(\mathrm{Hess}(Q))^{g-1}(x)w_{\Phi}^{1-2g}(x)\exp(\langle\lambda+\rho+\nu_{\det},x\rangle)]\left(-\sigma^{-1}\cdot[\theta_{-1}]_{\mathbf{B}}\right) - \\
&N_{r,k}\sum_{\mu} m_{\mu}\,\mu_1\sum_{\mathbf{B}\in\mathcal{D}} \underset{\mathbf{B},(k+r)K}{\mathrm{iBer}}\left[-w_{\Phi}^{1-2g}(x)\exp(\langle\widehat{\lambda}+\sigma^{-1}\cdot\mu,x\rangle-|\nu|x_r)\right]\left(-\sigma^{-1}\cdot[\theta_{-1}]_{\mathbf{B}}\right) = \\
&\qquad\qquad -F^{\nu}_{<}(k;\lambda).
\end{aligned}$$

For the last equality we used the Weyl-invariance of the multiplicities of weights μ of the irreducible representation ρ_ν.

The case of the last generator $\sigma = \alpha^{1,2}\circ s_{1,2}$ requires some extra observations. Applying Proposition 2.7.5, we obtain

$$\begin{aligned}
&\sum_{\mu} m_{\mu}\,\mu_1\sum_{\mathbf{B}\in\mathcal{D}} \underset{\mathbf{B},(k+r)K}{\mathrm{iBer}}\left[w_{\Phi}^{1-2g}(x)\exp(\langle\sigma.\lambda+\rho+\mu,x\rangle-|\nu|x_r)\right](-[\theta_{-1}]_{\mathbf{B}}) = \\
&\qquad -\sum_{\mu} m_{\mu}\,\mu_2\sum_{\mathbf{B}\in\mathcal{D}} \underset{\mathbf{B},(k+r)K}{\mathrm{iBer}}\left[w_{\Phi}^{1-2g}(x)\exp(\langle\widehat{\lambda}+\mu,x\rangle-|\nu|x_r)\right](-[\theta_{-1}]_{\mathbf{B}}). \qquad (3.29)
\end{aligned}$$

Since

$$\sigma.\lambda+\rho+\mu-\sum\nu_i x_r = s_{1,2}\cdot(\lambda+\rho)+\mu-|\nu|x_r+(k+r)(x_1-x_2)$$

and

$$s_{1,2}\cdot[\theta_{-1}] = [\theta_{-1}]-(x_1-x_2), \qquad (3.30)$$

we note that

$$\sigma.\lambda+\rho+\nu_{\det} = s_{1,2}\cdot(\lambda+\rho)+\nu_{\det}+(k+r)(x_1-x_2),$$

hence, using (3.29), we obtain

$$F^{\nu}_{<}(k;\sigma.\lambda) = N_r \frac{\partial}{\partial\delta}\Big|_{\delta=0} \sum_{\mathbf{B}\in\mathcal{D}} \underset{\mathbf{B},Q}{\mathrm{iBer}}[-\det(\mathrm{Hess}(Q))^{g-1}(x) w_{\Phi}^{1-2g}(x) \exp(\langle\lambda+\rho+\nu_{\det},x\rangle-(k+r)(x_1-x_2))]\,(-s_{1,2}\cdot[\theta_{-1}]_{\mathbf{B}}) + N_{r,k}\sum_{\mu} m_{\mu}\,\mu_2 \sum_{\mathbf{B}\in\mathcal{D}} \underset{\mathbf{B},(k+r)K}{\mathrm{iBer}}\left[w_{\Phi}^{1-2g}(x)\exp(\langle\widehat{\lambda}+\mu,x\rangle-|\nu|x_r)\right](-[\theta_{-1}]_{\mathbf{B}}). \quad (3.31)$$

Now using (3.30) and applying Lemma 3.2.2 with $w = x_1 - x_2$ to (3.31), we calculate that

$$F^{\nu}_{<}(k;\sigma.\lambda) = N_r \frac{\partial}{\partial\delta}\Big|_{\delta=0} \sum_{\mathbf{B}\in\mathcal{D}} \underset{\mathbf{B},Q}{\mathrm{iBer}}[-\det(\mathrm{Hess}(Q))^{g-1}(x) w_{\Phi}^{1-2g}(x)\exp(\langle\widehat{\lambda}+\nu_{\det},x\rangle)]\,(-[\theta_{-1}]_{\mathbf{B}}) - N_{r,k}\cdot\sum_{\mathbf{B}\in\mathcal{D}} \underset{\mathbf{B},(k+r)K}{\mathrm{iBer}}[-w_{\Phi}^{1-2g}(x)\phi_{\check{x}_{12}}(x)\exp(\langle\widehat{\lambda}+\nu_{\det},x\rangle)]\,(-[\theta_{-1}]_{\mathbf{B}}) + N_{r,k}\cdot\sum_{\mu} m_{\mu}\,\mu_2\sum_{\mathbf{B}\in\mathcal{D}} \underset{\mathbf{B},(k+r)K}{\mathrm{iBer}}\left[w_{\Phi}^{1-2g}(x)\exp(\langle\widehat{\lambda}+\mu,x\rangle-|\nu|x_r)\right](-[\theta_{-1}]_{\mathbf{B}}).$$

Finally, applying the following trivial equality

$$\phi_{\check{x}_{12}}(x)\exp(\langle\nu_{\det},x\rangle) = \sum_{\mu} m_{\mu}\,(\mu_1-\mu_2)\exp(\mu(x))$$

to the last two summands in our expression for polynomial $F^{\nu}_{<}(k;\sigma.\lambda)$, we conclude that $F^{\nu}_{<}(k;\sigma.\lambda) = -F^{\nu}_{<}(k;\lambda)$. This finishes the proof of the anti-invariance of the polynomial $F^{\nu}_{<}(k;\lambda)$; the proof for $F^{\nu}_{>}(k;\lambda)$ is similar. □

Note that the two differences $\chi^{\nu}_{<} - f^{\nu}_{<}$ and $\chi^{\nu}_{>} - f^{\nu}_{>}$ (cf. page 68) have the form of a linear combination of the Euler characteristics of line bundles on the moduli spaces of parabolic bundles; while the differences $R^{\nu}_{<} - F^{\nu}_{<}$ and $R^{\nu}_{>} - F^{\nu}_{>}$ (cf. page 70) may be written as an iterated residue of a meromorphic functions. Then using the residue formula for the Euler characteristic of line bundles, Theorem 2.3.8, we arrive at the following statement.

Proposition 3.4.6. *For polynomials $R^{\nu}_{>}, R^{\nu}_{<}$, $\chi^{\nu}_{>}, \chi^{\nu}_{<}$, $F^{\nu}_{>}, F^{\nu}_{<}$ and $f^{\nu}_{>}, f^{\nu}_{<}$ defined on pages 70 and 68, we have:*

$$\chi^{\nu}_{>}(k;\lambda) - f^{\nu}_{>}(k;\lambda) = R^{\nu}_{>}(k;\lambda) - F^{\nu}_{>}(k;\lambda);$$
$$\chi^{\nu}_{<}(k;\lambda) - f^{\nu}_{<}(k;\lambda) = R^{\nu}_{<}(k;\lambda) - F^{\nu}_{<}(k;\lambda).$$

3.5. Proof of Theorem 3.2.3 and some generelizations of our result

In this section, we finish the proof of our main result and present some of its generalizations.

3.5.1. *Proof of Theorem 3.2.3*

The proof of Theorem 3.2.3 follows the logic of Chapter 2. In this section, we repeat the argument from §2.7.4 with only minor changes.

Recall that in §2.1.2 we introduced a chamber structure on $\Delta \subset V^*$ created by the walls $S_{\Pi,l}$, where $\Pi = (\Pi', \Pi'')$ is a nontrivial partition, and $l \in \mathbb{Z}$. Denote by

$$\check{\Delta} = \{(k; a) | \, a/k \in \Delta\} \subset \mathbb{R}_{>0} \times V^*$$

the cone over $\Delta \subset V^*$, and let

$$\check{\Delta}^{\mathrm{reg}} = \{(k; a) | \, a/k \in \Delta \text{ is regular}\} \subset \check{\Delta}$$

be the set of its regular points. Denote by $\check{S}_{\Pi,l} \subset \check{\Delta}$ the cone over the wall $S_{\Pi,l} \subset \Delta$; then $\check{\Delta}^{\mathrm{reg}}$ is the complement of the union of walls $\check{S}_{\Pi,l}$ in $\check{\Delta}$. Finally, denote by $\check{\Delta}^{\mathrm{reg}}_{\Lambda}$ the intersection of the lattice $\mathbb{Z}_{>0} \times \Lambda$ with $\check{\Delta}^{\mathrm{reg}}$.

By substituting $c = \lambda/k$, we can consider the left-hand side and the right-hand side of the equation in Theorem 3.2.3 as functions in $(k, \lambda) \in \check{\Delta}^{\mathrm{reg}}_{\Lambda}$. We denote by $\chi(k; \lambda)$ and $R(k; \lambda)$ the left-hand side and the right-hand side, correspondingly.

We showed that $\chi(k; \lambda)$ and $R(k; \lambda)$ are *polynomials* on the cone over each chamber in Δ (cf. §2.1.2, §3.2.2). We proved that the wall-crossing terms, i.e. the differences between polynomials on neighbouring chambers, for $\chi(k; \lambda)$ (cf. Proposition 3.3.6) and for $R(k; \lambda)$ (cf. Corollary 3.2.7) coincide, hence there exists a polynomial $\Theta(k; \lambda)$ on $\mathbb{Z}_{>0} \times \Lambda$, such that the restriction of $\Theta(k; \lambda)$ to $\check{\Delta}^{\mathrm{reg}}_{\Lambda}$ is equal to the difference $\chi(k; \lambda) - R(k; \lambda)$.

Now for $r > 2$, we can conclude that

$$\Theta(k; \lambda) = \chi^{\nu}_{>}(k; \lambda) - R^{\nu}_{>}(k; \lambda) = \chi^{\nu}_{<}(k; \lambda) - R^{\nu}_{<}(k; \lambda),$$

where $\chi^{\nu}_{\gtrless}(k; \lambda)$ and $R^{\nu}_{\gtrless}(k; \lambda)$ are the restrictions of $\chi(k; \lambda)$ and $R(k; \lambda)$ to two specific chambers defined in Lemma 2.7.3. Then, according to Proposition 3.4.6,

$$\Theta(k; \lambda) = f^{\nu}_{>}(k; \lambda) - F^{\nu}_{>}(k; \lambda) = f^{\nu}_{<}(k; \lambda) - F^{\nu}_{<}(k; \lambda).$$

It follows from Propositions 3.4.2 and 3.4.5 that the polynomial $\Theta(k; \lambda)$ is anti-invariant with respect to the action of the subgroups $\Sigma^{\pm}_{r,\nu}$ (cf. the end of §3.4.2), and hence by Lemma 3.4.4, it is anti-invariant under the action of the entire affine Weyl group $\tilde{\Sigma}[k]$. It is easy to see that any such polynomial function has to vanish, and thus $\chi(k; \lambda) = R(k; \lambda)$.

As marked above, the argument does not work for $r = 2$, since in this case the groups $\Sigma^{+}_{r,\nu}$ and $\Sigma^{-}_{r,\nu}$ (cf. §3.4.2) coincide, and thus they do not generate the entire affine Weyl group. A solution is to consider the 2-punctured case, treated in §3.1.1-§3.1.3; this finishes the proof of Theorem 3.2.3.

3.5.2. *Generalization*

Now we formulate a mild generalization of our result, Theorem 3.5.1, and explain, following an idea of Teleman and Woodward [24], how our formulas can be used to calculate the Euler characteristic of a more general class of vector bundles on the moduli spaces of parabolic vector bundles.

Let $\nu[1], \ldots, \nu[m]$ be dominant weights of GL_r. Replacing Q and $\nu_{\det}$ in Theorem 3.2.3 by the multi-parameter version

$$\mathbf{Q} = (k + r)K - \sum_j \delta_j \phi^{\nu[j]}, \quad \mathbf{v}_{\det} = \sum_{j=1}^{m} (1, \ldots, 1, 1 - r) \frac{\sum_i \nu[j]_i}{r},$$

we can deduce the following Theorem.

Theorem 3.5.1. *Let* $\mathbf{Q}$ *and* $\mathbf{v}_{\det}$ *be as above, let* $\mathcal{K}$ *be the canonical class of the curve* C, $\lambda \in \Lambda$, $k \in \mathbb{Z}_{>0}$, $\nu = (\nu_1 \geqslant \nu_2 \ldots \geqslant \nu_r) \in \mathbb{Z}^r$, $\widehat{\lambda} = \lambda + \rho$, *and let* $c \in \Delta$ *be a regular element (cf. page 7). Then for any diagonal basis* $\mathcal{D} \in \mathcal{B}$, *the following equality holds:*

$$\chi(P_0(c), \mathcal{L}(k;\lambda) \otimes \pi_!(U_{\nu[1]} \otimes \mathcal{K}^{\frac{1}{2}}) \otimes \pi_!(U_{\nu[2]} \otimes \mathcal{K}^{\frac{1}{2}}) \otimes \ldots \otimes \pi_!(U_{\nu[m]} \otimes \mathcal{K}^{\frac{1}{2}})) =$$
$$N_r \cdot \frac{\partial^m}{\partial \delta_1 \ldots \partial \delta_m}\Big|_{\delta_1 = \ldots = \delta_m = 0} \sum_{\mathbf{B} \in \mathcal{D}} \underset{\mathbf{B},\mathbf{Q}}{\mathrm{iBer}} \left[\det(\mathrm{Hess}(\mathbf{Q}))^{g-1}(x) w_{\Phi}^{1-2g}(x) \exp(\langle \widehat{\lambda} + \mathbf{v}_{\det}, x \rangle)\right] (-[c]_{\mathbf{B}}).$$

The proof of this theorem is analogous to our proof of Theorem 3.2.3.

Using Theorem 3.5.1 one can also obtain formulas for the Euler characteristics of vector bundles, which involve the exterior powers $\bigwedge^l \pi_!(U_\nu \otimes \mathcal{K}^{\frac{1}{2}})$. Let us briefly explain the case

$$\chi\left(P_0(c), \mathcal{L}(k;\lambda) \otimes \bigwedge^2 \pi_!(U_\nu \otimes \mathcal{K}^{\frac{1}{2}})\right). \tag{3.32}$$

Recall that the n-th Adams operator ψ^n is defined by $\psi^n L = L^n$ for a line bundle L and extends to K-theory additively by the splitting principle. It follows from the Groethendieck-Riemann-Roch theorem and equation (3.17) that

$$\mathrm{ch}(\psi^n(\pi_!(U_\nu \otimes \mathcal{K}^{\frac{1}{2}}))) = \sum_{i \geqslant 0} n^i \cdot \mathrm{ch}_i(\pi_!(U_\nu \otimes \mathcal{K}^{\frac{1}{2}})) =$$
$$\frac{1}{n} \sum_{i \geqslant 1} n^i \cdot \pi_*(\mathrm{ch}_i(U_\nu)) = \frac{1}{n} \pi_* \mathrm{ch}(\psi^n(U_\nu)) = \frac{1}{n} \mathrm{ch}(\pi_!(\psi^n(U_\nu) \otimes \mathcal{K}^{\frac{1}{2}})). \tag{3.33}$$

Since for any vector bundle V

$$\mathrm{ch}\left(\bigwedge^2 V\right) = \frac{\mathrm{ch}(V^{\otimes 2}) - \mathrm{ch}(\psi^2 V)}{2},$$

the Euler characteristic (3.32) equals

$$\frac{1}{2}\chi(P_0(c), \mathcal{L}(k;\lambda) \otimes (\pi_!(U_\nu \otimes \mathcal{K}^{\frac{1}{2}}))^2) - \frac{1}{4}\chi(P_0(c), \mathcal{L}(k;\lambda) \otimes \pi_!(\psi^2(U_\nu) \otimes \mathcal{K}^{\frac{1}{2}})).$$

Finally, note that the character function (cf. page 54) for $\psi^n(U_\nu)$ is $\phi^\nu(x^n)$, hence using Theorem 3.5.1, we obtain the formula for the Euler characteristic (3.32).

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