Equivariant Jeffrey-Kirwan theorem in non-compact settings

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This publication URL: https://archive-ouverte.unige.ch/unige:35402
Publication DOI: 10.13097/archive-ouverte/unige:35402

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Equivariant Jeffrey-Kirwan theorem in non-compact settings

Thèse

présentée à la Faculté des Sciences de l'Université de Genève
pour obtenir le grade de Docteur ès sciences, mention matématiques

par

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de
Roumanie

Thèse N° 4636

Genève
Atelier d'impression ReproMail de l'Université de Genève
2014

UNIVERSITÉ DE GENÈVE

FACULTÉ DES SCIENCES

Doctorat ès sciences
Mention mathématiques

Thèse de *Monsieur Géza Zsolt SZILAGYI*

intitulée :

" Equivariant Jeffrey-Kirwan theorem in non-compact settings "

La Faculté des sciences, sur le préavis de Messieurs A. SZENES, professeur ordinaire et directeur de thèse (Section de mathématiques), A. ALEXEEV, professeur ordinaire (Section de mathématiques) et T. HAUSEL, professeur (Département de mathématiques, Ecole Polytechnique Fédérale de Lausanne, Suisse), autorise l'impression de la présente thèse, sans exprimer d'opinion sur les propositions qui y sont énoncées.

Genève, le 30 novembre 2013

Thèse - 4636 -

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Résumé

Dans cette thèse, nous généralisons le théorème de Jeffrey-Kirwan pour le cas non compact et équivariant. Nous l'appliquons au calcul de volumes équivariants symplectiques et d'anneaux de cohomologie des quotients symplectiques ou hyper-Kähler non compacts.

Nous considérons un quotient symplectique $M/\!/G$ non compact d'une G-variété Hamiltonienne M. Nous suivons l'approche de Hausel-Proudfoot [19] pour surmonter le problème de non convergence des intégrales sur les variétés non compactes M et $M/\!/G$ en supposant qu'il existe une action Hamiltonienne sur M d'un tore S tel que l'ensemble des points fixes M^S est compact. Nous définissons ensuite les intégrales $\oint_M \beta$ et $\oint_{M/\!/G} \kappa_S(\beta)$ d'une classe de cohomologie équivariante $\beta \in H_{G\times S}(M)$ formellement par la formule d'Atiyah-Bott-Berline-Vergne. De plus, motivés par Prato-Wu [40], Proudfoot [41] et Martens [31], nous posons la condition additionnelle que le tore S contient un sous-tore de dimension 1 admettant une application moment propre et bornée par en bas.

Nous introduisons EqRes^Λ, une version équivariante du résidu de Jeffrey-Kirwan et nous montrons qu'il admet des propriétés similaires à la version classique. Nous le comparons à la version de Martens [31]. Avec les conditions mentionnées plus haut, nous montrons la formule suivante (Théorème 4.5),

$$\oint_{M/\!/G} \kappa_S(\beta e^{\omega-\mu_{G\times S}}) = \lim_{\epsilon\to 0} \mathrm{EqRes}^\Lambda \left(\frac{\varpi}{vol(T)|W|} \oint_M \beta e^{\omega-\mu_{T\times S}+\epsilon\rho} \right),$$

c'est-à-dire qu'on peut calculer les intégrales formelles sur $M/\!/G$ à partir des intégrales formelles sur M par le résidu équivariant si on choisit une polarisation Λ compatible avec l'action de K. Nous donnons une variante de la formule ci-dessus pour les quotients hyper-Kähler $M/\!/\!/\!/G$ (Théorème 4.14).

Dans l'hypothèse où l'application de Kirwan κ_S est surjective, on peut calculer les anneaux de cohomologie ordinaire et équivariant des quotients non compacts en utilisant la formule ci-dessus combinée à la forme bilinéaire non dégénérée de Hausel-Proudfoot [19] et à la propriété de formalité équivariante, cf. [26, 41]. On remarque qu'avec nos conditions, l'application de Kirwan est surjective dans le cas symplectique, cf. [41], mais n'est en général pas connue dans le cas hyper-Kähler. Ce principe est illustré par le schéma de Hilbert sur le plan complexe.

Par le théorème de Prato-Wu [40], la formule ci-dessus peut aussi être utilisée pour le calcul de

volumes symplectiques équivariants de quotients symplectiques ou hyper-Kähler non compacts. On illustre ce type d'applications par le calcul de la fonction de partition de Nekrasov [38] sur l'espace des modules des faisceaux sans torsion sur $\mathbb{CP}^2$ avec rang et deuxième classe de Chern fixés. Dans ce cas-là, nous arrivons au même résultat que dans Nakajima-Yoshioka [37].

En général, on obtient des variétés plus compliquées par la réduction symplectique ou hyper-Kähler que les variétés initiales, donc on peut s'attendre à ce que les calculs par notre formule soient plus simples que les calculs directs sur les quotients.

Preface

In the thesis we generalize the Jeffrey-Kirwan theorem to the non-compact and equivariant setting. We demonstrate how it can be applied to compute equivariant symplectic volumes and cohomology rings of non-compact symplectic or hyperKähler quotients.

We consider a non-compact symplectic quotient $M/\!\!/G$ of a Hamiltonian G-manifold M. We follow the approach of Hausel-Proudfoot [19] to overcome the non-convergence problem of integrals on non-compact manifolds M and $M/\!\!/G$ by presuming the existence of an auxiliary Hamiltonian torus action S on M with compact fixed point set M^S and by defining the integrals $\oint_M \beta$ and $\oint_{M/\!\!/G} \kappa_S(\beta)$ of equivariant cohomology classes $\beta \in H_{G\times S}(M)$ formally by the Atiyah-Bott-Berline-Vergne localization formula. Motivated by Prato-Wu [40], Proudfoot [41] and Martens [31] we also pose the additional condition that S contains an 1-dimensional subtorus K with proper and bounded below moment map.

We introduce our main computational tool EqRes^Λ, an equivariant version of the Jeffrey-Kirwan residue and we show that it shares similar properties with the classical one. We also compare it with the version given by Martens [31]. Under the above assumption we prove the following formula (Theorem 4.5)

$$\oint_{M/\!\!/G} \kappa_S(\beta e^{\omega-\mu_{G\times S}}) = \lim_{\epsilon\to 0} \mathrm{EqRes}^\Lambda \left(\frac{\varpi}{vol(T)|W|} \oint_M \beta e^{\omega-\mu_{T\times S}+\epsilon\rho} \right).$$

That is, we can compute formal integrals on the quotient $M/\!\!/G$ out of formal integrals on the original space M using the equivariant Jeffrey-Kirwan residue when the polarization Λ is compatible with the K-action. We also give a similar formula when the symplectic quotient $M/\!\!/G$ is replaced by the hyperKähler quotient $M/\!\!/\!\!/\!\!/G$ (Theorem 4.14).

Under the assumption that the Kirwan map κ_S is surjective the above formula can be used to compute the equivariant and ordinary cohomology rings of non-compact quotients if we couple it with the non-degenerate bilinear pairing of Hausel-Proudfoot [19] and equivariant formality, cf. [26, 41]. This principle also works in the hyperKähler case and we demonstrate it on the example of Hilbert scheme of points on the plane. We remark that the Kirwan surjectivity holds in our setup for symplectic quotients, cf. [41], but in the hyperKähler case is generally not known.

By the Prato-Wu theorem [40] the above formula can be also used for equivariant symplectic volume computations on symplectic or hyperKähler quotients. As an illustration we compute

Nekrasov's partition function [38] on the framed moduli space of torsion free sheaves on $\mathbb{CP}^2$ with fixed rank and second Chern class using Nakajima quiver model [36]. We get back the result of Nakajima-Yoshioka [37] computed with Atiyah-Bott-Berline-Vergne localization formula on the quotient.

In general, we get topologically more complicated spaces from simpler ones by symplectic or hyperKähler reduction. Therefore, we can expect that the computation of the formal integrals on M and the evaluation of the equivariant Jeffrey-Kirwan residue is easier than the direct computation on the quotients $M/\!\!/G$ or $M/\!\!/\!\!/\!\!/G$.

Acknowledgements

Most importantly I would like to express my deepest gratitude to my advisor András Szenes for his time, patience and support during my work. I am very thankful for the knowledge he shared with me about localization techniques.

I am grateful to Prof. Anton Alekseev and Prof. Tamás Hausel who kindly agreed to be the referees of my thesis. The financial support of University of Geneva and Swiss National Foundation is gratefully acknowledged.

I would like to thank to Gergő Bérczi, Michèle Vergne and Richárd Rimányi for helpful discussions. I wish to thank to all the people having a great influence to become a mathematician. Especially I am thankful to my teachers and colleges Szilárd András, Csaba Varga, András Márkus, Andor Lukács and Árpád Baricz.

Ezúton szeretném megköszönni szüleimnek, hogy áldozatos segítségükkel mindez megvalósulhatott.

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1

Introduction

In this chapter we recall basic notions, constructions and results about equivariant cohomology, characteristic classes, orbifolds and symplectic geometry. Here we also present conventions and examples which will be used later in the text.

1.1 Equivariant cohomology

1.1.1 Group actions and fundamental vector fields

Let G be a compact connected Lie group and M be a manifold. A right group action is a (smooth) map $G \times M \to M$, $(g, x) \mapsto x \cdot g$ with property $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2)$ for all $g_1, g_2 \in G$ and $x \in M$. To any $\xi \in \mathfrak{g}$ we can associate a vector field $\underline{\xi} \in \mathfrak{X}(M)$ on M such that

$$\underline{\xi}_x = \frac{d}{dt}\Big|_{t=0} x \cdot \exp(t\xi), \qquad \forall\, x \in M$$

and $\exp : \mathfrak{g} \to G$ is the exponential map. This association defines a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{X}(M)$, $\xi \mapsto \underline{\xi}$.

Similarly, a left group action of G on M is a map $G \times M \to M$, $(g, x) \to g \cdot x$ satisfying relation $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ for all $g_1, g_2 \in G$ and $x \in M$. We can also associate a vector field $\underline{\xi} \in \mathfrak{X}(M)$ to any $\xi \in \mathfrak{g}$ such that

$$\underline{\xi}_x = -\frac{d}{dt}\Big|_{t=0} \exp(-t\xi) \cdot x, \qquad \forall\, x \in M.$$

It induces a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{X}(M)$, $\xi \mapsto \underline{\xi}$.

Definition 1.1. The vector field $\underline{\xi}$ is called the *fundamental vector field* of ξ.

Remark 1.2. (i) From a right action $G \times M \to M$, $(g, m) \mapsto m \cdot g$ we can construct a left action by setting $g \cdot m = m \cdot g^{-1}$. The fundamental vector field of this left action will be the opposite of the fundamental vector field of the original right action.

(ii) For G abelian the same action can be considered both as left and as right action. In this case both actions yield the same fundamental vector field.

◇

We say that G acts on M locally freely if for any $\xi \in \mathfrak{g}$ the fundamental vector field $\underline{\xi}$ is nowhere vanishing. A left G-action on M induces a left action on $\mathfrak{X}(M)$

$$(g, v) \mapsto dg(v), \qquad \forall\, g \in G,\ \forall\, v \in \mathfrak{X}(M),$$

and on differential forms $\Omega(M)$ on M

$$(g, \beta) \mapsto \left(g^{-1}\right)^{*} \beta, \qquad \forall\, g \in G,\ \forall\, \beta \in \Omega(M),$$

where we consider $g^{-1} \in G$ as a map $M \to M$, $x \mapsto g^{-1}x$.

1.1.2 Cartan model

Denote $S\mathfrak{g}^*$ the symmetric algebra of $\mathfrak{g}^*$ with grading induced by $\deg w = 2$ for any non-zero $w \in \mathfrak{g}^*$. We can consider homogeneous element of $S\mathfrak{g}^*$ as homogeneous polynomials in $\mathbb{R}[\mathfrak{g}]$ and their degree in $S\mathfrak{g}^*$ is twice of the polynomial degree in $\mathbb{R}[\mathfrak{g}]$. Moreover, $S\mathfrak{g}^*$ has a natural G-action induced by the coadjoint action on $\mathfrak{g}^*$. If M is a G-manifold then we consider on $\Omega(M) \otimes S\mathfrak{g}^*$ the diagonal G-action

$$g^*(\alpha \otimes \mathrm{p}) = g^*\alpha \otimes \mathrm{Ad}_g^*(\mathrm{p}), \qquad \forall\, g \in G,$$

and the contraction by a vector field

$$\iota_v(\alpha \otimes \mathrm{p}) = (\iota_v \alpha) \otimes \mathrm{p}, \qquad \forall\, v \in \mathfrak{X}(M).$$

On $\Omega(M) \otimes S\mathfrak{g}^*$ we take the total grading and the graded commutative multiplication

$$(\beta_1 \otimes \mathrm{p}_1) \cdot (\beta_2 \otimes \mathrm{p}_2) = (\beta_1\beta_2) \otimes (\mathrm{p}_1\mathrm{p}_2).$$

Definition 1.3. The graded differential algebra of *G-equivariant differential forms* on M is $\Omega_G(M) = (\Omega(M) \otimes S\mathfrak{g}^*)^G$ with *equivariant differential*

$$D_G(\alpha \otimes \mathrm{p}) = d\alpha \otimes \mathrm{p} - \sum_{i=1}^{r} \iota_{\underline{\xi}^i} \alpha \otimes u_i \mathrm{p},$$

where d is the exterior differential on M and $\{\xi^1, \ldots, \xi^r\}$ is a basis of $\mathfrak{g}$ with dual basis $\{u_1, \ldots, u_r\}$. The *equivariant cohomology* $H_G(M)$ of M is the cohomology of the chain complex $(\Omega_G(M), D_G)$.

Remark 1.4. (i) If $G = T$ is a torus then $\Omega_G(M) = \Omega(M)^T \otimes S\mathfrak{t}^*$.

(ii) We can consider equivariant differential forms $\beta \in \Omega_G(M)$ as G-equivariant polynomial maps $\beta : \mathfrak{g} \to \Omega(M)$ with equivariant differential $(D_G\beta)(\xi) = d(\beta(\xi)) - \iota_{\underline{\xi}}(\beta(\xi))$ for all $\xi \in \mathfrak{g}$ (cf. [45]).

◇

Theorem 1.5. *If G is a connected compact Lie group with maximal torus T and Weyl group W then*

$$H_G(M) \simeq H_T(M)^W$$

induced by restriction $\mathfrak{g}^* \to \mathfrak{t}^*$ (cf. [15], Theorem 6.8.2).

1.1.3 Connection and curvature forms

Let $P \to M$ be a (left) principal G-bundle (i.e. G acts on P from left).

Definition 1.6. A *connection form* θ on P is a Lie algebra valued 1-form with the following properties

(1) $\theta \in (\Omega(P) \otimes \mathfrak{g})^G$, where we consider on $\mathfrak{g}$ the adjoint action and on $\Omega(P) \otimes \mathfrak{g}$ the diagonal G-action,

(2) $\iota_{\underline{\xi}}\theta = \xi$ for all $\xi \in \mathfrak{g}$.

Any principal bundle admits a connection form.

Example 1.1. Let $G \subset GL_n(\mathbb{R})$ be a Lie group considered as a (right) principal G-bundle over the point. Then the Maurer-Cartan form $(\theta_{MC})_g = g^{-1}dg$ is a connection form on G as principal bundle. $\diamond$

Remark 1.7. If $\{\xi^1, \dots, \xi^r\}$ is a basis of $\mathfrak{g}$ then we can write $\theta = \sum_{i=1}^{r} \theta_i \otimes \xi^i$, where θ_i are 1-forms on P. Then the first condition is equivalent to $\sum_{i=1}^{r} (g^*\theta_i) \otimes \xi^i = \sum_{i=1}^{r} \theta_i \otimes \mathrm{Ad}_g \xi^i$, and the second condition is equivalent to $\iota_{\underline{\xi}^i}\theta_j = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker symbol. $\diamond$

Let K be a compact Lie group. $\pi : P \to M$ is a *K-equivariant principal G-bundle* if K acts on P and M, the G- and K-action on P commute and π is K-equivariant. A connection form $\theta \in (\Omega(P) \otimes \mathfrak{g})^G$ is *K-invariant* if $k^*\theta = \sum_{i=1}^{r} k^*\theta_i \otimes \xi^i = \theta$ for all $k \in K$. We can get an invariant connection form out of any connection form by averaging over K. That is, $\int_K k^*\theta dk$ is a K-invariant connection for any θ connection form on P.

Definition 1.8. The *curvature form* Θ of the connection form θ is defined as

$$\Theta = d\theta + \frac{1}{2}[\theta, \theta],$$

where $[\theta, \theta] = \left[\sum_{i=1}^{r} \theta_i \otimes \xi^i, \sum_{j=1}^{r} \theta_j \otimes \xi^j \right] = \sum_{i,j=1}^{r} \theta_i\theta_j \otimes \left[\xi^i, \xi^j\right]$.

The curvature form has the following properties.

(1) It is G-invariant, i.e. $\Theta \in (\Omega^2(P) \otimes \mathfrak{g})^G$.

(2) It is horizontal, i.e. $\iota_{\underline{\xi}}\Theta = 0$ for all $\xi \in \mathfrak{g}$.

(3) $d\Theta = [\Theta, \theta]$ (Bianchi identity or Cartan structure equation).

Definition 1.9. The *equivariant curvature* Θ_K of a K-invariant connection form θ is defined as

$$\Theta_K = D_K\theta + \frac{1}{2}[\theta, \theta].$$

1.1.4 Cartan isomorphism

Let $\pi : P \to M$ be a K-equivariant principal G-bundle and let θ be a K-invariant connection form on P.

Definition 1.10. The set of *horizontal forms* on P is defined as

$$\Omega(P)_{hor} = \{\alpha \in \Omega(P) \,|\, \iota_{\underline{\xi}}\alpha = 0,\ \forall \xi \in \mathfrak{g}\}.$$

We call G-invariant and horizontal forms on P *basic* and we use notation $\Omega(P)_{bas} = \Omega(P)^G_{hor}$. Moreover, we define $\Omega_K(P)_{bas} = \big(\Omega(P)_{bas} \otimes S\mathfrak{k}^*\big)^K$.

Remark 1.11. We can identify $\Omega(M)$ with basic forms $\Omega(P)_{bas}$ via the pull-back $\pi^* : \Omega(M) \to \Omega(P)$. $\diamond$

Definition 1.12. We define the *horizontal morphism* $\mathrm{Hor}_\theta : \Omega(P) \to \Omega(P)_{hor}$ by

$$\mathrm{Hor}_\theta(\alpha) = \prod_{i=1}^{r}(1 - \theta_i \iota_{\underline{\xi}^i})\alpha,$$

where $\{\xi^1, \dots, \xi^r\}$ is a basis of $\mathfrak{g}$ and $\theta = \sum_{i=1}^{r} \theta_i \otimes \xi^i$.

Remark 1.13. The horizontal morphism Hor_θ is a projection to $\Omega(P)_{hor}$, that is, $\mathrm{Hor}_\theta(\alpha) = \alpha$ for all $\alpha \in \Omega(P)_{hor}$. $\diamond$

Theorem 1.14 (Equivariant Cartan isomorphism)**.** *The chain homomorphism*

$$\mathcal{C}^K_\theta : (\Omega_{G\times K}(P),\, D_{G\times K}) \to (\Omega_K(P)_{bas},\, D_K), \quad \mathcal{C}^K_\theta\Big(\sum_j \alpha_j \otimes \mathrm{p}_j\Big) = \mathrm{Hor}_\theta\Big(\sum_j \alpha_j \otimes \mathrm{p}_j(\Theta_K)\Big)$$

induces an isomorphism $(\mathcal{C}^K_\theta)_* : H_{G\times K}(P) \to H_K(M)$*. Moreover,* $\mathcal{C}^K = \big(\mathcal{C}^K_\theta\big)_*$ *does not depend on the choice of* θ *and it is called the (equivariant) Cartan isomorphism (cf. [15], Theorem 5.2.1).*

1.1.5 Associated bundles

Let $P \to M$ be a (right) principal G-bundle and let $\rho : G \to GL(V)$ be a representation. Consider the G-action on $P \times V$ defined by

$$(p, v) \cdot g = (pg,\, g^{-1}v), \qquad \forall g \in G,\ \forall p \in P,\ \forall v \in V.$$

The quotient space $P \times_G V = (P \times V)/G$ is a vector bundle over M. Denote by $[p, v]$ the class of (p, v) and remark that $[pg, v] = [p, gv]$. If $P \to M$ is a K-equivariant principal G-bundle then $P \times_G V \to M$ becomes a K-equivariant vector bundle.

1.1.6 Frame bundles

Let $E \to M$ be a K-equivariant complex vector bundle of rank n. We construct an associated vector bundle which is equivariantly isomorphic to E. Choose a K-invariant Hermitian metric on E and on $\mathbb{C}^n$ consider the standard Hermitian inner product. The unitary frame bundle of E is defined as $F_{\mathbb{C}}(E) = \biguplus_{x \in M} F_{\mathbb{C}}(E)_x$, where $F_{\mathbb{C}}(E)_x = \{\varphi_x : \mathbb{C}^n \to E_x \,|\, \varphi_x \text{ isometry}\}$. It is a principal $U(n)$-bundle over M

$$(\varphi_x \cdot g)(v) = \varphi_x(gv), \qquad \forall\, g \in U(n),\ \forall\, v \in \mathbb{C}^n.$$

It admits a K-action which commutes with the $U(n)$-action

$$(k \cdot \varphi_x)(v) = k \cdot (\varphi_x(v)), \qquad \forall\, k \in K,\ \forall\, v \in \mathbb{C}^n. \tag{1.1}$$

Moreover, we have K-equivariant isomorphism of vector bundles

$$\Phi : F_{\mathbb{C}}(E) \times_{U(n)} \mathbb{C}^n \to E, \quad \Phi([\varphi_x, v]) = \varphi_x(v).$$

Choosing a different K-invariant Hermitian metric on E yields an isomorphic principal bundle.

Let $E \to M$ be a K-equivariant real orientable vector bundle of rank n. Similarly, we can construct it as an associated vector bundle. Choose a K-invariant Riemannian metric on E and the standard scalar product on $\mathbb{R}^n$. On $\mathbb{R}^n$ we choose the orientation given by the standard basis. Let $F^+_{\mathbb{R}}(E) = \biguplus_{x \in M} F^+_{\mathbb{R}}(E)_x$, where $F^+_{\mathbb{R}}(E)_x = \{\varphi_x : \mathbb{R}^n \to E_x \,|\, \varphi_x \text{ orientation preserving isometry}\}$. Then $F^+_{\mathbb{R}}(E)$ is a principal $SO(n)$-bundle over M and it is called the real oriented frame bundle of E. We also have a K-action on it defined by (1.1), which commutes with the $SO(n)$-action and

$$\Phi : F^+_{\mathbb{R}}(E) \times_{SO(n)} \mathbb{R}^n \to E, \quad \Phi([\varphi_x, v]) = \varphi_x(v)$$

is a K-equivariant isomorphism of vector bundles.

1.1.7 Euler class

Recall that the Lie algebra $\mathfrak{so}(n)$ of $SO(n)$ is the set of skew-symmetric real matrices. Let $\{e_1, \dots e_n\}$ be an orthonormal basis of $\mathbb{R}^n$ inducing the same orientation as the standard basis. To any $A \in \mathfrak{so}(n)$ we can associate a skew-symmetric form $\omega_A \in \Lambda^2\mathbb{R}^n$ by $\omega_A(v, w) = (Av, w)$ for all $v, w \in \mathbb{R}^n$.

Definition 1.15. The *Pfaffian* of $A \in \mathfrak{so}(n)$ is defined as the coefficient of $e_1 \wedge \ldots \wedge e_n$ in

$$\exp\left(\sum_{i<j} (Ae_i, e_j) \cdot e_i \wedge e_j\right).$$

Alternatively we can define it by

$$\mathrm{Pf}(A)(e_1 \wedge \ldots \wedge e_n) = \frac{\omega_A^{n/2}}{n!}$$

if n is even and $\mathrm{Pf}(A) = 0$ if n is odd. From the definition of ω_A follows that $\mathrm{Pf} \in (S\mathfrak{so}^*)^{SO(n)}$.

By the spectral theorem of skew-symmetric matrices any $A \in \mathfrak{so}(n)$ is conjugate in $O(n)$ to a matrix of form

$$\begin{pmatrix} 0 & -\lambda_1 & & & \\ \lambda_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -\lambda_k \\ & & & \lambda_k & 0 \end{pmatrix} \text{ if } n = 2k \text{ or } \begin{pmatrix} 0 & -\lambda_1 & & & & \\ \lambda_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & -\lambda_k & \\ & & & \lambda_k & 0 & \\ & & & & & 0 \end{pmatrix} \text{ if } n = 2k+1,$$

hence

$$\mathrm{Pf}(A) = \begin{cases} \lambda_1 \cdots \lambda_k & \text{if } n = 2k, \\ 0 & \text{if } n = 2k+1. \end{cases}$$

In particular, we have $\mathrm{Pf}(A)^2 = \det(A)$ for all $A \in \mathfrak{so}(n)$.

Definition 1.16. The *Euler class* of a real orientable vector bundle $E \to M$ of rank n is

$$e(E) = \mathcal{C}_\theta \left(\frac{\mathrm{Pf}}{(-2\pi)^{n/2}} \right) = \mathrm{Pf} \left(-\frac{\Theta}{2\pi} \right) \in H^n(M),$$

where θ is a connection form on the real oriented frame bundle of $F^+_{\mathbb{R}}(E)$ and Θ is its curvature [39]. The *equivariant Euler class* of a real orientable K-equivariant vector bundle $E \to M$ of rank n is equal to

$$e_K(E) = \mathcal{C}^K_\theta \left(\frac{\mathrm{Pf}}{(-2\pi)^{n/2}} \right) = \mathrm{Pf} \left(-\frac{\Theta_K}{2\pi} \right) \in H^n_K(M),$$

where θ is a K-invariant connection form on $F^+_{\mathbb{R}}(E)$ and Θ_K is its equivariant curvature [15].

1.1.8 Chern classes

Denote $\mathfrak{u}(n)$ the Lie algebra of $U(n)$. We define $\sigma_k \in (S\mathfrak{u}(n)^*)^{U(n)}$ by relation

$$\det \left(I + t \frac{\sqrt{-1}}{2\pi} A \right) = \sum_{k=0}^{n} \sigma_k(A) t^k.$$

By the spectral theory of skew-Hermitian matrices any $A \in \mathfrak{u}(n)$ is conjugate in $U(n)$ to a diagonal matrix $diag\big(\sqrt{-1}\lambda_1, \dots, \sqrt{-1}\lambda_n\big)$, hence

$$\sigma_k(A) = \frac{s_k(\lambda_1, \dots, \lambda_n)}{(-2\pi)^k},$$

where $s_k(\lambda_1, \dots, \lambda_n)$ is the k^{th} elementary symmetric polynomial in $\lambda_1, \dots, \lambda_n$.

Definition 1.17. The k^{th} *Chern class* of a rank n complex vector bundle $E \to M$ is defined by

$$c_k(E) = \mathcal{C}_\theta(\sigma_k) = \sigma_k(\Theta) \in H^{2k}(M),$$

where θ is a connection form on the unitary frame bundle $F_{\mathbb{C}}(E)$ and Θ is its curvature. The total Chern class of E is

$$c(E) = 1 + c_1(E) + \ldots + c_n(E).$$

We call $c_n(E)$ the top Chern class of E. We define the k^{th} *equivariant Chern class* of a K-equivariant complex vector bundle $E \to M$ of rank n by

$$c_k^K(E) = \mathcal{C}_\theta^K(\sigma_k) = \sigma_k(\Theta^K) \in H_K^{2k}(M),$$

where θ is a K-invariant connection form on the unitary frame bundle $F_{\mathbb{C}}(E)$ and Θ^K is its equivariant curvature.

1.1.9 Relation between the Euler class and the top Chern class

Recall that $\mathbb{C}^n$ considered as real vector space has a natural orientation. If $\{z_1, \ldots, z_n\}$ is a basis of $\mathbb{C}^n$ then the orientation is induced by the real basis $\{x_1, \sqrt{-1}y_1, \ldots, x_n, \sqrt{-1}y_n\}$ of $\mathbb{C}^n$. Therefore, any complex vector bundles $E \to M$ of rank n can be considered as oriented real vector bundle of rank $2n$. The following result can be found in [15] or [39].

Proposition 1.18. *The Euler class of E as real oriented vector bundle agrees with top Chern class of E as complex vector bundle. That is,*

$$e(E) = c_n(E).$$

It also holds for K-equivariant complex vector bundles $E \to M$, that is,

$$e_K(E) = c_n^K(E).$$

1.1.10 Properties of the Euler class

Let $E \to M$ be a K-equivariant (real orientable or complex) vector bundle and let $f : N \to M$ be a K-equivariant map which induces homomorphism $f^* : H_K(M) \to H_K(N)$. Then the pull-back bundle f^*E is also K-equivariant and we have

$$e_K(f^*E) = f^*(e_K(E)).$$

If $F \to M$ is another K-equivariant vector bundle then $e_K(E \oplus F) = e_K(E)e_K(F)$ (Whitney product formula).

For computations of Euler classes we will use the following correspondence. Let $P \to M$ be a K-equivariant principal G-bundle and let $E \to P$ be a $(K \times G)$-equivariant real oriented vector bundle. If $\mathcal{C}^K : H_{K\times G}(P) \to H_K(M)$ is the Cartan isomorphism then

$$\mathcal{C}^K(e_{K\times G}(E)) = e_K(E/G).$$

Let $\phi : G \to K$ be a Lie group homomorphism and denote $\phi^* : \mathfrak{k}^* \to \mathfrak{g}^*$ the induced map. The K-equivariant (real orientable) vector bundle $E \to M$ can be consider as a G-equivariant vector bundle via the homomorphism ϕ. We also have a homomorphism $\phi^* : H_K(M) \to H_G(M)$ between equivariant cohomologies, and moreover

$$e_G(E) = \phi^*(e_K(E)).$$

To make computation with the Euler classes we will use the Spitting Principle (cf. [15], Section 8.6 or [4], Section 21). If $E \to M$ is an K-equivariant complex vector bundle of rank r then we will assume that it splits to K-equivariant complex line bundles, that is, $E = \oplus_{i=1}^r L_i$ and by Whitney product formula

$$e_K(E) = \prod_{i=1}^r e_K(L_i).$$

Example 1.2. Let K be a torus acting on $\mathbb{C}$ by weight $\gamma \in \mathfrak{k}^*_{\mathbb{Z}}$, that is,

$$\exp(\xi) \cdot z = e^{2\pi\sqrt{-1}\gamma(\xi)} z, \qquad \forall\, \xi \in \mathfrak{k},\ \forall\, z \in \mathbb{C}.$$

We consider $\mathbb{C}$ as a K-equivariant vector bundle over the point and we denote it by $\mathbb{C}_\gamma$. The standard Hermitian metric on $\mathbb{C}$ is K-invariant and $F_{\mathbb{C}}(\mathbb{C}_\gamma) = U(1)$ is the unitary frame bundle. The Maurer-Cartan form $(\theta_{MC})_z = \dfrac{dz}{z}$, $z \in U(1) \subset \mathbb{C}$ is a K-invariant connection on $F_{\mathbb{C}}(\mathbb{C}_\gamma) \to \{pt\}$. Let $\{\xi^1, \dots, \xi^q\}$ be a basis of $\mathfrak{k}$ and $\{u_1, \dots, u_q\}$ be its dual basis. If we write $\gamma = \sum_{i=1}^q \gamma_i u_i$ then

$$\underline{\xi}^i_z = \frac{d}{dt}\Big|_{t=0} e^{2\pi\sqrt{-1}\gamma(t\xi^i)} \cdot z = 2\pi\gamma_i\sqrt{-1}z\frac{\partial}{\partial z}.$$

Moreover, $(\Theta_{MC})^K = -\sum_{i=1}^q \iota_{\underline{\xi}^i}\theta_{MC}u_i = -\sum_{i=1}^q 2\pi\sqrt{-1}\gamma_i u_i = -2\pi\sqrt{-1}\gamma$ and

$$e_K(\mathbb{C}_\gamma) = c_1^K(\mathbb{C}_\gamma) = \frac{\sqrt{-1}}{2\pi}(-2\pi\sqrt{-1}\gamma) = \gamma.$$

◇

1.2 Symplectic manifolds

1.2.1 Hamiltonian action

Definition 1.19. A 2-form ω on a manifold M is *symplectic* if it is closed and non-degenerate, i.e. $d\omega = 0$ and if $\omega(v, w) = 0$ for all $w \in \mathfrak{X}(M)$ then $v = 0$. The pair (M, ω) is called *symplectic manifold*.

Definition 1.20. An action of a compact Lie group G on a symplectic manifold (M, ω) is called *Hamiltonian* if $g^*\omega = \omega$ for all $g \in G$ (it preserves the symplectic form) and there is a G-equivariant map $\mu_G : M \to \mathfrak{g}^*$, called *moment map*, such that

$$d\langle \mu_G, \xi\rangle = -\iota_{\underline{\xi}}\omega, \qquad \forall\, \xi \in \mathfrak{g}. \tag{1.2}$$

On $\mathfrak{g}^*$ we have considered the coadjoint action.

The form $\omega - \mu_G \in \Omega^2(M) \oplus \Omega^0(M) \otimes \mathfrak{g}^*$ is invariant and equivariantly closed, that is,

$$D_G(\omega - \mu_G) = 0.$$

Hence, $\omega - \mu_G \in H^2_G(M)$ is called *equivariant symplectic form.* Let S be another compact connected Lie group which acts on M and preserves ω. If the G- and S-action commute then μ_G is S-invariant, that is, S preserves the fibers of μ_G.

Example 1.3. On $\mathbb{C}^n$ we have a natural symplectic form

$$\omega_{\mathbb{C}^n} = \frac{\sqrt{-1}}{2}\sum_{i=1}^n dz_i \wedge d\bar{z}_i = \sum_{i=1}^n dx_i \wedge dy_i = \sum_{i=1}^n r_i dr_i d\vartheta_i,$$

where $z_i = x_i + \sqrt{-1}y_i = r_i e^{\sqrt{-1}\vartheta_i}$. Let K be a torus acting on $\mathbb{C}^n$ with weights $\gamma_1, \dots, \gamma_n \in \mathfrak{k}^*_{\mathbb{Z}}$, that is,

$$\exp(\xi) \cdot z = \left(e^{2\pi\sqrt{-1}\gamma_1(\xi)}z_1, \dots, e^{2\pi\sqrt{-1}\gamma_n(\xi)}z_n\right), \qquad \forall\, \xi \in \mathfrak{k},\ \forall\, z = (z_1, \dots, z_n) \in \mathbb{C}^n.$$

Then $\underline{\xi}_z = \sum_{i=1}^n 2\pi\gamma_i(\xi)\frac{\partial}{\partial\vartheta_i}$ and from (1.2) follows that the moment map has of form

$$\mu(z) = 2\pi\sum_{i=1}^n \gamma_i \frac{r_i^2}{2} + \zeta, \qquad (\zeta \in \mathfrak{k}^*).$$

◇

1.2.2 Symplectic reduction and Kirwan map

The relation (1.2) and non-degeneracy of ω imply that $\zeta \in (\mathfrak{g}^*)^G$ is a regular value of μ_G if and only if G acts locally freely on $\mu_G^{-1}(\zeta)$. The quotient $M/\!\!/_\zeta G = \mu_G^{-1}(\zeta)/G$ is called *symplectic quotient.* Moreover, $M/\!\!/_\zeta G$ admits a symplectic form which is called *reduced symplectic form.* Hence, if G acts freely on $\mu_G^{-1}(\zeta)$ then $M/\!\!/_\zeta G$ is a symplectic manifold. If the action is only locally free then $M/\!\!/_\zeta G$ is a symplectic orbifold.

Let S be a connected compact Lie group. If the S-action on M preserving the symplectic form commutes with the G-action then we also have an S-action on $M/\!\!/_\zeta G$. Moreover, if the S-action is Hamiltonian with G-invariant moment map $\mu_S : M \to \mathfrak{s}^*$, then the S-action on $M/\!\!/_\zeta G$ is also Hamiltonian.

The *Kirwan map* $\kappa : H_G(M) \to H(M/\!\!/_\zeta G)$ is defined by

$$\kappa = \mathcal{C} \circ i^*,$$

where $i : \mu_G^{-1}(\zeta) \to M$ is the inclusion and $\mathcal{C} : H_G(\mu_G^{-1}(\zeta)) \to H(M/\!\!/_\zeta G)$ is the Cartan isomorphism. We remark that if $\zeta = 0$ then $\kappa(\omega - \mu_G)$ is the reduced symplectic form on $M/\!\!/_0 G$. It was proved in [26] that if M is compact then κ is surjective. Similarly, we can define the equivariant version of the Kirwan map $\kappa_S : H_{G\times S}(M) \to H_S(M/\!\!/_\zeta G)$,

$$\kappa_S = \mathcal{C}^S \circ i^*,$$

where $\mathcal{C}^S : H_{G\times S}(\mu_G^{-1}(\zeta)) \to H_S(M/\!\!/_\zeta G)$ is the equivariant Cartan isomorphism. If $\zeta = 0$ then $\kappa_S(\omega - \mu_G - \mu_S)$ is the reduced equivariant symplectic form on $M/\!\!/_0 G$. The Kirwan surjectivity was extended to the equivariant setting in [12].

1.2.3 Compatible triples and symplectic weights

M is a *complex manifold* of dimension n if admits an atlas $\{U_j\}_j$ with U_j open unit disk in $\mathbb{C}^n$ and holomorphic transition maps. The multiplication by $\sqrt{-1}$ on $\mathbb{C}^n$ defines an anti-involutive section $I \in \Gamma(\mathrm{End}(TM))$, that is $I^2 = -id_{TM}$, which is called complex structure on M.

Definition 1.21. A Riemannian metric ρ on a complex manifold M is a *Hermitian metric* if it is compatible with the complex structure

$$\rho(Iv, Iw) = \rho(v, w), \qquad \forall\, v, w \in \mathfrak{X}(M). \tag{1.3}$$

Definition 1.22. A complex manifold M is *Kähler* if it has a Hermitian metric ρ and a symplectic form ω such that

$$\omega(v, w) = \rho(Iv, w), \qquad \forall\, v, w \in \mathfrak{X}(M). \tag{1.4}$$

The triple (I, ρ, ω) is called *Kähler structure* on M.

Definition 1.23. An anti-involution $I \in \Gamma(\mathrm{End}(TM))$ is called *almost complex structure* on M. An *almost Kähler structure* on M is a triple (I, ρ, ω) such that I is an almost complex structure, ρ is a Riemannian metric and ω is a symplectic form on M satisfying compatibility relations (1.3) and (1.4). We call (I, ρ, ω) a *compatible triple.*

Proposition 1.24. *Let (M, ω) be a symplectic manifold then there exist a Riemannian metric ρ and an almost complex structure I on M such that (I, ρ, ω) is a compatible triple (cf. [7], Corollary 12.7 or [21], Lemma 3.16).*

Corollary 1.25. *If a compact Lie group G acts on a symplectic manifold M, preserving the symplectic form ω then there exist an invariant Riemannian metric ρ and an almost structure I such that (I, ρ, ω) is an invariant compatible triple.*

Proof. Following the proof of Lemma 3.16 of [21] we choose a G-invariant Riemannian metric ρ' on M. Then there exists a skew-adjoint operator $A \in \Gamma(\mathrm{End}(TM))$ (with respect to ρ') such that $\omega(u, v) = \rho'(Au, v)$ for all vector fields $u, v \in \mathfrak{X}(M)$. Since ω and ρ' are G-invariant, hence A is a G-invariant section of $\mathrm{End}(TM)$. Moreover, A^*A is a (G-equivariant) self-adjoint positive definite operator on TM (with respect to ρ'). The almost complex structure $I \in \Gamma(\mathrm{End}(TM))$ is defined by the relation $A = \sqrt{A^*A}I = I\sqrt{A^*A}$ and remark that it is preserved by the G-action. Finally, the Riemannian metric ρ is given by $\rho(u, v) = \rho'(\sqrt{A^*A}u, v)$. It is G-invariant and

$$\omega(u, v) = \rho'(Au, v) = \rho'(\sqrt{A^*A}Iu, v) = \rho(Iu, v)$$

for all $u, v \in \mathfrak{X}(M)$. □

Proposition 1.26. *Let (M,ω) be a symplectic manifold. The set of almost complex structures compatible with the symplectic structure on M*

$$\mathcal{I}_\omega = \left\{ I \in \Gamma(\mathrm{End}(TM)) \,|\, I^2 = -id_{TM},\ \exists\,(I,\rho,\omega) \text{ compatible triple} \right\}$$

is path connected (cf. [7], Proposition 12.8).

If a compact torus T acts on (M,ω) preserving the symplectic form and $x \in M^G$ is a fixed point then by choosing an invariant compatible triple we can talk about T-weights of T_xM and by Proposition 1.26 it does not depend on the choice of the compatible triple. Similarly, by choosing a compatible triple on a symplectic manifold (M,ω) we can consider Chern classes of TM.

1.2.4 HyperKähler manifolds

Definition 1.27. A Riemannian manifold (M,ρ) is *hyperKähler* if it admits three Kähler structures (I_i,ρ,ω_i), $i=1,2,3$ such that the three complex structures satisfy the quaternionic relations

$$I_1^2 = I_2^2 = I_3^2 = I_1I_2I_3 = -id_{TM}.$$

Definition 1.28. A G-action on a hyperKähler manifold is called *hyper-Hamiltonian* if it preserves the hyperKähler structure and it is Hamiltonian with respect to all three symplectic structures. If $\mu_i : (M,\omega_i) \to \mathfrak{g}^*$, $i=1,2,3$ are the three moment maps then we can compose *real* and *complex symplectic forms* and *moment maps* as follows

$$\begin{aligned} \omega_{\mathbb{R}} &= \omega_1, & \mu_{\mathbb{R}} &= \mu_1, \\ \omega_{\mathbb{C}} &= \omega_2 + \sqrt{-1}\omega_3, & \mu_{\mathbb{C}} &= \mu_2 + \sqrt{-1}\mu_3. \end{aligned}$$

Example 1.4. Denote $\mathbb{H}$ the quaternions with i,j,k such that $i^2 = j^2 = k^2 = ijk = -1$. Let $A = A_0 + iA_1 + jA_2 + kA_3 \in M_{n,r}(\mathbb{H})$ and denote $Z = A_0 + iA_1, W^t = A_2 + iA_3 \in M_{n,r}(\mathbb{C})$. We identify $M_{n,r}(\mathbb{H}) \xrightarrow{\sim} T^*M_{n,r}(\mathbb{C}) = M_{n,r}(\mathbb{C}) \oplus M_{r,n}(\mathbb{C})$, $A = Z + W^tj \mapsto (Z,W)$. The left multiplications by i,j,k on $M_{n,r}(\mathbb{H})$ define three complex structures on $T^*M_{n,r}(\mathbb{C})$

$$I_1(Z,W) = \left(\sqrt{-1}Z, \sqrt{-1}W\right), \quad I_2(Z,W) = (-W^*, Z^*), \quad I_3(Z,W) = \left(-\sqrt{-1}W^*, \sqrt{-1}Z^*\right).$$

Moreover, we also define Riemannian metric ρ, real and complex symplectic forms $\omega_{\mathbb{R}}$ and $\omega_{\mathbb{C}}$ on $T^*M_{n,r}(\mathbb{C})$ by

$$\begin{aligned} \rho &= \mathrm{Tr}(dZdZ^* + dWdW^*), \\ \omega_{\mathbb{R}} &= \frac{\sqrt{-1}}{2}\mathrm{Tr}\left(dZ \wedge dZ^* + dW \wedge dW^*\right), \\ \omega_{\mathbb{C}} &= \mathrm{Tr}(dZ \wedge dW). \end{aligned}$$

All these structures make $T^*M_{n,r}(\mathbb{C})$ a hyperKähler manifold. We consider the natural $U(n) \times U(r)$-action on $T^*M_{n,r}(\mathbb{C})$ given by

$$(U,V)\cdot(Z,W) = (UZV^*, VWU^*), \qquad \forall\, U \in U(n),\ \forall\, V \in U(r),\ \forall\, Z, W^t \in M_{n,r}(\mathbb{C}).$$

This action is hyper-Hamiltonian with real and complex moment maps

$$\mu_{\mathbb{R}} : (T^*M_{n,r}(\mathbb{C}), \omega_{\mathbb{R}}) \to \mathfrak{u}(n)^* \oplus \mathfrak{u}(r)^*, \qquad \mu_{\mathbb{R}}(Z, W) = \frac{\sqrt{-1}}{2} \begin{pmatrix} ZZ^* - W^*W \\ WW^* - Z^*Z \end{pmatrix},$$

$$\mu_{\mathbb{C}} : (T^*M_{n,r}(\mathbb{C}), \omega_{\mathbb{C}}) \to \mathfrak{u}(n)^*_{\mathbb{C}} \oplus \mathfrak{u}(r)^*_{\mathbb{C}}, \qquad \mu_{\mathbb{C}}(Z, W) = \begin{pmatrix} ZW \\ -WZ \end{pmatrix},$$

where we have identified $\mathfrak{u}(N) \simeq \mathfrak{u}(N)^*$ via the non-degenerate bilinear pairing $(u, v) = \mathrm{Tr}(u^*v)$ for all $u, v \in \mathfrak{u}(N)$ and $N = n$ or $N = r$. ◇

Similarly to the symplectic case, $\zeta = (\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}}) \in (\mathfrak{g}^* \oplus \mathfrak{g}^*_{\mathbb{C}})^G$ is a regular value of $(\mu_{\mathbb{R}}, \mu_{\mathbb{C}})$ if and only if G acts locally freely on $\mu_{\mathbb{R}}^{-1}(\zeta_{\mathbb{R}}) \cap \mu_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})$. Moreover, $M/\!\!/\!\!/\!\!/_{\zeta} G = \mu_{\mathbb{R}}^{-1}(\zeta_{\mathbb{R}}) \cap \mu_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})/G$ is again hyperKähler and it is called the *hyperKähler quotient* [22]. We also have a hyperKähler version of the Kirwan map $\kappa : H_G(M) \to H(M/\!\!/\!\!/\!\!/_{\zeta} G)$, $\kappa = \mathcal{C} \circ i^*$, where $i : \mu_{\mathbb{R}}^{-1}(\zeta_{\mathbb{R}}) \cap \mu_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}}) \to M$ is the inclusion and $\mathcal{C} : H_G(\mu_{\mathbb{R}}^{-1}(\zeta_{\mathbb{R}}) \cap \mu_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})) \to H(M/\!\!/\!\!/\!\!/_{\zeta} G)$ is the Cartan isomorphism. Similarly can be defined the equivariant version of the *hyperKähler Kirwan map.*

1.3 Orbifolds

We adopt the definition of orbifolds from [34], which extends the original definition of [42] for non-effective actions.

Let X be a paracompact Hausdorff topological space. An *orbifold chart* on X is a tuple $(U, \widetilde{U}, H, \varphi)$, where U is an open subset of X, H is a finite group acting linearly on an open subset $\widetilde{U}$ of $\mathbb{R}^n$ and $\varphi : \widetilde{U} \to U$ is a H-invariant continuous map, which induces homeomorphism $\widetilde{U}/H \to U$. We say that it is a chart around $x \in X$ if $x \in U$. An *embedding of orbifold charts* $\phi : (U_i, \widetilde{U}_i, H_i, \varphi_i) \to (U_j, \widetilde{U}_j, H_j, \varphi_j)$ consists of a group homomorphism $\widehat{\phi} : H_i \to H_j$ and a $\widehat{\phi}$-twisting embedding $\widetilde{\phi} : \widetilde{U}_i \to \widetilde{U}_j$ such that

$$\begin{tikzcd} \widetilde{U}_i \arrow[r, "\widetilde{\phi}"] \arrow[d, "\varphi_i"'] & \widetilde{U}_j \arrow[d, "\varphi_j"] \\ U_i \arrow[r] & U_j \end{tikzcd}$$

commutes, the bottom horizontal map is the inclusion and ϕ satisfies the additional condition

$$h_j \cdot \widetilde{\phi}(\widetilde{U}_i) \cap \widetilde{\phi}(\widetilde{U}_i) \neq \emptyset \quad \Rightarrow \quad h_j \in \widehat{\phi}(H_i). \tag{1.5}$$

We remark that if H_j acts effectively on $\widetilde{U}_j$ then the condition (1.5) is automatically satisfied.

Two orbifold charts $(U_i, \widetilde{U}_i, H_i, \varphi_i)$, $i = 1, 2$ on X are compatible if for any point $x \in U_1 \cap U_2$ there exists a third orbifold chart $(U_3, \widetilde{U}_3, H_3, \varphi_3)$ around x with chart embeddings $\phi_i : (U_3, \widetilde{U}_3, H_3, \varphi_3) \to (U_i, \widetilde{U}_i, H_i, \varphi_i)$, $i = 1, 2$.

Definition 1.29. An *orbifold structure* on X is an orbifold atlas $\{(U_i, \widetilde{U}_i, H_i, \varphi_i)\}_{i \in I}$ of compatible charts such that $X = \cup_{i \in I} U_i$.

Example 1.5. If a compact Lie group G acts locally freely on a manifold P then the quotient P/G is an orbifold. $\diamond$

Let $x \in X$ and let $(U, \widetilde{U}, H, \varphi)$ be a chart around x. Let $y \in \varphi^{-1}(x)$ be a lift of x and denote $H_y = \{h \in H \mid hy = y\}$ its isotropy group. Let $\phi : (V, \widetilde{V}, G, \psi) \to (U, \widetilde{U}, H, \varphi)$ be an embedding such that $x \in V$. If $z \in \psi^{-1}(x)$ is another lift then by condition (1.5) the isotropy groups H_y and G_z are isomorphic. Thus we can define up to isomorphism the *isotropy group* H_x of x as isotropy group of its lifts.

Definition 1.30. The *multiplicity* of a point x is the order of its isotropy group.

For any point $x \in X$ there is an orbifold chart $(U, \widetilde{U}, H, \varphi)$ around x such that H is the isotropy group of x, that is, $\varphi^{-1}(x) \in \widetilde{U}$ is a fixed point of H. Since the fixed point set $\widetilde{U}^H$ is a submanifold of $\widetilde{U}$, hence for a fixed finite group G the set $X_G = \{x \in X \mid H_x \simeq G\}$ is a submanifold of X. We can write $X = \cup_G X_G$. If X is connected then there is a unique open, dense submanifold $X_{H_\bullet} \subset X$ called *principal stratum* such that the order $|H_\bullet|$ is minimal for the isotropy groups.

Definition 1.31. The *orbifold multiplicity* of X is defined as the order of isotropy groups of any point in the principal stratum and we denote it by $\mathfrak{m}(X)$ [34].

Later in the text orbifolds will appear only as quotients of manifolds by compact groups acting locally freely. We can define differential form, equivariant cohomology and Euler classes on orbifolds using orbifold charts [30], but in our cases they will be computed as images of similar objects via Cartan isomorphism.

Finally, we recall the following orbifold version of the Atiyah-Bott-Berline-Vergne localization theorem [3, 2].

Theorem 1.32 ([34], Theorem 2.1). *Let X be an orbifold with a T torus action. For any $\beta \in H_T(X)$ we have*

$$\frac{1}{\mathfrak{m}(X)} \int_X \beta = \sum_{F \subset X^T} \frac{1}{\mathfrak{m}(F)} \int_F \frac{i_F^* \beta}{e_T \mathcal{N}(F \mid X)},$$

where $\mathfrak{m}(X)$ and $\mathfrak{m}(F)$ are the orbifold multiplicities of X and F (considering F as suborbifold of X), and $\mathcal{N}(F \mid X)$ is the equivariant normal orbibundle of F in X.

1.4 Proper and bounded below moment map

Let K be an 1-dimensional torus and let (M, ω) be a non-compact Hamiltonian K-manifold with moment map $\mu_K : M \to \mathfrak{k}^*$ which is assumed to be proper and non-surjective. Let γ be a generator of $\mathfrak{k}^*_{\mathbb{Z}}$ and define $\varphi : M \to \mathbb{R}$ by relation $\mu_K = \varphi \cdot \gamma$. Then φ is also proper and by Lemma 1.1 of [40] the image of φ is either $(-\infty, \eta]$ or $[\eta, +\infty)$ for some $\eta \in \mathbb{R}$. We assume that we have chosen γ such that φ is bounded below.

Definition 1.33. We call such an action as above a *PBB action* if M^K is compact. We refer to the data (μ_K, φ, γ) as the proper, bounded below moment map.

Remark 1.34. (i) Similarly to the compact case the fixed point set $M^K \neq \emptyset$ by Proposition 1.2 of [40].

(ii) M^K is compact if and only if the set of critical values of φ is finite. Indeed, φ is constant on K-fixed point components and $\varphi(M^K)$ is the set of critical values, since K is 1-dimensional. Therefore, M^K is compact if and only if $\varphi(M^K)$ is finite, since φ is proper.

$\diamond$

Let T be a compact torus with Hamiltonian action on (M, ω) which commutes with the K-action and denote $\mu_T : M \to \mathfrak{t}^*$ its moment map. We can approximate M by compact symplectic manifolds using symplectic cut technique [28] as follows. Let $\varepsilon \in \mathbb{R}$ be a regular value of φ. We consider the standard symplectic form $\omega_{\mathbb{C}} = \frac{\sqrt{-1}}{2} dz d\bar{z}$ on $\mathbb{C}$ and let K act on $\mathbb{C}$ by weight $-\gamma \in \mathfrak{k}^*_{\mathbb{Z}}$. It is a Hamiltonian action with moment map $\psi(z) = -2\pi\gamma\frac{|z|^2}{2}$. On the product space $M \times \mathbb{C}$ we consider the symplectic form $\omega + \omega_{\mathbb{C}}$ and $(T \times K)$-action

$$(t, k) \cdot (m, z) = (tk \cdot m,\, k^{-1} \cdot z) \qquad \forall\, (t, k) \in T \times K,\ \forall\, (m, z) \in M \times \mathbb{C}.$$

The K-action admits moment map $\Psi : M \times \mathbb{C} \to \mathfrak{k}^*$,

$$\Psi(m, z) = \mu_K(m) + \psi(z) = \big(\varphi(m) + \pi|z|^2\big)\,\gamma$$

and $\varepsilon\gamma$ is a regular value of it. Indeed, we have decomposition

$$\Psi^{-1}(\varepsilon\gamma) = \Big(\Psi^{-1}(\varepsilon\gamma) \cap (M \times \mathbb{C}^\times)\Big) \biguplus \Big(\mu_K^{-1}(\varepsilon\gamma) \times \{0\}\Big),$$

on which K acts locally freely, because K acts locally freely on $\mathbb{C}^\times$ and $\mu_K^{-1}(\varepsilon\gamma)$ by the assumption that $\varepsilon\gamma$ is a regular value of μ_K. Hence $X_{\leq\varepsilon} = \Psi^{-1}(\varepsilon\gamma)/K$ is a symplectic orbifold. Moreover, if $\operatorname{Im}\varphi = [\eta, +\infty)$ then

$$\Psi^{-1}(\varepsilon\gamma) \subset \varphi^{-1}\left([\eta, \varepsilon]\right) \times \left\{z \in \mathbb{C} \,\middle|\, \pi|z|^2 \leq \varepsilon - \eta\right\}$$

and since φ is proper, it follows that $\Psi^{-1}(\varepsilon\gamma)$ and consequently $X_{\leq\varepsilon}$ is compact. Denote $[m, z]$ the image of (m, z) by the quotient map $\Psi^{-1}(\varepsilon\gamma) \to X_{\leq\varepsilon}$ and let $X_{<\varepsilon} = \{[m, z] \in X_{\leq\varepsilon} \,|\, z \neq 0\}$. The torus T acts on $X_{\leq\varepsilon}$ with moment map $X_{\leq\varepsilon} \to \mathfrak{t}^*$, $[m, z] \mapsto \mu_T(m)$, which we will denote by ϕ_T. Moreover, there is a T-equivariant symplectomorphism

$$\varphi^{-1}(-\infty, \varepsilon) \to X_{<\varepsilon}, \quad m \mapsto \left[m, \sqrt{\frac{\varepsilon - \varphi(m)}{\pi}}\,\right].$$

We recall the following two theorems from [30], which are the orbifold versions of similar results of [1] and [16].

Theorem 1.35 ([30], Theorem 5.1)**.** *Let (X, ω) be a compact connected symplectic orbifold with Hamiltonian T-action and $\phi_T : X \to \mathfrak{t}^*$ moment map. Then for every $\zeta \in \mathfrak{t}^*$ the fiber $\phi_T^{-1}(\zeta)$ is connected.*

Theorem 1.36 ([30], Theorem 5.2)**.** *Let a T torus act on a compact connected symplectic orbifold (X,ω) with moment map $\phi_T : X \to \mathfrak{t}^*$. Then the image of the moment map $\phi_T(X) \subset \mathfrak{t}^*$ is a rational convex polytope. Moreover, $\phi_T(X)$ is the convex hull of $\phi_T\left(X^T\right)$.*

Approximating M by $X_{\leq\varepsilon}$ as ε tends to $+\infty$ we can show that

Corollary 1.37. *(a) For every $\zeta \in \mathfrak{t}^*$ the fibers $\mu_T^{-1}(\zeta) \subset M$ are connected.*

(b) The image of the moment map $\mu_T(M) \subset \mathfrak{t}^*$ is a convex polytope.

Let $H \subset T$ be a subtorus and let $N \subset M^H$ be an H-fixed point component. Assume that H is the maximal subtorus of T fixing every point of N. The restriction of φ to N is also proper and bounded below, therefore $\mu_T(N)$ is a subpolytope of $\mu_T(M)$. Since N is a H-fixed point component, the H-moment map is constant on N, therefore $\mu_T(N)$ lies in the affine subspace $\mu_T(x) + \ker(\mathfrak{t}^* \to \mathfrak{h}^*)$ for any $x \in N$. Moreover, the quotient group T/H acts (locally) freely on an open dense subset of N, hence the dimension of the polytope $\mu_T(N)$ agrees with the dimension of $\ker(\mathfrak{t}^* \to \mathfrak{h}^*)$.

Definition 1.38. The proper subpolytopes $\mu_T(N)$ are called *walls* of the moment polytope $\mu_T(M)$. Let $x \in N$. We will refer to $\ker(\mathfrak{t}^* \to \mathfrak{h}^*)$ and $\mu_T(x) + \ker(\mathfrak{t}^* \to \mathfrak{h}^*)$ respectively as *supporting plane* and *supporting affine plane* of the wall $\mu_T(N)$.

Example 1.6. For $x \in M^T$ let $\alpha_i \in \mathfrak{t}^*_{\mathbb{Z}}$, $i \in I$ be the weights of T_xM (with respect to an invariant compatible almost complex structure). Consider the subtorus $H \subset T$ with Lie algebra $\mathfrak{h} = \cap_{j=1}^k \ker \alpha_{i_j}$. Let $N \subset M^H$ the fixed point component containing x. Then the supporting affine plane of $\mu_T(N)$ is equal to $\mu_T(x) + \operatorname{span}\langle \alpha_{i_1}, \ldots, \alpha_{i_k}\rangle$. Indeed, if $T_xM = \oplus_{i\in I} W_{\alpha_i}$ is the decomposition to weight spaces then $T_xN = \oplus_{j\in J} W_{\alpha_j}$, where $J = \{i \in I \mid \alpha_i(\mathfrak{h}) = 0\}$. Thus, H is the maximal subtorus of T fixing every point of N and we have $\ker(\mathfrak{t}^* \to \mathfrak{h}^*) = \operatorname{span}\langle \alpha_{i_1}, \ldots, \alpha_{i_k}\rangle$. $\diamond$

Lemma 1.39. *The moment polytope $\mu_T(M)$ has finitely many walls.*

Proof. If $x \in N^{T\times K} \subset M^{T\times K}$ then $T_xN \subset T_xM$ is a T-invariant subspace and moreover it is a direct sum of weight spaces corresponding to weights $\alpha \in \mathfrak{t}^*_{\mathbb{Z}}$ such that $\alpha \in \ker(\mathfrak{t}^* \to \mathfrak{h}^*)$. T_xM has finitely many weights and they depend only on the connected components of $M^{T\times K}$, which are also finite in number, therefore $\mu_T(M)$ has finitely many walls. $\square$

The Kirwan surjectivity also extends to manifolds with proper bounded below moment maps.

Theorem 1.40 ([26], [12], [41])**.** *Let G be a compact Lie group and let S be a compact torus acting on M. Assume that the G-action is Hamiltonian with moment map $\mu_G : M \to \mathfrak{g}^*$ and let $\zeta \in (\mathfrak{g}^*)^G$ be a regular value of μ_G. Moreover, we suppose that the actions of G, S and K on M commute. Then the S-equivariant Kirwan map $\kappa_S : H_{G\times S}(M) \to H_S(M/\!\!/_\zeta G)$ is surjective.*

2

Symplectic cut

In this chapter we review in detail the symplectic cut technique with respect to a cone closely following [25]. In that paper the symplectic cut construction is used to get localization formulas as follows. From a Hamiltonian K-manifold M we can construct another Hamiltonian K-manifold (or orbifold) M_Γ which contains the torus symplectic quotient $M/\!\!/K$ as a K-fixed point component. Some of the K-fixed point components of M appear among the fixed point components of M, too. The Atiyah-Bott-Berline-Vergne localization formula on M_Γ yields a fraction of a particular form and the Jeffrey-Kirwan residue can be used to extract from this fractions relations between contributions of fixed point components. In this way we can compute integrals on the quotient space $M/\!\!/K$ in terms of the Jeffrey-Kirwan residue and fixed point data on M. Therefore, the goal of the chapter is to compute fixed point data on the symplectic cut space M_Γ such as fixed point components, Euler classes of their normal bundles and orbifold multiplicities in order to write up the Atiyah-Bott-Berline-Vergne localization formula on it (Theorem 2.13). Our setup differs from [25] solely in considering K as a subgroup of a bigger torus T.

Let (M,ω) be a connected symplectic manifold with Hamiltonian action of an n-dimensional compact torus T with moment map $\mu_T : M \to \mathfrak{t}^*$. We assume that M is compact or admits a PBB action which commutes with the T-action. Let $K \subset T$ be a q-dimensional subtorus and we denote by μ_K its moment map.

We also consider the auxiliary symplectic manifold $\mathbb{C}^q$ with the standard symplectic form $\omega_{\mathbb{C}^q} = \dfrac{\sqrt{-1}}{2}\displaystyle\sum_{i=1}^{q} dz_i d\bar{z}_i$ and let K act on $\mathbb{C}^q$ by linearly independent weights $\gamma_1, \ldots, \gamma_q \in \mathfrak{k}^*_{\mathbb{Z}}$. It is also a Hamiltonian action with moment map $\psi : \mathbb{C}^q \to \mathfrak{k}^*$,

$$\psi(z) = 2\pi \sum_{i=1}^{q} \gamma_i \frac{|z_i|^2}{2}.$$

The product space $M \times \mathbb{C}^q$ admits symplectic form $\omega + \omega_{\mathbb{C}^q}$ and Hamiltonian $(T \times K)$-action

$$(t,k)\cdot(m,z) = (tm, kz), \qquad \forall\,(t,k) \in T \times K,\ \forall\,(m,z) \in M \times \mathbb{C}^q$$

with moment map $\mu_T \times \psi : M \times \mathbb{C}^q \to \mathfrak{t}^* \oplus \mathfrak{k}^*$. The K-action will refer to the action of K on $\mathbb{C}^q$ and we will specify when the action of K on M as subgroup of T occurs. We consider the embedding $K \to T \times K$, $k \mapsto (k, k^{-1})$. We denote its image by K_{diag} and its Lie algebra by $\mathfrak{k}_{diag}$. The K_{diag}-moment map is $\Psi : M \times \mathbb{C}^q \to \mathfrak{k}^*_{diag}$,

$$\Psi(m, z) = \mu_K(m) - \psi(z).$$

We will investigate when is 0 a regular value of Ψ. By definition 0 is a regular value of Ψ if $\operatorname{Im} d_m\mu_K - \operatorname{Im} d_z\psi = \mathfrak{k}^*$ for all $(m, z) \in \Psi^{-1}(0)$. Let $K_m \subset K$ be the maximal subtorus fixing m and denote $\mathfrak{k}_m$ its Lie algebra. Remark that $\operatorname{Im} d_m\mu_K = \ker(\mathfrak{k}^* \to \mathfrak{k}^*_m)$, since $d_m\langle\mu_K, v\rangle = -\iota_{\underline{v}_m}\omega$ for all $v \in \mathfrak{k}$ and ω is non-degenerate. If $F_m \subset M^{K_m}$ is the connected component containing m then $\mu_K(F_m)$ is a convex polytope in $\mathfrak{k}^*$ with supporting plane $\operatorname{Im} d_m\mu_K$. Moreover, $\operatorname{Im} d_z\psi = \operatorname{span}\langle\gamma_j \,|\, z_j \neq 0\rangle$. Therefore, 0 is a regular value of Ψ if and only if the polytopes $\mu_K(F_m)$ and $Cone(\gamma_j \,|\, z_j \neq 0)$ intersect transversally for all $(m, z) \in \Psi^{-1}(0)$. That is, we got the following characterization of 0 being a regular value of Ψ from [25]:

(T) for every subset $I \subset \{1, \ldots, q\}$ the intersection of $Cone(\gamma_i \,|\, i \in I)$ with every wall of $\mu_K(M)$ is transverse.

Remark 2.1. (i) For $I = \emptyset$ condition (T) implies that 0 is a regular value of μ_K.

(ii) Recall that $\mu_K(M)$ has finitely many walls. Therefore, if 0 is a regular value of μ_K then for generic choice of $\gamma_1, \ldots, \gamma_q \in \mathfrak{k}^*_{\mathbb{Z}}$ the condition (T) holds.

$\diamond$

From now on we assume that the transversality condition (T) holds. We denote by $M/\!\!/K = \mu_K^{-1}(0)/K$ the symplectic quotient.

Definition 2.2. The *symplectic cut* of M with respect to the simplicial cone $\Gamma = Cone(\gamma_1, \ldots, \gamma_q)$ is the symplectic quotient

$$M_\Gamma = \Psi^{-1}(0)/K_{diag}.$$

We will use notations $[m]$ and $[m, z]$ for images of $m \in \mu_K^{-1}(0)$ and $(m, z) \in \Psi^{-1}(0)$ by quotient maps. As shown in [29], the T-action on $M \times \mathbb{C}^q$ descends to M_Γ and it is Hamiltonian with moment map $\phi_T : M_\Gamma \to \mathfrak{t}^*$,

$$\phi_T([m, z]) = \mu_T(m)$$

(ϕ_K for K-moment map). We denote by ω_Γ the reduced symplectic form on M_Γ.

Moreover, we have relations between moment polytopes $\phi_K(M_\Gamma) = \mu_K(M) \cap \Gamma$ and $\phi_T(M_\Gamma) = \mu_T(M) \cap \operatorname{pr}^{-1}_{\mathfrak{k}^*}(\Gamma)$, where $\operatorname{pr}_{\mathfrak{k}^*} : \mathfrak{t}^* \to \mathfrak{k}^*$.

2.1 T-fixed components on M_Γ

For $m \in M$ and $z \in \mathbb{C}^q$ denote $T_m \subset T$ and $K_z \subset K$ the maximal subtorus fixing m and z, respectively. Denote $\mathfrak{t}_m$ and $\mathfrak{k}_z$ their Lie algebras. Let $F_m \subset M^{T_m}$ and $F_z \subset (\mathbb{C}^q)^{K_z}$ be connected components containing m and z, respectively.

Lemma 2.3. *$[m,z] \in M_\Gamma$ is a T-fixed point if and only if $\mathfrak{t} = \mathfrak{t}_m \oplus \mathfrak{k}_z$. Moreover, if $F_{[m,z]} \subset (M_\Gamma)^T$ is the fixed point component containing $[m,z]$ then*

$$F_{[m,z]} = (F_m \times F_z)/\!\!/K_{diag} = (F_m \times F_z) \cap \Psi^{-1}(0)/K_{diag}.$$

Proof. Let $T'_m = \{t \in T \,|\, t \cdot m = m\}$ and $K'_z = \{k \in K \,|\, k \cdot z = z\}$ be isotropy subgroups of m and z, respectively. We have $[m,z] \in (M_\Gamma)^T$ if and only if for any $t \in T$ there exists $k \in K$ such that $(tk \cdot m,\, k^{-1} \cdot z) = (m,\, z)$, that is, $tk \in T'_m$ and $k^{-1} \in K'_z$. It implies that $T'_m \times K'_z \to T$, $(t',k') \to t'k'$ is surjective, which is equivalent to $\mathfrak{t}_m + \mathfrak{k}_z = \mathfrak{t}$ since T is connected. Moreover, (m,z) belongs to $\Psi^{-1}(0)$ on which K_{diag} acts locally freely, hence $\mathfrak{t}_m \cap \mathfrak{k}_z = \{0\}$. Therefore, $[m,z] \in (M_\Gamma)^T$ exactly when $\mathfrak{t}_m \oplus \mathfrak{k}_z = \mathfrak{t}$. In particular,

$$\Phi_T : T_m \times K_z \to T, \quad (t,k) \mapsto tk$$

is a finite cover, because $T_m \times K_z$ is compact.

We remark that $(F_m \times F_z) \cap \Psi^{-1}(0)$ is connected by [1] or Corollary 1.37, hence $(F_m \times F_z)/\!\!/K_{diag}$ is also connected. If $[m_1,z_1] \in (F_m \times F_z)/\!\!/K_{diag}$ then by surjectivity of Φ_T we can write any $\tau \in T$ as $\tau = tk$ with $t \in T_m$ and $k \in K_z$. Hence

$$\tau \cdot [m_1,z_1] = [tk \cdot m_1, z_1] = [t \cdot m_1, k \cdot z_1] = [m_1,z_1],$$

therefore $(F_m \times F_z)/\!\!/K_{diag} \subset F_{[m,z]}$ by connectedness.

Finally, we will show that $(F_m \times F_z)/\!\!/K_{diag}$ is a closed and open subset of the connected space $F_{[m,z]}$ which will imply the equality of these two sets. Remark that $F_m \times F_z$ is a closed subset of $M \times \mathbb{C}^q$, hence $(F_m \times F_z)/\!\!/K_{diag}$ is a closed subset of $F_{[m,z]}$. It remains to show that it is an open subset, too. If $[m_1,z_1] \in (F_m \times F_z)/\!\!/K_{diag}$ then $T_m \subset T_{m_1}$, $K_z \subset K_{z_1}$ and $\mathfrak{t}_m \oplus \mathfrak{k}_z = \mathfrak{t} = \mathfrak{t}_{m_1} \oplus \mathfrak{k}_{z_1}$, whence $T_m = T_{m_1}$ and $K_z = K_{z_1}$. The isotropy groups locally decrease (cf. [9], Tube theorem), thus there is an open neighborhood $\mathcal{U} \subset M \times \mathbb{C}^q$ of (m_1,z_1) such that for all $(m_2,z_2) \in \mathcal{U}$ we have $T'_{m_2} \subset T'_{m_1}$ and $K'_{z_2} \subset K'_{z_1}$. If $[m_2,z_2] \in F_{[m,z]} \subset (M_\Gamma)^T$ then $\mathfrak{t}_{m_2} \oplus \mathfrak{k}_{z_2} = \mathfrak{t}$, hence $T_{m_2} = T_{m_1} = T_m$ and $K_{z_2} = K_{z_1} = K_z$. It implies that $[m_2,z_2] \in (F_m \times F_z)/\!\!/K_{diag}$ and therefore $(F_m \times F_z)/\!\!/K_{diag}$ is an open subset of $F_{[m,z]}$. □

Remark 2.4. From the proof follows that for any $[m_1,z_1] \in F_{[m,z]}$ we have $T_m = T_{m_1}$ and $K_z = K_{z_1}$, hence the finite cover Φ_T depends only on the fixed point component $F_{[m,z]}$ not on the points m and z. ◇

Remark 2.5. The above lemma is reflected on the geometry of the moment polytope as follows. For $z = (z_1, \ldots, z_q) \in \mathbb{C}^q$ we introduce the index set $J_z = \{j = 1, \ldots, q \,|\, z_j = 0\}$.

(i) Since $\mu_K(m) = \phi_K(F_{[m,z]})$ and $\psi(F_z) = Cone(\gamma_j \,|\, j \notin J_z)$, we have

$$\phi_K(F_{[m,z]}) = \mu_K(F_m) \cap Cone(\gamma_j \,|\, j \notin J_z).$$

(ii) Moreover, $\dim \mu_T(F_m) = \dim \mathfrak{k}_z = \dim \mu_K(F_m)$ and $\mathrm{pr}_{\mathfrak{k}^*}(\mu_T(F_m)) = \mu_K(F_m)$, therefore $\mu_T(F_m)$ and $\mathrm{pr}_{\mathfrak{k}^*}^{-1} Cone(\gamma_j \,|\, j \notin J_z)$ also have complementary dimensions and intersect transversally in $\mathfrak{t}^*$, thus

$$\phi_T(F_{[m,z]}) = \mu_T(F_m) \cap \mathrm{pr}_{\mathfrak{k}^*}^{-1} Cone(\gamma_j \,|\, j \notin J_z).$$ ◇

We consider the following subsets of M and M_Γ:

$$\begin{aligned} M_{int} &= \{m \in M \mid \mu_K(m) \in \operatorname{int}\Gamma\}, \\ M_{\Gamma,int} &= \{[m,z] \in M_\Gamma \mid \mu_K([m,z]) \in \operatorname{int}\Gamma\}, \\ M_{\Gamma,0} &= \{[m,z] \in M_\Gamma \mid z = 0\}. \end{aligned}$$

We organize the T-fixed point components of M_Γ in three groups.

(F0) Fixed point components $D_0 \subset (M_{\Gamma,0})^T$. They are characterized by $\phi_K(D_0) = 0$, that is, $D_0 = F_{[m,0]}$ for some $m \in \mu_K^{-1}(0)$. They can be naturally identified to T-fixed point components $F_0 \subset M/\!\!/K$ via the T-equivariant diffeomorphism

$$\Upsilon_0 : M/\!\!/K \to M_{\Gamma,0}, \quad [m] \mapsto [m,0].$$

(F1) Fixed point components $D_1 \subset (M_{\Gamma,int})^T$ characterized by $\phi_K(D_1) \in \operatorname{int}\Gamma$, i.e. $D_1 = F_{[m,z]}$ for some $[m,z] \in (M_\Gamma)^T$ with $z \in (\mathbb{C}^\times)^q$. They correspond to fixed point components of M_{int} as follows. Let $\mathcal{K} \subset K$ be the maximal subgroup of K acting trivially on $\mathbb{C}^q$ and remark that it is finite since $\gamma_1, \ldots, \gamma_q$ are linearly independent. For any $m \in M_{int}$ we can write $\mu_K(m) = \sum_{i=1}^{q} \pi\gamma_i\mu_K(m)_i$ with $\mu_K(m)_i > 0$ for all $i = 1, \ldots, q$. The map $M_{int} \to M_{\Gamma,int}$, $m \mapsto \big[m, \big(\sqrt{\mu_K(m)_1}, \ldots, \sqrt{\mu_K(m)_q}\,\big)\big]$ induces T-equivariant isomorphism

$$\Upsilon_{int} : M_{int}/\mathcal{K} \to M_{\Gamma,int} \tag{2.1}$$

of orbifolds. Under this map the fixed point components D_1 correspond to suborbifolds $F_1/\mathcal{K}$ of $M_{int}/\mathcal{K}$, where F_1 is a T-fixed point component of M_{int}.

(F2) Other fixed point components $D \subset (M_\Gamma)^T$ which are characterized by $\phi_K(D) \neq 0$ and $\phi_K(D)$ lies on the boundary of Γ.

Remark 2.6. When $\dim K = 1$ and $\gamma_1 \in \mathfrak{k}^*_\mathbb{Z}$ is a generator then $\mathcal{K}$ is trivial and $\Upsilon_{int} : M_{int} \to M_\Gamma$ is an open embedding. Moreover, fixed components as in (F2) do not occur and we have decomposition

$$M_\Gamma = \Upsilon_{int}(M_{int}) \uplus \Upsilon_0(M/\!\!/K).$$

$\diamond$

2.2 Choice of cohomology classes and their restrictions

Definition 2.7. We define the homomorphism $\Delta : H_T(M) \to H_T(M_\Gamma)$ by the following diagram

$$\begin{tikzcd} H_T(M) \arrow[r, "\Delta"] \arrow[d] & H_T(M_\Gamma) \\ H_{T\times K}(M) \arrow[r, "\sim"] & H_{T\times K_{diag}}(M \times \mathbb{C}^q) \arrow[u, "\kappa_T"'] \end{tikzcd}$$

where the left vertical map is induced by $T \times K \to T$, $(t,k) \mapsto tk$, the bottom horizontal isomorphism is induced by projection $M \times \mathbb{C}^q \to M$ and κ_T is the Kirwan map.

More explicitly, if $i : \Psi^{-1}(0) \to M$ is the composition of the inclusion $\Psi^{-1}(0) \to M \times \mathbb{C}^q$ and projection $M \times \mathbb{C}^q \to M$ then for any $\beta \in H_T(M)$

$$(\Delta\beta)(u) = \mathrm{Hor}_\theta\big[(i^*\beta)(u + \Theta(u))\big],$$

where $u \in \mathfrak{t}$ and θ is a T-invariant connection form on the principal K_{diag}-bundle $\Psi^{-1}(0) \to M_\Gamma$ with equivariant curvature form $\Theta(u)$. We remark that the cohomology class of $\Delta(\beta)$ does not depend on the choice of θ.

Lemma 2.8.

(i) *If $i_{int} : M_{int} \to M$ and $i_{\Gamma,int} : M_{\Gamma,int} \to M_\Gamma$ are inclusions then*

$$i_{int}^* = \Upsilon_{int}^* \circ i_{\Gamma,int}^* \circ \Delta : H_T(M) \to H_T(M_{int}).$$

In particular, if $D_1 \subset (M_{\Gamma,int})^T$ and $F_1 \subset (M_{int})^T$ are fixed point components such that $\Upsilon_{int}(F_1) = D_1$ then for all $\beta \in H_T(M)$

$$\Upsilon_{int}^*(i_{D_1}^*\Delta(\beta)) = i_{F_1}^*\beta. \tag{2.2}$$

(ii) *If $i_{\Gamma,0} : M_{\Gamma,0} \to M_\Gamma$ denotes the inclusion then*

$$\kappa_{T/K} = \Upsilon_0^* \circ i_{\Gamma,0}^* \circ \Delta : H_T(M) \to H_T(M/\!\!/K).$$

In particular, if $D_0 \subset (M_{\Gamma,0})^T$ and $F_0 \subset (M/\!\!/K)^T$ are fixed point components such that $\Upsilon_0(F_0) = D_0$ then for all $\beta \in H_T(M)$

$$\Upsilon_0^*(i_{D_0}^*\Delta(\beta)) = i_{F_0}^*\kappa_{T/K}(\beta).$$

(iii) *$\Delta(\omega - \mu_T) = \omega_\Gamma - \phi_T \in H_T^2(M_\Gamma)$ is the class of the T-equivariant symplectic form on M_Γ.*

Proof.

(i) Denote $\pi : \Psi^{-1}(0) \to M_\Gamma$ the quotient map. To compute the restriction $i_{M_{\Gamma,int}}^*\Delta(\beta)$ we may choose any T-invariant connection form θ on the principal K_{diag}-bundle $\pi^{-1}(M_{\Gamma,int}) \to M_{\Gamma,int}$ by the naturality of the Cartan isomorphism with respect to restrictions. Consider the $K_{diag} \xrightarrow{=} K$ twisting map $\pi^{-1}(M_{\Gamma,int}) \to U(1)^q$,

$$(m, (z_1, \ldots, z_q)) \mapsto \left(\frac{z_1}{\sqrt{\mu_K(m)_1}}, \ldots, \frac{z_q}{\sqrt{\mu_K(m)_q}}\right)$$

and let θ be the pull-back of a connection form on the principal K-bundle $U(1)^q \to \{pt\}$. Remark that θ is T-invariant, moreover $d\theta = 0$ and $\iota_{\underline{u}}\theta = 0$ for all $u \in \mathfrak{t}$, hence its T-equivariant curvature $\Theta(u) = 0$. Denote $i_\Gamma : \pi^{-1}(M_{\Gamma,int}) \to M$ the restriction of $i : \Psi^{-1}(0) \to M$. We have

$$\Upsilon_{int}^*\big(i_{M_{\Gamma,int}}^*\Delta(\beta)\big)(u) = \Upsilon_{int}^*\left(\mathrm{Hor}_\theta\left[(i_\Gamma^*\beta)(u + \Theta(u))\right]\right) = \Upsilon_{int}^*\left(\mathrm{Hor}_\theta\left[(i_\Gamma^*\beta)(u)\right]\right) = (i_{int}^*\beta)(u),$$

because Υ_{int}^* is induced by $\nu : M_{int} \to \pi^{-1}(M_{\Gamma,int})$, $m \mapsto \big(m, (\sqrt{\mu_K(m)_1}, \ldots, \sqrt{\mu_K(m)_q})\big)$ and $\nu^*(\theta) = 0$, thus the operator $\Upsilon_{int}^*(\mathrm{Hor}_\theta)$ is the identity.

(ii) Again, to compute the cohomology class $i^*_{M_{\Gamma,0}}\Delta(\beta)$ we may use any T-invariant connection form θ on the principal K_{diag}-bundle $\pi^{-1}(M_{\Gamma,0}) = \mu_K^{-1}(0)\times\{0\} \to M_{\Gamma,0}$, by the naturality of the Cartan isomorphism. Moreover, the natural map $\mu_K^{-1}(0) \to \mu_K^{-1}(0)\times\{0\}$ induces a $T\times K \xrightarrow{=} T\times K_{diag}$ twisting isomorphism of principal bundles

$$\begin{tikzcd} \mu_K^{-1}(0) \arrow[r, "\sim"] \arrow[d] & \mu_K^{-1}(0)\times\{0\} \arrow[d] \\ M/\!\!/K \arrow[r, "\sim", "\Upsilon_0"'] & M_{\Gamma,0}. \end{tikzcd}$$

Let θ' the pull-back of θ and denote $\Theta'(u)$ and $\Theta(u)$ their T-equivariant connection forms. Hence

$$\Upsilon_0^*\big(i^*_{M_{\Gamma,0}}\Delta(\beta)\big) = \Upsilon_0^*\left(\mathrm{Hor}_\theta[(i_0^*\beta)(u+\Theta(u))]\right) = \mathrm{Hor}_{\theta'}[(j_0^*\beta)(u+\Theta'(u))] = \kappa_{T/K}(\beta),$$

where $j_0 : \mu_K^{-1}(0) \to M$ is the inclusion and $i_0 : \mu_K^{-1}(0)\times\{0\} \to M$ is the restriction of i.

(iii) Remark that $\omega_{\mathbb{C}^q} - \psi = D_K\left(\sum_{i=1}^q \dfrac{x_i dy_i - y_i dx_i}{2}\right)$ is exact, therefore

$$\Delta(\omega-\mu_T) = \kappa_T(\omega-\mu_K-\mu_T) = \kappa_T(\omega+\omega_{\mathbb{C}^q}-\Psi-\mu_T) = \kappa(\omega+\omega_{\mathbb{C}^q})-\mu_T = \omega_\Gamma - \phi_T.$$

□

Remark 2.9. In particular, $\Upsilon_0^*(\Delta(\omega-\mu_T))$ is the class of the reduced equivariant symplectic form $\kappa_{T/K}(\omega-\mu_T)$ on $M/\!\!/K$. ◇

2.3 T-equivariant Euler classes of normal bundles of fixed point components

We have described the T-fixed point components of M_Γ in section 2.1. To compute their normal bundles and Euler classes we will use the following lemma (cf. [25], Proposition 2.2).

Lemma 2.10 ([13], Proposition 3.1)**.** *Let G be a compact Lie group and let Z be an invariant symplectic submanifold of a Hamiltonian G-manifold X with moment map $\mu : X \to \mathfrak{g}^*$. Assume that 0 is a regular value of μ and let $X/\!\!/G = \mu^{-1}(0)/G$ be the symplectic quotient. If $Z\cap\mu^{-1}(0)\neq\emptyset$ then 0 is also a regular value of $\mu|_Z : Z \to \mathfrak{g}^*$ and let $Z/\!\!/G = (\mu|_Z)^{-1}(0)/G$. We have isomorphism of normal bundles*

$$\mathcal{N}(Z/\!\!/G \,|\, X/\!\!/G) \simeq \mathcal{N}(Z\,|\,X)/\!\!/G \tag{2.3}$$

and consequently $e\mathcal{N}(Z/\!\!/G\,|\,X/\!\!/G) = \kappa(e_G\mathcal{N}(Z\,|\,X))$, where $\kappa : H_G(Z) \to H(Z/\!\!/G)$ is the Kirwan map. In particular, if X has an additional S-action which commutes with G and preserves the fibers of μ then the isomorphism (2.3) is S-equivariant and $e_S\mathcal{N}(Z\,|\,X) = \kappa_S(e_{G\times S}\mathcal{N}(Z\,|\,X))$, where $\kappa_S : H_{G\times S}(Z) \to H_S(Z/\!\!/G)$ is the S-equivariant Kirwan map.

Proof. 0 is a regular value of μ is equivalent to locally free action of G on $\mu^{-1}(0)$. Hence G acts locally freely on the set $Z \cap \mu^{-1}(0) = (\mu|_Z)^{-1}(0)$. Therefore, 0 is a regular value of $\mu|_Z$ and we have symplectic quotient $Z\sslash G = (\mu|_Z)^{-1}(0)/G = (Z \cap \mu^{-1}(0))/G$.

Denote $\pi : \mu^{-1}(0) \to X\sslash G$ the quotient map. We have the following commutative diagram of G-equivariant vector bundles over $\pi^{-1}(Z\sslash G)$ with short exact sequences in rows and in the first two columns.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \pi^{-1}(Z\sslash G) \times \mathfrak{g} \arrow[r] \arrow[d] & (\pi^{-1}(X\sslash G) \times \mathfrak{g})|_{\pi^{-1}(Z\sslash G)} \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & TZ|_{\pi^{-1}(Z\sslash G)} \arrow[r] \arrow[d] & TX|_{\pi^{-1}(Z\sslash G)} \arrow[r] \arrow[d] & \mathcal{N}(Z \,|\, X)|_{\pi^{-1}(Z\sslash G)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi^* T(Z\sslash G) \arrow[r] \arrow[d] & \pi^*(T(X\sslash G)|_{Z\sslash G}) \arrow[r] \arrow[d] & \pi^*\mathcal{N}(Z\sslash G \,|\, X\sslash G) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The 9-lemma implies the exactness of the last column, thus $\mathcal{N}(Z \,|\, X)\sslash G \simeq \mathcal{N}(Z\sslash G \,|\, X\sslash G)$.

If we have an additional S-action commuting with G and preserving fibers of μ then the above diagram is $(G \times S)$-equivariant, hence the isomorphism (2.3) is also S-equivariant. $\square$

By Lemma 2.3 we have $F_{[m,z]} = (F_m \times F_z)\sslash K_{diag}$ and by Lemma 2.10 we also have a T-equivariant isomorphism of vector bundles

$$\mathcal{N}(F_{[m,z]} \,|\, M_\Gamma) \simeq \mathcal{N}(F_m \times F_z \,|\, M \times \mathbb{C}^q)\sslash K_{diag} \simeq \big(\mathrm{pr}_1^*\mathcal{N}(F_m \,|\, M) \oplus \mathrm{pr}_2^*\mathcal{N}(F_z \,|\, \mathbb{C}^q)\big)\sslash K_{diag},$$

therefore

$$e_T\mathcal{N}(F_{[m,z]} \,|\, M_\Gamma) = \kappa_T'\big[e_{T\times K_{diag}}(\mathrm{pr}_1^*\mathcal{N}(F_m \,|\, M))\big]\, \kappa_T'\big[e_{T\times K_{diag}}(\mathrm{pr}_2^*\mathcal{N}(F_z \,|\, \mathbb{C}^q))\big],$$

where $\kappa_T' : H_{T\times K_{diag}}(F_m \times F_z) \to H_T(F_{[m,z]})$ is the equivariant Kirwan map. The isomorphism of tori $\nu : T \times K_{diag} \to T \times K$, $\nu(t,k) = (tk, k^{-1})$ is compatible with their actions on $M \times \mathbb{C}^q$, therefore it induces isomorphism

$$\nu^* : H_{T\times K}(F_m \times F_z) \to H_{T\times K_{diag}}(F_m \times F_z)$$

in equivariant cohomology.

Denote $\mathbb{C}_{\gamma_j} \to \{pt\}$ the K-equivariant complex line bundle over a point on which K acts by weight $\gamma_j \in \mathfrak{k}_{\mathbb{Z}}^*$ and T acts trivially. If $\mathcal{L}_j$ is the pull-back of the bundle $\mathbb{C}_{\gamma_j}$ along $F_m \times F_z \to \{pt\}$ then $\mathrm{pr}_2^*\mathcal{N}(F_z \,|\, \mathbb{C}^q) = \oplus_{j\in J_z}\mathcal{L}_j$, where $J_z = \{j = 1, \ldots, q \,|\, z_j = 0\}$. Therefore, the Euler class can be computed as

$$e_{T\times K_{diag}}(\mathrm{pr}_2^*\mathcal{N}(F_z \,|\, \mathbb{C}^q)) = \prod_{j\in J_z} e_{T\times K_{diag}}(\mathcal{L}_j) = \prod_{j\in J_z} \nu^*(e_{T\times K}(\mathcal{L}_j)) = \prod_{j\in J_z} \nu^*(\gamma_j) = \prod_{j\in J_z} (-\gamma_j)\,.$$

The vector bundle $\mathrm{pr}_1^*\mathcal{N}(F_m\,|\,M)$ splits $(T_m\times K_z)$-equivariantly to T_m-weight bundles

$$\mathrm{pr}_1^*\mathcal{N}(F_m\,|\,M)=\bigoplus_i \mathcal{N}_i,$$

that is, T_m acts on the fibers of $\mathcal{N}_i$ by some weight $\alpha_i\in(\mathfrak{t}_m)^*_{\mathbb{Z}}$. To compute Euler classes by the Splitting Principle we suppose that $\mathcal{N}_i$ are complex line bundles with respect to a compatible almost complex structure on M. Recall that we have finite covers of tori $\Phi_T: T_m\times K_z\to T$ and $\Phi_K: K_m\times K_z\to K$ which induce splitting of Lie algebras $\mathfrak{t}=\mathfrak{t}_m\oplus\mathfrak{k}_z$ and $\mathfrak{k}=\mathfrak{k}_m\oplus\mathfrak{k}_z$. They yield the following commutative diagram

$$\begin{array}{ccc}
\mathfrak{t}^* & \xrightarrow[\sim]{\Phi_T^*} & \mathfrak{t}_m^*\oplus\mathfrak{k}_z^* \\
\Big\downarrow{\scriptstyle \mathrm{pr}_{\mathfrak{k}^*}} & & \Big\downarrow{\scriptstyle \mathrm{pr}_{\mathfrak{k}_m^*}\oplus id} \\
\mathfrak{k}^* & \xrightarrow[\sim]{\Phi_K^*} & \mathfrak{k}_m^*\oplus\mathfrak{k}_z^*
\end{array}$$

where all the maps are adjoint to inclusions. Denote

$$\varrho:\mathfrak{t}_m^*\oplus\mathfrak{k}_z^*\to\mathfrak{t}^*$$

and $\varrho_{\mathfrak{k}}:\mathfrak{k}_m^*\oplus\mathfrak{k}_z^*\to\mathfrak{k}^*$ the inverses of Φ_T^* and Φ_K^*, respectively. Moreover, let

$$\sigma=\varrho\circ\mathrm{pr}_{\mathfrak{k}_z^*}:\mathfrak{k}^*\to\mathfrak{t}^*,$$

where $\mathrm{pr}_{\mathfrak{k}_z^*}:\mathfrak{k}^*\to\mathfrak{k}_z^*$ is the projection. Hence

$$e_{T_m\times K_z}(\mathcal{N}_i)=\alpha_i+e_{K_z}(\mathcal{N}_i)$$

and

$$e_{T\times K}(\mathrm{pr}_1^*\mathcal{N}(F_m\,|\,M))=e_T\left(\mathrm{pr}_1^*\mathcal{N}(F_m\,|\,M)\right)=\varrho(e_{T_m\times K_z}(\oplus_i\mathcal{N}_i)).$$

We emphasize that here the K_z-action is induced by the T-action on $M\times\mathbb{C}^q$ and recall that the K-action is trivial on M. Choose a T-invariant connection form θ on the principal K_{diag}-bundle $\Psi^{-1}(0)\cap(F_m\times F_z)\to F_{[m,z]}$. Its T-equivariant curvature is $\Theta=\lambda+d\theta$, where $\lambda:\mathfrak{t}\to\Omega^0(\Psi^{-1}(0)\cap(F_m\times F_z))\otimes\mathfrak{k}$, $\lambda(u)=-\iota_{\underline{u}}\theta$.

Remark 2.11. (i) If $u\in\mathfrak{t}_m$ then its fundamental vector field $\underline{u}$ on $\Psi^{-1}(0)\cap(F_m\times F_z)$ vanishes since T_m acts trivially on $F_m\times F_z$.

(ii) If $u\in\mathfrak{k}_z\subset\mathfrak{t}$ then $\iota_{\underline{u}}\theta=u$ since K_z as subgroup of T acts the same way on $F_m\times F_z$ as subgroup of K_{diag}.

(iii) Recall that $\mathfrak{t}=\mathfrak{t}_m\oplus\mathfrak{k}_z$, thus by the above remarks λ is the projection $\mathfrak{t}^*\to\mathfrak{k}_z^*$ under identification of $\mathfrak{k}_z^*$ with $\ker(\mathfrak{k}^*\to\mathfrak{k}_m^*)$ via $\varrho_{\mathfrak{k}}$.

$\diamond$

Denote $\kappa' : H_{K_{diag}}(F_m \times F_z) \to H(F_{[m,z]})$ the ordinary Kirwan map. For any $u \in \mathfrak{t}$ and $j \in J_z$ we compute

$$\begin{aligned}\kappa'_T(e_{T\times K_{diag}}(\mathcal{L}_j))(u) &= \mathrm{Hor}_\theta\left[-\gamma_j(\Theta(u))\right] \\ &= \mathrm{Hor}_\theta\left[-\gamma_j(-\iota_{\underline{u}}\theta + d\theta)\right] \\ &= \gamma_j(\mathrm{pr}_{\mathfrak{k}_z} u) - \mathrm{Hor}_\theta\left[\gamma_j(d\theta)\right] \\ &= \gamma_j(\mathrm{pr}_{\mathfrak{k}_z} u) - \kappa'(\gamma_j),\end{aligned}$$

thus

$$\kappa'_T\left(e_{T\times K_{diag}}(\mathcal{L}_j)\right) = \sigma(\gamma_j) - \kappa'(\gamma_j) \tag{2.4}$$

Moreover, $\kappa(e_{K_{diag}}(\mathcal{N}_i)) = \mathrm{Hor}_\theta\left[(\alpha_i + e_{K_z}(\mathcal{N}_i))(d\theta)\right] = \mathrm{Hor}_\theta\left[\alpha_i(\mathrm{pr}_{\mathfrak{k}_m} d\theta) + e_{K_z}(\mathcal{N}_i)(\mathrm{pr}_{\mathfrak{k}_z} d\theta)\right]$ and

$$\begin{aligned}\kappa_T(e_{T\times K_{diag}}(\mathcal{N}_i))(u) &= \mathrm{Hor}_\theta[e_{T\times K_{diag}}(\mathcal{N}_i)(u,\Theta(u))] \\ &= \mathrm{Hor}_\theta[e_{T\times K}(\mathcal{N}_i)(\nu^*(u,\Theta(u)))] \\ &= \mathrm{Hor}_\theta[e_T(\mathcal{N}_i)(u+\Theta(u))] \\ &= \mathrm{Hor}_\theta\left[\alpha_i(\mathrm{pr}_{\mathfrak{t}_m}(u - \iota_{\underline{u}}\theta + d\theta)) + e_{K_z}(\mathcal{N}_i)(\mathrm{pr}_{\mathfrak{k}_z}(u - \iota_{\underline{u}}\theta + d\theta))\right] \\ &= \alpha_i(\mathrm{pr}_{\mathfrak{t}_m} u) + \mathrm{Hor}_\theta\left[\alpha_i(\mathrm{pr}_{\mathfrak{k}_m} d\theta) + e_{K_z}(\mathcal{N}_i)(\mathrm{pr}_{\mathfrak{k}_z} d\theta)\right] \\ &= \alpha_i(\mathrm{pr}_{\mathfrak{t}_m} u) + \kappa'(e_{K_{diag}}(\mathcal{N}_i)),\end{aligned}$$

thus

$$\kappa'_T(e_{T\times K_{diag}}(\mathcal{N}_i)) = \varrho(\alpha_i) + \kappa'(e_{K_{diag}}(\mathcal{N}_i)). \tag{2.5}$$

We have got

$$\begin{aligned}e_T\mathcal{N}(F_{[m,z]}|M_\Gamma) &= \prod_i \left[\varrho(\alpha_i) + \kappa'(e_{K_{diag}}(\mathcal{N}_i))\right] \prod_{j\in J_z}\left[\sigma(\gamma_j) - \kappa'(\gamma_j)\right] \\ &= \prod_i \left[\varrho(\alpha_i) + e(\mathcal{N}_i/\!\!/K_{diag})\right] \prod_{j\in J_z}\left[\sigma(\gamma_j) + e(\mathcal{L}_j/\!\!/K_{diag})\right].\end{aligned} \tag{2.6}$$

The geometry of the weight vectors of the normal bundle $\mathcal{N}(F_{[m,z]} \,|\, M_\Gamma)$ will be important in the proof of the equivariant Jeffrey-Kirwan theorem. Therefore, we make the following remarks. The K-weights of the normal bundle $\mathcal{N}(F_{[m,z]} \,|\, M_\Gamma)$ are either parallel to the supporting planes of $\mu_K(F_m)$ or $\psi(F_z)$. Indeed, the supporting plane of $\psi(F_z)$ is equal to $\varrho_{\mathfrak{k}}(\mathfrak{k}_m^*) = \ker(\mathfrak{k}^* \to \mathfrak{k}_z^*)$, which is the subspace of $\mathfrak{k}^*$ spanned by γ_j, $j \notin J_z$ since $\mathfrak{k}_z = \cap_{j\notin J_z} \ker\gamma_j$. Hence

$$\mathrm{pr}_{\mathfrak{k}^*}(\varrho(\alpha_i)) \in \varrho_{\mathfrak{k}}(\mathfrak{k}_m^*) = \mathrm{span}\langle\gamma_j \,|\, j \notin J_z\rangle, \qquad \forall\, i. \tag{2.7}$$

Moreover, the K-weight $\mathrm{pr}_{\mathfrak{k}^*}(\sigma(\gamma_j))$, $j \in J_z$ is the projection of γ_j to the supporting plane $\varrho_{\mathfrak{k}}(\mathfrak{k}_z^*) = \ker(\mathfrak{k}^* \to \mathfrak{k}_m^*)$ of $\mu_K(F_m)$ along the supporting plane $\varrho_{\mathfrak{k}}(\mathfrak{k}_m^*)$ of $\psi(F_z)$. It also implies that

$$\Gamma = Cone(\gamma_1,\ldots,\gamma_q) \subset \mathrm{span}\langle\gamma_j \,|\, j\notin J_z\rangle + Cone(\mathrm{pr}_{\mathfrak{k}^*}(\sigma(\gamma_j)) \,|\, j\in J_z). \tag{2.8}$$

Remark that $\mathfrak{k}_z^* = \mathrm{span}\langle \gamma_j \,|\, j \in J_z \rangle$, hence the supporting affine plane of the wall $\mu_T(F_m)$ is equal to $\mu_T(m) + \mathrm{span}\langle \sigma(\gamma_j) \,|\, j \in J_z \rangle$. Moreover, since $\mu_T(m) = \psi(z)$ is in the relative interior of $\mathrm{pr}_{\mathfrak{k}^*}^{-1}(Cone(\gamma_j \,|\, j \notin J_z))$, hence

$$\mu_T(F_m) \cap \mathrm{pr}_{\mathfrak{k}^*}^{-1}(\Gamma) \subset \mu_T(m) + Cone(\sigma(\gamma_j) \,|\, j \in J_z) \tag{2.9}$$

by (2.8).

For the rest of this section we assume that $T = K \times S$ and $z = 0$. Then $\mathfrak{k}_0 = \mathfrak{k}$ and $J_0 = \{1, \ldots, q\}$, hence $\sigma = \varrho$ and $\mathrm{pr}_{\mathfrak{k}^*} \circ \sigma = id_{\mathfrak{k}^*}$. Moreover, by (2.4)

$$\kappa'_T(e_{T \times K_{diag}}(\mathcal{L}_i)) = \gamma_j - \kappa'_S(\gamma_j) = \gamma_j + \zeta_j - \kappa'(\gamma_j),$$

where $\kappa'_S : H_{K \times S}(F_m \times \{0\}) \to H_S(F_{[m,0]})$ is the S-equivariant Kirwan map and $\zeta_j \in \mathfrak{s}^*$ is the projection of $\sigma(\gamma_j)$ to $\mathfrak{s}^*$.

If we regard $M/\!\!/K \subset M_\Gamma$ via the embedding Υ_0 then $i^*_{F_{[m,0]}}\mathcal{N}(M/\!\!/K \,|\, M_\Gamma) = \oplus_{j=1}^q \mathcal{L}_j/\!\!/K_{diag}$, thus

$$i^*_{F_{[m,0]}} e_{K \times S}\mathcal{N}(M/\!\!/K \,|\, M_\Gamma) = \prod_{j=1}^q \left(\gamma_j - i^*_{F_{[m,0]}}\kappa_S(\gamma_j)\right), \tag{2.10}$$

where $\kappa_S : H_{K \times S}(M) \to H_S(M/\!\!/K)$ and $i^*_{F_{[m,0]}}\kappa_S = \kappa'_S$. In particular, if K is 1-dimensional then

$$i^*_{F_{[m,0]}} e_{K \times S}\mathcal{N}(M/\!\!/K \,|\, M_\Gamma) = \gamma + i^*_{F_{[m,0]}} e_S\mathcal{N}(M/\!\!/K \,|\, M_\Gamma) = \varrho(\gamma) + i^*_{F_{[m,0]}} e\mathcal{N}(M/\!\!/K \,|\, M_\Gamma), \tag{2.11}$$

and $M/\!\!/K \subset (M_\Gamma)^K$ yields

$$e_{K \times S}\mathcal{N}(M/\!\!/K \,|\, M_\Gamma) = \gamma + e_S\mathcal{N}(M/\!\!/K \,|\, M_\Gamma). \tag{2.12}$$

Finally, $\mathcal{N}(F_{[m,0]} \,|\, M/\!\!/K) = \oplus_i \mathcal{N}_i/\!\!/K_{diag}$ and $M/\!\!/K$ is fixed by K, thus by (2.5)

$$e_S\mathcal{N}(F_{[m,0]} \,|\, M/\!\!/K) = e_{K \times S}\mathcal{N}(F_{[m,0]} \,|\, M/\!\!/K) = \prod_i [\varrho(\alpha_i) + \kappa'(e_{K_{diag}}(\mathcal{N}_i))], \tag{2.13}$$

furthermore $\mathcal{N}(F_{[m,0]} \,|\, M_\Gamma) = \mathcal{N}(F_{[m,0]} \,|\, M/\!\!/K) \oplus \mathcal{N}(M/\!\!/K \,|\, M_\Gamma)|_{F_{[m,0]}}$, therefore by (2.10) and (2.13) we have

$$e_{K \times S}\mathcal{N}(F_{[m,0]} \,|\, M_\Gamma) = e_S\mathcal{N}(F_{[m,0]} \,|\, M/\!\!/K) \prod_{j=1}^q [\gamma_j - i^*_{F_{[m,0]}}\kappa_S(\gamma_j)]. \tag{2.14}$$

2.4 Orbifold multiplicities

In general M_Γ is an orbifold and its fixed point components are suborbifolds. Next we will compute the orbifold multiplicities of fixed point components as in (F1) for Theorem 2.13.

Lemma 2.12. *Let $\tau = \{\tau_1, \ldots, \tau_q\}$ be a basis of the lattice $\mathfrak{k}^*_{\mathbb{Z}}$ and define $\delta_\Gamma = \left|\det([\gamma_{ij}]_{i,j=1}^q)\right|$, where $\gamma_i = \sum_{j=1}^q \gamma_{ij}\tau_j$. Then for any $D \subset (M_{\Gamma,int})^T$ fixed point component the orbifold multiplicity $\mathfrak{m}(D)$ of D equals δ_Γ.*

Proof. Recall that we have isomorphism of orbifolds $\Upsilon_{int} : M_{int}/\mathcal{K} \to M_{\Gamma,int}$, where $\mathcal{K}$ is the kernel of the K-action on $\mathbb{C}^q$. Hence there is a fixed point component $F \subset (M_{int})^T$ such that D is isomorphic to the orbifold $F/\mathcal{K}$, therefore $\mathfrak{m}(D) = \mathfrak{m}(F/\mathcal{K}) = |\mathcal{K}|$. Recall that the action of K on $\mathbb{C}^q$ is given by

$$\exp(u) \cdot (z_1, \ldots, z_q) = \left(e^{2\pi\sqrt{-1}\gamma_1(u)} z_1, \ldots, e^{2\pi\sqrt{-1}\gamma_q(u)} z_q\right),$$

for all $u \in \mathfrak{k}$ and $(z_1, \ldots, z_q) \in \mathbb{C}^q$. Then $|\mathcal{K}|$ is equal to the degree of the map $K \to U(1)^q$,

$$\exp(u) \mapsto \left(e^{2\pi\sqrt{-1}\gamma_1(u)}, \ldots, e^{2\pi\sqrt{-1}\gamma_q(u)}\right).$$

The basis τ yields an isomorphism $K \to U(1)^q$, $\exp(u) \mapsto \left(e^{2\pi\sqrt{-1}\tau_1(u)}, \ldots, e^{2\pi\sqrt{-1}\tau_q(u)}\right)$, thus $|\mathcal{K}|$ is the degree of the map $U(1)^q \to U(1)^q$,

$$(t_1, \ldots, t_q) \mapsto (t_1^{\gamma_{11}} \cdots t_q^{\gamma_{1q}}, \ldots, t_1^{\gamma_{q1}} \cdots t_q^{\gamma_{qq}}),$$

which is equal to $\delta_\Gamma = |\det([\gamma_{ij}])_{ij=1}^q|$. □

We also remark that since we have embedding of orbifolds $\Upsilon_0 : M/\!\!/K \to M_\Gamma$, the orbifold multiplicity of $M/\!\!/K$ is the same as the orbifold multiplicity of $\Upsilon_0(M/\!\!/K)$ as suborbifold of M_Γ.

2.5 Atiyah-Bott-Berline-Vergne theorem on M_Γ

We apply Theorem 1.32 on M_Γ, which incorporates most of the results of this chapter.

Theorem 2.13. *Suppose that $T = K \times S$ and M_Γ is compact. For any $\beta \in H_T(M)$ we have*

$$\frac{1}{\mathfrak{m}(M_\Gamma)} \int_{M_\Gamma} \Delta(\beta e^{\omega - \mu_T}) = \oint_{M/\!\!/K} \frac{\kappa_S(\beta e^{\omega-\mu_T})}{\prod_{j=1}^q [\gamma_j - \kappa_S(\gamma_j)]} + \sum_{\substack{F \subset M^T \\ \mu_K(F) \in \Gamma}} \frac{1}{\delta_\Gamma} \int_F \frac{i_F^*(\beta e^{\omega-\mu_T})}{e_T \mathcal{N}(F \,|\, M)}$$
$$+ \sum_{\substack{D \subset M_\Gamma^T \\ \mu_K(D) \in \partial\Gamma \setminus \{0\}}} \frac{1}{\mathfrak{m}(D)} \int_H \frac{i_D^*\left(\Delta(\beta) e^{\omega_\Gamma - \phi_T}\right)}{e_T \mathcal{N}(D \,|\, M_\Gamma)},$$

where the number δ_Γ is defined in Lemma 2.12 and we used notation

$$\oint_{M/\!\!/K} \frac{\kappa_S(\beta e^{\omega-\mu_T})}{\prod_{j=1}^q [\gamma_j - \kappa_S(\gamma_j)]} = \sum_{B \subset (M/\!\!/K)^S} \frac{1}{\mathfrak{m}(B)} \int_B \frac{i_B^* \kappa_S(\beta e^{\omega-\mu_T})}{e_S(B \,|\, M/\!\!/K) \prod_{j=1}^q \left[\gamma_j - i_B^* \kappa_S(\gamma_j)\right]}.$$

Proof. By the Atiyah-Bott-Berline-Vergne theorem on M_Γ (Theorem 1.32) we have

$$\frac{1}{\mathfrak{m}(M_\Gamma)} \int_{M_\Gamma} \Delta(\beta e^{\omega-\mu_T}) = \sum_{D \subset (M_\Gamma)^T} \frac{1}{\mathfrak{m}(D)} \int_D \frac{i_D^* \Delta(\beta e^{\omega-\mu_T})}{e_T \mathcal{N}(D \,|\, M_\Gamma)}.$$

If $D \subset (M_\Gamma)^T$ is as in (F1) then there is $F \subset M^T$ such that $\Upsilon_{int}(F) = D$ and

$$\frac{1}{\mathfrak{m}(D)} \int_D \frac{i_D^* \Delta(\beta e^{\omega-\mu_T})}{e_T \mathcal{N}(D \,|\, M_\Gamma)} = \frac{1}{\delta_\Gamma} \int_F \frac{i_F^*(\beta e^{\omega-\mu_T})}{e_T \mathcal{N}(F \,|\, M)}$$

by Lemma 2.8(i), Lemma 2.12 and $\Upsilon^*_{int}(e_T\mathcal{N}(D\,|\,M_\Gamma)) = e_T\mathcal{N}(F/\mathcal{K}\,|\,M/\mathcal{K}) = e_T\mathcal{N}(F\,|\,M)$. If $D \subset (M_\Gamma)^T$ is as in (F0) then there is $B \subset (M/\!\!/K)^S$ such that $\Upsilon_0(B) = D$ and

$$\frac{1}{\mathfrak{m}(D)}\int_D \frac{i_D^*\Delta(\beta e^{\omega-\mu_T})}{e_T\mathcal{N}(D\,|\,M_\Gamma)} = \frac{1}{\mathfrak{m}(B)}\int_B \frac{i_B^*\kappa_S(\beta e^{\omega-\mu_T})}{e_S\mathcal{N}(B\,|\,M/\!\!/K)\prod_{j=1}^{q}[\gamma_j - i_B^*\kappa_S(\gamma_j)]}$$

by Lemma 2.8(ii) and (2.14). Finally, $\Delta(\omega-\mu_T) = \omega_\Gamma - \phi_T$ by Lemma 2.8(iii). □

Corollary 2.14. *Let K be 1-dimensional and let $\Gamma = \mathbb{R}_{\geq 0}\gamma$ be such that γ is a generator of $\mathfrak{k}^*_{\mathbb{Z}}$. Assume that $T = K \times S$ and M_Γ is compact. Then for any $\beta \in H_T(M)$ we have*

$$\int_{M_\Gamma} \Delta(\beta e^{\omega-\mu_T}) = \oint_{M/\!\!/K} \frac{\kappa_S(\beta e^{\omega-\mu_T})}{\gamma - \kappa_S(\gamma)} + \sum_{\substack{F\subset M^T\\ \mu_K(F)\in\Gamma}} \int_F \frac{i_F^*(\beta e^{\omega-\mu_T})}{e_T\mathcal{N}(F\,|\,M)}.$$

3

Jeffrey-Kirwan residues

In this chapter we introduce our main tool an equivariant version of the Jeffrey-Kirwan residue. It is based on the classical Jeffrey-Kirwan residue [23, 24], therefore we start the discussion with the classical one. Besides new results (Proposition 3.20) we reprove classical ones ([24], Proposition 3.2) to make the chapter self-contained. We show that this equivariant version of the Jeffrey-Kirwan residue admits similar properties as the usual one (Proposition 3.35 and 3.39). Finally, we compare our equivariant version with the one in [31].

3.1 The classical Jeffrey-Kirwan residue

The Atiyah-Bott-Berline-Vergne formula on Hamiltonian manifolds formally yields fractions of form $\sum_I \frac{P_I e^{\lambda_I}}{\prod_{i\in I}\alpha_i}$, where the finite sum is over finite index sets I, $\lambda_I, \alpha_i \in \mathfrak{t}^*$, $i \in I$, and $P_I \in \mathbb{R}[\mathfrak{t}]$ are polynomial functions on the r-dimensional real vector space $\mathfrak{t}$. The construction will be explained in more detail in section 3.1.2. Therefore, we define the real vector space of functions

$$\mathfrak{F} = \mathfrak{F}[\mathfrak{t}] = \left\{ \sum_I \frac{P_I e^{\lambda_I}}{\prod\limits_{i\in I}\alpha_i} \;\middle|\; P_I \in \mathbb{R}[\mathfrak{t}],\ \lambda_I, \alpha_i, \in \mathfrak{t}^*,\ \alpha_i \neq 0,\ i \in I,\ I \text{ finite set} \right\}.$$

We will consider two subsets of $\mathfrak{F}$. In section 3.1.1 we will define the subset of regular fractions $\mathfrak{F}_{reg}$ which can be considered as subset of generic elements of $\mathfrak{F}$, while in section 3.1.2 we will introduce a geometrically motivated subset $\mathfrak{F}_{Ham}$.

We define the classical Jeffrey-Kirwan residue as linear functional on $\mathfrak{F}$ given in terms of iterated residues with respect to a fixed ordered basis x on $\mathfrak{t}^*$. However, we are mainly interested in its behavior on subsets $\mathfrak{F}_{reg}$ and $\mathfrak{F}_{Ham}$, in particular its dependence on the ordered basis x.

Definition 3.1. Let $x = \{x_1, \dots, x_r\}$ be an ordered basis of $\mathfrak{t}^*$. Let $\frac{Pe^{\lambda}}{\prod_{i\in I}\alpha_i} \in \mathfrak{F}$ and write

$\lambda = \lambda_1 x_1 + \ldots + \lambda_r x_r$. We define

$$\operatorname*{Res}_{x_1}^{+} \frac{P(x)e^{\lambda(x)}}{\prod\limits_{i \in I} \alpha_i(x)} dx_1 = \begin{cases} \operatorname*{Res}\limits_{x_1=\infty} \dfrac{P(x)e^{\lambda(x)}}{\prod\limits_{i \in I} \alpha_i(x)} dx_1 & \text{if } \lambda_1 \geq 0, \\ 0 & \text{if } \lambda_1 < 0 \end{cases}$$

considering $x_2, \ldots, x_r$ as constants while taking the residue with respect to x_1. We fix a scalar product on $\mathfrak{t}^*$ and we define the *Jeffrey-Kirwan residue* as

$$\operatorname*{JKRes}_{x} \frac{P(x)e^{\lambda(x)}}{\prod\limits_{i \in I} \alpha_i(x)} dx = \frac{1}{\sqrt{\det \left[(x_i, x_j)\right]_{i,j=1}^{r}}} \operatorname*{Res}_{x_r}^{+} \left(\ldots \left(\operatorname*{Res}_{x_1}^{+} \frac{P(x)e^{\lambda(x)}}{\prod\limits_{i \in I} \alpha_i(x)} dx_1 \right) \ldots \right) dx_r,$$

where $\det \left[(x_i, x_j)\right]_{i,j=1}^{r}$ is the Gram determinant. We will use the short notation $\operatorname*{Res}\limits_{x}^{+}$ for $\operatorname*{Res}\limits_{x_r}^{+} \ldots \operatorname*{Res}\limits_{x_1}^{+}$.

Remark 3.2. If $\tau = \{\tau_1, \ldots, \tau_r\}$ is an orthonormal basis of $\mathfrak{t}^*$ then

$$\operatorname*{JKRes}_{x} \frac{P(x)e^{\lambda(x)}}{\prod\limits_{i \in I} \alpha_i(x)} dx = \left| \det \left(\frac{\partial x_i(\tau)}{\partial \tau_j} \right)_{i,j=1}^{r} \right|^{-1} \operatorname*{Res}_{x}^{+} \frac{P(x)e^{\lambda(x)}}{\prod\limits_{i \in I} \alpha_i(x)} dx.$$

$\diamond$

Proposition 3.3. $\operatorname*{JKRes}\limits_{x}$ *can be extended to* $\mathfrak{F}$ *additively.*

Proof. Suppose that in $\mathfrak{F}$ there is a non-trivial relation $\sum\limits_{I} \dfrac{P_I(x)e^{\lambda_I(x)}}{\prod_{i \in I} \alpha_i(x)} = 0$ with $\lambda_I \neq \lambda_J$ for $I \neq J$. We may assume that it is a non-trivial relation with smallest number of summands. Bringing to common denominator we get an equation of form $\sum\limits_{I} Q_I(x)e^{\lambda_I(x)} = 0$, where $Q_I = P_I(x) \prod\limits_{J \neq I} \prod\limits_{j \in J} \alpha_j(x) \in \mathbb{R}[\mathfrak{t}]$. We fix a J and we write the latter equation in form

$$-Q_J(x) = \sum_{I \neq J} Q_I(x)e^{\lambda_I(x) - \lambda_J(x)}.$$

Let $\xi \in \mathfrak{t}$ be such that $\lambda_I(\xi) \neq \lambda_J(\xi)$ for all $I \neq J$. Take the derivative in direction ξ of both sides $(\deg Q_J + 1)$ times to get an equation of form

$$0 = \sum_{I \neq J} \left[c_I Q_I(x) + R_I(x)\right] e^{\lambda_I(x) - \lambda_J(x)},$$

where c_I's are non-zero constants and R_I are polynomials with degree smaller than $\deg Q_I$. Thus, we have got a non-trivial relation with less summand as the initial one, which leads to contradiction. $\square$

Example 3.1. Let $\mathfrak{t}^*$ be an one-dimensional real vector space with orthonormal basis $\{x\}$ and consider the analytic function $x \mapsto \dfrac{e^x}{x} + \dfrac{e^{-x}}{-x}$. Conflicting intuitions about analytic functions and residues we have

$$\operatorname*{JKRes}_x \left(\frac{e^x}{x} + \frac{e^{-x}}{-x}\right) dx = \operatorname*{JKRes}_x \frac{e^x}{x} dx = 1.$$

$\diamond$

The notion of polarization of non-zero elements of $\mathfrak{t}^*$ emerges inevitably when we investigate the properties of the classical Jeffrey-Kirwan residue and it will play a crucial role in the equivariant case.

Definition 3.4. Let $z = \{z_1, \dots, z_r\}$ be an ordered basis of $\mathfrak{t}^*$. For any non-zero $\alpha = \sum_{i=1}^{r} a_i z_i \in \mathfrak{t}^*$ we define its *polarization* as

$$\overline{\alpha} = \begin{cases} \alpha & \text{if } a_1 = \ldots = a_{k-1} = 0,\ a_k > 0, \\ -\alpha & \text{if } a_1 = \ldots = a_{k-1} = 0,\ a_k < 0. \end{cases}$$

We say that α is *polarized* with respect to z if $\alpha = \overline{\alpha}$. We define $\varepsilon(\alpha) \in \{\pm 1\}$ by $\alpha = \varepsilon(\alpha) \cdot \overline{\alpha}$.

It will be handy in the case of the equivariant residue to separate the polarization from the basis x according to which we take the residue. If the reader is interested only in the classical Jeffrey-Kirwan residue he/she may suppose that $x = z$.

The set of polarized vectors in $\mathfrak{t}^*$ with respect to the basis z form a cone $\mathcal{C}_z$ in $\mathfrak{t}^*$ which we call the cone of polarized vectors. Moreover, we have decomposition $\mathfrak{t}^* = \mathcal{C}_z \cup \{0\} \cup -\mathcal{C}_z$. Jeffrey and Kirwan implicitly defined the polarization of vectors in a slightly different way [24]. Let $\mathcal{A} = [\alpha_i \,|\, i \in I]$ be a finite collection of non-zero vectors in $\mathfrak{t}^*$. Let Λ be a connected component of $\{t \in \mathfrak{t} \,|\, \alpha_i(t) \neq 0,\ \forall i \in I\}$. For any $\xi \in \Lambda$ let

$$\widetilde{\alpha}_i = \begin{cases} \alpha_i & \text{if } \alpha_i(\xi) > 0, \\ -\alpha_i & \text{if } \alpha_i(\xi) < 0, \end{cases} \tag{3.1}$$

which does not depend on the choice of ξ. We call $\widetilde{\alpha}_i$ the *polarization of α_i with respect to the cone* Λ. The relationship between the two notions of polarization is as follows. For a cone Λ as above choose any $\xi \in \Lambda$ and consider an ordered basis $z = \{z_1, \dots, z_r\}$ of $\mathfrak{t}^*$ such that $z_1(\xi) = 1$, $z_2(\xi) = \ldots = z_r(\xi) = 0$. Then the polarization $\overline{\alpha}_i$ of α_i with respect to the ordered basis z agrees with $\widetilde{\alpha}_i$ for all $i \in I$. Conversely, for an ordered basis z of $\mathfrak{t}^*$ consider the open cone $\Lambda = \{t \in \mathfrak{t} \,|\, \overline{\alpha}_i(t) > 0,\ \forall i \in I\}$, which is non-empty and moreover, $\widetilde{\alpha}_i$ agrees with $\overline{\alpha}_i$ for all $i \in I$. The advantage of Definition 3.4 is that on addition of new vectors to $\mathcal{A}$ their polarization is induced automatically, while in the other case we may have to choose a subcone of Λ in order to the polarization of newly added vectors to be defined. However, by (3.1) it is clear that all possible simultaneous polarizations for a fixed collection of vectors $\mathcal{A}$ are parametrized by connected components of $\{t \in \mathfrak{t} \,|\, \alpha(t) \neq 0,\ \forall \alpha \in \mathcal{A}\}$.

We borrow the notation Λ for polarizations induced by ordered basis, too. A polarization Λ on $\mathfrak{t}^*$ induces polarizations on any subspace $V \subset \mathfrak{t}^*$. More precisely

Lemma 3.5. *There is an ordered basis v of V such that any non-zero vector $\alpha \in V$ is polarized with respect to Λ if and only if it is polarized with respect to v.*

Proof. Assume that Λ is induced by ordered basis $z = \{z_1, \ldots, z_r\}$ of $\mathfrak{t}^*$. Let $\{z^1, \ldots, z^r\} \subset \mathfrak{t}$ be its dual basis. Denote $i^* : \mathfrak{t} \to V^*$ the adjoint of the inclusion $i : V \hookrightarrow \mathfrak{t}^*$. Let $\{v^1 = i^*(z^{j_1}), \ldots, v^q = i^*(z^{j_q})\}$ be a basis of V^* such that $j_1 + \ldots + j_q$ is minimal and let $v = \{v_1, \ldots, v_q\} \subset V$ be its dual basis.

If $\alpha \in V$ is polarized with respect to Λ then there is a k such that $\alpha(z^1) = \ldots = \alpha(z^{k-1}) = 0$ and $\alpha(z^k) > 0$. Then we have $\alpha(i^*(z^1)) = \ldots = \alpha(i^*(z^{k-1})) = 0$ and $\alpha(i^*(z^k)) > 0$. Hence $i^*(z^1), \ldots, i^*(z^{k-1})$ cannot span V^*, therefore $i^*(z^k) = v^l$ for some l by minimality condition. We also have $v^1, \ldots, v^{l-1} \in \{i^*(z^1), \ldots, i^*(z^{k-1})\}$, thus α is polarized with respect to v.

Conversely, let $\alpha \in V$ be polarized with respect to v, i.e. $\alpha(v^1) = \ldots = \alpha(v^{l-1}) = 0$ and $\alpha(v^l) > 0$. We have $\alpha(z^{j_1}) = \ldots = \alpha(z^{j_{l-1}}) = 0$ and $\alpha(z^{j_l}) > 0$. By minimality, for all $h < j_l$ we have $i^*(z^h) \in \operatorname{span}\langle i^*(z^{j_1}), \ldots, i^*(z^{j_{l-1}})\rangle$, therefore $\alpha(z^h) = \alpha(i^*(z^h)) = 0$. Thus α is also polarized with respect to z. $\square$

To deduce properties of the Jeffrey-Kirwan residue we start to analyze iterated residues in more depth.

Definition 3.6. Denote $\Pi_{\langle\beta_1,\ldots,\beta_k\rangle} : \mathfrak{t}^* \to \operatorname{span}\langle x_{k+1}, \ldots, x_r\rangle$ the projection along $\operatorname{span}\langle\beta_1, \ldots, \beta_k\rangle$ when $\{\beta_1, \ldots, \beta_k, x_{k+1}, \ldots, x_r\}$ is a basis of $\mathfrak{t}^*$.

We fix two bases $x = \{x_1, \ldots, x_r\}$ and $z = \{z_1, \ldots, z_r\}$ of $\mathfrak{t}^*$. Denote Λ the polarization induced by z. For $\beta \in \mathfrak{t}^*$ we define the residue $\operatorname*{Res}^{\Lambda}_{x_1|\beta}$ as follows. If $\beta = \sum_{i=1}^{r} b_i x_i \notin \operatorname{span}\langle x_2, \ldots, x_r\rangle$ then $\beta(x) = 0$ it defines a pole $x_1 = -\sum_{i=2}^{r} \frac{b_i}{b_1} x_i$ and the residue at this pole

$$\operatorname*{Res}^{\Lambda}_{x_1|\beta} \frac{P(x)e^{\lambda(x)}}{\prod\limits_{i\in I} \alpha_i(x)} dx_1 = \frac{1}{|b_1|} \operatorname*{Res}_{u=0} \frac{P(u, x_2, \ldots, x_r)e^{\lambda^1 u + \Pi_{\langle\beta\rangle}\lambda(x)}}{\prod\limits_{i\in I} (a_i u + \Pi_{\langle\beta\rangle}\alpha_i(x))} du, \tag{3.2}$$

where $u = \overline{\beta}$ is the polarization of β with respect to Λ and $\alpha_i = a_i u + \Pi_{\langle\beta\rangle}\alpha_i$ for all $i \in I$. If $\beta \in \operatorname{span}\langle x_2, \ldots, x_r\rangle$ then $\beta(x) = 0$ does not define a pole with respect to x_1 and we set $\operatorname*{Res}^{\Lambda}_{x_1|\beta} \frac{P(x)e^{\lambda(x)}}{\prod_{i\in I} \alpha_i(x)} dx_1 = 0$. Similarly, if $\lambda = \lambda^1\overline{\beta} + \Pi_{\langle\beta\rangle}\lambda$ then we define

$$\operatorname*{Res}^{\Lambda,+}_{x_1|\beta} \frac{P(x)e^{\lambda(x)}}{\prod_{i\in I} \alpha_i(x)} dx_1 = \begin{cases} \operatorname*{Res}^{\Lambda}_{x_1|\beta} \dfrac{P(x)e^{\lambda(x)}}{\prod\limits_{i\in I} \alpha_i(x)} dx_1 & \text{if } \lambda^1 \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

In the case $z = x$ we will drop Λ from notations, that is,

$$\operatorname*{Res}_{x_1|\beta} = \operatorname*{Res}^{\Lambda}_{x_1|\beta} \qquad \text{and} \qquad \operatorname*{Res}^{+}_{x_1|\beta} = \operatorname*{Res}^{\Lambda,+}_{x_1|\beta}.$$

Example 3.2. Let $x = \{x_1, x_2\}$, $z = \{x_2, x_1\}$ and $\beta(x) = 2x_1 - x_2$. Then we set $u = \overline{\beta}(x) = x_2 - 2x_1$ and we compute

$$\operatorname*{Res}^{\Lambda}_{x_1|\beta} \frac{dx_1}{(2x_1 - x_2)^3 x_1} = \frac{1}{2} \operatorname*{Res}_{u=0} \frac{du}{-u^3(-\frac{u}{2} + \frac{x_2}{2})} = -\operatorname*{Res}_{u=0} \frac{1}{u^3 x_2} \sum_{l \geq 0} \left(\frac{u}{x_2}\right)^l du = -\frac{1}{x_2^3},$$

$$\operatorname*{Res}^{\Lambda,+}_{x_1|\beta} \frac{e^{2x_1}}{(2x_1 - x_2)^3 x_1} dx_1 = \frac{1}{2} \operatorname*{Res}^{+}_{u=0} \frac{e^{-u+x_2}}{-u^3(-\frac{u}{2} + \frac{x_2}{2})} du = 0.$$

If we take $z = x$ then $u = \overline{\beta}(x) = 2x_1 - x_2$ and

$$\operatorname*{Res}_{x_1|\beta} \frac{dx_1}{(2x_1 - x_2)^3 x_1} = \frac{1}{2} \operatorname*{Res}_{u=0} \frac{du}{u^3(\frac{u}{2} + \frac{x_2}{2})} = \operatorname*{Res}_{u=0} \frac{1}{u^3 x_2} \sum_{l \geq 0} \left(-\frac{u}{x_2}\right)^l du = \frac{1}{x_2^3},$$

$$\begin{aligned}\operatorname*{Res}^{+}_{x_1|\beta} \frac{e^{2x_1}}{(2x_1 - x_2)^3 x_1} dx_1 &= \frac{1}{2} \operatorname*{Res}^{+}_{u=0} \frac{e^{u+x_2}}{u^3(\frac{u}{2} + \frac{x_2}{2})} du = e^{x_2} \operatorname*{Res}_{u=0} \frac{e^u}{u^3 x_2} \sum_{l \geq 0} \left(-\frac{u}{x_2}\right)^l du \\ &= e^{x_2} \left(\frac{1}{x_2^3} - \frac{1}{x_2^2} + \frac{1}{2x_2}\right).\end{aligned}$$

◇

In computing the right hand side of (3.2) we expand every fraction

$$\frac{1}{a_i u + \Pi_{\langle\beta\rangle} \alpha_i} = \frac{1}{\Pi_{\langle\beta\rangle} \alpha_i} \sum_{l_i \geq 0} \left(-\frac{a_i u}{\Pi_{\langle\beta\rangle} \alpha_i}\right)^{l_i} \tag{3.3}$$

if $\Pi_{\langle\beta\rangle} \alpha_i \neq 0$, and remark that we can truncate every expansion at $l_i = |I|$. Thus, (3.2) yields a fraction of form $\dfrac{Q(x_2, \ldots, x_r) e^{\Pi_{\langle\beta\rangle} \lambda(x_2, \ldots, x_r)}}{\prod_{j \in J} \alpha'_j(x_2, \ldots, x_r)} \in \mathfrak{F}$.

Remark 3.7. Suitable truncations of expansions (3.3) will not affect $\operatorname*{Res}^{+}_{x_1}$. However, the same is not true for $e^{\lambda(x)}$. Suitable truncation of the expansion of $e^{\lambda(x)}$ does not change the usual residue, but it will affect Res^+. For example, $\operatorname*{Res}_{x} \dfrac{e^{-x}}{x^3} dx = \operatorname*{Res}_{x} \dfrac{1 - x + \frac{1}{2}x^2}{x^3} dx = \dfrac{1}{2}$, while $\operatorname*{Res}^{+}_{x} \dfrac{e^{-x}}{x^3} dx = 0$ and $\operatorname*{Res}^{+}_{x} \dfrac{1 - x + \frac{1}{2}x^2}{x^3} dx = \dfrac{1}{2}$. ◇

Two vectors $\beta, \gamma \in \mathfrak{t}^*$ yield the same pole if and only if $\beta = c \cdot \gamma$ for some $c \neq 0$. We will denote the class of β under this equivalence relation by $[\beta]$. Moreover, (3.2) vanishes if $\Pi_{\langle\beta\rangle} \alpha_i \neq 0$ for all $i \in I$, that is, $[\beta] \neq [\alpha_i]$ for all $i \in I$. With these notations we have relations

$$\operatorname*{Res}_{x_1 = \infty} \frac{P(x) e^{\lambda(x)}}{\prod\limits_{i \in I} \alpha_i(x)} dx_1 = \sum_{[\beta]} \operatorname*{Res}_{x_1|\beta} \frac{P(x) e^{\lambda(x)}}{\prod\limits_{i \in I} \alpha_i(x)} dx_1$$

and

$$\operatorname*{Res}^{+}_{x_1} \frac{P(x) e^{\lambda(x)}}{\prod\limits_{i \in I} \alpha_i(x)} dx_1 = \sum_{[\beta]} \operatorname*{Res}^{+}_{x_1|\beta} \frac{P(x) e^{\lambda(x)}}{\prod\limits_{i \in I} \alpha_i(x)} dx_1.$$

We define $\operatorname*{Res}^{\Lambda}_{x_1}$ and $\operatorname*{Res}^{\Lambda,+}_{x_1}$ such that they satisfy similar relations

$$\operatorname*{Res}^{\Lambda}_{x_1} \frac{P(x)e^{\lambda(x)}}{\prod\limits_{i\in I}\alpha_i(x)}dx_1 = \sum_{[\beta]} \operatorname*{Res}^{\Lambda}_{x_1|\beta} \frac{P(x)e^{\lambda(x)}}{\prod\limits_{i\in I}\alpha_i(x)}dx_1$$

and

$$\operatorname*{Res}^{\Lambda,+}_{x_1} \frac{P(x)e^{\lambda(x)}}{\prod\limits_{i\in I}\alpha_i(x)}dx_1 = \sum_{[\beta]} \operatorname*{Res}^{\Lambda,+}_{x_1|\beta} \frac{P(x)e^{\lambda(x)}}{\prod\limits_{i\in I}\alpha_i(x)}dx_1.$$

More generally, if a tuple $(\beta_1,\dots,\beta_k)$ satisfies flag like condition

$$\text{(Fl)} \qquad \Pi_{\langle\beta_1,\dots,\beta_{i-1}\rangle}\beta_i \in \operatorname{span}\langle x_i,\dots,x_r\rangle \setminus \operatorname{span}\langle x_{i+1},\dots,x_r\rangle, \qquad \forall\, i=1,\dots,k$$

then let

$$\begin{cases} u_1 = \overline{\beta_1}(x) \\ u_2 = \overline{\Pi_{\langle\beta_1\rangle}\beta_2}(x) \\ \vdots \\ u_k = \overline{\Pi_{\langle\beta_1,\dots,\beta_{k-1}\rangle}\beta_k}(x) \end{cases}, \tag{3.4}$$

where the polarization is taken with respect to Λ. We express this system in matrix form

$$(u_1,\dots,u_k)^t = B\cdot(x_1,\dots,x_r)^t, \tag{3.5}$$

where $B \in M_{k,r}(\mathbb{R})$ is an upper triangular matrix with non-zero diagonal entries and denote $\delta_B = |\det([B_{ij}]_{i,j=1}^k)|$. We set

$$\begin{aligned} &\operatorname*{Res}^{\Lambda}_{x_k|\beta_k}\dots\operatorname*{Res}^{\Lambda}_{x_1|\beta_1} \frac{P(x)e^{\lambda(x)}}{\prod\limits_{i\in I}\alpha_i(x)}dx_1\dots dx_k \\ &\quad= \operatorname*{Res}^{\Lambda}_{x_k|\Pi_{\langle\beta_1,\dots,\beta_{k-1}\rangle}\beta_k}\dots\operatorname*{Res}^{\Lambda}_{x_2|\Pi_{\langle\beta_1\rangle}\beta_2}\operatorname*{Res}^{\Lambda}_{x_1|\beta_1} \frac{P(x)e^{\lambda(x)}}{\prod\limits_{i\in I}\alpha_i(x)}dx_1\dots dx_k \\ &\quad= \frac{1}{\delta_B}\cdot\operatorname*{Res}_{u_k=0}\dots\operatorname*{Res}_{u_1=0} \frac{P(u_1,\dots,u_k,x_{k+1},\dots,x_r)e^{\lambda^1u_1+\dots+\lambda^ku_k+\Pi_{\langle\beta_1,\dots,\beta_{k-1}\rangle}\lambda(x)}}{\prod\limits_{i\in I}(a_{i1}u_1+\dots a_{ik}u_k+\Pi_{\langle\beta_1,\dots,\beta_k\rangle}\alpha_i(x))}du_1\dots du_k, \end{aligned} \tag{3.6}$$

where $\lambda = \lambda^1u_1+\dots+\lambda^ku_k+\Pi_{\langle\beta_1,\dots,\beta_k\rangle}\lambda$ and $\alpha_i = a_{i1}u_1+\dots+a_{ik}u_k+\Pi_{\langle\beta_1,\dots,\beta_k\rangle}\alpha_i$ for all $i\in I$. If $(\beta_1,\dots,\beta_k)$ does not satisfy condition (Fl) then we set $\operatorname*{Res}^{\Lambda}_{x_k|\beta_k}\dots\operatorname*{Res}^{\Lambda}_{x_1|\beta_1}\frac{P(x)e^{\lambda(x)}}{\prod_{i\in I}\alpha_i(x)}dx_1\dots dx_k = 0$.

Similarly, if $\lambda = \lambda^1\overline{\beta_1}+\dots+\lambda^k\overline{\Pi_{\langle\beta_1,\dots,\beta_{k-1}\rangle}\beta_k}+\Pi_{\langle\beta_1,\dots,\beta_k\rangle}\lambda$ then we set

$$\begin{aligned} &\operatorname*{Res}^{\Lambda,+}_{x_k|\beta_k}\dots\operatorname*{Res}^{\Lambda,+}_{x_1|\beta_1} \frac{P(x)e^{\lambda(x)}}{\prod\limits_{i\in I}\alpha_i(x)}dx_1\dots dx_k \\ &\qquad= \begin{cases} \operatorname*{Res}^{\Lambda}_{x_k|\beta_k}\dots\operatorname*{Res}^{\Lambda}_{x_1|\beta_1} \dfrac{P(x)e^{\lambda(x)}}{\prod\limits_{i\in I}\alpha_i(x)}dx_1\dots dx_k & \text{if } \lambda^1,\dots,\lambda^k \geq 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Definition 3.8. Tuples $(\beta_1,\dots,\beta_k)$ and $(\gamma_1,\dots,\gamma_k)$ satisfying condition (Fl) are equivalent if there are $c_i \neq 0$ such that $\Pi_{\langle\beta_1,\dots,\beta_{i-1}\rangle}\beta_i = c_i \cdot \Pi_{\langle\gamma_1,\dots,\gamma_{i-1}\rangle}\gamma_i$ for all $i=1,\dots,k$. We denote by $[\beta_1,\dots,\beta_k]$ the equivalence class of $(\beta_1,\dots,\beta_k)$.

Remark 3.9. To a tuple $(\beta_1,\dots,\beta_k)$ satisfying condition (Fl) we can associate a flag $V_1 \subset V_2 \subset \dots \subset V_k$ such that $V_i = \operatorname{span}\langle\beta_1,\dots,\beta_i\rangle$. Then $(\beta_1,\dots,\beta_k)$ and $(\gamma_1,\dots,\gamma_k)$ are equivalent if they have the same associated flag. $\diamond$

The tuples $(\beta_1,\dots,\beta_k)$ and $(\gamma_1,\dots,\gamma_k)$ satisfying condition (Fl) are equivalent if and only if

$$\operatorname*{Res}^{\Lambda}_{x_k|\beta_k}\dots\operatorname*{Res}^{\Lambda}_{x_1|\beta_1} F(x)dx = \operatorname*{Res}^{\Lambda}_{x_k|\gamma_k}\dots\operatorname*{Res}^{\Lambda}_{x_1|\gamma_1} F(x)dx$$

for all $F \in \mathfrak{F}$. Inductively we can also see that $\operatorname*{Res}^{\Lambda}_{x_k|\beta_k}\dots\operatorname*{Res}^{\Lambda}_{x_1|\beta_1} \frac{P(x)e^{\lambda(x)}}{\prod_{i\in I}\alpha_i(x)}dx_1\dots dx_k$ may not vanish only if there are $i_1,\dots,i_k \in I$ such that $[\beta_1,\dots,\beta_k] = [\alpha_{i_1},\dots,\alpha_{i_k}]$. In this case we may suppose that $\beta_l = \alpha_{i_l}$ for all $l = 1,\dots,k$. Then we have relation

$$\begin{aligned}\operatorname*{Res}^{\Lambda,+}_{x_k}\dots\operatorname*{Res}^{\Lambda,+}_{x_1} \frac{P(x)e^{\lambda(x)}}{\prod\limits_{i\in I}\alpha_i(x)}dx_1\dots dx_k &= \sum_{[\beta_1,\dots,\beta_k]}\operatorname*{Res}^{\Lambda,+}_{x_k|\beta_k}\dots\operatorname*{Res}^{\Lambda,+}_{x_1|\beta_1} \frac{P(x)e^{\lambda(x)}}{\prod\limits_{i\in I}\alpha_i(x)}dx_1\dots dx_k \\ &= \sum_{[\alpha_{i_1},\dots,\alpha_{i_k}]}\operatorname*{Res}^{\Lambda,+}_{x_k|\alpha_{i_k}}\dots\operatorname*{Res}^{\Lambda,+}_{x_1|\alpha_{i_1}} \frac{P(x)e^{\lambda(x)}}{\prod\limits_{i\in I}\alpha_i(x)}dx_1\dots dx_k. \qquad (3.7)\end{aligned}$$

We have the following vanishing result (cf. [24], Proposition 3.2(iii)).

Lemma 3.10. *Let $Q \in \mathbb{R}[x_1,\dots,x_k] \subset \mathbb{R}[x_1,\dots,x_r]$ be a homogeneous polynomial of degree d and let $\beta_1,\dots,\beta_k \in \operatorname{span}\langle x_1,\dots,x_k\rangle$. If $d > n-k$ then*

$$\operatorname*{Res}_{x_k|\beta_k}\dots\operatorname*{Res}_{x_1|\beta_1} \frac{Q(x)}{\prod\limits_{i=1}^{n}\alpha_i(x)}dx_1\dots dx_k = 0.$$

In particular, if $P \in \mathbb{R}[x_1,\dots,x_r]$ is a homogeneous polynomial of degree d such that $d > n-r$ then $\operatorname*{JKRes}_{x} \frac{P(x)e^{\lambda(x)}}{\prod_{i=1}^{n}\alpha_i(x)}dx = 0.$

Proof. We will prove it by induction on k. Assume that $\beta_1 \notin \operatorname{span}\langle x_2,\dots,x_k\rangle$ and let $u = \overline{\beta_1}$ with respect to the polarization induced by x. Write $\alpha_i = a_i u + \gamma_i$ with $a_i \in \mathbb{R}$ and $\gamma_i \in \operatorname{span}\langle x_2,\dots,x_r\rangle$ for all $i=1,\dots,n$. Moreover, suppose that $\gamma_i \neq 0$ if $i \leq m$ and $\gamma_i = 0$ if $i > m$. If $k=1$ the the lemma is trivial since $\operatorname*{Res}_{x_1|\beta_1} = \operatorname*{Res}_{x_1=0}$. If $k>1$ then

$$\operatorname*{Res}_{x_1|\beta_1}\frac{Q(x)}{\prod_{i=1}^{n}\alpha_i(x)}dx_1 = \left|\frac{\partial\beta_1}{\partial x_1}\right|^{-1}\sum_{l_1,\dots,l_m\geq 0}\operatorname*{Res}_{u=0}\frac{(-a_1u)^{l_1}\cdots(-a_mu)^{l_m}Q(u,x_2,\dots,x_k)}{\gamma_1(x)^{l_1+1}\cdots\gamma_m(x)^{l_m+1}(a_{m+1}u)\cdots(a_nu)}du.$$

On the right hand side there are finitely many non-zero terms, since summands with $l_1+\ldots+l_m \geq n-m$ vanish. Moreover, it yields a homogeneous fraction $\frac{R(x_2,\dots,x_k)}{\gamma_1(x)^{h_1}\dots\gamma_m(x)^{h_m}}$ of degree $d-n+1$

in variables $x_2, \ldots, x_r$. Hence

$$\operatorname*{Res}_{x_k|\beta_k} \ldots \operatorname*{Res}_{x_1|\beta_1} \frac{Q(x)}{\prod_{i=1}^{n} \alpha_i(x)} dx_1 \ldots dx_k = \operatorname*{Res}_{x_k|\Pi_{\langle\beta_1\rangle}\beta_k} \ldots \operatorname*{Res}_{x_2|\Pi_{\langle\beta_1\rangle}\beta_2} \frac{R(x_2, \ldots, x_k)}{\gamma_1(x)^{h_1} \ldots \gamma_m(x)^{h_m}} dx_2 \ldots dx_k = 0$$

by the induction hypothesis, since $\Pi_{\langle\beta_1\rangle}\beta_2, \ldots, \Pi_{\langle\beta_1\rangle}\beta_k \in \operatorname{span}\langle x_2, \ldots, x_k\rangle$ and $d - n + 1 > -(k-1)$. □

Corollary 3.11. *Let* $F = \sum_I \frac{P_I e^{\lambda_I}}{\prod_{i \in I} \alpha_i} \in \mathfrak{F}$ *and let* $\beta_1, \ldots, \beta_k \in V = \operatorname{span}\langle x_1, \ldots, x_k\rangle$. *Write* $\alpha_i = \alpha_i' + \alpha_i''$ *such that* $\alpha_i' \in V$ *and* $\alpha_i'' \in \operatorname{span}\langle x_{k+1}, \ldots, x_r\rangle$. *Denote* $F_V \in \mathfrak{F}$ *the fraction got from* F *by replacing every fraction* $\frac{1}{\alpha_i} = \frac{1}{\alpha_i' + \alpha_i''}$ *by* $\frac{1}{\alpha_i''} \sum_{l_i=0}^{|I|} \left(-\frac{\alpha_i'}{\alpha_i''}\right)^{l_i}$ *when* $\alpha_i'' \neq 0$. *Then*

$$\operatorname*{Res^+}_{x_k|\beta_k} \ldots \operatorname*{Res^+}_{x_1|\beta_1} F(x) dx_1 \ldots dx_k = \operatorname*{Res^+}_{x_k|\beta_k} \ldots \operatorname*{Res^+}_{x_1|\beta_1} F_V(x) dx_1 \ldots dx_k.$$

Proof. It is enough to show it for $F = \frac{Pe^{\lambda}}{\prod_{i=1}^{n} \alpha_i}$. Moreover, the construction of F_V does not change the exponential e^{λ}, hence it is enough to show that

$$\operatorname*{Res}_{x_k|\beta_k} \ldots \operatorname*{Res}_{x_1|\beta_1} F(x) dx_1 \ldots dx_k = \operatorname*{Res}_{x_k|\beta_k} \ldots \operatorname*{Res}_{x_1|\beta_1} F_V(x) dx_1 \ldots dx_k.$$

Assume that $\alpha_i'' = 0$ if and only if $i > m$. Let $u = \{u_1, \ldots, u_k\}$ be the basis of V given by (3.4) and $v = \{x_{k+1}, \ldots, x_r\}$. If $a \ll b \ll c$ then we can expand $\frac{1}{a+b+c}$ in two ways: first expand as $a \ll b + c$ followed by expansion with respect to $b \ll c$ and secondly we can expand as $a + b \ll c$. These two expansions are equal. Hence, the expansion of $\frac{1}{\alpha_i'(u) + \alpha_i''(v)}$ with respect to $\alpha_i'(u) \ll \alpha_i''(v)$ yields the same result as successive expansions $u_1 \ll u_2 \ll \ldots \ll u_k \ll v$. Therefore,

$$\begin{aligned}
\operatorname*{Res}_{x_k|\beta_k} \ldots \operatorname*{Res}_{x_1|\beta_1} F(x) dx_1 \ldots dx_k &= \frac{1}{\delta_B} \cdot \operatorname*{Res}_{u_k=0} \ldots \operatorname*{Res}_{u_1=0} \frac{P(u,v) e^{\lambda(u,v)}}{\prod_{i=1}^{n} (\alpha_i'(u) + \alpha_i''(v))} du_1 \ldots du_k \\
&= \frac{1}{\delta_B} \cdot \operatorname*{Res}_{u_k=0} \ldots \operatorname*{Res}_{u_1=0} \sum_{l_1, \ldots, l_m \geq 0} \frac{(-\alpha_1'(u))^{l_1} \cdots (-\alpha_m'(u))^{l_m} P(u,v) e^{\lambda(u,v)}}{\prod_{i=1}^{m} (\alpha_i''(v))^{l_i+1} \prod_{j=m+1}^{n} \alpha_j'(u)} du_1 \ldots du_k \\
&= \frac{1}{\delta_B} \cdot \operatorname*{Res}_{u_k=0} \ldots \operatorname*{Res}_{u_1=0} \sum_{l_1, \ldots, l_m = 0}^{n} \frac{(-\alpha_1'(u))^{l_1} \cdots (-\alpha_m'(u))^{l_m} P(u,v) e^{\lambda(u,v)}}{\prod_{i=1}^{m} (\alpha_i''(v))^{l_i+1} \prod_{j=m+1}^{n} \alpha_j'(u)} du_1 \ldots du_k \\
&= \frac{1}{\delta_B} \cdot \operatorname*{Res}_{u_k=0} \ldots \operatorname*{Res}_{u_1=0} F_V(u,v) du_1 \ldots du_k \\
&= \operatorname*{Res}_{x_k|\beta_k} \ldots \operatorname*{Res}_{x_1|\beta_1} F_V(x) dx_1 \ldots dx_k
\end{aligned}$$

by (3.6) and Lemma 3.10. □

We have the following base change formula, which will be used in section 3.1.2.

Lemma 3.12. *Let $x = \{x_1, \dots, x_r\}$ be an ordered basis and let $(\beta_1, \dots, \beta_k)$ satisfy condition (Fl). For $i \le k$ define v_i to be the projection of x_i to* $\operatorname{span}\langle\beta_1, \dots, \beta_k\rangle$ *along* $\operatorname{span}\langle x_{k+1}, \dots, x_r\rangle$ *and for $i > k$ let $v_i = x_i$. If $F \in \mathfrak{F}$ then*

$$\operatorname*{Res}^+_{x_k|\beta_k} \dots \operatorname*{Res}^+_{x_1|\beta_1} F(x)dx_1 \dots dx_k = \operatorname*{Res}^+_{v_k|\beta_k} \dots \operatorname*{Res}^+_{v_1|\beta_1} F(v)dv_1 \dots dv_k.$$

Proof. By construction $\operatorname{span}\langle x_{k+1}, \dots, x_r\rangle = \operatorname{span}\langle v_{k+1}, \dots, v_r\rangle$ and for all $i = 1, \dots, k$

$$x_i + \operatorname{span}\langle x_{k+1}, \dots, x_r\rangle = v_i + \operatorname{span}\langle v_{k+1}, \dots, v_r\rangle, \tag{3.8}$$

hence

$$\operatorname{span}\langle x_l, \dots, x_r\rangle = \operatorname{span}\langle v_l, \dots, v_r\rangle$$

for all $l = 1, \dots, r$. This implies that the tuple $(\beta_1, \dots, \beta_k)$ also satisfies the condition (Fl) with respect to the basis v. Moreover, for all $l = 1, \dots, k$ the projection of β_l to $\operatorname{span}\langle x_l, \dots, x_r\rangle$ and $\operatorname{span}\langle v_l, \dots, v_r\rangle$ along $\operatorname{span}\langle\beta_1, \dots \beta_{l-1}\rangle$ agree which we denote by $\Pi_{\langle\beta_1,\dots,\beta_{l-1}\rangle}\beta_l$. The relation (3.8) implies that the polarization of $\Pi_{\langle\beta_1,\dots,\beta_{l-1}\rangle}\beta_l$ for all $l = 1, \dots, k$ with respect to ordered bases x and v are the same, and we denote it by $\overline{\Pi_{\langle\beta_1,\dots,\beta_{l-1}\rangle}\beta_l}$. Furthermore, we consider systems of equations

$$\begin{cases} u_1 = \overline{\beta_1}(x) \\ u_2 = \overline{\Pi_{\langle\beta_1\rangle}\beta_2}(x) \\ \vdots \\ u_k = \overline{\Pi_{\langle\beta_1,\dots,\beta_{k-1}\rangle}\beta_k}(x) \end{cases} \quad \text{and} \quad \begin{cases} u_1 = \overline{\beta_1}(v) \\ u_2 = \overline{\Pi_{\langle\beta_1\rangle}\beta_2}(v) \\ \vdots \\ u_k = \overline{\Pi_{\langle\beta_1,\dots,\beta_{k-1}\rangle}\beta_k}(v) \end{cases},$$

which we express in matrix form

$$(u_1, \dots, u_k)^t = B \cdot (x_1, \dots, x_r)^t \quad \text{and} \quad (u_1, \dots, u_k)^t = B' \cdot (v_1, \dots, v_r)^t.$$

Matrices $B, B' \in M_{k,r}(\mathbb{R})$ are upper triangular with positive diagonal entries and we denote by δ_B and $\delta_{B'}$ the product of their diagonal entries, respectively. The relation (3.8) implies that $\delta_B = \delta_{B'}$ and finally, the lemma follows from (3.6). $\square$

3.1.1 Regular fractions

Definition 3.13. A vector $\lambda \in \mathfrak{t}^*$ is *regular* with respect to $\{\alpha_i \in \mathfrak{t}^* \,|\, i \in I\}$ if λ is not on any $(r-1)$- or less dimensional subspace of $\mathfrak{t}^*$ spanned by subsets of $\{\alpha_i \,|\, i \in I\}$. We call $\dfrac{Pe^\lambda}{\prod_{i\in I}\alpha_i} \in \mathfrak{F}$ *regular* if λ is regular with respect to $\{\alpha_i \,|\, i \in I\}$ and we say that $F = \sum_I \dfrac{P_I e^{\lambda_I}}{\prod_{i\in I}\alpha_i}$ is regular if each summand is regular. We denote by $\mathfrak{F}_{reg}$ the set of regular fractions.

Definition 3.14. A fraction $\dfrac{Pe^\lambda}{\prod_{i\in I}\alpha_i} \in \mathfrak{F}$ is called *generating* if $\{\alpha_i \,|\, i \in I\}$ spans $\mathfrak{t}^*$, otherwise it is called *non-generating*.

Remark 3.15. The relation (3.7) implies the vanishing of $\operatorname*{Res}_{x_r}^{\Lambda,+}\dots\operatorname*{Res}_{x_1}^{\Lambda,+}$, and therefore of $\operatorname*{JKRes}_{x}$, on non-generating fractions. $\diamond$

We have the following partial fraction decomposition. We will prove it in a bit more general form in Lemma 3.37.

Lemma 3.16 (cf. [6], Theorem 1)**.** *Any fraction* $\dfrac{Pe^{\lambda}}{\prod_{i\in I}\alpha_i}$ *can be written as linear combination of non-generating fractions and generating fractions of simple form* $\dfrac{e^{\lambda}}{\alpha_{i_1}^{n_1+1}\dots\alpha_{i_r}^{n_r+1}}$ *with* $i_1,\dots,i_r\in I$ *and* $n_1,\dots,n_r\geq 0$.

The following proposition gives a explicit formula for the Jeffrey-Kirwan residue if we apply it to a partial fraction decomposition as in Lemma 3.16. The second part of the proposition can be found in [24] (Proposition 3.2 (iv)) for $n_1=\dots=n_r=0$ and with some generic assumption on the basis x.

Proposition 3.17. *Let* Λ *be a polarization on* $\mathfrak{t}^*$ *and let* $x=\{x_1,\dots,x_r\}$ *be an ordered basis of* $\mathfrak{t}^*$*. Consider* k *vectors* $\alpha_i=a_{i1}x_1+\dots+a_{ir}x_r\in\mathfrak{t}^*$, $i=1,\dots,k$ *such that* $\det([a_{ij}]_{i,j=1}^k)\neq 0$*. Let* $\lambda=\lambda_1\overline{\alpha_1}+\dots+\lambda_k\overline{\alpha_k}+\eta$ *such that* $\lambda_1,\dots,\lambda_k\neq 0$ *and* $\eta\in\operatorname{span}\langle x_{k+1},\dots,x_r\rangle$*. Then*

$$\operatorname*{Res}_{x_k}^{\Lambda,+}\dots\operatorname*{Res}_{x_1}^{\Lambda,+}\frac{e^{\lambda(x)}}{\prod\limits_{i=1}^{k}\alpha_i(x)^{n_i+1}}dx_1\dots dx_k=\begin{cases}\dfrac{e^{\eta}}{|\det([a_{ij}]_{i,j=1}^k)|}\displaystyle\prod_{i=1}^{k}\frac{\varepsilon(\alpha_i)^{n_i+1}\lambda_i^{n_i}}{n_i!} & \lambda_1,\dots,\lambda_k>0\\ 0 & \text{otherwise,}\end{cases}$$

where $\varepsilon(\alpha_i)\in\{\pm1\}$ *such that* $\overline{\alpha_i}=\varepsilon(\alpha_i)\alpha_i$*. In particular, for* $k=r$ *we have*

$$\operatorname*{JKRes}_{x}\frac{e^{\lambda(x)}}{\prod\limits_{i=1}^{r}\alpha_i(x)^{n_i+1}}dx=\begin{cases}\dfrac{1}{\sqrt{\det\left[(\alpha_i,\alpha_j)\right]_{i,j=1}^r}}\displaystyle\prod_{i=1}^{r}\frac{\varepsilon(\alpha_i)^{n_i+1}\lambda_i^{n_i}}{n_i!} & \lambda_1,\dots,\lambda_r>0\\ 0 & \text{otherwise,}\end{cases}$$

where $\det\left[(\alpha_i,\alpha_j)\right]_{i,j=1}^r$ *is the Gram determinant.*

Proof. Remark that we want to show that

$$\operatorname*{Res}_{x_k}^{\Lambda,+}\dots\operatorname*{Res}_{x_1}^{\Lambda,+}\frac{e^{\lambda(x)}}{\prod\limits_{i=1}^{k}\alpha_i(x)^{n_i+1}}dx_1\dots dx_k$$
$$=\frac{e^{\eta}}{|\det([a_{ij}]_{i,j=1}^k)|}\prod_{i=1}^{k}\frac{\varepsilon(\alpha_i)^{n_i+1}\lambda_i^{n_i}}{n_i!}\chi_{Cone(\overline{\alpha_1},\dots,\overline{\alpha_k},\,\pm x_{k+1},\dots,\pm x_r)}(\lambda),$$

where $\chi_{Cone(\overline{\alpha_1},\dots,\overline{\alpha_k},\,\pm x_{k+1},\dots,\pm x_r)}$ is the characteristic function of the closed set

$$Cone(\overline{\alpha_1},\dots,\overline{\alpha_k},\pm x_{k+1},\dots,\pm x_r).$$

It is enough to prove the proposition for polarized α_i's, i.e. when $\varepsilon(\alpha_i)=1$. First we prove it for $n_1=\dots=n_k=0$ from which we will deduce the general case. We proceed by induction on k. For $k=1$ the statement is obvious.

Let $\sigma \in S_k$ be a permutation such that $\sigma(1) = j$. If $\alpha_j \notin \mathrm{span}\langle x_2, \dots, x_r\rangle$ then denote $\gamma_i = \Pi_{\langle \alpha_j \rangle} \alpha_i$ and $\nu = \Pi_{\langle \alpha_j \rangle} \lambda$ moreover, we compute

$$\begin{aligned}
&\operatorname*{Res}^{\Lambda,+}_{x_k|\alpha_{\sigma(k)}} \dots \operatorname*{Res}^{\Lambda,+}_{x_2|\alpha_{\sigma(2)}} \operatorname*{Res}^{\Lambda,+}_{x_1|\alpha_j} \frac{e^{\lambda(x)}}{\prod_{i=1}^{k} \alpha_i(x)} dx_1 \dots dx_k \\
&\qquad = \frac{1}{|a_{j1}|} \cdot \operatorname*{Res}^{\Lambda,+}_{x_k|\gamma_{\sigma(k)}} \dots \operatorname*{Res}^{\Lambda,+}_{x_2|\gamma_{\sigma(2)}} \operatorname*{Res}^{+}_{u=0} \frac{e^{\lambda'_j u + \nu(x)}}{u \prod_{i \neq j} (c_i u + \gamma_i(x))} du dx_2 \dots dx_k \\
&\qquad\qquad = \frac{1}{|a_{j1}|} \cdot \chi_{[0,+\infty)}(\lambda'_j) \operatorname*{Res}^{\Lambda,+}_{x_k|\gamma_{\sigma(k)}} \dots \operatorname*{Res}^{\Lambda,+}_{x_2|\gamma_{\sigma(2)}} \frac{e^{\nu(x)}}{\prod_{i \neq j} \gamma_i(x)} dx_2 \dots dx_k,
\end{aligned}$$

where $u = \alpha_j(x)$, $\lambda = \lambda'_j u + \nu$, $\chi_{[0,+\infty)}$ is the characteristic function of the set $[0, +\infty)$ and $c_i \in \mathbb{R}$ such that $\alpha_i = c_i \alpha_j + \gamma_i$. Assume that $\alpha_1, \dots, \alpha_q \notin \mathrm{span}\langle x_2, \dots, x_r\rangle$ and $\alpha_{q+1}, \dots, \alpha_k \in \mathrm{span}\langle x_2, \dots, x_r\rangle$. By induction we have

$$\begin{aligned}
&\operatorname*{Res}^{\Lambda,+}_{x_k} \dots \operatorname*{Res}^{\Lambda,+}_{x_1} \frac{e^{\lambda(x)}}{\prod_{i=1}^{k} \alpha_i(x)} dx_1 \dots dx_k \\
&\qquad = \sum_{j=1}^{q} \sum_{\substack{\sigma \in S_k \\ \sigma(1)=j}} \operatorname*{Res}^{\Lambda,+}_{x_k|\alpha_{\sigma(k)}} \dots \operatorname*{Res}^{\Lambda,+}_{x_2|\alpha_{\sigma(2)}} \operatorname*{Res}^{\Lambda,+}_{x_1|\alpha_j} \frac{e^{\lambda(x)}}{\prod_{i=1}^{k} \alpha_i(x)} dx_1 \dots dx_k \\
&\qquad = \sum_{j=1}^{q} \frac{\chi_{[0,+\infty)}(\lambda'_j)}{|a_{j1}|} \sum_{\substack{\sigma \in S_k \\ \sigma(1)=j}} \operatorname*{Res}^{\Lambda,+}_{x_k|\gamma_{\sigma(k)}} \dots \operatorname*{Res}^{\Lambda,+}_{x_2|\gamma_{\sigma(2)}} \frac{e^{\nu(x)}}{\prod_{i \neq j} \gamma_i(x)} dx_2 \dots dx_k \\
&\qquad = \sum_{j=1}^{q} \frac{\chi_{[0,+\infty)}(\lambda'_j)}{|a_{j1}|} \frac{\prod_{i \neq j} \varepsilon(\gamma_i)}{\left| \det \left(\left[\frac{\partial \gamma_i(x)}{\partial x_l} \right]_{i \neq j,\, l \neq 1} \right) \right|} \chi_{Cone(\overline{\gamma_i},\, \pm x_{k+1}, \dots, \pm x_r \,|\, 1 \leq i \leq k)}(\nu) \\
&\qquad = \frac{1}{\left| \det \left([a_{i,l}]_{i,l=1}^{k} \right) \right|} \sum_{j=1}^{q} \prod_{i \neq j} \varepsilon(\gamma_i)\, \chi_{Cone(\alpha_j,\, \overline{\gamma_i},\, \pm x_{k+1}, \dots, \pm x_r \,|\, 1 \leq i \leq k)}(\lambda)
\end{aligned}$$

We will show that

$$\chi_{Cone(\alpha_1, \dots \alpha_k, \pm x_{k+1}, \dots, \pm x_r)}(\lambda) = \sum_{j=1}^{q} \prod_{i \neq j} \varepsilon(\gamma_i)\, \chi_{Cone(\alpha_j,\, \overline{\gamma_i}, \pm x_{k+1}, \dots, \pm x_r \,|\, 1 \leq i \leq k)}(\lambda). \tag{3.9}$$

It is enough to show it when $\alpha_i(x) = x_1 + \beta_i(x)$, $1 \leq i \leq q$, $\alpha_i(x) = \beta_i(x)$, $q < i \leq k$ with $\beta_i \in \mathrm{span}\langle x_2, \dots, x_r\rangle$ and $\beta_i - \beta_j = \overline{\beta_i - \beta_j}$ if $j < i \leq q$. Then we can reformulate (3.9) as

$$\chi_{Cone(\alpha_1, \dots, \alpha_k, \pm x_{k+1}, \dots, \pm x_r)}(\lambda) = \sum_{j=1}^{q} (-1)^{j-1} \chi_{Cone(\alpha_j,\, \overline{\beta_i - \beta_j},\, \beta_l, \pm x_{k+1}, \dots, \pm x_r \,|\, i \leq q,\, q < l \leq k)}(\lambda). \tag{3.10}$$

For any $j \leq q$ we have

$$\lambda = \eta + \lambda_1\alpha_1 + \ldots + \lambda_k\alpha_k = \eta + (\lambda_1 + \ldots + \lambda_q)\alpha_j + \sum_{i=1}^{q} \lambda_i(\beta_i - \beta_j) + \sum_{q<l\leq k} \lambda_l\beta_l$$
$$= \eta + (\lambda_1 + \ldots + \lambda_q)\alpha_j - \sum_{i<j} \lambda_i(\overline{\beta_i - \beta_j}) + \sum_{j<i\leq q} \lambda_i(\overline{\beta_i - \beta_j}) + \sum_{q<l\leq k} \lambda_l\beta_l.$$

By hypothesis $\lambda_1, \ldots, \lambda_k \neq 0$, thus if we write $\eta = \eta_{k+1}x_{k+1} + \ldots + \eta_r x_r$ then we have

$$\chi_{Cone(\alpha_1,\ldots,\alpha_k,\pm x_{k+1},\ldots,\pm x_r)}(\lambda) = \chi_{(\mathbb{R}_{>0})^k\times\mathbb{R}^{r-k}}(\lambda_1, \ldots, \lambda_k, \eta_{k+1}, \ldots, \eta_r) = \chi_{(\mathbb{R}_{>0})^k}(\lambda_1, \ldots, \lambda_k)$$

and

$$\chi_{Cone\left(\alpha_j,\, \overline{\beta_i-\beta_j},\, \beta_l, \pm x_{k_1},\ldots,\pm x_r \,|\, i\leq q,\, q<l\leq k\right)}(\lambda)$$
$$= \chi_{\left\{s\in(\mathbb{R}^\times)^k\times\mathbb{R}^{r-k} \,\middle|\, \sum_{i=1}^{q} s_i\geq 0,\, s_i<0,\, s_l>0,\, i<j<l\leq k\right\}}(\lambda_1, \ldots, \lambda_k, \eta_{k+1}, \ldots, \eta_r),$$
$$= \chi_{\left\{s\in(\mathbb{R}^\times)^k \,\middle|\, \sum_{i=1}^{q} s_i\geq 0,\, s_i<0,\, s_l>0,\, i<j<l\leq k\right\}}(\lambda_1, \ldots, \lambda_k),$$

where $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$. Then (3.10) is equivalent to the following inclusion-exclusion relation

$$\begin{aligned}
\chi_{\{s\in(\mathbb{R}^\times)^k \,|\, s_1,\ldots,s_k>0\}} &= \chi_{\left\{s\in(\mathbb{R}^\times)^k \,\middle|\, \sum_{i=1}^{q} s_i\geq 0,\ s_1,\ldots,s_k>0\right\}} \\
&= \chi_{\left\{s\in(\mathbb{R}^\times)^k \,\middle|\, \sum_{i=1}^{q} s_i\geq 0,\ s_2,\ldots,s_k>0\right\}} \\
&\quad - \chi_{\left\{s\in(\mathbb{R}^\times)^k \,\middle|\, \sum_{i=1}^{q} s_i\geq 0,\ s_1<0,\ s_3,\ldots,s_k>0\right\}} \\
&\quad + \chi_{\left\{s\in(\mathbb{R}^\times)^k \,\middle|\, \sum_{i=1}^{q} s_i\geq 0,\ s_1,s_2<0,\ s_4,\ldots,s_k>0\right\}} \\
&\quad \ldots + (-1)^{q-1}\chi_{\left\{s\in(\mathbb{R}^\times)^k \,\middle|\, \sum_{i=1}^{q} s_i\geq 0,\ s_1,\ldots,s_{q-1}<0,\ s_{q+1},\ldots,s_k>0\right\}}.
\end{aligned}$$

Thus we have proved the proposition for $n_1 = \ldots = n_k = 0$. To deduce the general case we set $y_i = tx_i$, $(t > 0)$ and $\mathcal{R}_{n_1,\ldots,n_k} = \operatorname*{Res}_{x_k}^{\Lambda,+} \ldots \operatorname*{Res}_{x_1}^{\Lambda,+} \dfrac{e^{\lambda(x)}}{\prod_{i=1}^{k} \alpha_i(x)^{n_i+1}} dx_1 \ldots dx_k$. Then for $\lambda = \sum_{j=1}^{r} \lambda^j x_j$ and $N = n_1 + \ldots + n_k$ we have

$$\operatorname*{Res}_{x_k}^{\Lambda,+} \ldots \operatorname*{Res}_{x_1}^{\Lambda,+} \frac{e^{\eta + t\sum_{i=1}^{k} \lambda_i\alpha_i(x)}}{\prod_{i=1}^{k} \alpha_i(x)^{n_i+1}} dx_1 \ldots dx_k$$
$$= t^N \operatorname*{Res}_{y_k}^{\Lambda,+} \ldots \operatorname*{Res}_{y_1}^{\Lambda,+} \frac{e^{\eta+\lambda^1 y_1+\ldots+\lambda^r y_r}}{\prod_{i=1}^{k} (a_{i1}y_1 + \ldots + a_{ir}y_r)^{n_i+1}} dy_1 \ldots dy_k = t^N \mathcal{R}_{n_1,\ldots,n_k}.$$

Take the derivative of both sides with respect to t at $t=1$ to get

$$\sum_{i=1}^{k} \lambda_i \mathcal{R}_{n_1,\ldots,n_{i-1},\, n_i-1,\, n_{i+1},\ldots,n_k} = N\,\mathcal{R}_{n_1,\ldots,n_k}.$$

From this relation it follows the first part of the proposition by induction on N.

Finally, we show the last part of the proposition. Let $\tau = \{\tau_i, \ldots, \tau_r\}$ be an orthonormal basis of $\mathfrak{t}^*$. If we write $x_i = \sum\limits_{j=1}^{r} x_{ij}\tau_j$ then $\det\left[(x_i,\, x_j)\right]_{i,j=1}^{r} = \det\left([x_{ij}]_{i,j=1}^{r}\right)^2$ and

$$\det\left([a_{ij}]_{i,j=1}^{r}\right)^2 \det\left([x_{ij}]_{i,j=1}^{r}\right)^2 = \det\left[(\alpha_i,\, \alpha_j)\right]_{i,j=1}^{r},$$

thus the second part of the proposition follows. □

Corollary 3.18. *If* $F = \sum\limits_{I} \dfrac{P_I e^{\lambda_I}}{\prod_{i\in I}\alpha_i} \in \mathfrak{F}$ *is regular then* $\underset{x}{\mathrm{JKRes}}\, F(x)dx$ *depends only on the polarization induced by the ordered basis* x. *Moreover,* $\underset{x}{\mathrm{JKRes}}\, F(x)e^{\rho(x)}dx$ *depends continuously on* $\rho \in \mathfrak{t}^*$ *in a small neighborhood* $\mathcal{U}$ *of* 0. *The neighborhood* $\mathcal{U}$ *is such that* $\dfrac{P_I e^{\lambda_I+\rho}}{\prod_{i\in I}\alpha_i}$ *is regular for all* I *and* $\rho \in \mathcal{U}$.

The following property is very useful in showing vanishing of Jeffrey-Kirwan residues (property (1) in [25], section 3.4).

Corollary 3.19. *Let* x *be an ordered basis and let* $\dfrac{Pe^{\lambda}}{\prod_{i\in I}\alpha_i}$ *be regular. If* $\lambda \notin Cone(\overline{\alpha_i} \mid i \in I)$ *then* $\underset{x}{\mathrm{JKRes}}\, \dfrac{P(x)e^{\lambda(x)}}{\prod_{i\in I}\alpha_i(x)}dx = 0$.

Proof. By Lemma 3.16 we can decompose $\dfrac{P(x)e^{\lambda(x)}}{\prod_{i\in I}\alpha_i(x)}$ to sum of non-generating fractions and generating fractions of form $\dfrac{e^{\lambda(x)}}{\prod_{k=1}^{r}\alpha_{i_k}(x)^{n_k}}$. Recall that on non-generating fractions $\underset{x}{\mathrm{Res}^+}$ vanishes. Since $Cone(\overline{\alpha_{i_1}}, \ldots, \overline{\alpha_{i_r}}) \subset Cone(\overline{\alpha_i} \mid i \in I)$ for all $i_1, \ldots, i_r \in I$, hence the corollary follows from Proposition 3.17. □

The following proposition is a generalization of Lemma 3.3 in [25].

Proposition 3.20. *Let* $F = \sum\limits_{I} \dfrac{P_I e^{\lambda_I}}{\prod_{i\in I}\alpha_i}$ *be regular. If* $F(x)$ *is analytic then* $\underset{x}{\mathrm{JKRes}}\, F(x)dx$ *is independent on the choice of the ordered basis* x.

Proof. By Corollary 3.18 the $\underset{x}{\mathrm{JKRes}}\, F(x)dx$ may only depend on the polarizations induced by the ordered basis x. Polarizations on $\mathcal{A} = \cup_I \{\alpha_i \mid i \in I\}$ correspond to connected components of $\{t \in \mathfrak{t} \mid \alpha(t) \neq 0,\ \forall \alpha \in \mathcal{A}\}$. These components are open polyhedral cones. Let Λ and Λ' be two neighboring cones, separated by a hyperplane $\{t \in \mathfrak{t} \mid \alpha(t) = 0\}$ for some $\alpha \in \mathcal{A}$. We suppose that all elements of $\mathcal{A}$ are polarized with respect to Λ. Let $\xi \in \mathfrak{t}$ be in the relative interior of the intersection of closures $cl(\Lambda) \cap cl(\Lambda')$. Then $\mathcal{A} \cap \{\tau \in \mathfrak{t}^* \mid \tau(\xi) = 0\}$ contains only multiples of α. Let $x = \{x_1, \ldots, x_r\}$ be an ordered basis such that

- $x_1(\xi) = 1$, $x_2 = \alpha$, $x_3(\xi) = \ldots = x_r(\xi) = 0$,
- $\lambda_I \notin \operatorname{span}\langle x_1, x_3, \ldots, x_r\rangle$ for all I,
- every $\beta = b_1x_1 + b_2x_2 + \ldots + b_rx_r \in \mathcal{A} \setminus \mathbb{R}\alpha$ is in the opposite open half-space of $\operatorname{span}\langle x_1, x_3, \ldots, x_r\rangle$ as α, i.e. $b_2 < 0$.

We can modify an ordered basis satisfying the first two generic conditions to get a new basis which will satisfy all three conditions as follows. Denote $\pi : \mathfrak{t}^* \to \operatorname{span}\langle x_1, x_2\rangle$ the projection to the 2-dimensional plane $\operatorname{span}\langle x_1, x_2\rangle$ along $\operatorname{span}\langle x_3, \ldots, x_r\rangle$. Remark that $\pi(\beta) \neq 0$ and $b_1 > 0$ for all $\beta \in \mathcal{A} \setminus \mathbb{R}\alpha$, hence there exists a $\delta \in \mathbb{R}$ such that the line $\mathbb{R}(x_1 + \delta x_2)$ in $\operatorname{span}\langle x_1, x_2\rangle$ separates $\{\pi(\beta) \,|\, \beta \in \mathcal{A} \setminus \mathbb{R}\alpha\}$ from α and $\pi(\lambda_I) \notin \mathbb{R}(x_1 + \delta x_2)$ for all I. Then the ordered basis $\{x_1 + \delta x_2,\, x_2, \ldots, x_r\}$ satisfies all three conditions.

An ordered basis $x = \{x_1, x_2, \ldots, x_r\}$ satisfying the above properties induces the same polarization on elements of $\mathcal{A}$ as Λ, while $x' = \{x_1' = -x_2,\, x_2' = -x_1, x_3' = x_3, \ldots, x_r' = x_r\}$ induces the same polarization on $\mathcal{A}$ as Λ'. It is possible that there is I such that $\lambda_I \in \mathbb{R}\alpha$, therefore choose a small $\rho \in \mathfrak{t}^*$ such that for all $s \in (0, 1]$ and all I the following generic conditions are fulfilled

- $\lambda_I + s\rho$ is regular with respect to $\{\alpha_i \,|\, i \in I\}$,
- $\lambda_I + s\rho \notin \operatorname{span}\langle x_2, \ldots, x_r\rangle$,
- $\lambda_I + s\rho \notin \operatorname{span}\langle x_1, x_3, \ldots, x_r\rangle$.

We define Res^- by replacing "$\lambda_1 \geq 0$" by "$\lambda_1 \leq 0$" in Definition 3.1. Since F is analytic and $\lambda_I + s\rho \notin \operatorname{span}\langle x_2, \ldots, x_r\rangle$ for all I and $s \in (0, 1]$, we have that

$$\operatorname*{Res}_{x_1}^+ F(x)e^{s\rho(x)}dx_1 + \operatorname*{Res}_{x_1}^- F(x)e^{s\rho(x)}dx_1 = \operatorname*{Res}_{x_1} F(x)e^{s\rho(x)}dx_1 = 0,$$

hence

$$\operatorname*{Res}_{x_r}^+ \ldots \operatorname*{Res}_{x_2}^+ \operatorname*{Res}_{x_1}^+ F(x)e^{s\rho(x)}dx_1dx_2 \ldots dx_r = \operatorname*{Res}_{x_r}^+ \ldots \operatorname*{Res}_{x_2}^+ \operatorname*{Res}_{x_1}^- F(x)e^{s\rho(x)}d(-x_1)dx_2 \ldots dx_r.$$

Moreover, for all $\beta = b_1x_1 + b_2x_2 + \ldots + b_rx_r \in \mathcal{A} \setminus \mathbb{R}\alpha$ we have $b_1 > 0$ and $b_2 < 0$, hence the ordered basis $\{-x_1, x_2, \ldots, x_r\}$ and $\{x_2, -x_1, x_3 \ldots, x_r\}$ induce the same polarization on $\mathcal{A}$. Since $\operatorname*{Res}_{x_1}^- = \operatorname*{Res}_{-x_1}^+$, by Proposition 3.17 we have

$$\operatorname*{Res}_{x_r}^+ \ldots \operatorname*{Res}_{x_2}^+ \operatorname*{Res}_{x_1}^- F(x)e^{s\rho(x)}d(-x_1)dx_2 \ldots dx_r = \operatorname*{Res}_{x_r}^+ \ldots \operatorname*{Res}_{x_1}^- \operatorname*{Res}_{x_2}^+ F(x)e^{s\rho(x)}dx_2d(-x_1) \ldots dx_r.$$

Again, since F is analytic, and $\lambda_I + s\rho \notin \operatorname{span}\langle x_1, x_3, \ldots, x_r\rangle$ for all $s \in (0, 1]$ and I, thus we have

$$\operatorname*{Res}_{x_r}^+ \ldots \operatorname*{Res}_{x_1}^- \operatorname*{Res}_{x_2}^+ F(x)e^{s\rho(x)}dx_2d(-x_1) \ldots dx_r =$$

$$\operatorname*{Res}_{x_r}^+ \ldots \operatorname*{Res}_{x_1}^- \operatorname*{Res}_{x_2}^- F(x)e^{s\rho(x)}d(-x_2)d(-x_1) \ldots dx_r = \operatorname*{Res}_{x_r'}^+ \ldots \operatorname*{Res}_{x_1'}^+ F(x')e^{s\rho(x')}dx_1' \ldots dx_r'.$$

Summarizing the above results for all $s \in (0,1]$ we have

$$\operatorname*{Res}_{x_r}^+ \ldots \operatorname*{Res}_{x_1}^+ F(x) e^{s\rho(x)} dx_1 \ldots dx_r = \operatorname*{Res}_{x'_r}^+ \ldots \operatorname*{Res}_{x'_1}^+ F(x') e^{s\rho(x')} dx'_1 \ldots dx'_r.$$

Taking the limit when s approaches to 0 yields

$$\operatorname*{Res}_{x_r}^+ \ldots \operatorname*{Res}_{x_1}^+ F(x) dx_1 \ldots dx_r = \operatorname*{Res}_{x'_r}^+ \ldots \operatorname*{Res}_{x'_1}^+ F(x') dx'_1 \ldots dx'_r$$

by Corollary 3.18. Finally, by Remark 3.2 we get equality

$$\operatorname*{JKRes}_{x} F(x) dx = \operatorname*{JKRes}_{x'} F(x') dx'.$$

$\square$

Lemma 3.21. *Let* $F = \sum_I \frac{P_I e^{\lambda_I}}{\prod_{i \in I} \alpha_i}$ *be regular. If F is analytic and* 0 *is not contained in the convex hull* $conv(\lambda_I \,|\, I)$ *then for all ordered basis x we have* $\operatorname*{Res}_{x}^+ F(x) dx = 0$.

Proof. Since $\operatorname*{Res}_{x}^+ F(x) dx$ is a multiple of $\operatorname*{JKRes}_{x} F(x) dx$, it is enough to show the lemma for a particular ordered basis x by Proposition 3.20. Since $0 \notin conv(\lambda_I \,|\, I)$, there is a hyperplane $\mathcal{H}$ containing the set $conv(\lambda_I \,|\, I)$ in one of its open half-spaces. Choose an ordered basis $x = \{x_1, \ldots, x_r\}$ such that $\mathcal{H} = \operatorname{span}\langle x_2, \ldots, x_r \rangle$ and $conv(\lambda_I \,|\, I) \subset \left\{ \sum_{k=1}^r a_k x_k \,\middle|\, a_1 < 0 \right\}$. But all polarized vectors $\overline{\alpha_i}$, $i \in I$ lie in $\left\{ \sum_{k=1}^r a_k x_k \,\middle|\, a_1 \geq 0 \right\}$, therefore from Corollary 3.19 follows the lemma. $\square$

3.1.2 Hamiltonian fractions

Definition 3.22. We call a basis $x = \{x_1, \ldots, x_r\}$ of $\mathfrak{t}^*$ *generic with respect to* $F = \sum_I \frac{P_I e^{\lambda_I}}{\prod_{i \in I} \alpha_i} \in \mathfrak{F}$ if for any I the vector λ_I is regular with respect to a set $\{\alpha_j \,|\, j \in J\}$, $J \subset I$ then it remains regular for the set $\{x_1, \ldots, x_r, \alpha_j \,|\, j \in J\}$, too. That is, the affine planes $\lambda_I + \operatorname{span}\langle \alpha_{i_1}, \ldots, \alpha_{i_k} \rangle$, $i_1, \ldots, i_k \in I$ intersect the coordinate planes $\operatorname{span}\langle x_{j_1}, \ldots, x_{j_l} \rangle$ transversally.

Remark 3.23. Let x be a generic basis with respect to $\frac{P e^{\lambda}}{\prod_{i \in I} \alpha_i}$. If $(\alpha_{i_1}, \ldots, \alpha_{i_r})$ satisfies condition (Fl) then

$$\lambda + \operatorname{span}\langle \alpha_{i_1}, \ldots, \alpha_{i_k} \rangle \cap \operatorname{span}\langle x_{k+2}, \ldots, x_r \rangle \neq \emptyset \;\Leftrightarrow\; 0 \in \lambda + \operatorname{span}\langle \alpha_{i_1}, \ldots, \alpha_{i_k} \rangle,$$

hence $\Pi_{\langle \alpha_{i_1}, \ldots, \alpha_{i_k} \rangle} \lambda = 0$. It means that if we write $\lambda_I = \lambda^1 \overline{\alpha_{i_1}} + \ldots + \lambda^r \overline{\Pi_{\langle \alpha_{i_1}, \ldots, \alpha_{i_{r-1}} \rangle} \alpha_{i_r}}$ then $\lambda^1, \ldots, \lambda^k \neq 0$ and $\lambda^{k+1} = \ldots = \lambda^r = 0$ for some $k \in \{1, \ldots, r\}$. In particular, if λ is regular with respect to $\{\alpha_{i_1}, \ldots, \alpha_{i_r}\}$ then $\lambda^1, \ldots, \lambda^r \neq 0$. $\diamond$

Definition 3.24. Let x be a generic basis with respect to $\frac{Pe^{\lambda}}{\prod_{i\in I}\alpha_i} \in \mathfrak{F}$ and let $i_1, \ldots, i_r \in I$ such that $(\alpha_{i_1}, \ldots, \alpha_{i_r})$ satisfies condition (Fl). The *order* $\operatorname{ord}(i_1, \ldots, i_r; I)$ of the iterated residue

$$\operatorname*{Res^+}_{x_r|\alpha_{i_r}} \ldots \operatorname*{Res^+}_{x_1|\alpha_{i_1}} \frac{P(x)e^{\lambda(x)}}{\prod\limits_{i\in I}\alpha_i(x)}dx$$

is equal to k if $\lambda = \sum_{l=1}^{r} \lambda^l \, \overline{\Pi_{\langle\alpha_{i_1},\ldots,\alpha_{i_{l-1}}\rangle}\alpha_{i_l}}$ with $\lambda^1, \ldots, \lambda^k \neq 0$ and $\lambda^{k+1} = \ldots = \lambda^r = 0$.

Remark 3.25. If $\operatorname*{Res^+}_{x_r|\alpha_{i_r}} \ldots \operatorname*{Res^+}_{x_1|\alpha_{i_1}} \frac{P(x)e^{\lambda(x)}}{\prod_{i\in I}\alpha_i(x)}dx$ has order k, then

$$\operatorname*{Res^+}_{x_r|\alpha_{i_r}} \ldots \operatorname*{Res^+}_{x_1|\alpha_{i_1}} \frac{P(x)e^{\lambda(x)+\rho(x)}}{\prod\limits_{i\in I}\alpha_i(x)}dx \tag{3.11}$$

depends continuously on small $\rho \in \operatorname{span}\langle\alpha_{i_1}, \ldots, \alpha_{i_k}\rangle$ (small means that (3.11) stays of order k). ◇

Let T be a compact torus of rank r and let $\mathfrak{t}$ be its Lie algebra. Let (M, ω) be a compact Hamiltonian T-manifold with moment map $\mu : M \to \mathfrak{t}^*$. For any $\eta \in H_T(M)$ we have

$$\int\limits_M \eta e^{\omega-\mu} = \sum_{D\subset M^T} \int\limits_D \frac{i_D^*(\eta e^{\omega-\mu})}{e_T\mathcal{N}(D\,|\,M)} \tag{3.12}$$

by Atiyah-Bott-Berline-Vergne theorem. The integral $\int_D \frac{i_D^*(\eta e^{\omega-\mu})}{e_T\mathcal{N}(D\,|\,M)}$ yields a fraction $\frac{P_D e^{\lambda_D}}{\prod_{i\in I_D}\alpha_i}$ as follows. The moment map μ is constant on fixed point component, hence $\lambda_D = -\mu(D)$. Choose an invariant compatible almost complex structure on M to make $\mathcal{N}(D\,|\,M)$ a T-equivariant complex vector bundle and assume that it splits to T-equivariant complex line bundles $\mathcal{N}(D\,|\,M) = \oplus_l \mathcal{N}_l$ by the Splitting Principle. Then $e_T(\mathcal{N}_l) = \alpha_l + e(\mathcal{N}_l)$, where $\alpha_l \in \mathfrak{t}^*_{\mathbb{Z}}$ is the T-weight of fibers of $\mathcal{N}_l$ and $\frac{1}{\alpha_l + e(\mathcal{N}_l)} = \sum_{k\geq 0}(-1)^k \frac{e(\mathcal{N}_l)^k}{\alpha_l^{k+1}} = \frac{1}{\alpha_l^{d+1}}\sum_{k=0}^{d}(-1)^k\alpha_l^{d-k}e(\mathcal{N}_l)^k$, where $d = \frac{1}{2}\dim D$. Thus $P_D = \int\limits_D i_D^*(\eta e^{\omega}) \prod_l \left[\sum_{k=0}^{d}(-1)^k\alpha_l^{d-k}e(\mathcal{N}_l)^k\right]$ and $\prod\limits_{i\in I_D}\alpha_i = \prod\limits_l \alpha_l^{d+1}$. Denote

$$F = \sum_D \frac{P_D e^{\lambda_D}}{\prod\limits_{i\in I_D}\alpha_i} \tag{3.13}$$

and we emphasize that F is analytic by the Atiyah-Bott-Berline-Vergne formula.

Definition 3.26. We call (3.13) a *Hamiltonian fraction* if (M, ω) is a compact Hamiltonian T-manifold or orbifold with moment map μ having 0 as regular value. We denote the set of Hamiltonian fractions by $\mathfrak{F}_{Ham}$.

There are Hamiltonian manifolds M such that (3.13) is not a regular fraction, but 0 is a regular value of μ.

Example 3.3. Let a 2-dimensional torus $K = U(1)^2$ act on $\mathbb{CP}^3$ as

$$(t,s)\cdot(z_0 : z_1 : z_2 : z_3) = (t^{-1}sz_0 : (ts)^{-1}z_1 : tz_2 : t^2z_3).$$

It is a Hamiltonian action with moment map

$$\mu(z) = \frac{(\sigma-\tau)|z_0|^2 - (\tau+\sigma)|z_1|^2 + \tau|z_2|^2 + 2\tau|z_3|^2}{2(|z_0|^2+|z_1|^2+|z_2|^2+|z_3|^2)}$$

and 0 is a regular value of it. By Atiyah-Bott-Berline-Vergne formula

$$\int_{\mathbb{CP}^3} e^{\omega-\mu} = \frac{(2\pi)^3 e^{\tau-\sigma}}{2\sigma(\sigma-2\tau)(3\tau-\sigma)} + \frac{(2\pi)^3 e^{\tau+\sigma}}{2\sigma(2\tau+\sigma)(3\tau+\sigma)} + \frac{(2\pi)^3 e^{-\tau}}{(2\tau-\sigma)(2\tau+\sigma)\tau} + \frac{(2\pi)^3 e^{-2\tau}}{(\sigma-3\tau)(3\tau+\sigma)\tau},$$

which is not a regular fraction. ◇

We will show that $\operatorname*{JKRes}_x$ has similar properties on Hamiltonian fraction for generic basis x which have been shown for regular fractions. We will use walls of $\mu(M)$ to group iterated residues. Let $i_1,\dots,i_r \in I_D$ and assume that $(\alpha_{i_1},\dots,\alpha_{i_r})$ satisfies condition (Fl) for a generic basis x with respect to F. To each residue $\operatorname*{Res}^+_{x_r|\alpha_{i_r}} \dots \operatorname*{Res}^+_{x_1|\alpha_{i_1}} \frac{P_D(x)e^{\lambda_D(x)}}{\prod_{i\in I_D}\alpha_i(x)}dx$ we associate a series of walls of the moment polytope $\mu(M)$ as follows. Let $k<r$ and let $K=K_k$ be a subtorus of T with Lie algebra $\mathfrak{k} = \cap_{j=1}^k \ker\alpha_{i_j}$. We can identify

$$Lie(T/K)^* = (\mathfrak{t}/\mathfrak{k})^* = \ker(\mathfrak{t}^*\to\mathfrak{k}^*) = \operatorname{span}\langle\alpha_{i_1},\dots,\alpha_{i_k}\rangle.$$

Let $N=N_k$ be the fixed point component of M^K containing the fixed point component $D\subset M^T$. Then $\mu(N)$ is a k-dimensional convex subpolytope of $\mu(M)$ with supporting affine plane $-\lambda_D + \operatorname{span}\langle\alpha_{i_1},\dots,\alpha_{i_k}\rangle$. Remark that if $\operatorname{ord}(i_1,\dots,i_r;I_D)\le k$ then 0 is in the supporting affine plane of $\mu(N)$.

Consider a k-dimensional wall $\mu(N)$ such that $N\subset M^K$ is a fixed point component. Let $v_N = \{v_N^1,\dots,v_N^k\}$ be the projection of $\{x_1,\dots,x_k\}$ to $(\mathfrak{t}/\mathfrak{k})^* = \operatorname{span}\langle\alpha_{i_1},\dots,\alpha_{i_k}\rangle$ along the plane $\operatorname{span}\langle x_{k+1},\dots,x_r\rangle$ and let $w_i = x_i$ for all $i>k$. Choose $S\subset T$ subtorus such that $\Phi : K\times S\to T$, $\Phi(k,s)=ks$ is a finite cover. We identify $\mathfrak{t}^* = \mathfrak{s}^*\oplus\mathfrak{k}^*$ via Φ^* and remark that

$$\mathfrak{s}^* = \ker(\mathfrak{t}^*\to\mathfrak{k}^*) = \operatorname{span}\langle v_N^1,\dots,v_N^k\rangle.$$

Moreover, we also have isomorphism $\Phi^* : H_T(M)\to H_{K\times S}(M)$. Let $D\subset N^T = N^S$ be a fixed point component. We have a $(K\times S)$-equivariant isomorphism of vector bundles $\mathcal{N}(D\,|\,M) = \mathcal{N}(N\,|\,M)|_D \oplus \mathcal{N}(D\,|\,N)$ and assume that $\mathcal{N}(N\,|\,M)$ splits to $(K\times S)$-equivariant complex line bundles $\mathcal{N}(N\,|\,M) = \oplus_j\mathcal{L}_j$. Moreover, let

$$e_{K\times S}(\mathcal{L}_j) = \beta_j + e_S(\mathcal{L}_j),$$

where $\beta_j \in \mathfrak{k}^*_{\mathbb{Z}}$ is the K-weight of fibers of $\mathcal{L}_j$ and let $i^*_D e_S(\mathcal{L}_j) = \gamma^D_j + i^*_D e(\mathcal{L}_j)$, where $\gamma^D_j \in \mathfrak{s}^*_{\mathbb{Z}}$ are the S-weights of the fibers of $\mathcal{L}_j|_D$. We split

$$\beta_j = \beta'_j + \beta''_j \in \operatorname{span}\langle v^1_N, \ldots, v^k_N\rangle \oplus \operatorname{span}\langle w_{k+1}, \ldots, w_r\rangle.$$

Remark that $\dfrac{1}{\beta_j + i^*_D e_S(\mathcal{L}_j)} = \displaystyle\sum_{l_j \geq 0} (-1)^{l_j} \frac{[\beta'_j + i^*_D e_S(\mathcal{L}_j)]^{l_j}}{(\beta''_j)^{l_j+1}}$ is the expansion of $\displaystyle\sum_{l\geq 0} \frac{[-i^*_D e(\mathcal{L}_j)]^l}{(\beta_j + \gamma^D_j)^{l+1}}$ with respect to $v_N \ll w$, that is $\beta'_j + \gamma^D_j \ll \beta''_j$ for all j. Therefore,

$$\int_D \frac{i^*_D(\eta e^{\omega-\mu})}{e_S\mathcal{N}(D\,|\,M)} \prod_j \sum_{l_j\geq 0} (-1)^{l_j} \frac{[\beta'_j + i^*_D e_S(\mathcal{L}_j)]^{l_j}}{(\beta''_j)^{l_j+1}} \tag{3.14}$$

is the expansion of $\displaystyle\int_D \frac{i^*_D(\eta e^{\omega-\mu})}{e_T\mathcal{N}(D\,|\,M)} = \frac{P_D e^{\lambda_D}}{\prod_{i\in I_D}\alpha_i}$ with respect to $v_N \ll w$. We truncate every infinite sum in (3.14) at $l_j = n^2$, where $n = \dfrac{1}{2}\dim N$, and we set

$$F_N(v_N, w) = \sum_{D\subset N^T} \int_D \frac{i^*_D(\eta e^{\omega-\mu})}{e_S\mathcal{N}(D\,|\,M)} \prod_j \sum_{l_j=0}^{n^2} (-1)^{l_j} \frac{[\beta'_j + i^*_D e_S(\mathcal{L}_j)]^{l_j}}{(\beta''_j)^{l_j+1}}. \tag{3.15}$$

By Atiyah-Bott-Berline-Vergne formula

$$F_N(v_N, w) = \int_N i^*_N(\eta e^{\omega-\mu}) \prod_j \sum_{l_j=0}^{n^2} (-1)^{l_j} \frac{[\beta'_j + e_S(\mathcal{L}_j)]^{l_j}}{(\beta''_j)^{l_j+1}},$$

hence it is analytic in v_N. Moreover,
(3.16)

$$\operatorname*{Res}^+_{w_r|\alpha_{i_r}} \ldots \operatorname*{Res}^+_{w_{k+1}|\alpha_{i_{k+1}}} \operatorname*{Res}^+_{v^k_N|\alpha_{i_k}} \ldots \operatorname*{Res}^+_{v^1_N|\alpha_{i_1}} F_N(v_N, w)dv_N dw = \sum_{D\subset N^T} \operatorname*{Res}^+_{x_r|\alpha_{i_r}} \ldots \operatorname*{Res}^+_{x_1|\alpha_{i_1}} \frac{P_D(x)e^{\lambda_D(x)}}{\prod\limits_{i\in I_D}\alpha_i(x)}dx$$

by Corollary 3.11, Lemma 3.10 and 3.12.

Lemma 3.27. *Denote $\mathcal{W}_k(M)$ the set of k-dimensional walls $\mu(N)$ of $\mu(M)$ containing* 0 *in their supporting affine plane. Then*

$$\sum_{\operatorname{ord}(i_1,\ldots,i_r;I_D)\leq k} \operatorname*{Res}^+_{x_r|\alpha_{i_r}} \ldots \operatorname*{Res}^+_{x_1|\alpha_{i_1}} \frac{P_D(x)e^{\lambda_D(x)}}{\prod\limits_{i\in I_D}\alpha_i(x)}dx = \sum_{\mu(N)\in\mathcal{W}_k(M)} \operatorname*{Res}^+_{w} \operatorname*{Res}^+_{v_N} F_N(v_N, w)dv_N dw.$$

Proof. Remark that all linear terms in the denominator of $F_N(v_N, w)$ involving v_N are S-weights of a normal bundle $\mathcal{N}(D\,|\,N)$ for a fixed point component $D \subset N^S$, i.e. they are equal to an $\alpha_i \in \mathfrak{s}^* = \operatorname{span}\langle v^1_N, \ldots, v^k_N\rangle$ for some $i \in I_D$. By (3.16) we have

$$\operatorname*{Res}^+_{w} \operatorname*{Res}^+_{v_N} F_N(v_N, w)dv_N dw = \sum_{D\subset N_T} \sum_{[\alpha_{i_1},\ldots,\alpha_{i_r}]} \operatorname*{Res}^+_{x_r|\alpha_{i_r}} \ldots \operatorname*{Res}^+_{x_1|\alpha_{i_1}} \frac{P_D(x)e^{\lambda_D(x)}}{\prod\limits_{i\in I_D}\alpha_i(x)}dx,$$

where the second sum is over all classes of tuples satisfying condition (Fl) such that $i_1, \ldots, i_r \in I_D$ and $\alpha_{i_1}, \ldots, \alpha_{i_k}$ is in the supporting plane of $\mu(N) \in \mathcal{W}_k(M)$, hence the residues on the right hand side have order $\operatorname{ord}(i_1, \ldots, i_r; I_D) \leq k$.

Conversely, let $\mu(N)$ be the associated k-dimensional wall of

$$\operatorname*{Res^+}_{x_r|\alpha_{i_r}} \ldots \operatorname*{Res^+}_{x_1|\alpha_{i_1}} \frac{P_D(x)e^{\lambda_D(x)}}{\prod\limits_{i \in I_D} \alpha_i(x)} dx \tag{3.17}$$

in $\mu(M)$. If (3.17) has order $\operatorname{ord}(i_1, \ldots, i_r; I_D) \leq k$ then $\lambda_D \in \operatorname{span}\langle \alpha_{i_1}, \ldots \alpha_{i_k} \rangle$, hence 0 is in the supporting affine plane of $\mu(N)$, that is, $\mu(N) \in \mathcal{W}_k(M)$. From (3.16) follows that (3.17) is a summand of $\operatorname*{Res^+}_{w} \operatorname*{Res^+}_{v_N} F_N(v_N, w) dv_N dw$. □

We have the following vanishing result.

Proposition 3.28. *Let x be a generic basis with respect to $F = \sum_D \frac{P_D e^{\lambda_D}}{\prod_{i \in I_D} \alpha_i} \in \mathfrak{F}_{Ham}$. Then for any $k < r$*

$$\sum_{\operatorname{ord}(i_1, \ldots, i_r; I_D) = k} \operatorname*{Res^+}_{x_r|\alpha_{i_r}} \ldots \operatorname*{Res^+}_{x_1|\alpha_{i_1}} \frac{P_D(x)e^{\lambda_D(x) + \rho(x)}}{\prod\limits_{i \in I_D} \alpha_i(x)} dx = 0$$

for any ρ in a small neighborhood of 0.

Proof. By hypothesis there is a Hamiltonian T-manifold M such that $F = \sum_{D \subset M^T} \int_D \frac{i_D^*(\eta e^{\omega - \mu})}{e_T \mathcal{N}(D \mid M)}$ and 0 is a regular value of μ. That is, 0 is not on any (proper) wall of the moment polytope $\mu(M)$. First, we will show by induction on k that for any $\mu(N) \in \mathcal{W}_k(M)$, $k < r$ we have

$$\operatorname*{Res^+}_{v_N} F_N(v_N, w) e^{\rho(v_N, w)} dv_N = 0$$

for any ρ in a small neighborhood of 0.

Let k_0 be the smallest number such that $\mathcal{W}_{k_0}(M) \neq \emptyset$. By Lemma 3.27 this is equal to the smallest order. Since x is generic, we have $k_0 > 0$. Consider w as a fixed parameter and remark that $F_N(v_N, w)$ is regular as fraction in v_N for $\mu(N) \in \mathcal{W}_{k_0}(M)$. Furthermore, $F_N(v_N, w)e^{\rho'(v_N)}$ is also regular for small $\rho' \in \operatorname{span}\langle v_N^1, \ldots, v_N^{k_0} \rangle$. Since 0 is on the supporting affine plane of $\mu(N)$, but it is not contained in the convex polytope $-\mu(N) + \rho'(v_N)$, and $F_N(v_N, w)$ is analytic in v_N, we have $\operatorname*{Res^+}_{v_N} F_N(v_N, w) e^{\rho'(v_N)} dv_N = 0$ by Atiyah convexity theorem [1] and Lemma 3.21. Moreover, we can write any small ρ as $\rho(v_N, w) = \rho'(v_N) + \rho''(w)$, thus

$$\operatorname*{Res^+}_{v_N} F_N(v_N, w) e^{\rho(v_N, w)} dv_N = e^{\rho''(w)} \operatorname*{Res^+}_{v_N} F_N(v_N, w) e^{\rho'(v_N)} dv_N = 0.$$

For general $k < r$ the residue $\operatorname*{Res^+}_{v_N} F_N(v_N, w) dv_N$ can be written as sum of order k terms and lower order terms. By Lemma 3.27 the sum of lower order terms is equal to

$$\sum_{\mu(N') \in \mathcal{W}_l(N),\, l < k} \operatorname*{Res^+}_{w'} \operatorname*{Res^+}_{v_{N'}} F_{N'}(v_{N'}, w', w) dv_{N'} dw',$$

where $w' = \{v_N^{l+1}, \ldots, v_N^k\}$. By induction hypothesis

$$\sum_{\mu(N')\in\mathcal{W}_l(N),\, l<k} \operatorname*{Res}^+_{w'} \operatorname*{Res}^+_{v_{N'}} F_{N'}(v_{N'}, w', w) e^{\rho'(v_{N'}, w')} dv_{N'} dw' = 0$$

for all $\rho' \in \langle v_N^1, \ldots, v_N^k \rangle$ small. It implies by Remark 3.25 that

$$\operatorname*{Res}^+_{v_N} F_N(v_N, w) e^{\rho'(v_N)} dv_N$$

depends continuously on small ρ'. Fix ρ' small and let $\varrho_N(v_N)$ such that $F_N(v_N, w)e^{\rho'(v_N)+\varrho_N(v_N)}$ is regular as fraction in v_N. Then

$$\operatorname*{Res}^+_{v_N} F_N(v_N, w) e^{\rho'(v_N)} dv_N = \lim_{s\to 0} \operatorname*{Res}^+_{v_N} F_N(v_N, w) e^{\rho'(v_N)+s\varrho_N(v_N)} dv_N = 0$$

by Atiyah convexity theorem [1] and Lemma 3.21. We can write any small ρ as $\rho(v_N, w) = \rho'(v_N) + \rho''(w)$ and we have

$$\operatorname*{Res}^+_{v_N} F_N(v_N, w) e^{\rho(v_N, w)} dv_N = e^{\rho''(w)} \operatorname*{Res}^+_{v_N} F_N(v_N, w) e^{\rho'(v_N)} dv_N = 0.$$

In particular, for any $\mu(N) \in \mathcal{W}_k(M)$, $k < r$ and for any ρ small we have

$$\operatorname*{Res}^+_{w} \operatorname*{Res}^+_{v_N} F_N(v_N, w) e^{\rho(v_N, w)} dv_N dw = 0.$$

Together with Lemma 3.27 it implies that for small ρ we have

$$\begin{aligned} \sum_{\operatorname{ord}(i_1,\ldots,i_r;I_D)=k} & \operatorname*{Res}^+_{x_r|\alpha_{i_r}} \ldots \operatorname*{Res}^+_{x_1|\alpha_{i_1}} \frac{P_D(x) e^{\lambda_D(x)+\rho(x)}}{\prod_{i\in I_D} \alpha_i(x)} dx \\ &= \sum_{\mu(N)\in\mathcal{W}_k(M)} \operatorname*{Res}^+_{w} \operatorname*{Res}^+_{v_N} F_N(v_N, w) e^{\rho(v_N, w)} dv_N dw \\ &\quad - \sum_{\mu(N)\in\mathcal{W}_{k-1}(M)} \operatorname*{Res}^+_{w} \operatorname*{Res}^+_{v_N} F_N(v_N, w) e^{\rho(v_N, w)} dv_N dw = 0. \end{aligned}$$

□

Proposition 3.29. *Let x be a generic basis with respect to $F \in \mathfrak{F}_{Ham}$. Then*

(i) $\operatorname*{JKRes}_x F(x) e^{\rho(x)} dx$ *depends continuously on ρ in a small neighborhood of* 0.

(ii) $\operatorname*{JKRes}_x F(x) dx$ *does not depend on the choice of generic basis x. That is, if y is another generic basis with respect to F then* $\operatorname*{JKRes}_x F(x)dx = \operatorname*{JKRes}_y F(y)dy$.

Proof. (i) By Proposition 3.28 we have that for small ρ the residue $\operatorname*{Res}^+_x F(x)e^{\rho(x)}dx$ is equal to the sum of residues of order $r = \dim \mathfrak{t}^*$. By Remark 3.25 the residues of order r depend continuously on small ρ.

(ii) If F is regular then it follows from Proposition 3.20, because F is also analytic. If F is not regular then there exist arbitrarily small ρ such that Fe^{ρ} is regular. Then by the continuity and Corollary 3.18 we have

$$\operatorname*{JKRes}_x F(x)dx = \lim_{s\to 0} \operatorname*{JKRes}_x F(x)e^{s\rho(x)}dx = \lim_{s\to 0} \operatorname*{JKRes}_y F(x)e^{s\rho(y)}dy = \operatorname*{JKRes}_y F(y)dy.$$

□

Definition 3.30. For $F = \sum_D \frac{P_D e^{\lambda_D}}{\prod_{i\in I_D}\alpha_i}$ in $\mathfrak{F}_{reg}$ or $\mathfrak{F}_{Ham}$ and polarization Λ on $\mathfrak{t}^*$ we define

$$\mathrm{JKRes}^{\Lambda} F(t)dt = \operatorname*{JKRes}_x F(x)dx,$$

where x is any generic ordered basis with respect to F, inducing the same polarization as Λ on $\cup_D\{\alpha_i \,|\, i\in I_D\}$. The polarization Λ can be thought as an ordered basis on $\mathfrak{t}^*$ or in this case as a connected component of $\cap_D\{\tau\in\mathfrak{t}\,|\,\alpha_i(\tau)\neq 0,\ \forall i\in I_D\}$.

We remark that if $F\in\mathfrak{F}_{reg}$ then non-generic basis x is also allowed in the above definition, which can make computations easier. However, if $F\in\mathfrak{F}_{Ham}\setminus\mathfrak{F}_{reg}$ then let $\rho\in\mathfrak{t}^*$ such that $Fe^{\rho}\in\mathfrak{F}_{reg}$ and in this case

$$\mathrm{JKRes}^{\Lambda} F(t)dt = \lim_{s\to 0}\operatorname*{JKRes}_x F(x)e^{s\rho(x)}dx$$

for any ordered basis x (not necessarily a generic basis) inducing the same polarization as Λ on $\cup_D\{\alpha_i \,|\, i\in I_D\}$.

3.2 Equivariant Jeffrey-Kirwan residue

The equivariant Jeffrey-Kirwan residue can be thought as a parametric version of the usual one, but the additional freedom in the choice of polarization makes it more flexible.

Let $\mathfrak{k}^*$ and $\mathfrak{s}^*$ be real vector spaces of dimension q and $r-q$, respectively. Set $\mathfrak{t}^*=\mathfrak{k}^*\oplus\mathfrak{s}^*$.

Definition 3.31. A $\mathfrak{k}^*$*-pole* in $\mathfrak{t}^*$ is a q-dimensional subspace V such that $V\oplus\mathfrak{s}^*=\mathfrak{t}^*$. If Λ is a polarization on $\mathfrak{t}^*$ then we will denote by Λ_V the polarization induced on V.

Definition 3.32. An $F=\sum_I \frac{P_I e^{\lambda_I}}{\prod_{i\in I}\alpha_i}\in\mathfrak{F}$ is called $\mathfrak{k}^*$*-regular* if for all I the vector $\mathrm{pr}_{\mathfrak{k}^*}(\lambda_I)$ is regular with respect to $\{\mathrm{pr}_{\mathfrak{k}^*}(\alpha_i)\,|\,i\in I\}$, where $\mathrm{pr}_{\mathfrak{k}^*}:\mathfrak{t}^*\to\mathfrak{k}^*$ is the projection. We denote the set of $\mathfrak{k}^*$-regular fractions by $\mathfrak{F}_{\mathfrak{k}^*\text{-}reg}$.

Let K and S be two compact tori with Lie algebras $\mathfrak{k}$ and $\mathfrak{s}$, respectively. Let (M,ω) be a compact Hamiltonian $(K\times S)$-manifold with moment map $\mu=\mu_K\times\mu_S: M\to\mathfrak{k}^*\oplus\mathfrak{s}^*$. We assume that $0\in\mathfrak{k}^*$ is a regular value of μ_K. We denote by $\mathfrak{F}_{\mathfrak{k}^*\text{-}Ham}$ the set of $F=\sum_{D\in M^{K\times S}}\int_D \frac{i_D^*(\eta e^{\omega-\mu})}{e_{K\times S}\mathcal{N}(D\,|\,M)}$ for such M and for some $\eta\in H_{K\times S}(M)$.

We fix a scalar product on $\mathfrak{k}^*$. It defines a symmetric bilinear pairing on $\mathfrak{k}^* \oplus \mathfrak{s}^*$ by

$$(\alpha, \beta) = (\mathrm{pr}_{\mathfrak{k}^*}(\alpha), \mathrm{pr}_{\mathfrak{k}^*}(\beta)), \qquad \forall\, \alpha, \beta \in \mathfrak{k}^* \oplus \mathfrak{s}^*.$$

In particular, it induces scalar product on every $\mathfrak{k}^*$-pole V. Let $F = \sum_I \dfrac{P_I e^{\lambda_I}}{\prod_{i \in I} \alpha_i}$. For each $\mathfrak{k}^*$-pole V we define F_V by replacing every $\dfrac{1}{\alpha_i} = \dfrac{1}{\alpha_i' + \alpha_i''}$ by $\dfrac{1}{\alpha_i''} \sum_{l_i=0}^{|I|} \left(-\dfrac{\alpha_i'}{\alpha_i''}\right)^{l_i}$ in F, where $\alpha_i' \in V$ and $0 \neq \alpha_i'' \in \mathfrak{s}^*$. Remark that linear terms in the denominators of F_V are either in V or $\mathfrak{s}^*$.

Definition 3.33. We define the *equivariant Jeffrey-Kirwan residue* of $F \in \mathfrak{F}_{\mathfrak{k}^*\text{-reg}}$ or $F \in \mathfrak{F}_{\mathfrak{k}^*\text{-Ham}}$ as

$$\mathrm{EqRes}^{\Lambda} F = \sum_{\mathfrak{k}^*\text{-pole } V} \mathrm{JKRes}^{\Lambda_V} F_V(v, s) dv,$$

where v and s are basis of V and $\mathfrak{s}^*$, respectively.

Remark 3.34. It is enough to take $\mathfrak{k}^*$-poles of form $V = \mathrm{span}\langle \alpha_{i_1}, \ldots, \alpha_{i_q} \rangle$ with $i_1, \ldots, i_q \in I$, otherwise $F_V(v, s)$ will be non-generating as fraction in v and $\mathrm{JKRes}^{\Lambda_V}$ vanishes on non-generating fractions. $\diamond$

Proposition 3.35. *Let F be in $\mathfrak{F}_{\mathfrak{k}^*\text{-}reg}$ or $\mathfrak{F}_{\mathfrak{k}^*\text{-}Ham}$. Then*

(i) $\mathrm{EqRes}^{\Lambda} F$ *is well defined.*

(ii) If F is analytic then $\mathrm{EqRes}^{\Lambda} F$ *does not depend on* Λ.

(iii) $\mathrm{EqRes}^{\Lambda} F e^{\rho}$ *depends continuously on $\rho \in \mathfrak{t}^*$ in a small neighborhood of* 0.

Proof. If F is $\mathfrak{k}^*$-regular then for every $\mathfrak{k}^*$-pole V the $F_V(v, s)$ is regular as fraction in v, hence (i) and (iii) follows by Corollary 3.18. If in addition F is analytic then $F_V(v, s)$ is analytic in v and (ii) follows from Proposition 3.20.

If $F \in \mathcal{F}_{\mathfrak{k}^*\text{-Ham}}$ then

$$F = \sum_D \frac{P_D e^{\lambda_D}}{\prod\limits_{i \in I_D} \alpha_i} = \sum_{D \subset M^{K \times S}} \int_D \frac{i_D^*(\eta e^{\omega - \mu})}{e_{K \times S} \mathcal{N}(D \,|\, M)}$$

for a Hamiltonian $(K \times S)$-manifold M such that 0 is regular value of the K-moment map μ_K and $\eta \in H_{K \times S}(M)$. Let $V = \mathrm{span}\langle \alpha_{i_1}, \ldots, \alpha_{i_q} \rangle$ be a $\mathfrak{k}^*$-pole, where $i_1, \ldots, i_q \in I_D$. Let v be a basis of V, generic with respect to F_V and inducing the same polarization as Λ_V on $\cup_{i \in I_D} \{\alpha_i \,|\, i \in I_D\} \cap V$. We will show that $\underset{v}{\mathrm{JKRes}}\, F_V(v, s) dv$ does not depend on v, hence $\mathrm{JKRes}^{\Lambda_V} F_V(v, s) dv$ is well defined.

Recall that $\alpha_i \in (\mathfrak{k} \oplus \mathfrak{s})^*_{\mathbb{Z}}$ for all i. Consider the subtorus $G \subset K \times S$ with Lie algebra $\mathfrak{g} = \cap_{j=1}^{q} \ker \alpha_{i_j}$. Then $\Phi : K \times G \to K \times S$, $(k, g) \mapsto (k, 1)g$ is a finite cover and it induces splitting $\Phi^* : \mathfrak{k}^* \oplus \mathfrak{s}^* \to \mathfrak{k}^* \oplus \mathfrak{g}^*$ such that $\Phi^*(\mathfrak{s}^*) = \mathfrak{g}^*$ and $\Phi^*(V) = \mathfrak{k}^*$. Moreover, it also induces isomorphism in cohomology $\Phi^* : H_{K \times S}(M) \to H_{K \times G}(M)$ and we have relation between equivariant symplectic forms $\Phi^*(\omega - \mu) = \omega - \mu_{K \times G}$.

Let $N \subset M^G$ be a fixed point component. Assume that its normal bundle splits to $(K \times G)$-equivariant line bundles $\mathcal{N}(N \,|\, M) = \oplus_j \mathcal{L}_j$ with respect to an invariant compatible almost complex structure on M and let $e_{K\times G}(\mathcal{L}_j) = \beta_j + e_K(\mathcal{L}_j)$, where $\beta_j \in \mathfrak{g}^*$. Let $D \subset N^K$ be a fixed point component and observe that

$$\int_D \frac{i_D^* \Phi^*(\eta e^{\omega-\mu})}{e_K \mathcal{N}(D \,|\, N)} \prod_j \sum_{l_j \geq 0} (-1)^{l_j} \frac{i_D^* e_K(\mathcal{L}_j)^{l_j}}{\beta_j^{l_j+1}}$$

corresponds to the expansion of $\left(\int_D \frac{i_D^*(\eta e^{\omega-\mu})}{e_{K\times S}\mathcal{N}(D \,|\, M)} \right)(v,s)$ with respect to $v \ll s$ under Φ^*. We truncate expansions at $l_j = n^2$, where $n = \frac{1}{2}\dim N$ and we set

$$F_N(v,s) = (\Phi^*)^{-1} \left(\sum_{D \in N^K} \int_D \frac{i_D^* \Phi^*(\eta e^{\omega-\mu})}{e_K \mathcal{N}(D \,|\, N)} \prod_j \sum_{l_j = 0}^{n^2} (-1)^{l_j} \frac{i_D^* e_K(\mathcal{L}_j)^{l_j}}{\beta_j^{l_j+1}} \right),$$

which is equal to

$$(\Phi^*)^{-1} \left(\int_N i_N^* \Phi^*(\eta e^{\omega-\mu}) \prod_j \sum_{l_j=0}^{n^2} (-1)^{l_j} \frac{e_K(\mathcal{L}_j)^{l_j}}{\beta_j^{l_j+1}} \right)$$

by Atiyah-Bott-Berline-Vergne formula, hence $F_N(v,s)$ is a Hamiltonian fraction in v. Remark that if a $D \subset M^{K\times S}$ is not a fixed point component of some $N \subset M^G$ then $\frac{P_D(v,s)e^{\lambda_D(v,s)}}{\prod_{i\in I_D} \alpha_i(v,s)}$ is non-generating as fraction in v. Hence

$$\operatorname*{JKRes}_v F_V(v,s)dv = \sum_{N \subset M^G} \operatorname*{JKRes}_v F_N(v,s)dv$$

by Lemma 3.10, therefore the proposition follows from Proposition 3.29. □

Definition 3.36. A fraction $\frac{P_I e^{\lambda_I}}{\prod_{i\in I}\alpha_i}$ is called $\mathfrak{k}^*$*-generating* if $\{\mathrm{pr}_{\mathfrak{k}^*}(\alpha_i) \,|\, i \in I\}$ spans $\mathfrak{k}^*$, otherwise we call it *non-*$\mathfrak{k}^*$*-generating.*

Similarly to Lemma 3.16 we can decompose any $F \in \mathfrak{F}$ to sum of $\mathfrak{k}^*$-generating and non-$\mathfrak{k}^*$-generating fractions.

Lemma 3.37. *Any fraction* $\frac{Pe^\lambda}{\prod_{i\in I}\alpha_i} \in \mathfrak{F}[\mathfrak{t}]$ *can be written as linear combination of non-*$\mathfrak{k}^*$*-generating fractions and* $\mathfrak{k}^*$*-generating fractions of form* $Q\frac{e^\lambda}{\prod_{j=1}^q \alpha_j^{n_j+1}}$*, where* $Q \in \mathfrak{F}[\mathfrak{s}]$*.*

Proof. We assume that $\lambda = 0$ and we consider $P \in \mathbb{R}[\mathfrak{k} \oplus \mathfrak{s}]$ as polynomial function on $\mathfrak{k}$ with coefficients in $\mathbb{R}[\mathfrak{s}]$. We denote its degree by $\deg_{\mathfrak{k}} P$. We reduce the problem first to the $\deg_{\mathfrak{k}} P = 0$ case, i.e. $P \in \mathbb{R}[\mathfrak{s}]$ by induction on degree $\deg_{\mathfrak{k}} P$ of P. If $\frac{Pe^\lambda}{\prod_{i\in I}\alpha_i}$ is $\mathfrak{k}^*$-generating then there are $i_1, \dots, i_q \in I$ such that $\mathrm{pr}_{\mathfrak{k}^*}(\alpha_{i_1}), \dots, \mathrm{pr}_{\mathfrak{k}^*}(\alpha_{i_q})$ generate $\mathfrak{k}^*$. If $\deg_{\mathfrak{k}} P \geq 1$ then we can write $P = P_0 + \sum_{k=1}^{q} \alpha_{i_k} P_k$ such that $P_0 \in \mathbb{R}[\mathfrak{s}]$ and $\deg_{\mathfrak{k}} P_k < \deg_{\mathfrak{k}} P$ for all $k = 1, \dots, q$. Moreover,

$$\frac{P}{\prod_{i \in I} \alpha_i} = \frac{P_0}{\prod_{i \in I} \alpha_i} + \sum_{k=1}^{q} \frac{P_k}{\prod_{i \in I \setminus \{i_k\}} \alpha_i}.$$

Assume that $P = 1$ and $I = \{1, \dots, N\}$. We may also suppose that $\alpha_i \notin \mathfrak{s}^*$ for all $i \in I$. We consider ordering on fractions $\dfrac{Q}{\alpha_1^{n_1} \dots \alpha_N^{n_N}}$ with $Q \in \mathfrak{F}[s]$ by associating it the lexicographical order $(n_1, \dots, n_N) \in \mathbb{N}^N$. Suppose that $\dfrac{1}{\alpha_1^{n_1} \dots \alpha_N^{n_N}}$ is $\mathfrak{k}^*$-generating. Let $i_1, \dots, i_m \in \{i \in I \,|\, n_i > 0\}$ such that $i_1 < \dots < i_m$ and $\sum_{k=1}^{m} a_k \alpha_{i_k} = \beta \in \mathfrak{s}^*$ with $a_k \neq 0$ for all $k = 1, \dots, m$. We distinguish two cases. If $\beta = 0$ then

$$\frac{1}{\alpha_1^{n_1} \dots \alpha_N^{n_N}} = \sum_{k=1}^{m-1} \frac{-a_k a_m^{-1}}{\alpha_1^{n_1} \dots \alpha_{i_k}^{n_k - 1} \dots \alpha_{i_m}^{n_m + 1} \dots \alpha_N^{n_N}}. \tag{3.18}$$

If $\beta \neq 0$ then

$$\frac{1}{\alpha_1^{n_1} \dots \alpha_N^{n_N}} = \sum_{k=1}^{m} \frac{a_k}{\beta} \cdot \frac{1}{\alpha_1^{n_1} \dots \alpha_{i_k}^{n_k - 1} \dots \alpha_N^{n_N}}. \tag{3.19}$$

Remark that the fractions on the right hand side of (3.18) and (3.19) are also $\mathfrak{k}^*$-generating and have order strictly less than the fraction on the left hand side. We continue the decomposition on fraction on the right hand sides. This algorithm stops in finite steps because the lexicographical order is a well-order. Moreover, it yields fractions $\dfrac{Q}{\alpha_1^{m_1} \dots \alpha_N^{m_N}}$ with $m_j = 0$ unless $j = i_1, \dots, i_q$ such that $\{\mathrm{pr}_{\mathfrak{k}^*}(\alpha_{i_1}), \dots, \mathrm{pr}_{\mathfrak{k}^*}(\alpha_{i_q})\}$ is a basis of $\mathfrak{k}^*$ and $Q \in \mathfrak{F}[\mathfrak{s}]$. □

Remark 3.38. Since non-$\mathfrak{k}^*$-generating fractions yield non-generating fractions $F_V(v, s)$ in v (considering s as real parameter), therefore the equivariant Jeffrey-Kirwan residue vanishes on non-$\mathfrak{k}^*$-generating fractions. ◇

We have the following analogue of Proposition 3.17.

Proposition 3.39. *Let Λ be a polarization on $\mathfrak{t}^* = \mathfrak{k}^* \oplus \mathfrak{s}^*$. Consider $\alpha_1, \dots, \alpha_q, \lambda \in \mathfrak{t}^*$ such that $\{\mathrm{pr}_{\mathfrak{k}^*}(\alpha_i) \,|\, i = 1, \dots, q\}$ spans $\mathfrak{k}^*$ and $\mathrm{pr}_{\mathfrak{k}^*}(\lambda)$ is regular with respect to it. Write $\lambda = \lambda_0 + \lambda_1 \overline{\alpha_1} + \dots + \lambda_q \overline{\alpha_q}$ with $\lambda_0 \in \mathfrak{s}^*$ and $\lambda_1, \dots, \lambda_q \in \mathbb{R}$. Then we have*

$$\mathrm{EqRes}^{\Lambda} \frac{e^{\lambda}}{\prod_{i=1}^{q} \alpha_i^{n_i+1}} = \begin{cases} \dfrac{e^{\lambda_0}}{\sqrt{\det\left[(\alpha_i, \alpha_j)\right]_{i,j=1}^{q}}} \displaystyle\prod_{i=1}^{q} \frac{\varepsilon(\alpha_i)^{n_i+1} \lambda_i^{n_i}}{n_i!} & \text{if } \lambda_1, \dots, \lambda_q > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have only one relevant $\mathfrak{k}^*$-pole $V = \mathrm{span}\langle \alpha_1, \dots, \alpha_q \rangle$ and let v be a basis of V inducing the same polarization on $\alpha_1, \dots, \alpha_q$ as Λ. Then

$$\mathrm{EqRes}^{\Lambda} \frac{e^{\lambda}}{\prod_{i=1}^{q} \alpha_i^{n_i+1}} = \underset{v}{\mathrm{JKRes}} \frac{e^{\lambda_0(s) + \lambda_V(v)}}{\prod_{i=1}^{q} \alpha_i(v)^{n_i+1}} dv,$$

where $\lambda_V = \lambda_1\overline{\alpha_1} + \ldots + \lambda_1\overline{\alpha_q}$ and s is a basis of $\mathfrak{s}^*$. Hence the proposition follows from Proposition 3.17. $\square$

We also have an analogue of Corollary 3.19.

Corollary 3.40. *Let* $F = \dfrac{Pe^{\lambda}}{\prod_{i\in I}\alpha_i} \in \mathfrak{F}_{\mathfrak{k}^*\text{-}reg}$. *If* $\mathrm{pr}_{\mathfrak{k}^*}(\lambda) \notin Cone(\mathrm{pr}_{\mathfrak{k}^*}(\overline{\alpha_i}) \,|\, i \in I)$ *with respect to the polarization* Λ *then* $\mathrm{EqRes}^{\Lambda} F = 0$.

Proof. Let V be a $\mathfrak{k}^*$-pole. Write $\lambda = \lambda_0 + \lambda_V$, where $\lambda_0 \in \mathfrak{s}^*$ and $\lambda_V \in V$. Remark that λ_V is regular with respect to $V \cap \{\alpha_i \,|\, i \in I\}$. Moreover, $\mathrm{pr}_{\mathfrak{k}^*}(\lambda) \notin Cone(\mathrm{pr}_{\mathfrak{k}^*}(\overline{\alpha_i}) \,|\, i \in I)$ implies that $\lambda_V \notin Cone(\overline{\alpha_i} \,|\, \alpha_i \in V,\, i \in I)$, hence by Corollary 3.19 follows that $\mathrm{JKRes}^{\Lambda_V} F_V(v,s)dv = 0$. $\square$

Corollary 3.41. *Let* $F = \dfrac{Pe^{\lambda}}{\prod_{i\in I}\alpha_i} \in \mathfrak{F}_{\mathfrak{k}^*\text{-}reg}$ *and let* Λ *be the polarization induced by an ordered basis* $\{x_1, \ldots, x_q, s_1, \ldots, s_{r-q}\}$ *such that* $x = \{x_1, \ldots, x_q\}$ *and* $s = \{s_1, \ldots, s_{r-q}\}$ *are basis of* $\mathfrak{k}^*$ *and* $\mathfrak{s}^*$, *respectively. If* $\overline{\lambda} = -\lambda$ *with respect to the polarization* Λ *then* $\mathrm{EqRes}^{\Lambda} F = 0$.

Proof. Since F is $\mathfrak{k}^*$-regular, we have $\lambda \notin \mathfrak{s}^*$. For any $\beta \notin \mathfrak{s}^*$ we have $\mathrm{pr}_{\mathfrak{k}^*}(\overline{\beta}) = \overline{\mathrm{pr}_{\mathfrak{k}^*}(\beta)}$, which implies that $\mathrm{pr}_{\mathfrak{k}^*}(\lambda) \notin Cone(\overline{\mathrm{pr}_{\mathfrak{k}^*}(\alpha_i)} \,|\, i \in I) = Cone(\mathrm{pr}_{\mathfrak{k}^*}(\overline{\alpha_i}) \,|\, i \in I)$, hence the corollary follows from Corollary 3.40. $\square$

We can also compute EqRes^{Λ} using residue $\mathrm{Res}^{\Lambda,+}$.

Proposition 3.42. *If* $F \in \mathfrak{F}_{\mathfrak{k}^*\text{-reg}}$ *then*

$$\mathrm{EqRes}^{\Lambda} F = \frac{1}{\sqrt{\det[(x_i, x_j)]_{i,j=1}^q}} \operatorname*{Res}_x^{\Lambda,+} F(x,s)dx,$$

where $x = \{x_1, \ldots, x_q\}$ *and* s *are basis of* $\mathfrak{k}^*$ *and* $\mathfrak{s}^*$, *respectively.*

Proof. Decompose F to partial fraction as in Lemma 3.37. On non-$\mathfrak{k}^*$-generating fractions both EqRes^{Λ} and $\operatorname*{Res}_x^{\Lambda,+}$ vanish by Remark 3.38 and (3.7). For $\mathfrak{k}^*$-generating fractions of form $G = \dfrac{Qe^{\lambda(x,s)}}{\prod_{j=1}^q \alpha_j(x,s)^{n_j+1}}$ with $Q \in \mathfrak{F}[s]$ we have

$$\mathrm{EqRes}^{\Lambda} G = \frac{1}{\sqrt{\det[(x_i, x_j)]_{i,j=1}^q}} \operatorname*{Res}_x^{\Lambda,+} G(x,s)dx$$

by Propositions 3.17 and 3.39. $\square$

Example 3.4. Let $\{x, y\}$ be an orthonormal basis of $\mathfrak{k}^*$ and let $\{s\}$ be a basis of $\mathfrak{s}^*$. On $\mathfrak{k}^* \oplus \mathfrak{s}^*$ we consider the polarization Λ induced by the ordered basis $\{s, x, y\}$ and let

$$F(x,y,s) = \frac{e^x}{(x-y+s)^2(x+y+s)(y-2s)}.$$

We will compute $\mathrm{EqRes}^{\Lambda} F$ in two ways: by definition and by Proposition 3.42, since F is $\mathfrak{k}^*$-regular.

In the first case we remark that F has three $\mathfrak{k}^*$-poles $V_1 = \mathrm{span}\langle x-y+s, x+y+s\rangle$, $V_2 = \mathrm{span}\langle x-y+s, y-2s\rangle$ and $V_3 = \mathrm{span}\langle x+y+s, y-2s\rangle$. We consider basis $\{u_i, v_i\}$ on poles V_i inducing the same polarization as Λ_{V_i} on $\{x-y+s, x+y+s, y-2s\} \cap V_i$, hence

$$\mathrm{EqRes}^{\Lambda} F = \sum_{i=1}^{3} \mathrm{JKRes}^{\Lambda_{V_i}} F(u_i, v_i, s) du_i dv_i.$$

(a) Let $u_1 = s+x-y$ and $v_1 = s+x+y$. Then

$$\mathrm{JKRes}^{\Lambda_{V_1}} F(u_1, v_1, s) du_1 dv_1 = \mathrm{JKRes}^{\Lambda_{V_1}} \frac{e^{\frac{u_1+v_1}{2}-s}}{u_1^2 v_1 (\frac{v_1-u_1}{2} - 2s)} du_1 dv_1 = \frac{e^{-s}}{2}\left(\frac{1}{8s^2} - \frac{1}{4s}\right).$$

(b) Let $u_2 = s+x-y$ and $v_2 = 2s-y$. Then

$$\mathrm{JKRes}^{\Lambda_{V_2}} F(u_2, v_2, s) du_2 dv_2 = \mathrm{JKRes}^{\Lambda_{V_2}} \frac{e^{u_2-v_2+s}}{u_2^2 (u_2 - 2v_2 + 4s)(-v_2)} du_2 dv_2 = 0.$$

(c) Let $u_3 = s+x+y$ and $v_3 = 2s-y$. Then

$$\mathrm{JKRes}^{\Lambda_{V_3}} F(u_3, v_3, s) du_3 dv_3 = \mathrm{JKRes}^{\Lambda_{V_3}} \frac{e^{u_3+v_3-3s}}{(u_3 + 2v_3 - 4s)^2 u_3 (-v_3)} du_3 dv_3 = \frac{e^{-3s}}{-16s^2}.$$

Therefore,

$$\mathrm{EqRes}^{\Lambda} F = \frac{e^{-s}}{2}\left(\frac{1}{8s^2} - \frac{1}{4s}\right) + \frac{e^{-3s}}{-16s^2}.$$

Now we compute $\mathop{\mathrm{Res}}_{y}^{\Lambda,+} \mathop{\mathrm{Res}}_{x}^{\Lambda,+} F(x,y,s)dxdy$. We have two linear terms in the denominator of F containing x, namely $u_1 = s+x-y$ and $u_2 = s+x+y$. Both of them are polarized with respect to Λ. Hence

$$\begin{aligned}
\mathop{\mathrm{Res}}_{x}^{\Lambda,+} \frac{e^x}{(x-y+s)^2(x+y+s)(y-2s)} dx &= \mathop{\mathrm{Res}^{+}}_{u_1=0} \frac{e^{u_1+y-s}}{u_1^2(u_1+2y)(y-2s)} du_1 \\
&\quad + \mathop{\mathrm{Res}^{+}}_{u_2=0} \frac{e^{u_2-y-s}}{(u_2-2y)^2 u_2 (y-2s)} du_2 \\
&= \frac{(2y-1)e^{y-s}}{(2y)^2(y-2s)} + \frac{e^{-y-s}}{4y^2(y-2s)}.
\end{aligned}$$

We compute first $\mathop{\mathrm{Res}}_{y}^{\Lambda,+} \dfrac{(2y-1)e^{y-s}}{(2y)^2(y-2s)} dy$. Again, $v_1 = y$ and $v_2 = 2s-y$ are the polarized linear terms in the denominator involving y. Hence,

$$\begin{aligned}
\mathop{\mathrm{Res}}_{y}^{\Lambda,+} \frac{(2y-1)e^{y-s}}{(2y)^2(y-2s)} dy &= \mathop{\mathrm{Res}^{+}}_{v_1=0} \frac{(2v_1-1)e^{v_1-s}}{4v_1^2(v_1-2s)} dv_1 + \mathop{\mathrm{Res}^{+}}_{v_2=0} \frac{(4s-2v_2-1)e^{s-v_2}}{4(2s-v_2)^2(-v_2)} dv_2 \\
&= e^{-s}\left(\frac{1}{16s^2} - \frac{1}{8s}\right) + 0.
\end{aligned}$$

Finally, we compute $\operatorname*{Res}_{y}^{\Lambda,+}\dfrac{e^{-y-s}}{4y^2(y-2s)}dy$. Again, $w_1 = y$ and $w_2 = 2s - y$ are the polarized vectors in the denominator containing y. Thus,

$$\begin{aligned}\operatorname*{Res}_{y}^{\Lambda,+}\frac{e^{-y-s}}{4y^2(y-2s)}dy &= \operatorname*{Res}_{w_1=0}^{+}\frac{e^{-w_1-s}}{4w_1^2(w_1-2s)}dw_1 + \operatorname*{Res}_{w_2=0}^{+}\frac{e^{w_2-3s}}{4(2s-w_2)^2(-w_2)}dw_2\\ &= 0 + \frac{e^{-3s}}{-16s^2}.\end{aligned}$$

Therefore,

$$\operatorname*{Res}_{y}^{\Lambda,+}\operatorname*{Res}_{x}^{\Lambda,+}F(x,y,s)dxdy = e^{-s}\left(\frac{1}{16s^2}-\frac{1}{8s}\right)+\frac{e^{-3s}}{-16s^2}.$$

◇

Remark 3.43. We compare our version of equivariant Jeffrey-Kirwan residue with the one in [31]. Let $x = \{x_1, \ldots, x_q\}$ and $s = \{s_1, \ldots, s_{r-q}\}$ be an ordered basis of $\mathfrak{k}^*$ and $\mathfrak{s}^*$, respectively. Denote Λ and Λ' the polarizations induced by ordered basis $\{x_1, \ldots, x_q, s_1, \ldots, s_{r-q}\}$ and x on $\mathfrak{k}^* \oplus \mathfrak{s}^*$ and $\mathfrak{k}^*$, respectively. Then EqRes^{Λ} corresponds to $\mathrm{JKRes}^{\Lambda'}$ the Jeffrey-Kirwan residue adapted to the equivariant setting in [31]. It is enough to check it on $\mathfrak{k}^*$-regular fractions of form $\dfrac{e^{\lambda}}{\prod_{i=1}^{q}\alpha_i^{n_i+1}}$. For simplicity we demonstrate it when $\dim\mathfrak{k}^* = 1$ and the same computation can be carried out in the general case. Let $\beta \in \mathfrak{k}^*$, $\gamma \in \mathfrak{s}^*$ such that $\alpha = \beta - \gamma$ and suppose that $\beta = \overline{\beta}$. Then we can write $\lambda = \lambda_0 + \lambda_1\beta$ with $\lambda_0 \in \mathfrak{s}^*$ and assume that $\lambda_1 > 0$. In [31] $\mathrm{JKRes}^{\Lambda'}\dfrac{e^{\lambda(x,s)}}{\alpha^{n+1}(x,s)}dx$ is reduced to the usual Jeffrey-Kirwan residue by expansion $x \gg s$ as follows

$$\mathrm{JKRes}^{\Lambda'}\frac{e^{\lambda(x,s)}}{\beta(x)^{n+1}\Big(1-\frac{\gamma(s)}{\beta(x)}\Big)^{n+1}}dx = \mathrm{JKRes}^{\Lambda'}\sum_{k_1,\ldots,k_{n+1}\geq 0}\frac{\gamma(s)^{k_1+\ldots+k_{n+1}}e^{\lambda(x,s)}}{\beta(x)^{n+1+k_1+\ldots+k_{n+1}}}dx$$

$$= \mathrm{JKRes}^{\Lambda'}\sum_{m\geq 0}\binom{m+n}{n}\frac{\gamma(s)^m e^{\lambda_0(s)+\lambda_1\beta(x)}}{\beta(x)^{m+n+1}}dx = \sum_{m\geq 0}\frac{\gamma(s)^m\lambda_1^{m+n}e^{\lambda_0(s)}}{\sqrt{(\beta,\beta)}\,m!\,n!} = \frac{\lambda_1^n e^{\lambda_0(s)+\lambda_1\gamma(s)}}{n!\sqrt{(\beta,\beta)}}$$

by Proposition 3.20. In the case of EqRes^{Λ} there is a single $\mathfrak{k}^*$-pole $V = \operatorname{span}\langle\alpha\rangle$, moreover $\alpha = \overline{\alpha}$, thus

$$\mathrm{EqRes}^{\Lambda}\frac{e^{\lambda}}{\alpha^{n+1}} = \mathrm{JKRes}^{\Lambda_V}\frac{e^{\lambda_0(s)+\lambda_1\gamma(s)+\lambda_1\alpha}}{\alpha^{n+1}}d\alpha = \frac{\lambda_1^n e^{\lambda_0(s)+\lambda_1\gamma(s)}}{n!\sqrt{(\alpha,\alpha)}}$$

by Proposition 3.39. Finally, we have equality of norm squares $(\alpha,\alpha) = (\beta,\beta)$ by definition. ◇

4

Equivariant Jeffrey-Kirwan theorem

In this chapter we give a generalization of Jeffrey-Kirwan theorem to non-compact symplectic quotients. We use the Atiyah-Bott-Berline-Vergne formula to define integrals on non-compact spaces and the usual Jeffrey-Kirwan residue is replaced by the equivariant version introduced in section 3.2. At the end of the chapter we give an hyperKähler version of our theorem.

4.1 Symplectic version

Let G be a compact connected Lie group with maximal torus T of rank r and let S be a q-dimensional torus. Let (M, ω) be a possibly non-compact connected symplectic manifold with Hamiltonian $(G \times S)$-action and denote by $\mu_{G\times S} : M \to \mathfrak{g}^* \oplus \mathfrak{s}^*$ its moment map. For any subgroup $H \subset G \times S$ we denote by $\mu_H = \mathrm{pr}_{\mathfrak{h}^*} \circ \mu_{G\times S}$ the corresponding moment map, where $\mathrm{pr}_{\mathfrak{h}^*} : \mathfrak{g}^* \oplus \mathfrak{s}^* \to \mathfrak{h}^*$ is the natural projection.

We assume that there is an 1-dimensional subtorus $K \subset S$ such that the K-action on M is PBB, that is, M^K is compact, moreover there is $\gamma \in \mathfrak{k}^*_{\mathbb{Z}}$ generator such that the K-moment map $\mu_K = \varphi \cdot \gamma$ with $\varphi : M \to \mathbb{R}$ proper, bounded below.

Remark 4.1. M^K is compact is equivalent to $M^{T\times S}$ being compact. Indeed, since $M^{T\times S} \subset M^K$ is closed, therefore $M^{T\times S}$ is compact. Conversely, assume that $M^{T\times S}$ is compact. Each fixed point component $F \subset M^K$ is compact since μ_K is proper, moreover F contains a $(T \times S)$-fixed point component. Then M^K has finitely many connected components, because $M^{T\times S}$ has finitely many, too. $\diamond$

Motivated by [19] we define equivariant integration on M formally by the Atiyah-Bott-Berline-Vergne formula.

Definition 4.2. For any $\beta \in H_{G\times S}(M) \simeq H_{T\times S}(M)^W$ we define

$$\oint_M \beta = \sum_{F\subset M^{T\times S}} \int_F \frac{i_F^*\beta}{e_{T\times S}\mathcal{N}(F\,|\,M)}.$$

It is well defined due to the compactness assumption and Stokes theorem. If M is compact then the definition is compatible with the usual equivariant integration by the Atiyah-Bott-Berline-Vergne theorem.

Let $0 \in \mathfrak{g}^*$ be a regular value of μ_G and denote $M/\!\!/G = \mu_G^{-1}(0)/G$ the symplectic quotient. It is a possibly non-compact symplectic manifold (orbifold) with Hamiltonian S-action induced by the action of the same group on M. In order to define integration on $M/\!\!/G$ similarly to Definition 4.2 we have to show that $(M/\!\!/G)^S$ is compact.

Proposition 4.3. *If M^K is compact then $(M/\!\!/G)^S$ is also compact.*

Proof. Since $(M/\!\!/G)^S \subset (M/\!\!/G)^K$ is a closed subset, it suffices to show that $(M/\!\!/G)^K$ is compact. Denote $\pi : \mu_G^{-1}(0) \to M/\!\!/G$ the quotient map. Let $m \in (M/\!\!/G)^K$ and $x \in \pi^{-1}(m)$. Let $(G\times K)_x$ be the maximal connected subgroup of $G\times K$ which fixes x. Since G acts locally freely on $\pi^{-1}(m)$, therefore $(G \times K)_x$ is 1-dimensional, i.e. it is a torus. For any $y \in (G \times K) \cdot x$ the subgroups $(G\times K)_x$ and $(G\times K)_y$ are conjugate in $G\times K$, therefore there is y such that $(G\times K)_y \subset T\times K$. Since $y \in \pi^{-1}(m) = (G \times K) \cdot x$, we have that $(G \times K)_y \not\subset T$ and $\pi^{-1}(m) \subset G \cdot M^{(G\times K)_y}$. Let $\mathcal{T}$ be the set of all 1-dimensional subtori $T' \subset T \times K$ with properties

(T1) $T' \not\subset T$,

(T2) there is $F' \subset M^{T'}$ connected component such that for any $T'' \neq T'$ torus such that $T' \subset T'' \subset T \times K$ we have that $(F')^{T''}$ is strictly smaller than F'.

We show that the set $\mathcal{T}$ is finite. Indeed, let $T' \in \mathcal{T}$ and let F' be as in (T2). Then F' contains a $(T \times K)$-fixed point since F' is a K-invariant symplectic submanifold of M with K-moment map $\mu_K|_{F'}$, which is also proper and non-surjective. We choose an invariant compatible almost complex structure on M and let $D' \subset (F')^{T\times K}$ be a fixed point component. For any $u \in D'$ the tangent space T_uF' is an $(T \times K)$-invariant complex subspace of T_uM. Denote $\{\eta_i \in (\mathfrak{t}\oplus\mathfrak{k})^*_{\mathbb{Z}} \,|\, i \in I\}$ the set of $(T\times K)$-weights on T_uF' and remark that they do not depend on u. The Lie algebra $\mathfrak{t}'$ of T' satisfies $\mathfrak{t}' \subset \cap_{i\in I} \ker \eta_i$. Moreover, the subalgebra $\mathfrak{t}'' = \cap_{i\in I} \ker \eta_i$ is the Lie algebra of a subtorus $T'' \subset T \times K$ and we have $T' \subset T''$. Furthermore, T'' acts trivially on T_uF', hence $(F')^{T''}$ is an open and closed subset of the connected component F', therefore $(F')^{T''} = F'$. By (T2) we must have $T' = T''$, hence $\mathfrak{t}' = \cap_{i\in I} \ker \eta_i$. Since $M^{T\times K}$ has finitely many components, therefore we also have finitely many subalgebras of form $\mathfrak{t}' = \cap_{i\in I} \ker \eta_i$, hence $\mathcal{T}$ is finite.

We have inclusions of closed subsets

$$\pi^{-1}\left((M/\!\!/G)^K\right) \subset G \cdot \left(\bigcup_{T'\in\mathcal{T}} M^{T'} \cap \mu_G^{-1}(0)\right) \subset G \cdot \left(\bigcup_{T'\in\mathcal{T}} M^{T'} \cap \mu_T^{-1}(0)\right).$$

We conclude our proof by showing that $M^{T'} \cap \mu_T^{-1}(0)$ is compact for any $T' \in \mathcal{T}$. Let $F' \subset M^{T'}$ be a connected component. Recall that $\mu_{T\times K}(F')$ lies in an affine hyperplane $\mathcal{H}$ of $\mathfrak{t}^* \oplus \mathfrak{k}^*$, where $\mathcal{H}$ is the inverse image of the point $\mu_{T'}(F')$ under the projection $\mathfrak{t}^* \oplus \mathfrak{k}^* \to (\mathfrak{t}')^*$. The intersection $\mathcal{H} \cap \mathfrak{k}^*$ is a point since $T' \not\subset T$ by (T1). Finally,

$$F' \cap \mu_T^{-1}(0) \;\subset\; F' \cap \mu_{T\times K}^{-1}(\mathfrak{k}^*) \;\subset\; \mu_{T\times K}^{-1}(\mathcal{H} \cap \mathfrak{k}^*)$$

and latter set is compact because $\mu_{T\times K}$ is proper. $M^{T'}$ has finitely many connected components since each of them contains a connected component of $M^{T\times K} \subset M^K$, hence $M^{T'} \cap \mu_T^{-1}(0)$ is compact. $\square$

Thus we can define equivariant integration on $M/\!\!/G$ as

$$\oint_{M/\!\!/G} \beta = \sum_{F\subset (M/\!\!/G)^S} \frac{1}{\mathfrak{m}(F)} \int_F \frac{i_F^*\beta}{e_S\mathcal{N}(F\,|\,M/\!\!/G)}$$

for any $\beta \in H_S(M/\!\!/G)$ (cf. Theorem 1.32). Let $\chi : \mathfrak{t}^* \oplus \mathfrak{s}^* \to \mathbb{R}$ be the linear function defined by

$$\mathrm{pr}_{\mathfrak{k}^*} = \chi \cdot \gamma,$$

where $\mathrm{pr}_{\mathfrak{k}^*} : \mathfrak{t}^* \oplus \mathfrak{s}^* \to \mathfrak{k}^*$ is the projection.

Definition 4.4. A polarization Λ on $\mathfrak{t}^* \oplus \mathfrak{s}^*$ is *compatible with the proper and bounded below moment map* (μ_K, φ, γ) *on* M if it is induced by an ordered basis $\{y_1, \ldots, y_{r+q}\}$ such that $\chi(y_1) > 0$ and $y_2, \ldots, y_{r+q} \in \ker\chi$.

The main result of this chapter is the following generalization of the Jeffrey-Kirwan theorem to non-compact symplectic quotients.

Theorem 4.5. *Let* (M, ω) *be a Hamiltonian* $(G\times S)$*-manifold with moment map* $\mu_{G\times S}$*. Assume that* $0 \in \mathfrak{g}^*$ *is a regular value of* μ_G *and denote* $M/\!\!/G = \mu_G^{-1}(0)/G$ *the symplectic quotient. Moreover, assume that* S *has an* 1*-dimensional subtorus with proper, bounded below moment map and let* Λ *be a polarization on* $\mathfrak{t}^* \oplus \mathfrak{s}^*$ *compatible with it. Then for any* $\beta \in H_{G\times S}(M)$ *we have*

$$\oint_{M/\!\!/G} \kappa_S(\beta e^{\omega-\mu_{G\times S}}) = \lim_{\epsilon\to 0} \mathrm{EqRes}^\Lambda\left(\frac{\varpi}{|W| vol(T)} \oint_M \beta e^{\omega-\mu_{T\times S}+\epsilon\rho}\right),$$

where $\kappa_S : H_{G\times S}(M) \to H_S(M/\!\!/G)$ *is the Kirwan map,* $|W|$ *is the order of the Weyl group of* G*,* $vol(T)$ *is the volume induced by the scalar product on* $\mathfrak{t}^*$ *used in* EqRes^Λ*,* ϖ *is the product of roots of* G *and* ρ *is a small regular value of* μ_T*.*

Remark 4.6. If 0 is a regular value of μ_T then we may choose $\rho = 0$ and the limit is unnecessary. $\diamond$

The strategy of the proof is as follows. First we prove Theorem 4.5 for M compact and $G = T$ abelian, then we prove it for M non-compact and $G = T$ still abelian. Finally, we deduce the general case by Martin's method [32].

4.1.1 Compact and abelian case

As first step to prove Theorem 4.5 we will show the following theorem, which is the generalization of the abelian version of the Jeffrey-Kirwan theorem for the residue EqRes.

Theorem 4.7. *Let T and S be two compact tori and let (M,ω) be a compact Hamiltonian $(T\times S)$-manifold. Assume that $0\in\mathfrak{t}^*$ is a regular value of the T-moment map μ_T and denote $M/\!\!/T=\mu_T^{-1}(0)/T$ the symplectic quotient. Then for any polarization Λ on $\mathfrak{t}^*\oplus\mathfrak{s}^*$ and for any $\beta\in H_{T\times S}(M)$ we have*

$$\oint\limits_{M/\!\!/T}\kappa_S(\beta e^{\omega-\mu_{T\times S}})=\mathrm{EqRes}^{\Lambda}\left(\frac{1}{vol(T)}\oint\limits_{M}\beta e^{\omega-\mu_{T\times S}}\right),$$

where $\kappa_S: H_{T\times S}(M)\to H_S(M/\!\!/T)$ is the Kirwan map.

Proof. Since M is compact, it is enough to prove it for a particular polarization Λ by Proposition 3.35(ii). Let $t=\{t_1,\ldots,t_r\}$ and $s=\{s_1,\ldots,s_q\}$ be an ordered basis of $\mathfrak{t}^*$ and $\mathfrak{s}^*$, respectively such that $\mu_T(M^T)\cap\mathrm{span}\langle t_2,\ldots,t_r\rangle=\emptyset$. Let Λ be the polarization induced by the ordered basis $\{t_1,\ldots,t_r,s_1,\ldots,s_q\}$ of $\mathfrak{t}^*\oplus\mathfrak{s}^*$. In this case the proof goes the same way as the proof of Theorem A in [25]. Denote $\{t^1,\ldots,t^r,s^1,\ldots,s^q\}\subset\mathfrak{t}\oplus\mathfrak{s}$ the dual basis and let $\Gamma=Cone(\gamma_1,\ldots,\gamma_r)\subset\mathfrak{t}^*$ such that

(Γ1) $\gamma_1,\ldots,\gamma_r\in(t^1)^{<0}\cap\mathfrak{t}^*_{\mathbb{Z}}$ are linearly independent, where $(t^1)^{<0}=\{u\in\mathfrak{t}^*\oplus\mathfrak{s}^*\,|\,u(t^1)<0\}$,

(Γ2) for any $I\subset\{1,\ldots,r\}$ the $Cone(\gamma_i\,|\,i\in I)$ intersects every wall of $\mu_T(M)$ transversally,

(Γ3) $(t^1)^{<0}\cap\mu_T(M^T)\subset\Gamma$.

By assumptions (Γ1) and (Γ2) we can construct the symplectic cut M_Γ. We will denote by $\mu_{T\times S}$ the $(T\times S)$-moment map and by ω_Γ the symplectic form on M_Γ. By Theorem 2.13 for any $\beta\in H_{T\times S}(M)$ we have

$$\text{(4.1)}\qquad \frac{1}{\mathfrak{m}(M_\Gamma)}\int\limits_{M_\Gamma}\Delta(\beta e^{\omega-\mu_{T\times S}})=\oint\limits_{M/\!\!/T}\frac{\kappa_S(\beta e^{\omega-\mu_{T\times S}})}{\prod_{j=1}^{r}(\gamma_j-\kappa_S(\gamma_j))}$$

$$\text{(4.2)}\qquad +\sum_{\substack{F\subset M^{T\times S}\\ \mu_T(F)\in\Gamma}}\frac{1}{\delta_\Gamma}\int\limits_F\frac{i_F^*(\beta e^{\omega-\mu_{T\times S}})}{e_{T\times S}\mathcal{N}(F\,|\,M)}$$

$$\text{(4.3)}\qquad +\sum_{\substack{D\subset(M_\Gamma)^{T\times S}\\ \mu_T(D)\in\partial\Gamma\setminus\{0\}}}\frac{1}{\mathfrak{m}(D)}\int\limits_D\frac{i_D^*\Delta(\beta)e^{i_D^*\omega_\Gamma-\mu_{T\times S}(D)}}{e_{T\times S}\mathcal{N}(D\,|\,M_\Gamma)}.$$

Denote the three summands on the right hand side of (4.1), (4.2), (4.3) by $\mathcal{I}_{red}$, $\mathcal{I}_{old}$ and $\mathcal{I}_{new}$, respectively. We choose $\rho\in\mathfrak{t}^*$ near 0 in the interior of Γ which is not on the supporting plane of any wall of $\mu_T(M_\Gamma)$ and with property

$$\text{(4.4)}\qquad 0>\langle\rho,t^1\rangle>\langle\mu_T(F'),t^1\rangle$$

for any $F'\subset(M_\Gamma)^{T\times S}$ with $\mu_T(F')\neq0$. We will show the following relations:

(i) $\mathrm{EqRes}^{-\Lambda}\mathcal{I}_{old}e^{\epsilon\rho}+\mathrm{EqRes}^{-\Lambda}\mathcal{I}_{new}e^{\epsilon\rho}=0$,

(ii) $\mathrm{EqRes}^{\Lambda}\mathcal{I}_{red}e^{\epsilon\rho} = 0$,

(iii) $\mathrm{EqRes}^{\Lambda}\mathcal{I}_{old}e^{\epsilon\rho} = \mathrm{EqRes}^{\Lambda}\dfrac{1}{\delta_\Gamma}\oint\limits_M \beta e^{\omega-\mu_{T\times S}+\epsilon\rho}$,

(iv) $\mathrm{EqRes}^{\Lambda}\mathcal{I}_{new}e^{\epsilon\rho} = 0$,

(v) $\lim\limits_{\epsilon\to 0^+}\mathrm{EqRes}^{-\Lambda}\mathcal{I}_{red}e^{\epsilon\rho} = \dfrac{vol(T)}{\delta_\Gamma}\oint\limits_{M/\!\!/T}\kappa_S(\beta e^{\omega-\mu_{T\times S}})$.

Then by Proposition 3.35(ii)

$$\mathrm{EqRes}^{\Lambda}(\mathcal{I}_{red}+\mathcal{I}_{old}+\mathcal{I}_{new})e^{\epsilon\rho} = \mathrm{EqRes}^{-\Lambda}(\mathcal{I}_{red}+\mathcal{I}_{old}+\mathcal{I}_{new})e^{\epsilon\rho}$$

for all $\epsilon\in(0,1]$, hence by (i), (ii), (iv) we have

$$\mathrm{EqRes}^{\Lambda}\mathcal{I}_{old}e^{\epsilon\rho} = \mathrm{EqRes}^{-\Lambda}\mathcal{I}_{red}e^{\epsilon\rho}$$

for all $\epsilon\in(0,1]$. By (iii), (v) and Proposition 3.35(iii) we get

$$\begin{aligned}\mathrm{EqRes}^{\Lambda}\frac{1}{\delta_\Gamma}\oint\limits_M \beta e^{\omega-\mu_{T\times S}} &= \lim_{\epsilon\to 0^+}\mathrm{EqRes}^{\Lambda}\mathcal{I}_{old}e^{\epsilon\rho} = \lim_{\epsilon\to 0^+}\mathrm{EqRes}^{-\Lambda}\mathcal{I}_{red}e^{\epsilon\rho}\\ &= \frac{vol(T)}{\delta_\Gamma}\oint\limits_{M/\!\!/T}\kappa_S(\beta e^{\omega-\mu_{T\times S}}),\end{aligned}$$

thus the theorem follows.

Now we start to prove the above relations. From (Γ1) and (4.4) follows that $-\mu_{T\times S}(F')+\epsilon\rho$ are not polarized with respect to $-\Lambda$ for all $F'\subset M^{T\times S}$ with $\mu_T(F')\in\Gamma$ and for all $F'\in(M_\Gamma)^{T\times S}$ with $\mu_T(F')\in\partial\Gamma\setminus\{0\}$, thus the relation (i) follows from Corollary 3.41.

Remark that $\mathcal{I}_{red}e^{\epsilon\rho}$ is a $\mathfrak{t}^*$-regular fraction of form $\sum\limits_I \dfrac{P_I e^{\lambda_I+\epsilon\rho}}{\prod_{i\in I}\beta_i}$, where $\lambda_I\in\mathfrak{s}^*$, moreover $\lambda_I+\epsilon\rho$ is not polarized with respect to Λ for any I by (4.4). Hence, the relation (ii) follows again from Corollary 3.41.

To prove relation (iii) remark that for any $F\subset M^{T\times S}$ we have either $\mu_T(F)\in\Gamma$ or $-\mu_T(F)\in(t^1)^{<0}$ by (Γ3). In the latter case $-\mu_T(F)+\epsilon\rho\in(t^1)^{<0}$, therefore $-\mu_{T\times S}(F)+\epsilon\rho = -\mu_T(F)-\mu_S(F)+\epsilon\rho$ is not polarized with respect to Λ. By Corollary 3.41 follows that

$$\begin{aligned}\mathrm{EqRes}^{\Lambda}\frac{1}{\delta_\Gamma}\oint\limits_M \beta e^{\omega-\mu_{T\times S}+\epsilon\rho} &= \mathrm{EqRes}^{\Lambda}\frac{1}{\delta_\Gamma}\sum_{F\subset M^{T\times S}}\int\limits_F \frac{i_F^*(\beta e^{\omega-\mu_{T\times S}+\epsilon\rho})}{e_{T\times S}\mathcal{N}(F\,|\,M)}\\ &= \mathrm{EqRes}^{\Lambda}\frac{1}{\delta_\Gamma}\sum_{\substack{F\subset M^{T\times S}\\ \mu_T(F)\in\Gamma}}\int\limits_F \frac{i_F^*(\beta e^{\omega-\mu_{T\times S}+\epsilon\rho})}{e_{T\times S}\mathcal{N}(F\,|\,M)}\\ &= \mathrm{EqRes}^{\Lambda}\mathcal{I}_{old}e^{\epsilon\rho}.\end{aligned}$$

To prove (iv) recall that

$$\mathcal{I}_{new}e^{\epsilon\rho} = \sum_{\substack{D\subset (M_\Gamma)^{T\times S}\\ \mu_T(D)\in\partial\Gamma\setminus\{0\}}} \frac{1}{\mathfrak{m}(D)} \int_D \frac{i_D^*\Delta(\beta)e^{i_D^*\omega_\Gamma-\mu_{T\times S}(D)+\epsilon\rho}}{e_{T\times S}\mathcal{N}(D\,|\,M_\Gamma)}.$$

By Lemma 2.3 any $D\subset (M_\Gamma)^{T\times S}$ fixed point component is of form $D=F_{[m,z]}=(F_m\times F_z)/\!\!/T_{diag}$ such that $F_m\subset M^{(T\times S)_m}$ and $F_z\subset(\mathbb{C}^r)^{T_z}$ are connected components containing m and z, respectively, where $(T\times S)_m\subset T\times S$ and $T_z\subset T$ are maximal subtori fixing m and z, respectively. Recall that $\mu_T(D)=\mu_T(m)$. Let $J_z=\{j=1,\dots,r\,|\,z_j=0\}$, where $z=(z_1,\dots,z_r)$. The condition $\mu_T(D)\in\partial\Gamma\setminus\{0\}$ implies that J_z is a proper subset of $\{1,\dots,r\}$. Recall that we have splitting $\varrho:(\mathfrak{t}\oplus\mathfrak{s})^*_m\oplus\mathfrak{t}^*_z\to(\mathfrak{t}\oplus\mathfrak{s})^*$ and let $\sigma=(\varrho\circ\mathrm{pr}_{\mathfrak{t}^*_z}):\mathfrak{t}^*\to(\mathfrak{t}\oplus\mathfrak{s})^*$, where $\mathrm{pr}_{\mathfrak{t}^*_z}:\mathfrak{t}^*\to\mathfrak{t}^*_z$. By Splitting Principle and (2.6) we have

$$e_{T\times S}\mathcal{N}(F_{[m,z]}\,|\,M_\Gamma)=\prod_{i\in I}(\varrho(\alpha_i)+e(\mathcal{N}_i/\!\!/T_{diag}))\prod_{j\in J_z}(\sigma(\gamma_j)+e(\mathcal{L}_j/\!\!/T_{diag}),$$

assuming the splitting

$$\mathcal{N}(F_{[m,z]}\,|\,M_\Gamma)=\bigoplus_{i\in I}\mathcal{N}_i/\!\!/T_{diag}\bigoplus_{j\in J_z}\mathcal{L}_j/\!\!/T_{diag}$$

to complex line bundles with respect to an invariant compatible almost complex structure on M. Recall that $\alpha_i\in(\mathfrak{t}\oplus\mathfrak{s})^*_m$ for all $i\in I$ and $\gamma_j\in\mathfrak{t}^*_z$ for all $j\in J_z$. Hence, the integral $\frac{1}{\mathfrak{m}(D)}\int_D\frac{i_D^*\Delta(\beta)e^{i_D^*\omega_\Gamma-\mu_{T\times S}(D)+\epsilon\rho}}{e_{T\times S}\mathcal{N}(D\,|\,M_\Gamma)}$ is of form $\frac{Pe^{-\mu_{T\times S}(m)+\epsilon\rho}}{\prod_{l\in L}\eta_l}$, where $P\in\mathbb{R}[\mathfrak{t}\oplus\mathfrak{s}]$ and $\eta_l\in\{\varrho(\alpha_i),\sigma(\gamma_j)\,|\,i\in I,\ j\in J_z\}$ for all $l\in L$. Denote $\overline{\eta_l}$ the polarization of η_l with respect to Λ. Remark that by the choice of Λ

$$\mathrm{pr}_{\mathfrak{t}^*}(\overline{\sigma(\gamma_j)})=\overline{\mathrm{pr}_{\mathfrak{t}^*}(\sigma(\gamma_j))},\tag{4.5}$$

since $\mathrm{pr}_{\mathfrak{t}^*}(\sigma(\gamma_j))\neq 0$. By Corollary 3.40 it is enough to show that

$$-\mu_T(m)+\epsilon\rho\notin Cone(\mathrm{pr}_{\mathfrak{t}^*}(\overline{\eta_l})\,|\,l\in L),$$

that is,

$$0\notin\mu_T(m)-\epsilon\rho+Cone(\mathrm{pr}_{\mathfrak{t}^*}(\overline{\varrho(\alpha_i)})\,|\,i\in I)+Cone(\mathrm{pr}_{\mathfrak{t}^*}(\overline{\sigma(\gamma_j)})\,|\,j\in J_z).\tag{4.6}$$

Moreover, by (2.7) we have $\mathrm{span}\big\langle\mathrm{pr}_{\mathfrak{t}^*}(\overline{\varrho(\alpha_i)})\,|\,i\in I\big\rangle\subset\mathrm{span}\big\langle\gamma_j\,|\,j\notin J_z\big\rangle$, therefore (4.6) holds if

$$0\notin\mu_T(m)-\epsilon\rho+\mathrm{span}\langle\gamma_j\,|\,j\notin J_z\rangle+Cone\big(\mathrm{pr}_{\mathfrak{t}^*}(\overline{\sigma(\gamma_j)})\,|\,j\in J_z\big).\tag{4.7}$$

On the other hand, by (2.8) we have

$$\epsilon\rho\in\operatorname{int}\Gamma\subset\mathrm{span}\langle\gamma_j\,|\,j\notin J_z\rangle+\operatorname{int}Cone(\mathrm{pr}_{\mathfrak{t}^*}(\sigma(\gamma_j))\,|\,j\in J_z)$$

and by Remark 2.5(i) we have $\mu_T(m)\in Cone(\gamma_j\,|\,j\notin J_z)$, therefore

$$0\in\mu_T(m)-\epsilon\rho+\mathrm{span}\langle\gamma_j\,|\,j\notin J_z\rangle+\operatorname{int}Cone(\mathrm{pr}_{\mathfrak{t}^*}(\sigma(\gamma_j))\,|\,j\in J_z).\tag{4.8}$$

Comparing (4.7) and (4.8) it is enough to show that

$$Cone(\mathrm{pr}_{\mathfrak{t}^*}(\overline{\sigma(\gamma_j)})\,|\,j\in J_z)\cap \operatorname{int} Cone(\mathrm{pr}_{\mathfrak{t}^*}(\sigma(\gamma_j))\,|\,j\in J_z)=\emptyset. \tag{4.9}$$

Since $Cone(\mathrm{pr}_{\mathfrak{t}^*}(\sigma(\gamma_j))\,|\,j\in J_z)$ is simplicial, by (4.5) we only need to show that $\overline{\sigma(\gamma_j)}=-\sigma(\gamma_j)$ for some $j\in J_z$.

Since $F_m\subset M^{(T\times S)_m}$ is a compact symplectic submanifold, the wall $\mu_{T\times S}(F_m)$ is the convex hull of $\mu_{T\times S}\big((F_m)^{T\times S}\big)$ by [1]. From Remark 2.5(ii) follows that $\mu_{T\times S}(m)\in \mathrm{pr}_{\mathfrak{t}^*}^{-1}Cone(\gamma_j\,|\,j\notin J_z)\cap\mu_{T\times S}(F_m)$, and by (Γ3) we have that $\mu_{T\times S}(F_m)\cap(t^1)^{<0}\neq\emptyset$ and $\mu_{T\times S}(F_m)\cap(t^1)^{\geq 0}\neq\emptyset$, where $(t^1)^{\geq 0}=\{u\in\mathfrak{t}^*\oplus\mathfrak{s}^*\,|\,u(t^1)\geq 0\}$. Therefore, $\nu:\mu_{T\times S}(F_m)\to\mathbb{R}$, $\nu(p)=\langle p,t^1\rangle$ is a non-trivial convex function. It takes its minimum on a proper face $\mathcal{F}$ of $\mu_{T\times S}(F_m)$. Moreover, $\mu_{T\times S}(m)$ is in the relative interior of the polytope $\mu_{T\times S}(F_m)$, hence it cannot be a minimum point of ν. By (2.9) we have

$$\mu_{T\times S}(F_m)\cap\mathrm{pr}_{\mathfrak{t}^*}^{-1}(\Gamma)\subset\mu_{T\times S}(m)+Cone(\sigma(\gamma_j)\,|\,j\in J_z).$$

If $p\in\mathcal{F}$ then $p=\mu_{T\times S}(m)+\sum\limits_{j\in J_z}a_j\sigma(\gamma_j)$ with $a_j\geq 0$ for all $j\in J_z$, hence

$$\nu(p)=\nu(\mu_{T\times S}(m))+\sum_{j\in J_z}a_j\nu(\sigma(\gamma_j)).$$

If $\sigma(\gamma_j)=\overline{\sigma(\gamma_j)}$ for all $j\in J_z$ then $\nu(\sigma(\gamma_j))\geq 0$ for all $j\in J_z$, thus $\nu(p)\geq\nu(\mu_{T\times S}(m))$, which leads to contradiction that $\mu_{T\times S}(m)$ is an inner point of $\mu_{T\times S}(F_m)$.

We conclude the proof of the theorem by proving the relation (v). We have

$$\oint\limits_{M/\!\!/T}\frac{\kappa_S(\beta e^{\omega-\mu_{T\times S}})}{\prod\limits_{j=1}^r(\gamma_j-\kappa_S(\gamma_j))}=\sum_{B\subset(M/\!\!/T)^S}\frac{1}{\mathfrak{m}(B)}\int\limits_B\frac{i_B^*\kappa_S(\beta e^{\omega-\mu_{T\times S}})}{e_S\mathcal{N}(B\,|\,M/\!\!/T)\prod\limits_{j=1}^r(\gamma_j-i_B^*\kappa_S(\gamma_j))}.$$

We can write $\gamma_j-i_B^*\kappa_S(\gamma_j)=\gamma_j+\zeta_j-i_B^*\kappa(\gamma_j)$, where $\zeta_j\in\mathfrak{s}^*$ and $\kappa:H_T(M)\to H(M/\!\!/T)$ is the Kirwan map. Then

$$\begin{aligned}&\frac{1}{\mathfrak{m}(B)}\int\limits_B\frac{i_B^*\kappa_S(\beta e^{\omega-\mu_{T\times S}})}{e_S\mathcal{N}(B\,|\,M/\!\!/T)\prod\limits_{j=1}^r(\gamma_j-i_B^*\kappa_S(\gamma_j))}\\&\quad=\sum_{k_1,\dots,k_r\geq 0}\frac{1}{\mathfrak{m}(B)}\int\limits_B\frac{i_B^*\kappa_S(\beta e^{\omega-\mu_{T\times S}})}{e_S\mathcal{N}(B\,|\,M/\!\!/T)}\prod_{j=1}^r\frac{i_B^*\kappa(\gamma_j)^{k_j}}{(\gamma_j+\zeta_j)^{k_j+1}}=\sum_{k_1,\dots,k_r\geq 0}\frac{P_{B,k_1,\dots,k_r}e^{-\mu_S(B)}}{\prod\limits_{j=1}^r(\gamma_j+\zeta_j)^{k_j+1}},\end{aligned}$$

since $\mu_{T\times S}(B)=\mu_S(B)$ for $B\subset(M/\!\!/T)^S$, and where $P_{B,k_1,\dots,k_r}$ are rational functions on $\mathfrak{s}$ such that

$$P_{B,k_1,\dots,k_r}e^{-\mu_S(B)}=\frac{1}{\mathfrak{m}(B)}\int\limits_B\frac{i_B^*\Big[\kappa_S\big(\beta e^{\omega-\mu_{T\times S}}\big)\kappa\Big(\prod\limits_{j=1}^r\gamma_j^{k_j}\Big)\Big]}{e_S\mathcal{N}(B\,|\,M/\!\!/T)}.$$

We remark that for degree reason only finitely many of these functions are non-zero. By (Γ1) the vectors $\gamma_1+\zeta_1,\dots,\gamma_r+\zeta_r$ are polarized with respect to $-\Lambda$ and we can write

$$\epsilon\rho = \epsilon\rho_1(\gamma_1+\zeta_1)+\ldots+\epsilon\rho_r(\gamma_r+\zeta_r)-\epsilon\sum_{j=1}^{r}\rho_j\zeta_j$$

with $\rho_1,\dots,\rho_r>0$. Therefore, by Proposition 3.39 we get

$$\mathrm{EqRes}^{-\Lambda}\left(\frac{P_{B,k_1,\dots,k_r}e^{-\mu_S(B)+\epsilon\rho}}{\prod\limits_{j=1}^{r}(\gamma_j+\zeta_j)^{k_j+1}}\right)=\frac{P_{B,k_1,\dots,k_r}e^{-\mu_S(B)-\epsilon\sum\limits_{j=1}^{r}\rho_j\zeta_j}}{\sqrt{\det[(\gamma_i+\zeta_i,\,\gamma_j+\zeta_j)]_{i,j=1}^{r}}}\cdot\frac{(\epsilon\rho_1)^{k_1}\cdots(\epsilon\rho_r)^{k_r}}{k_1!\cdots k_r!}.$$

Let $\{\tau_1,\dots,\tau_r\}$ be a basis of the lattice $\mathfrak{t}^*_{\mathbb{Z}}$ and let $\{\vartheta_1,\dots,\vartheta_r\}$ be an orthonormal basis of $\mathfrak{t}^*$. By definition $(\gamma_i+\zeta_i,\,\gamma_j+\zeta_j)=(\gamma_i,\,\gamma_j)$ for all $i,j=1,\dots,r$, hence

$$\sqrt{\det[(\gamma_i+\zeta_i,\,\gamma_j+\zeta_j)]_{i,j=1}^{r}}=\sqrt{\det[(\gamma_i,\,\gamma_j)]_{i,j=1}^{r}}=\left|\det\left(\frac{\partial\gamma_i}{\partial\tau_j}\right)\right|\cdot\left|\det\left(\frac{\partial\tau_j}{\partial\vartheta_l}\right)\right|=\delta_\Gamma\cdot vol(T)^{-1}.$$

Finally, we compute

$$\begin{aligned}
\lim_{\epsilon\to0^+}\mathrm{EqRes}^{-\Lambda}\mathcal{I}_{red}e^{\epsilon\rho} &= \lim_{\epsilon\to0^+}\mathrm{EqRes}^{-\Lambda}\sum_{B\subset(M/\!\!/T)^S}\frac{1}{\mathfrak{m}(B)}\int_B\frac{i_B^*\kappa_S(\beta e^{\omega-\mu_{T\times S}})e^{\epsilon\rho}}{e_S\mathcal{N}(B\,|\,M/\!\!/T)\prod\limits_{j=1}^{r}(\gamma_j-i_B^*\kappa_S(\gamma_j))}\\
&=\lim_{\epsilon\to0^+}\sum_B\sum_{k_1,\dots,k_r\ge0}\mathrm{EqRes}^{-\Lambda}\frac{P_{B,k_1,\dots,k_r}e^{-\mu_S(B)+\epsilon\rho}}{\prod\limits_{j=1}^{r}(\gamma_j+\zeta_j)^{k_j+1}}\\
&=\lim_{\epsilon\to0^+}\sum_B\sum_{k_1,\dots,k_r\ge0}\frac{vol(T)}{\delta_\Gamma}\frac{(\epsilon\rho_1)^{k_1}\cdots(\epsilon\rho_r)^{k_r}}{k_1!\cdots k_r!}P_{B,k_1,\dots,k_r}e^{-\mu_S(B)-\epsilon\sum\limits_{j=1}^{r}\rho_j\zeta_j}\\
&=\frac{vol(T)}{\delta_\Gamma}\sum_{B\subset(M/\!\!/T)^S}P_{B,0,\dots,0}e^{-\mu_S(B)}\\
&=\frac{vol(T)}{\delta_\Gamma}\sum_{B\subset(M/\!\!/T)^S}\frac{1}{\mathfrak{m}(B)}\int_B\frac{i_B^*\kappa_S(\beta e^{\omega-\mu_{T\times S}})}{e_S\mathcal{N}(B\,|\,M/\!\!/T)}\\
&=\frac{vol(T)}{\delta_\Gamma}\oint_{M/\!\!/T}\kappa_S(\beta e^{\omega-\mu_{T\times S}}).
\end{aligned}$$

□

4.1.2 Non-compact and abelian case

In this section we will prove Theorem 4.7 in the abelian case, namely

$$\oint_{M/\!\!/T}\kappa_S(\beta e^{\omega-\mu_{T\times S}})=\frac{1}{vol(T)}\mathrm{EqRes}^{\Lambda}\oint_M\beta e^{\omega-\mu_{T\times S}}. \tag{4.10}$$

We will approximate M by a compact symplectic space using symplectic cut with respect to the moment map μ_K. Denote $\varphi':M/\!\!/T\to\mathbb{R}$ and $\varphi'':M/\!\!/G\to\mathbb{R}$ the functions induced by φ on the quotients. They are also proper and bounded below.

Lemma 4.8. *There exists $\varepsilon \in \mathbb{R}$ such that*

(E1) $M^{T\times S} \subset \varphi^{-1}(-\infty, \varepsilon)$,

(E2) $(M/\!\!/T)^S \subset (\varphi')^{-1}(-\infty, \varepsilon)$,

(E3) $(M/\!\!/G)^S \subset (\varphi'')^{-1}(-\infty, \varepsilon)$,

(E4) ε is a regular value of φ, φ' and φ'',

(E5) if the supporting affine plane $\mathcal{W}$ of a wall of $\mu_{T\times S}(M)$ intersects $\mathfrak{s}^*$ in a point, that is $\mathcal{W} \cap \mathfrak{s}^* = \{p\}$, then we have $\chi(p) < \varepsilon$.

Proof. Since φ, φ' and φ'' are K-moment maps and K is 1-dimensional, therefore $\varphi(M^K)$, $\varphi'((M/\!\!/T)^K)$ and $\varphi''((M/\!\!/G)^K)$ are the set of critical values of φ, φ' and φ'', respectively. Moreover, these three functions are constant on K-fixed point components, hence

$$\varphi(M^K) = \varphi\left(M^{T\times S}\right), \qquad \varphi'((M/\!\!/T)^K) = \varphi'\left((M/\!\!/T)^S\right), \qquad \varphi''((M/\!\!/G)^K) = \varphi\left((M/\!\!/G)^S\right)$$

and they are finite by Proposition 4.3.

By Lemma 1.39 we have only finitely many supporting affine planes $\mathcal{W}$, and they yield finitely many values $\chi(p)$ with $\mathcal{W} \cap \mathfrak{s}^* = \{p\}$. Any value ε bigger than all above will satisfy the required properties. $\square$

Consider an $S' \subset S$ subtorus such that $S' \times K \to S$, $(s', k) \mapsto s'k$ is a finite cover. It yields a finite cover $T \times S' \times K \to T \times S$, $(t, s', k) \mapsto (t, s'k)$, which induces isomorphisms $\Phi^* : \mathfrak{t}^* \oplus \mathfrak{s}^* \to \mathfrak{t}^* \oplus (\mathfrak{s}')^* \oplus \mathfrak{k}^*$ and $\Phi^* : H_{T\times S}(M) \to H_{T\times S'\times K}(M)$. This latter isomorphism commutes with equivariant integration, reduction and equivariant residue, more precisely

$$\Phi^*\left(\oint_{M/\!\!/T} \kappa_S(\beta e^{\omega-\mu_{T\times S}})\right) = \oint_{M/\!\!/T} \Phi^*\big(\kappa_S(\beta e^{\omega-\mu_{T\times S}})\big) = \oint_{M/\!\!/T} \kappa_S\big(\Phi^*(\beta e^{\omega-\mu_{T\times S}})\big)$$

and

$$\Phi^*\left(\mathrm{EqRes}^{\Lambda} \oint_M \beta e^{\omega-\mu_{T\times S}}\right) = \mathrm{EqRes}^{\Lambda}\Phi^*\left(\oint_M \beta e^{\omega-\mu_{T\times S}}\right) = \mathrm{EqRes}^{\Lambda} \oint_M \Phi^*(\beta e^{\omega-\mu_{T\times S}}),$$

therefore we may suppose that $S = S' \times K$. Moreover, for any $\xi \in \mathfrak{s}^*$ multiplication by e^{ξ} also commutes with equivariant integration, reduction and equivariant residue, more precisely

$$\oint_{M/\!\!/T} \kappa_S(\beta e^{\omega-\mu_{T\times S}+\xi}) = e^{\xi} \oint_{M/\!\!/T} \kappa_S(\beta e^{\omega-\mu_{T\times S}})$$

and

$$\mathrm{EqRes}^{\Lambda} \oint_M \beta e^{\omega-\mu_{T\times S}+\xi} = e^{\xi}\mathrm{EqRes}^{\Lambda} \oint_M \beta e^{\omega-\mu_{T\times S}},$$

therefore we may suppose that $\varepsilon = 0$ satisfies conditions of Lemma 4.8 by replacing the moment map $\mu_{T\times S}$ by $\mu_{T\times S} - \varepsilon\gamma$.

From now on we will suppose that $S = S' \times K$ and $\varepsilon = 0$. Consider the cone $\Sigma = \mathbb{R}_{\leq 0}\gamma$ and construct the symplectic cut $X = M_\Sigma$. It is a $(G \times S)$-Hamiltonian manifold (orbifold) with moment map $\phi_{G\times S} : X \to \mathfrak{g}^* \oplus \mathfrak{s}^*$ induced by $\mu_{G\times S}$ and denote ω_X the reduced symplectic form on X.

Lemma 4.9. *(i) If $0 \in \mathfrak{t}^*$ is a regular value of μ_T then it is a regular value of ϕ_T, too.*

(ii) If $0 \in \mathfrak{g}^*$ is a regular value of μ_G then it is a regular value of ϕ_G, too.

Proof. We will only prove the second statement. It is enough to check that $0 \in \mathfrak{g}^*$ is a regular value of ϕ_G on $\varphi^{-1}(0)/K$, which is equivalent to G acts locally freely on $\phi_G^{-1}(0) \cap \varphi^{-1}(0)/K$. This latter holds if and only if $G \times K$ acts locally freely on $\mu_G^{-1}(0) \cap \varphi^{-1}(0)$, that is, when $(0, 0) \in \mathfrak{g}^* \oplus \mathbb{R}$ is a regular value of $\mu_G \times \varphi : M \to \mathfrak{g}^* \oplus \mathbb{R}$. By a similar argument we can show that this holds exactly when $0 \in \mathbb{R}$ is a regular value of $\varphi'' : M/\!\!/G \to \mathbb{R}$, which is assumed by (E4) of Lemma 4.8. $\square$

Denote $X/\!\!/T = \phi_T^{-1}(0)/T$ and $M/\!\!/K = \varphi^{-1}(0)/K = \mu_K^{-1}(0)/K$ the symplectic quotients. They admit S- and $(T \times S)$-actions induced by the actions of the same groups on X and M, respectively. By Corollary 2.14 and (E1) we have

$$\int_X \Delta(\beta e^{\omega-\mu_{T\times S}}) = \oint_M \beta e^{\omega-\mu_{T\times S}} + \oint_{M/\!\!/K} \frac{\kappa_{T\times S'}(\beta e^{\omega-\mu_{T\times S}})}{e_{T\times S}\mathcal{N}(M/\!\!/K \,|\, X)}. \tag{4.11}$$

We have a similar formula on $X/\!\!/T$ by (E2)

$$\begin{aligned}\oint_{X/\!\!/T} \kappa_S\Delta(\beta e^{\omega-\mu_{T\times S}}) &= \oint_{M/\!\!/T} \kappa_S(\beta e^{\omega-\mu_{T\times S}}) + \oint_{(M/\!\!/K)/\!\!/T} \frac{\kappa_{S'}\kappa_{T\times S'}(\beta e^{\omega-\mu_{T\times S}})}{e_S\mathcal{N}\big((M/\!\!/K)/\!\!/T \,|\, X/\!\!/T\big)}\\ &= \oint_{M/\!\!/T} \kappa_S(\beta e^{\omega-\mu_{T\times S}}) + \oint_{(M/\!\!/K)/\!\!/T} \kappa_S\left(\frac{\kappa_{T\times S'}(\beta e^{\omega-\mu_{T\times S}})}{e_{T\times S}\mathcal{N}(M/\!\!/K \,|\, X)}\right),\end{aligned} \tag{4.12}$$

since $e_S\mathcal{N}\big((M/\!\!/K)/\!\!/T \,|\, X/\!\!/T\big) = \kappa_S\big(e_{T\times S}\mathcal{N}(M/\!\!/K \,|\, X)\big)$ by Lemma 2.10. Moreover, by Theorem 4.7 we have

$$\begin{aligned}\oint_{X/\!\!/T} \kappa_S\Delta(\beta e^{\omega-\mu_{T\times S}}) &= \oint_{X/\!\!/T} \kappa_S\big(\Delta(\beta)e^{\omega_X-\phi_{T\times S}}\big) = \mathrm{EqRes}^\Lambda \frac{1}{vol(T)} \oint_X \Delta(\beta)e^{\omega_X-\phi_{T\times S}}\\ &= \mathrm{EqRes}^\Lambda \frac{1}{vol(T)} \oint_X \Delta(\beta e^{\omega-\mu_{T\times S}}),\end{aligned}$$

which yields by (4.11) and (4.12)

$$\begin{aligned}\oint_{M/\!\!/T} \kappa_S(\beta e^{\omega-\mu_{T\times S}}) = \mathrm{EqRes}^\Lambda \frac{1}{vol(T)} \oint_M \beta e^{\omega-\mu_{T\times S}} + \mathrm{EqRes}^\Lambda \frac{1}{vol(T)} \oint_{M/\!\!/K} \frac{\kappa_{T\times S'}(\beta e^{\omega-\mu_{T\times S}})}{e_{T\times S}\mathcal{N}(M/\!\!/K \,|\, X)}\\ - \oint_{(M/\!\!/K)/\!\!/T} \kappa_S\left(\frac{\kappa_{T\times S'}(\beta e^{\omega-\mu_{T\times S}})}{e_{T\times S}\mathcal{N}(M/\!\!/K \,|\, X)}\right).\end{aligned}$$

Therefore, it is enough to show that

$$\mathrm{EqRes}^{\Lambda}\frac{1}{vol(T)}\oint\limits_{M/\!\!/K}\frac{\kappa_{T\times S'}(\beta e^{\omega-\mu_{T\times S}})}{e_{T\times S}\mathcal{N}(M/\!\!/K\,|\,X)}=\oint\limits_{(M/\!\!/K)/\!\!/T}\kappa_S\left(\frac{\kappa_{T\times S'}(\beta e^{\omega-\mu_{T\times S}})}{e_{T\times S}\mathcal{N}(M/\!\!/K\,|\,X)}\right).$$

Despite its resemblance, the formula of Theorem 4.7 cannot be applied directly. By definition (4.13)

$$\oint\limits_{M/\!\!/K}\frac{\kappa_{T\times S'}(\beta e^{\omega-\mu_{T\times S}})}{e_{T\times S}\mathcal{N}(M/\!\!/K\,|\,X)}=\sum_{D\subset(M/\!\!/K)^{T\times S'}}\frac{1}{\mathfrak{m}(D)}\oint\limits_{D}\frac{i_D^*\kappa_{T\times S'}(\beta e^{\omega-\mu_{T\times S}})}{i_D^*e_{T\times S}\mathcal{N}(M/\!\!/K\,|\,X)e_{T\times S'}\mathcal{N}(D\,|\,M/\!\!/K)}.$$

Recall from section 2.1 that each fixed point component $D\subset(M/\!\!/K)^{T\times S'}$ is of form $D=D_m/\!\!/K$, where $m\in\varphi^{-1}(0)$ is a point such that $(\mathfrak{t}\oplus\mathfrak{s})_m\oplus\mathfrak{k}=\mathfrak{t}\oplus\mathfrak{s}$ and $D_m\subset M$ is the $(T\times S)_m$-fixed point component containing m. We have splitting $\varrho:(\mathfrak{t}\oplus\mathfrak{s})_m^*\oplus\mathfrak{k}^*\to(\mathfrak{t}\oplus\mathfrak{s})^*$ and remark that $\varrho((\mathfrak{t}\oplus\mathfrak{s})_m^*)=(\mathfrak{t}\oplus\mathfrak{s}')^*=\ker\chi$. Then by (2.11)

$$i_D^*e_{T\times S}\mathcal{N}(M/\!\!/K\,|\,X)=\varrho(\gamma)+i_D^*e\mathcal{N}(M/\!\!/K\,|\,X)$$

and by the Splitting Principle we can assume that $\mathcal{N}(D_m\,|\,M)=\oplus_{i\in I_D}\mathcal{N}_i$ is a $(T\times S)$-equivariant splitting to sum of complex line bundles (using an invariant compatible almost complex structure on M), thus by (2.13)

$$e_{T\times S'}\mathcal{N}(D\,|\,M/\!\!/K)=\kappa'_{T\times S'}(e_{T\times S}\mathcal{N}(D_m\,|\,M))=\prod_{i\in I_D}[\varrho(\alpha_i)+\kappa'(e_K(\mathcal{N}_i))],$$

where $\alpha_i\in(\mathfrak{t}\oplus\mathfrak{s})_m^*$ is the $(T\times S)_m$-weight of fibers of $\mathcal{N}_i$ and $\kappa'_{T\times S'}:H_{T\times S}(D_m)\to H_{T\times S'}(D)$, $\kappa':H_K(D_m)\to H(D)$ are Kirwan maps. Therefore, (4.13) has two kinds of $\mathfrak{t}^*$-poles

(1) $V=\mathrm{span}\,\langle\varrho(\alpha_{i_1}),\dots,\varrho(\alpha_{i_r})\rangle$ with $i_1,\dots,i_r\in I_D$, i.e. $V\subset\ker\chi$.

(2) $V=\mathrm{span}\,\langle\varrho(\gamma),\varrho(\alpha_{i_2}),\dots,\varrho(\alpha_{i_r})\rangle$ with $i_2,\dots,i_r\in I_D$, i.e. $V\not\subset\ker\chi$.

Recall that

$$\begin{aligned}\mathrm{EqRes}^{\Lambda}&\frac{1}{vol(T)}\oint\limits_{M/\!\!/K}\frac{\kappa_{T\times S'}(\beta e^{\omega-\mu_{T\times S}})}{e_{T\times S}\mathcal{N}(M/\!\!/K\,|\,X)}\\&=\sum_{V\subset\ker\chi}\mathrm{JKRes}^{\Lambda_V}\left(\frac{1}{vol(T)}\oint\limits_{M/\!\!/K}\frac{\kappa_{T\times S'}(\beta e^{\omega-\mu_{T\times S}})}{e_{T\times S}\mathcal{N}(M/\!\!/K\,|\,X)}\right)(v,s)dv\\&\qquad+\sum_{V\not\subset\ker\chi}\mathrm{JKRes}^{\Lambda_V}\left(\frac{1}{vol(T)}\oint\limits_{M/\!\!/K}\frac{\kappa_{T\times S'}(\beta e^{\omega-\mu_{T\times S}})}{e_{T\times S}\mathcal{N}(M/\!\!/K\,|\,X)}\right)(v,s)dv.\end{aligned}$$

Lemma 4.10. *For all $\mathfrak{t}^*$-poles V of (4.13) such that $V\not\subset\ker\chi$ we have*

$$\mathrm{JKRes}^{\Lambda_V}\left(\frac{1}{vol(T)}\oint\limits_{M/\!\!/K}\frac{\kappa_{T\times S'}(\beta e^{\omega-\mu_{T\times S}})}{e_{T\times S}\mathcal{N}(M/\!\!/K\,|\,X)}\right)(v,s)dv=0.$$

Proof. Let $V = \operatorname{span}\langle \varrho(\gamma), \varrho(\alpha_{i_2}), \dots, \varrho(\alpha_{i_r})\rangle$. There is a wall of $\mu_{T\times S}(M)$ such that its supporting affine plane is $\mu_{T\times S}(m)+V$. We construct this wall as follows. Let $H \subset (T\times S)_m \subset T\times S$ be a subtorus such that $\mathfrak{h} = \cap_{j=2}^{r} \ker \alpha_{i_j}$ and let $N \subset M^H$ be the fixed point component containing m. By Example 1.6 the supporting affine plane of $\mu_{T\times S}(N)$ is $\mu_{T\times S}(m) + \ker((\mathfrak{t}\oplus\mathfrak{s})^* \to \mathfrak{h}^*) = \mu_{T\times S}(m) + V$.

Finally, since V is a $\mathfrak{t}^*$-pole, we have $(\mu_{T\times S}(m)+V)\cap\mathfrak{s}^* = \{p\}$. If we write $\mu_{T\times S}(m) = \lambda_s+\lambda_v$ with $\lambda_s \in \mathfrak{s}^*$ and $\lambda_v \in V$ then $\lambda_s = p$. Since $m \in \varphi^{-1}(0)$, we have $\chi(\lambda_s)+\chi(\lambda_v) = \chi(\mu_{T\times S}(m)) = 0$, thus $\chi(\lambda_v) = -\chi(p) > 0$ by (E5) and assumption $\varepsilon = 0$. Hence $-\mu_{T\times S}(m) = -\mu_{T\times S}(D)$ is not polarized with respect to Λ and the lemma follows by Corollary 3.41. $\square$

Lemma 4.11.

(4.14)
$$\sum_{V\subset\ker\chi} \operatorname{JKRes}^{\Lambda_V}\left[\frac{1}{vol(T)}\oint_{M/\!\!/K} \frac{\kappa_{T\times S'}(\beta e^{\omega-\mu_{T\times S}})}{e_{T\times S}\mathcal{N}(M/\!\!/K\,|\,X)}\right](v,s)dv = \oint_{(M/\!\!/K)/\!\!/T} \kappa_S\left[\frac{\kappa_{T\times S'}(\beta e^{\omega-\mu_{T\times S}})}{e_{T\times S}\mathcal{N}(M/\!\!/K\,|\,X)}\right].$$

Proof. Let $s' = \{s'_1, \dots, s'_{q-1}\}$ be a basis of $(\mathfrak{s}')^*$ and recall that $\{\gamma\}$ is a basis of $\mathfrak{k}^*$. We prove the equality of fractions on $\mathfrak{s}'\oplus\mathfrak{k}$ on both sides of (4.14) by showing that for any fixed $\delta \in (\mathfrak{s}')^*$ their expansions with respect to $\gamma+\delta \ll s'_1, \dots, s'_{q-1}$ are equal. Moreover, by (2.12) we have

$$e_{T\times S}\mathcal{N}(M/\!\!/K\,|\,X) = \gamma + e_{T\times S'}\mathcal{N}(M/\!\!/K\,|\,X) = \gamma+\delta-\delta+e_{T\times S'}\mathcal{N}(M/\!\!/K\,|\,X),$$

hence let

$$\frac{1}{e_{T\times S}\mathcal{N}(M/\!\!/K\,|\,X)} = \frac{1}{(\gamma+\delta)}\sum_{n\geq 0}\left(\frac{\delta - e_{T\times S'}\mathcal{N}(M/\!\!/K\,|\,X)}{\gamma+\delta}\right)^n. \tag{4.15}$$

Now we can apply Theorem 4.7 on $M/\!\!/K$ with respect to the $(T\times S')$-action, considering $\gamma+\delta$ as non-zero constant. Denote Λ' the polarization induced by Λ on $(\mathfrak{t}\oplus\mathfrak{s}')^*$. Remark that $\kappa_{T\times S'}(\omega-\mu_{T\times S}) = \omega' - \mu'_{T\times S'}$ is the $(T\times S')$-equivariant symplectic form on $M/\!\!/K$, since we assume that $\varepsilon = 0$. Then we have equality of expansions with respect to $\gamma+\delta \ll s'$ by

$$\begin{aligned}
&\sum_{V\subset\ker\chi} \operatorname{JKRes}^{\Lambda'_V}\left(\frac{1}{vol(T)}\oint_{M/\!\!/K}\sum_{n\geq 0}\frac{(\delta - e_{T\times S'}\mathcal{N}(M/\!\!/K\,|\,X))^n}{(\gamma+\delta)^{n+1}}\kappa_{T\times S'}(\beta)e^{\omega'-\mu'_{T\times S'}}\right)(v,s')dv\\
&\qquad = \operatorname{EqRes}^{\Lambda'}\left(\frac{1}{vol(T)}\oint_{M/\!\!/K}\sum_{n\geq 0}\frac{(\delta - e_{T\times S'}\mathcal{N}(M/\!\!/K\,|\,X))^n}{(\gamma+\delta)^{n+1}}\kappa_{T\times S'}(\beta)e^{\omega'-\mu'_{T\times S'}}\right)\\
&\qquad\qquad = \oint_{(M/\!\!/K)/\!\!/T}\kappa_S\left(\sum_{n\geq 0}\frac{(\delta - e_{T\times S'}\mathcal{N}(M/\!\!/K\,|\,X))^n}{(\gamma+\delta)^{n+1}}\kappa_{T\times S'}(\beta)e^{\omega'-\mu'_{T\times S'}}\right).
\end{aligned}$$

$\square$

Hereby we have showed (4.10).

4.1.3 Passing from abelian to the non-abelian case

We deduce the non-abelian version following closely [32]. Recall that $0 \in \mathfrak{g}^*$ is a regular value of $\mu_G : M \to \mathfrak{g}^*$. First we assume that $0 \in \mathfrak{t}^*$ is a regular value of $\mu_T : M \to \mathfrak{t}^*$, too. Then from Lemma 4.9 follows that 0 is also a regular value of ϕ_G and ϕ_T. Denote $X/\!\!/G = \phi_G^{-1}(0)/G$ and $X/\!\!/T = \phi_T^{-1}(0)/T$ the symplectic quotients.

We make a choice of positive roots of G. Denote $\mathcal{R}^+$ and $\mathcal{R}^-$ the set of positive and negative roots, respectively. Set $\mathfrak{v}^* = \ker(\mathfrak{g}^* \to \mathfrak{t}^*)$ and fix an isomorphism of real G-representations

$$\mathfrak{v}^* \simeq \bigoplus_{\alpha \in \mathcal{R}^-} \mathbb{C}_\alpha, \tag{4.16}$$

where $\mathbb{C}_\alpha$ is the 1-dimensional representation on which T-acts by weight α. We have a natural orientation on each $\mathbb{C}_\alpha$, therefore isomorphism (4.16) fixes the orientation of $\mathfrak{v}^*$. Moreover, isomorphism (4.16) induces an isomoprhism of real G-representations

$$\mathfrak{v} \simeq \bigoplus_{\alpha \in \mathcal{R}^+} \mathbb{C}_\alpha, \tag{4.17}$$

which also fixes the orientation of $\mathfrak{v}$. Consider the $(T \times S)$-equivariant map $\sigma : X \to \mathfrak{v}^*$ defined by

$$\begin{tikzcd} X \arrow[r, "\phi_G"] \arrow[dr, "\sigma"'] & \mathfrak{g}^* \arrow[d] \\ & \mathfrak{v}^* \end{tikzcd}$$

where S acts trivially on $\mathfrak{g}^*$ and $\mathfrak{g}^* \to \mathfrak{v}^*$ is the orthogonal projection induced by a T-invariant scalar product on $\mathfrak{g}^*$. This scalar product also yields splitting $\mathfrak{g}^* = \mathfrak{t}^* \oplus \mathfrak{v}^*$, and under this splitting $\phi_G = (\phi_T, \sigma)$. The map σ induces an S-equivariant section $\widetilde{\sigma}$ of the associated bundle

$$E^- = (X \times \mathfrak{v}^*)/\!\!/T \simeq X \times \bigoplus_{\alpha \in \mathcal{R}^-} \mathbb{C}_\alpha.$$

over $X/\!\!/T$. It is a transversal section, because $0 \in \mathfrak{g}^*$ is a regular value of ϕ_G. If we denote $Z = \phi_G^{-1}(0)$ then Z/T is the zero set of $\widetilde{\sigma}$ and it is a submanifold of $X/\!\!/T$. We consider the following diagram

$$\begin{tikzcd} Z/T \arrow[r, "i"] \arrow[d, "\pi"] & X/\!\!/T \\ Z/G = X/\!\!/G & \end{tikzcd}$$

Remark that on $X/\!\!/T$ we have a canonical orientation induced by the reduced symplectic form and the orientation of $\mathfrak{v}^*$ determines the orientations of the fibers of E^-. These two orientations yield the orientation on Z/T. Moreover, we also have a canonical orientation on $X/\!\!/G$ induced by the reduced symplectic form and it fixes the orientation of fibers of π. The vertical subbundle $\ker d\pi \subset T(Z/T)$ is isomorphic to $(E^+|_{Z/T})$, where

$$E^+ = (X \times \mathfrak{v})/\!\!/T \simeq X \times \bigoplus_{\alpha \in \mathcal{R}^+} \mathbb{C}_\alpha.$$

Remark that this isomorphism is orientation preserving.

We will use different style of notations for Kirwan maps. Before, in the subscript we have indicated the group which acts on the quotient, now we will indicate the group by which we divide. Denote $\mathcal{K}'_T : H_{T\times S}(X) \to H_S(X/\!\!/T)$ and $\mathcal{K}'_G : H_{G\times S}(X) \to H_S(X/\!\!/G)$ the Kirwan maps. As in [32] for any $\beta_X \in H_{G\times S}(X) \simeq H_{T\times S}(X)^W$ we compute

$$\begin{aligned}
\int_{X/\!\!/G} \mathcal{K}'_G(\beta_X) &= \frac{1}{|W|}\int_{Z/T} e_S(E^+|_{Z/T})\pi^*\mathcal{K}'_G(\beta_X) && \text{by Lemma 4.12}\\
&= \frac{1}{|W|}\int_{Z/T} i^*\big[e_S(E^+)\mathcal{K}'_T(\beta_X)\big] && \pi^*\mathcal{K}'_G = i^*\mathcal{K}'_T\\
&= \frac{1}{|W|}\int_{X/\!\!/T} e_S(E^-)e_S(E^+)\mathcal{K}'_T(\beta_X) && \text{by Lemma 4.13}\\
&= \frac{1}{|W|}\int_{X/\!\!/T} \mathcal{K}'_T(\varpi\,\beta_X), && \text{by } e_S(E^-E^+) = \mathcal{K}'_T(\varpi),
\end{aligned} \tag{4.18}$$

where ϖ is the product of all roots of G. By (4.12) for any $\beta \in H_{G\times S}(M)$ we have

$$\oint_{M/\!\!/T} \mathcal{K}_T(\varpi\,\beta) = \int_{X/\!\!/T} \mathcal{K}'_T\Delta(\varpi\,\beta) - \oint_{(M/\!\!/K)/\!\!/T} \frac{\mathcal{K}''_T\mathcal{K}''_K(\varpi\,\beta)}{e_S\mathcal{N}((M/\!\!/K)/\!\!/T\,|\,X/\!\!/T)}, \tag{4.19}$$

where $\mathcal{K}_T : H_{T\times S}(M) \to H_S(M/\!\!/T)$, $\mathcal{K}''_T : H_{T\times S'}(M/\!\!/K) \to H_{S'}((M/\!\!/K)/\!\!/T)$ and $\mathcal{K}''_K : H_{T\times S}(M) \to H_{T\times S'}(M/\!\!/K)$ are Kirwan maps. Similarly,

$$\oint_{M/\!\!/G} \mathcal{K}_G(\beta) = \int_{X/\!\!/G} \mathcal{K}'_G\Delta(\beta) - \oint_{(M/\!\!/K)/\!\!/G} \frac{\mathcal{K}''_G\mathcal{K}''_K(\beta)}{e_S\mathcal{N}((M/\!\!/K)/\!\!/G\,|\,X/\!\!/G)}, \tag{4.20}$$

where $\mathcal{K}_G : H_{G\times S}(M) \to H_S(M/\!\!/G)$ and $\mathcal{K}''_G : H_{G\times S'}(M/\!\!/K) \to H_{S'}((M/\!\!/K)/\!\!/G)$. Let $s' = \{s'_1, \ldots, s'_{q-1}\}$ be a basis of $(\mathfrak{s}')^*$. For any $\delta \in (\mathfrak{s}')^*$ and expansion $\gamma + \delta \ll s'_1, \ldots, s'_{q-1}$ of $\dfrac{1}{e_{G\times S}\mathcal{N}(M/\!\!/K\,|\,X)}$ we have by (4.18)

$$\begin{aligned}
&\int_{(M/\!\!/K)/\!\!/G} \mathcal{K}''_G\Big(\sum_{n\ge 0}\frac{(\delta - e_{G\times S'}\mathcal{N}(M/\!\!/K\,|\,X))^n}{(\gamma+\delta)^{n+1}}\mathcal{K}''_K(\beta)\Big)\\
&\qquad\qquad = \frac{1}{|W|}\int_{(M/\!\!/K)/\!\!/T} \mathcal{K}''_T\Big(\varpi\sum_{n\ge 0}\frac{(\delta - e_{T\times S'}\mathcal{N}(M/\!\!/K\,|\,X))^n}{(\gamma+\delta)^{n+1}}\mathcal{K}''_K(\beta)\Big),
\end{aligned}$$

therefore

$$\begin{aligned}
\oint_{(M/\!\!/K)/\!\!/G} \frac{\mathcal{K}''_G\mathcal{K}''_K(\beta)}{e_S\mathcal{N}((M/\!\!/K)/\!\!/G\,|\,X/\!\!/G)} &= \oint_{(M/\!\!/K)/\!\!/G} \mathcal{K}''_G\left(\frac{\mathcal{K}''_K(\beta)}{e_{G\times S}\mathcal{N}(M/\!\!/K\,|\,X)}\right)\\
= \frac{1}{|W|}\oint_{(M/\!\!/K)/\!\!/T} \mathcal{K}''_T\left(\frac{\varpi\mathcal{K}''_K(\beta)}{e_{T\times S}\mathcal{N}(M/\!\!/K\,|\,X)}\right) &= \frac{1}{|W|}\oint_{(M/\!\!/K)/\!\!/T} \frac{\mathcal{K}''_T\mathcal{K}''_K(\varpi\beta)}{e_S\mathcal{N}((M/\!\!/K)/\!\!/T\,|\,X/\!\!/T)},
\end{aligned} \tag{4.21}$$

because $\mathcal{K}''_K(\varpi\beta) = \varpi\,\mathcal{K}''_K(\beta)$, since $\varpi \in (S\mathfrak{t}^*)^W$. Moreover, $K \subset S$ implies that the homomorphism $\Delta : H_{T\times S}(M) \to H_{T\times S}(X)$ is H_T-linear, hence

$$\varpi\Delta(\beta) = \Delta(\varpi\beta). \tag{4.22}$$

Furthermore,

$$\begin{aligned}
\oint_{M/\!/G} \mathcal{K}_G(\beta) &= \int_{X/\!/G} \mathcal{K}'_G\Delta(\beta) - \oint_{(M/\!/K)/\!/G} \frac{\mathcal{K}''_G\mathcal{K}''_K(\beta)}{e_S\mathcal{N}((M/\!/K)/\!/G\,|\,X/\!/G)} && \text{by (4.20)}\\
&= \frac{1}{|W|}\int_{X/\!/T} \mathcal{K}'_T(\varpi\Delta(\beta)) - \frac{1}{|W|}\oint_{(M/\!/K)/\!/T} \frac{\mathcal{K}''_T\mathcal{K}''_K(\varpi\beta)}{e_S\mathcal{N}((M/\!/K)/\!/T\,|\,X/\!/T)} && \text{by (4.18)\&(4.21)}\\
&= \frac{1}{|W|}\int_{X/\!/T} \mathcal{K}'_T\Delta(\varpi\beta) - \frac{1}{|W|}\oint_{(M/\!/K)/\!/T} \frac{\mathcal{K}''_T\mathcal{K}''_K(\varpi\beta)}{e_S\mathcal{N}((M/\!/K)/\!/T\,|\,X/\!/T)} && \text{by (4.22)}\\
&= \frac{1}{|W|}\oint_{M/\!/T} \mathcal{K}_T(\varpi\beta) && \text{by (4.19).} \qquad (4.23)
\end{aligned}$$

By (4.10) and (4.23) follows that

$$\oint_{M/\!/G} \mathcal{K}_G(\beta e^{\omega-\mu_{G\times S}}) = \frac{1}{|W|}\oint_{M/\!/T} \mathcal{K}_T(\varpi\beta e^{\omega-\mu_{T\times S}}) = \mathrm{EqRes}^{\Lambda}\frac{\varpi}{|W|vol(T)}\oint_M \beta e^{\omega-\mu_{T\times S}}.$$

If $0 \in \mathfrak{t}^*$ is not a regular value of μ_T then we choose a regular value ρ close to 0 such that $(\rho, 0) \in \mathfrak{t}^* \oplus \mathfrak{v}^*$ remains a regular value of $\mu_T \times \sigma$. Replacing $Z/T = \phi_G^{-1}(0)/T$ by $Z_{\epsilon\rho}/T = \phi_T^{-1}(\epsilon\rho) \cap \sigma^{-1}(0)/T$ in (4.18) and taking the limit $\epsilon \to 0$ we get

$$\int_{X/\!/G} \mathcal{K}'_G(\beta_X) = \lim_{\epsilon\to 0}\frac{1}{|W|}\int_{X/\!/_{\epsilon\rho}T} \mathcal{K}'_T(\varpi\beta_X)$$

for any $\beta_X \in H_{G\times S}(X)$, and where $X/\!/_{\epsilon\rho}T = \phi_T^{-1}(\epsilon\rho)/T$. Thus, we can similarly show as in the regular case that

$$\oint_{M/\!/G} \mathcal{K}_G(\beta) = \lim_{\epsilon\to 0}\frac{1}{|W|}\oint_{M/\!/_{\epsilon\rho}T} \mathcal{K}_T(\varpi\beta)$$

for all $\beta \in H_{G\times S}(M)$, and where $M/\!/_{\epsilon\rho}T = \mu_T^{-1}(\epsilon\rho)/T$. Finally, by (4.10) we get

$$\begin{aligned}
\oint_{M/\!/G} \mathcal{K}_G(\beta e^{\omega-\mu_{G\times S}}) &= \lim_{\epsilon\to 0}\frac{1}{|W|}\oint_{M/\!/_{\epsilon\rho}T} \mathcal{K}_T(\varpi\beta e^{\omega-\mu_{T\times S}+\epsilon\rho})\\
&= \lim_{\epsilon\to 0}\mathrm{EqRes}^{\Lambda}\frac{\varpi}{|W|vol(T)}\oint_M \beta e^{\omega-\mu_{T\times S}+\epsilon\rho},
\end{aligned}$$

which concludes the proof of Theorem 4.5.

We close this section by showing the following two lemmas, which were used earlier.

Lemma 4.12.
$$\int_{Z/G} \mathcal{K}'_G(\beta) = \frac{1}{|W|} \int_{Z/T} e_S(\ker d\,\pi)\pi^*\mathcal{K}'_G(\beta).$$

Proof. We follow the proof of Theorem B in [32]. $\pi : Z/T \to Z/G$ is a fibration with fiber G/T and π is S-equivariant. For $D \subset (Z/G)^S$ fixed point component we have $\pi^{-1}(D) = F \subset (Z/T)^S$ is also a fixed point component. Moreover, we have S-equivariant isomorphism of normal bundles $\mathcal{N}(F \,|\, Z/T) \simeq \pi^*\mathcal{N}(H \,|\, Z/G)$. Finally, we compute

$$\begin{aligned}
\int_{Z/G} \mathcal{K}'_G(\beta) &= \sum_{D\subset(Z/G)^S} \int_D \frac{i_D^*\mathcal{K}'_G(\beta)}{e_S\mathcal{N}(D\,|\,Z/G)} \\
&= \frac{1}{|W|} \sum_{F\subset(Z/T)^S} \int_F \frac{i_F^*\left(e(\ker d\,\pi)\pi^*\mathcal{K}'_G(\beta)\right)}{e_S\pi^*\mathcal{N}(D\,|\,Z/G)} \qquad \text{by } |W| = \chi(G/T) = \int_{G/T} e(\ker d\pi) \\
&= \frac{1}{|W|} \sum_{F\subset(Z/T)^S} \int_F \frac{i_F^*\left(e_S(\ker d\,\pi)\pi^*\mathcal{K}'_G(\beta)\right)}{e_S\mathcal{N}(F\,|\,Z/T)} \qquad \text{by } e(\ker d\,\pi|_F) = e_S(\ker d\,\pi|_F), \\
&= \frac{1}{|W|} \int_{Z/T} e_S(\ker d\,\pi)\,\pi^*(\mathcal{K}'_G(\beta)).
\end{aligned}$$

□

We have the following equivariant version of Proposition 12.8 of [4].

Lemma 4.13. *Let $E \to X$ be an S-equivariant vector bundle over a compact manifold (orbifold) X. Let σ be an S-equivariant section, transverse to the zero section. Denote $Z = \sigma^{-1}(0)$ the zero set of the section and $i_Z : Z \hookrightarrow X$ the inclusion. For any $\eta \in H_S(X)$ we have*
$$\int_Z i_Z^*\eta = \int_X e_S(E)\eta.$$

Proof. Let $F \subset X^S$ and $D \subset F \cap Z \subset Z^S$ be fixed point components. Denote $i_F : F \hookrightarrow X$, $i_D : D \hookrightarrow Z$ and $j_D : D \hookrightarrow F$ the inclusions. By transversality of σ we have equivariant isomorphism of vector bundles

$$E|_Z \simeq \mathcal{N}(Z\,|\,X). \tag{4.24}$$

We have equivariant decomposition $E|_F = (E|_F)^S \oplus E'$, hence $E|_D = (E|_D)^S \oplus E'|_D$. Moreover, by (4.24) we get

$$\mathcal{N}(Z\,|\,X)|_D \simeq (E|_D)^S \oplus E'|_D. \tag{4.25}$$

The inclusions $D \subset Z \subset X$ give

$$\mathcal{N}(D\,|\,X) \simeq \mathcal{N}(D\,|\,Z) \oplus \mathcal{N}(Z\,|\,X)|_D, \tag{4.26}$$

and by (4.25) we have

$$\mathcal{N}(D\,|\,X) \simeq \mathcal{N}(D\,|\,Z) \oplus (E|_D)^S \oplus E'|_D. \tag{4.27}$$

Moreover, the inclusions $D \subset F \subset X$ yield

$$\mathcal{N}(D\,|\,X) \simeq \mathcal{N}(D\,|\,F) \oplus \mathcal{N}(F\,|\,X)|_D, \tag{4.28}$$

and finally we also have a decomposition

$$\mathcal{N}(D\,|\,X) \simeq \mathcal{N}(D\,|\,X)^S \oplus \mathcal{N}(D\,|\,X)'. \tag{4.29}$$

By equations (4.28) and (4.29) we have $\mathcal{N}(D\,|\,F) \simeq \mathcal{N}(D\,|\,X)^S$, hence

$$\mathcal{N}(F\,|\,X)|_D \simeq \mathcal{N}(D\,|\,X)'. \tag{4.30}$$

D is a fixed component of Z, therefore from equations (4.27) and (4.29) follows that

$$\mathcal{N}(D\,|\,X)' \simeq \mathcal{N}(D\,|\,Z) \oplus E'|_D. \tag{4.31}$$

The isomorphisms (4.30) and (4.31) imply

$$e_S\mathcal{N}(D\,|\,Z) = \frac{j_D^*(e_S\mathcal{N}(F\,|\,X))}{j_D^*(e_S(E'))} \tag{4.32}$$

Finally, we compute

$$\begin{aligned}
\int_Z i_Z^*\eta &= \sum_{D\subset Z^S} \int_D \frac{i_D^*\eta}{e_S\mathcal{N}(D\,|\,Z)} \\
&= \sum_{D\subset Z^S} \int_D \frac{j_D^*(e_S(E'))i_D^*\eta}{j_D^*(e_S\mathcal{N}(F\,|\,X))} && \text{by (4.32)} \\
&= \sum_{F\subset X^S} \int_F \frac{e\left((E|_F)^S\right) e_S(E')i_F^*\eta}{e_S\mathcal{N}(F\,|\,X)} && \text{by Proposition 12.8 of [4]} \\
&= \sum_{F\subset X^S} \int_F \frac{e_S(E|_F)i_F^*\eta}{e_S\mathcal{N}(F\,|\,X)} && \text{by } e((E|_F)^S) = e_S((E|_F)^S) \\
&= \int_X e_S(E)\eta.
\end{aligned}$$

□

4.2 HyperKähler version

We formulate an analogue of Theorem 4.5 for hyperKähler quotients. First we compare the torus hyperKähler quotients to symplectic quotients, then we conclude the formula for non-abelian quotients by [19].

Let M be a hyperKähler manifold with real symplectic form $\omega_{\mathbb{R}}$ and complex symplectic form $\omega_{\mathbb{C}}$. Assume that M admits an action of a compact connected Lie group G which acts on it in a hyper-Hamiltonian manner with hyperKähler moment map $\mu = (\mu_{\mathbb{R}}, \mu_{\mathbb{C}}) : M \to \mathfrak{g}^* \oplus \mathfrak{g}^*_{\mathbb{C}}$. We also consider an additional Hamiltonian S-action on $(M, \omega_{\mathbb{R}})$ which commutes with the G-action and denote $\mu_S : M \to \mathfrak{s}^*$ the S-moment map. As in the symplectic case we assume that there is an 1-dimensional subtorus $K \subset S$ such that its moment map μ_K is proper and non-surjective. Then we can write $\mu_K = \varphi \cdot \gamma$ with a generator $\gamma \in \mathfrak{k}^*_{\mathbb{Z}}$ such that $\varphi : M \to \mathbb{R}$ is proper and bounded below. In hyperKähler case we impose the additional conditions that $\mathfrak{g}^*_{\mathbb{C}}$ is an S-representation such that $\mu_{\mathbb{C}} : M \to \mathfrak{g}^*_{\mathbb{C}}$ is $(G \times S)$-equivariant and $(\mathfrak{t}^*_{\mathbb{C}})^S = \{0\}$, where $\mathfrak{t}$ is the Lie algebra of the maximal torus $T \subset G$ of rank r. We introduce notations $\mu^T_{\mathbb{R}} : M \to \mathfrak{t}^*$ and $\mu^T_{\mathbb{C}} : M \to \mathfrak{t}^*_{\mathbb{C}}$ for the abelian real and complex moment maps, that is, $\mu^T_{\mathbb{R}} = \mathrm{pr}_{\mathfrak{t}^*} \circ \mu_{\mathbb{R}}$ and $\mu^T_{\mathbb{C}} = \mathrm{pr}_{\mathfrak{t}^*} \circ \mu_{\mathbb{C}}$, where $\mathrm{pr}_{\mathfrak{t}^*} : \mathfrak{g}^* \to \mathfrak{t}^*$ is the projection.

Consider a regular value of μ of form $(\xi, 0) \in (\mathfrak{g}^* \oplus \mathfrak{g}^*_{\mathbb{C}})^G$. Then the quotient $M/\!\!/\!\!/\!\!/_{(\xi,0)}G = \mu_{\mathbb{R}}^{-1}(\xi) \cap \mu_{\mathbb{C}}^{-1}(0)/G$ is again a hyperKähler manifold (orbifold) [22].

Theorem 4.14. *For any $\beta \in H_{G\times S}(M) \simeq H_{T\times S}(M)^W$ we have*

$$\oint\limits_{M/\!\!/\!\!/\!\!/_{(\xi,0)}G} \kappa_S(\beta e^{\omega_{\mathbb{R}} - \mu_{\mathbb{R}} - \mu_S + \xi}) = \lim_{\epsilon \to 0} \mathrm{EqRes}^{\Lambda} \frac{\vartheta \varpi_{\mathbb{R}} \varpi_{\mathbb{C}}}{|W| vol(T)} \oint\limits_{M} \beta e^{\omega_{\mathbb{R}} - \mu^T_{\mathbb{R}} - \mu_S + \xi + \epsilon\rho},$$

where $\kappa_S : H_{G\times S}(M) \to H_S(M/\!\!/\!\!/\!\!/_{(\xi,0)}G)$ is the Kirwan map, Λ is a polarization compatible with the proper bounded below moment map, $|W|$ is the rank of the Weyl group, $\vartheta\varpi_{\mathbb{C}}$ is the product of $(T \times S)$-weights on $\mathfrak{g}^*_{\mathbb{C}}$, $\varpi_{\mathbb{R}}$ is the product of roots of G and $\rho \in \mathfrak{t}^*$ such that $\xi + \epsilon\rho$ is a regular value of $\mu^T_{\mathbb{R}}$ for small ϵ.

Proof. First we examine the $G = T$ abelian case. Assume that $\xi \in \mathfrak{t}^*$ is a regular value of $\mu_{\mathbb{R}}$. If $M/\!\!/_\xi T = (\mu^T_{\mathbb{R}})^{-1}(0)/T$ denotes the symplectic quotient then $\mu^T_{\mathbb{C}} : M \to \mathfrak{t}^*_{\mathbb{C}}$ induces an S-equivariant map $\widetilde{\mu}^T_{\mathbb{C}} : M/\!\!/_\xi T \to \mathfrak{t}^*_{\mathbb{C}}$. Since $(\xi, 0) \in \mathfrak{t}^* \oplus \mathfrak{t}^*_{\mathbb{C}}$ is a regular value of $(\mu^T_{\mathbb{R}}, \mu^T_{\mathbb{C}})$, therefore $0 \in \mathfrak{t}^*_{\mathbb{C}}$ is a regular value of $\widetilde{\mu}^T_{\mathbb{C}}$ and we can identify $(\widetilde{\mu}^T_{\mathbb{C}})^{-1}(0) = M/\!\!/\!\!/\!\!/_{(\xi,0)}T$. Consequently, we have an S-equivariant isomorphism of vector bundles

$$\mathcal{N}\big(M/\!\!/\!\!/\!\!/_{(\xi,0)}T \,|\, M/\!\!/_\xi T\big) \simeq M/\!\!/\!\!/\!\!/_{(\xi,0)}T \times \mathfrak{t}^*_{\mathbb{C}}. \tag{4.33}$$

Since $\widetilde{\mu}^T_{\mathbb{C}}$ is S-equivariant, the condition $(\mathfrak{t}^*_{\mathbb{C}})^S = \{0\}$ implies that

$$\big(M/\!\!/_\xi T\big)^S = \big(M/\!\!/\!\!/\!\!/_{(\xi,0)}T\big)^S. \tag{4.34}$$

By (4.33) for any $F \subset (M/\!\!/\!\!/\!\!/_{(\xi,0)}T)^S$ fixed point component we have

$$\mathcal{N}(F \,|\, M/\!\!/_\xi T) = \mathcal{N}(F \,|\, M/\!\!/\!\!/\!\!/_{(\xi,0)}T) \oplus (F \times \mathfrak{t}^*_{\mathbb{C}}),$$

hence

$$e_S\mathcal{N}(F \,|\, M/\!\!/_\xi T) = e_S\mathcal{N}(F \,|\, M/\!\!/\!\!/\!\!/_{(\xi,0)}T)\, \vartheta, \tag{4.35}$$

where ϑ is the product of all S-weights on $\mathfrak{t}_{\mathbb{C}}^*$. Remark that $\vartheta \neq 0$ by condition $(\mathfrak{t}_{\mathbb{C}}^*)^S = \{0\}$. If $\kappa_S : H_{T\times S}(M) \to H_S(M/\!\!/\!\!/\!\!/_{(\xi,0)}T)$ and $\kappa'_S : H_{T\times S}(M) \to H_S(M/\!\!/_{\xi}T)$ denote the hyperKähler and the symplectic Kirwan maps, respectively, then $i^*_{M/\!\!/\!\!/\!\!/_{(\xi,0)}T}\kappa'_S = \kappa_S$ and we compute

$$\begin{aligned}
\oint\limits_{M/\!\!/\!\!/\!\!/_{(\xi,0)}T} \kappa_S(\beta e^{\omega_{\mathbb{R}}-\mu_{\mathbb{R}}^T-\mu_S+\xi}) &= \sum_{F\subset(M/\!\!/\!\!/\!\!/_{(\xi,0)}T)^S} \frac{1}{\mathfrak{m}(F)} \int\limits_F \frac{i_F^*\kappa_S(\beta e^{\omega_{\mathbb{R}}-\mu_{\mathbb{R}}^T-\mu_S+\xi})}{e_S\mathcal{N}(F\,|\,M/\!\!/\!\!/\!\!/_{(\xi,0)}T)} \\
&= \sum_{F\subset(M/\!\!/\!\!/\!\!/_{(\xi,0)}T)^S} \frac{1}{\mathfrak{m}(F)} \int\limits_F \frac{\vartheta i_F^*\kappa'_S(\beta e^{\omega_{\mathbb{R}}-\mu_{\mathbb{R}}^T-\mu_S+\xi})}{e_S\mathcal{N}(F\,|\,M/\!\!/_{\xi}T)} \quad \text{by (4.35)} \\
&= \sum_{F\subset(M/\!\!/\!\!/\!\!/_{(\xi,0)}T)^S} \frac{1}{\mathfrak{m}(F)} \int\limits_F \frac{i_F^*\kappa'_S(\vartheta\beta e^{\omega_{\mathbb{R}}-\mu_{\mathbb{R}}^T-\mu_S+\xi})}{e_S\mathcal{N}(F\,|\,M/\!\!/_{\xi}T)} \quad (\kappa'_S \text{ is } H_S\text{-linear}) \\
&= \oint\limits_{M/\!\!/_{\xi}T} \kappa'_S(\vartheta\beta e^{\omega_{\mathbb{R}}-\mu_{\mathbb{R}}^T-\mu_S+\xi}) \quad \text{by (4.34)} \\
&= \mathrm{EqRes}^{\Lambda} \frac{1}{vol(T)} \oint\limits_M \vartheta\beta e^{\omega_{\mathbb{R}}-\mu_{\mathbb{R}}^T-\mu_S+\xi} \quad \text{by Theorem 4.5.}
\end{aligned}$$

If ξ is not regular value of $\mu_{\mathbb{R}}^T$ then we perturb ξ to a regular value $\xi+\epsilon\rho$ and we take the limit $\epsilon \to 0$, hence

$$\begin{aligned}
(4.36)\qquad \oint\limits_{M/\!\!/\!\!/\!\!/_{(\xi,0)}T} \kappa_S(\beta e^{\omega_{\mathbb{R}}-\mu_{\mathbb{R}}^T-\mu_S+\xi}) &= \lim_{\epsilon\to0} \oint\limits_{M/\!\!/\!\!/\!\!/_{(\xi+\epsilon\rho,0)}T} \kappa_S(\beta e^{\omega_{\mathbb{R}}-\mu_{\mathbb{R}}^T-\mu_S+\xi+\epsilon\rho}) \\
= \lim_{\epsilon\to0} \oint\limits_{M/\!\!/_{\xi+\epsilon\rho}T} \kappa'_S(\vartheta\beta e^{\omega_{\mathbb{R}}-\mu_{\mathbb{R}}^T-\mu_S+\xi+\epsilon\rho}) &= \lim_{\epsilon\to0} \mathrm{EqRes}^{\Lambda} \frac{1}{vol(T)} \oint\limits_M \vartheta\beta e^{\omega_{\mathbb{R}}-\mu_{\mathbb{R}}^T-\mu_S+\xi+\epsilon\rho}.
\end{aligned}$$

Now consider the non-abelian case. Suppose that $(\xi,0) \in (\mathfrak{g}^*\oplus\mathfrak{g}_{\mathbb{C}}^*)^G$ is a regular value of $(\mu_{\mathbb{R}},\mu_{\mathbb{C}})$ such that $(\xi,0)\in\mathfrak{t}^*\oplus\mathfrak{t}_{\mathbb{C}}^*$ is also a regular value of $(\mu_{\mathbb{R}}^T,\mu_{\mathbb{C}}^T)$ via the identification $(\mathfrak{g}^*)^G \simeq (\mathfrak{t}^*)^W$. By Theorem 2.2 of [19] and (4.36) for any $\beta \in H_{G\times S}(M) \simeq H_{T\times S}(M)^W$ we have

$$\begin{aligned}
(4.37)\qquad \oint\limits_{M/\!\!/\!\!/\!\!/_{(\xi,0)}G} \kappa_S(\beta e^{\omega_{\mathbb{R}}-\mu_{\mathbb{R}}-\mu_S+\xi}) &= \frac{1}{|W|} \oint\limits_{M/\!\!/\!\!/\!\!/_{(\xi,0)}T} \kappa_S(\varpi_{\mathbb{R}}\varpi_{\mathbb{C}}\beta e^{\omega_{\mathbb{R}}-\mu_{\mathbb{R}}^T-\mu_S+\xi}) \\
= \frac{1}{|W|} \oint\limits_{M/\!\!/_{\xi}T} \kappa_S(\vartheta\varpi_{\mathbb{R}}\varpi_{\mathbb{C}}\beta e^{\omega_{\mathbb{R}}-\mu_{\mathbb{R}}^T-\mu_S+\xi}) &= \mathrm{EqRes}^{\Lambda} \frac{\vartheta\varpi_{\mathbb{R}}\varpi_{\mathbb{C}}}{|W|vol(T)} \oint\limits_M \beta e^{\omega_{\mathbb{R}}-\mu_{\mathbb{R}}^T-\mu_S+\xi}.
\end{aligned}$$

If $(\xi,0)\in\mathfrak{t}^*\oplus\mathfrak{t}_{\mathbb{C}}^*$ is not a regular value of $(\mu_{\mathbb{R}}^T,\mu_{\mathbb{C}}^T)$ then we choose a small $\rho\in\mathfrak{t}^*$ such that $\xi+\epsilon\rho$ and $(\xi+\epsilon\rho,0)$ are regular values of $\mu_{\mathbb{R}}^T$ and $(\mu_{\mathbb{R}}^T,\mu_{\mathbb{C}}^T)$, respectively for all $\epsilon\in(0,1]$. Similarly to the symplectic case we can show that

$$\oint\limits_{M/\!\!/\!\!/\!\!/_{(\xi,0)}G} \kappa_S(\beta e^{\omega_{\mathbb{R}}-\mu_{\mathbb{R}}-\mu_S+\xi}) = \lim_{\epsilon\to0} \oint\limits_{M/\!\!/\!\!/\!\!/_{(\xi+\epsilon\rho,0)}T} \kappa_S(\varpi_{\mathbb{R}}\varpi_{\mathbb{C}}\beta e^{\omega_{\mathbb{R}}-\mu_{\mathbb{R}}^T-\mu_S+\xi+\epsilon\rho})$$

and the theorem follows by (4.37). $\square$

5

Applications

In this chapter we give two applications of results of Chapter 4. In the first one we describe a method for computation of cohomology ring of the Hilbert scheme of points on the plane [27, 44]. We give a description of the equivariant cohomology of this Hilbert scheme of points from which the ordinary cohomology ring can be computed in terms of generators and relations.

In the second example we compute Nekrasov's partition function [38] on the moduli space of framed torsion free sheaves on $\mathbb{CP}^2$. We get back the formula by Nakajima and Yoshioka [37].

5.1 Equivariant cohomology of the Hilbert scheme of points on the plane

Consider $\mathcal{M} = T^*(\mathrm{End}(\mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}, \mathbb{C}^n)) = M_n(\mathbb{C}) \oplus M_{n,1}(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus M_{1,n}(\mathbb{C})$ with $U(n)$-action

$$g \cdot (A, a, B, b) = (gAg^{-1}, ga, gBg^{-1}, bg^{-1}),$$

for all $g \in U(n)$, $A, B \in M_n(\mathbb{C})$ and $a, b^t \in M_{n,1}(\mathbb{C})$. This action is hyper-Hamiltonian with real and complex moment maps

$$\begin{aligned} &\mu_{\mathbb{R}} : (\mathcal{M}, \omega_{\mathbb{R}}) \to \mathfrak{u}(n)^*, \quad \mu_{\mathbb{R}}(A, a, B, b) = \frac{\sqrt{-1}}{2}\left([A, A^*] + [B, B^*] + aa^* - b^*b\right), \\ &\mu_{\mathbb{C}} : (\mathcal{M}, \omega_{\mathbb{C}}) \to \mathfrak{u}(n)^*_{\mathbb{C}}, \quad \mu_{\mathbb{C}}(A, a, B, b) = [A, B] + ab, \end{aligned}$$

where we identify $\mathfrak{u}(n)^* \simeq \mathfrak{u}(n)$ via the non-degenerate bilinear form $(u, v) = \mathrm{Tr}(u^*v)$ for all $u, v \in \mathfrak{u}(n)$. The hyperKähler moment map $\mu = (\mu_{\mathbb{R}}, \mu_{\mathbb{C}})$ has $(\xi, 0) = \left(\frac{\sqrt{-1}}{2} I,\, 0\right) \in \mathfrak{u}(n)^* \oplus \mathfrak{u}(n)^*_{\mathbb{C}}$ as regular value and the hyperKähler quotient $\mathcal{M} /\!\!/\!\!/\!\!/_{(\xi,0)} U(n)$ is diffeomorphic to $\mathrm{Hilb}^n(\mathbb{C}^2)$ the Hilbert scheme of n points on $\mathbb{C}^2$ [36].

Let $T = \{g \in U(n) \,|\, g \text{ diagonal}\}$ be a maximal torus of $U(n)$ and we choose the basis $\{u_1, \ldots, u_n\}$ of $\mathfrak{t}^*$ such that $u_i\left(diag(\sqrt{-1}\tau_1, \ldots, \sqrt{-1}\tau_n)\right) = \tau_i$ for all $i = 1, \ldots, n$. Then the real and complex abelian moment maps are as follows

$$\mu_{\mathbb{R}}^T : (\mathcal{M}, \omega_{\mathbb{R}}) \to \mathfrak{t}^*, \quad \mu_{\mathbb{R}}^T(A, a, B, b) = \sum_{k,j=1}^{n} (u_k - u_j) \frac{|A_{kj}|^2 + |B_{kj}|^2}{2} + \sum_{k=1}^{n} u_k \frac{|a_k|^2 - |b_k|^2}{2},$$

$$\mu_{\mathbb{C}}^T : (\mathcal{M}, \omega_{\mathbb{C}}) \to \mathfrak{t}_{\mathbb{C}}^*, \quad \mu_{\mathbb{C}}^T(A, a, B, b) = -\sqrt{-1} \sum_{k=1}^{n} u_k \left(\sum_{j=1}^{n} A_{kj} B_{jk} - B_{kj} A_{jk} + a_k b_k \right).$$

ξ is a regular value of $\mu_{\mathbb{R}}^T$, because $u_1 + \ldots + u_n$ is regular with respect to $\{u_k - u_j,\, u_k \,|\, k, j = 1, \ldots, n\}$. Moreover, $(\xi, 0)$ is also regular value of the abelian hyperKähler moment map $\mu^T = (\mu_{\mathbb{R}}^T, \mu_{\mathbb{C}}^T)$. Therefore, both abelian symplectic and hyperKähler quotients $\mathcal{M}/\!\!/_{\xi} T$ and $\mathcal{M}/\!\!/\!\!/\!\!/_{(\xi,0)} T$ exist. We note that $\mathcal{M}/\!\!/\!\!/\!\!/_{(\xi,0)} T$ is a hypertoric variety [20], [18].

We also consider the following circle action of $S = U(1)$ on $\mathcal{M}$

$$s \cdot (A, a, B, b) = (sA, sa, s^N B, s^N b), \qquad (N > n)$$

for all $s \in S$, $A, B \in M_n(\mathbb{C})$, $a, b^t \in M_{n,1}(\mathbb{C})$, which commutes with the $U(n)$-action and admits moment map $\mu_S : (\mathcal{M}, \omega_{\mathbb{R}}) \to \mathfrak{s}^*$, $\mu_S = \varphi \cdot z$, where

$$\varphi(A, a, B, b) = \frac{1}{2} \mathrm{Tr}(AA^* + NBB^* + a^* a + N b b^*)$$

is proper, bounded below and $z \in \mathfrak{s}^*$ is such that $z(\sqrt{-1}\sigma) = \sigma$ for all $(\sqrt{-1}\sigma) \in \mathfrak{s} = \mathfrak{u}(1)$. Finally, we note that $\mu_{\mathbb{C}}$ is S-equivariant, i.e. $\mu_{\mathbb{C}}(s \cdot (A, a, B, b)) = s^{N+1} \mu_{\mathbb{C}}(A, a, B, b)$, and $(\mathfrak{t}_{\mathbb{C}}^*)^S = \{0\}$.

The cohomology ring $H(\mathrm{Hilb}^n(\mathbb{C}^2))$ is generated by the Chern classes of the tautological vector bundle $\Xi_n = (\mathcal{M} \times \mathbb{C})/\!\!/\!\!/\!\!/_{(\xi,0)} U(n)$, where $U(n)$ acts on $\mathbb{C}^n$ via the standard representation [10]. This is equivalent to the surjectivity of the Kirwan map $\kappa : H_{U(n)}(\mathcal{M}) \to H(\mathrm{Hilb}^n(\mathbb{C}^2))$.

We use the same idea as in [18], Lemma 4.9 to show that

Lemma 5.1. *The S-equivariant Kirwan map $\kappa_S : H_{U(n) \times S}(\mathcal{M}) \to H_S(\mathrm{Hilb}^n(\mathbb{C}^2))$ is also surjective.*

Proof. Since the S-moment map on $\mathcal{M}/\!\!/\!\!/\!\!/_{(\xi,0)} U(n) \simeq \mathrm{Hilb}^n(\mathbb{C}^2)$ is proper, bounded below, hence $H_S(\mathrm{Hilb}^n(\mathbb{C}^2)) \simeq H(\mathrm{Hilb}^n(\mathbb{C}^2)) \otimes H_S(pt)$ as $H_S(pt)$-modules, i.e. $H_S(\mathrm{Hilb}^n(\mathbb{C}^2))$ is equivariantly formal ([26], [43] Theorem 4.2, [41] Proposition 2.10). Recall that $H_S(pt) = \mathbb{R}[z]$, $H_{U(n) \times S}(\mathcal{M}) = \mathbb{R}[u_1, \ldots, u_n, z]^{S_n}$ and $H_{U(n)}(\mathcal{M}) = \mathbb{R}[u_1, \ldots, u_n]^{S_n}$. We have the following commutative diagram

$$\begin{tikzcd} \mathbb{R}[u_1, \ldots, u_n, z]^{S_n} \arrow[r, "\kappa_S"] \arrow[d, "\pi_1"'] & H(\mathrm{Hilb}^n(\mathbb{C}^2)) \otimes \mathbb{R}[z] \arrow[d, "\pi_2"] \\ \mathbb{R}[u_1, \ldots, u_n]^{S_n} \arrow[r, "\kappa"'] & H(\mathrm{Hilb}^n(\mathbb{C}^2)), \end{tikzcd} \tag{5.1}$$

where $\pi_i(\alpha_i \otimes 1) = \alpha_i$ and $\pi_i(1 \otimes z) = 0$ for all $i = 1, 2$, $\alpha_1 \in \mathbb{R}[u_1, \ldots, u_n]^{S_n}$ and $\alpha_2 \in H(\mathrm{Hilb}^n(\mathbb{C}^2))$. Since κ_S is $\mathbb{R}[z]$-linear, therefore $H^0(\mathrm{Hilb}^n(\mathbb{C}^2)) \otimes \mathbb{R}[z]$ is in the image of κ_S.

Assume that $\oplus_{i<k} H^i(\mathrm{Hilb}^n(\mathbb{C}^2)) \otimes \mathbb{R}[z] \subset \mathrm{Im}\, \kappa_S$ and let $\beta_k \in H^k(\mathrm{Hilb}^n(\mathbb{C}^2))$. By surjectivity of κ there is $\alpha_k \in \mathbb{R}[u_1, \ldots, u_n]^{S_n}$ of degree k such that $\beta_k = \kappa(\alpha_k)$. By commutativity of (5.1) we have $\beta_k \otimes 1 - \kappa_S(\alpha_k) \in \ker \pi_2 = H(\mathrm{Hilb}^n(\mathbb{C}^2)) \otimes z\mathbb{R}[z]$, hence $\beta_k \otimes 1 = \kappa_S(\alpha_k) + \sum_{i=0}^{k-1} \beta_i \otimes z^{k-i}$ with $\beta_i \in H^i(\mathrm{Hilb}^n(\mathbb{C}^2))$ for all $i = 0, \ldots, k-1$. From the inductive hypothesis follows that $\beta_k \otimes 1 \in \mathrm{Im}\, \kappa_S$ and κ_S is surjective. $\square$

The surjectivity of the Kirwan map provides the generators of the cohomology ring of the Hilbert scheme, that is, we have isomorphism of rings

$$H_S(\mathrm{Hilb}^n(\mathbb{C}^2)) \simeq \mathbb{R}[u_1, \ldots, u_n, z]^{S_n} \big/ \ker \kappa_S.$$

The Jeffrey-Kirwan formula can be used as follows to compute the relations between generators. First, we remark that $(\mathrm{Hilb}^n(\mathbb{C}^2))^S$ is compact by $\mathcal{M}^S = \{0\}$ and Proposition 4.3. By [19] there is a non-degenerate bilinear pairing on the rationalized ring $\widehat{H}_S(\mathrm{Hilb}^n(\mathbb{C}^2)) = H_S(\mathrm{Hilb}^n(\mathbb{C}^2)) \otimes \mathbb{R}(z)$, which will play the role of Poincaré duality and it is given by

$$\langle \widehat{\eta}_1, \widehat{\eta}_2 \rangle = \oint\limits_{\mathrm{Hilb}^n(\mathbb{C}^2)} \widehat{\eta}_1 \widehat{\eta}_2$$

for all $\widehat{\eta}_1, \widehat{\eta}_2 \in \widehat{H}_S(\mathrm{Hilb}^n(\mathbb{C}^2))$. This pairing is $\mathbb{R}(z)$-linear and the natural map $H_S(\mathrm{Hilb}^n(\mathbb{C}^2)) \to \widehat{H}_S(\mathrm{Hilb}^n(\mathbb{C}^2))$ is injective by the equivariant formality. Hence $\eta \in H_S(\mathrm{Hilb}^n(\mathbb{C}^2))$ is zero exactly when

$$\oint\limits_{\mathrm{Hilb}^n(\mathbb{C}^2)} \eta\gamma = 0, \qquad \forall \gamma \in H_S(\mathrm{Hilb}^n(\mathbb{C}^2)).$$

If we couple it with the Kirwan surjectivity then the kernel of κ_S can be described as

$$\ker \kappa_S = \Big\{ \beta \in \mathbb{R}[u_1, \ldots, u_n, z]^{S_n} \;\Big|\; \oint_{\mathrm{Hilb}^n(\mathbb{C}^2)} \kappa_S(\beta\gamma) = 0,\ \forall \gamma \in \mathbb{R}[u_1, \ldots, u_n, z]^{S_n} \Big\}.$$

We have the following integration formula on $\mathrm{Hilb}^n(\mathbb{C}^2)$.

Theorem 5.2. *For any $\beta \in H_{U(n)\times S}(\mathcal{M}) = \mathbb{R}[u_1, \ldots, u_n, z]^{S_n}$ we have*

$$\oint\limits_{\mathrm{Hilb}^n(\mathbb{C}^2)} \kappa_S(\beta) = \sum_{\lambda \vdash n} b_\lambda(z)\, \beta(p_\lambda(z), z),$$

where

$$p_\lambda(z) = -\left(z,\, (1+N)z,\, \ldots,\, (1+(\lambda_1 - 1)N)z,\, \ldots, kz,\, (k+N)z,\, \ldots,\, (k + (\lambda_k - 1)N)z\right)$$

and $0 \neq b_\lambda(z) \in \mathbb{R}[z^{\pm 1}]$ for any $\lambda = (\lambda_1, \ldots, \lambda_k)$ partition of n with $\lambda_1 \geq \ldots \geq \lambda_k$.

Remark 5.3. For our purpose only the non-vanishing of $b_\lambda(z)$ is relevant, nevertheless the exact value is equal

$$b_\lambda(z) = \frac{(2\pi)^n z^{-2n}}{\prod\limits_{i \in Y_\lambda} [-A_\lambda(i) + (L_\lambda(i) + 1)N][A_\lambda(i) + 1 - L_\lambda(i)N]},$$

where Y_λ is the Young diagram of the partition λ, $A_\lambda(i)$ and $L_\lambda(i)$ are the arm-length and leg-length of the box i in Y_λ, respectively (Definition 5.16). We can deduce this formula as follows. We observe that $H'(u,z)$ in (5.4) is equal to $G'_V(u,z,Nz,-z)$ in (5.10) for $r=1$, and which is computed in Lemma 5.23 via the relation (5.13). $\diamond$

The following lemma shows that how can we compute the kernel of κ_S from Theorem 5.2.

Lemma 5.4. *Let $\mathbb{K}$ be a field and let $\{q_1,\dots,q_m\}\subset\mathbb{K}^n$. If $\mathcal{L}:\mathbb{K}[x_1,\dots,x_n]\to\mathbb{K}$, $\mathcal{L}(P)=\sum_{i=1}^m b_iP(q_i)$ with $b_i\in\mathbb{K}\setminus\{0\}$ for all $i=1,\dots,m$ then*

$$\{P\in\mathbb{K}[x_1,\dots,x_n]\,|\,\mathcal{L}(PQ)=0,\ \forall Q\}=\{P\in\mathbb{K}[x_1,\dots,x_n]\,|\,P(q_i)=0,\ \forall i=1,\dots,m\}.$$

Proof. Consider the polynomial $Q_i(x)=\prod_{1\le k\le n}\prod_{\substack{1\le j\le m\\ q_{jk}\ne q_{ik}}}(x_k-q_{jk})$, where $q_j=(q_{j1},\dots,q_{jn})$. Then $Q_i(q_j)=0$ if $j\ne i$ and $Q_i(q_i)\ne 0$. If $P\in\{P\in\mathbb{K}[x_1,\dots,x_n]\,|\,\mathcal{L}(PQ)=0,\ \forall Q\}$ then for all $i=1,\dots,m$ we have $\mathcal{L}(PQ_i)=b_iP(q_i)Q_i(q_i)=0$, hence $P(q_i)=0$. The other inclusion is obvious. $\square$

We got the following description of the equivariant cohomology ring of the Hilbert scheme of points on the plane.

Theorem 5.5. *We have isomorphism of rings*

$$H_S(\mathrm{Hilb}^n(\mathbb{C}^2))\simeq\mathbb{R}[u_1,\dots,u_n,z]^{S_n}\Big/\{P\in\mathbb{R}[u_1,\dots,u_n,z]^{S_n}\,\big|\,P(p_\lambda(z),z)=0,\ \forall\lambda\vdash n\}.$$

Furthermore, the ordinary cohomology ring can be computed from the equivariant one as

$$H(\mathrm{Hilb}^n(\mathbb{C}^2))\simeq H_S(\mathrm{Hilb}^n(\mathbb{C}^2))\Big/zH_S(\mathrm{Hilb}^n(\mathbb{C}^2)),$$

since $H_S(\mathrm{Hilb}^n(\mathbb{C}^2))$ is equivariantly formal [11]. Therefore, if the ideal $\ker\kappa_S$ is generated by $P_1(u,z),\dots,P_r(u,z)\in\mathbb{R}[u_1,\dots,u_n,z]^{S_n}$ then $\ker\kappa$ is generated by $P_1(u,0),\dots,P_r(u,0)\in\mathbb{R}[u_1,\dots,u_n]^{S_n}$.

Proof of Theorem 5.2. By Theorem 4.14 for any $\beta\in\mathbb{R}[u_1,\dots,u_n,z]^{S_n}$ we have

$$\oint_{\mathrm{Hilb}^n(\mathbb{C}^2)}\kappa_S(\beta)=\lim_{\epsilon\to0^+}\oint_{\mathcal{M}/\!\!/\!\!/\!\!/_{(\xi,0)}U(n)}\kappa_S\left(\beta e^{\epsilon(\omega_\mathbb{R}-\mu_\mathbb{R}-\mu_S+\xi)}\right)= \tag{5.2}$$

$$\lim_{\epsilon\to0^+}\frac{(2\pi)^{3n}}{n!\,vol(T)}\mathrm{EqRes}^\Lambda\left[\frac{\beta(u,z)((N+1)z)^n\prod_{1\le i\ne j\le n}(u_i-u_j)((N+1)z+u_i-u_j)e^{\epsilon\sum_{i=1}^n u_i}}{N^nz^{2n}\prod_{1\le i\ne j\le n}(z+u_i-u_j)(Nz+u_i-u_j)\prod_{1\le k\le n}(z+u_k)(Nz-u_k)}\right]=$$

$$\lim_{\epsilon\to0^+}\frac{(2\pi)^{2n}}{n!}\left[\frac{N+1}{Nz}\right]^n\mathrm{EqRes}^\Lambda\left[\frac{\beta(u,z)\prod_{1\le i\ne j\le n}(u_i-u_j)((N+1)z+u_i-u_j)e^{\epsilon\sum_{i=1}^n u_i}}{\prod_{1\le i\ne j\le n}(z+u_i-u_j)(Nz+u_i-u_j)\prod_{1\le k\le n}(z+u_k)(Nz-u_k)}\right],$$

where Λ is the polarization induced by the ordered basis $\{z, u_1, \ldots, u_n\}$ of $\mathfrak{t}^* \oplus \mathfrak{s}^*$ and we choose the scalar product on $\mathfrak{t}^*$ such that $\{u_1, \ldots, u_n\}$ becomes an orthonormal basis, whence $vol(T) = (2\pi)^n$.

Let

$$F(u,z) = \frac{\prod\limits_{1\le i\ne j\le n}(u_i-u_j)((N+1)z+u_i-u_j)}{\prod\limits_{1\le i\ne j\le n}(z+u_i-u_j)(Nz+u_i-u_j)\prod\limits_{1\le k\le n}(z+u_k)(Nz-u_k)}.$$

To compute $\mathrm{EqRes}^{\Lambda}\beta(u,z)F(u,z)e^{\epsilon\sum\limits_{i=1}^{n}u_i}$ we consider only $\mathfrak{t}^*$-poles V spanned by subsets of $\mathcal{A} = \{z+u_i-u_j,\, Nz+u_i-u_j,\, z+u_k,\, Nz-u_k \,|\, i\ne j,\, i,j,k = 1,\ldots,n\}$ which contains linear terms from the denominator of F. Remark that all elements of $\mathcal{A}$ are polarized with respect to Λ.

Lemma 5.6. *Let $\mathcal{B}_V = \{\alpha_1, \ldots, \alpha_n\} \subset \mathcal{A}\cap V$ be a basis of $V = \mathrm{span}\langle u_1 - p_1 z,\, \ldots,\, u_n - p_n z\rangle$.*

(i) There is k such that $Nz-u_k$ or $z+u_k$ is in $\mathcal{B}_V$.

(ii) For any $i = 1,\ldots,n$ we can write $u_i - p_i z = \sum\limits_{i=1}^{n} q_i\alpha_i$ such that $q_i \in \{0,\pm1\}$.

(iii) For all $i = 1,\ldots,n$ we can present $-p_i = a_i + b_i N$ uniquely such that $-n \le a_i, b_i \le n$.

Proof. Let $\mathcal{N}_0$ and $\mathcal{N}_1$ be subsets of $\{1,\ldots,n\}$ such that

(1) $i \in \mathcal{N}_1$ if $z+u_i$ or $Nz-u_i$ is in $\mathcal{B}_V$,

(2) $i \in \mathcal{N}_1$ if $j \in \mathcal{N}_1$ and $\varepsilon z + u_i - u_j$ or $\varepsilon z + u_j - u_i$ is in $\mathcal{B}_V$ for any $\varepsilon \in \{1, N\}$.

(3) any element unsorted by (1) and (2) is in $\mathcal{N}_0$.

We can also define subsets $\mathcal{B}_V^0, \mathcal{B}_V^1 \subset \mathcal{B}_V$ such that $\alpha_i \in \mathcal{B}_V^l$ if $\pm u_j$ is a summand of α_i for some $j \in \mathcal{N}_l$. Remark that $\mathcal{N}_l = \emptyset$ if and only if $\mathcal{B}_V^l = \emptyset$. By construction $\{1,\ldots,n\} = \mathcal{N}_0 \uplus \mathcal{N}_1$, hence $\mathcal{B}_V = \mathcal{B}_V^0 \uplus \mathcal{B}_V^1$. If $\mathrm{pr}_{\mathfrak{t}^*} : \mathfrak{t}^* \oplus \mathfrak{s}^* \to \mathfrak{t}^*$ is the projection then $\mathrm{pr}_{\mathfrak{t}^*}(\mathcal{B}_V^l) \subset \mathrm{span}\langle u_i \,|\, i \in \mathcal{N}_l\rangle$ for any $l = 0, 1$. Moreover, $\mathcal{B}_V^0$ may only contain elements of form $\varepsilon\sigma + u_i - u_j$ ($\varepsilon \in \{1, N\}$), hence $\sum\limits_{i\in\mathcal{N}_0} u_i \notin \mathrm{span}(\mathrm{pr}_{\mathfrak{t}^*}(\mathcal{B}_V^0))$. If $\mathcal{N}_0 \ne \emptyset$ then

$$\mathrm{pr}_{\mathfrak{t}^*}(V) = \mathrm{span}(\mathrm{pr}_{\mathfrak{t}^*}(\mathcal{B}_V^0)) + \mathrm{span}(\mathrm{pr}_{\mathfrak{t}^*}(\mathcal{B}_V^1)) \ne \mathrm{span}\langle u_i \,|\, i\in\mathcal{N}_0\rangle + \mathrm{span}\langle u_i \,|\, i \in \mathcal{N}_1\rangle = \mathfrak{t}^*,$$

which leads to contradiction that V is a $\mathfrak{t}^*$-pole. Therefore, we have $\mathcal{N}_1 = \{1,\ldots,n\}$, which implies (i), and moreover, for any $k \in \{1,\ldots,n\}$ there exists a sequence $i_1,\ldots,i_m$ such that $u_{i_1}, u_{i_2} - u_{i_1}, \ldots, u_{i_m} - u_{i_{m-1}} \in \pm\mathrm{pr}_{\mathfrak{t}^*}(\mathcal{B}_V)$ and $i_m = k$, which yields (ii). Finally, by (ii) we have $-p_i = \sum\limits_{i=1}^{n} q_i\varepsilon_i$, where $q_i \in \{0,\pm1\}$ and $\varepsilon_i \in \{1, N\}$, therefore $-p_i = a_i + b_i N$ such that $-n \le a_i, b_i \le n$ and the uniqueness of a_i, b_i follows from $n < N$. $\square$

We associate to a $\mathfrak{t}^*$-pole V an oriented graph Γ_V with vertices $\{0,1,\ldots,n\}$ on the sublattice $\mathbb{Z}\times N\mathbb{Z} \subset \mathbb{Z}^2$ as follows. The vertex 0 has coordinates $(0,0) \in \mathbb{Z}^2$ and the coordinates of a non-zero vertex i are $(a_i, b_i N)$, where a_i, b_i are defined by Lemma 5.6(iii). For each element of $\mathcal{A}_V = \mathcal{A}\cap V$ we draw on edge according to the following table.

$\alpha \in \mathcal{A}_V$	edges of Γ_V
$Nz + u_i - u_j$	vertical edge from j to i
$Nz - u_i$	vertical edge from i to 0
$z + u_i - u_j$	horizontal edge from j to i
$z + u_i$	horizontal edge from 0 to i

From the proof of Lemma 5.6 follows that the graph Γ_V is connected. Moreover, it is complete in the following sense. If the vertex i has coordinates (a, b) and j has coordinates $(a+1, b)$ (respectively $(a, b+N))$) then there is an edge $i \to j$ if $j \neq 0$ (respectively $i \neq 0$). Furthermore, if vertices i and j have the same coordinates and there is an edge $k \to i$ (or $i \to k$) then we also have an edge $k \to j$ (or $j \to k$) in Γ_V.

By decomposing F to partial fractions we will get fractions which can be derived from F by removing linear terms from its nominator and denominator. The removal of linear terms will be encoded in a graph which we get from $\Gamma_V = \Gamma_{V,F}$ by replacing edges $i \to j$ with $i \dashrightarrow j$ or by adding additional dotted arrows by the following rules. Let G be a fraction derived from F.

(D1) The partial fraction decomposition

$$G = G' \frac{((N+1)z + u_k - u_i)}{(z + u_k - u_l)(Nz + u_l - u_i)} = G' \frac{1}{z + u_k - u_l} + G' \frac{1}{Nz + u_l - u_i} = G_1 + G_2$$

with $z + u_k - u_l, Nz + u_l - u_i \in V$ will be pictured as

$$\begin{tikzcd} l \arrow[r] & k \\ i \arrow[u] & \end{tikzcd} = \begin{tikzcd} l \arrow[r] & k \\ i \arrow[u, dotted] \arrow[ur, dotted] & \end{tikzcd} + \begin{tikzcd} l \arrow[r, dotted] & k \\ i \arrow[u] \arrow[ur, dotted] & \end{tikzcd} .$$

We can get G_1 from G by removing $Nz + u_l - u_i$ and $(N+1)z + u_k - u_i$ respectively from the denominator and nominator. We translate it in terms of graphs. We get Γ_{V,G_1} from $\Gamma_{V,G}$ by replacing the edge $i \to l$ with $i \dashrightarrow l$ (corresponding to removal of $Nz + u_l - u_i$) and by adding diagonal arrow $i \dashrightarrow k$ (corresponds to removal of $(N+1)z + u_k - u_i$). Similarly, we get Γ_{V,G_2} from $\Gamma_{V,G}$ by replacing $l \to k$ with $l \dashrightarrow k$ (removal of $z + u_k - u_l$) and by adding diagonal arrow $i \dashrightarrow k$ (removal of $(N+1)z + u_k - u_i$).
In general, dotted diagonal arrows $i \dashrightarrow k$ corresponds to removal of terms $(N+1)z + u_i - u_k$ from the nominator, hence we cannot have double dotted diagonal arrows from i to k. Moreover, on pictures we only draw the subgraph, where a mutation occurs in the initial graph. In the case of $l = 0$ we set $u_0 = 0$.

(D2) Similarly, for decomposition

$$G = G' \frac{(N+1)z + u_k - u_i}{(z + u_j - u_i)(Nz + u_k - u_j)} = G' \frac{1}{Nz + u_k - u_j} + G' \frac{1}{z + u_j - u_i} = G_1 + G_2$$

with $z + u_j - u_i, Nz + u_k - u_j \in V$, $i \neq 0$ we draw

$$\begin{tikzcd} & k \\ 0 \neq i \arrow[r] & j \arrow[u] \end{tikzcd} = \begin{tikzcd} & k \\ 0 \neq i \arrow[r, dotted] \arrow[ur, dotted] & j \arrow[u] \end{tikzcd} + \begin{tikzcd} & k \\ 0 \neq i \arrow[r] \arrow[ur, dotted] & j \arrow[u, dotted] \end{tikzcd} .$$

(D3) The decomposition

$$G = G' \frac{u_i - u_j}{(z + u_i - u_l)(z + u_j - u_l)} = G' \frac{1}{z + u_j - u_l} - G' \frac{1}{z + u_i - u_l} = G_1 - G_2$$

with $z + u_i - u_l, z + u_j - u_l \in V$ yields

$$l \overset{li}{\underset{lj}{\rightrightarrows}} i,j \qquad = \qquad l \overset{li}{\underset{lj}{\rightrightarrows}} i,j \circlearrowleft^{ji} \qquad - \qquad l \overset{li}{\underset{lj}{\rightrightarrows}} i,j \circlearrowleft^{ji}$$

where the arrow labeled by li (respectively by lj) goes from l to i (respectively to j). The dotted hook arrow labeled by ji corresponds to removal of $u_i - u_j$ from the nominator of G. Thus we get the graph Γ_{V,G_1} (corresponding to G_1) from $\Gamma_{V,G}$ by replacing $l \xrightarrow{li} i$ by $l \overset{li}{\dashrightarrow} i$ (removal of $z + u_l - u_i$) and by adding hook arrow $i,j \circlearrowleft^{ji}$ (removal of $u_i - u_j$). Similarly, we get Γ_{V,G_2} from $\Gamma_{V,G}$ by replacing $l \xrightarrow{lj} j$ with $l \overset{lj}{\dashrightarrow} j$ and by adding hook arrow $i,j \circlearrowleft^{ji}$.

(D4) Similarly, the decomposition

$$G = G' \frac{u_i - u_j}{(Nz + u_i - u_l)(Nz + u_j - u_l)} = G' \frac{1}{Nz + u_j - u_l} - G' \frac{1}{Nz + u_i - u_l} = G_1 - G_2$$

with $Nz + u_i - u_l, Nz + u_j - u_l \in V$ corresponds to mutation of subgraphs

$$\begin{tikzcd} i,j & & i,j \arrow[loop above, dotted, "ji"] & & i,j \arrow[loop above, dotted, "ji"] \\ 0 \neq l \arrow[u, shift left, "lj"'] \arrow[u, shift right, "li"] & = & 0 \neq l \arrow[u, shift left, "lj"'] \arrow[u, shift right, dotted, "li"] & - & 0 \neq l \arrow[u, shift left, dotted, "lj"'] \arrow[u, shift right, "li"] \end{tikzcd} .$$

Definition 5.7. We say that a fraction G derived from F (or a graph $\Gamma_{V,G}$ got from Γ_V) *does not contribute* (at pole V) if

$$\mathrm{JKRes}^{\Lambda_V} \beta(u(v,z), z) G(u(v,z), z) e^{\epsilon \sum_{i=1}^{n} u_i(v,z)} dv = 0$$

for all $\beta \in \mathbb{R}[u_1, \ldots, u_n, z]^{S_n}$ and $\epsilon > 0$. If $G = F$ then we say that the pole V does not contribute.

Definition 5.8. We call (horizontal or vertical) edges $j \to i$ *tails* of the vertex i.

Lemma 5.9. *(i) A fraction G derived from F does not contribute if $\Gamma_{V,G}$ has a vertex $i \neq 0$ without a tail.*

(ii) If there is a vertex $i \neq 0$ in Γ_V with coordinates not in $\{(a, b,) \in \mathbb{Z}^2 \mid a > 0, b \geq 0\}$ then the pole V does not contribute.

Proof. (i) Denote $\mathcal{G}$ the set of $\alpha \in \mathcal{A}_V$ which appears in the denominator of G. Since i has no tail, therefore the only elements of $\mathcal{G}$ involving u_i may be of form $Nz - u_i$, $Nz + u_j - u_i$ or $z + u_j - u_i$, that is, any polarized element in $\mathcal{G}$ has non-positive u_i coefficient. Therefore, $\epsilon \sum_{i=1}^{n} u_i \notin Cone(\mathrm{pr}_{\mathfrak{t}^*}(\overline{\alpha}) \,|\, \alpha \in \mathcal{G})$ for any $\epsilon > 0$, hence by Corollary 3.19

$$\mathrm{JKRes}^{\Lambda_V} \beta(u(v,z),z) G(u(v,z),z) e^{\epsilon \sum\limits_{i=1}^{n} u_i(v,z)} dv = 0$$

for all $\beta \in \mathbb{R}[u_1, \ldots, u_n, z]^{S_n}$ and $\epsilon > 0$.

(ii) If there is a non-zero vertex i with coordinates (a_i, b_i) such that $a_i \leq 0$ then we can assume that for any other vertex $j \neq 0$ with coordinates (a_j, b_j) we have $a_i \leq a_j$ and if $a_i = a_j$ then $b_i \leq b_j$. Hence there is no edge $0 \neq j \to i$ in Γ_V and remark that if there is an edge $0 \to i$ then $a_i > 0$. If $b_i < 0$ then similarly we can show that there is a vertex in Γ_V without tail, hence V does not contribute by (i).

$\square$

Lemma 5.10. *A pole V does not contribute if any of the following holds:*

(i) Γ_V contains a subgraph
$$\begin{tikzcd} & k \\ 0 \neq i \arrow[r] & j \arrow[u] \end{tikzcd}$$
and there is no vertex l such that
$$\begin{tikzcd} l \arrow[r] & k \\ 0 \neq i \arrow[r] \arrow[u] & j \arrow[u] \end{tikzcd}$$
is also in Γ_V,

(ii) Γ_V contains a subgraph
$$\begin{tikzcd} j \arrow[r] & k \\ i \arrow[u] & \end{tikzcd}$$
and there is no vertex l such that
$$\begin{tikzcd} j \arrow[r] & k \\ i \arrow[u] \arrow[r] & l \arrow[u] \end{tikzcd}$$
is in Γ_V,

(iii) there are two vertices in Γ_V with same coordinates.

Proof. Suppose that all non-zero vertex i in Γ_V has coordinates (a_i, b_i) with $a_i > 0$ and $b_i \geq 0$.

(i) Assume that if (a_j, b_j) are coordinates of the vertex j then b_j is minimal. That is, if there is a vertex j' with coordinates $(a_{j'}, b_{j'})$ and it admits horizontal and vertical edges $0 \neq i' \to j'$ and $j' \to k'$ respectively, then $b_{j'} \geq b_j$. If there is no edge $l \to k$ then by decomposition (D2) we get $F = G_1 + G_2$ such that in Γ_{V,G_2} the vertex k has no tails, hence F contributes exactly when G_1 does. By assumption any vertex j' below j, i.e. with coordinates $a_{j'} = a_j$ and $b_{j'} \leq b_j$, may only have vertical tail, thus the last one cannot have tails. Therefore, G_1 does not contribute either.
If the horizontal edge $l \to k$ exists then the vertical edge $i \to l$ also exists in Γ_V, because

$$Nz + u_l - u_i = (Nz + u_k - u_j) - (z + u_k - u_l) + (z + u_j - u_i) \in V.$$

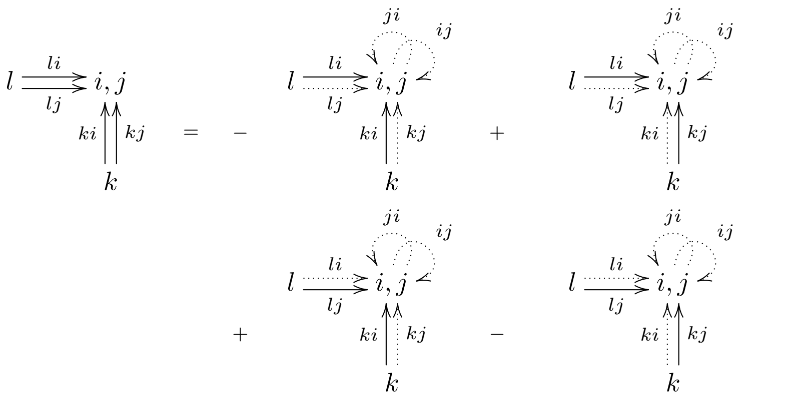
(ii) Similarly, as above assume that if (a_j, b_j) are coordinates of the vertex j then a_j is minimal. If there is no vertical edge $l \to k$ in Γ_V then by decomposition (D1) we get $F = G_1 + G_2$ such that the vertex k has no tails in Γ_{V,G_2}. Hence F contributes if and only if G_1 does. Moreover, in Γ_{V,G_1} the vertex j has no vertical tail and by assumption any vertex j' with coordinates $a_{j'} \leq a_j$, $b_{j'} = b_j$ may only admit horizontal tail. But the first vertex in the row cannot have any tail. If this last vertex is 0 than i has coordinates (a_i, b_i) with $b_i < 0$, which leads to contradiction. Therefore, G_2 does not contribute either.
If the vertical edge $l \to k$ exists in Γ_V then the horizontal edge $i \to l$ also exists, because

$$z + u_l - u_i = (Nz + u_j - u_i) + (z + u_k - u_j) - (Nz + u_k - u_l) \in V.$$

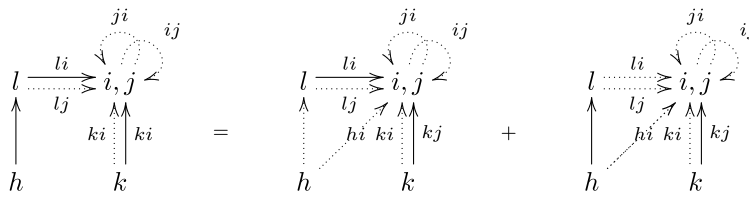
(iii) Suppose that there are two vertices i and j with same coordinates (a, b). We may suppose that there are no more lattice point $(x, y) \in \mathbb{Z}^2$ with double vertices such that $x \leq a$, $y \leq b$ and $(x, y) \neq (a, b)$. If there is only horizontal or vertical tail to i (and therefore to j, too) then by decomposition (D3) or (D4) we get $F = G_1 - G_2$ such that in Γ_{V,G_1} and Γ_{V,G_2} vertices i and j, respectively have no tails. Suppose that i (and therefore j) has both vertical and horizontal tails. By the assumption that all non-zero vertex i has coordinates in $\{(x, y) \in \mathbb{Z}^2 \,|\, x > 0,\, y \geq 0\}$ we can assume that $(a, b) \neq (1, 0)$. By decompositions (D3) and (D4) we get

$$\begin{aligned}
F = G' & \frac{(u_i - u_j)(u_j - u_i)}{(z + u_i - u_l)(z + u_j - u_l)(Nz + u_i - u_k)(Nz + u_j - u_k)} \\
= & - G' \frac{1}{(z + u_i - u_l)(Nz + u_i - u_k)} + G' \frac{1}{(z + u_i - u_l)(Nz + u_j - u_k)} \\
& + G' \frac{1}{(z + u_j - u_l)(Nz + u_i - u_k)} - G' \frac{1}{(z + u_j - u_l)(Nz + u_j - u_k)} \\
= & - G_1 + G_2 + G_3 - G_4
\end{aligned}$$

or in picture



Remark that in Γ_{V,G_1} and Γ_{V,G_4} the vertex i and j respectively has no tail, hence G_1 and G_4 do not contribute. By symmetry we consider only G_2. Suppose that l has a vertical tail $h \to l$. By decomposition (D1) we get $G_2 = H_1 + H_2$, that is,



and in Γ_{V,H_2} the vertex i has no tails, thus H_2 does not contribute. Iterating decomposition (D1) we can assume that all vertices l' before l (i.e. with coordinates $(a_{l'}, b_{l'})$ such that $a_{l'} \leq a_l$ and $b_{l'} = b_l$) may have only horizontal tail. Therefore, the first vertex in the row has no tails and it is not the zero vertex, hence H_1, and thus G_2, does not contribute. We can show similarly that G_3 does not contribute either.

$\square$

From Lemma 5.9 and 5.10 follows

Corollary 5.11. *A pole V does not contribute unless*

(i) there are no double vertices in Γ_V, that is, no vertices with same coordinates,

(ii) all non-zero vertex in Γ_V has coordinates in $\mathbb{Z}_{>0} \times N\mathbb{Z}_{\geq 0}$,

(iii) if there is a vertex in Γ_V with coordinates $(a, b) \in \mathbb{Z}_{>0} \times N\mathbb{Z}_{\geq 0}$ then for any $(x, y) \in \mathbb{Z}_{>0} \times N\mathbb{Z}_{\geq 0}$ such that $x \leq a$ and $y \leq b$ there is a vertex with coordinates (x, y).

Remark 5.12. Graphs Γ_V satisfying properties Corollary 5.11(i)-(iii) are in one-to-one correspondence with Young diagrams Y_λ of partitions $\lambda \vdash n$ with boxes labeled by $1, \ldots, n$. The correspondence is as follows.

$$\begin{array}{|c|c|cc} \hline 3 & 1 \\ \cline{1-2} \end{array} \qquad\qquad \begin{tikzcd} & 3 \arrow[r] & 1 & & \\ 0 \arrow[r] & 5 \arrow[r] \arrow[u] & 2 \arrow[r] \arrow[u] & 4 \arrow[r] & 6 \end{tikzcd}$$

$$\begin{array}{|c|c|c|c|} \hline 5 & 2 & 4 & 6 \\ \hline \end{array}$$

To the box of Y_λ in column x_i and row y_i labeled by i it corresponds the vertex i of Γ_V with coordinates $(x_i, (y_i - 1)N) \in \mathbb{Z}^2$. $\diamond$

Remark 5.13. There is a natural action of the symmetric group S_n on Γ_V permuting the non-zero vertices, which induces S_n-action on poles. Poles in the same S_n-orbit yield the same contribution. More precisely, if $V = \operatorname{span}\langle u_1 - p_1 z, \ldots, u_n - p_n z\rangle$ and $\pi \in S_n$ then $V' = \pi \cdot V = \operatorname{span}\langle u_{\pi(1)} - p_{\pi(1)} z, \ldots, u_{\pi(n)} - p_{\pi(n)} z\rangle$ and we have

$$\mathrm{JKRes}^{\Lambda_V} \beta(u(v,z),z) F(u(v,z),z) e^{\epsilon \sum\limits_{i=1}^{n} u_i(v,z)} dv =$$
$$\mathrm{JKRes}^{\Lambda_{V'}} \beta(u(v',z),z) F(u(v',z),z) e^{\epsilon \sum\limits_{i=1}^{n} u_i(v',z)} dv',$$

because $\beta(u,z)F(u,z)e^{\epsilon \sum_{i=1}^{n} u_i}$ is S_n-invariant. Moreover, the S_n-action is free on contributing poles, because for any $i = 1, \ldots, n$ we have $p_i = -x_i - (y_i - 1)N$ and $p_i = p_j$ only if $(x_i, y_i) = (x_j, y_j)$ since $n < N$. $\diamond$

Lemma 5.14. *Let V be a pole such that Γ_V is associated to a labeled Young diagram Y_λ of a partition $\lambda = (\lambda_1 \geq \ldots \geq \lambda_k)$ of n. Then*

$$\lim_{\epsilon \to 0^+} \mathrm{JKRes}^{\Lambda_V} \beta(u(v,z),z)F(u(v,z),z)e^{\epsilon \sum_{i=1}^{n} u_i(v,z)} dv = \beta(p_\lambda(z),z)b_\lambda(z),$$

where $0 \neq b_\lambda(z) \in \mathbb{R}[z^{\pm 1}]$.

Proof. Starting with $G = F$ we pick the top-left subgraph $\begin{tikzcd} l \arrow[r] & k \\ i \arrow[u] \arrow[r] & j \arrow[u] \end{tikzcd}$ in Γ_V and we apply decomposition (D1) to get $G = G_1 + G_2$. That is, in picture

$$\begin{tikzcd} l \arrow[r] & k \\ i \arrow[u] \arrow[r] & j \arrow[u] \end{tikzcd} = \begin{tikzcd} l \arrow[r] & k \\ i \arrow[u, dotted] \arrow[ur, dotted] \arrow[r] & j \arrow[u] \end{tikzcd} + \begin{tikzcd} l \arrow[r, dotted] & k \\ i \arrow[u] \arrow[ur, dotted] \arrow[r] & j \arrow[u] \end{tikzcd}.$$

Since we have picked the top-left rectangle, the vertex l has no tail in Γ_{V,G_1} by Corollary 5.11(iii), hence

$$\mathrm{JKRes}^{\Lambda_V} \beta(u(v,z),z)F(u(v,z),z)e^{\epsilon \sum_{i=1}^{n} u_i(v,z)} dv$$
$$= \mathrm{JKRes}^{\Lambda_V} \beta(u(v,z),z)G_2(u(v,z),z)e^{\epsilon \sum_{i=1}^{n} u_i(v,z)} dv$$

by Lemma 5.9(i). We replace F by G_2 and we continue this decomposition until no rectangle is left. Finally, we arrive to a graph $\Gamma_{V,H}$, where any horizontal edge not in the bottom row is replaced by a dotted arrow and in every rectangle there is a dotted diagonal arrow.

$$\begin{tikzcd} & 8 \arrow[r] & 9 & & \\ & 5 \arrow[u] \arrow[r] & 6 \arrow[u] \arrow[r] & 7 & \\ 0 \arrow[r] & 1 \arrow[u] \arrow[r] & 2 \arrow[u] \arrow[r] & 3 \arrow[u] \arrow[r] & 4 \end{tikzcd} \qquad \begin{tikzcd} & 8 \arrow[r, dotted] & 9 & & \\ & 5 \arrow[u] \arrow[ur, dotted] \arrow[r, dotted] & 6 \arrow[u] \arrow[r, dotted] & 7 & \\ 0 \arrow[r] & 1 \arrow[u] \arrow[ur, dotted] \arrow[r] & 2 \arrow[u] \arrow[ur, dotted] \arrow[r] & 3 \arrow[u] \arrow[r] & 4 \end{tikzcd}$$

In particular,

$$\mathrm{JKRes}^{\Lambda_V} \beta(u(v,z),z)F(u(v,z),z)e^{\epsilon \sum_{i=1}^{n} u_i(v,z)} dv \tag{5.3}$$
$$= \mathrm{JKRes}^{\Lambda_V} \beta(u(v,z),z)H(u(v,z),z)e^{\epsilon \sum_{i=1}^{n} u_i(v,z)} dv.$$

To each non-zero vertex there is exactly one (solid) edge, hence the set $\mathcal{H}$ of linear terms in the denominator of H which belong to V has exactly n elements. We have $\epsilon \sum_{i=1}^{n} u_i \in Cone(\mathrm{pr}_{\mathfrak{t}^*}(\overline{\alpha}) \,|\, \alpha \in \mathcal{H})$ since

$$\sum_{i=1}^{n} u_i = \sum_{(l \to k) \in \Gamma_{V,H}} C_{kl}(u_k - u_l)$$

with $C_{kl} > 0$. The coefficients C_{kl} can be computed as follows. We introduce a partial order on vertices of $\Gamma_{V,H}$ such that $l \prec k$ if there is an oriented (solid) path in $\Gamma_{V,H}$ from l to k. Then $C_{kl} = \#\{h \,|\, l \prec h\}$.

Denote α_i the element of $\mathcal{H}$ with positive u_i-coefficient. Sort non-zero vertices $\{i_1, \ldots, i_n\}$ of $\Gamma_{V,H}$ such that $k < l$ when $i_k \prec i_l$. Hence the matrix $\left[\dfrac{\partial \alpha_{i_k}(u,z)}{\partial u_{i_l}}\right]_{k,l=1}^{n}$ is lower triangular with 1's on the diagonal. Therefore, we have

$$\sqrt{\det[(\alpha_k, \alpha_l)]_{k,l=1}^{n}} = \left| \det \left[\frac{\partial \alpha_{i_k}(u,z)}{\partial u_{i_l}}\right]_{k,l=1}^{n} \right| = 1,$$

and in particular $\mathcal{H}$ spans V. Denote x_i and y_i respectively the column and row of the box labeled by i in Y_λ. Then we have $H = H' \dfrac{1}{\prod_{i=1}^{n} \alpha_i}$, where

$$H'(u,z) = \frac{\prod\limits_{1 \le i \ne j \le n} (u_i - u_j) \prod\limits_{\substack{1 \le i \ne j \le n \\ (x_i, y_i) \ne (x_j+1, y_j+1)}} ((N+1)z + u_i - u_j)}{\prod\limits_{\substack{1 \le i \ne j \le n \\ (x_i, y_i) \ne (x_j+1, y_j)}} (z + u_i - u_j) \prod\limits_{\substack{1 \le i \ne j \le n \\ (x_i, y_i) \ne (x_j, y_j+1)}} (Nz + u_i - u_j)} \cdot \frac{1}{\prod\limits_{\substack{1 \le k \le n \\ x_k \ne 1}} (z + u_k) \prod\limits_{1 \le k \le n} (Nz - u_k)}. \tag{5.4}$$

By the correspondence between labeled Young diagrams and graphs the vertex i in Γ_V has coordinates $(x_i, (y_i - 1)N) \in \mathbb{Z}^2$, hence $V = \mathrm{span}\langle u_1 - p_1 z, \ldots, u_n - p_n z\rangle$ such that $p_i = -(x_i + (y_i - 1)N)$ for all $i = 1, \ldots, n$. Let

$$b_\lambda(z) = H'(p_1 z, \ldots, p_n z, z) \tag{5.5}$$

and remark that $b_\lambda(z) \ne 0$. By Proposition 3.39 and (5.3) we conclude

$$\begin{aligned} \lim_{\epsilon \to 0^+} \mathrm{JKRes}^{\Lambda_V} \beta(u(v,z),z) F(u(v,z),z) e^{\epsilon \sum_{i=1}^{n} u_i(v,z)} dv &= \frac{\beta(u(0,z),z) H'(u(0,z),z)}{\sqrt{\det[(\alpha_k, \alpha_l)]_{k,l=1}^{n}}} \\ &= \beta_\lambda(p_\lambda(z), z) b_\lambda(z). \end{aligned}$$

□

Finally, the theorem follows from Corollary 5.11, Remarks 5.12 and 5.13, Lemma 5.14 and (5.2). □

5.2 Nekrasov's partition function

Consider $\mathcal{M}(r,n) = T^*(\mathrm{End}(\mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)) = M_n(\mathbb{C}) \oplus M_{n,r}(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus M_{r,n}(\mathbb{C})$ with $U(n)$-action

$$g \cdot (A, a, B, b) = (gAg^{-1}, ga, gBg^{-1}, bg^{-1})$$

for all $g \in U(n)$, $A, B \in M_n(\mathbb{C})$, $a, b^t \in M_{n,r}(\mathbb{C})$. This action is hyper-Hamiltonian with hyperKähler moment map $\mu = (\mu_{\mathbb{R}}, \mu_{\mathbb{C}}) : \mathcal{M}(r,n) \to \mathfrak{u}(n)^* \oplus \mathfrak{u}(n)^*_{\mathbb{C}}$, where

$$\begin{aligned}
&\mu_{\mathbb{R}} : (\mathcal{M}(r,n), \omega_{\mathbb{R}}) \to \mathfrak{u}(n)^*, \quad \mu_{\mathbb{R}}(A, a, B, b) = \frac{\sqrt{-1}}{2} \left([A, A^*] + [B, B^*] + aa^* - b^*b\right), \\
&\mu_{\mathbb{C}} : (\mathcal{M}(r,n), \omega_{\mathbb{C}}) \to \mathfrak{u}(n)^*_{\mathbb{C}}, \quad \mu_{\mathbb{C}}(A, a, B, b) = [A, B] + ab
\end{aligned}$$

with identification $\mathfrak{u}(n) \simeq \mathfrak{u}(n)^*$ induced by non-degenerate bilinear pairing $(u, v) = \mathrm{Tr}(u^*v)$ for all $u, v \in \mathfrak{u}(n)$. Moreover, $(\xi, 0) = (\sqrt{-1}I, 0) \in \mathfrak{u}(n)^* \oplus \mathfrak{u}(n)^*_{\mathbb{C}}$ is a regular value of μ and the hyperKähler quotient $\mathcal{M}(r,n)/\!\!/\!\!/\!\!/_{(\xi,0)}U(n)$ is isomorphic to the framed moduli space of torsion free sheaves on $\mathbb{CP}^2$ with rank r and second Chern class $c_2 = n$ [36]. We consider $T = \{g \in U(n) \,|\, g \text{ diagonal}\}$ maximal torus of G and we choose a basis $\{u_1, \ldots, u_n\}$ of $\mathfrak{t}^*$ such that $u_i\left(\mathrm{diag}(\sqrt{-1}\tau_1, \ldots, \sqrt{-1}\tau_n)\right) = \tau_i$ for all $i = 1, \ldots, n$. The real and complex abelian moment maps are as follows

$$\mu^T_{\mathbb{R}} : (\mathcal{M}(r,n), \omega_{\mathbb{R}}) \to \mathfrak{t}^*,\ \mu^T_{\mathbb{R}}(A, a, B, b) = \sum_{i,j=1}^{n} (u_i - u_j) \frac{|A_{ij}|^2 + |B_{ij}|^2}{2} + \sum_{i=1}^{n} u_i \sum_{j=1}^{r} \frac{|a_{ij}|^2 - |b_{ji}|^2}{2},$$

$$\mu^T_{\mathbb{C}} : (\mathcal{M}(r,n), \omega_{\mathbb{C}}) \to \mathfrak{t}^*_{\mathbb{C}},\ \mu^T_{\mathbb{C}}(A, a, B, b) = -\sqrt{-1} \sum_{i=1}^{n} u_j \left(\sum_{j=1}^{n} A_{ij}B_{ji} - B_{ij}A_{ji} + \sum_{j=1}^{r} a_{ij}b_{ji} \right).$$

Moreover, ξ is a regular value of $\mu^T_{\mathbb{R}}$ because $u_1 + \ldots + u_n$ is regular with respect to the set $\{u_i - u_j,\, u_j \,|\, i, j = 1, \ldots, n\}$, thus the symplectic quotient $\mathcal{M}(r,n)/\!\!/_{\xi}T$ exists. Furthermore, $(\xi, 0)$ is also a regular value of the abelian hyperKähler moment map $\mu^T = (\mu^T_{\mathbb{R}}, \mu^T_{\mathbb{C}})$ and the abelian hyperKähler quotient $\mathcal{M}(r,n)/\!\!/\!\!/\!\!/_{(\xi,0)}T$ also exists.

We also consider a Hamiltonian torus action of $S = U(1)^{r+2}$ on $(\mathcal{M}(r,n), \omega_{\mathbb{R}})$ given by

$$s \cdot (A, a, B, b) = \left(s_{r+1} \cdot A,\ a \cdot \mathrm{diag}(s_1, \ldots, s_r)^{-1},\ s_{r+2} \cdot B,\ s_{r+1}s_{r+2}\mathrm{diag}(s_1, \ldots, s_r) \cdot b\right)$$

for all $s = (s_1, \ldots, s_{r+2}) \in U(1)^{r+2}$, $A, B \in M_n(\mathbb{C})$, $a, b^t \in M_{n,r}(\mathbb{C})$ (cf. [37], Lemma 2.8). If $\{x, y, z_1, \ldots, z_r\}$ is a basis of $\mathfrak{s}^*$ such that for $\sigma = \mathrm{diag}(\sqrt{-1}\sigma_1, \ldots, \sqrt{-1}\sigma_{r+2})$ we have $x(\sigma) = \sigma_{r+1}$, $y(\sigma) = \sigma_{r+2}$ and $z_i(\sigma) = \sigma_i$ for all $i = 1, \ldots, r$ then the S-action has moment map $\phi_S : (\mathcal{M}(r,n),\, \omega_{\mathbb{R}}) \to \mathfrak{s}^*$,

$$\phi_S(A, a, B, b) = \frac{1}{2}\mathrm{Tr}(AA^* + bb^*)x + \frac{1}{2}\mathrm{Tr}(BB^* + bb^*)y + \sum_{j=1}^{r}\sum_{i=1}^{n} \frac{|b_{ij}|^2 - |a_{ji}|^2}{2} z_i.$$

Let $K = \{(k, k, \mathrm{diag}(k^{-1}, \ldots, k^{-1})) \in S \,|\, k \in U(1)\}$ be 1-dimensional subtorus of S which acts on $\mathcal{M}(r,n)$ as

$$k \cdot (A, a, B, b) = (kA, ka, kB, kb)$$

for all $k \in U(1)$, $A, B \in M_n(\mathbb{C})$, $a, b^t \in M_{n,r}(\mathbb{C})$. It admits moment map $\phi_K = \varphi \cdot (x + y - z_1 - \ldots - z_r)$ with $\varphi : \mathcal{M}(r, n) \to \mathbb{R}$,

$$\varphi(A, a, B, b) = \frac{1}{2}\mathrm{Tr}(AA^* + BB^* + a^*a + bb^*)$$

proper and bounded below. Finally, we remark that the complex moment map is S-equivariant $\mu_{\mathbb{C}}(s \cdot (A, a, B, b)) = s_1 s_2 \mu_{\mathbb{C}}(A, a, B, b)$ and $(\mathfrak{t}_{\mathbb{C}}^*)^S = \{0\}$.

Definition 5.15. *Nekrasov's partition function* ([38], cf. [37]) is defined as

$$Z(x, y, z, \mathfrak{q}) = \sum_{n=0}^{\infty} \mathfrak{q}^n \oint_{\mathcal{M}(r,n)/\!\!/\!\!/\!\!/_{(\xi,0)}U(n)} 1.$$

Let $\delta = (\delta_1, \ldots, \delta_k)$ be a partition with $\delta_1 \geq \ldots \geq \delta_k$. In the Young diagram Y_δ associated to the partition δ the i^{th} column contains δ_i boxes.

Definition 5.16. For a box s in the column c_s and row r_s of Y_δ we define the *arm-length* $A_\delta(s) = \#\{h \,|\, h > c_s,\, \delta_h \geq r_s\}$, and the *leg-length* $L_\delta(s) = \delta_{c_s} - r_s$. That is, $A_\delta(s)$ and $L_\delta(s)$ are respectively the number of boxes on the right and on the top of the box s in Y_δ.

The definition of arm-length and leg-length also extends to boxes which are off Y_δ. Thus for any two partitions δ and η we define

$$K_{\delta,\eta}(x, y, \zeta) = \prod_{i \in Y_\delta} [-A_\delta(i)x + (L_\eta(i) + 1)y + \zeta] \prod_{j \in Y_\eta} [(A_\eta(j) + 1)x - L_\delta(j)y + \zeta]. \tag{5.6}$$

In order to compute Nekrasov's partition function we reprove the following formula using results of Chapter 4.

Theorem 5.17 ([38, 37])**.**

$$\oint_{\mathcal{M}(r,n)/\!\!/\!\!/\!\!/_{(\xi,0)}U(n)} 1 = (2\pi)^{2rn} \sum_{(Y_{\lambda_1},\ldots,Y_{\lambda_r})} \frac{1}{\prod_{k,l=1}^{r} K_{\lambda_k,\lambda_l}(x, y, z_k - z_l)},$$

where the sum is over tuples of Young diagrams $(Y_{\lambda_1}, \ldots, Y_{\lambda_r})$ such that $n = \sum_{i=1}^{r} n_i$ and $\lambda_i \vdash n_i$ for all $i = 1, \ldots, r$.

Remark 5.18. Nekrasov computes the integral $\oint_{\mathcal{M}(r,n)/\!\!/\!\!/\!\!/_{(\xi,0)}U(n)} 1$ via contour integrals and in [38] (3.20) arrives to a similar result as in (5.14). His method is close to ours in spirit. Nakajima and Yoshioka use the Atiyah-Bott-Berline-Vergne formula on the quotient space $\mathcal{M}(r, n)/\!\!/\!\!/\!\!/_{(\xi,0)}U(n)$ to arrive to the same formula as above in [37], (6.2). ◇

Proof. Let Λ be the polarization on $\mathfrak{t}^* \oplus \mathfrak{s}^*$ induced by ordered basis

$$\{x + y - z_1 \ldots - z_r,\, x - y,\, x + z_1, \ldots, x + z_r, u_1, \ldots, u_r\},$$

which is compatible with the proper bounded below moment map $(\phi_K, x+y-z_1-\ldots-z_r, \varphi)$. By Theorem 4.14 we have

$$\oint_{\mathcal{M}(r,n)/\!/\!/\!/_{(\xi,0)}U(n)} 1 = \lim_{\epsilon\to 0^+} \mathrm{EqRes}^{\Lambda}\left(\frac{\vartheta\varpi_{\mathbb{R}}\varpi_{\mathbb{C}}}{n!\,vol(T)}\oint_{\mathcal{M}(r,n)} e^{\epsilon(\omega_{\mathbb{R}}-\mu_{\mathbb{R}}^T-\phi_S+\xi)}\right)$$

$$= \lim_{\epsilon\to 0^+} \mathrm{EqRes}^{\Lambda}\left(\frac{(2\pi)^{(2r+1)n}}{n!\,vol(T)}\frac{\prod\limits_{1\le i\neq j\le n}(u_i-u_j)\prod\limits_{1\le i,j\le n}(x+y+u_i-u_j)e^{\epsilon\sum\limits_{i=1}^{n}u_i}}{\prod\limits_{\substack{1\le i,j\le n\\ 1\le k\le r}}(x+u_i-u_j)(y+u_i-u_j)(u_i-z_k)(x+y+z_k-u_j)}\right)$$

$$= \lim_{\epsilon\to 0^+}\frac{(2\pi)^{(2r+1)n}}{n!\,vol(T)}\frac{(x+y)^n}{x^ny^n}\mathrm{EqRes}^{\Lambda}\left(F(u,x,y,z)e^{\epsilon\sum\limits_{i=1}^{r}u_i}\right),$$

where

$$F(u,x,y,z) = \prod_{1\le i\neq j\le n}\frac{(u_i-u_j)(x+y+u_i-u_j)}{(x+u_i-u_j)(y+u_i-u_j)}\prod_{\substack{1\le i,j\le n\\ 1\le k\le r}}\frac{1}{(u_i-z_k)(x+y+z_k-u_j)}.$$

Denote $\mathcal{A} = \{x+u_i-u_j,\, y+u_i-u_j,\, u_i-z_k,\, x+y+z_k-u_j \,|\, 1\le i\neq j\le n, 1\le k\le r\}$ the set of linear terms in the denominator of F and remark that all elements of $\mathcal{A}$ are polarized with respect to Λ.

To compute the above residue we consider $\mathfrak{t}^*$-poles

$$V = \mathrm{span}\left\langle u_1-p_V^1(x,y,z),\,\ldots,\,u_n-p_V^n(x,y,z)\right\rangle$$

spanned by subsets of $\mathcal{A}$, and where $p_V^i(x,y,z)\in\mathfrak{s}^*$ for all $i=1,\ldots,n$. We can describe these $\mathfrak{t}^*$-poles in terms of graphs as follows. Fix a $\mathfrak{t}^*$-pole V and define subsets $\mathcal{N}_V^0,\ldots,\mathcal{N}_V^r$ of $\{1,\ldots,n\}$ by the following rules

(1) $i\in\mathcal{N}_V^k$ if u_i-z_k or $x+y+z_k-u_i$ is in V,

(2) $i\in\mathcal{N}_V^k$ if $j\in\mathcal{N}_V^k$ and $\varepsilon+u_i-u_j$ or $\varepsilon+u_j-u_i$ belongs to V for any $\varepsilon\in\{x,y\}$,

(3) elements of $\{1,\ldots,n\}$ unsorted by (1) and (2) are listed in $\mathcal{N}_V^0$.

We also define subsets $\mathcal{A}_V^0,\ldots,\mathcal{A}_V^r$ of $\mathcal{A}_V=\mathcal{A}\cap V$ as follows: $\alpha\in\mathcal{A}_V^k$ if u_i or $-u_i$ is a summand of $\alpha\in\mathcal{A}_V$ and $i\in\mathcal{N}_V^k$.

Lemma 5.19. *We have decomposition $\{1,\ldots,n\}=\mathcal{N}_V^1\uplus\ldots\uplus\mathcal{N}_V^r$ and consequently, $\mathcal{A}_V=\mathcal{A}_V^1\uplus\ldots\uplus\mathcal{A}_V^r$. In particular, $\mathcal{N}_V^0=\mathcal{A}_V^0=\emptyset$.*

Proof. We have $\{1,\ldots,n\}=\mathcal{N}_V^0\cup\ldots\cup\mathcal{N}_V^r$ and $\mathcal{A}_V=\mathcal{A}_V^0\cup\ldots\cup\mathcal{A}_V^r$. Moreover, $\mathcal{N}_V^l=\emptyset$ if and only if $\mathcal{A}_V^l=\emptyset$. Therefore, it is enough to show that $\mathcal{N}_V^0=\emptyset$ and $\mathcal{N}_V^k\cap\mathcal{N}_V^l=\emptyset$ for all

$1 \leq k \neq l \leq r$. If $i \in \mathcal{N}_V^k$ then $p_V^i(x,y,z)$ has of form $q_1x+q_2y+z_k$, hence $\mathcal{N}_V^k \cap \mathcal{N}_V^l = \emptyset$ if $k \neq l$. By construction, $\mathcal{N}_V^0 \cap (\mathcal{N}_V^1 \cup \ldots \cup \mathcal{N}_V^r) = \emptyset$. If $\mathcal{N}_V^0 \neq \emptyset$ then $\sum\limits_{i \in \mathcal{N}_V^0} u_i \notin \operatorname{span}\langle \operatorname{pr}_{\mathfrak{t}^*}(\mathcal{A}_V^0)\rangle$, therefore

$$\begin{aligned}\operatorname{pr}_{\mathfrak{t}^*}(V) &= \operatorname{span}\langle \operatorname{pr}_{\mathfrak{t}^*}(\mathcal{A}_V^0)\rangle + \operatorname{span}\langle \operatorname{pr}_{\mathfrak{t}^*}(\mathcal{A}_V^1 \cup \ldots \cup \mathcal{A}_V^r)\rangle \\ &\neq \operatorname{span}\langle u_i \,|\, i \in \mathcal{N}_V^0\rangle + \operatorname{span}\langle u_i \,|\, i \in \mathcal{N}_V^1 \cup \ldots \cup \mathcal{N}_V^r\rangle = \mathfrak{t}^*,\end{aligned}$$

which leads to contradition that V is a $\mathfrak{t}^*$-pole. □

To a $\mathfrak{t}^*$-pole V we associate a tuple of graphs $\Gamma_V = (\Gamma_V^1, \ldots, \Gamma_V^r)$ as follows. The vertices of Γ_V^k lie on the lattice $\mathbb{Z}^2$ and they are labeled by elements of $\{0\} \cup \mathcal{N}_V^k$. The coordinates of 0 and i are respectively $(0,0)$ and (a_i, b_i), where

$$a_i x + b_i y = -p_V^i(x, y, -(x, \ldots, x)) \tag{5.7}$$

for all $i \in \mathcal{N}_V^k$. The oriented edges of Γ_V^k are drawn according to the table

$\alpha \in \mathcal{A}_V^k$	edges of Γ_V^k
$x + u_i - u_j$	horizontal edge from j to i
$y + u_i - u_j$	vertical edge from j to i
$u_i - z_k$	horizontal edge from 0 to i
$x + y + z_k - u_j$	vertical edge from j to 0.

The graph is complete in the following sense. Let i and j be two vertices in Γ_V^k with coordinates (a_i, b_i) and (a_j, b_j), respectively. If $(a_j, b_j) = (a_i, b_i + 1)$ and $i \neq 0$ then there is an edge $i \to j$ in Γ_V^k. If $(a_j, b_j) = (a_i + 1, b_i)$ and $j \neq 0$ then we have an edge $i \to j$ in Γ_V^k.

We compute $\operatorname{JKRes}^{\Lambda_V} F(u(v,x,y,z),x,y,z)e^{\epsilon \sum_{i=1}^n u_i(v,x,y,z)} dv$ by decomposing F to partial fractions. We only use the following type of decompositions:

$$\text{(D1)} \quad \frac{x+y+u_l-u_i}{(x+u_j-u_i)(y+u_l-u_j)} = \frac{1}{y+u_l-u_j} + \frac{1}{x+u_j-u_i} \text{ if } x+u_j-u_i,\, y+u_l-u_j \in V,$$

$$\text{(D2)} \quad \frac{x+y+u_l-u_i}{(y+u_j-u_i)(x+u_l-u_j)} = \frac{1}{x+u_l-u_j} + \frac{1}{y+u_j-u_i} \text{ if } y+u_j-u_i,\, x+u_l-u_j \in V,$$

$$\text{(D3)} \quad \frac{x+y+u_i-u_j}{(x+y+z_k-u_j)(u_i-z_k)} = \frac{1}{u_i-z_k} + \frac{1}{x+y+z_k-u_j} \text{ if } x+y+z_k-u_j,\, u_i-z_k \in V,$$

$$\text{(D4)} \quad \frac{u_i-u_j}{(x+u_i-u_l)(x+u_j-u_l)} = \frac{1}{x+u_j-u_l} - \frac{1}{x+u_i-u_l} \text{ if } x+u_i-u_l,\, x+u_j-u_l \in V,$$

$$\text{(D5)} \quad \frac{u_i-u_j}{(y+u_i-u_l)(y+u_j-u_l)} = \frac{1}{y+u_j-u_l} - \frac{1}{y+u_i-u_l} \text{ if } y+u_i-u_l,\, y+u_j-u_l \in V.$$

We will keep track of fractions arising from these partial fraction decompositions on mutations of the tuple of graphs Γ_V. Applying these decompositions to F several times it yields fractions G which can be got from F by removing certain linear terms from the nominator and denominator of F. Moreover, G can be encoded in a tuple of graphs $\Gamma_{V,G} = (\Gamma_{V,G}^1, \ldots, \Gamma_{V,G}^r)$ by modifying $\Gamma_V = (\Gamma_V^1, \ldots, \Gamma_V^r)$ as follows:

- if $i, j \in \Gamma_V^k$ have same coordinates and $u_i - u_j$ is missing from the nominator of G then we draw a hook $\begin{tikzcd} i,j \arrow[loop right, dotted, "ji"] \end{tikzcd}$ in $\Gamma_{V,G}^k$,
- if $i, j \in \Gamma_V^k$ are such that $(a_i, b_i) = (a_j + 1, b_j + 1)$ and $x + y + u_i - u_j$ is missing from the nominator of G then we have diagonal arrow $j \dashrightarrow i$ in $\Gamma_{V,G}^k$,
- if $i, j \in \Gamma_V^k$ are such that $x + u_i - u_j$, $y + u_i - u_j$, $u_i - z_k$, $x + y + z_k - u_j \in V$ respectively are missing from the denominator of G then the corresponding arrows $j \longrightarrow i$, $0 \longrightarrow i$, $j \longrightarrow 0$ are replaced by $j \dashrightarrow i$, $0 \dashrightarrow i$, $j \dashrightarrow 0$ respectively in $\Gamma_{V,G}^k$.

The above five types of partial fraction decomposition correspond to following mutations of subgraphs (only the part is drawn, where the mutation happens):

(M1)
$$\begin{tikzcd} & l \\ i \arrow[r] & j \arrow[u] \end{tikzcd} = \begin{tikzcd} & l \\ i \arrow[r, dotted] \arrow[ur, dotted] & j \arrow[u] \end{tikzcd} + \begin{tikzcd} & l \\ i \arrow[r] \arrow[ur, dotted] & j \arrow[u, dotted] \end{tikzcd} \quad \text{if } i, j, l \in \Gamma_V^k \text{ are non-zero vertices,}$$

(M2)
$$\begin{tikzcd} j \arrow[r] & l \\ i \arrow[u] & \end{tikzcd} = \begin{tikzcd} j \arrow[r] & l \\ i \arrow[u, dotted] \arrow[ur, dotted] & \end{tikzcd} + \begin{tikzcd} j \arrow[r, dotted] & l \\ i \arrow[u] \arrow[ur, dotted] & \end{tikzcd} \quad \text{if } i, j, l \in \Gamma_V^k \text{ are non-zero vertices,}$$

(M3)
$$\begin{tikzcd} 0 \arrow[r] & i \\ j \arrow[u] & \end{tikzcd} = \begin{tikzcd} 0 \arrow[r] & i \\ j \arrow[u, dotted] \arrow[ur, dotted] & \end{tikzcd} + \begin{tikzcd} 0 \arrow[r, dotted] & i \\ j \arrow[u] \arrow[ur, dotted] & \end{tikzcd} \quad \text{if } i, j \in \Gamma_V^k,$$

(M4)
$$\begin{tikzcd} l \arrow[r, shift left, "li"] \arrow[r, shift right, "lj"'] & i,j \end{tikzcd} = \begin{tikzcd} l \arrow[r, shift left, dotted, "li"] \arrow[r, shift right, "lj"'] & i,j \arrow[loop right, dotted, "ji"] \end{tikzcd} - \begin{tikzcd} l \arrow[r, shift left, "li"] \arrow[r, shift right, dotted, "lj"'] & i,j \arrow[loop right, dotted, "ji"] \end{tikzcd} \quad \text{if } i, j, l \in \Gamma_V^k,$$

(M5)
$$\begin{tikzcd} i,j \\ l \arrow[u, shift right, "li"] \arrow[u, shift left, "lj"'] \end{tikzcd} = \begin{tikzcd} i,j \arrow[loop above, dotted, "ji"] \\ l \arrow[u, shift right, dotted, "li"] \arrow[u, shift left, "lj"'] \end{tikzcd} - \begin{tikzcd} i,j \arrow[loop above, dotted, "ji"] \\ l \arrow[u, shift right, "li"] \arrow[u, shift left, dotted, "lj"'] \end{tikzcd} \quad \text{if } i, j, l \in \Gamma_V^k.$$

Definition 5.20. We say that G *does not contribute* if

$$\lim_{\epsilon \to 0^+} \mathrm{JKRes}^{\Lambda_V} G(u(v,x,y,z),x,y,z) e^{\epsilon \sum_{i=1}^n u_i(v,x,y,z)} dv = 0.$$

When $G = F$ we say that the $\mathbf{t}^*$-pole V does not contribute.

Definition 5.21. A *tail* of a vertex $j \in \Gamma_{V,G}^k$ is an edge $i \to j$.

Similarly to Lemma 5.9 and Corollary 5.11 we can show that

Lemma 5.22. *G does not contribute unless* *(i) every non-zero vertex $i \in \Gamma^k_{V,G}$ has a tail,*

(ii) every non-zero vertex i has coordinates $(a_i, b_i) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0}$,

(iii) for every vertex $i \in \Gamma^k_V$ having coordinates (a_i, b_i) and for any $(x, y) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0}$ such that $x \leq a_i$, $y \leq b_i$ there is a vertex $j \in \Gamma^k_V$ with coordinates (x, y),

(iv) for any $k = 1, \ldots, r$ the graph Γ^k_V has no double vertex, i.e. two vertices with same coordinates.

Tuples of graphs $\Gamma_V = (\Gamma^1_V, \ldots, \Gamma^r_V)$ satisfying Lemma 5.22 (i)-(iv) are in one-to-one correspondence with tuples of labeled Young diagrams $Y_\lambda = (Y_{\lambda_1}, \ldots, Y_{\lambda_r})$ of partitions $\lambda_k \vdash |\mathcal{N}^k_V|$ such that boxes of Y_{λ_k} are labeled by elements of $\mathcal{N}^k_V$ for all $k = 1, \ldots, r$. To a Young diagram Y_{λ_k} labeled by elements of $\mathcal{N}^k_V$ we associated Γ^k_V as follows. To the box labeled by i in the column c_i and row r_i of Y_{λ_k} it corresponds the vertex i of Γ^k_V with coordinates $(c_i, r_i - 1) \in \mathbb{Z}^2$. We draw all horizontal and vertical edges between vertices on neighboring lattice points. We denote by $V_{(\lambda_1,\ldots,\lambda_r)}$ the $\mathfrak{t}^*$-pole corresponding to the tuple of labeled Young diagrams $(Y_{\lambda_1}, \ldots, Y_{\lambda_r})$.

To evaluate $\text{JKRes}^{\Lambda_V} F(u(v,x,y,z),x,y,z)e^{\epsilon \sum_{i=1}^n u_i(v,x,y,z)} dv$ for $V = V_{(\lambda_1,\ldots,\lambda_r)}$ we choose the top-leftmost rectangle $ijlm \in \Gamma^k_V$ for all $k = 1, \ldots, r$ and we apply decomposition (D2), that is, we mutate Γ^k_V as

$$\begin{tikzcd} l \arrow[r] & m \\ i \arrow[u] \arrow[r] & j \arrow[u] \end{tikzcd} \quad = \quad \begin{tikzcd} l \arrow[r] & m \\ i \arrow[u, dotted] \arrow[ur, dotted] \arrow[r] & j \arrow[u] \end{tikzcd} \quad + \quad \begin{tikzcd} l \arrow[r, dotted] & m \\ i \arrow[u] \arrow[ur, dotted] \arrow[r] & j \arrow[u] \end{tikzcd} . \tag{5.8}$$

Remark that in the first graph on the right hand side of (5.8) the vertex l has no tail, thus the corresponding fraction does not contribute. We continue this procedure with the second graph of the right hand side of (5.8) until no rectangle is left. We arrive to a graph Γ^k_{V,G_V} which can be constructed from Γ^k_V by replacing all horizontal edges $l \longrightarrow m$ with $l \dashrightarrow m$ except in the bottom row of Γ^k_V and by adding diagonal arrow $i \dashrightarrow m$ for each rectangle $ijlm$ as above. In particular, $\deg G_V = \deg F$. Moreover,

$$\begin{aligned} \text{JKRes}^{\Lambda_V} & F(u(v,x,y,z),x,y,z)e^{\epsilon \sum_{i=1}^n u_i(v,x,y,z)} dv \\ & = \text{JKRes}^{\Lambda_V} G_V(u(v,x,y,z),x,y,z)e^{\epsilon \sum_{i=1}^n u_i(v,x,y,z)} dv. \end{aligned} \tag{5.9}$$

Remark that any non-zero vertex $i \in \Gamma^k_{V,G_V}$ has a unique tail $j \longrightarrow i$, denote by α^i_V the corresponding element of $\mathcal{A}_V$. That is,

$$\alpha^i_V = \begin{cases} x + u_i - u_j & \text{if } j \neq 0 \text{ and } j \longrightarrow i \text{ is horizontal,} \\ y + u_i - u_j & \text{if } j \neq 0 \text{ and } j \longrightarrow i \text{ is vertical,} \\ u_i - z_k & \text{if } j = 0. \end{cases}$$

We introduce a partial order on $\{1,\dots,n\}$. Let $i \prec j$ when $i \in \mathcal{N}_V^k$, $j \in \mathcal{N}_V^l$, $k \le l$ and in the $k = l$ case there is an oriented path from i to j in Γ^k_{V,G_V}. Then we can write

$$\sum_{i=1}^n u_i = \sum_{i=1}^n C_i \mathrm{pr}_{\mathfrak{t}^*}(\alpha_V^i) = \sum_{k=1}^r \sum_{(j\to i)\in\Gamma^k_{V,G_V}} C_i(u_i - u_j),$$

where $u_0 = 0$ and $C_i = \#\left\{h \in \mathcal{N}_V^k \,|\, i \prec h,\, i \in \mathcal{N}_V^k\right\} + 1$ is positive for any $i \in \Gamma_V^k$. It also implies that $\{\alpha_V^i \,|\, i = 1,\dots,n\}$ is a basis of V since $\sum\limits_{i=1}^n u_i$ is regular with respect to the set $\{u_i - u_j,\, u_j \,|\, i \ne j = 1,\dots,n\}$.

We choose a scalar product on $\mathfrak{t}^*$ such that $\{u_1,\dots,u_n\}$ is an orthonormal basis, hence

$$vol(T) = (2\pi)^n.$$

Let $\nu : \{1,\dots,n\} \to \{1,\dots,n\}$ be a bijection such that $i < j$ if $\nu(i) \prec \nu(j)$. Then $\left(\dfrac{\partial\alpha_V^{\nu(i)}}{\partial u_{\nu(j)}}\right)_{i,j=1}^n$ is lower triangular with 1's on the diagonal, thus

$$\sqrt{\det[(\alpha_V^i,\, \alpha_V^j)]_{i,j=1}^n} = \left|\det\left(\frac{\partial\alpha_V^{\nu(i)}}{\partial u_{\nu(j)}}\right)_{i,j=1}^n\right| = 1.$$

Denote $(a_i, b_i) \in \mathbb{Z}^2$ the coordinates of the vertex i. Then $G_V = G'_V \cdot \dfrac{1}{\prod\limits_{i=1}^n \alpha_V^i}$, where

$$G'_V(u,x,y,z) = \prod_{1\le k\ne l\le r} \prod_{\substack{i\in\mathcal{N}_V^k \\ j\in\mathcal{N}_V^l}} \frac{(u_i - u_j)(x+y+u_i-u_j)}{(x+u_i-u_j)(y+u_i-u_j)(u_i - z_l)(x+y+z_k-u_j)} \tag{5.10}$$
$$\prod_{k=1}^r \frac{\prod\limits_{i\ne j\in\mathcal{N}_V^k}(u_i-u_j) \prod\limits_{\substack{i\ne j\in\mathcal{N}_V^k \\ (a_i,b_i)\ne(a_j+1,b_j+1)}} (x+y+u_i-u_j)}{\prod\limits_{\substack{i\ne j\in\mathcal{N}_V^k \\ (a_i,b_i)\ne(a_j+1,b_j)}} (x+u_i-u_j) \prod\limits_{\substack{i\ne j\in\mathcal{N}_V^k \\ (a_i,b_i)\ne(a_j,b_j+1)}} (y+u_i-u_j)}$$
$$\prod_{k=1}^r \prod_{\substack{i,j\in\mathcal{N}_V^k \\ (a_i,b_i)\ne(1,0)}} \frac{1}{(u_i - z_k)(x+y+z_k-u_j)}.$$

Remark that G'_V does not contain linear terms in V. By Proposition 3.39 we get

$$\lim_{\epsilon\to 0^+} \mathrm{JKRes}^{\Lambda_V} G_V(u(v,x,y,z),x,y,z)e^{\epsilon\sum\limits_{i=1}^n u_i(v,x,y,z)}dv = G'_V(p_V(x,y,z),x,y,z), \tag{5.11}$$

where $p_V(x,y,z) = (p_V^1(x,y,z),\dots,p_V^n(x,y,z))$. If i labels a box of the Young diagram Y_{λ_k} then i is a vertex of the graph Γ_V^k, hence $p_V^i(x,y,z)$ is of form $q_1x + q_2y + z_k$. Moreover, by the Young diagram-graph correspondence and (5.7) we have

$$p_V^i(x,y,z) = (1-a_i)x - b_iy + z_k = (1-c_i)x + (1-r_i)y + z_k,$$

where c_i and r_i are respectively the column and the row of the box labeled by i in Y_{λ_k} of $(Y_{\lambda_1}, \dots, Y_{\lambda_r})$, and (a_i, b_i) are the coordinates of the vertex i in Γ_V^k.

We express $G'_V(p_V(x,y,z), x, y, z)$ in a more comprehensible manner. Therefore, we define fractions

$$H_{\lambda_k,\lambda_l}(x,y,\zeta) = \prod_{\substack{i\in Y_{\lambda_k}\\ j\in Y_{\lambda_l}}} \frac{[(c_j-c_i)x+(r_j-r_i)y+\zeta][(1+c_j-c_i)x+(1+r_j-r_i)y+\zeta]}{[(1+c_j-c_i)x+(r_j-r_i)y+\zeta][(c_j-c_i)x+(1+r_j-r_i)y+\zeta]} \prod_{\substack{i\in Y_{\lambda_k}\\ j\in Y_{\lambda_l}}} \frac{1}{[(1-c_i)x+(1-r_i)y+\zeta][c_jx+r_jy+\zeta]} \tag{5.12}$$

and

$$E_\zeta(u,x,y,z) = \prod_{1\le i,j\le n} \frac{(u_i-u_j+\zeta)(x+y+u_i-u_j+\zeta)}{(x+u_i-u_j+\zeta)(y+u_i-u_j+\zeta)} \prod_{\substack{1\le i,j\le n\\ 1\le k\le r}} \frac{1}{(u_i-z_k+\zeta)(x+y+z_k-u_j+\zeta)}.$$

Up to a ζ^{-n} factor $E_\zeta(u,x,y,z)$ is the deformation of $\frac{(x+y)^n}{x^ny^n}F(u,x,y,z)$ by adding ζ to each linear term, hence $\lim_{\zeta\to 0}\frac{E_\zeta(u,x,y,z)}{\zeta^n} = \frac{(x+y)^n}{x^ny^n}F(u,x,y,z)$. The fraction G'_V can be obtained from F by removing all linear terms which lie in V and remark that $\deg G'_V = \deg F - n$. To get G'_V we have removed exactly those linear terms from F which vanish under evaluation $u = p_V(x,y,z)$. The number of such linear terms in the denominator of F is bigger by n than in the nominator. Therefore,

$$\begin{aligned}
\frac{(x+y)^n}{x^ny^n}G'_V(p_V(x,y,z),x,y,z) &= \lim_{\zeta\to 0} E_\zeta(p_V(x,y,z),x,y,z)\\
&= \lim_{\zeta\to 0}\prod_{1\le k,l\le r} H_{\lambda_k,\lambda_l}(x,y,z_k-z_l+\zeta)\\
&= \prod_{1\le k\ne l\le r} H_{\lambda_k,\lambda_l}(x,y,z_k-z_l)\lim_{\zeta\to 0}\prod_{k=1}^{r} H_{\lambda_k,\lambda_k}(x,y,\zeta),
\end{aligned} \tag{5.13}$$

which does not depend on the labeling of $(Y_{\lambda_1}, \dots, Y_{\lambda_r})$.

Moreover, the symmetric group S_n acts on the set of $\mathfrak{t}^*$-poles such that

$$\sigma\cdot V = \operatorname{span}\left\langle u_{\sigma(1)} - p_V^{\sigma(1)}(x,y,z), \dots, u_{\sigma(n)} - p_V^{\sigma(n)}(x,y,z)\right\rangle, \qquad \forall\,\sigma\in S_n.$$

This action is free on the set of $\mathfrak{t}^*$-poles associated to tuples of labeled Young diagrams and it corresponds to permutation of labels. Poles in the same orbit yield the same contribution by (5.9), (5.11) and (5.13).

We summarize our calculations so far

$$\oint_{\mathcal{M}(r,n)/\!/\!/\!/_{(\xi,0)}U(n)} 1 = \lim_{\epsilon\to 0^+}\frac{(2\pi)^{(2r+1)n}}{n!\,vol(T)}\frac{(x+y)^n}{x^ny^n}\operatorname{EqRes}^\Lambda\left(F(u,x,y,z)e^{\epsilon\sum_{i=1}^n u_i}\right)$$

$$= \lim_{\epsilon\to 0^+} \sum_{\mathfrak{t}^*\text{-poles } V} \frac{(2\pi)^{(2r+1)n}}{n!\,vol(T)} \frac{(x+y)^n}{x^n y^n} \mathrm{JKRes}^{\Lambda_V}\left(F(u(v,x,y,z),x,y,z)e^{\epsilon\sum_{i=1}^{n} u_i(v,x,y,z)}\right)dv$$

$$= \lim_{\epsilon\to 0^+} \sum_{V=V_{(\lambda_1,\ldots,\lambda_r)}} \frac{(2\pi)^{2rn}}{n!} \frac{(x+y)^n}{x^n y^n} \mathrm{JKRes}^{\Lambda_V}\left(G_V(u(v,x,y,z),x,y,z)e^{\epsilon\sum_{i=1}^{n} u_i(v,x,y,z)}\right)dv$$

$$= \sum_{V=V_{(\lambda_1,\ldots,\lambda_r)}} \frac{(2\pi)^{2rn}}{n!} \frac{(x+y)^n}{x^n y^n} G'_V(p_V(x,y,z),x,y,z)$$

$$(5.14) \qquad = \sum_{(Y_{\lambda_1},\ldots,Y_{\lambda_r})} (2\pi)^{2rn} \prod_{1\le k\ne l\le r} H_{\lambda_k,\lambda_l}(x,y,z_k-z_l) \lim_{\zeta\to 0} \prod_{k=1}^{r} H_{\lambda_k,\lambda_k}(x,y,\zeta),$$

where the last sum is over unlabeled tuples of Young diagrams $(Y_{\lambda_1},\ldots,Y_{\lambda_r})$ such that $n=\sum_{k=1}^{r} n_k$ and $\lambda_k \vdash n_k$ for all $k=1,\ldots,r$. The following lemma concludes the proof of the theorem. $\square$

Lemma 5.23. *For any two partition δ, η all linear terms in the nominator of $H_{\delta,\eta}(x,y,\zeta)$ cancel out such that we get*

$$H_{\delta,\eta}(x,y,\zeta) = K_{\delta,\eta}(x,y,\zeta)^{-1}.$$

Proof. We prove it first in the special case when partitions $\delta=(\delta_1,\ldots,\delta_k)$ and $\eta=(\eta_1,\ldots,\eta_l)$ have Young diagrams Y_δ and Y_η of rectangular shape, that is, $\delta_1=\ldots=\delta_k$ and $\eta_1=\ldots=\eta_l$. We introduce notation $w_\delta=k$, $w_\eta=l$, $h_\delta=\delta_1$ and $h_\eta=\eta_1$. Recall that c_i and r_i denote the column and the row of the box $i\in Y_\delta$.

If $Y_\delta=\emptyset$ then

$$\begin{aligned} H_{\delta,\eta}(x,y,\zeta)^{-1} &= \prod_{j\in Y_\eta}[c_j x + r_j y+\zeta] = \prod_{\substack{1\le c_j\le w_\eta\\ 1\le r_j\le h_\eta}} [c_j x + r_j y+\zeta] \\ &= \prod_{\substack{1\le c_j\le w_\eta\\ 1\le r_j\le h_\eta}} [(w_\eta - c_j+1)x-(-r_j)y+\zeta] = \prod_{j\in Y_\eta}[(A_\eta(j)+1)x - L_\delta(j)y+\zeta]. \end{aligned}$$

If $Y_\eta=\emptyset$ then

$$\begin{aligned} H_{\delta,\eta}(x,y,\zeta)^{-1} &= \prod_{i\in Y_\delta}[(1-c_i)x+(1-r_i)y+\zeta] = \prod_{\substack{1\le c_i\le w_\delta\\ 1\le r_i\le h_\delta}} [(1-c_i)x+(1-r_i)y+\zeta] \\ &= \prod_{\substack{1\le c_i\le w_\delta\\ 1\le r_i\le h_\delta}} [-(w_\delta-c_i)x+(1-r_i)y+\zeta] = \prod_{i\in Y_\delta}[-A_\delta(i)x+(L_\eta(i)+1)y+\zeta]. \end{aligned}$$

Assume that $Y_\delta, Y_\eta \neq \emptyset$. To simplify $H_{\delta,\eta}(x,y,\zeta)$ first we compute

$$\prod_{\substack{1\le c_j\le w_\eta\\ 1\le r_i\le h_\delta}} \frac{[(c_j-c_i)x+(r_j-r_i)y+\zeta][(1+c_j-c_i)x+(1+r_j-r_i)y+\zeta]}{[(1+c_j-c_i)x+(r_j-r_i)y+\zeta][(c_j-c_i)x+(1+r_j-r_i)y+\zeta]}$$

$$= \prod_{1\le r_i\le h_\delta} \frac{[(1-c_i)x+(r_j-r_i)y+\zeta][(1+w_\eta-c_i)x+(1+r_j-r_i)y+\zeta]}{[(1+w_\eta-c_i)x+(r_j-r_i)y+\zeta][(1-c_i)x+(1+r_j-r_i)y+\zeta]}$$

$$= \frac{[(1-c_i)x+(r_j-h_\delta)y+\zeta][(1+w_\eta-c_i)x+r_jy+\zeta]}{[(1+w_\eta-c_i)x+(r_j-h_\delta)y+\zeta][(1-c_i)x+r_jy+\zeta]},$$

hence

$$\begin{aligned} H_{\delta,\eta}(x,y,\zeta) &= \prod_{\substack{1\le c_i\le w_\delta\\ 1\le r_j\le h_\eta}} \frac{[(1-c_i)x+(r_j-h_\delta)y+\zeta][(1+w_\eta-c_i)x+r_jy+\zeta]}{[(1+w_\eta-c_i)x+(r_j-h_\delta)y+\zeta][(1-c_i)x+r_jy+\zeta]} \\ &\quad \cdot \frac{1}{\prod\limits_{\substack{1\le c_i\le w_\delta\\ 1\le r_i\le h_\delta}} [(1-c_i)x+(1-r_i)y+\zeta] \prod\limits_{\substack{1\le c_j\le w_\eta\\ 1\le r_j\le h_\eta}} [c_jx+r_jy+\zeta]} \\ &= \frac{\prod\limits_{\substack{1-w_\delta\le\gamma\le 0\\ 1-h_\delta\le\rho\le h_\eta-h_\delta}} [\gamma x+\rho y+\zeta] \prod\limits_{\substack{1+w_\eta-w_\delta\le\gamma\le w_\eta\\ 1\le\rho\le h_\eta}} [\gamma x+\rho y+\zeta]}{\prod\limits_{\substack{1+w_\eta-w_\delta\le\gamma\le w_\eta\\ 1-h_\delta\le\rho\le h_\eta-h_\delta}} [\gamma x+\rho y+\zeta] \prod\limits_{\substack{1-w_\delta\le\gamma\le 0\\ 1\le\rho\le h_\eta}} [\gamma x+\rho y+\zeta]} \\ &\quad \cdot \frac{1}{\prod\limits_{\substack{1-w_\delta\le\gamma\le 0\\ 1-h_\delta\le\rho\le 0}} [\gamma x+\rho y+\zeta] \prod\limits_{\substack{1\le\gamma\le w_\eta\\ 1\le\rho\le h_\eta}} [\gamma x+\rho y+\zeta]}. \end{aligned}$$

To further simplify the last fraction, denote $\boldsymbol{\chi}(E)$ the characteristic function of the set $E\cap\mathbb{Z}^2$ and we distinguish two cases.

Case I. $w_\delta \le w_\eta$. Then the fraction $H_{\delta,\eta}(x,y,\zeta)$ can be encoded in the function

$$\begin{gathered} \boldsymbol{\chi}([1-w_\delta,\,0]\times[1-h_\delta,\,h_\eta-h_\delta]) + \boldsymbol{\chi}([1+w_\eta-w_\delta,\,w_\eta]\times[1,\,h_\eta]) \\ -\boldsymbol{\chi}([1+w_\eta-w_\delta,\,w_\eta]\times[1-h_\delta,\,h_\eta-h_\delta]) - \boldsymbol{\chi}([1-w_\delta,\,0]\times[1,\,h_\eta]) \\ -\boldsymbol{\chi}([1-w_\delta,\,0]\times[1-h_\delta,\,0]) - \boldsymbol{\chi}([1,\,w_\eta]\times[1,\,h_\eta]) \\ = \boldsymbol{\chi}([1-w_\delta,\,0]\times[1-h_\delta,\,h_\eta-h_\delta]) - \boldsymbol{\chi}([1-w_\delta,\,0]\times[1,\,h_\eta]) - \boldsymbol{\chi}([1-w_\delta,\,0]\times[1-h_\delta,\,0]) \\ +\boldsymbol{\chi}([1+w_\eta-w_\delta,\,w_\eta]\times[1,\,h_\eta]) - \boldsymbol{\chi}([1,\,w_\eta]\times[1,\,h_\eta]) \\ -\boldsymbol{\chi}([1+w_\eta-w_\delta,\,w_\eta]\times[1-h_\delta,\,h_\eta-h_\delta]) \\ = -\boldsymbol{\chi}([1-w_\delta,\,0]\times[1+h_\eta-h_\delta,\,h_\eta]) - \boldsymbol{\chi}([1,\,w_\eta-w_\delta]\times[1,\,h_\eta]) \\ -\boldsymbol{\chi}([1+w_\eta-w_\delta,\,w_\eta]\times[1-h_\delta,\,h_\eta-h_\delta]). \end{gathered}$$

Now we decode the latter function to get the following simplication

$$
\begin{aligned}
H_{\delta,\eta}(x,y,\zeta)^{-1} = &\prod_{\substack{1-w_\delta\le\gamma\le 0\\ 1+h_\eta-h_\delta\le\rho\le h_\eta}}[\gamma x+\rho y+\zeta]\prod_{\substack{1+w_\eta-w_\delta\le\gamma\le w_\eta\\ 1-h_\delta\le\rho\le h_\eta-h_\delta}}[\gamma x+\rho y+\zeta]\prod_{\substack{1\le\gamma\le w_\eta-w_\delta\\ 1\le\rho\le h_\eta}}[\gamma x+\rho y+\zeta]\\
= &\prod_{\substack{1\le c_i\le w_\delta\\ 1\le r_i\le h_\delta}}[-(w_\delta-c_i)x+(h_\eta-r_i+1)y+\zeta]\prod_{\substack{1\le c_j\le w_\delta\\ 1\le r_j\le h_\eta}}[(w_\eta-c_j+1)x-(h_\delta-r_j)y+\zeta]\\
&\prod_{\substack{w_\delta<c_j\le w_\eta\\ 1\le r_j\le h_\eta}}[(w_\eta-c_j+1)x-(-r_j)y+\zeta]\\
= &\prod_{i\in Y_\delta}[-A_\delta(i)x+(L_\eta(i)+1)y+\zeta]\prod_{j\in Y_\eta}[(A_\eta(j)+1)x-L_\delta(j)y+\zeta].
\end{aligned}
$$

Case II. $w_\eta<w_\delta$. Similarly, we encode the fraction $H_{\delta,\eta}(x,y,\zeta)$ in the following function and we compute

$$
\begin{aligned}
&\boldsymbol{\chi}([1-w_\delta,\,0]\times[1-h_\delta,\,h_\eta-h_\delta])+\boldsymbol{\chi}([1+w_\eta-w_\delta,\,w_\eta]\times[1,\,h_\eta])\\
&-\boldsymbol{\chi}([1+w_\eta-w_\delta,\,w_\eta]\times[1-h_\delta,\,h_\eta-h_\delta])-\boldsymbol{\chi}([1-w_\delta,\,0]\times[1,\,h_\eta])\\
&-\boldsymbol{\chi}([1-w_\delta,\,0]\times[1-h_\delta,\,0])-\boldsymbol{\chi}([1,\,w_\eta]\times[1,\,h_\eta])\\
=&\,\boldsymbol{\chi}([1-w_\delta,\,0]\times[1-h_\delta,\,h_\eta-h_\delta])-\boldsymbol{\chi}([1-w_\delta,\,w_\eta-w_\delta]\times[1,\,h_\eta])\\
&-\boldsymbol{\chi}([1+w_\eta-w_\delta,\,w_\eta]\times[1-h_\delta,\,h_\eta-h_\delta])-\boldsymbol{\chi}([1-w_\delta,\,0]\times[1-h_\delta,\,0])\\
=&\,\big(\boldsymbol{\chi}([1-w_\delta,\,w_\eta-w_\delta]\times[1-h_\delta,\,h_\eta-h_\delta])+\boldsymbol{\chi}([1+w_\eta-w_\delta,\,0]\times[1-h_\delta,\,h_\eta-h_\delta])\big)\\
&-\boldsymbol{\chi}([1-w_\delta,\,w_\eta-w_\delta]\times[1,\,h_\eta])\\
&+\big(-\boldsymbol{\chi}\big([1+w_\eta-w_\delta,\,0]\times[1-h_\delta,\,h_\eta-h_\delta]\big)-\boldsymbol{\chi}([1,\,w_\eta]\times[1-h_\delta,\,h_\eta-h_\delta])\big)\\
&+\big(-\boldsymbol{\chi}\big([1-w_\delta,\,w_\eta-w_\delta]\times[1-h_\delta,\,0])\big)-\boldsymbol{\chi}([1+w_\eta-w_\delta,\,0]\times[1-h_\delta,\,0])\big)\\
=&\,\boldsymbol{\chi}([1-w_\delta,\,w_\eta-w_\delta]\times[1-h_\delta,\,h_\eta-h_\delta])-\boldsymbol{\chi}([1-w_\delta,\,w_\eta-w_\delta]\times[1-h_\delta,\,h_\eta])\\
&-\boldsymbol{\chi}([1,\,w_\eta]\times[1-h_\delta,\,h_\eta-h_\delta])-\boldsymbol{\chi}([1+w_\eta-w_\delta,\,0]\times[1-h_\delta,\,0])\\
=&\,-\boldsymbol{\chi}([1-w_\delta,\,w_\eta-w_\delta]\times[1+h_\eta-h_\delta,\,h_\eta])-\boldsymbol{\chi}([1,\,w_\eta]\times[1-h_\delta,\,h_\eta-h_\delta])\\
&-\boldsymbol{\chi}([1+w_\eta-w_\delta,\,0]\times[1-h_\delta,\,0])\,.
\end{aligned}
$$

Again, decoding the result we get the simplification

$$
\begin{aligned}
H_{\delta,\eta}(x,y,\zeta)^{-1} = &\prod_{\substack{1-w_\delta\le\gamma\le w_\eta-w_\delta\\ 1+h_\eta-h_\delta\le\rho\le h_\eta}}[\gamma x+\rho y+\zeta]\prod_{\substack{1\le\gamma\le w_\eta\\ 1-h_\delta\le\rho\le h_\eta-h_\delta}}[\gamma x+\rho y+\zeta]\prod_{\substack{1+w_\eta-w_\delta\le\gamma\le 0\\ 1-h_\delta\le\rho\le 0}}[\gamma x+\rho y+\zeta]\\
= &\prod_{\substack{1\le c_i\le w_\eta\\ 1\le r_i\le h_\delta}}[-(w_\delta-c_i)x+(h_\eta-r_i+1)y+\zeta]\prod_{\substack{1\le c_j\le w_\eta\\ 1\le r_j\le h_\eta}}[(w_\eta-c_j+1)x-(h_\delta-r_j)y+\zeta]\\
&\prod_{\substack{w_\eta<c_i\le w_\delta\\ 1\le r_i\le h_\delta}}[-(w_\delta-c_i)x+(-r_i+1)y+\zeta]\\
= &\prod_{i\in Y_\delta}[-A_\delta(i)x+(L_\eta(i)+1)y+\zeta]\prod_{j\in Y_\eta}[(A_\eta(j)+1)x-L_\delta(j)y+\zeta].
\end{aligned}
$$

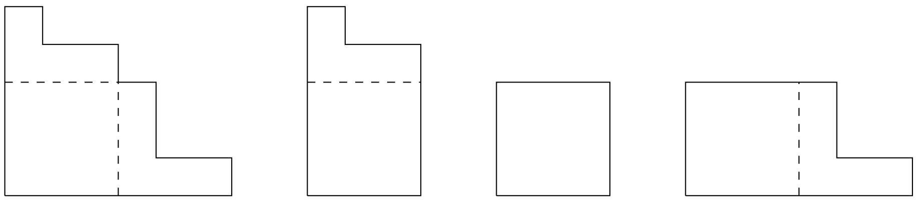
We have showed that

$$H_{\delta,\eta}(x,y,\zeta) = K_{\delta,\eta}(x,y,\zeta)^{-1} \tag{5.15}$$

for partitions δ and η with rectangular shape Young diagrams.

Let $\delta = (\delta_1, \dots, \delta_k)$ and η be two partitions and assume that the Young diagram of δ is not of rectangular shape, i.e. there is m such that $\delta_m > \delta_{m+1}$. Then we can define three new partitions

$$\begin{aligned}
\delta^1 &= (\delta_1, \dots, \delta_m),\\
\delta^{1,2} &= \big(\underbrace{\delta_{m+1}, \dots, \delta_{m+1}}_{m \text{ times}}\big),\\
\delta^2 &= \big(\underbrace{\delta_{m+1}, \dots, \delta_{m+1}}_{m+1 \text{ times}}, \delta_{m+2}, \dots, \delta_k\big).
\end{aligned}$$



Y_δ $\quad$ Y_{δ^1} $\quad$ $Y_{\delta^{1,2}}$ $\quad$ Y_{δ^2}

Formally, we have the inclusion-exclusion formula $Y_\delta = Y_{\delta^1} - Y_{\delta^{1,2}} + Y_{\delta^2}$. From (5.12) follows that

$$H_{\delta,\eta} = H_{\delta^1,\eta}\left(H_{\delta^{1,2},\eta}\right)^{-1} H_{\delta^2,\eta}, \tag{5.16}$$

$$H_{\eta,\delta} = H_{\eta,\delta^1}\left(H_{\eta,\delta^{1,2}}\right)^{-1} H_{\eta,\delta^2}. \tag{5.17}$$

$K_{\delta,\eta}$ has the same properties, more precisely

$$K_{\delta,\eta} = K_{\delta^1,\eta}\left(K_{\delta^{1,2},\eta}\right)^{-1} K_{\delta^2,\eta}, \tag{5.18}$$

$$K_{\eta,\delta} = K_{\eta,\delta^1}\left(K_{\eta,\delta^{1,2}}\right)^{-1} K_{\eta,\delta^2}. \tag{5.19}$$

Indeed, (5.18) follows from

$$\begin{aligned}
K_{\delta^1,\eta}(x,y,\zeta) &= \prod_{i\in Y_{\delta^1}} [-A_{\delta^1}(i)x + (L_\eta(i)+1)y + \zeta] \prod_{j\in Y_\eta} [(A_\eta(j)+1)x - L_{\delta^1}(j)y + \zeta]\\
&= \prod_{i\in Y_{\delta^1}\setminus Y_{\delta^{1,2}}} [-A_\delta(i)x + (L_\eta(i)+1)y + \zeta] \prod_{i\in Y_{\delta^{1,2}}} [-A_{\delta^{1,2}}(i)x + (L_\eta(i)+1)y + \zeta]\\
&\quad \prod_{\substack{j\in Y_\eta\\ c_j\le m}} [(A_\eta(j)+1)x - L_\delta(j)y + \zeta] \prod_{\substack{j\in Y_\eta\\ c_j> m}} [(A_\eta(j)+1)x - L_{\delta^{1,2}}(j)y + \zeta]
\end{aligned}$$

and

$$\begin{aligned} K_{\delta^2,\eta}(x,y,\zeta) = & \prod_{i\in Y_{\delta^2}} [-A_{\delta^2}(i)x + (L_\eta(i)+1)y+\zeta] \prod_{j\in Y_\eta} [(A_\eta(j)+1)x - L_{\delta^2}(j)y+\zeta] \\ = & \prod_{i\in Y_{\delta^2}} [-A_{\delta}(i)x + (L_\eta(i)+1)y+\zeta] \\ & \prod_{\substack{j\in Y_\eta\\ c_j\le m}} [(A_\eta(j)+1)x - L_{\delta^{1,2}}(j)y+\zeta] \prod_{\substack{j\in Y_\eta\\ c_j> m}} [(A_\eta(j)+1)x - L_{\delta}(j)y+\zeta]. \end{aligned}$$

By a similar computation follows (5.19).

Any two partitions δ and η can be written as

$$\delta = \big(\underbrace{\delta_1,\ldots,\delta_1}_{m_1},\ldots,\underbrace{\delta_k,\ldots,\delta_k}_{m_k}\big), \qquad \delta_1>\ldots>\delta_k,$$

$$\eta = \big(\underbrace{\eta_1,\ldots,\eta_1}_{o_1},\ldots,\underbrace{\eta_l,\ldots,\eta_l}_{o_l}\big), \qquad \eta_1>\ldots>\eta_l.$$

Then we define the following partitions with rectangular shape Young diagrams

$$\delta^i = \big(\underbrace{\delta_i,\ldots,\delta_i}_{m_1+\ldots+m_i}\big), \qquad \forall\, i=1,\ldots,k,$$

$$\delta^{i,i+1} = \big(\underbrace{\delta_{i+1},\ldots,\delta_{i+1}}_{m_1+\ldots+m_i}\big), \qquad \forall\, i=1,\ldots,k-1,$$

and

$$\eta^j = \big(\underbrace{\eta_j,\ldots,\eta_j}_{o_1+\ldots+o_j}\big), \qquad \forall\, j=1,\ldots,l,$$

$$\eta^{j,j+1} = \big(\underbrace{\eta_{j+1},\ldots,\eta_{j+1}}_{o_1+\ldots+o_j}\big), \qquad \forall\, j=1,\ldots,l-1.$$

We get the following formal inclusion-exclusion formulas

$$Y_\delta = Y_{\delta^1} - Y_{\delta^{1,2}} + Y_{\delta^2} - \ldots - Y_{\delta^{k-1,k}} + Y_{\delta^k}, \tag{5.20}$$

$$Y_\eta = Y_{\eta^1} - Y_{\eta^{1,2}} + Y_{\eta^2} - \ldots - Y_{\eta^{l-1,l}} + Y_{\eta^l}. \tag{5.21}$$

Finally, we compute

$$\begin{aligned} H_{\delta,\eta} &= H_{\delta^1,\eta}\left(H_{\delta^{1,2},\eta}\right)^{-1} H_{\delta^2,\eta}\cdots\left(H_{\delta^{k-1,k},\eta}\right)^{-1} H_{\delta^k,\eta} && \text{by (5.16), (5.20)}\\ &= \prod_{i=1}^{k}\prod_{j=1}^{k-1}\prod_{p=1}^{l}\prod_{q=1}^{l-1} H_{\delta^i,\eta^p}\left(H_{\delta^i,\eta^{q,q+1}}\right)^{-1}\left(H_{\delta^{j,j+1},\eta^p}\right)^{-1} H_{\delta^{j,j+1},\eta^{q,q+1}} && \text{by (5.17), (5.21)}\\ &= \prod_{i=1}^{k}\prod_{j=1}^{k-1}\prod_{p=1}^{l}\prod_{q=1}^{l-1} \left(K_{\delta^i,\eta^p}\right)^{-1} K_{\delta^i,\eta^{q,q+1}} K_{\delta^{j,j+1},\eta^p}\left(K_{\delta^{j,j+1},\eta^{q,q+1}}\right)^{-1} && \text{by (5.15)}\\ &= \left(K_{\delta,\eta}\right)^{-1}, \end{aligned}$$

by (5.18), (5.19), (5.20) and (5.21). □

References

[1] M.F. Atiyah, *Convexity and commuting Hamiltonians*, Bull. London Math. Soc. 14 (1982), pp. 1-15.

[2] M.F. Atiyah, R. Bott, *The moment map and equivariant cohomology*, Topology, 23(1) (1984), pp. 1-28.

[3] N. Berline, M. Vergne, *Classes caractéristiques équivariantes. Formule de localisation en cohomologie équivariante,* C. R. Acad. Sci. Paris Sér. I Math., 295(9) (1982), pp. 539-541.

[4] R. Bott, L. W. Tu, *Differential forms in algebraic topology.* Springer, 1995.

[5] R. Bott, L. W. Tu, *Equivariant characteristic classes in the Cartan model*, arXiv preprint math/0102001 (2001).

[6] M. Brion, M. Vergne, *Arrangements of hyperplane. I. Rational functions and the Jeffrey-Kirwan residue*, Annales scientifiques de l'É.N.S. 4e série, tome 32, n° 5 (1999), pp. 715 - 741.

[7] A. Cannas da Silva, *Lectures on symplectic geometry.* Lecture Notes in Mathematics 1764, Springer-Verlag 2001.

[8] J.J. Duistermaat and G.J. Heckman, *On the variation in the cohomology of the symplectic form of the reduced phase space*, Invent. Math., 69(2) (1982), pp. 259-268.

[9] J.J. Duistermaat, J.A.C. Kolk, *Lie groups.* Springer, 1999.

[10] G. Ellingsrud and S.A. Strømme, *On the homology of the Hilbert scheme of points in the plane*, Invent. Math. 87 (1987), pp. 343-352.

[11] V.A.Ginzburg, *Equivariant cohomologies and Kählers geometry*, Functional Anal. Appl. 21 (1987), no. 4, pp. 271-283.

[12] R.F. Goldin *An effective algorithm for the cohomology ring of symplectic reductions*, Geometric & Functional Analysis GAFA 12 (2002), pp. 567-583.

[13] V. Guillemin, J. Kalkman, *The Jeffrey-Kirwan localization theorem and residue operations in equivariant cohomology*, J. reine angew. Math. 470 (1996), pp. 123-142.

[14] V. Guillemin, Y. Karshon, V.L. Ginzburg, *Moment maps, cobordisms, and Hamiltonian group actions*. Vol. 98. AMS, 2002.

[15] V. Guillemin, S. Sternberg, *Supersymmetry and equivariant de Rham theory.* Vol. 2. Springer, 1999.

[16] V. Guillemin, S. Sternberg, *Convexity properties of the moment mapping I.*, Invent. Math. 67 (1982), pp. 491-513.

[17] V. Guillemin, S. Sternberg, *Symplectic techniques in physics.* Cambridge University Press, 1984.

[18] M. Harada, N. Proudfoot, *Properties of the residual circle action on a hypertoric variety*, Pacific J. Math 214.2 (2004), pp. 263-284.

[19] T. Hausel, N. Proudfoot, *Abelianization for hyperkähler quotients*, Topology 44(1) (2005), pp. 231-248.

[20] T. Hausel, B. Sturmfels, *Toric hyperkhler varieties*, Doc. Math. 7 (2002), pp. 495-534.

[21] G. Heckman, *Symplectic Geometry*, (lecture note).

[22] N. J. Hitchin, A. Karlhede, U. Lindström, M. Roček, *Hyperkähler metrics and supersymmetry*, Communications in Mathematical Physics 108 (1987), pp. 535-589.

[23] L.C. Jeffrey, F.C. Kirwan, *Localization for nonabelian group actions*, Topology 34 (1995), no. 2, pp. 291 - 327.

[24] L.C. Jeffrey, Frances C. Kirwan, *Localization and the quantization conjecture*, Topology, Volume 36, No. 3, pp. 647 - 693, (1997).

[25] L.C. Jeffrey, M. Kogan, *Localization theorems by symplectic cuts*, in "The breadth of symplectic and Poisson geometry" (2005), pp. 303-326.

[26] F.C. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, volume 31 of Mathematical Notes. Princeton University Press, Princeton, NJ, 1984.

[27] M. Lehn, C. Sorger. *Symmetric groups and the cup product on the cohomology of Hilbert schemes*, Duke Mathematical Journal 110.2 (2001), pp. 345-357.

[28] E. Lerman, *Symplectic cuts*, Math. Res. Lett 2.3 (1995), pp. 247-258.

[29] E. Lerman, E. Meinrenken, S. Tolman, C. Woodward, *Nonabelian convexity by symplectic cuts*, Topology 37 (1998), no. 2, pp. 245-259.

[30] E. Lerman, S. Tolman, *Hamiltonian torus actions on symplectic orbifolds and toric varieties*, Transactions of the American Mathematical Society 349 (1997), pp. 4201-4230.

[31] J. Martens, *Equivariant volumes of non-compact quotients and instanton counting*, Communications in Mathematical Physics 281.3 (2008), pp. 827-857.

[32] S. Martin, *Symplectic quotients by a nonabelian group and by its maximal torus*, arXiv preprint math/0001002 (2000).

[33] D. McDuff, D. Salamon, *Introduction to symplectic topology.* Oxford University Press, USA, 1999.

[34] E. Meinrenken, *Symplectic surgery and the Spin^c-Dirac operator*, Advances in mathematics 134.2 (1998), pp. 240-277

[35] G. Moore, N. Nekrasov, S. Shatashvili. *Integrating over Higgs branches*, Communications in Mathematical Physics 209.1 (2000), pp. 97-121.

[36] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces.* Vol. 18. AMS, 1999.

[37] H. Nakajima, K. Yoshioka, *Instanton counting on blow-up. I. 4-dimensional pure gauge theory*, Invent. Math. 162 (2005), pp. 313-355.

[38] N. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, Adv. Theor. Math. Phys. 7 (2003), pp. 831-864.

[39] L.I. Nicolaescu, *Lectures on the geometry of manifolds.* World Scientific, 2007.

[40] E. Prato, S. Wu. *Duistermaat-Heckman measures in a non-compact setting*, Compositio Math. 94.2 (1994), pp. 113-128.

[41] N. Proudfoot, *Hyperkähler Analogues of Kähler Quotients*, PhD Thesis, 2000.

[42] I. Satake. *The Gauss-Bonnet theorem for V-manifolds*, J. Math. Soc. Japan, 9 (1957), pp. 464-492.

[43] S. Tolman, J. Weitsman, *The cohomology rings of symplectic quotients.* Communications in Analysis and Geometry 11.4 (2003), pp. 751-774.

[44] E. Vasserot, *Sur l'anneau de cohomologie du schéma de Hilbert de* $\mathbb{C}^2$, Comptes rendus de l'Académie des sciences. Série 1, Mathématique 332.1 (2001), pp. 7-12.

[45] M. Vergne, *A note on the Jeffrey-Kirwan-Witten localisation formula*, Topology, 35(1) (1996), pp. 243-266.